## CHAPTER I

## CONCERNING THE MINIMAL DISTURBANCES OF AIR IN TUBES OF UNIFORM CROSS-SECTION

## PROBLEM 67

1. *While the air in a* [*closed*] *tube* (Fig. 70) *may be disturbed in some manner, whether it be straight or curved, with a uniform cross-section and placed horizontally, to arrive at the equations by which its motion may be determined.*

## SOLUTION

*AB* shall be the proposed tube, which we represent in the diagram as straight, whether it be straight or curved, since we have seen the motion not to be disturbed by the curvature. Therefore if its cross-section shall be constant $= ff$. Moreover for this problem requiring to be solved it will be agreed for the latter method to be used [see Part 3. Ch.1, E409], where the state of the air in the closed tube may be compared with the initial state.

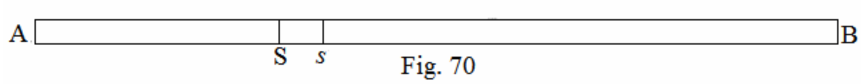


Fig. 70

Therefore we may require the solution to follow from problem 45, which we consider in terms of a particle of air, which initially will have been at *S*, when the time was $t = 0$, and we put the distance $AS = S$, and truly the density of this particle of air $= Q$; but the cross-section of the tube, which before was $= \Omega$, becomes for us here $= ff$. Now in the elapsed time *t* the same particle will arrive at *s* and we may set the distance $As = s$, the density of which particle $= q$, the pressure $= p$ and the speed along the direction $sB = \mathfrak{T} = \left(\frac{ds}{dt}\right)$, with the cross-section of the tube being $\omega = ff$. With these put in place the first equation found provides $q\left(\frac{ds}{dS}\right) = Q$; then, since on account of the tube being placed horizontally, gravity does not affect the motion, here the other equation found will be adapted to this form $\frac{2gdp}{q} = -ds\left(\frac{dds}{dt^2}\right)$, which since here we may regard the time as a constant, thus there can be put

$$\frac{2g}{q}\left(\frac{dp}{dS}\right) + \left(\frac{ds}{dS}\right)\left(\frac{dds}{dt^2}\right) = 0;$$

where it will help to remember that while quantities *p, q, s* are functions of the two variables *S* and *t*, the quantity *Q* truly to be a function only of *S* itself. But in addition here the nature of the air itself may be introduced, where we know the pressure *p* always to be proportional to the density $q$ ; from which if it may be agreed the pressure $=a$ of a given density *b,* there will become $p=\frac{aq}{b}$, from which the latter equation becomes

$$\frac{2ga}{bq}\left(\frac{dq}{dS}\right)+\left(\frac{ds}{dS}\right)\left(\frac{dds}{dt^2}\right)=0,$$

with the first being $q\left(\frac{ds}{dS}\right)=Q.$ Hence therefore the density $q$ will be allowed to be deduced, since there shall become

$$\left(\frac{dq}{dS}\right)=\frac{dQ}{dS\left(\frac{ds}{dS}\right)}-\frac{Q\left(\frac{dds}{dt^2}\right)}{\left(\frac{ds}{dS}\right)^2}\quad \text{on account of}\quad q=\frac{Q}{\left(\frac{ds}{dS}\right)}$$

and thus

$$\frac{1}{q}\left(\frac{dq}{dS}\right)=\frac{dQ}{QdS}-\frac{\left(\frac{dds}{dS^2}\right)}{\left(\frac{ds}{dS}\right)}.$$

Whereby we will have this equation, from which the determination of the motion will be contained :

$$\frac{2gadQ}{bQdS}\left(\frac{dq}{dS}\right)-\frac{2ga}{b}\left(\frac{dds}{dS^2}\right)+\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right)=0,$$

which equation thus is required to be resolved, so that on putting $t=0$ there may become $s=S$, but then there will become $\left(\frac{ds}{dS}\right)=1$ and thus $q=Q$, as the nature of the matter demands.

COROLLARY 1

2. Therefore if this equation may be allowed to be resolved thus, so that, whatever the function *Q* shall be going to become, it shall be able to designate the magnitude *s* of the two variables *S* and *t* , then all the motion may be able to be defined, which may fall on the air held in the tube. And thus the whole concern for the resolution of this equation has been resolved.

COROLLARY 2

3. Since this equation may involve differentials of the second order, the complete integral of this must contain two indefinite functions, which thence will require to be determined from the initial state, and since not only the initial position of each particle is assumed given, but also the motion, for this requiring to be effected certainly there is a need for the two indeterminate functions.

## SCHOLIUM 1

4. Therefore the determination of the motion of the air is by far the most difficult task, since the simplest case, where we may put the air to be moved into a tube with equal cross-section and with the effect of gravity removed, to be endowed with the same degree of heat, for an equation of this kind to be produced, the resolution of which cannot be expedited by any known artifice at this stage: which therefore deserves to be considered less, since that part of the infinitesimal analysis, in whatever manner this equation may be going to be found, recently at last has began to be improved, nor yet scarcely anything there has been produced outstanding at this stage beyond the first elements. Therefore all the questions are required to be judges as especially hard, which are established regarding the motion of the air, even if perhaps they may be considered the most easy, just as whether the air in the tube may be condensed or rarefied with the aid of a pump, certainly the determination of which motion may be undertaken in vain. Indeed when the force of the compressed air is accustomed to be defined by a pneumatic pump, the motion of the packets of air generally in no manner is considered to be the motion of the air itself, but this rather may be considered by the movement, as if the air itself may be at rest, with which ignored even if in the motion of the packets scarcely any error may be seen to arise, yet least this example may be allowed to advance , in which the motion of the air will have been defined in some manner: why not rather confess even now what we ourselves know about this part of the theory of motion of fluids, that we are living in a state of the maximum ignorance.

