CHAPTER VI

CONCERNED WITH THE MINIMAL DISTURBANCES

OF THE AIR IN CONICAL TUBES

PROBLEM 92

128. *If the tube shall have had a conical shape and the air contained in that may be disturbed everywhere from the state of equilibrium, to describe both the scales, from which henceforth the state of the air may be able to be defined henceforth at any time.*

SOLUTION

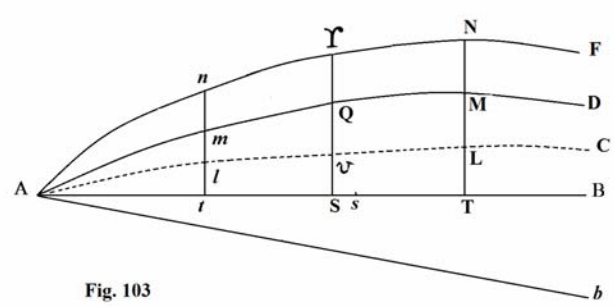

Fig. 103

The vertex of the cone shall be (Fig. 103) at A and the right line AB its axis, on which with some point taken S and with the distance put $AS = S$, the initial state of the air introduced shall be of this kind at S, so that the density were $= Q$, with the natural state being $= B$, truly the speed in the direction $AB = \Upsilon$. Now since the amplitude of the tube at S shall be $\Omega = nnSS$, from §. 88 adapted to this case we will have $\alpha = 1,\ \beta = 0$, and hence $M = \frac{-1}{S}$ and $L = \frac{1}{SS}$. So that if now we may transfer the solution of problem 84 here and we may ignore the force of gravity, initially we obtain :

$$O = \frac{1}{SS}\int SSdSl\frac{Q}{B},$$

then truly after some elapsed time t the state of the air, which initially was at S, is defined thus, so that the distance of the movement shall become :

$$Ss = \frac{1}{SS}\int SSdSl\frac{Q}{B} - \frac{1}{S}\Gamma':(S+ct) + \frac{1}{SS}\Gamma:(S+ct)$$
$$-\frac{1}{S}\Delta':(S-ct) + \frac{1}{SS}\Delta:(S-ct) = \upsilon,$$

from which we deduce for the density q and for the speed $\mathfrak{T}$:

$$\left(\frac{d\upsilon}{dS}\right) = \frac{-2}{S^3}\int SSdSl\frac{Q}{B} - \frac{1}{S}\Gamma'':(S+ct) + \frac{2}{SS}\Gamma':(S+ct)$$
$$-\frac{2}{S^3}\Gamma:(S+ct) + l\frac{Q}{B} - \frac{1}{S}\Delta''(S-ct) - \frac{2}{S^3}\Delta:(S-ct);$$

and on account of $\frac{d\Omega}{\Omega dS}=\frac{2}{S}$ there is :

$$\frac{\upsilon d\Omega}{\Omega dS}=\frac{2}{S^3}\int SSdSl\frac{Q}{B}-\frac{2}{SS}\Gamma'':(S+ct)+\frac{2}{S^3}\Gamma:(S+ct)$$
$$-\frac{2}{SS}\Delta':(S-ct)+l\frac{Q}{B}+\frac{2}{S^3}\Delta:(S-ct),$$

whereby, since there shall become $q=Q\left(1-\frac{\upsilon d\Omega}{\Omega dS}-\left(\frac{d\upsilon}{dS}\right)\right)$, we deduce

$$q=Q\left(1-l\frac{Q}{B}+\frac{1}{S}\Gamma'':(S+ct)+\frac{1}{S}\Delta'':(S-ct)\right)$$

and finally on account of

$$\mathfrak{T}=\left(\frac{d\upsilon}{dt}\right)$$

there will become

$$\mathfrak{T}=-\frac{c}{S}\Gamma'':(S+ct)+\frac{c}{SS}\Gamma':(S+ct)$$
$$+\frac{c}{S}\Delta'':(S-ct)-\frac{c}{SS}\Delta':(S-ct).$$

Hence therefore for the initial state on putting $t=0$ we arrive at :

$$\upsilon=\frac{1}{SS}\int SSdSl\frac{Q}{B}-\frac{1}{S}(\Gamma':S+\Delta':S)+\frac{1}{SS}(\Gamma:S+\Delta:S)$$
$$q=Q\left(1-l\frac{Q}{B}+\frac{1}{S}(\Gamma'':S+\Delta'':S)\right)$$
$$\mathfrak{T}=-\frac{c}{S}(\Gamma'':S-\Delta'':S)+\frac{c}{SS}(\Gamma':S-\Delta':S).$$

Therefore since then there will have been $\upsilon=0,\ \ q=Q$ et $\mathfrak{T}=\Upsilon$, we obtain the following equations, from which it will be required to find the nature of the two function Γ and Δ :

$$\text{I.}\qquad \int SSdSl\frac{Q}{B}-S(\Gamma':S+\Delta':S)+\Gamma:S+\Delta:S=0$$
$$\text{II.}\qquad Sl\frac{Q}{B}=\Gamma'':S+\Delta'':S$$
$$\text{III.}\qquad \frac{1}{c}SS\Upsilon=-S(\Gamma'':S-\Delta'':S)+\Gamma':S-\Delta':S,$$

the first differentiation of which agrees with the second, indeed just as the nature of the matter demands, thus so that it shall require to be satisfied by the two conditions only. For the sake of brevity we may put :

$$\Gamma':S+\Delta':S=y\ \text{ and }\ \Gamma':S-\Delta':S=z,$$

so that there may become :

$$\text{II.} \quad SdSl\tfrac{Q}{B} = dy \text{ and}$$

$$\text{III.} \quad \tfrac{1}{c}SS\Upsilon dS = -Sdz + z : dS$$

and hence

$$y = \int SdSl\tfrac{Q}{B} \quad \text{and} \quad \tfrac{-z}{S} = \tfrac{1}{c}\int \Upsilon dS \quad \text{or} \quad z = \tfrac{-S}{c}\int \Upsilon dS,$$

from which the functions are defined thus, so that there shall become:

$$\Gamma' : S = \tfrac{1}{2}\int SdSl\tfrac{Q}{B} - \tfrac{S}{2c}\int \Upsilon dS$$

and

$$\Delta' : S = \tfrac{1}{2}\int SdSl\tfrac{Q}{B} + \tfrac{S}{2c}\int \Upsilon dS$$

and hence again on differentiating

$$\Gamma'' : S = \tfrac{1}{2}Sl\tfrac{Q}{B} - \tfrac{1}{2c}\int \Upsilon dS - \tfrac{1}{2c}S\Upsilon$$

$$\Delta'' : S = \tfrac{1}{2}Sl\tfrac{Q}{B} + \tfrac{1}{2c}\int \Upsilon dS + \tfrac{1}{2c}S\Upsilon,$$

truly on integrating there will be found:

$$\Gamma : S = \tfrac{1}{2}S\int SdSl\tfrac{Q}{B} - \tfrac{1}{2}\int SSdSl\tfrac{Q}{B} - \tfrac{1}{4c}SS\int \Upsilon dS + \tfrac{1}{4c}\int SS\Upsilon dS$$

$$\Delta : S = \tfrac{1}{2}S\int SdSl\tfrac{Q}{B} - \tfrac{1}{2}\int SSdSl\tfrac{Q}{B} + \tfrac{1}{4c}SS\int \Upsilon dS - \tfrac{1}{4c}\int SS\Upsilon dS.$$

Therefore the two curved lines AQD and ATF may be constructed on the axis from the given initial conditions by taking the applied lines

$$SQ = Sl\tfrac{Q}{B} \quad \text{and} \quad S\Upsilon = \tfrac{1}{c}\int \Upsilon dS + \tfrac{1}{c}S\Upsilon,$$

and the nature of the functions Γ and Δ will be determined thus, so that there shall become:

$$\Gamma'' : S = \tfrac{1}{2}SQ - \tfrac{1}{2}S\Upsilon; \quad \Delta'' : S = \tfrac{1}{2}SQ + \tfrac{1}{2}S\Upsilon$$

and hence again by the areas

$$\Gamma' : S = \tfrac{1}{2}ASQ - \tfrac{1}{2}AS\Upsilon; \quad \Delta' : S = \tfrac{1}{2}ASQ + \tfrac{1}{2}AS\Upsilon,$$

then truly, if this form $M : ASQ$ may denote that quantity, which expressed the integral $\int dS \cdot ASQ$, there will be had :

$$\Gamma : S = +\tfrac{1}{2}M : ASQ - \tfrac{1}{2}M : AS\Upsilon$$
$$\Delta : S = +\tfrac{1}{2}M : ASQ + \tfrac{1}{2}M : AS\Upsilon.$$

From these curves described after the elapsed time $=t$, the intervals $ST = St = ct$ may be removed from the point *S* on both sides and the state of the air thus now is defined from the applied lines from these corresponding points, which initially was present around *S*, so that there shall become :

$$q = Q\left(1 - \tfrac{SQ}{S} + \tfrac{1}{2S}\left(TM - TN + tm + tn\right)\right)$$
$$\mathfrak{T} = \tfrac{c}{2S}\left(tm + tn - TM + TN\right) + \tfrac{1}{2SS}\left(ATM - ATN - Atm - Atn\right),$$

while the small distance of translation $Ss = \upsilon$ may be expressed thus:

$$Ss = \tfrac{1}{SS}\int SSdSl\tfrac{Q}{B} - \tfrac{1}{2S}\left(ATM - ATN + Atm + Atn\right)$$
$$+\tfrac{1}{2SS}\left(M : ATM - M : ATN\right) + \tfrac{1}{2SS}\left(M : Atm + M : Atn\right)$$

or

$$Ss = \tfrac{1}{SS}\int SSdSl\tfrac{Q}{B} - \tfrac{1}{2S}\left(ATM - ATN + Atm + Atn\right)$$
$$+\tfrac{1}{2SS}\left(M : ATM - M : ATN + M : Atm + M : Atn\right),$$

where there becomes:

$$\int SSdSl\tfrac{Q}{B} = \int S \cdot SQ \cdot dS = S \cdot ASQ - M : ASQ.$$

COROLLARY 1

128a. Therefore for some time *t*, with the intervals of the air taken $ST = St = ct$, which initially was at *S*, the state of the air is defined thus, so that there shall be

I. density $q = Q\left(1 + \tfrac{1}{2S}\left(TM + tm - 2SQ\right) - \tfrac{1}{2S}\left(TN - tn\right)\right)$

II. speed $\mathfrak{T} = \tfrac{c}{2S}\left(tm - TM\right) + \tfrac{c}{2S}\left(TN + tn\right) + \tfrac{c}{2SS} \cdot tmTM - \tfrac{c}{2SS}\left(ATN + Atn\right)$

III. distance $Ss = \tfrac{1}{2S}\left(2ASQ - ATM - Atm\right) + \tfrac{1}{2S} \cdot tnTN$

$$-\tfrac{1}{2SS}\left(2M : ASQ - M : AT\ M - M : Atm\right) - \tfrac{1}{2SS}\left(M : ATN - M : Atn\right).$$

COROLLARY 2

129. If the distance *AS* may be considered as infinite, the conical tube will be changed into a cylinder; and since then there becomes :

$$\tfrac{1}{S} \cdot SQ = l\tfrac{Q}{B}, \quad \tfrac{1}{S} \cdot S\Upsilon = \tfrac{\Upsilon}{c}, \quad \tfrac{1}{S} \cdot ASQ = \int dsl\tfrac{Q}{B},$$

and thus

$$\tfrac{1}{SS}\cdot ASQ = 0,\ \text{ but }\ \tfrac{1}{SS}M\cdot ASQ = \int dsl\tfrac{Q}{B},$$

then truly $\tfrac{1}{S}\cdot AS\Upsilon = \tfrac{1}{c}\int \Upsilon dS,$

hence

$$\tfrac{1}{SS}AS\Upsilon = 0 \quad\text{and}\quad \tfrac{1}{SS}M : AS\Upsilon = \tfrac{1}{c}\int \Upsilon dS,$$

these formulas agree with these, which have been found above for cylindrical tubes.

