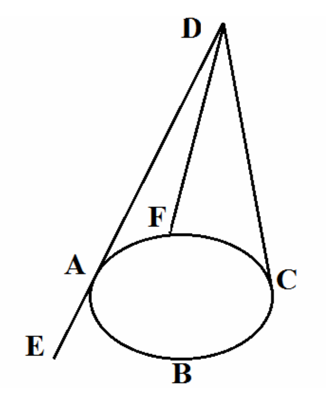
PROLEGOMENA TO CONIC SECTIONS.

A careful study of conic sections is going to be investigated, prompted by an affection for the subject and the extraordinary usefulness of these sections, however, to some extend, we approach the matter by changing from the deliberations of the ancients: and indeed these properties, which refer to several sections, will be attended to in common, since they will be involved likewise with a common demonstration, nor without good reason, with such an exacting account maintaining the teaching itself, with regard to order and elegance: but another goal restricts us, for the sake of those who are influenced by geometry, we intend to explain each one on its own account, where by an easier and less confused method, beginners may be encouraged and enticed to be introduced to the contemplation of the explorations of conics. For, as Federico Commandino has rightly said, if there shall be some branch of Mathematics which is unknown to our philosophers, it shall require the light of interpretation, that certainly may be called to be the case for conics: indeed any texts from the ancients should be treated with care, since the original records of these either have not come down to us at all, or thus will have scarcely arrived, on account of the injuries suffered from antiquity, and may only be understood with the greatest difficulty. But thus we will attempt to restore these conical matters which we have at hand uniformly, so that these may be able to be understood without difficulty, or as arranged lightly according to the elements of Euclid, without any tedium, indeed with the desire arising of being extended further, & these nocturnal studies required to be extended without end, nor indeed can the science of geometry be able to be contained within bounds, and much less able to be exhausted, since the depths shall be the most vast, an ocean absorbing all the speculations of whatever kind. And thus we will begin with these known accounts, with the ornamental parts omitted, which will indicate by their description the nomenclature & nature of the cone, or indeed which may be specified from the account to be established.

DEFINITION.

What shall become a cone?

I call the body a cone having the origin at a point at an elevated position from a plane, for a line traced out from the point stretched out indefinitely around the perimeter of a circle, which shall be different from the same point put in place in the plane.



Exposition.

To this end a certain circle ABC may be considered, beyond the plane of which some point D may be supposed to lie, from which the line AE may be drawn touching the perimeter of the circle at some point A: truly the line DE may be moved around the whole periphery of the circle ABC until it shall return to A, yet with this condition that the point D may remain fixed, that is, in order that the end point of the line DA, which is the start of the revolution applied from the point

D, shall never depart from that, I say for such a revolution of the line DE, the figure DABC will arise having two surfaces, formed by a single motion of the right line DE, the other truly the circle ABC. It is customary to call the first the surface of the cone besides the following base, moreover that to be called the base of the cone; the point D is called the vertex of the cone, the axis truly the line from the vertex D to the centre of the base.

From this description of the cone, it follows to be able to draw a right line from some point designated on the surface of the cone to the vertex D itself, since indeed the circular course made by the right line DA around the base ABC, thus produced the surface of the cone, it is necessary that from some point of the perimeter ABC it shall be able to draw a right line to the vertex D.

Hence I believe it to be unnecessary to demonstrate this result because all the right lines shall be lines drawn from the vertex to some point on the surface of the cone, and hence to be evident enough the cone requiring to be constructed by that account, every section which shall be through the vertex, to show a triangle the sides of which shall lie on the surface of the cone.

I suppose likewise to be known equally, every line assumed drawn from some point on the surface of the cone assumed drawn to some other point, because in the case of a right line it can only be extended through the vertex, and to pass through the interior of the cone itself it would have to be produced indefinitely on both sides, it may have pleased to permit all of this to be shown by Apollonius : the demonstrations of which if the reader may desire, you will be able to recognize in the same.

In this location several other noteworthy ways occur which are able to describe a cone ; for example from that same point D, by drawing an elliptical figure at a higher position around the right line AD, or some other from the conics : but not without reason for the circle alone to be used by the ancients, since that, as will become apparent from the discourse of the book, the figure shall be more convenient for demonstrating the required form : add to this also, so that the figure of another cone may be supposed for the formation of the first cone; from which hardly any need arises except for the ellipse, which as Serenus showed correctly, can be excluded from the cylinder also.

How many kinds of cone shall arise.

The cone is of two kinds; either isosceles, or as if with equal sides, which is called a right cone ; otherwise it is called a scalene cone. If a line dropped normally from the vertex of the cone shall reach the centre of the base, then it is called an isosceles cone. But every cone of that other kind is called scalene, which with the perpendicular drawn from the vertex to the circular base, has the point, which is called the middle of the cone, put in place beyond the centre, also if there is a need for the point to be extended indefinitely.

From which also people call a cone right which is isosceles, and oblique if the cone is scalene: and truly distinguishes especially between each kind in that situation, since you may distinguish the right cone by the axis drawn to the plane: Isosceles cones will result in parts of the triangle divided by the axis, for which the parts of each become equal and similar to each other : moreover a scalene cone varied indefinitely gives rise to the sections returned by the axis, which shall neither be similar nor equal to each other ;

concerning which we present these following propositions. To which you ought to refer before you proceed, Dear Reader.

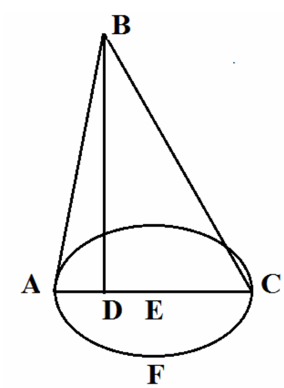
NOTE

Most of the theorems, which we bring forth in this introduction, also were proposed and demonstrated by Serenus of Antinoupolis: *Philosophical and Geometrical Works*, *Book* II, *Concerning the Section of a Cone*. Everything arising from the same material treated by Serenus has disappeared from my memory: for now it shall be twenty years or more since I have read that author. And thus I may wish that anyone who may read this work may not consider me guilty of plagiarism; but I may be thought, if perhaps certain things occurred to me in common with Serenus, to have reached the same conclusions as occurred to Serenus formerly.

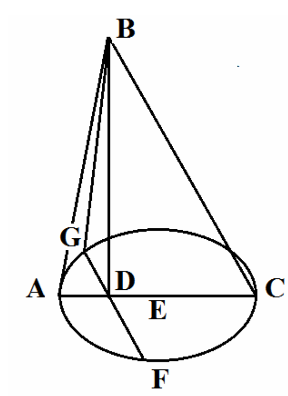
FIRST PROPOSITION.

In a scalene cone, if a perpendicular line may be dropped from the vertex, & it may act through the vertex, the centre of the base, and a point denoted by the perpendicular to the plane.
I say the triangle made by this section to contain the maximum & minimum of the lines which can be shown on the conical surface.

Demonstration.

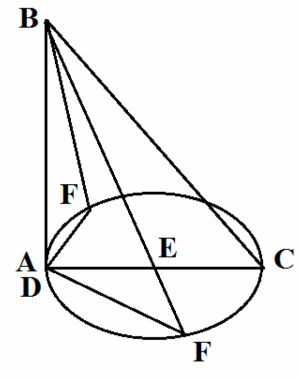


This case is threefold: either the point denoted by the perpendicular may be present within the periphery of the base, on the periphery itself, or finally outside the circle itself. Therefore for the first case the cone ABC may be put in place & the base centre E: moreover the vertex B: from which with the perpendicular dropped, it may meet with the base within the perimeter at the point D, through which point D, & and through the centre E, a plane may be extended as far as the vertex of the cone B ; this section of the triangle ABC will be shown: for from the construction of the cone BA, BC are right lines; and thus AB is required to be shown to be the minimum of all the lines, BC truly the maximum, which are present on the conical surface.



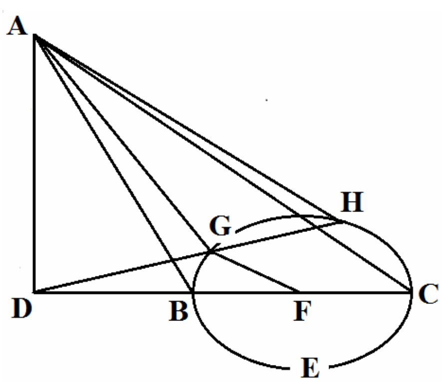
With some line BG drawn, the right line GDF may be put in place from G, since ADC passes through the centre, AD will be smaller than GD [§.7.*3*rd], & the square AD, smaller than the square GD, and by adding the common square DB: AD and DB, will be smaller than the squares GD, DB. But the angle GDB is right [*def*.3.*11*th], just as the angle ADB: therefore the square

AB is equal to the squares AD, DB[§.47.*1st*], truly the square GB is equal to the squares GD, DB; therefore the square AB, is smaller than the square GB. Therefore the line AB is smaller than the line BG.



With the same agreed on the right line BC will be shown to be the maximum of all the lines BC: indeed from whatever point F, if some line FB may be drawn: & FD may be placed through D, since ADC passes through the centre, DC [§.7.*3rd*] is greater than EF; from which the squares CD, DB are greater than the squares FD, DB, but since again the angles CDB, FDB shall be right, the square CB is equal to the squares CD, DB, & and the square FB is equal to the squares FD, DB. Therefore the square CB is greater than the square FB, and thus the right line CB is greater than the right line FB, from which BC is the maximum of these which can be assigned to the cone ; AB truly the minimum. Which was required to be demonstrated.

Truly now the perpendicular BD dropped from the vertex of the cone may fall on the perimeter of the circle AFC;
I say again AB to be the minimum line, & BC the maximum of these which can be present on the surface of the cone.
For some line BF may be drawn, and AF may be joined, therefore again the square BF is equal to the squares [§.47.*1st*] AF, AB, and thus is greater than the square AB. From which the line BF is greater than the right line AB: in addition it is clear the right line BC to be the maximum of all the lines, for the square BC equal to the squares CA, AB is greater than the square BF equal to the squares FA, AB.

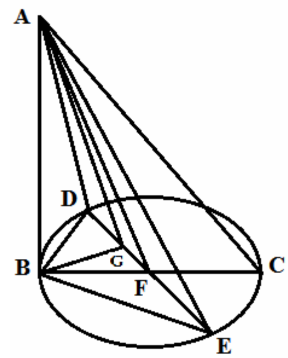


The remaining case is in which with a scalene cone ABC given, the line AD dropped perpendicularly to the plane BEC produced, designates a point D outside the cone itself D: where again I say in the case AB to be the minimum, AC truly the maximum. Indeed some right line AG may be drawn ; BD smaller than DG [§.8.*3rd*], therefore the square AG is equal to the squares GD, DA is greater than now some the square AB; therefore AB is smaller than AG, now some line AH, may be drawn; CD is greater than HD, therefore the square CA, equal to the squares CD, DA, is greater than the square HA [§.8.*3rd*], equal to the squares HD, DA. Therefore CA is greater than HA.
Therefore in the scalene cone, &c. which was required to be shown.

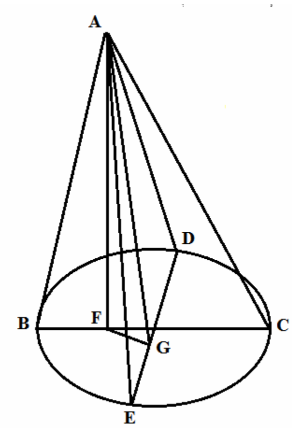
PROPOSITION II.

A scalene cone (from the vertex of which the perpendicular line to the base does not fall outside the base) may be cut through the centre and vertex, with the section showing a triangle, to the base of which a right line may be sent from the vertex of the cone. I say that to fall within the cone.

Demonstratio.



The case is twofold ; the first where the cone ABC, of which the base BDC, has the perpendicular BA erected from some perimeter point; while the other with the line drawn perpendicularly from the vertex to the base falls within the base. Therefore the section may be made through the centre F, and the apex A, therefore it will be required to show the normal presented to the triangle ADE sent from A to the base DE to fall within the cone, that is crossing the diameter DE between the points D & E. Indeed BD & BE may be drawn: then from the point B, BG may be put in place normal to the diameter DE (this by necessity falls between the points D and E) & AG may be joined, since AB from the hypothesis is normal to the base, the right line EB in base drawn to the same is normal by defin.3.11. : therefore the square AE is equal to the squares AB, BE: that is [§.47.*1*st] to the squares AB, BG, GE but the square AG, since the angle ABG also shall be right, is equal to the squares AB, BG , therefore the square AE exceeds the square AG, by the square GB; from which the square AE is equal to the squares AG, GE, therefore AGE is right, & AG perpendicular. And this angle of the squares AG, GE therefore the angle AGE is right, & AG perpendicular. And this has been drawn from the vertex of the cone & meets with the base of the triangle within the cone. Therefore the truth of the proposition had been evident in the first case.



The demonstration proceeds in the same way if in place of this AE, we may use the line AD.

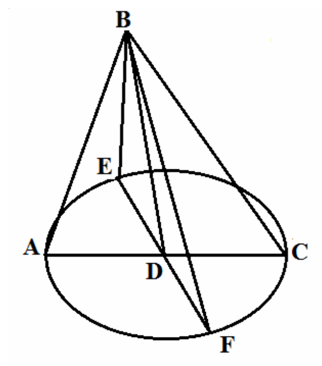
So that if the perpendicular sent from the vertex of the cone to the base of the cone may fall within the cone as AF, FG actually may be drawn normal to ED and AG may be joined. By remaining in the same plane we have demonstrated the square AE to equal the squares AG; EG ; and thus the angle AGE to be right, & AG perpendicular to the base of the triangle ADE, from which it is evident in this case the proposition is true.

PROPOSITION III.

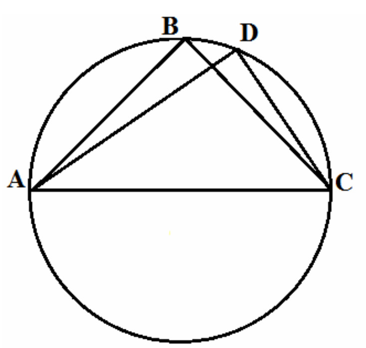
The squares of the sides of any triangle in a scalene cone made through the axis are equal to the squares of the sides of any other triangle through the same axis.

Demonstration.

Indeed ABC shall be a scalene cone of which the axis BD: through which some section BEF may be made: but another section may be made of the triangle ABC through the axis BD. I say the squares AB, BC, to be equal to the squares BF, BE ; for the squares AB, BC have been shown in the book that we have written concerning the powers of lines, to be equal to the squares AD, DC: & the square BD taken twice ; but also it has been shown by the discourse for the squares BF, BD to be equal to the squares FD, DE together with the square BD taken twice : therefore since the lines FE, AC are equal and the halves of these FD, DE, AD, CD shall be equal to each other, the truth of the demonstration is clear.



Corollary.



Hence the proof will be allowed to be deduced easily, required to be understood for the sides of the triangles which arise through the section made of the axis, and the proportions of these among themselves ; indeed for that semicircle ABC, the triangle shall be inscribed having the equal sides AB, BC for the largest triangle of any section of the cone through the axis.

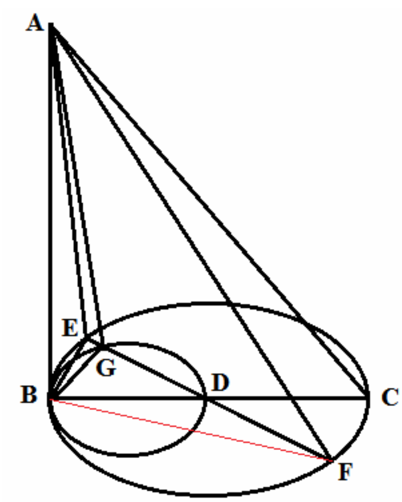
[*i.e.* this semicircle is normal to the base of the cone, and B corresponds to the vertex of the right cone, while D corresponds to the vertex of some other scalene cone.]