## SCHOLIUM 2

5. Since I have approached the solution of the problem by the latter method set out above, lest anyone may suspect the first method to be used perhaps with a more fruitful outcome, I shall put in place here the solution thence demanded by both. Therefore following problem 44 without any regard to the initial state had for the time $=t$, we will consider the small portion of the air moving at $s$ : I call the interval $As=s$ and there the density $=q$, the pressure $=p$ and the speed $=\mathfrak{T}$ in the direction *sB*, and since the cross-section of the tube is constant or $\omega = ff$ and no forces are allowed to be acting, these two equations will be had :

$$\left(\frac{d\cdot q\mathfrak{T}}{ds}\right)+\left(\frac{dq}{dt}\right)=0 \quad \text{and} \quad \frac{2gdp}{q}=-\mathfrak{T}d\mathfrak{T}-ds\left(\frac{d\mathfrak{T}}{dt}\right).$$

But since from the nature of air there shall be $p=\frac{aq}{b}$, the latter equation will become:

$$\frac{2gadq}{bq}+\mathfrak{T}d\mathfrak{T}+ds\left(\frac{d\mathfrak{T}}{dt}\right)=0,$$

which, since the time $t$ may be put constant, is reduced to this form:

$$\frac{2ga}{bq}\left(\frac{dq}{ds}\right)+\mathfrak{T}\left(\frac{d\mathfrak{T}}{ds}\right)+\left(\frac{d\mathfrak{T}}{dt}\right)=0.$$

Therefore since the first equation expanded out provides

$$\mathfrak{T}\left(\frac{dq}{ds}\right)+q\left(\frac{d\mathfrak{T}}{ds}\right)+\left(\frac{dq}{dt}\right)=0,$$

we may put $q=e^{y}$, then there will become:

$$\mathfrak{T}\left(\frac{dy}{ds}\right)+\left(\frac{d\mathfrak{T}}{ds}\right)+\left(\frac{dy}{dt}\right)=0,$$

Truly from the other equation:

$$\frac{2ga}{b}\left(\frac{dy}{ds}\right)+\mathfrak{T}\left(\frac{d\mathfrak{T}}{ds}\right)+\left(\frac{d\mathfrak{T}}{dt}\right)=0.$$

Hence there is elicited:

$$\left(\frac{dy}{ds}\right)=-\frac{b\mathfrak{T}}{2ga}\left(\frac{d\mathfrak{T}}{ds}\right)-\frac{b}{2ga}\left(\frac{d\mathfrak{T}}{dt}\right),$$

truly from that:

$$\left(\frac{dy}{dt}\right)=-\left(\frac{d\mathfrak{T}}{ds}\right)+\frac{b\mathfrak{T}\mathfrak{T}}{2ga}\left(\frac{d\mathfrak{T}}{ds}\right)+\frac{b\mathfrak{T}}{2ga}\left(\frac{d\mathfrak{T}}{dt}\right).$$

Whereby, since there shall become

$\left(\frac{ddy}{dsdt}\right)=\left(\frac{ddy}{dtds}\right)$, we conclude, [from the respective diff. of the two above equations]

$$\left(\mathfrak{T}\mathfrak{T}-\frac{2ga}{b}\right)\left(\frac{dd\mathfrak{T}}{ds^2}\right)+2\mathfrak{T}\left(\frac{d\mathfrak{T}}{ds}\right)^2+2\left(\frac{d\mathfrak{T}}{ds}\right)\left(\frac{d\mathfrak{T}}{dt}\right)+2\mathfrak{T}\left(\frac{dd\mathfrak{T}}{dtds}\right)+\left(\frac{dd\mathfrak{T}}{dt^2}\right)=0,$$

from which it will be required to investigate now, what kind of function $\mathfrak{T}$ shall become of the two variables *t* and *s*. But this equation, as we have given there in the solution of the problem, not only cannot be treated easily, but also that has been given here conveniently, so that it may be able to be applied to air disturbed minimally, just as we will demonstrate in the following problem.

## PROBLEM 68

6. *In the case of the preceding problem, if we may know the motion of the air thus prepared, so that the individual particles may not depart except minimally from the initial location, to determine these minimal disturbances of the air.*

## SOLUTION

The solution of the preceding problem may be adapted to this case, if the distance *Ss* (Fig. 70), so that we may treat the particle of air situated at *s* removed from its original position *S* in the calculation as minimal. To this end we may put $s = S + z$ , thus so that $z$ shall be required to be considered as a minimal quantity, and the equation determining the motion adopts this form:

$$\frac{2gadQ}{bQdS}\left(1+\left(\frac{dz}{dS}\right)\right)-\frac{2ga}{b}\left(\frac{ddz}{dS^2}\right)+\left(1+\left(\frac{dz}{ds}\right)\right)^2\left(\frac{ddz}{dt^2}\right)=0,$$

which, since the formula $\left(\frac{dz}{dS}\right)$ shall vanish besides unity, is contracted into this form :

$$\frac{2gadQ}{bQdS}-\frac{2ga}{b}\left(\frac{ddz}{dS^2}\right)+\left(\frac{ddz}{dt^2}\right)=0,$$

of which if only the first term may be absent, the integral from these, which now in this new calculation has been prepared generally, may be able to be given ; indeed there will become :

$$z=\varGamma:\left(S+t\sqrt{\frac{2ga}{b}}\right)+\varDelta:\left(S-t\sqrt{\frac{2ga}{b}}\right).$$