SCHOLIUM 1

130. The construction of the first scale *AQD* formed from the initial density is not subjected to any difficulty, but the other scale $A\Upsilon F$ is not simply constructed from the initial speeds Υ, but in addition involves a certain integral formula; on account of which hence we will be required to consider that construction with care. Therefore the curved lines $A\upsilon C$ described, the applied lines $S\upsilon$ of which shall show the initial speed Υ in place of S, thus to be constructed from that scale $A\Upsilon F$, so that its applied line shall be

$$S\Upsilon = \tfrac{1}{c}AS\cdot S\upsilon + \tfrac{1}{c}AS\upsilon,$$

then truly the area of this curve $AS\Upsilon = \tfrac{1}{c}AS\cdot AS\upsilon,$ since there shall be, as we have seen,

$$\int dS(\Upsilon S + \int \Upsilon dS) = S\int \Upsilon dS.$$

Therefore by the introduction of this curve $A\upsilon C$ in place of that, we will have

$$TN = \tfrac{1}{c}AT\cdot TL + \tfrac{1}{c}ATL,\ \ tn = \tfrac{1}{c}At\cdot tl + \tfrac{1}{c}Atl,$$

$$ATN = \tfrac{1}{c}AT\cdot ATL,\ \ Atn = \tfrac{1}{c}At\cdot Atl,$$

from which the formula for the speed

$$\mathfrak{T} = \tfrac{c}{2S}\left(tm - TM\right) + \tfrac{c}{2SS}\cdot tmTM + \tfrac{1}{2S}\left(AT\cdot TL + At\cdot tl\right) - \tfrac{ct}{2SS}TLtl\,,$$

on account of $ST = St = ct$. And hence now it is clear, if through the whole distance *Tt* both the initial density will have been natural as well as the speed zero, then also to become $q = Q$ and $\mathfrak{T} = 0$, which latter is less evident from the first formula. If indeed if indeed the initial motion were within the distance *At* and the curve $A\upsilon C$ will have enclosed a certain area there, even if henceforth this whole curve shall lie on the axis, this area will present an applied line for the curve *ATF* also through the whole axis following *tT*, so that neither the areas *ATN* and *Atn* nor the applied lines *TN* and *tn* will vanish: and thus some doubt remains, whether the formula

$$\tfrac{c}{2S}(TN+tn)-\tfrac{c}{2SS}(ATN+Atn)$$

may vanish in this case; so that now moreover it is understood by necessity a certain scale $A\upsilon C$ must arise to be introduced into the calculation.

SCHOLION 2

131. In place of the index *M*, which we have put in place in front of the area, so that the written expression $M:ASQ$ may denote the integral $\int dS\cdot ASQ$, we will use more conveniently with the summation sign $\int$, so that $\int ASQ$ will denote the same, as $\int dS\cdot ASQ$, since the differential of the abscissa *dS* is easily supplied mentally. With this allowed also I may write $\int AS\Upsilon$ in place of $M:ASQ$ for the scale $A\Upsilon F$, which form is determined thus from the scale of the scale of the natural speed $A\upsilon G$, so that there shall become

$$\int AS\Upsilon=\tfrac{1}{c}\left(AS\int AS\upsilon-\int\int AS\upsilon\right)$$

and thus

$$M:ATN=\tfrac{1}{c}\left(AT\int ATL-\int\int ATL\right)$$

$$M:Atn=\tfrac{1}{c}\left(At\int Atl-\int\int Atl\right).$$

Whereby our determinations by means of the scale of the density *AQD* and of the speed $A\upsilon C$ themselves will be had thus :

$$q=Q\left(1+\tfrac{1}{2S}(TM+tm-2SQ)-\tfrac{1}{2cS}(AT\cdot TL-At\cdot tl+tlTL)\right)$$

$$\mathfrak{T}=\tfrac{c}{2S}(tm-TM)+\tfrac{1}{2SS}TMtm+\tfrac{1}{2S}(AT\cdot TL+At\cdot tl)-\tfrac{ST}{2SS}\cdot tlTL$$

$$Ss=\tfrac{1}{2S}(SQtm-SQTM)-\tfrac{1}{2SS}\left(\int SQtm-\int SQTM\right)$$
$$+\tfrac{1}{2cS}(\mathrm{AT}\cdot ATL-At\cdot Atl)-\tfrac{1}{2cSS}\left(AT\cdot\int ATL-AT\cdot\int Atl-\int tlTL\right),$$

which are seen to be more useful.

PROBLEM 93

132. *If a pulse may be excited through the minimum distance GH somewhere* (Fig. 104) *in the conical tube ABb extended indefinitely, to determine the propagation of this pulse arising in the tube towards Bb* .

SOLUTION

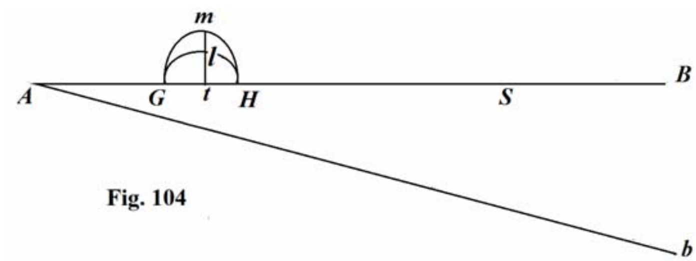

Fig. 104

Since the pulse may be contained in the small interval *GH* , it is required at each end *G* and *H* to be the natural density and the speed to be zero. Therefore *GlH* shall be the scale of the speed, the applied line of which *tl* may express the initial speed at the location *t*, so that there shall be $tl = \Upsilon$ in the direction towards *B* : for the scale of the density truly *GmH* shall be the applied line $tm = At \cdot l\frac{Q}{B}$ with *Q* denoting the initial density at *t*, and *B* the natural density. With these in place we will consider some location in the tube *S*, with there being $AS = S$, and since both scales outside the interval *GH* lie on the axis, so that the air in the tube at *S* will persist in equilibrium for a long time, so that the pulse will flow from the start for the time $= \frac{SH}{c}$ sec., moreover the pulse may be considered to cease again after the elapsed time $\frac{SG}{c}$, thus so that no pulse shall be going to be present except for the time $\frac{GH}{c}$. Therefore we may put the elapsed time to be $t = \frac{St}{c}$ from the beginning for the time $\frac{GH}{c}$ to pass , and since for the other part towards the equal other part *B* with the abscissa distance $ST = St$ [Fig. 103] there both the applied lines *TL* and *TM* vanish, the density of the air *q* and the speed $\mathfrak{T}$ tending towards *B* at the location *S* will be expressed thus:

$$q = B\left(1+\frac{tm}{2S}+\frac{At\cdot tl}{2cS}-\frac{Htl}{2cS}\right)$$

since the initial density at *S* was $= B$, then truly there will become :

$$\mathfrak{T} = \frac{c\cdot tm}{2S}+\frac{c\cdot Htm}{2SS}+\frac{At\cdot tl}{2S}-\frac{St\cdot Htl}{2SS}.$$

Bu since the interval *GH* may be assumed minimal, the areas *Htl* and *Htm* can be considered as vanishing; whereby on account of $tl = \Upsilon$ and $tm = Atl\frac{Q}{B} = At\left(\frac{Q}{B}-1\right)$ we will have :

$$q = B\left(1+\tfrac{At}{2AS}\left(\tfrac{Q}{B}-1\right)+\tfrac{At\cdot\Upsilon}{2c\cdot AS}\right) \text{ or } l\,\tfrac{Q}{B} = \tfrac{At}{2AS}\left(l\,\tfrac{Q}{B}+\tfrac{\Upsilon}{c}\right)$$

$$\mathfrak{T} = \tfrac{c\cdot At}{2AS}\left(\tfrac{Q}{B}-1\right)+\tfrac{At\cdot\Upsilon}{2AS} = \tfrac{c\cdot At}{2AS}\left(l\,\tfrac{Q}{B}+\tfrac{\Upsilon}{c}\right).$$

COROLLARY 1

133. If we may not wish to neglect *Htl* and *Htm*, we will have

$$l\,\tfrac{Q}{D} = \tfrac{At}{2AS}\left(l\,\tfrac{Q}{B}+\tfrac{T}{c}\right)-\tfrac{Ar\cdot Htl}{2c\cdot AS}$$

and

$$\mathfrak{T} = \tfrac{c\cdot At}{2AS}\left(l\,\tfrac{Q}{B}+\tfrac{\Upsilon}{c}\right)+\tfrac{c}{2AS^2}\left(\text{Ar.}Htm-\tfrac{St}{c}\,\text{Ar.}Htl\right),$$

where $\frac{St}{c}$ denotes the time *t* elapsed from the beginning expressed in seconds.

COROLLARY 2

134. But these areas therefore are less likely to be neglected, where the small distance *GH* were greater and where it will have been closer to the vertex of the cone *A*, since then on account of $tl = \Upsilon$ the area *Htl* by no means can be had as vanishing before the rectangle $At\cdot tl$.

COROLLARY 3

135. If the elapsed time *t* were greater than $\frac{SG}{c}$, thus so that the point *t* may lie beyond *G* towards *A*, where initially there was $l\,\frac{Q}{B}=0$ and $\Upsilon=0$, for the location *S* at this stage there will become :

$$l\,\tfrac{q}{B} = -\tfrac{\text{Ar.}GlH}{2\cdot cAS}B \text{ and } \mathfrak{T} = \tfrac{c}{2AS^2}\left(\text{Ar.}GmH - t\cdot GlH\right),$$

then therefore neither is the density natural nor is the motion completely finished.

SCHOLIUM

136. This phenomenon is quite unique, because, even if the pulse may be considered now to have been released at *S*, yet the equilibrium of the air shall not yet be fully restored, but it may be able to happen, that even now both the density of the air *q* may differ from the natural density *B*, as well as the air may be moved forwards by a certain motion. Certainly this distinction can be viewed as zero in very great distances *AS* ; truly in smaller distances there it is more noteworthy, which may be seen to endure always, since the calculation again may indicate no change. Truly here it is required to have noted not to have extended our scales beyond the vertex *A* and the continuation of these from the other part of *A* to involve a similar pulse *GH* at an equal distance, where if it may

pertain to the time *St*, at *S* a new secondary pulse may be excited, with which finished the air at last may be restored at *S* to equilibrium. Indeed here as a similar repetition must arise, if the tube were cylindrical and closed at *A* : but now we see this distinction to intercede, because meanwhile in the case of a conical tube, while a secondary pulse will be returned to *S*, the air here retains a certain disturbance at this stage, and otherwise it happens in a cylinder. Whereby, just as both scales ought to be continued beyond the vertex of the cone, we will investigate in the following problem.

PROBLEM 94

137. *If the air around the vertex A of the cone* (Fig.105) *may be disturbed from the state of equilibrium in some manner, or some pulse may be excited there, both the scales, for which there is a need requiring to be defined for the propagation, to be continued backwards beyond the vertex A and thence to determine the propagation in the tube for some location S.*

SOLUTION

In the first place the exciting pulse may be extended through the distance *AD*, and at some location *T* with the interval put $AT = S$ the density of the air will have been $= Q$, with the natural density $= B$, truly the speed $= \Upsilon$ along the direction *AB*, hence at first the curve *AMD* may be constructed