Therefore since the squares of the sides of the minimum triangle through the axis of the same cone will be able to be equal to the squares of the sides of the largest AB, BC, (that is to the square AC) & the smallest sides inflected into the semicircle, which shall be AD, DC. But if the third side shall become some section through the same axis, the proportion of the sides of that triangle, by necessity will be found in some one of the triangles which will have the base AC, and with the vertex at one of the points which are in the part of the perimeter BD. Indeed since the sum of the squares of the sides shall equal the sum of the squares of any another triangle, which may result from a section made through the axis, and the squares AB, BC of the triangle ABC shall be equal to the squares of the triangle with the maximum sides ; truly the squares AD, DC shall be equal to the squares of the smallest sides; it follows the sides of any other triangle, which are produced through the axis of the cone, must be found in some other triangles which may have the base AC and the remaining sides themselves arranged crosswise in some of the points which can be assigned to the arc BD, which will be elucidated more fully by running through the following proposition.

PROPOSITION IV.

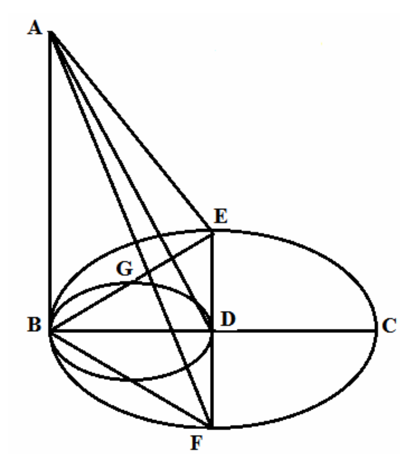
In a scalene cone a perpendicular sent from the vertex of which falls on the perimeter of the base, if some triangle may be shown by the section made through the axis.

I say that a normal be drawn from the vertex of the cone to the base of the triangle, which falls on the perimeter of the circle which may be described by the diameter, to be intercepted between the perpendicular drawn from the vertex to the base, and the centre of the same base.



Demonstration.

Therefore ABC shall be a cone, with the centre of the base D. From the vertex A of the cone ABC, the right line AB sent perpendicularly, shall fall at the point B of the perimeter of the base BEC; thence with the circle BGD described on the diameter BD, some plane AEF may be introduced through the axis, producing the triangle AEF; the base of which, in the first place, cuts the circle at the points D and G: the line AG may be drawn from the vertex point A; I say that line to be standing normally to the base EF ; for the right line BG may be drawn, and BF may be joined [not drawn in the original text]. Therefore, since by hypothesis, AB is at right angles to the plane of the base BE, CF. The angle ABF is right [§.11.*def.3*] : and thus the square AF is equal to the squares AB, BF; that is, since the angle BGF in the semicircle also is right, [this applies also] to the squares AB, BG, GF; but the square AG is equal to the squares AB, BG, where the angle ABG similarly shall be right. Therefore the square AF exceeds the square AG, by the square FG: whereby the square AF is equal to the squares AG, GF. From which the right line AG stands normally to GF itself at the point G, as in the perimeter of the circle having the diameter BD.



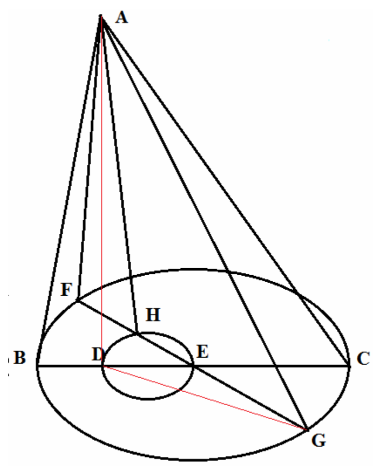
In the second case the base of the triangle EF shall touch the circle BDG at the point D; and AD may be drawn to the point of contact : since the angles BDE, BDF are right, the point of contact is with the centre of the base of the cone BDC; AE, AF shall be joined: since the angles BDE, BDF are right, & the side ED to the side DF, and BD itself is equal to the same, also the bases BE, BF are equal. Therefore the squares BE, BA are equal to the squares BF, BA. And the square FA is equal to the squares BF, BA, & the square EA to the squares BE, BA, since the right line AB shall be normal to the plane of the base, and thus the angles ABF, ABE to be right : therefore the squares FA, EA and hence also the right lines FA, EA are equal. Whereby since in the isosceles triangle FAE, AD bisects the base AD, this will be perpendicular to the base. Thus also in this case AD is perpendicular to the circle BGD.

In the third case if the triangle through the axis may pass through the normal AB drawn from the vertex to the base of the cone, and then its base shall be the same as with BDC: moreover for the perpendicular AB which is drawn from the vertex to the base of the cone. Whereby also in this third case the perpendicular to the base of the triangle will be present for the periphery of the circle; therefore in a scalene cone, &c. Q.f.d.

PROPOSITION V.

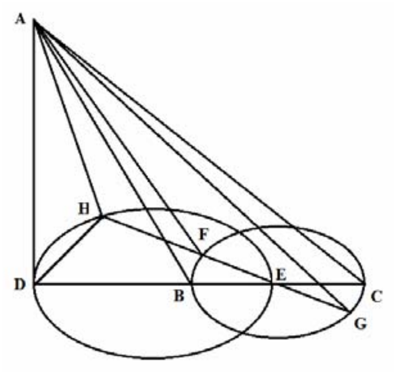
From the vertex of a scalene cone its perpendicular falls within the cone, if a section may show the triangle ABC through the axis, proceeding from the vertex normal to its base :

I say the diameter of that circle going to meet its perimeter will be equal to the line intercepted between the perpendicular from the vertex of the cone to the centre of the base of the same cone.



Demonstration.

Let the cone be ABC, from the vertex of which the perpendicular AD dropped shall fall within the cone, crossing the base at D: from which the right line DE may be drawn to the centre of the base, so that upon which the diameter of the circle DHE may be described: thereafter through the axis some section may be made shown by the triangle AFG: of which the base FG in the first place cuts the circle in E & H, then AH may be sent from the vertex A to the right line FG. I say that to be incident on the perimeter of the circle DHE, indeed to be drawn to the point H, in which the right line GE crosses the circle DH; and DG may be joined: therefore the square AG is equal to the squares AD, DG, that is to the squares AD, DH, HG: but the square AH is equal to the squares AD, DH: there the square AG exceeds the square AH by the square HG: therefore the square AG is equal to the squares AH, HG, and thus the right line AH is normal to the line FG. From which it is clear in this case all the normals from the vertex A fall to the bases of the triangles sent through the axis in the circle DHE.

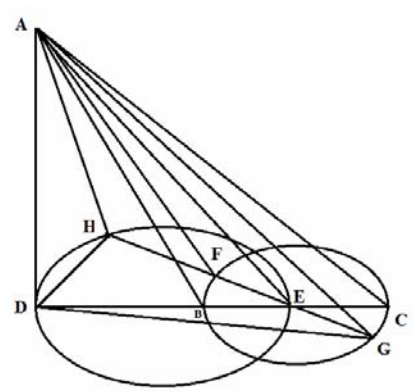


The same is demonstrated in the following third case when the base of the triangle either touches the circle DHE, or it falls at BEC, which were brought forth for the same cases proposed above ; therefore the scalene cone, &c. Q.f.d.

PROPOSITION VI.

If, in the scalene cone, the perpendicular drawn from the vertex to the base may fall outside the base, moreover some triangle may be produced by a section made through the axis: I say the normal drawn from the vertex to the base of the triangle, to be to the periphery of the circle, of which the diameter is the right line between the centre of the base of the cone and the perpendicular dropped from the vertex of the cone to the base joined together.

Demonstration.



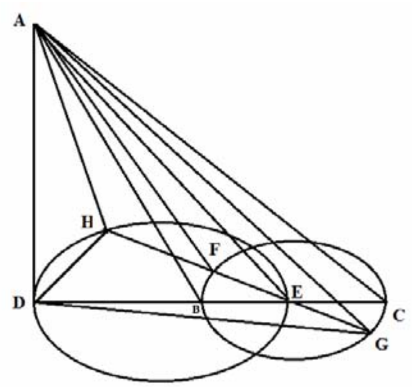
The perpendicular right line AD may fall beyond the base of the scalene cone ABC, and, passing through the centre of the base E, the circle DHE may be described around the right line DE ; some other triangle FAG may be put in place through the axis : and with GF produced as far as to the periphery of the circle DHE (for initially we put the base EF to cut the circle DHE), then the right line AH may be sent from the point A, to the point where the circle DHE is cut by the line GF. I say that to be normal to the right line GF produced, indeed DG, DH may be drawn : therefore the square AG is equal to the two squares AD, DG: (since AD is normal to the plane of the base produced, and thus also to DG), that is to the three squares AD, DH, HG, since DHE shall be the right angle in a semicircle ; but the square HA is equal to the squares AD, DH, since again the angle ADH is right; therefore the square AG will exceed the square AH by the square HG, from which the square AG is present, equal to the two squares AH, HG : whereby the angle AHG is right, and from the construction, the point H lies on the perimeter of the circle DHE; therefore it is clear in the first case from the construction, all the normals from the vertex of the scalene cone, for triangles through the base axis, to lie on the perimeter of the circle DHE.

In the second and third case when the base of the triangle either touches the circle DHE, or lies on the line DEC, the demonstration agrees with that, which we have brought forwards for the same cases in prop. 4.

Therefore in the scalene cone, &c. Q.f.d.

First Corollary.

Hence, it is apparent in every case where, from the vertex of a scalene cone for the base of the same brought forwards, a perpendicular dropped outside the base of the cone, is resolved into three different kinds of triangles being produced, which are drawn through the axis of the cone from the vertex normal to the base of the triangles, certain points of the circle DHE occur, which are outside the cone, with certain points put in place within, but with the number two themselves from the perimeter of the base, clearly I and K, which is apparent from the parts of the circle DHE: several of which are present outside the base, certainly indeed some within, truly the points I and K, which clearly belong to the parts of the circle DHE: of which several emerge outside the base, truly some inside, the points I and K, since by necessity they shall be present at the intersection of both the circles.



Second Corollary.

The perpendicular AD incident on the perimeter DHE is the maximum of all the perpendiculars at D; the point E, which likewise is the centre of the base of the cone, since indeed any other AH, which itself also is incident on the perimeter DHE, and DH may be joined : the square EA is equal to the squares ED, DA; but since ED is the diameter of the circle, some other right line EH may be drawn within the circle, and hence with the greater squares HD, DA, that is, equal to the square HA: Whereby it is evident also, that EA shall be greater than AD, since in the triangle EAD, EA will be required to be placed opposite the right angle, AD opposite the acute angle, then EA to be greater than any other, e.g. HA; thus we will show briefly also how ; since HA, from the hypothesis, is normal to the base GF of the triangle FAG, AHE is right, and thus the other angle AEH is acute. Whereby EA opposite the right angle is greater than HA which is placed opposite the acute angle.

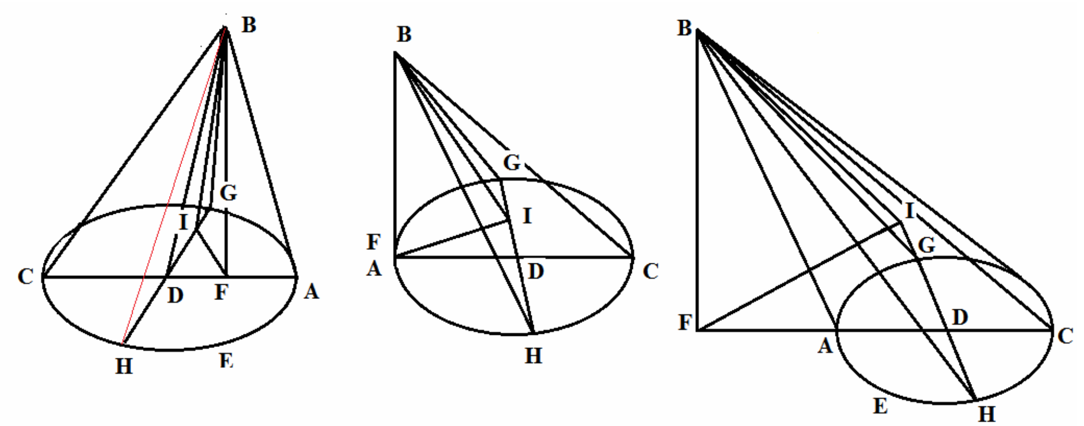
Truly the perpendicular which is incident at the point D, and hence it is likewise with the right line AD which is drawn normal from the vertex to the base of the cone, is the minimum of all the perpendiculars. For some other perpendicular may be drawn AH, (for which AD shall be less than EA, now it has been shown) this will be perpendicular also the periphery DHE: DH may be joined: in the triangle ADH, AD is placed opposite the acute angle AHD; AH truly opposite the right angle ADH, therefore AD is smaller than AH.

This corollary to be true also in the cases of the two preceding theorems, clearly will be approved by the same demonstration.

PROPOSITION VII.

In a given scalene cone, to show the minimum of the triangles made by a section through the axis.

Construction & demonstration.

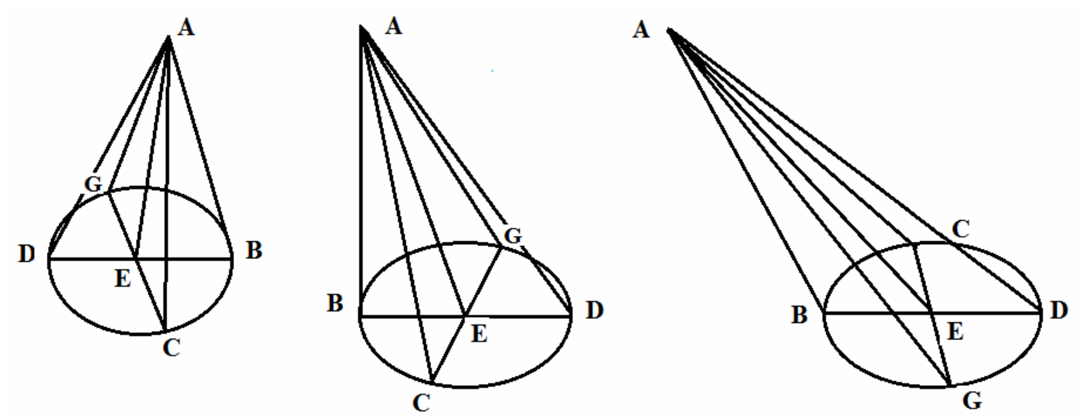


A scalene cone ABC shall be given, the vertex of which is B: the centre of the base D : truly the axis BD; the perpendicular BF may be dropped to the base of the cone, and FD may be joined, which may be continued as far as to the perimeter of the base on each side at A and C, therefore with a section of the cone made through the apex B, and the line AC, the triangle ABC is produced through the axis. This, I say, to be the minimum of these which can be produced from the cone ABC from the section made through the axis. The demonstration is evident from Cor. 2. of the preceding proposition; for some other section may be made through the axis, namely BGH: and from the vertex B, BI may be sent to the diameter GH, and the right line FI may be joined. Since the line BF shall be normal to the plane AHC, the angle BFI will be right; whereby the angle BIF is acute; therefore BI is greater than BF; and thus since the triangles GBH, ABC shall have the equal bases GH, AC, the smaller triangle ABC having the altitude BF, smaller than the triangle GBH, by having the greater altitude BI. We may show similarly the triangle ABC to be smaller with any other. Therefore we have shown, &c. Q.e.f.

PROPOSITION VIII.

In a scalene cone, to assign the maximum of the triangles which can be drawn by a section though the axis.

Construction & demonstration.



The scalene cone shall be ABCD, truly the minimum triangle of these may be found by the preceding proposition, whereby the minimum of these to be done through the axis, without doubt ABD: then the diameter CEG may be arranged normally to the right line BD through the centre E, and following which the vertex A drawn in the plane the triangle ACG may be put in place. I say here everything to be a maximum, indeed from the second case of the fourth proposition it is apparent AE to be normal to the right line BD, and following which the vertex A drawn in the plane, the triangle ACG may be put in place. I say here everything to be a maximum, indeed from the case following from the fourth preposition AE will be apparent to be normal to the base CG: and from the corollary following the sixth to be apparent the perpendicular AE to the base of the triangle CAG, to be the maximum of all the perpendiculars, which are drawn to the bases of all the remaining through the axis of the triangles. Whereby since the bases of all the triangles drawn through the axis shall be diameters of the base of the cone, and hence equal, the triangle CAG having the maximum perpendicular, this is the altitude which is the greatest of all ; therefore we have shown, &c. Q.E.D.

First Corollary.

Hence from the second corollary of the sixth proposition of this part, it is apparent the maximum triangle from the section through the axis to be in proportion to the minimum triangle, made from the same section, the axis drawn to the centre of the base for the perpendicular AF, in the diagram of proposition seven; since these shall be proportional to the perpendiculars of the equal base of each triangle standing on each side.