But since the first term alone may contain the variable *S*, of which *Q* is a given function, the integration is not disturbed by that and our complete integral equation becomes :

$$z=\int dSl\,\frac{Q}{B}+\varGamma:\left(S+t\sqrt{\frac{2ga}{b}}\right)+\varDelta:\left(S-t\sqrt{\frac{2ga}{b}}\right),$$

with which found so that the conditions of the remaining motion may be elicited, on account of $s = S + z$ , there will become

$$\left(\frac{ds}{dS}\right)=1+l\,\frac{Q}{B}+\varGamma':\left(S+t\sqrt{\frac{2ga}{b}}\right)+\varDelta':\left(S-t\sqrt{\frac{2ga}{b}}\right)$$

and

$$\left(\frac{ds}{dt}\right)=\sqrt{\frac{2ga}{b}}\varGamma':\left(S+t\sqrt{\frac{2ga}{b}}\right)-\varDelta':\left(S-t\sqrt{\frac{2ga}{b}}\right),$$

from those of which forms the density of the air at *s* is deduced at the elapsed time *t*, which is $q=\frac{Q}{\left(\frac{ds}{dS}\right)}$ ; and hence again the pressure $p=\frac{aq}{b}$ , truly from this the speed at the same location $\mathfrak{T}=\left(\frac{ds}{dt}\right)$ , and thus for whatever time given, in which the air will be moved around, it will be able to be assigned perfectly, but only if the two indefinite functions $\varGamma$ and $\varDelta$ will be determined duly from the start: which must happen in the following manner. The initial status is contained by the two conditions, of which the one is given for the initial positions *S* of the air density *Q*, the other truly the initial motion impressed on that; therefore then we may put the speed of the particle at *S* in the region *SB* to become $\varUpsilon$ , thus so that *Q* and $\varUpsilon$ shall be given functions of *S*. Hence in the general formulas found on putting the time $t=0$, initially it is necessary that $s=S$ or $z=0$, from which there is had :

$$0=\int dS\, l\tfrac{Q}{B}+\Gamma:S+\Delta:S$$

and in a similar manner for that condition to be satisfied, whereby on putting $t=0$ there must be produced $q=Q$. Now generally, if $z$ were a function of *S* and *t*, so that on putting $t=0$ there may become $z=0$, then also to become $\left(\frac{dz}{dS}\right)=0$ , and thus by necessity $\left(\frac{ds}{dS}\right)=1$. Then for the motion, which were impressed on the air from the beginning, this equation will be obtained

$$\Upsilon=\sqrt{\tfrac{2ga}{b}}\Gamma':S-\sqrt{\tfrac{2ga}{b}}\Delta':S,$$

therefore, from which two conditions, the natures of each of the functions $\Gamma$ and $\Delta$ are determined. Then from which done, for any time given, the state and the motion of the air will be able to be defined.

## COROLLARY 1

7. Since by hypothesis the magnitude *z* must be a minimum, so that the initial state may be adapted to this state, the density of the air *Q* everywhere must be required to differ minimally from the equilibrium density, as I have put as *B* ; indeed if the density may be changed minimally, the magnitude *z* then may accept so great a value, which may be opposed to the hypothesis.

## COROLLARY 2

8. Then lest the functions $\Gamma$ and $\Delta$ of $z$ may lead to exceedingly large values, it will be convenient for these to be multiplied by some small fraction $\alpha$, so that there may be put

$$z=s-S=\int dS\, l\tfrac{Q}{B}+\alpha\Gamma:\left(S+t\sqrt{\tfrac{2ga}{b}}\right)+\Delta:\left(S-t\sqrt{\tfrac{2ga}{b}}\right),$$

by this way nothing stands in the way, however large or small the values of these functions may be come upon.

## COROLLARY 3

9. With this minimum multiplier introduced for the initial state, it will be required for these two equations to be satisfied :

$$0=\int dS\, l\tfrac{Q}{B}+\alpha\Gamma:S+\alpha\Delta:S$$

and

$$\Upsilon = \alpha\sqrt{\tfrac{2ga}{b}}\Gamma' : S - \alpha\sqrt{\tfrac{2ga}{b}}\Delta' : S,$$

from which it is apparent also in the initial state only the minimum speeds to be allowed.

## COROLLARY 4

10. So that then if for some time $t$ the state of the air must be defined, in the first place for the density $q = \frac{Q}{\left(\frac{ds}{dS}\right)}$ there will be had :

$$\left(\tfrac{ds}{dS}\right) = 1 + l\tfrac{Q}{B} + \alpha\Gamma' : \left(S + t\sqrt{\tfrac{2ga}{b}}\right) + \alpha\Delta' : \left(S - t\sqrt{\tfrac{2ga}{b}}\right),$$

then truly for the speed $\mathfrak{T} = \left(\frac{ds}{dt}\right)$ there will become:

$$\mathfrak{T} = \alpha\sqrt{\tfrac{2ga}{b}}\Gamma' : \left(S + t\sqrt{\tfrac{2ga}{b}}\right) - \alpha\sqrt{\tfrac{2ga}{b}}\Delta' : \left(S - t\sqrt{\tfrac{2ga}{b}}\right),$$

and from this the speed will be a minimum always.

## SCHOLIUM 1

11. If we may investigate the matter more carefully, for the determination of the motion it is not absolutely necessary, that the quantity $z$ shall be a minimum on putting $s = S + z$ , provided that may be prepared thus, so that the formula thence arising $\left(\frac{dz}{dS}\right)$ may become very small, which happens, if to the value of $z$ given before we may add the term $\beta t$, with $\beta$ being some quantity, however great. Indeed hence since the value neither of the formula $\left(\frac{dz}{dS}\right)$ nor of this $\left(\frac{dds}{dt^2}\right)$ is changed, likewise the value of the proposed equation is satisfied. But thence nevertheless the speeds $\Upsilon$ and $\mathfrak{T}$ will be increased by the constant amount $\beta$ and the whole matter will return to this, as if the whole tube with the air included to be conveyed with a uniform motion, or if the total motion of the whole air in the tube were impressed with a uniform motion from the start, which will be maintained thereafter. But since the phenomena arising shall themselves be evident, it is not worth the effort to pursue the cases of this kind arising separately.