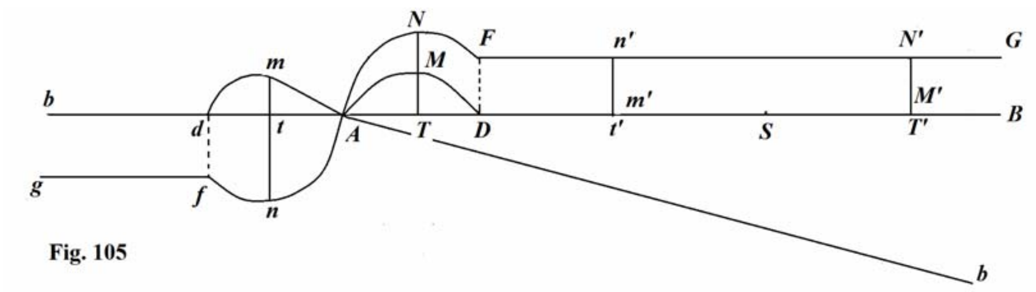

Fig. 105

on assuming the applied line $TM = Sl\frac{Q}{B}$, which therefore beyond *D* falls on the axis *DB* itself. Thence also the curve *ANF* may be constructed also on assuming the applied line $TN = \frac{S\Upsilon + \int \Upsilon dS}{c}$, of which indeed the first member vanishes at *D*, where $\Upsilon = 0$, truly the other part $\frac{1}{c}\int \Upsilon dS$ there will gain a certain value, to which all the following applied lines towards *B* are equal, thus so that this scale *ANF* beyond *F* may become the right line *FG* parallel to the axis *AB* itself; but the area of this curve thus will be defined, so that there shall become $ATN = \frac{S}{c}\int \Upsilon dS$: from which, since with the point *T* advanced to *D* the applied line may become $DF = \frac{1}{c}\int \Upsilon dS$, the area will become $ADF = AD \cdot DF$, and likewise if the whole scale *ANFG* shall become parallel to *GF* itself and above the vertex *A* as far as can be made continuous. With these put in place there is sought at first, in what manner will it be able for these scales to be continued beyond *A*, where before everything it is required to be observed the air at the vertex *A* itself plainly cannot partake of any motion. Whereby, if the point *S* of the tube, in which we have defined the speed $\mathfrak{T}$ generally above , we may transfer to *A*, the formula there for the speed shown:

$$\mathfrak{T} = \tfrac{c}{2S}\left(tm - TM + TN + tn\right) + \tfrac{c}{2SS}\left(ATM - Atm - ATN - Atn\right)$$

in this case must be zero. For this case we will have $S = 0$, since in this formula we have put $AS = S$, from which it is necessary the continuation sought to be prepared thus, so that there shall be $tm = TM$ and $tn = -TN$. On account of which in our figure it is convenient to describe the scale *AMD* without any variation above the axis *Ad*, so that that shall be *Amd*, truly the other scale *ANFG* must be described there on the other part of the axis, so that it shall be *Anfg*. Now we may consider that both these scales be continued to some location *S* in the tube beyond *A*, by requiring now the distance from the $AS = S$, where, since the air initially will have been in equilibrium, thus there may be put $\Upsilon = 0$ and $Q = B$ and hence $SQ = 0$, with the elapsed time $= t$ we may take both the distance from *S* to be $ST' = St' = ct$, and the applied lines at the points *T'* and *t'* of both the scales shall be *T'M', T'N'* and *t'm', t'n';* and from these the state of the air *S* thus may be defined, so that there the density shall become

$$q = B\left(1 + \tfrac{1}{2S}\left(T'M' + t'm'\right) - \tfrac{1}{2S}\left(T'N' - t'n'\right)\right)$$

and the speed

$$\mathfrak{T} = \tfrac{c}{2S}\left(t'm' - T'M'\right) + \tfrac{c}{2S}\left(T'N' + t'n'\right) - \tfrac{c}{2SS}\left(At'm' - AT'M'\right) - \tfrac{c}{2SS}\left(AT'N' + At'n'\right),$$

from which in the first place it is clear, how much less the elapsed time *t* will have been from the beginning than $\frac{SD}{c}$, then on account of

$$T'M' = 0,\ \ t'm' = 0 \ \text{ and } \ T'N' = t'n' = DF$$

to become $q = B$, and on account of

$$At'm' = AT'M',\ \ AT'N' = AT' \cdot DF,\ \text{ and } \ At'n' = At' \cdot DF$$

to produce

$$\mathfrak{T} = \tfrac{c}{S} \cdot DF - \tfrac{c \cdot DF}{2SS}\left(AT' + At'\right) = 0$$

on account of $AT' + At' = 2AS = 2S$; as long therefore as the air will remain there in equilibrium .

But immediately the elapsed time *t* shall become greater than $\frac{SD}{c}$, the air at *S* will begin to be disturbed ; therefore now we may put the elapsed time to become $t=\frac{AS}{c}$, and since now there is :

$$T'M'=0,\ \ t'm'=0,\ \ T'N'=DF,t'n'=0,$$
$$AT'M'=AMD,\ \ At'm'=0,\ \ AT'N'=AT'\cdot DF=2S\cdot DF\ \text{ et }\ At'n'=0,$$

then at the location S

the density will become $q=B\left(1-\frac{DF}{2S}\right)$

and

the speed will be $\mathfrak{T}=\frac{C\cdot DF}{2S}+\frac{c\cdot AMD}{2SS}-\frac{c\cdot DF}{S}=\frac{c\cdot AMD}{2SS}-\frac{c\cdot DF}{2S}.$

But for some elapsed time in the middle, $t=\frac{ST}{c}$ the density is determined to be

$$q=B\left(1+\tfrac{TM+TN-DF}{2S}\right)$$

and

the speed $\mathfrak{T}=\frac{c}{2S}\left(TM+TN-DF\right)+\frac{c}{2SS}\left(DTM+DFTN-DT\cdot DF\right).$

Finally we may observe also, how the state of the air at *S* shall itself be going to be had, since in the passage of time $t=\frac{Sd}{c}=\frac{AS+AD}{c}$; then truly there will become

$$T'M'=0,\ \ AT'M'=AMD,\ \ T'N'=\ DF,$$
$$AT'N'=AT'\cdot DF=2S\cdot DF+AD\cdot DF,$$

again

$$t'm'=0,\ \ At'm'=Amd=AMD,\ t'n'=-df\ =-DF,$$
$$At'n'=-Adf\ =-AD\cdot DF,$$

and thus

the density $q=B\left(1-\frac{DF}{S}\right)$

and

the speed $\mathfrak{T}=-\frac{c\cdot DF}{S}.$

COROLLARY 1

138. Therefore the speed of the pulse, when it shall have arrived at some position *S*, is agreed to consist of two parts, of which the former decreases inversely in the ratio of the distance $AS=S$, the latter truly in the ratio of twice the distance, but which latter part vanishes before the former part, and once the distance $AS=S$ may become small.

COROLLARY 2

139. Since the impulse made on the ear may depend without doubt on the speed of the disturbance in whatever pulse, hence it is evident, how the sound heard may be diminished continually; and since the force of the impulse requiring to be put in place will be seen to be proportional to the square of the speed, the weakening of the sound will follow in the inverse square ratio of the distances.

SCHOLIUM 1

140. Here a significant difficulty occurs, because, after a pulse will have passed completely through the location S , the air there still may not be restored to equilibrium, unless the initial pulse were prepared thus, so that the interval *DF* may vanish. Certainly such a disturbance after the passage of the pulse in no way can be allowed, and thus our rather defective calculation, is seen to be required to be proved in some way, unless we may wish to say all the disturbance of the air first induced always must be necessarily prepared thus, so that the interval *DF* thence may be produced to be vanishing. Truly since this cannot be proved by any account, the origin of this inconvenience is seen to be required to be sought in that, because, since the interval *DF* shall arise from the value of the integral $\int \Upsilon dS$, in this integration itself just the constant may be ignored ; therefore provided we may have introduced such a constant, by which this integral shall have been reduced to zero extended through the whole scale of the speed, all this difficulty may have vanished completely, since then also the curve *ANF* beyond *D* may intersect the axis at *DB*. Therefore there is no doubt, why the origin of this defect of the calculation may not be placed here, and the whole calculation may be agreed to be set out in this manner. But such inconveniences in a new straightforward argument are the least to be a source of concern and it may be permitted to hope, when that were completed more carefully, then all to be going to be removed at once. But this continuation of the scale beyond the vertex *A* is seen deservedly with suspicion, because therefore in this case there shall be $S = 0$ and these terms, which were required to be removed, to become infinite; to which inconvenience I will try to affect a cure in the following problem.

SCHOLIUM 2

141. From this problem it is allowed to gather, in whatever way some pulse may be propagated in some manner backwards and forwards; indeed as if the centre of the pulse propagated shall be at *A*, all the air moving around can be considered to be distributed in an infinitude of cones, the vertices of which concur at *A*; and it is evident likewise a pulse is going to be propagated through any of these, and through the free air, because, where two tubes of this kind touch each other, there the density and therefore also the pressure is the same, thus so that also the propagation of distant tubes shall be going to follow the same law. Now therefore we understand pulses in free air to be propagated forwards with the same speed, as well as in cylindrical tubes, but truly on account of the

increase in the width of the tubes the pulses to be continually weakened, thus so that the speed of each disturbance may be diminished on account of the distances from the initial pulse *A*, truly the width of the pulses and the other qualities of the sound not to be altered in the propagation.

PROBLEM 95

142. *If the conical tube end* (Fig. 106) *nearer the vertex I shall be the end Aa and there to be either open or closed, to show the formulas requiring to be defined from the initial disturbance induced, for the following motion of the air contained in the tube AaBb.*

SOLUTION

With the distance put $IA = a$, of the end A from the vertex of the cone I, some point S may be taken in the cone with the distance called $AS = s$, so that the distance, as we have put before to be *S*, here shall be $= a+s$, moreover the initial density at *S* will have been $= Q$, with the natural density being $= B$, and the speed along $SB = \Upsilon$. But for some time elapsed *t* the density of the same air, which initially was at *S*, shall become $= q$, the speed $= \mathfrak{T}$ and the translation of the small distance $Ss = \upsilon$. Now since the amplitude of the tube at *S* shall be $\Omega = nn(a+s)^2$, there will become $\frac{d\Omega}{\Omega dS} = \frac{2}{a+s}$, and hence from the formulas found above we will obtain:

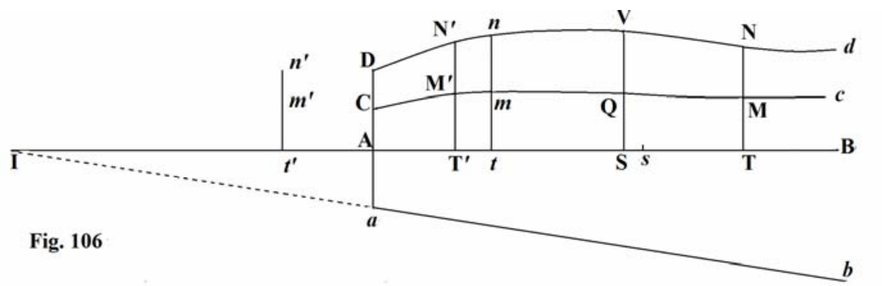

Fig. 106

$$\upsilon = \frac{1}{(a+s)^2}\int (a+s)^2\, ds l \frac{Q}{B} - \frac{1}{a+s}\Gamma' : (s+ct) + \frac{1}{(a+s)^2}\Gamma : (s+ct)$$
$$- \frac{1}{a+s}\Delta' : (s+ct) + \frac{1}{(a+s)^2}\Delta : (s+ct)$$

$$q = Q\left(1 - l\frac{Q}{B} + \frac{1}{a+s}\Gamma'' : (s+ct) + \frac{1}{a+s}\Delta'' : (s-ct)\right)$$

and

$$\mathfrak{T} = \frac{-c}{a+s}\Gamma'' : (s+ct) + \frac{c}{a+s}\Delta'' : (s-ct) + \frac{c}{(a+s)^2}\Gamma' : (s+ct) - \frac{c}{(a+s)^2}\Delta' : (s-ct),$$

from which we may gather for the initial known state:

$$0=\tfrac{1}{(a+s)^2}\int(a+s)^2\,dsl\tfrac{Q}{B}-\tfrac{1}{a+s}\left(\Gamma':s+\Delta':s\right)+\tfrac{1}{(a+s)^2}\left(\Gamma:s+\Delta:s\right)$$
$$0=-l\tfrac{Q}{B}+\tfrac{1}{a+s}\left(\Gamma'':s+\Delta'':s\right)$$
$$\Upsilon=\tfrac{-c}{a+s}\left(\Gamma'':s-\Delta'':s\right)+\tfrac{c}{(a+s)^2}\left(\Gamma':s-\Delta':s\right).$$

From that there becomes

$$\Gamma'':s+\Delta'':s=(a+s)l\tfrac{Q}{B}\quad\text{and}\quad\Gamma':s+\Delta':s=\int(a+s)\,dsl\tfrac{Q}{B}$$

and

$$\Gamma':s+\Delta':s=\int(a+s)\,dsl\tfrac{Q}{B},$$

truly from the same:

$$\int\Upsilon ds=\tfrac{-c}{a+s}\left(\Gamma':s-\Delta':s\right)\ \text{ or }\ c\left(\Gamma':s-\Delta':s\right)=-(a+s)\int\Upsilon ds$$

and hence

$$c\left(\Gamma'':s-\Delta'':s\right)=-\int\Upsilon ds-(a+s)\Upsilon\ \text{ and }\ c\left(\Gamma:s-\Delta:s\right)=-\int(a+s)\,ds\int\Upsilon ds.$$

Which values substituted into the first equation provide:

$$0=\int(a+s)^2\,dsl\tfrac{Q}{B}-(a+s)\int(a+s)\,dsl\tfrac{Q}{B}+\int ds\int(a+s)\,dsl\tfrac{Q}{B}$$

from which the ambiguity arising on account of the constant entering by integration is removed, since there must become :

$$\int ds\int(a+s)\,dsl\tfrac{Q}{B}=(a+s)\int(a+s)\,dsl\tfrac{Q}{B}-\int(a+s)^2\,dsl\tfrac{Q}{B},$$

which two simple integrals may be taken thus, so that they may vanish on putting $s=0$. [*i.e.* on integrating by parts.]