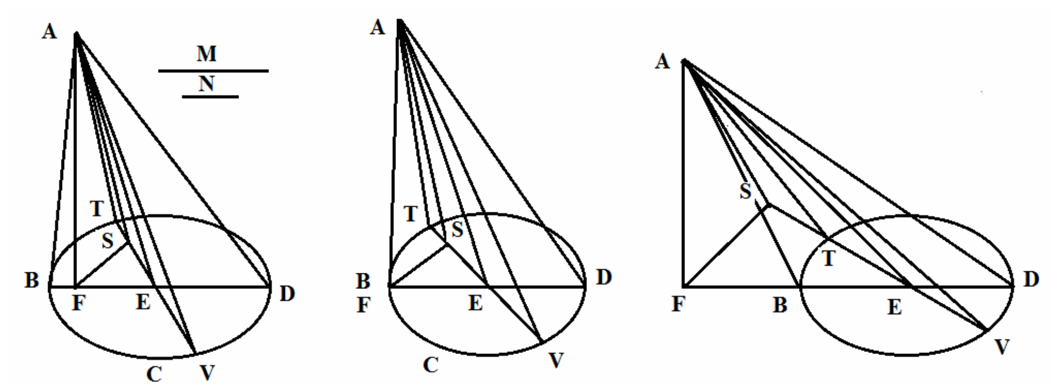
Second Corollary.

Then to be apparent the maximum triangle, arising from the section made in the scalene cone through the axis, to be Isosceles, since the perpendicular AE shown shall cut the line CG into equal parts.

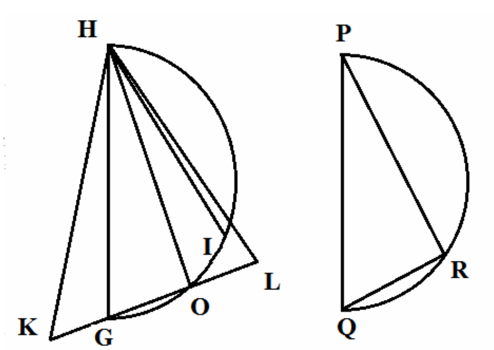
PROPOSITION IX.

The proposed triangle through the axis in the scalene cone shall be to show how, which with the smallest of the triangles made through the same, a given ratio may prevail: yet that ratio shall not exist which is greater than that which is going to be produced between the maximum and the minimum of the triangles, with the section made through the axis.

Construction & Demonstration.



A scalene cone shall be given, having the base BCD, into which the normal AF falls: the axis shall be AE, the centre of the base E. Moreover the ratio given shall be M to N. And since that cannot be put greater than the ratio of the maximum to the minimum triangle,



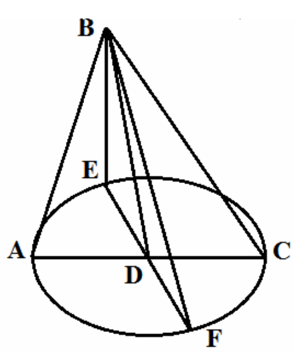
nor will the ratio of AE to AF itself be greater, as is deduced from the preceding corollary: Now with the plane drawn through the points A, F, E, the triangle ABD is produced with the points in the plane, which by Prop 7 above will be the minimum of the triangles through the axis ; then with the right line GH assumed, which shall be equal to EA, the semicircle GIH may be described, of which the diameter GH, and for the right line AF equal to HI to be inscribed in the semicircle: finally so that N to M, thus may become HI to HO, which can be adapted from the point H in the semicircle between the points I, G, since the given ratio M to N, shall not be greater than the ratio AE to AF, that is , on account of the equality of the lines from the construction, HG to HI. Then with OG joined, which with HO will constitute a right angle, as in the semicircle, GL and GK shall become equal to the radius

EB, and HL, HK may be joined. Therefore since the BD, LK are equal, the triangle LHK will be to the triangle BAD, as the perpendicular HO to the perpendicular AF, that is, as HO to HI, that is, as M to N; therefore so that the situation remains of this triangle we have assigned in the cone ABCD , with the line FG assumed, which shall be equal to HO, and the semicircle PRQ may be described with the interval PQ, thence it may be adapted to the semicircle PR equal to HI; that is AF, and RQ may be joined : finally without doubt the triangle FSE may be made on the base FE from the lines OG, RQ, so that ES shall be equal to OG, and FS to QR, then the diameter TV may be acting through E and S, and AT, AV and AS may be put in place. I say the triangle AT to be similar and equal to the triangle LHK ; Since AF has been drawn normal to the base of the cone, the angle AFS will be right, and the square AS equal to the squares AF, FS, that is (since from the construction AF is PR, and FS is RQ) to the squares PR, RQ. And also the square PQ is equal to these same squares PR, RQ ; therefore the square AS is equal to the square PQ, and thus the right line AS is equal to the right line PQ, that is, to the right line HO. Now truly thus I show, AS to be normal to the base TV, as HO is to the base LK. The square HG is equal to the squares HO, GO: whereby since from the construction AE shall be equal to HG, and SE to OG, and from the demonstration, AS shall be equal to HO itself, also AE shall be equal to the squares AS, SE: and hence the angle ASE is right, and AS perpendicular to the base. Whereby since in the triangles LHK, TAV, from the construction the bases LK, TV shall be equal, and the perpendiculars or altitudes HO, AS, are equal, also these triangles will be equal. In addition since OG shall be equal to SE, and GK itself to EV, OK will be equal to SV, and HO has been shown to equal AS, and the angles HOG, ASV are equal and right. Therefore HK is equal to AV. Similarly I may show HL to be equal to AT. And thus the triangles LHK, TAV are similar. Whereby we have designated the triangle ATV in the scalene cone through the axis which will have the given ratio M to N to the triangle ABD.
Which was required to be shown.

PROPOSITION X.

If the triangle through the axis of a scalene cone, may contain a right angle about the vertex: I say all such triangles through the axis to contain a right angle.

Demonstration.



The scalene cone may be cut through the axis by the section ABC, so that it may produce a right angle at the vertex B : but some other triangle may be cut through the axes in the plane shown by the triangle BFE; I say the angle FBE to be right. Because the angle ABC is right, the square AC is equal to the squares AB, BC; but above it has been shown the squares AB, BC to be equal to the squares FB, EB. Therefore the square AC is equal to the squares FB, EB; but FE is equal to the right line AC, and therefore the square FE will be equal to the same squares FB, BE, whereby the angle FBE is right. And thus if a scalene cone, &c. Which was required to be shown.

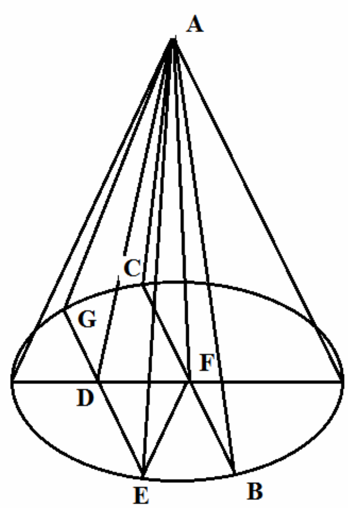
Scholion.

Besides the triangles which are elicited from the section of the cone made by the axis, others also are assigned which indeed will pass through the vertex, truly minimally through the axis ; which since they shall be without number, hence are removed from the determination, so that rightly they may be extricated from the solution of the problem, with which they are concerned.

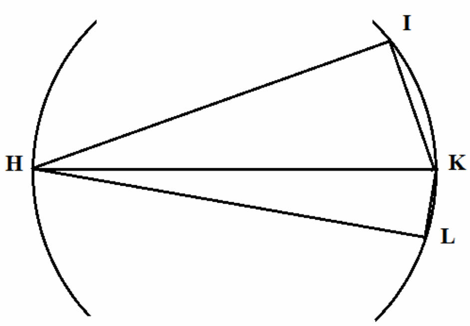
PROPOSITION XI.

In an acute-angled right cone, and with a right-angled triangle made through the axis, it is impossible to show an equal triangle not passing through the axis, or the triangle through the axis must be greater than any triangle you please, not drawn through the axis.

Demonstration.

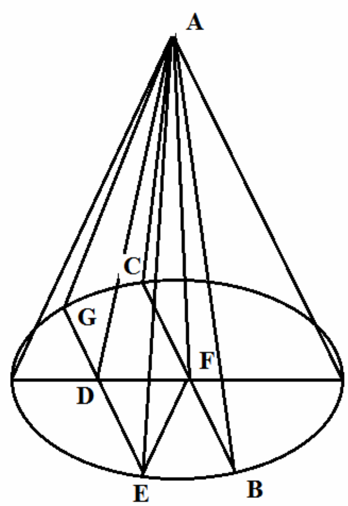


For if it can be done, some triangle AEG may be put in place not passing through the axis AF, equal to the triangle drawn through the axis BAC, which thus coincides with the cone drawn, so that its base BC shall be parallel to the base EG, which may be agreed to be able to happen, since all the triangles in a right cone through the axis shall be equal. From the centre F the right line FD may be drawn bisecting the base EG, and AD, FE, may be joined and with the above right line HK to be equal to AE itself, with the two triangles HIK, HLK put in place, equal and similar to the triangles AEF, ADE, thus so that the sides HI, IK shall be equal to the sides AD, DE , and thus I and L shall be equal, with the angles AFE, ADE. Therefore since by the hypothesis the axis AF is normal to the plane of the base, the angle AFE will be right. Also the angle ADE is right, as may be deduced from the demonstration in the second case of the fourth proposition presented here. Whereby the angles I and L are right also, therefore the points belong to the circle I, L, H, K of which the diameter is HK. And since the given cone is right angled or acute angled, the axis AF either will be equal to the radius FE, or greater. Whereby since AI to AF, and IK to FE shall equal, also there will become either HI equal to IK, or greater. Therefore the vertex I of the triangle AIK, either will bisect the arc HIK, or there will be another



point of the bisection towards K. Then since ADC, that is AL, is greater than AF, that is HI, the arc HL will be greater than the arc HI, and therefore the index L of the triangle HLK not only falls beyond the point by which the arc HLK may be bisected, but also closer still towards K, than the other vertex I of the triangle. Therefore the altitude of triangle HLK is smaller than that of the triangle HIK, and thus the triangle ALK is smaller than triangle ADE then triangle HIK, that is AFE. And triangle AFE is equal to triangle AFB, since all the sides of each, shall be equal in turn. Therefore triangle ADE also is smaller than triangle AFB: and hence triangle EAG to be the double of ADE (for indeed, from the construction, EG to be bisected in D), to be smaller than triangle BAC by twice AFB. Similarly, we may demonstrate whatever other triangle, not to be made through the axis, to be smaller than a triangle through the axis. Therefore in the right cone, &c. Which was required to be demonstrated.

Because moreover it has been assumed in the demonstration, (for the right cone) truly with the acute angled cone, the axes with the radius of the base to be greater, thus I will show to be less in the right cone with a lesser angle.

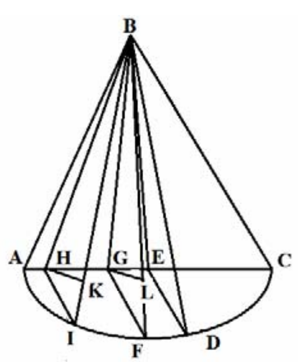


The acute-angled right cone section shall be made with the triangle BAC through the axis, since in the triangles BFA, CFA, the sides BA, CA, BF, FC are equal, and with FA common, the angles FAB, FAC also are equal, but the whole angle BAC as being acute is less than a right angle, therefore the angle FAB of half of this, is less than half a right angle . Whereby since the angle BFA shall be right, the remaining angle ABF will be less than a right angle, that is with the greater angle FAB. Therefore the axis AF is smaller with the radius FB.

So that if the right cone has a right angle, then the angle BAC will be right, and hence BAF half of the whole BAC, is a semi-right angle. Whereby since the angle BFA shall be right, it is necessary, that the remaining FBA also shall be semi-right, and thus equal to the angle BAF: from which the axis AF and the radius of the base of the cone will be equal.

PROPOSITION XII.

The preceding theorem will be demonstrated otherwise in a more expeditious manner.



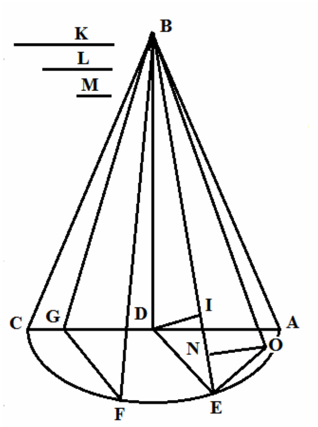
Demonstration.

In a right angled or acute angled cone the triangle through the axis shall be BFG and some other beyond the axis IBH, but the diameter of the base of the cone shall be AEC, with the normals GL, HK may be drawn from the points G and H to BF, BI. Since the cone shall be right-angled or acute-angled, the axis BG shall be as shown above, equal to the radius GF, or greater. From which it is evident the segment BL not to be less than the segment LF, and hence the point L to fall either between F and the midpoint of the line FB, or at most to coincide with the same midpoint. Then since the angles BGF, BHI are right, as to be apparent from the said Prop.11. and from these normals GK, HK sent, the rectangles LFB, KIB will be equal to the squares FG, GL. And the square GF is greater than the square HI, therefore the square LFB is greater than the rectangle KIB. Therefore since BI, BF shall be equal, IK is less than FL therefore since, as shown before, the point L falls between F and the mid-point of the line BF, the point K now falls between I and the midpoint of the line BI, that is, BF, and indeed closer towards I, than L towards F. Therefore it will be clear the rectangle BKI to be less than the rectangle BLF. And the squares LG, KH are equal to the rectangles BLF, BKI, therefore the square KH is smaller than the square LG, therefore with the normal KH to be smaller than the normal LG. Whereby since the bases BI, BF shall be equal, the triangle IBH is smaller than the triangle BFG. Similarly I may show some other triangle beyond the axis to be smaller than the triangle through the axis FBG, which therefore is the maximum. Which was required to be demonstrated.

PROPOSITION XIII.

A cone shall be given having the axis BD, and the ratio of the greater inequality K to L. It shall be required to show a triangle beyond the axis for which the triangle BED through the axis has the given ratio K to L.

Construction and demonstration.

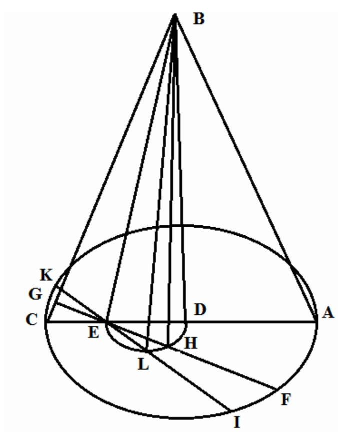


With DI drawn normal to BE from the point D, and it shall be made so that as K to L thus DI to M. Then with the square M made equal to the rectangle BNE, and since the square M is smaller than the square DI, that is to the rectangle BIE, (since the normal DI is drawn from the right angle to the base BDE) , also the rectangle BNE will be smaller than BIE. Then from the point N the normal NO may be raised equal to the right line M, and BO, EO may be joined. Therefore since the square NO is equal to the square M, that is, to the rectangle BNE, BN, NO, NE will be three continued proportional parts. Therefore the angle BOE is right. Whereby since the triangles BOE, BDE each shall be triangles on the same base BE, and the height of one, surely BOE, shall be with a smaller height NO than the other DI, it may be apparent one to another E with the sides BO, OE the greater to be of the other triangle with the side BD, which likewise is with the axis of the cone. And the side BO will be greater than the axis of the cone BD; this nevertheless to be smaller than the side of the cone BE, or BC; Therefore since the side BO is greater than the axis BD, and smaller than the side BC; within the cone it will be able to adapt, from the point B to the diameter AC with the right line AG equal to the side BO. This may be done, and from G to the normal BG draw GF and join BF. Since the angles BOE, BGF are right, the squares BE, BF therefore are equal to the squares BO, OE; BG, GF ; but the squares BE, BF are equal. Therefore since the squares BO, BG on account of the right angles from the equality of the construction, shall be equal, also the remaining OE, GF will be will be equal; therefore the right lines OE, GF are equal; therefore in the triangles BOE, BGF the individual sides are equal. Therefore also the triangles also are equal. And the triangle through the axis BED will have the same ratio to the triangle BOE as the normal DI to the normal NO, that is as DI to M, so as K to L: therefore the triangle through the axis ED normal to NO, that is as DI to M, that is as K to L: therefore the triangle through the axis BED to some triangle BGF, has the same ratio as K ad L. Therefore we have shown the triangle through the axis, &c. Which was required to be done.