## SCHOLIUM 2

12. But if the reckoning of the functions $\Gamma$ and $\Delta$ may assumed for argument's sake , then the initial state will be defined easily, but in turn it will be more difficult from the initial state given to elicit the nature of the functions of these. But since it will be necessary for the satisfaction of this condition:

$$0=\int dS\, l\tfrac{Q}{B}+\alpha\varGamma:S+\alpha\varDelta:S\ ,$$

and at once the other function $\varGamma:S$ will have been assumed, hence at the same time the other is defined, from that account where the functions are accustomed to be represented by the applied lines of the curves, that which will be shown most clearly in this manner. The right line *AB* will refer to the length of the tube (Fig. 71), in which the abscissa $AS=S$ may be taken, upon that the curved line *DQE* may be constructed, the applied line of which $SQ=\int dS\, l\tfrac{Q}{B}$, then truly for argument's sake another curve

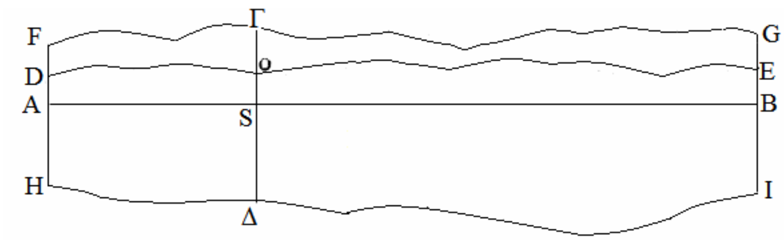



Fig. 71

$F\varGamma G$, of which the applied line shall be $S\varGamma=\alpha\varGamma:S$. Now below the axis *AB* the new curve $H\varDelta I$ may be constructed from this law, so that everywhere its applied line shall be $S\varDelta=SQ+S\varGamma$, and there will become $S\varDelta=-\alpha\varDelta:S$ *;* and thus in this manner from the two first curves this third reference function $\varDelta:S$ may be constructed. But hence again the speed of the air in the initial state for any location *S* will become known from this equation:

$$\varUpsilon\sqrt{\tfrac{b}{2ga}}=\alpha\varGamma':S-\alpha\varDelta':S,$$

since there becomes:

$$\alpha\varGamma':S=\tfrac{d\cdot S\varGamma}{d\cdot AS}\ \text{ and }\ \alpha\varDelta':S=\tfrac{-d\cdot S\varDelta}{d\cdot AS},$$

thus so that there shall become:

$$\varUpsilon\sqrt{\tfrac{b}{2ga}}=\tfrac{d(S\varGamma+S\varDelta)}{d\cdot AS}=\tfrac{d\cdot\varGamma\varDelta}{d\cdot AS},$$

and thus is easily assigned from the tangents drawn. But when in turn the speed $\varUpsilon$ may be given for the initial state at the individual points *S* , just as thence in turn both the curves $F\varGamma G$ and $H\varDelta I$ may be required to be defined, we will investigate in the following problem.

## PROBLEM 69

13. *With the initial state given for some location $S$ in the tube both of the air density $Q$ as well as with the speed $\Upsilon$ directed into the region SB, then in the elapsed time t to define the state and motion of the air in the tube, if indeed the initial state were to differ very little from the equilibrium state.*

## SOLUTION

Now the whole business leads to this, so that the nature of the functions $\Gamma$ and $\Delta$ will be determined from these two equations:

$$0=\int dS\, l\tfrac{Q}{B}+\alpha\Gamma:S+\alpha\Delta:S$$

and

$$\Upsilon\sqrt{\tfrac{b}{2ga}}=\alpha\Gamma':S-\alpha\Delta':S.$$

Now the first differentiated provides :

$$-l\tfrac{Q}{B}=\alpha\Gamma':S+\alpha\Delta':S,$$

and from that combined with the other equation we deduce :

$$\alpha\Gamma':S=\tfrac{1}{2}\Upsilon\sqrt{\tfrac{b}{2ga}}-\tfrac{1}{2}l\tfrac{Q}{B}$$

and

$$\alpha\Delta':S=-\tfrac{1}{2}\Upsilon\sqrt{\tfrac{b}{2ga}}-\tfrac{1}{2}l\tfrac{Q}{B},$$

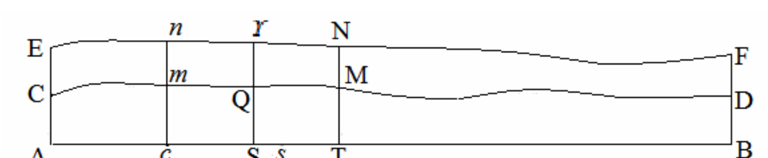


Fig. 72

which values will be allowed to be constructed in the following manner (Fig. 72). From the density given at the point $S$ $Q$ differing very little from the density $B$, which exists in equilibrium, $l\tfrac{Q}{B}$ may be deduced, from which $\tfrac{Q-B}{B}$ will suffice to be assumed, and hence the curved line $CQD$ will be described above the axis $AB$, and each of the abscissa

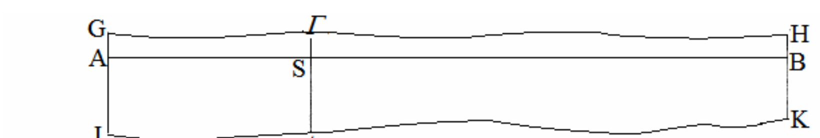


Fig. 73

$AS = S$ shall agree with the applied line $SQ = l\frac{Q}{B} = \frac{Q-B}{B}$ . Then since also the speed may be given at *S*, by which a distance $=\Upsilon$ may be completed in one second, at *S* the applied line $S\Upsilon = \Upsilon\sqrt{\frac{b}{2ga}}$ and thus the curve $E\Upsilon F$ is described . Now from these two lines *CQD* and $E\Upsilon F$ constructed on the same axis *AB* (Fig. 73) two new lines will be formed, the first evidently the line $G\Upsilon H$ upon the axis *AB* by taking the applied line $S\Gamma = \frac{1}{2}S\Upsilon - \frac{1}{2}SQ,$ then truly the line $I\Delta K$ below the axis by taking the applied line $S\Delta = \frac{1}{2}S\Upsilon + \frac{1}{2}SQ$ ; with which done there will become $\alpha\Gamma' : S = S\Gamma$ and $\alpha\Delta' : S = -S\Delta,$ and hence the integrals shown by the areas :

$$\alpha\Gamma : S = AGS\Gamma \text{ and } \alpha\Delta : S = -AIS\Delta.$$

Moreover, for these two curves above only we will have (Fig. 72) :

$$\alpha\Gamma' : S = \tfrac{1}{2}\left(S\Upsilon - SQ\right), \quad \alpha\Delta' : S = -\tfrac{1}{2}(S\Upsilon + SQ),$$
$$\alpha\Gamma : S = \tfrac{1}{2}\left(AES\Upsilon - ACSQ\right), \ \alpha A : S = -\tfrac{1}{2}(AES\Upsilon + ACSQ).$$