Therefore the functions Γ and Δ are defined thus, so that there shall become:

$$\Gamma'':s=\tfrac{1}{2}(a+s)l\tfrac{Q}{B}-\frac{(a+s)\Upsilon+\int\Upsilon ds}{2c}$$
$$\Delta'':s=\tfrac{1}{2}(a+s)l\tfrac{Q}{B}+\frac{(a+s)\Upsilon+\int\Upsilon ds}{2c}$$

then truly

$$\Gamma':s=\tfrac{1}{2}(a+s)l\tfrac{Q}{B}-\frac{(a+s)\Upsilon+\int\Upsilon ds}{2c}$$
$$\Delta':s=\tfrac{1}{2}(a+s)l\tfrac{Q}{B}+\frac{(a+s)\Upsilon+\int\Upsilon ds}{2c}$$

and

$$\varGamma : s = \tfrac{1}{2}(a+s)\int (a+s)\,dsl\,\tfrac{Q}{B} - \tfrac{1}{2}\int (a+s)^2\,dsl\,\tfrac{Q}{B} - \tfrac{\int (a+s)ds\int \Upsilon ds}{2c}$$
$$\varDelta : s = \tfrac{1}{2}(a+s)\int (a+s)\,dsl\,\tfrac{Q}{B} - \tfrac{1}{2}\int (a+s)^2\,dsl\,\tfrac{Q}{B} + \tfrac{\int (a+s)ds\int \Upsilon ds}{2c}.$$

Therefore we may take the two applied lines at S :

$$SQ = (a+s)\,l\,\tfrac{Q}{B} \text{ and } SV = \tfrac{(a+s)\Upsilon + \int \Upsilon ds}{c},$$

and with the two scales CQc and DVd described, there will become:

$$\varGamma'' : s = \tfrac{1}{2}SQ - \tfrac{1}{2}SV, \qquad \varDelta'' : s = \tfrac{1}{2}SQ + \tfrac{1}{2}SV,$$
$$\varGamma' : s = \tfrac{1}{2}ACSQ - \tfrac{1}{2}ADSV, \quad \varDelta' : s = \tfrac{1}{2}ACSQ + \tfrac{1}{2}ADSV.$$

Therefore with the intervals taken, $ST = St = ct$, for the time elapsed t, the density q and the speed $\mathfrak{T}$ will be defined thus, so that there shall become:

$$q = Q\left(1 + \tfrac{1}{2(a+s)}\left(TM - TN + tm + tn - 2SQ\right)\right)$$
$$\mathfrak{T} = \tfrac{c}{2(a+s)}\left(tm + tn - TM + TN\right) + \tfrac{c}{2(a+s)^2}\left(ACTM - ADTN - ACtm - ADtn\right),$$

which two elements to be known to suffice, since then the third, truly the small interval *Ss*, may be included readily.

So that now we may investigate, how each scale ought to be continued beyond A , we may take the point S at the term A itself, so that there shall become $s = 0$, and with the abscissas on both ends $AT' = At' = ct$, since we have computed the areas of the curves on the right from the point *A*, we must now the ones which now fall to the left to be negative, and for this point *A* there will become:

$$q = Q\left(1 + \tfrac{1}{2a}\left(T'M' + t'm' - 2AC\right) - \tfrac{1}{2a}\left(T'N' - t'n'\right)\right)$$
$$\mathfrak{T} = \tfrac{c}{2a}\left(t'm' - T'M'\right) + \tfrac{c}{2a}\left(T'N' + t'n'\right) + \tfrac{c}{2aa}\left(ACT'M' + ACt'm'\right)$$
$$- \tfrac{c}{2a^2}\left(ADT'N' - ADt'n'\right).$$

Now the two cases are required to be set out, accordingly as the tube were open or close at the end A :

I. The tube shall be open at *Aa*, and since there the density remains natural always, so that there shall be $Q = B$ and $q = B$, there will become $AC = 0$ and it is necessary to become :

$$T'M' + t'm' = 0 \quad \text{and} \quad T'N' = t'n'\,;$$

therefore in this case both scales may be continued in a straight forwards manner as in cyulindrical tubes; evidently the scale of the density *CQc* to the other part of the axis, truly the speed *DVd* will be described to the same.

II. If the tube shall be closed at *Aa*, and since there the speed must remain zero always, in the first place for the scale of the density there must be:

$$a\left(t'm' - T'M'\right) + ACT'M' + ACt'm' = 0;$$

we may put

$$AT' = At' = x,\ T'M' = y \ \text{ and } \ t'm' = z,$$

so that we may have:

$$a\left(z-y\right)+\int\left(y+z\right)dx = 0 \ \text{ or } \ adz+zdx-ady+ydx = 0,$$

which integrated gives:

$$e^{\frac{x}{a}}az = \int e^{\frac{x}{a}}\left(ady - ydx\right) = e^{\frac{x}{a}}ay - 2\int e^{\frac{x}{a}}ydx = -e^{\frac{x}{a}}ay + 2a\int e^{\frac{x}{a}}dx$$

and hence

$$z = t'm' = y - \tfrac{2}{a}e^{\frac{-x}{a}}\int e^{\frac{x}{a}}ydx = -y + 2e^{\frac{-x}{a}}\int e^{\frac{x}{a}}dy.$$

Then for the scale of the speed there shall be required to become :

$$a\left(T'N' + t'n'\right) - ADT'N' + ADt'n' = 0\,,$$

we may make

$$AT' = At' = x, \quad T'N' = y, \ \ t'n' = z$$

and the equation

$$a\left(y+z\right) - \int ydx + \int zdx = 0 \ \text{ or } \ adz+zdx+ady-ydx = 0$$

integrated will provide:

$$e^{\frac{x}{a}}az = \int e^{\frac{x}{a}}\left(ydx - ady\right) = -e^{\frac{x}{a}}ay + 2\int e^{\frac{x}{a}}ydy = e^{\frac{x}{a}}ay - 2a\int e^{\frac{x}{a}}dy$$

and hence

$$z = t'n' = -y + \tfrac{2}{a}e^{\frac{-x}{a}}\int e^{\frac{x}{a}}ydy = y - 2e^{\frac{-x}{a}}\int e^{\frac{x}{a}}dy,$$

thus in each case the integrals must be defined, so that on putting $x = 0$ there may become $z = y$.

COROLLARY 1

143. Since in the case, where the tube is open at *Aa*, both scales may be continued in the same manner, where with cylindrical tubes, also the same law of continuation will be used, also if the conical tube were open at the other end *Bb*, from which it follows in a conical tube with both ends open the disturbances of the air to agree at once with these, which we have defined above for cylindrical tubes.

COROLLARY 2

144. Therefore concerning the sounds we have noted above, which the apertures of flutes produce, the same also have a place, if a conical shape may be attributed to the flutes. Nor yet can exceedingly different shapes be allowed, since then the disturbances through each part no longer shall be equal, as our theory demands.

COROLLARY 3

145. The theory of cylindrical tubes is deduced from this problem, if the distance $IA = a$ may be established to be infinite. Moreover then for the case, where the tube is closed at *Aa*, the same law of each scale continued is deduced, since on account of $a = \infty$ there will become $t'm' = T'M'$ and $t'n' = -T'N'$.

SCHOLIUM 1

146. Now the doubts are removed completely, which were remaining above about the propagation of a pulse excited in the vertex of the cone itself; indeed since the whole business may be returned to that, in what manner may each scale be required to be continued beyond the vertex of the cone, we have adapted our problem to this case, if we may establish the distance $IA = a$ to be vanishing and likewise we may put in place a tube closed at *Aa*, for since the cone itself is closed at the vertex, certainly there the air can be considered to have no motion. From which at first for the continuation of the scale of the density here the question returns, so that this equation

$$z = -y + 2e^{\frac{-x}{a}} \int e^{\frac{x}{a}} dy$$

may define the value of *z* in the case, where $a = 0$, which since on account of the infinite exponent $\frac{x}{a}$ it may appear less, we may depart from the differential equation

$$adz + zdx - ady + ydx = 0,$$

so that on account of $a = 0$, clearly it follows $z = -y$, so that the scale of the density axis must be applied to the opposite part, otherwise as above we may become deceived by the vanishing interval $S=0$. Then truly for our scale of the speed from the differential equation

$$adz+zdx+ady-ydx=0 \text{ on account of } a=0$$

we deduce $z=y$, thus so that this scale ought to constitute a part of the same axis, evidently each continuation is performed by the same rule, as if the tube were open. Hence therefore (Fig. 105) the latter part set out in problem 94 thus will be corrected, so that, since the scales may lie in opposite parts, and have been shown in the figure, in the elapsed time $t=\frac{Sd}{c}$ there shall become $t'm'=0,\ t'n'=+DF$, the area $At'm'=+Amd=AMD,$ since here a change must be made to the two signs, the one in as much as this area lies beyond *A*, the other in as much as it falls below the axis. Finally truly the area becomes

$$At'n'=-Adf=-AD\cdot DF;$$

with which we follow :

$$T'M'+t'm'=0,\quad T'N'-t'n'=0,\quad At'm'-AT'M'=0,$$
$$t'm'-T'M'=0,\ T'N'+t'n'=2DF,\ AT'N'+At'n'=2S\cdot DF$$

and hence again the density $q=B$ and the speed

$$\mathfrak{T}=\tfrac{c}{2S}\cdot 0+\tfrac{c}{2S}\cdot 2DF-\tfrac{c}{2SS}\cdot 0-\tfrac{c}{2SS}\cdot 2S\cdot 2DF$$

or

$$\mathfrak{T}=\tfrac{c}{S}\cdot DF-\tfrac{c}{S}\cdot DF=0,$$

from which it is evident in the lapse of time $t=\frac{Sd}{c}$ the air at *S* will be restored completely to equilibrium, and thus the truth has been saved particularly well.

SCHOLIUM 2

147. Since we have observed conical flutes for the same length produce the same notes as cylindrical flutes, if indeed they shall be open, it is required for this agreement to be noted properly by no means to be applicable to closed flutes. Since indeed, if a conical tube were closed at *Aa*, both scales must be contained in greatly dissimilar lines, hence no regular oscillations will be able to be produce, but the sounds produced from these will be exceedingly rough and rude and not in the least adapted to bringing forth harmony.