PROPOSITION XIV.

In a right cone the triangles not drawn through the axis, the bases of which intersect each other in the same point, have perpendiculars drawn to the bases from the vertex, in the periphery of the circle of which the diameter is a right line between the centre of the base and the common point of intersection.

Demonstration.



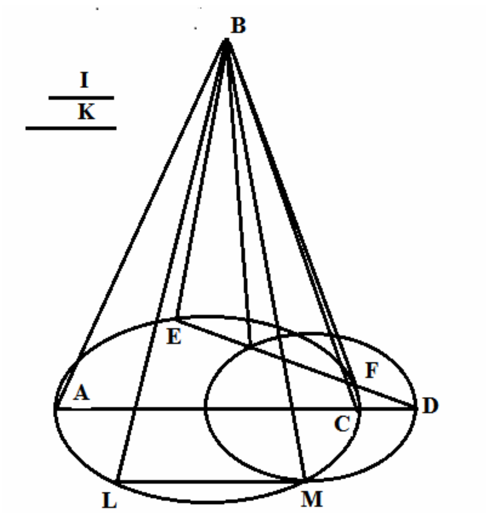
In the right cone ABC, several bases FEG, IEK of the triangles FBG, IBK, not through the axes, intersect each other mutually at some point E, and around DE, between the centre D and the point E, describe the circle DHLE : I say the perpendiculars drawn from the vertex B to FEG, IEK, to be incident on the periphery DLE. Indeed BL, BH, DL, DH may be joined, DLE will be a right angle in the semicircle, from which the right lines IL and LK will be equal; but the right line BI to be equal to BK ; therefore also BL is normal to the right line IK. We may show similarly at H, BH to be normal to FG. Therefore all the triangles drawn through the point E and with their normals to the base through the vertex, will be found on the perimeter of the small circle DLE. Which it was required to demonstrate.

Scholium.

Since the point of intersection E may be able to be given either, within the base, on the perimeter of the base, or outside the base, it shall be so that here three cases may arise: in the first case: where the small circle DEH is taken to be the whole base; secondly: where the same point is required to be touching the perimeter AIC *: in which case it is a fourth part of the circle, which acts in turn as the base of the cone ABC; truly in the third case where a wider circle outside the base may cut the circle of the base at two points; but since here neither the construction nor the demonstration of the three different cases will be undertaken, hence it has been suppressed with silence both in the proposition as well as in the demonstration.*

PROPOSITION XV.

In a right cone to show the triangle which may pass through the apex of the cone and of a point given either within or outside the base of the cone, truly so that it may have a given ratio to the triangle through the given axis.



Construction and demonstration.

The right cone shall be ABC, and with the triangle ABC made through its axis, a certain point D may be given, truly it will be required to put in place the triangle EBF through the point D, and the vertex of the cone, so that it may have the given ratio I to K to the triangle ABC through the axis. The triangle LBM may be found by the 12th Proposition, which may have the given ratio to triangle ABC. Then from the point D, DFE may be drawn, thus so that FE shall be equal to LM, through that which can be written down concerned with the properties of circle [§.44 *Concerning Circles*] : and the plane may be acting, following EF & B, this will show the triangle BEF postulated, indeed the triangle EBF will be equal to the triangle BLM, since the base LM shall be equal to EF from the construction, and the lines EB, FB not only will be equal to each other, but also they shall be equal to the right lines LB, BM. Therefore we have completed what was commanded.

Scholium.

Up to this point in some manner, we have tried to explain the sections of cones which pass through the apex, to be either in a right cone or in a scalene cone: truly since the time of the antiquaries, especially Archimedes, the remaining conic sections found were signified by another nomenclature, the explanation of which we intend to pursue in the following books; hence I have declared there be a need to send something ahead, which will be seen to pertain to such matter.

We have placed the cones to be different in two ways, namely some to be right cones, others to be scalene cones: truly in antiquity these were considered to be different in three ways. Some cones were said to be right-angled, some acute-angled, others finally to be obtuse-angled, with the denomination taken from the angle which the section made through the axis would pertain to the vertex of the cone ; from these the then unknown distinction between right cones and scalene could be seen, since the scalene cone would allow all kinds of angles at the vertex, with the only variation , following the diverse situations being made by the section through the axis. And thus they called the right-angled cone an Isosceles cone, which allowed no variation in the triangles arising through the axis, truly all the other angles being excluded, apart from the right angle, which were separated from the initial right-angled cone by the division through the axis, *and thence elsewhere they were cutting the side of the triangle arising by another plane at right angles which in addition would be normal to the triangle itself; as the antiquaries called the section of the cone right-angled, but more recently they are called parabolic. Truly the acute-angled cone and likewise the Isosceles cone were divided into two parts by the axis, and indeed in the first place they established a section through the axis ; then a triangle would be produced from the section through the axis, otherwise they will be cut by a plane, so that both in the plane as well as the sides of the triangle may be right; with what being done so that with one of the legs being crossed, they may present a figure of the other form, and be far different from a parabola, which evidently may be completely closed by the interval of the right lines of the triangles through the axis. Then the resulting section of the acute-angled cone will be called Isosceles, and finally the obtuse-angled cone, will be divided by the same practice and by the same axis, then with the plane for one of the sides, and a right triangle and for the triangle itself with the right angle acting through the cone, they will form a figure the antiquaries called an obtuse-angled cone, which in more recent times the same to be reconciled with the hyperbola. Therefore this account of the antiquarians of requiring how to divide the cone, with more recent accounts, truly especially from Apollonius, has change to some extent: for a cone of any kind by its nature shall be such a body that the figures of the individual sections shall be able to be elicited ; for any triangle shown by the axis can be divided in three ways by a plane established at right angles to this plane; or indeed by the common intersection of these planes: evidently the situation of the triangle through the axis, and of a plane placed orthogonally to this same plane, but also parallel to one of the sides of the vertex triangle, in this situation will produce a parabolic section ; or if the intersection of the normal plane with the surface of the cone occurs completely within the triangle with its sides produced, then in this case an elliptic section emerges; or the normal plane placed parallel to the axis intersects one of the sides of the produced beyond the triangle of the cone, and thus in this case a hyperbolic section is shown; all these sections to be the same as with these created by the antiquarians, which you will understand by going over these books: yet thus so that any section made from the more recent investigations may not be able to be shown in the conic section of which made in antiquity; for the antiquaries were able to offer only one kind of ellipse, and one kind of hyperbola in a given cone, thus so that if they were to require a different ellipse or hyperbola, they had to recourse to a more or less acute or obtuse cone. But just as an infinitude of varieties arise in any conic section on a par with the more recent method, just as the benign reader will be able to know, if he may wish to pursue the following.*

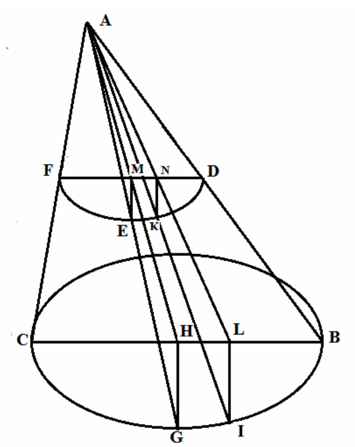
In addition, the account of the more recent cone requiring to be cut, presents a circular section not made parallel to the base of the cone; which circle the antiquarians were unable to elicit, since this circle does not prevail to be shown either a right or isosceles cone, which the ancestors only had examined, but it was to be elicited only from a scalene cone, as we will show in the following.

Prior to this discourse I may impose a limit, for a little word requiring to be insinuated by me is why the axis of the cone be determined by the line dropped from the vertex of the cone to the centre of the circle rather than to the centre of the ellipse, since the ellipse may not obtain a lesser centre than may be obtained by the centre of the circle, and following the perimeter of the ellipse from a point put in the highest position, if that same *right line may be turned about the same point it shall describe at once that same conical surface, just as if from the same point with circles of the same cone, to be produced by the line you may lead around. In short the response is, indeed neither the daughter precedes the mother, indeed the ellipse is begat from the cone. In the above, the properties of the circle with everything noted to be presumed from the elements, attend correctly and conveniently to the demonstrations of the propositions which arise from the cone: but ellipses shall occur with unknown qualities, in vain these may be attributed to the truths of theorems or problems being evident, truly chiefly it may be considered in my account, so that ellipses may be able to be assigned in a cone to be different from each other without number and hence for the individual ellipses it will be necessary to change the axis of the cone: indeed they are not able to have the same axes from the vertex of the cone to the centre of the circle, since with the same axes from the same vertex to the centre of the ellipse, therefore these inimical variations deservedly exclude the ellipse deservedly lest with the aid of the base it may perform on the cone, yet it may be seen more perfectly in the scalene cone the right line which besides one axis does not allow the other, to be used in the scalene cone, from which it is included in the chapter on the imperfections.*

PROPOSITION XVI.

In any cone every section parallel to the base is a circle.

Demonstration.



The cone ABC may be given, the base of which is BGC, and the section EF to be equidistant from the plane of the base; I say DEF to be a circle; indeed AH may be drawn through the centre of the base, and AL some other line in the plane of the triangle through the axes ABC, crossing the right line DF at the points M, N and HG, LI may be erected normal to the diameter BC, and with AG, AI joined with the crossings of the section DEF at E and K, ME, NK may be joined : since the planes DEF, BGC are equidistant, also ME, HG; NK, LI ; and MN, HL are equidistant ; therefore ME to HG, as AM to AH, that is: as AN to AL, that is as MK to LI. And thus on interchanging, ME is to NK as HG to LI , and indeed the square ME to the square NK as the square HG to the square LI; but the square HG is equal to the rectangle BHC , and similarly the square LI is equal to the rectangle BLC. Therefore the square ME is to the square NK as the rectangle BHC is equal to the rectangle BLC, but as the rectangle BHC is to the rectangle BLC, thus the rectangle DMF to the rectangle DNF, since they shall be composed from the same ratio, thus the rectangle DMF to the rectangle DNF; therefore the square ME is to the square NK, as the rectangle DMF to the rectangle DNF, since they shall be composed from the same ratios ; therefore the square ME is to the square NK, as the rectangle DMF to the rectangle DNF. Now truly DM is to BH, as AM to AH, that is : as ME to HG. Therefore on interchanging DM is to ME, as BH to HG. But BH, HG are equal, therefore DM are equal ME.

Therefore the square DM, that is the rectangle DMF (for since BC shall be bisected at H, and DF in M) is equal to the square ME. Whereby since the square ME, as we have shown above, shall be to the square NK, thus the rectangle DMF, to the rectangle DNF, also the rectangle DNF, will be equal to the square NK. Finally since DF, BC shall be parallel and likewise GH, EM, and the single GH shall be normal to the single BC, the other, EM, will be normal to the other DF; similarly I may show KN to be normal to DF. And thus since of the normals EM, KN the squares shall be equal to the rectangles DMF, DNF, the section DE, EF is a circle. Which was required to be shown.

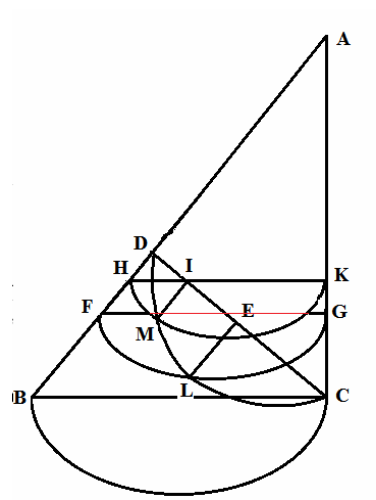
Corollary.

From the discourse of the demonstration it is clear the centre of the circle DEF to be at the point E on the axis, at which without doubt the axis crosses the right line DF, which is the common section of the triangle by the axis and the circle DEF.

PROPOSITION XVII.

In a scalene cone to show a circle which shall not be parallel to the base.

Construction & demonstration.

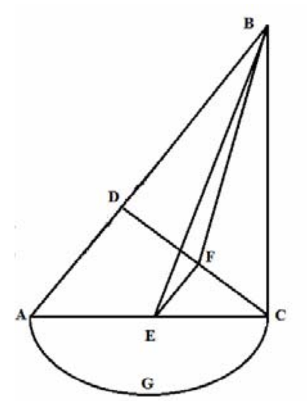


The scalene cone ABC may be put in place with the plane through the axis bringing forth the minimum triangle ABC of these which are able to be produced will be that scalene cone, since every other scalene triangle shall be greater besides the maximum through the axis by Cor. 2. of the 8th Prop.; therefore the angle ABC may be placed smaller than the angle ACB: whereby the angle ABC may become equal to the angle ACD and through the right line DC draw a plane at right angles to the triangle ABC : I say the section thence produced DLC to be a circle. Indeed two other planes may be drawn parallel to the base FLG, HMK, the common sections of which with the plane DLC shall be the right lines EL, IM, from Prop.7 above, it is apparent the base of the cone to be perpendicular to the minimum triangle ABC ; and therefore the other parallel planes FLG, HMK will be perpendicular to the triangle ABC : but the plane DLC is normal to the triangle ABC ; therefore the common sections LE, MI, shall be normal to the triangle, and thus to the line DC. But since FLG, HMK shall be circles by the preceding, and EL, IM to be normals to DC, then both the rectangle FEG shall be equal to the square EL, as well as the rectangle HIK to be equal to the square IM, since truly the triangles DIH and IKC are similar, since by the construction the angle ACD shall be equal to the angle ABC, that is, DHI: therefore the points HDKC will lie in the same plane: also I may show similarly the points FD, GC to belong to the circle: whereby the rectangles DIC, HIK are equal, just as the rectangles FEG and DEC: and hence also the equality of the squares IM and EL of the perpendiculars to DC. Therefore the points DMLC is a circle of which the right line DC is the diameter; therefore, &c.

PROPOSITION XVIII.

It is required to be shown how there shall be two axes in any scalene cone.

Demonstration.



Thus the scalene cone ABC may be cut by a plane through the axis BE, which may be extended to the centre of the base of the circle AGC : the triangle ABC produced shall be cut by the right line CD, which shall show the angle BCD equal to the angle BAC: and DC may be bisected at F, and BF may be joined; I say that line to be the second axis of this cone: which shall be apparent thus, from the preceding proposition the section made through the right line DC, thus in the manner in which that was perceived, to produce a circle: therefore the right line dropped from the apex B to the point F of this circle, must obtain the name of the axis from the definition of that same axis; therefore it remains to show the right line BFE not to be the line or BF produced not to intersect the centre of the base at E, and hence BF to be a different axis from the axis BE, which thence will be apparent, because the right line EF shall be parallel to the line AB, because it will bisect the two right lines CD, CA. Therefore if the line EFB is right, it follows the two parallel lines themselves to intersect, which is absurd. Therefore it is agreed every scalene cone allows two axes. Which was required to be shown.

PROLEGOMENA AD SECTIONES CONI.

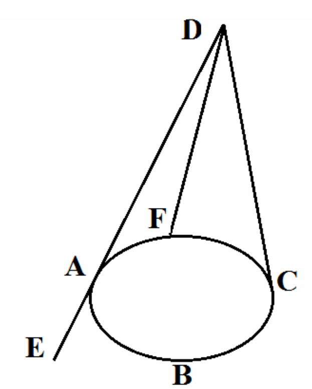
Conicarum sectionum affèctiones & accidentia indagaturi, alieno nonnihil ab antiquis consilio rem aggredimur: illi etenim proprietates quas pluribus sectionibus communes esse adverterunt, communi quoque demonstratione involuerunt, nec immerito, exigente talem scribendi rationem ipso doctrinae tenore, ordine ac nitore: nos autem in alium scopum collimates, in gratiam eorum, qui Geometriae afficiuntur, singillatim singulas explanare intendimus, quo faciliore methode & minus confusa, tyrones ad conicarum speculationum contemplationem invitentur & alliciantur. Nam ut recte Fredericus Commandinus ait, quod si aliqua pars est Matheseos quae nostris incognita Philosophis, interpretationis lumen aliquod postulet,ea profecto est quae de Conicus appellatur: quanquam enim a veteribus diligenter tractata sit, tamen eorum monumenta aut ad nos non pervenerunt, aut ita pervenerunt ut vix propter vetustatis iniurias maximasque difficultates intelligantur. conabimur autem ita planas reddere quas prae manibus havemus conicas materias , ut vel leuiter in elementis Euclidis versati eas percipere queant, sine ullo taedio, imo vero cum orexi ad ulteriora tendendi , & has lucubrationes in immensum extendendi, neque enim Geometriae scientia terminis contineri potest, multoque minus exhaurire, cum abyssus sit vastissima, omnem Oceanum quarumcumque speculationum absorbens. Ordiemur itaque ab ipsius nominis notione, phaleris omissis, exponentes, quid Coni nomenclatura & natura ex ipsius descriptione aut construendi ratione denotetur.