Now after the passage of some time *t* seconds each may be taken from some point *S* on the axis *AB* (Fig. 72) the distance $ST = St = t\sqrt{\frac{2ga}{b}}$ , so that there shall become :

$$AT = S + t\sqrt{\tfrac{2ga}{b}} \text{ and } At = S - t\sqrt{\tfrac{2ga}{b}},$$

and hence it is evident to become

$$\alpha\Gamma' : \left(S + t\sqrt{\tfrac{2ga}{b}}\right) = \tfrac{1}{2}\left(TN - TM\right)$$
$$\alpha\Delta' : \left(S - t\sqrt{\tfrac{2ga}{b}}\right) = -\tfrac{1}{2}\left(tn + tm\right)$$
$$\alpha\Gamma : \left(S + t\sqrt{\tfrac{2ga}{b}}\right) = \tfrac{1}{2}\left(AETN - ACTM\right)$$
$$\alpha\Delta : \left(S - t\sqrt{\tfrac{2ga}{b}}\right) = -\tfrac{1}{2}\left(AEtn + ACtm\right).$$

With these found after this time $=t$ for a particle of air, which was initially at *S*, now truly will be moved in the tube to the location *s*, so that there shall become

$$Ss = ACSQ + \tfrac{1}{2}AETN - \tfrac{1}{2}ACTM - \tfrac{1}{2}AEtn - \tfrac{1}{2}ACtm,$$

which distances are returned to this form:

$$Ss = \tfrac{1}{2}TNtn - \tfrac{1}{2}SQTM + \tfrac{1}{2}SQtm.$$

Thereupon the density found of the particle of air , $q = \frac{Q}{\left(\frac{ds}{dS}\right)}$, which is present now at *s*, but from the formula elicited above related to the figure, becomes

$$\left(\tfrac{ds}{dS}\right) = 1 + sQ + \tfrac{1}{2}TN - \tfrac{1}{2}TM - \tfrac{1}{2}tn - \tfrac{1}{2}tm,$$

from which, since the latter members of the expression shall be minimal besides unity, there will become approximately

$$q = Q\left(1 - SQ - \tfrac{1}{2}TN + \tfrac{1}{2}TM + \tfrac{1}{2}tn + \tfrac{1}{2}tm\right).$$

Finally for the motion, with which this particle may be moved, its speed along the direction *sB* will become :

$$\mathfrak{T} = \tfrac{1}{2}(TN - TM + tn + tm)\sqrt{\tfrac{2ga}{b}}.$$

COROLLARY 1

14. Whereby from the given disturbance, by which the air contained in the tube will have been disturbed initially from its state of equilibrium and which may be represented by the two lines *CQD* and $E\Upsilon F$ , thence to be moved for some time, it will be able to designate the density and the motion of each particle of the air in the tube.

COROLLARY 2

15. Evidently if the initial equilibrium of the air were not disturbed, so that there would become $Q = B$ everywhere and there would be no motion or $\Upsilon = 0$, then both the lines *CQD* and $E\Upsilon F$ will become straight and lie on the axis *AB* itself. Therefore since both all the areas as well as the applied lines vanish, no change will arise either in the position or in the density of the individual particles, and thus equilibrium will persist.

COROLLARY 3

16. If initially only a certain change of the natural density without any motion were induced, the line $E\Upsilon F$ lies in the line *AB*, and thence there will become :

$$Ss = -\tfrac{1}{2}SQTM + \tfrac{1}{2}SQtm;$$

then truly

$$q = Q\left(1 - SQ + \tfrac{1}{2}TM + \tfrac{1}{2}tm\right) \text{ and } \mathfrak{T} = \tfrac{1}{2}(tm - TM)\sqrt{\tfrac{2ga}{b}}.$$

## COROLLARY 4

17. If initially whatever motion were impressed on the air in the tube, not at once would any change be allowed at least in the first instant, since the line *CQD* will agree with the axis, and for the disturbance following for some later time *t* there will be had :

$$Ss = \tfrac{1}{2}TNtn,\ q = Q\left(1 - \tfrac{1}{2}TN + \tfrac{1}{2}tn\right) \text{ and } \mathfrak{T} = \tfrac{1}{2}\left(TN + tn\right)\sqrt{\tfrac{2ga}{b}}.$$

## SCHOLIUM 1

18. It will need to be observed here the applied lines *CQD* and $E\Upsilon F$ of both our curves not to be expressed by linear quantities but by absolute numbers, from which the construction of these demands, that certain right lines may be taken as it pleases for unity ; from which henceforth the magnitude of the individual applied lines duly will be determined; therefore that right line will be agreed to be put in place so great, so that the changes even of the smallest induced to the air may be referred to sensibly in the figure. Hence it will help to mention, whatever letters introduced into the calculation refer to absolute numbers and denote certain linear quantities. Moreover at first the time *t* as expressed in seconds denotes an absolute number, then truly also the letters *b*, *B*, *Q* and *q*, with which we may indicate densities are absolute numbers, since they are referred to some defined density designated by unity, which I use in defining pressures. The remaining letters entering into the calculations are linear magnitudes ; namely, in the first place the letter *g* denotes the height, which weights will fall in one second, which is estimated to be 15,625 Rhenish feet. Truly the letters $\Upsilon$ et $\mathfrak{T}$ for the speeds denote the linear distances used up, which they may run through with these speeds in one second; and finally the letters *a* and *p* introduced for the pressures introduce the heights and thus also signify linear quantities. Indeed from these the height of a corresponding column of uniform matter, of which the density is put $=1$, of which the weight is equal to the pressure acting on an equal base of unit area. Therefore since the scales of our curves, of which the one *CQD* may be allowed to be called the scale of the density, the other $E\Upsilon F$ the scale of the speeds, the applied lines of which shall be absolute numbers, the abscissas truly linear quantities, and indeed the areas taken from these also will be linear quantities. With these considerations it is evident the formula formed from the time $t\sqrt{\frac{2ga}{b}}$ to be a linear quantity, so that it may be added to the abscissa $AS = S$ and may be able to be subtracted from that; then truly the expression found for the translation *Ss* likewise to be a linear quantity and that, which has been shown for the speed $\mathfrak{T}$; finally the ratio of the density *q* and *Q* to be expressed by an absolute number, as the nature of the ratio demands. Finally, each scale established properly must be constructed thus, so that, if initially for *s* the density were $=Q$ and the speed were $=\Upsilon$, for the scale of the density taken, the applied line must become