CAPUT VI

DE MINIMIS AERIS AGITATIONIBUS IN TUBIS CONICIS

PROBLEMA 92

128. *Si tubus habuerit figuram conicam aërque in eo contentus utcunque de statu aequilibrii deturbetur, describere ambas scalas, ex quibus deinceps status aëris ad quodvis tempus definiri queat.*

SOLUTIO

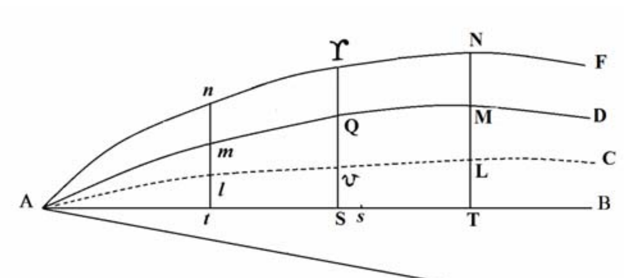

Fig. 103

Sit (Fig. 103) vertex coni in *A* et recta *AB* eius axis, in quo sumto puncto quocunque *S* positoque intervallo $AS = S$, aëri in *S* initio eiusmodi inductus sit status, ut densitas fuerit $= Q$, naturali existente $= B$, celeritas vero in directione $AB = \Upsilon$. Cum nunc tubi amplitudo in *S* sit $\Omega = nnSS$, ex paragrapho 88 ad hunc casum accommodato habebimus $\alpha = 1,\ \beta = 0$, hincque $M = \frac{-1}{S}$ et $L = \frac{1}{SS}$. Quodsi iam solutionem problematis 84 huc transferamus et gravitatis actionem negligamus, huc transferamus et gravitatis actionem negligamus, obtinimus primo

$$O = \tfrac{1}{SS}\int SSdSl\,\tfrac{Q}{B},$$

tum vero elapso tempore quocunque *t* status aëris, qui initio fuerat in S, ita definitur, ut sit spatium translationis

$$Ss = \tfrac{1}{SS}\int SSdSl\,\tfrac{Q}{B} - \tfrac{1}{S}\Gamma' : (S+ct) + \tfrac{1}{SS}\Gamma : (S+ct)$$
$$-\tfrac{1}{S}\Delta' : (S-ct) + \tfrac{1}{SS}\Delta : (S-ct) = \upsilon,$$

unde pro densitate *q* et celeritate $\mathfrak{T}$ deducimus:

$$\left(\tfrac{d\upsilon}{dS}\right) = \tfrac{-2}{S^3}\int SSdSl\,\tfrac{Q}{B} - \tfrac{1}{S}\Gamma'' : (S+ct) + \tfrac{2}{SS}\Gamma' : (S+ct)$$
$$-\tfrac{2}{S^3}\Gamma : (S+ct) + l\,\tfrac{Q}{B} - \tfrac{1}{S}\Delta''(S-ct) - \tfrac{2}{S^3}\Delta : (S-ct);$$

ac ob $\frac{d\Omega}{\Omega dS} = \frac{2}{S}$ est

$$\frac{\upsilon d\Omega}{\Omega dS}=\frac{2}{S^3}\int SSdSl\frac{Q}{B}-\frac{2}{SS}\Gamma'':(S+ct)+\frac{2}{S^3}\Gamma:(S+ct)$$
$$-\frac{2}{SS}\Delta':(S-ct)+l\frac{Q}{B}+\frac{2}{S^3}\Delta:(S-ct),$$

quare, cum sit $q=Q\left(1-\frac{\upsilon d\Omega}{\Omega dS}-\left(\frac{d\upsilon}{dS}\right)\right)$, colligimus

$$q=Q\left(1-l\frac{Q}{B}+\frac{1}{S}\Gamma'':(S+ct)+\frac{1}{S}\Delta'':(S-ct)\right)$$

ac denique ob

$$\mathfrak{T}=\left(\frac{d\upsilon}{dt}\right)$$

erit

$$\mathfrak{T}=-\frac{c}{S}\Gamma'':(S+ct)+\frac{c}{SS}\Gamma':(S+ct)$$
$$+\frac{c}{S}\Delta'':(S-ct)-\frac{c}{SS}\Delta':(S-ct).$$

Hinc ergo pro statu initiali ponendo $t=0$ adipiscimur:

$$\upsilon=\frac{1}{SS}\int SSdSl\frac{Q}{B}-\frac{1}{S}\left(\Gamma':S+\Delta':S\right)+\frac{1}{SS}\left(\Gamma:S+\Delta:S\right)$$
$$q=Q\left(1-l\frac{Q}{B}+\frac{1}{S}\left(\Gamma'':S+\Delta'':S\right)\right)$$
$$\mathfrak{T}=-\frac{c}{S}\left(\Gamma'':S-\Delta'':S\right)+\frac{c}{SS}\left(\Gamma':S-\Delta':S\right).$$

Cum igitur tum fuerit $\upsilon=0$, $q=Q$ et $\mathfrak{T}=\Upsilon$, sequentes nanciscimur aequationes, ex quibus naturam binarum functionum Γ et Δ definiri oportet

$$\text{I.}\quad \int SSdSl\frac{Q}{B}-S\left(\Gamma':S+\Delta':S\right)+\Gamma:S+\Delta:S=0$$
$$\text{II.}\quad Sl\frac{Q}{B}=\Gamma'':S+\Delta'':S$$
$$\text{III.}\quad \frac{1}{c}SS\Upsilon=-S\left(\Gamma'':S-\Delta'':S\right)+\Gamma':S-\Delta':S,$$

quarum prima differentiata cum secunda convenit, uti quidem rei natura postulat, ita ut duabus tantum conditionibus sit satisfaciendum. Ponamus brevitatis gratia:

$$\Gamma':S+\Delta':S=y \text{ et } \Gamma':S-\Delta':S=z,$$

ut fiat

$$\text{II.}\quad SdSl\frac{Q}{B}=dy \text{ et}$$
$$\text{III.}\quad \frac{1}{c}SS\Upsilon dS=-Sdz+z:dS$$

hincque

$$y=\int SdSl\frac{Q}{B} \text{ et } \frac{-z}{S}=\frac{1}{c}\int\Upsilon dS \text{ seu } z=\frac{-S}{c}\int\Upsilon dS,$$

unde functiones ita definiuntur, ut sit:

$$\Gamma' : S = \tfrac{1}{2}\int SdSl\,\tfrac{Q}{B} - \tfrac{S}{2c}\int \Upsilon dS$$

et

$$\Delta' : S = \tfrac{1}{2}\int SdSl\,\tfrac{Q}{B} + \tfrac{S}{2c}\int \Upsilon dS$$

hincque porro differentiando

$$\Gamma'' : S = \tfrac{1}{2}Sl\,\tfrac{Q}{B} - \tfrac{1}{2c}\int \Upsilon dS - \tfrac{1}{2c}S\Upsilon$$
$$\Delta'' : S = \tfrac{1}{2}Sl\,\tfrac{Q}{B} + \tfrac{1}{2c}\int \Upsilon dS + \tfrac{1}{2c}S\Upsilon,$$

integrando vero reperitur:

$$\Gamma : S = \tfrac{1}{2}S\int SdSl\,\tfrac{Q}{B} - \tfrac{1}{2}\int SSdSl\,\tfrac{Q}{B} - \tfrac{1}{4c}SS\int \Upsilon dS + \tfrac{1}{4c}\int SS\Upsilon dS$$
$$\Delta : S = \tfrac{1}{2}S\int SdSl\,\tfrac{Q}{B} - \tfrac{1}{2}\int SSdSl\,\tfrac{Q}{B} + \tfrac{1}{4c}SS\int \Upsilon dS - \tfrac{1}{4c}\int SS\Upsilon dS.$$

Construantur ergo super axe ex dato statu initiali duae lineae curvae *AQD* et *ATF* sumendo applicatas

$$SQ = Sl\,\tfrac{Q}{B} \quad \text{et} \quad S\Upsilon = \tfrac{1}{c}\int \Upsilon dS + \tfrac{1}{c}S\Upsilon,$$

et functionum Γ et Δ natura inde ita determinabitur, ut sit

$$\Gamma'' : S = \tfrac{1}{2}SQ - \tfrac{1}{2}S\Upsilon;\quad \Delta'' : S = \tfrac{1}{2}SQ + \tfrac{1}{2}S\Upsilon$$

hincque porro per areas

$$\Gamma' : S = \tfrac{1}{2}ASQ - \tfrac{1}{2}AS\Upsilon;\quad \Delta' : S = \tfrac{1}{2}ASQ + \tfrac{1}{2}AS\Upsilon,$$

tum vero, si haec forma $M : ASQ$ denotet eam quantitatem, quae exprimit integrale $\int dS\cdot ASQ$, habebitur:

$$\Gamma : S = +\tfrac{1}{2}M : ASQ - \tfrac{1}{2}M : AS\Upsilon$$
$$\Delta : S = +\tfrac{1}{2}M : ASQ + \tfrac{1}{2}M : AS\Upsilon.$$

His curvis descriptis post tempus elapsum $= t$ a puncto *S* utrinque abscindantur intervalla $ST = St = ct$ et ex applicatis his punctis respondentibus status aëris, qui initio ad *S* versabatur, nunc ita definitur, ut sit

$$q = Q\left(1 - \tfrac{SQ}{S} + \tfrac{1}{2S}\left(TM - TN + tm + tn\right)\right)$$
$$\mathfrak{T} = \tfrac{c}{2S}\left(tm + tn - TM + TN\right) + \tfrac{1}{2SS}\left(ATM - ATN - Atm - Atn\right),$$

at spatiolum translationis $Ss = \upsilon$ ita exprimetur:

$$Ss = \tfrac{1}{SS}\int SSdSl\,\tfrac{Q}{B} - \tfrac{1}{2S}\left(ATM - ATN + Atm + Atn\right)$$
$$+\tfrac{1}{2SS}\left(M:ATM - M:ATN\right) + \tfrac{1}{2SS}\left(M:Atm + M:Atn\right)$$

seu

$$Ss = \tfrac{1}{SS}\int SSdSl\,\tfrac{Q}{B} - \tfrac{1}{2S}\left(ATM - ATN + Atm + Atn\right)$$
$$+\tfrac{1}{2SS}\left(M:ATM - M:ATN + M:Atm + M:Atn\right),$$

ubi est

$$\int SSdSl\,\tfrac{Q}{B} = \int S\cdot SQ\cdot dS = S\cdot ASQ - M:ASQ.$$

COROLLARIUM 1

128a. Ad quodvis ergo tempus t sumtis intervallis $ST = St = ct$ aëris, qui initio erat in S, status ita definitur, ut sit

I. densitas $q = Q\left(1 + \tfrac{1}{2S}\left(TM + tm - 2SQ\right) - \tfrac{1}{2S}\left(TN - tn\right)\right)$

II. celeritas $\mathfrak{T} = \tfrac{c}{2S}\left(tm - TM\right) + \tfrac{c}{2S}\left(TN + tn\right) + \tfrac{c}{2SS}\cdot tmTM - \tfrac{c}{2SS}\left(ATN + Atn\right)$

III. spatiolum $Ss = \tfrac{1}{2S}\left(2ASQ - ATM - Atm\right) + \tfrac{1}{2S}\cdot tnTN$

$$-\tfrac{1}{2SS}\left(2M:ASQ - M:AT\ M - M:Atm\right) - \tfrac{1}{2SS}\left(M:ATN - M:Atn\right).$$

COROLLARIUM 2

129. Si distantia AS consideretur ut infinita, tubus conicus abibit in cylindricum; et quia tum fit

$$\tfrac{1}{S}\cdot SQ = l\,\tfrac{Q}{B},\quad \tfrac{1}{S}\cdot S\Upsilon = \tfrac{\Upsilon}{c},\ \tfrac{1}{S}\cdot ASQ = \int dsl\,\tfrac{Q}{B}$$

ideoque

$$\tfrac{1}{SS}\cdot ASQ = 0,\ \text{ at }\ \tfrac{1}{SS}M\cdot ASQ = \int dsl\,\tfrac{Q}{B},$$

tum vero $\tfrac{1}{S}\cdot AS\Upsilon = \tfrac{1}{c}\int \Upsilon dS,$

hinc

$$\tfrac{1}{SS}AS\Upsilon = 0\quad \text{et}\quad \tfrac{1}{SS}M:AS\Upsilon = \tfrac{1}{c}\int \Upsilon dS,$$

hae formulae cum illis, quae supra pro tubis oylindricis sunt inventae, convenient.