DEFINITIO.

Quid sit Conus.

Conum appello corpus habens originem ex circumducta lineae a puncto in sublimi posito in infinitum protensae circa perimetrum circularem, quae in diverso ab eodem puncto plano constituta sit.

Expositio.



Concipiatur in hunc finem circulus quidam ABC , extra cuius planum in quo iacet, assumatur quodvis punctum D, à quo ducta sit linea AE contingens perimetrum circularem in puncto aliquo AE agatur vero linea DE circa peripheriam integram circuli ABC donec in A redeat, hac tamen conditione ut D punctum fixam permaneat, hoc est, ut extremum punctum lineae DA, quod applicatum est initio circumvolutionis puncto D, ab ipso numquam recedat, tali inquam circumlatione lineae DE, orietur figura DABC binas habens superficies, una motu rectae DE efformatam, alteram vero circulum ABC. Priorem appellare solent superficiem praeter basum secundam autem ipsam coni basim; punctum D vertex coni nuncupatur, axis vero linea à vertice D ad centrum baseos pertingens.

Ex hac coni descriptione sequitur à quovis puncto in superficie conica assignato posse duci rectam lineam ad ipsum verticem D cùm enim per rectam DA circa basim ABC circulatione facta, prôducta sic coni superficies, necesse est ut à quovos puncto perrimetri ABC duci possit recta ad verticem D.
Hinc superuacaneum credo demostrare quod rectae omnes sint lineae quae a vertica ad quodlibet punctum in conica superficie ducuntur, proindeque satis esse manfestum ex ipsa ratione construendi conum, omnem sectionem quae sit per verticem , triangulum exhibere cuius latera fiat in conisuperficie.

Suppono quoque aequè notum , omnem lineam à puncto in superficie coni assumpto ductam ad quodvis aliud quod in recta non existat quae ad verticem tendit intra conum ipsum penetrare si infinitè ut rimque producatur, licet haec omnia placuerit Apollonio demonstrare : quarum demonstrationes si lector desideras, in eodem poteris cognoscere.

Hoc loco occurrit notandum pluribus aliis viis conum decribi posse ; puta circumducendo ex eodem puncto D, in sublimi posito rectam AD circa figuram ellipticam aut quamlibet aliam è conicis: sed placuit non sine causa antiquis circulo solo uti, eo quod, ut in dècursu libri apparebit, commodior figura sit ad demonstrationes formandas : adde quod figurarum aliarum efformatio conum priùs supponat; ex quo ferè solo ortum habent, praeter ellipsin, quae ut recte Serenus ostendit,ex Cylindro quoque excludi potest.

Quotuplex sit conus.

Duplex est conus; alius Isoscelius est sive Aequicruris, qui & rectus dicitur; alius Scalenus. Aequicruris est à cuius vertice linea ad basim normaliter demissa in baseos centrum pertingit. Scalenus autem omnis ille conus appellatur, qui perpendicularem ductam a vertice ad circularem basum, si opus est etiam extensam infinite, extra centri punctum constitutam habet, quod basis medium est.

Unde rectum quoque dicunt conum qui aequicruris est, obliquum qui scalenus discrimen inter verumque in eo maximè situm, quod rectum conum si per axem plano dispescas, Isoscelium resultabit in partibus divisis triangulum, cuilibet alteri aequale & simile quod per axem sit : scalenus autem sine numero varia producit triangula sectionibus per axem repetitis, quae inter se neque similia sint, neque aequalia; circa quae has sequentes propositiones praemittemus. Ad quas perlegendas priusquam accedas amice Lector,

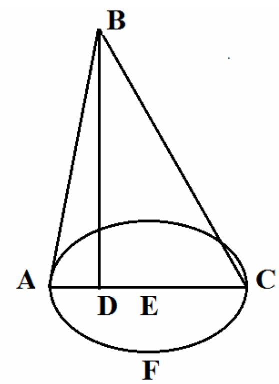
NOTA

Pleraque theoremata, quae in hisce prolegomenis adferemus, proponi etiam ac demonstrari à Sereno Antinsensi Philosopho ac Geometra, libro secundo de sectione coni. Omnino mihi memoria exciderat tractatam à Sereno eandem esse materiam : iam enim anni sunt viginti & amplius quod authorem illum non legetim. Velim itaque, qui haec leget plagii reum me ne faciat; sed cogiter, si deinceps fortasse quaedam mihi cum Sereno communia occurrent, mihi eadem, quae Sereno olim, potuisse incidere.

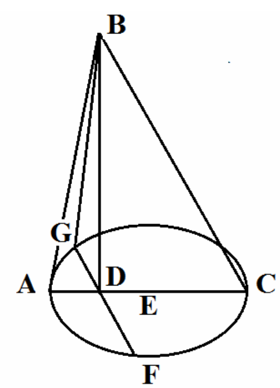
PROPOSITIO PRIMA.

In cono scaleno si à vertice linea ad perpendiculum demittatur, & per verticem, centrum basis, ac punctum à perpendiculo denotatum planum agatur, dico triangulum per hanc sectionem factum continere maximam & minimam linearum quae in superficie conica exhiberi possunt.

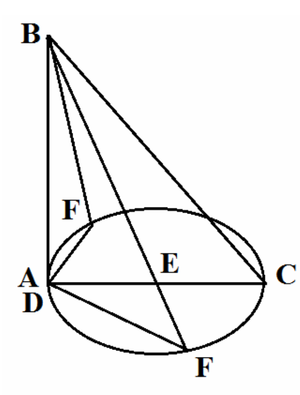
Demonstratio.



Triplex est hic casus: vel punctum perpendiculo denotatum intra peripheriam baseos existit, vel in ipsa peripheria, vel denique extra ipsum circulum. Pro primo igitur casu ponatur conus ABC & baseos centrum E: vertex autem B: à quo demissa perpendicularis, occurrat basi intra peripheram in puncto D, per quod, & per centrum, agatur plannum quod pertingat usque ad B coni verticem; exhibebit haec sectio triangulum ABC: nam ex constructione coni BA, BC sunt rectae lineae; ostendendum est itaque AB esse minimam omnium linearum, BC vero maximam quae in superficie conica existunt.



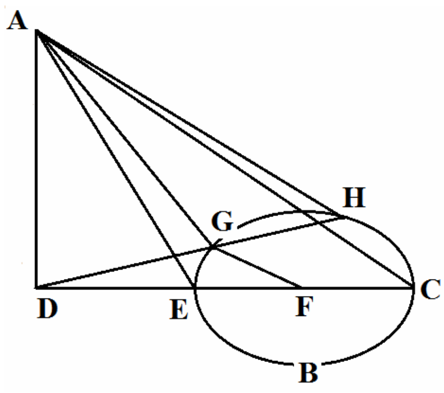
Ducta quavis BG, ponatur ex G recta GDF, quoniam ADC per centrum transit, erit AD minor quam GD, & quadratum AD, minus quadrato GD, additoque communi quadrato DB: atque AD , DB, minora sunt quadratis GD, DB. angulus autem GDB rectus est, quemadmodum & angulus ADB: igitur quadratum AB, aequatur quadratis AD, DB, quadratum vero GB aequatur quadratis GD, DB; ergo quadratum AB, minus est quadrato GB. ergo linea AB minor recta BG.



Eodem pacto ostendetur recta BC omnium maxima: ex puncto quippe quovis F, si ducatur quaevis FB: & per D ponatur FD, quia ADC per centrum transit, DC maior est quam EF; unde quadrata CD, DB maiora sunt quadratis FD, DB, sed cum rursum anguli CDB, FDB sint recti, quadram CD, DB, & quadratum FB quadratis FD, DB aequale est. Ergo quadratum CB maius est quadrato FB, adeoque recta CB maior quàm FB, unde maxima BC est earum que in superficie coni assignari possunt ; AB vero minima. Quod demonstrari oportuit.

Iam vero BD perpendicularis è coni vertice demissa cadat in perimetrum circuli AFC;
dico iterum AB lineam esse minimam,& BC maximam illarum quae in superficie coni existunt.

Ducatur enim quaevis BF, iungaturque AF, rursum igitur quadratum BF aequatur quadratis AF, AB, adeoque maius est quadrato AB. Unde BF, linea maior est recta AB: clarum est insuper BC rectam omnium esse maximam, nam quadratum BE aequale quadratis CA, AB maius est quadrato BF aequali quadratis FA, AB.



Reliquus casus est in quo dato scaleno cono ABC, linea AD demissa perpendiculariter ad planun BEC productum, extra ipsum conum signat punctum D: quo , casu dico rursum AB minimam esse, AC vero maximam. Ducatur enim quaecunque recta AG ; BD minor est DG , igitur quadratum AG aequale quadratis GD; DA maius est quadrato AB; ergo AB minor est quam AG, ducatur iam quaevis AH;CD maior est quàm HD, ergo quadratum CA, aequale quadratis CD, DA, maius est quadrato HA, aequali quadratis HD; DA, ergo CA maior HA.
In cono igitur scaleno &c. quod fuit demonstrandum.

PROPOSITION II.

Secetur conus scalenus (à cuius vertice linea ad basim perpendiculo demissa extra basim non cadit) per centrum & verticem, sectione exhibente triangulum, ad cuius basim demittatur normaliter recta a vertice coni.
Dico illam intra conum cadere.

Demonstratio.

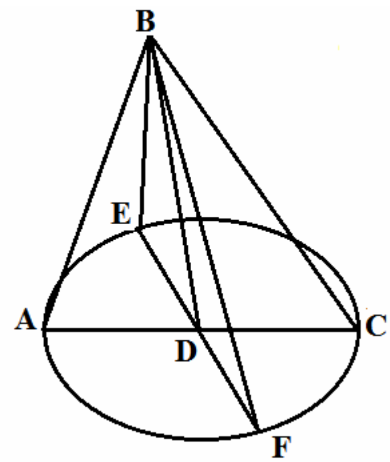
Duplex est casus; primus quo conus ABC , cuius basis BDC, habet BA perpendicularem a perimetroo aliquo puncto ad verticem erectam; alter dum linea a vertice ad basim ducta perpendiculariter intra basim cadit. Igitur fiat sectio per centrum F, & apicem A, exhibens triangulum ADE ostendere igitur oportet, normalem ex A ad basim DE demissam intra conum, hoc est inter puncta D & E occursuram diametrum DE. Ducantur enim BD & BE: deinde ex puncto B ponatur BG, normalis ad diametrum DE (cadet haec necessario inter puncta D,E) & iungantur AG, quoniam AB ex hyp. normalis est basi, recta EB in base ad ipsam ducta per defin.3.11. normalis est : ergo quadratum AE aequatur quadratis AB, BE: hoc est quadratis AB, BG, GE quadratum autem AG quod angulus ABG etiam rectus sit aequatur quadratis AB, BG , igitur quadratum AE excedit quadratum AG, quadrato GB; unde quadratum AE aequale est quadratis AG, GE, ergo AGE rectus est, & Asi perpendicularis. Atqui haec quadratis AG, GE angulus ergo AGE, rectus est, & Asi perpendicularis. Atqui haec ducta est a vertice coni & occurrit basi trianguli intra conum. Perspicua est igitur veritas propositionis in casu primo.
Eodem modo procedet demonstratio si loco haec AE, utamur linea AD.
Quod si perpendicularis a vertice coni ad coni basim demissa cadat intra conum ut AF rursum ducatur FG normalis ad ED iungaturque AG. Eodem plane discursu demonstrabimus quadratum AE aequari quadratis AG; EG ; adeoque angulum AGE rectum esse, & Asi perpendicularem ad basim trianguli ADE, ex quo manifesta in hoc etiam casu propositionis est veritas.

PROPOSITIO III.

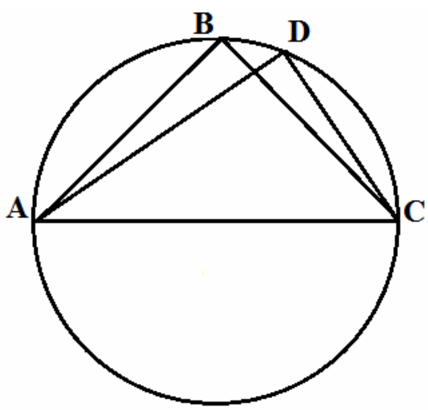
Laterum quadrata cuiuscunque trianguli in cono scaleno per axem facti aequalia sunt quadratis laterum cuiuscunque alterius trianguli per eundem axem.

Demonstratio.

Sit enim ABC conus scalenus cuius axis BD: per quem fiat quaevis sectio BEF: fiat autem ABC triangulum aliud sectum per axem BD. Dico quadrata AB, BC, quadratis BF, BE esse aequalia, quadrata enim AB, BC ostensa sunt libro quem de linearum potentiis scripsimus, aequalia esse quadratis AD, DC: & BD bis sumpto; at etiam discursu ostensum est quadratis BF, BD equari quadrata. FD, DE una cum quadrato BD bis sumpto: igitur cum FE.AC lineae earumque dimidiae FD, DE, AD, CD sint inter se aequales, patet veritas demonstrationis.



Corollarium.

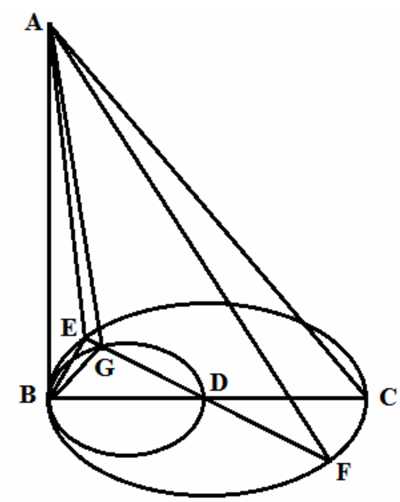


Hinc colligere licet facilem praxim cognoscendi latera triangulorum quae per axem sectione facta emergunt, & proportionem illorum inter se ponatur enim semicirculus ABC triangulum sit inscriptum habens latera AB, BC aequalia lateribus maximi trianguli alicuius coni per axem secti : Quoniam ergo quadrata laterum minimi trianguli per axem eiusdem coni aequantur quadratis laterum maximi AB, BC, (hoc est quadrato AC) poterunt & minimi latera inflecti in semicirculo, quae sint AD, DC. quodsi tertia quaedam sectio fiat per eundem axem, proportio laterum illius trianguli, reperietur necessario in aliquo triangulorum quod habebit basim AC, & verticé in uno punctorum quae sunt in parte perimetri BD. cum enim aequalia sint quadrata laterum unius trianguli quadratis laterum alterius, quod per axem facta sectione resultat, sintque ABC trianguli quadrata AB, BC aequalia quadratis laterum maximi; quadrata vero AD, DC aequalia sint quadratis laterum minimi; sequitur latera cuiuscunque alterius, quod per axem coni producitur, in aliquo triangulorum reperiri debere quod basim habeat AC & latera reliqua sese in aliquo punctorum decussantia quae in arcu BD, assignari possunt, quod ex decursu sequentium propositionum magis elucidabitur.

PROPOSITIO IV.

In cono scaleno à cuius vertice demissa perpendicularis in perimetrum baseos cadit, si quodvis triangulum per axem sectione facta exhibeatur.
dico quod normalis à vertice coni ducta ad basim trianguli cadet in circuli perimetrum qui describetur diametro intercepta inter perpendicularem a vertice ad basim ducta & centrum baseos eiusdem.

Demonstratio.