$$SQ = l\,\tfrac{Q}{B} = \tfrac{Q-B}{B},$$

and for the scale of the speed indeed the applied line

$$S\Upsilon = \Upsilon\sqrt{\tfrac{b}{2ga}}.$$

SCHOLIUM 2

19. Here the most convenient use to have arisen is to be wondered at, since from the two given scales of the density and speed related to the initial state so set out for some elapsed time the disturbance of the air thence arising may be able to be assigned, since still the question shall be seen to be overcome initially by looking at the forces to be overcome analytically, and also certainly may be overcome, only if the disturbances may be assumed as minimal; truly besides also the hypothesis, by which we may attribute the same cross-section to the tube always, is to be agreed to have contributed the most to this convenient solution, since for tubes with unequal cross-sections at this stage the most difficult obstacles occur. But nevertheless this solution is restricted to a specific kind of maximum motion, yet the greatest use prevails in physical investigations, and thence now two phenomena , which until now were especially abstruse and which frustrated the treatment of the natural investigators, now can be explained with the greatest success, only if we may make the exception of the sharpest Geometer, Ludovic de Lagrange of Turin. The one phenomenon consists in the propagation of sound, where it was required to explain, in what manner, while a minimal disturbance of the air induced at one place, thence similar successive disturbances may be discovered at great distances. Truly the other phenomenon, in which at this stage much less has been presented by the authors, is concerned with the investigation of sound, which flutes produce on being blown, of which indeed a little has been observed by me at one time with the likeness of vibrating strings ; but in no way could the movement of the air itself be allowed to be defined, by which these sounds were being produced. Therefore each phenomenon, provided it is produced in equally wide tubes, I am going to pursue with all care.

SCHOLIUM 3

20. But before I may approach this work, certain circumstances of the greatest importance are required to be set out concerning both the scales of the density and of the speed and the continuation of each. Indeed at first I observe, if each tube may be extended indefinitely, the application of our solution to be subject to no difficulty; since indeed then for the initial state both scales themselves may be continued indefinitely; with the elapse of some very great time *t*, if from some point *S* of the axis the intervals $ST = St = t\sqrt{\tfrac{2ga}{b}}$ may be cut off on both sides, from these points *T* and *t* on each scale the corresponding applied lines always will correspond, from which the state of the air everywhere will be able to be defined for this time. But if the tube either were terminated at each end or at least if the other part were terminated, and there to be either closed or open , there also it is necessary both scales may be terminated according to the initial

state of the air extracted; hence by necessity it will eventuate, so that in the time spent passing through the intervals *ST* and *st* at least one may fall beyond the end of the scale, thus so that then plainly no part of these scales will support the supplied applied lines, from which the state of the air at this time may be able to be defined. Therefore since the nature of the given solution will demand always, that the scales of each shall be continued indefinitely, even if the tube may have a finite length, the maximum importance of this solution depends on this, so that we may define, by which law for these cases each scale may be able to be continued, so that thence truly a solution may be elicited. And the same also must be outstanding, when the tube has a figure returning on itself; though I represent the directrix of a tube as right line, yet here it has been shown well enough thus curvature cannot be excluded, since hence the motion is not altered.

PROBLEM 70

21. *If a tube of uniform cross-section may be terminated somewhere at the point B and this shall be an aperture, each scale both of the density as well as of the speed, which were constructed from that which formed the initial state, will continue to be produced beyond the point B on the axis AB.*

SOLUTION

Since *B* is the aperture of the tube and thus the air contained in the tube is shared with the external air, at that extremity of the tube *B* the internal pressure of the air cannot be different from the external pressure, whereby also the density of the internal air at this location must agree with the external density, which if it may be called $= B$, there will become not only initially, but also perpetually for this extremity of the tube

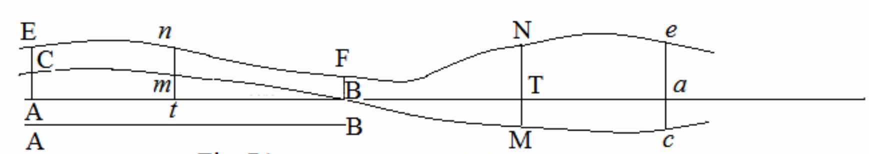


Fig. 74

$B$ both the initial density $Q = B$ as well as that after some elapsed time, $q = B$. Therefore for the scale of the density (Fig. 74) *CQD* formed from the initial state at the extremity *B* on account of $Q = B$ the applied line shall become $BD = 0$. Therefore the proposed tube *AB* terminated and open at $B$, but extended in some manner beyond *A*, and the scale of the density formed from the initial state shall be *CmB* concurring with the axis *AB* at *B* ; but the scale of the speed shall be *EnF*. Now after some elapsed time $= t$ the density of the air at *B* must always be the same $= B$, in the first place I note that this condition likewise must be fulfilled, whatever the scale of the speed were ; that is, whether the pressure of the air in the tube were the beginning of some motion, or a whole disturbance of such a size were put in place from the variation of the speed. But whatever this property may be applied to that element of the air, which may be turning around at any time in the opening *BB*, it will be agreed, since the translation of the individual elements is as minimal, that it can be attributed to that element also, which will have occupied the opening *BB* constantly. Therefore the outflow at whatever time $=t$ , from some point *B* taken in each interval

$$Bt = BT = t\sqrt{\tfrac{2ag}{b}},$$

thus the continuation of the scale must be prepared thus, so that there may become $q = Q = B$ , whereby on account of $SQ = 0$ in this case there will be required to become

$$-\tfrac{1}{2}TN + \tfrac{1}{2}TM + \tfrac{1}{2}tn + \tfrac{1}{2}tm = 0$$

or

$$TM + tm - TN + tn = 0,$$

which since it shall be necessary to arise, whatever were the scale of the speed, there must become separately:

$$\text{both } TM + tm = 0 \text{ and } TN - tn = 0 .$$

From the first condition it is apparent the scale of the density *CmB* thus must be continued beyond *B*, so that the curve *BMc* may become similar and equal to the curve *BmC*, but which may be put in place at a different part of the axis. Thence the continuation *FNe* for the scale of the speed equally similar and equal to the curve *FnE* may be put in place and likewise put at the same part of the axis, and by this rule each scale of the density and speed will be agreed to be continued beyond the end *B*.