SCHOLION 1

130. Constructio prioris scalae *AQD* ex densitate initiali formatae nulli est subiecta difficultati, at altera scala $A\Upsilon F$ non simpliciter ex celeritatibus initialibus Υ construitur, sed insuper formulam quandam integralem involvit; quemadmodum ergo hinc eam construi oporteat, perpendamus. Descripta ergo linea curva $A\upsilon C$, cuius applicatae $S\upsilon$ ipsam celeritatem initialem Υ in loco *S* exhibeant, ex ea scala illa $A\Upsilon F$ ita construitur, ut sit eius applicata

$$S\Upsilon = \tfrac{1}{c} AS \cdot S\upsilon + \tfrac{1}{c} AS\upsilon,$$

tum vero erit huius curvae area $AS\Upsilon = \tfrac{1}{c} AS \cdot AS\upsilon,$ cum sit, ut vidimus,

$$\int dS(\Upsilon S + \int \Upsilon dS) = S\int \Upsilon dS.$$

Hanc ergo curvam $A\upsilon C$ loco illius indroducendo habebimus

$$TN = \tfrac{1}{c} AT \cdot TL + \tfrac{1}{c} ATL, \quad tn = \tfrac{1}{c} At \cdot tl + \tfrac{1}{c} Atl,$$

$$ATN = \tfrac{1}{c} AT \cdot ATL, \quad Atn = \tfrac{1}{c} At \cdot Atl,$$

unde fit pro formula celeritatum

$$\mathfrak{T} = \tfrac{c}{2S}\left(tm - TM\right) + \tfrac{c}{2SS} \cdot tmTM + \tfrac{1}{2S}\left(AT \cdot TL + At \cdot tl\right) - \tfrac{ct}{2SS} TLtl$$

ob $ST = St = ct$. Atque hinc iam perspicitur, si per totum spatium *Tt* tam densitas initialis fuerit naturalis quam celeritas nulla, tum etiam fore $q = Q$ et $\mathfrak{T} = 0$, quod posterius ex formula priori minus perspicitur. Si enim intra spatium *At* initio fuerit motus et curva $A\upsilon C$ ibi incluserit aream quandam, etiamsi deinceps haec curva tota in axem incidat, haec area praebebit applicatas pro curva *ATF* etiam per totum axem sequentem *tT*, unde neque areae *ATN* et *Atn* neque applicatae *TN* et *tn* evanescent: sicque dubium foret, utrum forma

$$\tfrac{c}{2S}\left(TN + tn\right) - \tfrac{c}{2SS}\left(ATN + Atn\right)$$

hoc casu evanesceret; quod autem nunc quidem scala $A\upsilon C$ in calculum introducta necessario evenire debere intelligitur.

SCHOLION 2

131. Loco indicis *M*, quem supra areis praefiximus, ut scriptio $M : ASQ$ denotet integrale $\int dS \cdot ASQ$, commodius signo summatorio $\int$ utemur, ut $\int ASQ$ idem denotet, quod $\int dS \cdot ASQ$, quoniam differentiale abscissae *dS* facile mente suppletur. Hoc

praemisso etiam pro scala $A\Upsilon F$ loco $M : ASQ$ scribam $\int AS\Upsilon$, quae forma autem ex scala celeritatum naturali *AvG* ita determinatur, ut sit

$$\int AS\Upsilon = \tfrac{1}{c}\left(AS\int AS\upsilon - \int\int AS\upsilon\right)$$

ideoque

$$M : ATN = \tfrac{1}{c}\left(AT\int ATL - \int\int ATL\right)$$
$$M : Atn = \tfrac{1}{c}\left(At\int Atl - \int\int Atl\right).$$

Quare per scalam densitatum *AQD* et celeritatum $A\upsilon C$ nostrae determinationes ita se habebunt

$$q = Q\left(1 + \tfrac{1}{2S}\left(TM \;\; + tm - 2SQ\right) - \tfrac{1}{2cS}\left(AT\cdot TL - At\cdot tl + tlTL\right)\right)$$

$$\mathfrak{T} = \tfrac{c}{2S}\left(tm - TM\right) + \tfrac{1}{2SS}TMtm + \tfrac{1}{2S}\left(AT\cdot TL + At\cdot tl\right) - \tfrac{ST}{2SS}\cdot tlTL$$

$$Ss = \tfrac{1}{2S}\left(SQtm - SQTM\right) - \tfrac{1}{2SS}\left(\int SQtm - \int SQTM\right)$$
$$+ \tfrac{1}{2cS}(\mathrm{AT}\cdot ATL - At\cdot Atl) - \tfrac{1}{2cSS}\left(AT\cdot\int ATL - AT\cdot\int Atl - \int tlTL\right),$$

quae ad usum magis videntur accommodatae.

PROBLEMA 93

132. *Si* (Fig.104) *in tubo conico AB b in infinitum extenso alicubi per spatiolum minimum GH pulsus excitetur, propagationem huius pulsus in tubo versus Bb ortam determinare.*

SOLUTIO

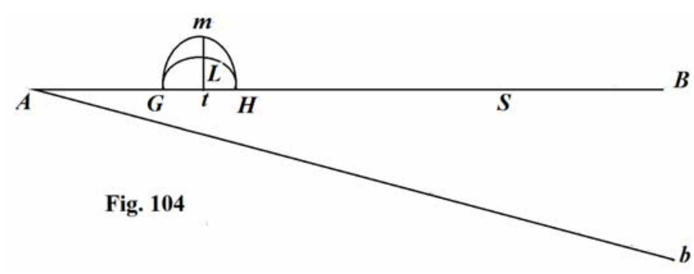

Fig. 104

Cum pulsus in spatiolo *G H* contineatur, in utroque termino *G* et *H* densitatem naturalem et celeritatem nullam esse oportet. Sit igitur *GlH* scala celeritatum, cuius applicata *tl* celeritatem ipsam initialem in loco *t* exprimat, ut sit $tl = \Upsilon$ directione ad *B* versa: pro scala densitatum *GmH* vero applicata $tm = At\cdot l\frac{Q}{B}$ denotante *Q* densitatem in *t* initialem et *B* naturalem. His positis consideremus locum tubi quemcunque *S*, existente $AS = S$, et quia ambae scalae extra spatium *GH* in axem incidunt, aër in *S* tamdiu in aequilibrio perseverabit, quoad ab initio effluxerit tempus $=\frac{SH}{c}$ min. sec., elapso autem tempore $\frac{SG}{c}$ pulsus iterum cessasse est existimandus, ita ut totus pulsus ibi nonnisi per tempus $\frac{GH}{c}$ sit duraturus. Ponamus ergo ab initio elapsum esse $\frac{GH}{c}$ tempus $t=\frac{St}{c}$, et quia ad alteram partem *B* versus aequali spatio $ST=St$ abscisso ibi ambae applicatae *TL* et *TM* evanescunt, densitas aëris *q* et celeritas $\mathfrak{T}$ versus *B* tendens in loco *S* ita exprimentur:

$$q=B\left(1+\tfrac{tm}{2S}+\tfrac{At\cdot tl}{2cS}-\tfrac{Htl}{2cS}\right)$$

quia initio densitas in *S* erat $=B$, tum vero erit:

$$\mathfrak{T}=\tfrac{c\cdot tm}{2S}+\tfrac{c\cdot Htm}{2SS}+\tfrac{At\cdot tl}{2S}-\tfrac{St\cdot Htl}{2SS}.$$

Cum autem intervallum *GH* minimum assumatur, areae *Htl* et *Htm* pro evanescentibus haberi possunt; quare ob $tl=\Upsilon$ et $tm=Atl\,\frac{Q}{B}=At\left(\frac{Q}{B}-1\right)$
habebimus:

$$q=B\left(1+\tfrac{At}{2AS}\left(\tfrac{Q}{B}-1\right)+\tfrac{At\cdot\Upsilon}{2c\cdot AS}\right)\ \text{ seu }\ l\,\tfrac{Q}{B}=\tfrac{At}{2AS}\left(l\,\tfrac{Q}{B}+\tfrac{\Upsilon}{c}\right)$$
$$\mathfrak{T}=\tfrac{c\cdot At}{2AS}\left(\tfrac{Q}{B}-1\right)+\tfrac{At\cdot\Upsilon}{2AS}=\tfrac{c\cdot At}{2AS}\left(l\,\tfrac{Q}{B}+\tfrac{\Upsilon}{c}\right).$$

COROLLARIUM 1

133. Si arcolas *Htl* et *Htm* negligere nolimus, habebimus

$$l\,\tfrac{Q}{D}=\tfrac{At}{2AS}\left(l\,\tfrac{Q}{B}+\tfrac{T}{c}\right)-\tfrac{Ar\cdot Htl}{2c\cdot AS}$$

et

$$\mathfrak{T}=\tfrac{c\cdot At}{2AS}\left(l\,\tfrac{Q}{B}+\tfrac{\Upsilon}{c}\right)+\tfrac{c}{2AS^2}\left(\text{Ar.}Htm-\tfrac{St}{c}\,\text{Ar.}Htl\right),$$

ubi $\frac{St}{c}$ denotat tempus ab initio elapsum *t* in min. sec. expressum.

COROLLARIUM 2

134. Areas autem has eo minus negligere licet, quo maius spatiolum *GH* et quo propius fuerit vertici coni *A*, quoniam tum ob $tl=\Upsilon$ area *Htl* prae rectangulo $At\cdot tl$ neutiquam pro evanescente haberi potest.

COROLLARIUM 3

135. Si tempus elapsum *t* maius fuerit quam $\frac{SG}{c}$, ita ut punctum *t* ultra *G* versus *A* cadat, ubi initio erat $l\frac{Q}{B}=0$ et $\Upsilon=0$, pro loco *S* adhuc erit:

$$l\frac{q}{B}=-\frac{\text{Ar}.GlH}{2\cdot cAS}B \text{ et } \mathfrak{T}=\frac{c}{2AS^2}\left(\text{Ar}.GmH-t\cdot GlH\right),$$

tum ergo neque densitas est naturalis neque motus penitus extinctus.

SCHOLION

136. Omnino singulare est hoc phaenomenon, quod, etiamsi pulsus in *S* iam cessavisse est aestimandus, tamen aëris aequilibrium nondum prorsus sit restitutum, sed fieri possit, ut etiamnum tam densitas *q* a naturali *B* discrepet, quam aër ipse motu quodam proferatur. Discrimen quidem hoc in valde magnis distantiis *AS* ut nullum spectari potest; in minoribus vero eo magis est notatu dignum, quod perpetuo durare videatur, cum calculus nullam porro mutationem indicet. Verum hic observari oportet nostras scalas hic non ultra verticem *A* porrigi earumque continuationem ex altera parte ipsius *A* similem pulsum *GH* ad parem distantiam implicare, quo si tempus *St* pertigerit, in *S* novus pulsus secundarius excitetur, quo finito demum aër ad *S* prorsus in aequilibrium restituatur. Hic enim similis repetitio oriri debet, ac si tubus esset cylindricus et in *A* clausus: nunc autem videmus hoc discrimen intercedere, quod in casu tubi conici interea, dum pulsus secundarius ad *S* appellit, aërem ibi quandam adhuc agitationem retinere, secus ac fit in cylindricis. Quare, quemadmodum ambae scalae ultra coni verticem continuari debeant, in sequente Problemate investigabimus.

PROBLEMA 94

137. *Si* (Fig.105) *circa verticem coni A aër utcunque de statu, aequilibrii deturbetur seu pulsus ibi quicunque excitetur, scalas ambas, quibus ad propagationem definiendam opus est, retro ultra verticem A continuare indeque propagationem in tubo pro quovis loco S determinare.*

SOLUTIO

Extendatur pulsus primo excitatus per spatium *AD*, et in loco quocunque *T* posito intervallo $AT=S$ fuerit densitas aëris $=Q$, naturali existente $=B$, celeritas vera $=\Upsilon$ secundum directionem *AB*, hinc construatur primo