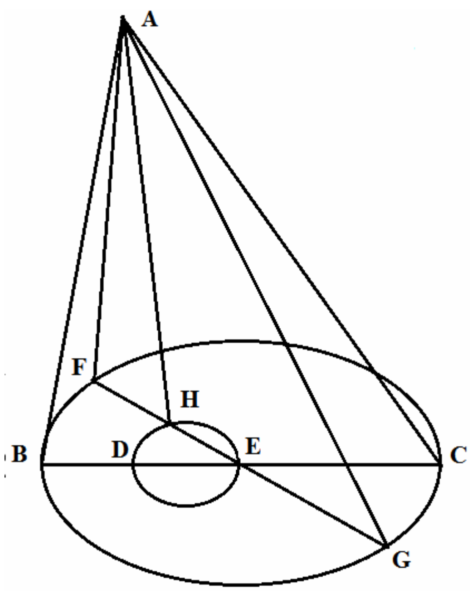


Sit igitur ABC conus centrum baseos D a coni vertice A recta AB demissa perpendiculariter, cadat in B punctum perimetri baseos BEC, descripto deinde circula BGD, super diametro BD, agatur planum per axem quodcumque AEF, exhibens triangulum AEF; cuius basis primo secet circulum in punctis DG: ex A puncto verticis, demittatur AG. dico illam esse quae basi EF normaliter insistit ; ducatur enim recta linea BG, iungaturque BF. Quoniam igitur ex hypothest AB recta est plano basis BE, CF. angulus ABF rectus est: adeoque quadratum AF aequale quadratis AB, BF; hoc est, quoniam angulus BGF in semicirculo etiam rectus est, quadratis AB, BG, GF. quadratum autem AG aequale est quadratis AB, BG, quo angulus ABG similiter rectus sit. Igitur excedit quadratum AF quadratum AG, quadrato FG: quare quadratum AF aequatur quadratis AG, GF. unde normaliter insistit recta AG, ipsi GF in puncto G. quod in perimetro circuli habenti diametrum BD.
Secundo basis trianguli EF contingat circulum BDG in puncto D; ducaturque AD, AF: quoniam anguli BDE, BDF recti sunt , punctum contactus , quod idem est cum centro basis coni BHC; iungantur AE, AF: quoniam anguli BDE, BDF recti sunt, & ED latus lateri DF, ac BD sibi ipsi aequale est, etiam bases BE, BF aequales erunt.ergo quadrata BE, BA quadratis BF, BA aequalia sunt. Atqui quadratum FA aequatur quadratis BF, BA, & quadratum EA quadratiis BE, BA, quod recta AB plano basis com sit recta, adeoque anguli ABF,ABE recti: quadrata igitur FA, EA ac proinde etiam rectae FA, EA aequantur. Quare cum in Isoscele FAE, basim bisecet AD, erit haec perpendicularis ad basim. Itaque in hoc etiam casu perpendicularis AD est ad circulum BGD.
Tertio si triangulum per axem, transeat per AB normalem a vertice ad basim coni ductam, ac proinde eius basis eadem sit cum BDC: tum perpendiculari AB quae ducitur a vertice ad basim coni. Quare etiam hoc casu tertioperpendicularis ad basim trianguli ad circuli peripheriam existet; in cono igitur scaleno, &c. Quod fuit demonstrandum.

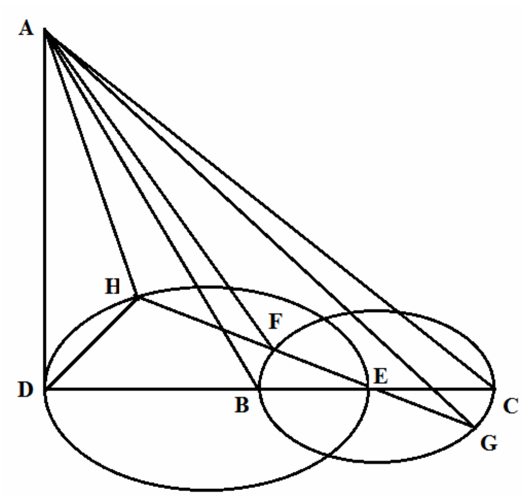
PROPOSITIO V.

Conus scalenus a cuius vertice perpendicularis cadit intra conum, si secetur per axem exhibens triangulum, ad cuius basim normalis ducatur a vertice procedens:
dico illam perimetro circuli occursuram cuius diameter aequalis erit lineae inter perpendicularem a vertice coni & centrum baseos eiusdem coni interceptae.

Demonstratio.



Esto conus ABC, a cuius vertice perpendicularis AD demissa cadat intra conum, occurrens basi in D: a quo ducta sit DE recto ad centrum baseos, super qua ut diametro circulus DHE describatur: dein per axem fiat quaecunque sectio exhibes triangulum AFG : cuius basis FG primo secet circulum in E & H, tum ex vertice A demittatur AH ad rectam FG. dico illam in perimetrum circuli DHE incidere. ducatur enim ad punctum H, in quo GE circulo occurrit recta DH; iunganturque DG: erit igitur quadratum AG aequale quadratis AD, DG, hoc est quadratis AD, DH, HG: quadratum autem AH aequale est quadratis AD, DH: igitur excedit AGquadratum, quadratum AH, quadrato HG: igitur quadratum AG, quadratis AH, HG est aequale, adeoque rectas AH normalis ad lineam FG. unde patet normales omnes a vertice A hoc in casu ad bases triangulorum per axem demissas in circulum DHE cadere.



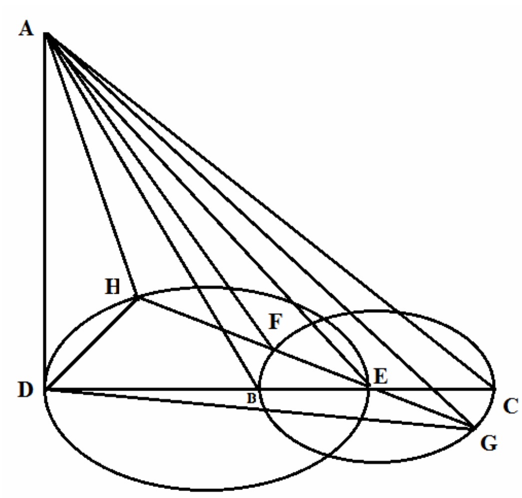
Casus secundi & tertii quando basis trianguli aut contingit circulum DHE, aut incidit in BEC, eadem est demonstratio, quae pro iisdem casibus superiori propositione allate est ; conus igitur scalenus, &c. Quod fuit demonstrandum.

PROPOSITIO VI.

Si in cono scaleno perpendicularis a vertice ad basim ducta extra basim cadat, productum sit autem sectione per axem facta quodcunque triangulum:
dico normalem a vertice ad basim trianguli ductam , esse ad peripheriam circuli, cuius diameter est recta inter centrum basis coni & perpendicularem a vertice ad basim coni interiecta.

Demonstratio.

Extra basim coni scaleni ABC perpendicularis a vertice cadat recta AD actaque per centrum baseos E, recta DE circa quam circulus describatur DHE, ponatur quodcunque triangulum per axem FAG: & protracta GF usque ad peripheriam circuli DHE (ponimus enim primo basim EF secare circulum DHE) demittatur deinde recta AH ex puncto verticis A, ad punctum quo secatur circulus DHE a linea G F. dico illam fore normalem.



ad rectam GF productam ducantur enim DG, DH: erit igitur quadratum AG aequale duobus quadratis AD, DG: (quoniam AD normalis est ad planum baseos productum, adeoque: etiam ad DG) hoc est tribus quadratis AD, DH, HG, cum DHE sit in semicirculo angulus rectus; sed quadratum HA aequatur quadratis AD, DH, quod angulus ADH iterum rectus sit; igitur quadratum AG excedit quadratum AH quadrato HG, unde AG quadratum, duobus quadratis AH, HG aequale existit : quare angulus rectus sit AHG. estque punctum H in perimetro circuli DHE ex constructione; manifestum igitur est in primo casu omnem normalem à vertice scaleni ad trianguli per axem basim in circuli DHE perimetrum incidere.
Secundi casus tertiique quando basis trianguli aut contingit circulum DHE, aut incidit in rectam DEC, demonstratio conuenit cum ea, quam propos.4. pro iisdem casibus attulimus.
Si igitur in cono scaleno, &c. Quod fuit demonstrandum.

Corollarium primum.

Hinc patet in omni casu quo è vertice coni scaleni ad basim eiusdem protractam, demittitur perpendicularis, extra basim coni, in triplici differentia producenda triangulo per axem conicum linearum quae à vertice ad bases triangulorum normaliter ducuntur, quaedam occurrant punctis circuli DHE, quae extra conum sunt, nonnullae punctis intra cum constitutis , duae autem numero ipsi perimetro baseos, videlicet I & K , quod patet ex partibus circuli DHE: quarum nonnulla extra basim existunt, quaedam vero intus, puncta vero I & K, quod patet ex partibus circuli DHE: quarum nonnullae extra basim existunt, quaedam vero intus, puncta vero I & K , cum sint intersectionum necessario in utroque circulorum consistunt.

Corollarium secundum.

Perpendicularis incidens in perimetri DHE, punctum E, quod idem est cum centro basis coni, est omnium maxima, sit enim alia quaevis AH, quae ipsa etiam incidet in perimetrum DHE, & iunga DH: quadratum EA aequatur quadratis ED, DA; quia autem ED diameter circuli quavis alia recta ducta intra circulum, ac proinde & recta DH maior est, erunt quadrata ED, DA maiora quadratis HD, DA, hoc est quadrato HA: Quare quadratum EA maius quoque est quam AD, manifestum est, cum in triangulo EAD, opponatur EA recto, AD acuto, EA maiorem esse quavis alia v.g. HA; sic quoque & brevius ostendemus; quoniam HA ex hypothest normalis est ad basim GF trianguli FAG, AHE rectus est, adeoque alter AEH acutus. Quare EA opposita angulo recto maior est quam HA quae acuto opponitur.

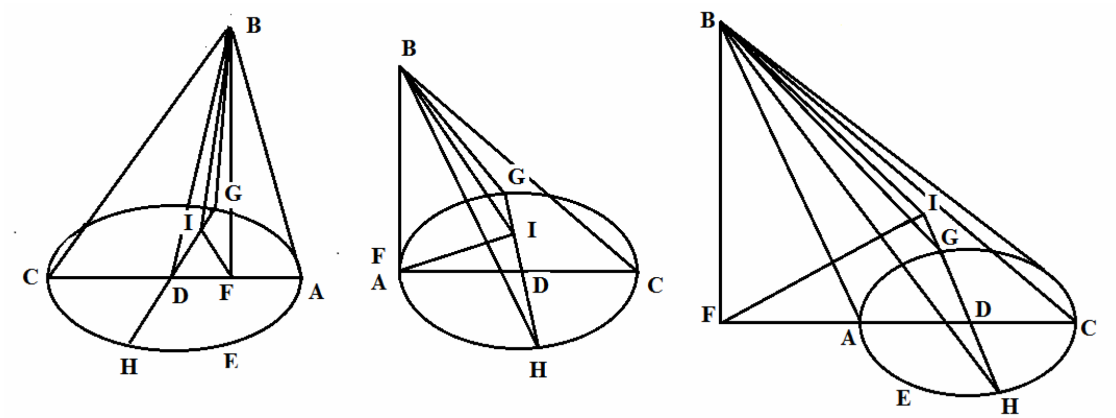
Perpendicularis vero quae incidit in punctum D, ac proinde eadem est cum recta AD quae a vertice ad basim coni normalis ducitur, omnium perpendicularium est minima. Ducatur enim quaevis alia perpendicularis AH, (nam quod AD minor sit quam EA, iam ostensum est) erit haec etiam ad peripheriam DHE: iungatur DH: in triangula ADH, AD opponitur angulo acuto AHD; AH vero angulo recto ADH, ergo minor est AD quam AH.

Hoc corollarium verum quoque esse in casibus duarum praecedentium theorematum, eadem plane demonstratione probabitur.

PROPOSITIO VII.

In cono scaleno dato exhibere minimum triangulorum sectione per axem facta.

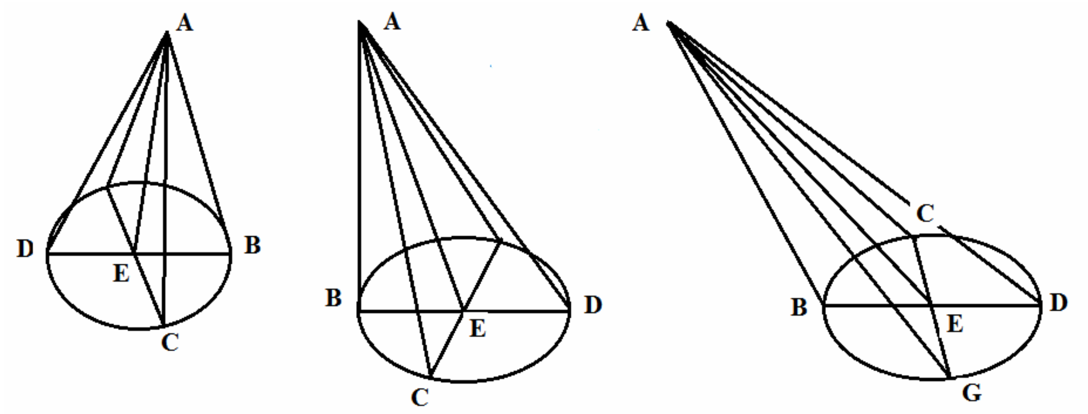
Constructio & demonstratio.



Conus scalenus ABC, datus sit, cuius vertex B: centrum baseos D : axis vero BD. demittatur ad basim coni perpendicularis BF, & iungantur FD, quae continuetur utque ad perimetrum baseos utrimque in A & C, facta igitur sectione coni per apicem B, & lineam AC, producetur triangulum per axem ABC. dico hoc esse minimum eorum quae ex cono ABC possunt educi per axem sectione facta. Demonstratio est manifesta ex corollario 2. propos. praeced.; fiat enim quaevis alia sectio per axem, scilicet BGH: & e vertice B demittature normaliter BI ad diametrum GH, iungaturque recta FI. Cum BF recta sit plano AHC, angulus BIF rectus erit; quare angulus BIF acutus est. maior ergo est BI quam BF. itaque cum triangula GBH, ABC aequales habeant bases GH, AC, erit ABC triangulum minorem habens altitudinem BF, minus triangulo GBH, maiorem habente altitudinem BI. Similiter ostendemus triangulum ABC quovis alio minus esse. exhibuimus ergo, &c. Quod erat faciendum

PROPOSlTIO VIII.

In cono scaleno assignarc maximum triangulorum quod sectione
quae per axem facta duci potest.



Constructio & demonstratio.

Sit conus scalenusABCD, per praecedentem vero propositionem inveniatur triangulum minimum eorum quare per axem fiunt, nimirum ABD: deinde per centrum E collocetur CEG diameter normaliter ad rectam BD, secundum quam & verticem A plano ducto statuatur triangulum ACG. dico hoc omnium esse maximum, ex casu enim secundo propositionis quartae patet AE normalem esse ad rectam BD, secundum quam & verticem A plano ducto statuatur triangulum ACG. dico hoc omnium esse maximum, ex casu enim secundo propositionis quartae patet AE normalem esse ad basim CG: & ex corollario secundo propositionis sexta patet perpendicularem AE ad basim trianguli CAG, esse maximam omnium perpendicularium, quae ad bases reliquorum per axem triangularum ducuntur. Quare cum omnium per axem triangulorum bases sint diametri basis coni, ac proinde aequales, triangulum CAG maximam habens perpendicularem, hoc est altitudinem, omnium est maximum; in cono igitur scaleno exhibuimus, &c. Quod erat faciendum.

Corollarium premum.

Hinc & ex corollario secundo sextae huius patet triangulum maximum sectione per axem ad triangulum minimum, factum eadem sectione, esse in proportione, axis ad centrum basis ducti ad perpendicularem AF, in fig. propositionis septimae; cum ille utriusque trianguli aequali basi insistentis perpendiculares sint.

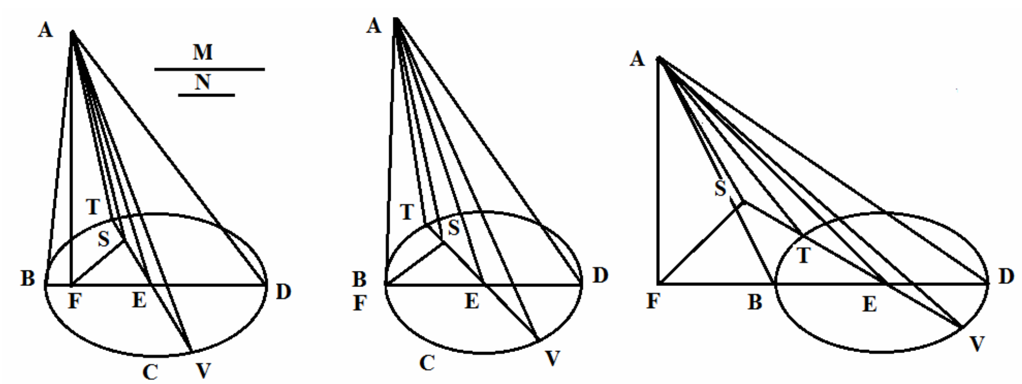
Corollarium secundum.