COROLLARY 1

22. If the tube may be extended indefinitely from the part *A*, with this construction each scale *B* will be continued indefinitely. But likewise if the tube shall be terminated at *A*, then in this way scales will not be allowed to continue beyond the end *a* with there being $Ba = AB$ . Truly since a similar continuation must be established beyond *A*, on this account we will be able to progress indefinitely on both sides.

COROLLARY 2

23. Therefore in this continuation of the scales by no means shall the internal nature of these be observed, neither if these were algebraic curves nor to be understood from a certain equation, the continuation from this equation would be desired, but only an external figure may be provided drawn from that continuation, which is needed here. And likewise the rule is required to be observed, whether these scales shall be themselves continued in line, or shall be irregular of whatever kind, which are drawn freely by hand .

## SCHOLlUM

24. This circumstance therefore is worthy of greater attention, since generally in analysis no other curved lines to be allowed, unless they may be contained by a certain equation expressing the nature of these, and no discontinuous lines to be permitted according to this law, and thence they are accustomed to be excluded completely. For the analysis both of the finite, as well as that part of the infinite, which hitherto for the most part has been improved, certainly cannot be applied to other curved lines, except for those which may be included in a certain law of continuity, since in these cases an equation expressing their nature must always be introduced into the calculation; nor previously with discontinuous lines, certainly with no law introduced, a place will have to be conceded in the analysis for the higher part of the infinitesimal analysis, which depends on functions of two variables, which has begun to be developed, of which indeed I have observed the first principles to be prepared thus, so that discontinuous lines of this kind may be able to be referred to that equally, and regular curved lines determined by a certain equation expressed, nor thus here with these latter cases shall any prerogative be put in place, since as if by no means may be internal structure be observed. Whereby, perhaps if the line *CmB* were the arc of a circle, in our set up clearly the nature of the circle need not be considered, but in the continuation to that of another equal inverted arc *BMc* in place added on, just as must be able to be used, if the line *CmB* may not be drawn according to a determined account, which also is concerned with the other scale *EnF* to be put in place, the continuation of which *FNe* always must be established similar and equal to that, even if perhaps the nature of that may involve another continuation in length. Since this consideration of curved lines evidently is new, and by those Analysts who are not yet accustomed to this kind of calculation, it is accustomed to be considered from the principles received to be especially abhorrent, however, this circumstance more often is required to be considered the least superfluous to be forced upon them.

## PROBLEM 71

25. *If an equally wide tube* (Fig. 75) *is terminated at B and there it shall be closed, each scale both of the density as well as of the speed, constructed for the initial state, to continue to be produced continually beyond the end B on the axis AB.*

## SOLUTION

Since the tube *AB* is closed at *B*, at this extremity itself of the air contained in the tube plainly no motion can be present and the final layer of the air adjacent to the cover *BB* itself must continue in a state of rest, from which not only at the beginning, but also for all the time the speed of the air at the point *B* must be equal to zero.

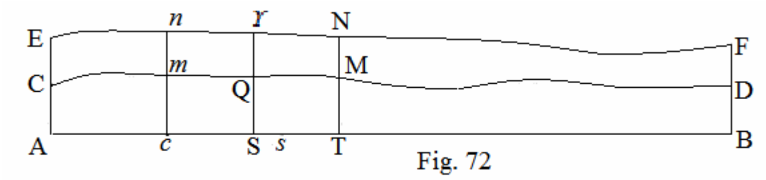


Fig. 72

Hence it is necessary that the extreme point *F* of the scales of the speed $E\Upsilon F$ (Fig. 72) must fall on the axis point *B* ; and therefore this figure may be had with the scale *EnB*, truly the scale of the density shall be *CmD.* Now since in some elapsed time *t*, the formula for the speed found above, if it may be applied to the point *B* , must vanish always and that for any scale of the density, the continuation of each scale to

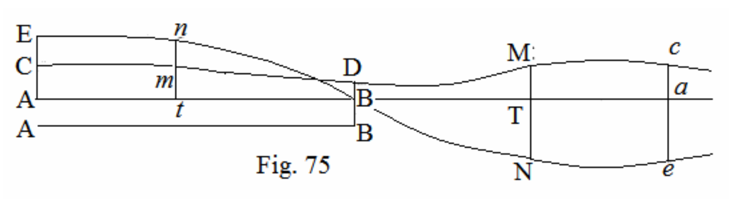


Fig. 75

will need to be adapted for this requirement. Therefore on both sides of the axis from the end *B* equal intervals must be taken

$$Bt = BT = t\sqrt{\tfrac{2ga}{b}},$$

and since the speed at the point *B* above is found to be expressed thus, so that there shall become :

$$\mathfrak{T} = \tfrac{1}{2}(TN + tn - TM + tm)\sqrt{\tfrac{2ga}{b}},$$

hence we elicit the two fold equation, the one for the scale of the speed:

$$TN + tn = 0,$$

the other for the scale of the density $TM = tm$*;* from which we learn the scale of the speed *EnB* must be made continuous thus, so that there shall become $TN = -tn$, and thus the continued part *BNe* to become similar and equal to the scale *BnE,* but for the contrary part of the axis to be adjusted. Moreover, since for the scale of the density there shall be $TM = tm$, its continuation *DMc* will be entirely of the same and equal parts for the line *DmC,* for these same parts of the axis adjusted. Each likeness evidently is referred to the point *B,* thus so thence with the equal parts $BT = Bt$ taken on both sides, in each scale also the applied lines shall be made equal $TN = tn$ and $TM = tm$, indeed with that for the opposite, truly this shall be placed on part of the same axis.