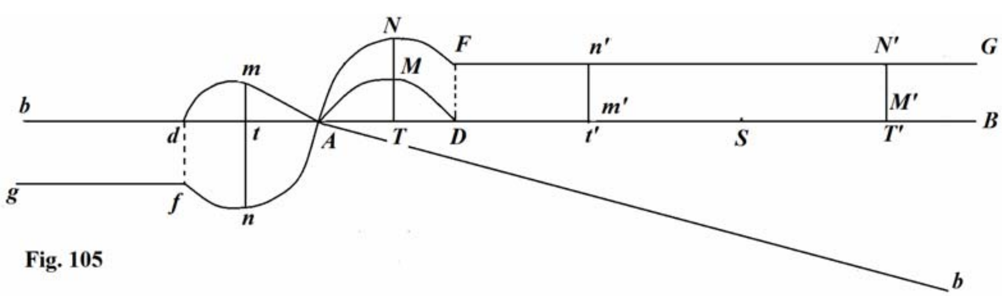

Fig. 105

curva *AMD* sumendo applicatas $TM = Sl\frac{Q}{B}$, quae ergo ultra *D* in ipsum axem *DB* incidit. Deinde construatur etiam curva *ANF* sumendo applicatas $TN = \frac{S\Upsilon + \int \Upsilon dS}{c}$, cuius quidem primum membrum in *D*, ubi $\Upsilon = 0$, evanescit, alterum vero $\frac{1}{c}\int \Upsilon dS$ ibi certum valorem adipiscetur, cui sequentes applicatae omnes versus *B* erunt aequales, ita ut haec scala *ANF* ultra *F* abeat in rectam *FG* ipsi axi *AB* parallelam; at area huius curvae ita definietur, ut sit $ATN = \frac{S}{c}\int \Upsilon dS$: unde, cum puncto *T* in *D* promoto fiat applicata $DF = \frac{1}{c}\int \Upsilon dS$, erit area $ADF = AD \cdot DF$, perinde ac si tota scala *ANFG* esset recta *GF* ipsi axi parallela et supra verticem *A* usque continuata. His positis primo quaeritur, quomodo has scalas ultra *A* continuari oporteat, ubi ante omnia est observandum aërem in ipso vertice *A* nullum plane motum concipere posse. Quare, si tubi punctum *S*, in quo supra generatim celeritatem $\mathfrak{T}$ definivimus, in *A* transferamus, formula ibi pro celeritate exhibita

$$\mathfrak{T} = \frac{c}{2S}\left(tm - TM + TN + tn\right) + \frac{c}{2SS}\left(ATM - Atm - ATN - Atn\right)$$

hoc casu nulla esse debet. At hoc casu habebimus $S = 0$, quia in hac formula posuimus $AS = S$, ex quo necesse est continuationem quaesitam ita esse comparatam, ut sit $tm = TM$ et $tn = -TN$. Quamobrem in nostra figura scalam *AMD* sine ulla variatione super axe *Ad* describi convenit, ut sit ea *Amd*, altera vero scala *ANFG* ibi ad contrariam axis partem describi debet, ut sit *Anfg*. His iam ambabus scalis ultra *A* continuatis consideremus in tubo locum quemcunque *S*, statuendo nunc distantiam a vertice $AS = S$, ubi, cum aër initio fuerit in aequilibrio, ideoque $\Upsilon = 0$ et $Q = B$ hincque $SQ = 0$, elapso tempore $= t$ capiamus utrinque ab *S* intervalla $ST' = St' = ct$, sintque in punctis *T'* et *t'* applicatae ambarum scalarum *T'M'*, *T'N'* et *t'm'*, *t'n';* atque ex his status aëris in *S* ita definietur, ut sit ibi

$$\text{densitas } q = B\left(1 + \tfrac{1}{2S}\left(T'M' + t'm'\right) - \tfrac{1}{2S}\left(T'N' - t'n'\right)\right)$$

et

$$\text{celeritas } \mathfrak{T} = \tfrac{c}{2S}\left(t'm' - T'M'\right) + \tfrac{c}{2S}\left(T'N' + t'n'\right)$$
$$- \tfrac{c}{2SS}\left(At'm' - AT'M'\right) - \tfrac{c}{2SS}\left(AT'N' + A\,t'n'\right),$$

unde primo patet, quamdiu tempus ab initio elapsum *t* minus fuerit quam $\frac{SD}{c}$, tum ob

$$T'M' = 0,\quad t'm' = 0 \text{ et } T'N' = t'n' = DF$$

fore $q = B$ et ob $At'm' = AT'M'$ et $AT'N' = AT' \cdot DF$ atque $At'n' = At' \cdot DF$ prodire

$$\mathfrak{T} = \tfrac{c}{S} \cdot DF - \tfrac{c \cdot DF}{2SS}\left(AT' + At'\right) = 0$$

ob $AT' + At' = 2AS = 2S$; tamdiu ergo aër ibi in aequilibrio manebit.
Statim autem ac tempus elapsum *t* maius fit quam $\frac{SD}{c}$, aër in *S* agitari incipiet; ponamus ergo elapsum iam esse tempus $t = \frac{AS}{c}$, et quia nunc est

$$T'M' = 0,\ \ t'm' = 0,\ \ T'N' = DF, t'n' = 0,$$
$$AT'M' = AMD,\ \ At'm' = 0,\ \ AT'N' = AT'\cdot DF = 2S\cdot DF \ \text{ et } \ At'n' = 0,$$

erit tum in loco *S*

$$\text{densitas} \quad q = B\left(1 - \tfrac{DF}{2S}\right)$$

et

$$\text{celeritas} \ \ \mathfrak{T} = \tfrac{C\cdot DF}{2S} + \tfrac{c\cdot AMD}{2SS} - \tfrac{c\cdot DF}{S} = \tfrac{c\cdot AMD}{2SS} - \tfrac{c\cdot DF}{2S}.$$

Pro tempore autem quovis medio elapso $t = \frac{ST}{c}$ elicitur
densitas $q = B\left(1 + \frac{TM + TN - DF}{2S}\right)$
et

celeritas $\mathfrak{T} = \frac{c}{2S}\left(TM + TN - DF\right) + \frac{c}{2SS}\left(DTM + DFTN - DT\cdot DF\right).$

Videamus denique etiam, quomodo status aëris in *S* se sit habiturus, cum effluxerit tempus $t = \frac{Sd}{c} = \frac{AS+AD}{c}$; tum vero erit

$$T'M' = 0,\ \ AT'M' = AMD,\ \ T'N' = DF,$$
$$AT'N' = AT'\cdot DF = 2S\cdot DF + AD\cdot DF,$$

porro

$$t'm' = 0,\ \ At'm' = Amd = AMD,\ t'n' = -df = -DF,$$
$$At'n' = -Adf = -AD\cdot DF,$$

ideoque

$$\text{densitas} \quad q = B\left(1 - \tfrac{DF}{S}\right)$$

et

$$\text{celeritas} \ \ \mathfrak{T} = -\tfrac{c\cdot DF}{S}.$$

COROLLARIUM 1

138. Celeritas ergo pulsus, cum ad locum quemvis *S* pervenerit, duabus constat partibus, quarum prior decrescit in ratione distantiae $AS = S$, posterior vero in ratione duplicata distantiae, quae autem posterior pars prae priori evanescit, statim ac distantia $AS = S$ fit modica.

COROLLARIUM 2

139. Cum impulsio in organo auditus facta pendeat sine dubio a celeritate agitationis in quovis pulsu, hinc patet, quomodo sonus aucta distantia continuo diminuatur; et quia vis impulsionis quadrato celeritatis proportionalis statuenda videtur, soni debilitatio sequetur rationem duplicatam distantiarum.

SCHOLION 1

140. Insignis hic difficultas occurrit, quod, postquam pulsus penitus per locum *S* transierit, aër tamen ibi non in aequilibrium restituatur, nisi pulsus initialis ita fuerit comparatus, ut intervallum *DF* evanescat. Talis certe commotio post transitum pulsus nullo modo admitti potest, ideoque calculus noster defectus potius, cuiuspiam arguendus videtur, nisi dicere velimus omnem agitationem aëri primum inductam necessario ita semper esse comparatam, ut intervallum *DF* inde prodeat evanescens. Quoniam vero hoc nulla ratione probari potest, origo huius incommodi in eo quaerenda videtur, quod, cum intervallum *DF* ex valore integrali $\int \Upsilon dS$ sit natum, in hac ipsa integratione iusta constans sit neglecta; dummodo ergo talem constantem introduxissemus, qua hoc integrale per totam scalam celeritatum extensam ad nihilum fuisset reductum, omnis haec difficultas penitus evanuisset, cum inde etiam curva *ANF* ultra *D* tota in axem *DB* incidisset. Nullum ergo est dubium, quin hic origo istius defectus calculi sit sita totumque calculum hoc modo expediri conveniat. Talia autem incommoda in argumento prorsus novo minime sunt miranda et sperare licet, cum id diligentius fuerit excultum, tum omnia sponte dissipatum iri. Hic autem continuatio scalarum ultra verticem *A* merito suspecta videtur, propterea quod in hoc loco fit $S=0$ iique termini, quos destrui oportet, infiniti; cui incommodo in sequenti problemate medelam afferre conabor.

SCHOLION 2

141. Ex hoc problemate colligere licet, quemadmodum pulsus quicunque in libero aëre quaquaversus propagetur; si enim quasi centrum pulsus propagati sit in *A*, universus aër circumfusus in infinitos conos, quorum vertices in *A* concurrant, distributus concipi potest; et manifestum est per quemlibet eorum pulsum perinde propagatum iri, ac per liberum aërem, quoniam, qua duo huiusmodi tubi se mutuo tangunt, ibi densitas ac propterea etiam pressio est eadem, ita ut etiam tubis remotis propagatio eandem legem sit secutura. Nunc ergo intelligimus pulsus in libero aëre eadem prorsus celeritate propagari, atque in tubis cylindricis, at vero ob tuborum amplificationem pulsus continuo debilitari, ita ut celeritas cuiusque agitationis diminuatur in ratione distantiarum a pulsu initiali *A*, latitudinem vero pulsuum ceterasque soni qualitates in propagatione non alterari.

PROBLEMA 95

142. *Si* (Fig. 106) *tubus conicus citra verticem I in Aa sit terminatus ibique sive apertus sive clausus, ex agitatione initiali aëri in tubo A aB b contento inducta formulas pro motu sequente definiendo exhibere.*

SOLUTIO

Posita termini *A* a vertice coni *I* distantia $IA = a$, sumatur in cono locus quicunque *S* vocata distantia $AS = s$, ut distantia, quam ante posuimus *S,* hic sit $= a+s$, initio autem in *S* fuerit densitas $= Q$, naturali existente $= B$, et celeritas secundum $SB = \Upsilon$. Elapso autem tempore quocunque *t* eiusdem aëris, qui initio fuerat in *S,* sit densitas $= q$, celeritas $= \mathfrak{T}$ et translationis spatiolum $Ss = \upsilon$. Cum iam amplitudo tubi in *S* sit $\Omega = nn(a+s)^2$, erit $\frac{d\Omega}{\Omega dS} = \frac{2}{a+s}$, hincque ex formulis supra inventis obtinebimus:

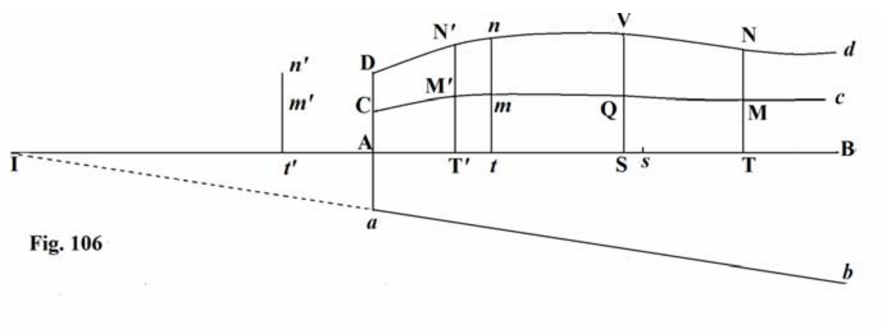

Fig. 106

$$\upsilon = \frac{1}{(a+s)^2}\int (a+s)^2\, ds l \frac{Q}{B} - \frac{1}{a+s}\Gamma' : (s+ct) + \frac{1}{(a+s)^2}\Gamma : (s+ct)$$
$$- \frac{1}{a+s}\Delta' : (s+ct) + \frac{1}{(a+s)^2}\Delta : (s+ct)$$

$$q = Q\left(1 - l\frac{Q}{B} + \frac{1}{a+s}\Gamma'' : (s+ct) + \frac{1}{a+s}\Delta'' : (s-ct)\right)$$

et

$$\mathfrak{T} = \frac{-c}{a+s}\Gamma'' : (s+ct) + \frac{c}{a+s}\Delta'' : (s-ct) + \frac{c}{(a+s)^2}\Gamma' : (s+ct) - \frac{c}{(a+s)^2}\Delta' : (s-ct),$$

unde pro statu initiali cognito colligimus:

$$0 = \frac{1}{(a+s)^2}\int (a+s)^2\, ds l \frac{Q}{B} - \frac{1}{a+s}\left(\Gamma' : s + \Delta' : s\right) + \frac{1}{(a+s)^2}\left(\Gamma : s + \Delta : s\right)$$
$$0 = -l\frac{Q}{B} + \frac{1}{a+s}\left(\Gamma'' : s + \Delta'' : s\right)$$
$$\Upsilon = \frac{-c}{a+s}\left(\Gamma'' : s - \Delta'' : s\right) + \frac{c}{(a+s)^2}\left(\Gamma' : s - \Delta' : s\right).$$