Patet deinde in cono scalene triangulum maximum facta sectione per axem natum, Isoscelium esse, cum AE perpendicularis ostensae sit secare lineam CG in partes aequales .

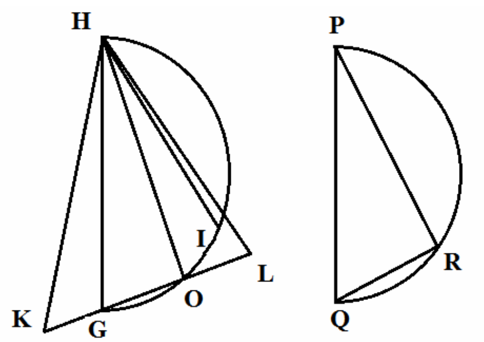
PROPOSlTIO IX.

Propositum sit in cono scaleno triangulum per axem exhibere quod cum minimo triangulorum per eundem facto datam obtineat rationem: quae tamen maior non existat illa quae est inter triangulorum maximum & minimum sectione per axem facta productorum.

Constructio & demonstratio.



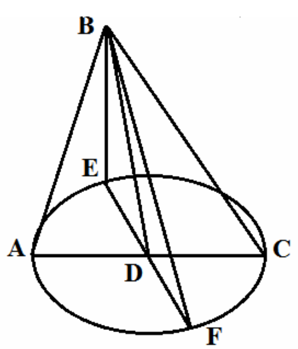
Datus sit conus scalenus, basim habens BCD, in quam normalis cadat AF: axis sit AE, centrum basis E. Ratio autem data sit M ad N. Et quoniam illa ponitur non maior ratione maximi per axem trianguli ad minimum, neque maior erit ratione ipsius AE ad AF, ut colligitur ex coroll. praeced: Ducto iam per A, F, E, puncta plano producatur triangulum ABD quod per *7*.huius erit minimum per axem triangulorum; assumpta deinde recta GH, quae sit aequalis EA, describatur semicirculus GIH, cuius diametcr GH, &: rectae AF aequalem HI inscribe semicirculo: denique ut N ad M, ita fiat HI ad HO, quae ex puncto H aptati poterit in semicirculo inter puncta I, G, cum ratio data M ad N, non sit maior ratione AE ad AF, hoc est, ob linearum ex constr. aequalitatem, HG ad HI. Deinde iuncta OG, quae cum HO angulum rectum constituet, utpote in semicirculo, fiant GL,GK aequales semidiametro EB, iungaturque HL, HK. Quoniam igitur bases BD, LK, aequales sunt, triangulum LHK erit ad triangulum BAD, ut perpendicularis HO ad perpendiculateris AF, hoc est ut HO ad HI, hoc est ut M ad N restat igitur ut situm huius trianguli in cono ABCD assignemus, assumpta linea FG quae sit aequalis HO, describatur intervallo PQ semicirculus PRQ, deinde aptetur in semicirclo PR aequalis HI; hoc est AF, & iungatur RQ: tandem super base FE fiat triangulum ex lineis OG, RQ nimirum FSE, ut ES ipsi OG, & FS aequalis sit QR, tum per E & S agatur diameter TV, ponanturque AT, AV & AS. dico triangulum AT aequale triangulo LHK & simile ; Quoniam AF ducta est normalis ad basim coni, erit angulus AFS rectus, & quadratum AS aequale quadratis AF, FS, hoc est (quia ex constructione AF est PR, & FS est RQ*)* quadratis PR, RQ. Atqui etiam quadratum PQ aequatur iisdem quadratis PR, RQ; igitur quadratum AS aequale est quadrato PQ, adeoque & recta AS aequalis est recte PQ, hoc est rectae HO. Iam vero AS normalem esse basi TV, ut HO est basi LK, sic ostendo. Quadratum HG aequatur quadratis HO, GO: quare cum ex constructione AE ipsi HG, & SE ipsi OG, & ex demonstrat. AS ipsi HO sit aequalis, aequabitur etiam AE quadratis AS, SE: ac proinde angulus ASE rectus est, & AS perpendicularis basi. Quare cum in triangulis LHK, TAV & bases LK, TV ex constructione, & perpendiculares sive altitudines HO, AS, aequales sunt, ipsa quoque triangula erunt aequalia. lnsuper cum OG ipsi SE, & GK ipsi EV aequales sint, erit OK aequalis SV, sed & HO ostensa est aequalis AS, angulique HOG, ASV recti sunt & aequales. Ergo sunt aequalis est AV. similiter ostendam HL, aequari AT. Itaque similia etiam sunt triangula LHK, TAV. Quare assignavimus in cono scaleno triangulum per axem ATV quod ad triangulum ABD, habet datam rationem M ad N. Quod erat faciendum.



PROPOSlTIO X.

Si coni scaleni triangulum per axem, ad verticem angulum rectum contineat: dico omnia per axem facta angulum rectum continere.

Demonstratio.



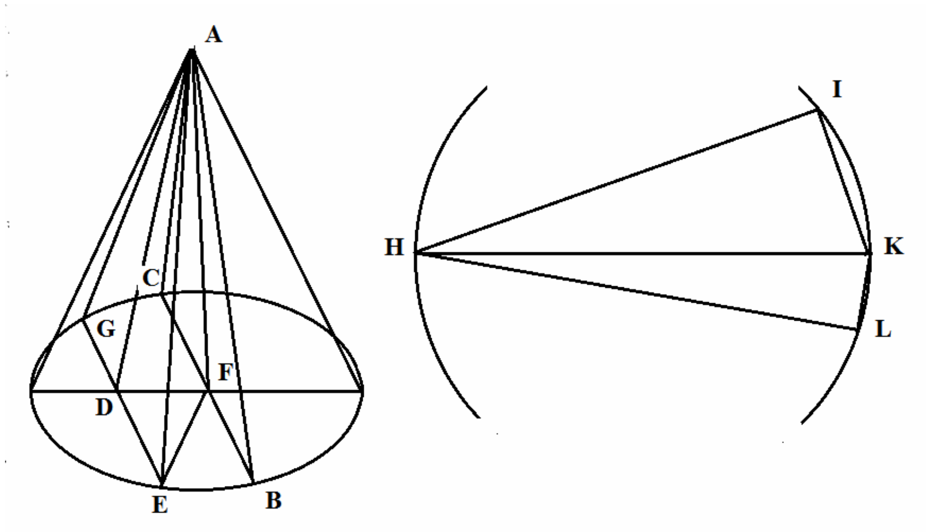
Scalenus conus secetur per axem sectione ABC, quod triangulum producat habens angulum ad verticem B rectum: sedetur autem quovis alio per axem plano exhibente triangulum BFE. dico angulum FBE, rectum esse. Quoniam angulus BC est rectus, quadratum AC aequale est quadratis AB, BC; sed supra ostensum est quadrata AB, BC aequari quadratis FB, EB. Ergo quadratum AC aequatur quadratis FB, EB est autem FE aequalis rectae AC, ergo & FE quadratum iisdem quadratis FB, BE aequale erit, quare angulus FBE rectus. Itaque si coni scaleni, &c. Quod fuit demonstrandum.

Scholion.

Praeter triangula qua eruuntur ex cono sectione per axem facta, alia quoque assignantur quae per verticem quidem transeunt, minime vero per axem ; quae cum sine *numero sint, hinc determinatione exigunt ut solutiones problematum quae circa ea versantur, recte expediantur*.

PROPOSlTIO XI.

In cono recto acutanangulo & rectangulo impossibile est triangulo per axem facto, aequale triangulum exhibere non per axem sive triangulum per axem maius est quolibet non per axem.



Demonstratio.

Nam si fieri potest, ponatur triangulum aliquod AEG non transiens per axem AF, aequale triangulo per axem ducto BAC, quod ita in cono ductum concide, ut eius basis BC sit parallela basi EG, quod fieri posse constat, cum omnia in conu recto perr axem triangula sint aequalia. Ducatur ex centro F rectas FD bisecans basim EG, iunganturque AD, FE, & super recta HK aequali ipsi AE, constitue bina triangula HIK, HLK, aequalia & similia trianglis AEF, ADE, sic ut latera HI, IK ; lateribus AD, DE aequalia sint, adeoque I & L, angulis AFE, ADE sint aequales. Quoniam igitur axis AF ex hypot. rectus est plano basis, erit angulus AFE rectus. Angulus quoque ADE rectus est, ut colligitur ex demonstratis in secundo casu quartae huius. Quare recti etiam sunt anguli I & L, puncta igitur I, L, H, K sunt ad circulum cuius diameter HK. Et quoniam conus datus est rectangulus vel acutangulus, axis AF erit vel aequalis semidiametro FE, vel maior. Quare cum AI ipsi AF, & IK ipsi FE sint aequales , erit quoque HI aut aequalis IK, aut major. Ergo vertex I trianguli AIK,vel bisecabit arcum HIK, vel erit alterum punctum bisectionis versus K. Deinde quia ADC, hoc est AL, maioir est quam AF hoc est HI, erit arcus HL maior arcu HI, ac proinde vertex L trianguli HLK non solum cadet ultra punctum quo bisecatur arcus HLK, sed etiam propius adhuc incidet versus K, quam alterius trianguli vertex I. Minor igitur est altidudo trianguli HLK, quam trianguli HIK, adeoque minus est ALK hoc est ADE, triangulum, triangulo HIK hoc est AFE. Atque minus est ALK hoc ADE, triangulum, triangulo HIK hoc est AFE. Atque triangulum AFE aequatur triangulo AFB, cum omnia utriuisque latera, vicissim sint aequalia. Ergo triangulum ADE minus etiam est triangulo AFB: ac proinde triangulum EAG duplum ipsius ADE. (est enim EG ex constructione bisecta in D) minus est triangulo BAC duplo ipsius AFB. similiter demonstrabimus quodvis aliud triangulum quod per axem non sit factum, minus esse triangulo per axem. In cono igitur recto, &c. Quod erat demonstrandum.

Quod autem in demonstratione fuit assumptum, (in recto) nempè cono acutangulo axem semidiametro basis esse maiorem, in cono vèro recto rectangulo aequalem, paucis sic demonstrabo.

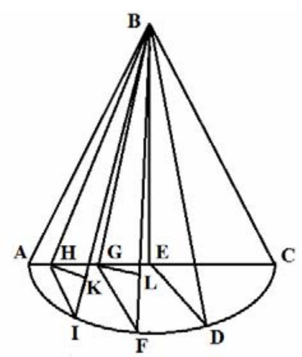
Sit conus rectus acutangulus sectus triangulo per axem BAC quoniam in triangulis BFA, CFA, latera BA, CA, BF, FC aequalia sunt, & FA commune, anguli quoque FAB, FAC aequalia sunt, sed totus BAC est minor recto utpote acutus, angulus igitur FAB ipsius dimidius minor est semirecto. Quare cum angulus BFA rectus sit, erit reliquus ABF minor semirecto, hoc est maior angulo FAB. Ergo axis AF minor est semidiametro FB.

Quod si conus fuerit rectus rectanguluus, tunc angulus BAC erit rectus, ac proinde BAF dimidius totius BAC, semirectus est. Quare cum angulus BFA rectus sit, necesse est, ut reliquus FBA etiam sit semirectos, adeoque aequalis angulo BAF: unde axis AF & semidiameter basis coni aequales erunt.

PROPOSlTIO XII.

Aliter & expeditius praecedens theorema hunc in modum demonstrabitur

Demonstratio.

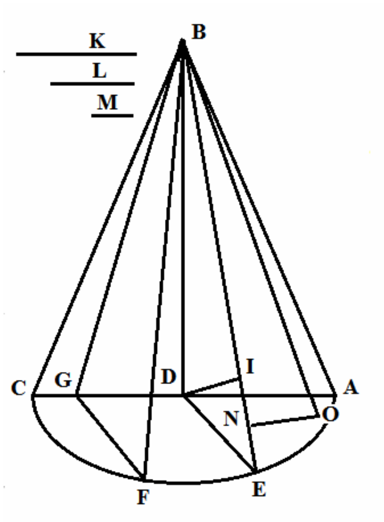


In cono rectangulo vel acutangulo sit triangulum per axem BFG & quodvis aliud extra axem IBH, diameter autem basis coni sit AEC ex punctis G ac H ducantur GL, HK normales ad BF, FI. Quandoquidem conus sit rectangulus vel acutangulus, erit axis BG ut ostendi supra, aequalis semidiametro GF, aut maior. Ex quo liquet segmentum BL non esse minus segmento LF, ac proinde punctum L cadere vel inter & F punctum medium rectum FB, vel ad summum in ipsam medium punctum incidere. Deinde quia angulus BGF, BHI, recte sunt, ut patet ex dictis prop.11. & ab iis normales dimissae GK, HK erunt rectangula LFB, KIB aequalia quadratis FG, GL. Atqui quadratum GF maius est quadrato HI, rectangulum igitur LFB. maius est rectangulo KIB. Ergo cum BI, BF aequales sint, IK minor est quam FL quoniam igitur, ut ante ostendi, punctum L cadit inter F & punctum medium rectae BF, punctum K
iam cadet inter I & punctum medium rectae BI, id est BF, & quidem propius versus I, quam L versus F. Liquet igitur rectangulum BKI minus esse rectangulo BLF.Atqui quadrata LG, KH, aequantur rectangulis BLF, BKI, ergo quadratum KH minus est cum quadrato LG, ergo normalis KH minor normali LG. Quare cum bases BI, BF aequales sint, triangulum IBH minus est triangulo BFG. Similer ostendam quodvis aliud extra axem triangulum minus esse triangulo per axem FBG, illud igitur maximum est. Quod fuerat demonstrandum.

PROPOSlTIO XIII.

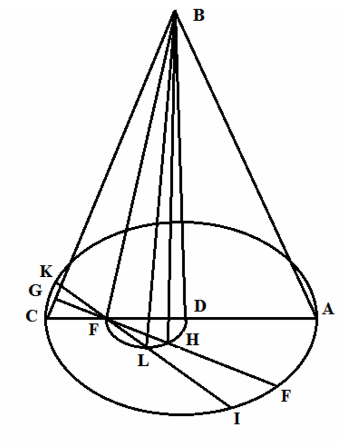
Datus sit conus rectus axem habens BD, & ratio maioris inaequalitatis K ad L. Oporteat triangulum exhibere extra axem ad quod BED triangulum per axem datam habet rationem K ad L.

Constructio & demonstratio.



Ex puncto D ad BE normalem ducito DI, fiatque ut K ad L sic DI ad M. Tum quadrato M aequale fac rectangulum BNE, & quoniam quadratum M minus est quadrato DI, hoc est (quia ex angulo recto BDE ad basim ducta est normalis DI) rectangulo BIE, erit quoque BNE rectangulum minus rectangulo BIE. Erigatur inde ex puncto N normalis NO par rectae M, iungaturque BO, EO. Quoniam igitur quadratum NO aequatur quadrato M, hoc est rectangulo BNE, erunt BN, NO, NE tres continuae. Angulus igitur BOE rectus est. Quare cum triangula BOE, BDE super eadem basi BE utraque sint triangula, & unius nempe BOE altitudo NO minor sit altitudine alterius DI, paret alterutrum E lateribus BO, OE maius esse alterius trianguli latere BD, quod idem est cum axe coni. Sit latus BO maius axe BD: eritque; hoc nihilominus minus latere coni, BE, sive BC; Quoniam igitur latus BO maius est axe BD, & minus latere BC; intra conum aptare poterit ex puncto BO ad diametrum AC, recta AG aequalis lateri BO. Factum sit, & ex G ad BG normalem duc GF & iunge BF. Quandoquidem anguli BOE, BGF recti sunt , quadrata BE, BF quadratis BO, OE; BG, GF aequalia sunt; sed quadrata BE, BF aequantur, aequantur igitur quadrata BO, OE, quadratis BG, GF. Cum igitur quadrata BO, BG ob rectarum ex constructione aequalitatem, sint aequalia, reliqua etiam OE, GF aequalia erunt; aequantur igitur rectae OE, GF; in triangulis ergo BOE, BGF singula singulis latera sunt aequalia. Ergo ipsa quoque triangula sunt aequalia. Atqui triangulum per axem BED ad triangulum BOE eandem habet rationem quam normalis DI ad normalem NO, hoc est quam DI ad M, hoc est quam K ad L: ergo triangulum per axem ED ad normalem NO, hoc est quam DI ad M, hoc est quam DI ad M, hoc est quam K ad L: ergo triangulum per axem BED ad triangulum quoque BGF rationem eandem habet quam K ad L. Exhibuimus ergo triangulo per axem , &c. Quod erat faciendum.

PROPOSITIO XIV.



In cono recto triangula non per axem ducta, quorum bases in eodem puncto se intersecant, habent perpendiculares ad bases e vertice ductas, In peripheria circuli cuius diameter est recta inter centrum basis coniae & punctum intersectionis interiecta.

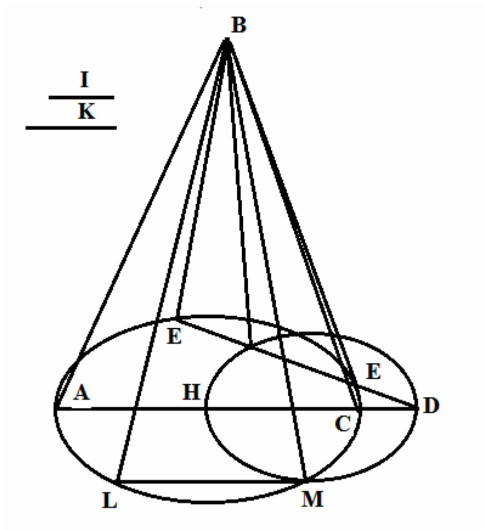
Demonstratio.

In cono recto ABC intersecent sese mutuo in E puncto quaevis bases FIG, IEK triangulorum non per axem FBG, IBK, & circa DE, inter centrum D & E punctum interiectum describe circulum DHLE : dico perpendiculares a vertice B ductas ad FEG, IEK, incidere in peripheriam DLE. iungantur enim BL, BH, DL, DH, erit angulus DLE in semicirculo rectus unde rectae IL & LK aequales erunt; est autem recta BI aequalis BK ; igitur etiam BL normalis est ad rectam IK. similiter ostendemus L normale esse BH ad FG. ergo omnia triangula per E punctum & verticem ducta normales suas invenient in perimetro circelli DLE. quod oportuit demonstrare.

Scholion.

Cum intersectionis punctum E aut intra basim dari possit, aut in perimetro basis, aut extra basim, sit ut hic triplex casus oriatur: primus, quo circellus DEH totus base comprehenditur; secundus quo idem punctum perimetri AIC *eundem contingendo : in quo casu quarta est pars circuli, qui fungitur basis vice coni ABC; tertius vero quo extrabasim difffusus circulum baseos secat in tuobus punctis; sed quia triplex hic casus diversitatem nec in ipsi constructione, nec in demonstratione inducit, hinc silentio tam in propositione quam demonstratione suppressus est.*

PROPOSITIO XV.



In cono recto exhibere triangulum quod per apicem coni & punctum datum sive intra, sive extra coni basim transeat, habeat vero ad triangulum per axem datum rationem.

Constructio & demonstratio.

Conus esto ABC rectos, & per eius axem facto triangulo ABC detur punctum quoddam D, oporteat vero per punctum D, & verticem coni constituere triangulum EBF quod ad triangulum per axem ABC rationem habeat I ad K datum. Inveniatur per duodecimam triangulum LBM, quod habeat ad triangulum ABC rationem I ad K. Deinde ex D puncto ducatur DFE, sic ut FE sit aequalis ipsi LM, per ea quae de circulorum proprietatibus conscripsi: & secundum EF & B, agatur planum, hoc ipsum exhibebit triangulum postulatum BEF, erit enim EBF aequale triangula BLM, cùm basis LM aequalis sit EF ex constructione, & lineae EB, FB non solum sint aequales inter se, sed etiam sint aequales rectis LB, BM. Igitur perfecimus quod imperatum fuit.

Scholion.

Hactenus conati utcunque sumus sectiones coni explicare qua per apicem transeunt sive in recto, sive in scaleno cono: quia vero apud antiquiores, maxime Archimedem, alia nomenclatura insignitas reperies reliquas coni sectiones, quarum explanationem sequentibus libis prosequi intendimus; hinc operae pretium iudicaui praemittere quae ad rem hanc pertinere videbuntur.

In duplici differentia conos esse possuimus, alios scilicet rectos, alios scalenos: antiqui vero eos in triplici differentia posuerunt. Quosdam dixerunt conos rectangulos, nonnullos acutiangulos, alios denique obtsiangulos, denominatione sumpta ab angulis quos sectio per axes facta contineret ad verticem coni; unde illis distinctio conorum rectorum & scaleorum ignota fuisse videtur, cum scalenus conus, omnis generis anglos admittat ad verticem, variata solummodo, secundum diversos situs, sectione per axem. Rectangulum itaque conum dixerunt conum Isoscelium qui admitterit nullam varietatem in triangulis per axem exsurgentibus, omnes alios angulos excludens praeter rectum Conum vero rectangulum prius dispescebant divisione per axem, ortique inde trianguli latus alio plano secabant ad angulos rectos quod insuper ipsi triangulo esset orthogonum; quam sectionem appellabant coni rectanguli, recentiores autem parabolam dicunt. Conum vero acutiangulum similiter Isoscelium partiebantur bifariam per axem, ac primo quidem sectionem per axem coni instituebant; deinde triangulum ex sectione per axem productum, alio secabant plano, quod tam plano quam lateri trianguli rectum esset; quo fiebat ut alterit quoque crurum occurrens, figuram alterius formae proferrent, & a *parabola longe diversam, quae scilicet tota clauderetur intercapedine linearum rectarum trianguli per axem sectione inde resultantem coni acutianguli dicebant Isoscelem, denique conum obtusiangulum, eadem praxi & per axem dividebant, deinde plano ad unum laterum, & ad ipsum triangulum recto per conum acto, figuram formabant quam coni obtusianguli vocabant, quae in idem recidit cumhyperbola recentiorum. Haec igitur antiquorum ration divident conum, a recentioribus maxime vero ab Apollonio, nonnihil est imutata: conus enim qualiscunque sit ex natura sua tale corpus est, ex quo singulae harum sectonum erui possint; quodvis enim triangulum per axem exhibitum trifariam dividi potest a plano huic ad angulos rectos constituto; aut enim communis intersectio horum planorum, trianguli scilicet per axem, & plani orthogonaliter eidem insistentis aequidistat uni laterum trianguli, & hoc situ parabolam gignit; aut utrique occurrit intra triangulum producto, atque ita hyperbolam exhibet; has omnes sectiones easdem esse cum illis quas antiqui creabat, horum librorum decursu intelliges: non ita tamen ut quaecunque sectio a recentioribus facta exhiberi nata sit in coni alicuius sectione ab antiquis facta; nam antiqui unam solam Ellipseos speciem proferre poterant unamque speciem Hyperbolae in dato cono, ita ut si diversam requirerent Ellipsim aut hyperbolam, ad conum confugere deberent magis minusve acutum aut obtusum. Infinitae autem varietates oiuntur in quovis cono secto iuxta methodum recentiorum, prout cognoscere poteris benigne Lector si sequentia prosequaris.*

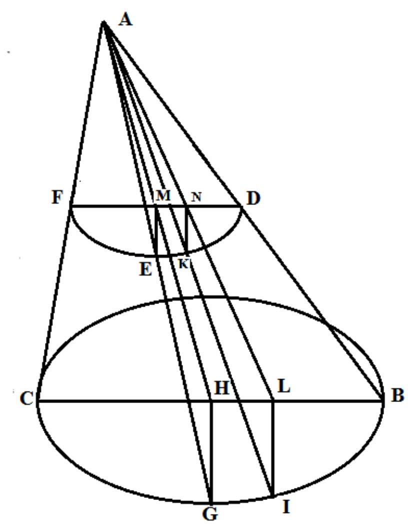
Insuper recentiorum ratio secandi conum, perhibet circulum sectione non facta ad basim coni equidistante; quem circulum antiqui eruere nequibant, quia hunc circulum exhibere non valet conus rectus sive Isoscelius, quem antiqui solum cosiderauere, sed solummodo ex cono scaleno eruitur, uti sequentia docebunt.

Priusquam huic discursui finem imponam,verbulo insinuandum mihi est cur axem coni determinent lineam a vertice coni ad circuli centrum demissam potius quam ad centrum Ellipseos, cum Ellipsis non minus centrum obtineat quam circulus, & secundum Ellipseos perimetrum a puncto in sublimi posito si circumagatur recta linea eandem prorsus conicam superficiem describat quam si ab eodem puncto circa circulis eiusdem coni producti lineam circumduxeris. In promptu responsum est, neque enim filia matrem praecedit, gignitur quippe e cono Ellipsis. Insuper circuli proprietates cum ex elementis supponantur omnibus notae, inseruiunt recte & commode demonstrationibus propositionum quae ex cono eruuntur: Ellipseos autem cum accidentia ignota sint, frustra ea adhiberetur ad theorematum aut problematum veritates manifestandas, potissima vero ratio mihi videtur, quod Ellipses in cono assignari queant sine numero inter se diversae ac proinde pro singulis deberet mutari axis coni: non enim iidem esse possunt axes a vertice conni ad circuli centrum, cum axibus ab eodem vertice ad centra Ellipseon, varietas ergo haec inimica doctrinae ellipsin merito excludit ne baseos munere fungeretur in cono, perfectior tamen videtur cono scaleno Rectus qui praeter unum axem alterum non admittit, uti conus scalenus, ex quo capite imperfectionem includit.

PROPOSlTIO XVI.

IN cono quocunque omnis sectio basi parallela circulus est.

Demonstratio.



Detur conus ABC cuius basis BGC, & sectio EF aequidistans plano baseos dico DEF esse circulum ducantur enim AH per *c*entrum & AL quaevis altera in plano trianguli per axem ABC occurrences rectae DF in punctis M, N & HG, LI normaliter erigantur ad BC diametrum, & iunctis AG, AI occurrentibus sectioni DEF in E & K, iungantur ME, NK: quoniam plana aequidistantia sunt DEF, BGC, erunt a quoq: ME, HG, & NK, LI & MN, HL aequidistantes ; igitur ME ad HG, ut AM ad AH, hoc est: ut AN ad AL, hoc est ut MK ad LI. Itaque permutando ME est ad NK ut HG ad LI adeoque & quadratum ME ad quadratum NK ut quadratum HG ad quadratum LI; sed quadrato HG aequale est BHC rectangulum, & LI similiter quadratum aequale rectangulo BLC. Igitur ME quadratum est ad NK quadratum ut BHC rectangulum ad rectangulum BLC, est autem ut BHC rectangulum ad BLC rectangulum, ita rectangulum DMF ad DNF rectangulum, cum ex iisdem rationibus composita sint; ita rectangulum DMF ad DNF rectangulum, cum ex iusdem rationibus composita sint; igitur quadratum ME est ad NK quadrarum, ut rectangulum DMF ad DNF, rectangulum, cum ex iisdem rationibus composita sint; igitur quadratum ME est ad NK quadratum, ut rectangulum DMF ad DNF, rectangulum. Iam vero DM est ad BH, ut AM ad AH, hoc est: ut ME ad HG. Ergo permutando DM est ad ME, ut BH ad HG. Sed BH, HG sunt aequales, ergo & DM, ME aequales sunt. Igitur quadratum DM, hoc est rectangulum DMF (nam cum BC sit bisecta in H, erit & DF in M) aequatur quadratum ME. Quare cum quadratum ME; ut supra ostendimus, sit ad quadratum NK, ut rectangulum DMF, ad rectangulum DNF, etiam rectangulum DNF, aequabitur quadrato NK. Denique cum DF, BC sint parallelae itemque GH, EM, & una GH sit normalis ad unam BC, erit altera EM normalis ad alteram DF; similiter ostendam KN normalem esse ad DF. Itaque cum normalium EM, KN quadrata aequalia sint rectangulis DMF, DNF, sectio DE, EF circulus est. Quod fuit demonstrandum.

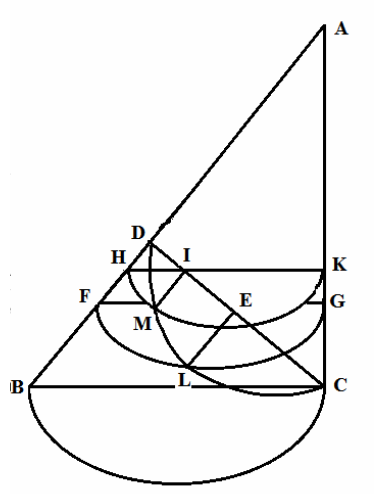
Corollarium.

Ex discursu demonstrationis liquet centrum circuli DEF esse in axis puncto E, in quo puncto E, in quo nimirum axis occurrit rectae DF, quae est communis sectio trianguli per axem & circui DEF.

PROPOSITIO XVII.

In cono scaleno circulum exhibere qui basi aequidistans non sit.

Constructio & demonstratio.

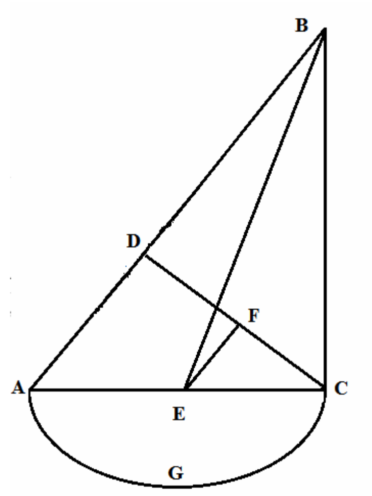


Scalenus conus ponatur ABC sectus plano per axem proferente triangulum ABC minimum illorum quae per axem fieri possunt erit illud scalenum, cum omne scalenum sit quod per axem sit praeter maximum per cor.2.octanae; angulus igitur ABC minor ponatur angulo ACB: quare fiat angulo ABC aequalis ACD & per recta DC duc planum rectum ad triangulum ABC : dico sectionem inde natam DLC esse circulum. Ducantur enim duo alia plana basi parallela FLH, HMK, quorum communes sectiones cum plano DLC sint rectae EL, IM, ex 7.huius patet basim coni ad minimum triangulum ABC rectum esse; ergo & alia plana parallela FLH, HMK triangulo ABC recta erunt : sed & planum DLC triangulo rectum est ; ergo communes sectiones LE, MI, rectae sunt triangulo, adeoque & lineae DC. Quoniam autem FLH, HMK, circuli sunt per praecedentem, & EL, IM normales DC, erit tam rectangulum FEG aequale quadrato EL, quam rectangulum HIK, quadrato IM, quia vero similia sunt triangula DIH, & IKC, cum angulo ACD, sit ex constructione aequalis ABC, hoc est DHI: erunt puncta HDKC in eodem circulo: similiter ostendam puncta quoque FD, GC esse ad circulum: quare rectangula DIC, HIK aequalia sunt , uti & rectangula FEG, & DEC: ac proinde aequalia etiam quadratis IM & ELperpendicularium ad DC. Igitur puncta DMLC, circulus est cuius recta DC est diameter; igitur, &c.

PROPOSITIO XVIII.

Ostendendum modo est in quovis cono scaleno duos axes esse.

Demonstratio.



Conus itaque scalenus ABC secetur plano per axem BE, quae ad centrum circuli AGC baseos protenditur: triangulum productum ABC secetur recta CD, quae angulum BCD aequalem exhibeat angulo BAC: & dividatur DC bifariam in F, & iungatur BF. dico illam esse axem secundum huius coni: quod ita patebit, praecedenti proposition sectionem factam per rectum DC, eo modo quo illic praeceptum fuit, circulum producere: igitur recta ex apice B ad punctum F quod huius circul est demissa, axis nomen obtinere debet ex ipsa axeos definitione; restat igitur ostendere BFE rectam non esse lineam sive BF productam non incidere in E centrum basis, ac proinde BF axem esse alium ab axe BE, quod inde patet, quod EF recta sit aequidistands lineae AB, cum secet duas rectas CD, CA bifariam. Igitur si recta est linea EFB, sequetur duas parallelas sese intersecare, quod est absurdum.Constat igitur omnem conum scalenum duos axes admittere. Quod demonstrandum fuit.