Ex illa fit

$$\Gamma'' : s + \Delta'' : s = (a+s) l \tfrac{Q}{B} \quad \text{et} \quad \Gamma' : s + \Delta' : s = \int (a+s) dsl \tfrac{Q}{B}$$

atque

$$\Gamma' : s + \Delta' : s = \int (a+s) dsl \tfrac{Q}{B},$$

ex ista vero

$$\int \Upsilon ds = \tfrac{-c}{a+s}(\Gamma' : s - \Delta' : s) \quad \text{seu} \quad c(\Gamma' : s - \Delta' : s) = -(a+s)\int \Upsilon ds$$

hincque

$$c(\Gamma'' : s - \Delta'' : s) = -\int \Upsilon ds - (a+s)\Upsilon \quad \text{et} \quad c(\Gamma : s - \Delta : s) = -\int (a+s) ds \int \Upsilon ds.$$

Qui valores in prima aequatione substituti praebent:

$$0 = \int (a+s)^2 dsl \tfrac{Q}{B} - (a+s)\int (a+s) dsl \tfrac{Q}{B} + \int ds \int (a+s) dsl \tfrac{Q}{B}$$

unde ambiguitas ob constantem per integrationem ingressam oriunda tollitur, cum debeat esse:

$$\int ds \int (a+s) dsl \tfrac{Q}{B} = (a+s)\int (a+s) dsl \tfrac{Q}{B} - \int (a+s)^2 dsl \tfrac{Q}{B},$$

quae duo integralia simplicia ita capiantur, ut posito $s = 0$ evanescant.
Functiones ergo Γ et Δ ita definiuntur, ut sit

$$\Gamma'' : s = \tfrac{1}{2}(a+s) l \tfrac{Q}{B} - \frac{(a+s)\Upsilon + \int \Upsilon ds}{2c}$$

$$\Delta'' : s = \tfrac{1}{2}(a+s) l \tfrac{Q}{B} + \frac{(a+s)\Upsilon + \int \Upsilon ds}{2c}$$

tum vero

$$\Gamma' : s = \tfrac{1}{2}(a+s) l \tfrac{Q}{B} - \frac{(a+s)\Upsilon + \int \Upsilon ds}{2c}$$

$$\Delta' : s = \tfrac{1}{2}(a+s) l \tfrac{Q}{B} + \frac{(a+s)\Upsilon + \int \Upsilon ds}{2c}$$

et

$$\Gamma : s = \tfrac{1}{2}(a+s)\int (a+s) dsl \tfrac{Q}{B} - \tfrac{1}{2}\int (a+s)^2 dsl \tfrac{Q}{B} - \frac{\int (a+s) ds \int \Upsilon ds}{2c}$$

$$\Delta : s = \tfrac{1}{2}(a+s)\int (a+s) dsl \tfrac{Q}{B} - \tfrac{1}{2}\int (a+s)^2 dsl \tfrac{Q}{B} + \frac{\int (a+s) ds \int \Upsilon ds}{2c}.$$

Statuamus ergo in *S* duas applicatas

$$SQ = (a+s) l \tfrac{Q}{B} \text{ et } SV = \frac{(a+s)\Upsilon + \int \Upsilon ds}{c},$$

descriptisque binis scalis *CQc* et *DVd* erit

$$\Gamma'' : s = \tfrac{1}{2}SQ - \tfrac{1}{2}SV, \qquad \Delta'' : s = \tfrac{1}{2}SQ + \tfrac{1}{2}SV,$$
$$\Gamma' : s = \tfrac{1}{2}ACSQ - \tfrac{1}{2}ADSV, \quad \Delta' : s = \tfrac{1}{2}ACSQ + \tfrac{1}{2}ADSV.$$

Pro tempore ergo elapso t sumtis intervallis $ST = St = ct$ densitas q et celeritas $\mathfrak{T}$ ita definiuntur, ut sit:

$$q = Q\left(1 + \tfrac{1}{2(a+s)}\left(TM - TN + tm + tn - 2SQ\right)\right)$$
$$\mathfrak{T} = \tfrac{c}{2(a+s)}\left(tm + tn - TM + TN\right) + \tfrac{c}{2(a+s)^2}\left(ACTM - ADTN - ACtm - ADtn\right),$$

quae duo elementa nosse sufficit, cum inde tertium, nempe spatiolum Ss facile concludatur.

Ut iam investigemus, quomodo utramque scalam ultra A continuari oporteat, sumamus punctum S in ipso termino A, ut sit $s = 0$, et abscissis utrinque $AT' = At' = ct$, quoniam areas curvarum a puncto A dextrorsum computavimus, quae nunc sinistrorsum cadunt, negative capi debent eritque pro hoc puncto A

$$q = Q\left(1 + \tfrac{1}{2a}\left(T'M' + t'm' - 2AC\right) - \tfrac{1}{2a}\left(T'N' - t'n'\right)\right)$$
$$\mathfrak{T} = \tfrac{c}{2a}\left(t'm' - T'M'\right) + \tfrac{c}{2a}\left(T'N' + t'n'\right) + \tfrac{c}{2aa}\left(ACT'M' + ACt'm'\right)$$
$$- \tfrac{c}{2aa}\left(ADT'N' - ADt'n'\right).$$

Nunc duo casus sunt expediendi, prout tubus in termino A fuerit apertus vel clausus:

I. Sit tubus in Aa apertus, et quia ibi densitas perpetuo manet naturalis, ut sit $Q = B$ et $q = B$, erit $AC = 0$ fierique necesse est

$$T'M' + t'm' = 0 \quad \text{et} \quad T'N' = t'n';$$

hoc ergo casu utraque scala prorsus ut in tubis cylindricis continuatur; scala scilicet densitatum CQc ad alteram axis partem, celeritatum vero *DVd* ad eandem describitur.

II. Sit tubus in Aa clausus, et quia ibi celeritas perpetuo manet nulla, pro scala densitatum primo esse debet:

$$a\left(t'm' - T'M'\right) + ACT'M' + ACt'm' = 0;$$

statuamus

$$AT' = At' = x,\ T'M' = y \ \text{ et } \ t'm' = z,$$

ut habeamus:

$$a\left(z - y\right) + \int\left(y + z\right)dx = 0 \ \text{ seu } \ adz + zdx - ady + ydx = 0,$$

quae integrata dat:

$$e^{\frac{x}{a}}az=\int e^{\frac{x}{a}}\left(ady-ydx\right)=e^{\frac{x}{a}}ay-2\int e^{\frac{x}{a}}ydx=-e^{\frac{x}{a}}ay+2a\int e^{\frac{x}{a}}dx$$

hincque

$$z=t'm'=y-\tfrac{2}{a}e^{\frac{-x}{a}}\int e^{\frac{x}{a}}ydx=-y+2e^{\frac{-x}{a}}\int e^{\frac{x}{a}}dy.$$

Deinde pro scala celeritatum oportet sit:

$$a\left(T'N'+t'n'\right)-ADT'N'+ADt'n'=0\,,$$

faciamus

$$AT'=At'=x,\quad T'N'=y,\quad t'n'=z$$

et aequatio

$$a\left(y+z\right)-\int ydx+\int zdx=0\ \text{ seu }\ adz+zdx+ady-ydx=0$$

integrata praebet:

$$e^{\frac{x}{a}}az=\int e^{\frac{x}{a}}\left(ydx-ady\right)=-e^{\frac{x}{a}}ay+2\int e^{\frac{x}{a}}ydy=e^{\frac{x}{a}}ay-2a\int e^{\frac{x}{a}}dy$$

hincque

$$z=t'n'=-y+\tfrac{2}{a}e^{\frac{-x}{a}}\int e^{\frac{x}{a}}ydy=y-2e^{\frac{-x}{a}}\int e^{\frac{x}{a}}dy,$$

utroque casu integralia ita definiri debent, ut posito $x=0$ fiat $z=y$.

COROLLARIUM 1

143. Cum casu, quo tubus in *Aa* est apertus, ambae scalae eodem modo continuentur, quo in tubis cylindricis, eadem quoque lex continuationis locum habebit, si tubus conicus etiam in altero termino B b fuerit apertus, unde sequitur in tubo conico utrinque aperto agitationes aëris prorsus convenire cum iis, quas supra pro tubis cylindricis definivimus.

COROLLARIUM 2

144. Quae igitur supra de sonis, quos tibiae apertae edunt, annotavimus, eadem quoque locum habent, si tibiis figura conica tribuatur. Neque tamen figura nimis divergens admitti potest, quia tum agitationes per unamquamque sectionem non amplius forent aequabiles, uti Theoria nostra postulat.

COROLLARIUM 3

145. Ex hoc problemate theoria tuborum cylindricorum deducitur, si distantia $IA=a$ statuatur infinita. Tum autem pro casu, quo tubus in *Aa* est clausus, eadem lex pro utriusque scalae continuatione colligitur, quia ob $a=\infty$ erit $t'm'=T'M'$ et $t'n'=-T'N'$.

SCHOLION 1

146. Nunc dubia, quae supra circa propagationem pulsus in ipso vertice coni excitati supererant, perfecte tolluntur; cum enim totum negotium eo redeat, quomodo utramque scalam ultra verticem coni oporteat continuari, ad hunc casum nostrum problema accommodabimus, si distantiam $IA = a$ evanescentem simulque tubum in A a clausum statuamus, quoniam enim conus in vertice per se clauditur, ibi certe aër nullum motum concipere potest. Ex quo primum pro continuatione scalae densitatum quaestio huc redit, ut ex hac aequatione

$$z = -y + 2e^{\frac{-x}{a}} \int e^{\frac{x}{a}} dy$$

valor ipsius z definiatur casu, quo $a = 0$, quod cum ob exponentem $\frac{x}{a}$ infiniatum minus pateat, ad aequationem differentialem

$$adz + zdx - ady + ydx = 0$$

confugiamus, unde ob $a = 0$ manifesto sequitur $z = -y$, ita ut scala densitatum axi ad partem contrariam applicari debeat, secus ac supra per evanescentiam intervalli $S = 0$ decepti fecimus. Deinde vero pro scala celeritatum ex aequatione differentiali

$$adz + zdx + ady - ydx = 0 \text{ ob } a = 0$$

deducimus $z = y$, ita ut haec scala ad eandem axis partem constitui debeat, utraque scilicet continuatio eadem lege peragitur, ac si tubus esset apertus. Hinc ergo (Fig. 105) postrema evolutio in problemate 94 ita emendabitur, ut, cum scalae in contrarias plagas cadant, atque in figura sunt repraesentatae,
Sd elapso tempore $t = \frac{Sd}{c}$ fiat $t'm' = 0,\ t'n' = +DF$, area $At'm' = +Amd = AMD,$ quia hic duplex signi mutatio fieri debet, altera quatenus haec area ultra A, altera quatenus infra axem cadit. Denique vero fit area

$$At'n' = -Adf = -AD \cdot DF;$$

unde consequimur :

$$T'M' + t'm' = 0,\ \ T'N' - t'n' = 0,\ \ At'm' - AT'M' = 0,$$
$$t'm' - T'M' = 0,\ T'N' + t'n' = 2DF,\ AT'N' + At'n' = 2S \cdot DF$$

hincque porro densitatem $q = B$ et celeritatem

$$\mathfrak{T} = \frac{c}{2S} \cdot 0 + \frac{c}{2S} \cdot 2DF - \frac{c}{2SS} \cdot 0 - \frac{c}{2SS} \cdot 2S \cdot 2DF$$

seu

$$\mathfrak{T} = \tfrac{c}{S} \cdot DF - \tfrac{c}{S} \cdot DF = 0,$$

ex quo manifestum est elapso tempore $t = \frac{Sd}{c}$ aërem in S penitis in aequilibrium restitui, sicque veritatem egregie salvari.

SCHOLION 2

147. Quoniam vidimus tibias conicas pro longitudine eosdem edere sonos atque cylindricos, siquidem sint apertae, probe notandum est hanc convenientiam neutiquam in tibiis clausis locum habere. Cum enim, si tubus conicus ad *A* a fuerit clausus, ambae scalae in lineas ipsis maxime dissimiles continuari debeant, hinc nullae oscillationes regulares oriri poterunt, sed sonus iis editus admodum erit rudis et inconditus minimeque ad harmoniam efficiendam accommodatus.