## COROLLARY 1

26. Therefore exactly as the tube were either open or closed at *B*, the continuation of each scale indeed to be diverse on account of the axis, but which must be put in place in a like manner on consideration of the applied lines themselves. Evidently in that case of the scale of the density, truly for that case of the scale of the speed continued to the contrary part of the axis is required to be adjusted.

## COROLLARY 2

27. Since the case, where the tube is closed at $BB$, the speed of the air situated at *B* is zero always, hence it follows at once that at no time to depart from its place: from which the interval of the movement *Ss* defined above in general here also must disappear. But with the point *S* moved to *B* there shall become certainly:

$$Ss = \tfrac{1}{2}TNtn - \tfrac{1}{2}BDTM + \tfrac{1}{2}BDtm = \tfrac{1}{2}TNtn = 0,$$

since there becomes $TNtn = Btn - BTN = 0,$ for the area $BTN$ falling on the opposite part of the axis must be taken negative.

## PROBLEM 72

28. *If a tube of equal cross-section were returning on itself, whatever shape it might have, each scale of the density and speed to add up to the same initial state on each side and to continue indefinitely.*

## SOLUTION

The length of the tube (Fig. 76) extended straight in the table will be represented by the right line $ASA'$, thus so that the point $A'$ by running through the whole perimeter may be considered to return to the point *A*. Therefore accumulating on this line $ASA'$ both the scale of the density $CQC'$, as well as the scale of the speed $E\Upsilon E'$, thus so that, if at *S*

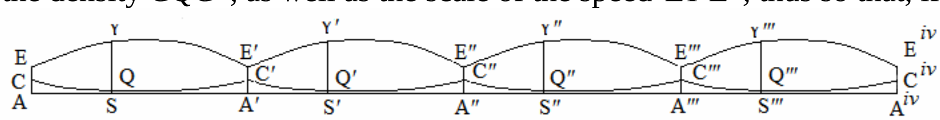


Fig. 76

initially the density were $= Q$ and the speed along the direction $SA' = \Upsilon$, the applied lines will become $SQ = l\frac{Q}{B} = \frac{Q-B}{B}$ and $S\Upsilon = \Upsilon\sqrt{\frac{b}{2ga}},$ it is evident at the point $A'$ there must become $A'C' = AC$ and $A'E' = AE,$ since the point $A'$ agrees with the point *A*. In a similar manner by traversing the tube many times for the individual revolutions repeated on the axis, the intervals may be taken $A'A''$, $A''A'''$, $A'''A''''$ etc. equal to the length of the tube $AA'$, and since the individual points $A, A', A'', A''', A''''$ etc. actually will represent the same point *A* of the tube, where the continuation of both scales must be prepared thus, so that the applied lines at all these points shall be the same. The same also is required to be understood with regard to all the points $S$, $S'$, $S''$, $S'''$ etc. for the intervals taken $AS = A'S' = A''S'' = A'''S'''$ etc., which all shall be considered to show the same point, for all also both the applied lines must agree. Whereby whatever both the curves $CQC'$ and $E\Upsilon E'$ may have been, the same will be repeated continually upon the elongated axis, which equally is required to be considered on the other axis to become elongated indefinitely. But since in this manner both scales were to be continued in each direction indefinitely, if for any elapsed time an element of the air, which initially was at $S$, following the above precepts given both the translation $Ss$ as well as its density and speed $\Upsilon$ may be defined, the same values are going to be produced, and if the same investigation may be put in place for the points $S'$, $S''$, $S'''$ etc.

## COROLLARY 1

29. Nor therefore in this case for the natural continuation of each scale $CQC'$ and $E\Upsilon E'$, as they will be going to have from their proper nature, it is required to be considered, moreover, that each continued on the same axis $AA'$ must be constructed extended on both sides.

## COROLLARY 2

30. Moreover in this indefinite continuation of each scale made on both sides, both in this case as well as in the preceding cases, thus it is necessary, that in some time $t$ elapsed, however great, from some point $S$, such great distances $t\sqrt{\frac{2ga}{b}}$ may be able to be opened up, and at these points the corresponding applied lines may be found.

# CAPUT I

## DE AERIS AGITATIONIBUS MINIMIS IN TUBIS AEQUALITER AMPLIS

### PROBLEMA 67

1. *Dum aër* (Fig. 70) *in tubo aequaliter amplo horizontaliter posito, sive sit rectus sive curvus, utcunque agitatur, aequationes in venire, quibus eius motus determinatur*.

### SOLUTIO

Sit *AB* tubus propositus, qui, sive sit rectus sive curvus, tanquam rectus in figura repraesentatur, quoniam vidimus ab eius curvamine motum non perturbari. Sit ergo eius amplitudo constans $= ff$. Ad hoc autem problema resolvendum uti conveniet methodo

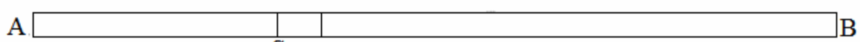


Fig. 70

posteriori, qua status aëris in tubo quicunque cum statu initiali comparatur. Solutionem ergo ex problemate 45 petamus, quem in finem consideremus aëris particulam, quae initio, ubi erat
tempus $t = 0$, fuerit in *S*, ac ponamus spatium $AS = s$, eius particulae vero densitatem $= Q$; at amplitudo tubi, quae ibi posita erat $= \Omega$, hic nobis est $= ff$. Iam elapso tempore *t* eadem particula pervenerit in *s* statuaturque spatium $As = s$, eius densitas $= q$, pressio $= p$ et celeritas secundum directionem $sB = \mathfrak{T} = \left(\frac{ds}{dt}\right)$, amplitudine tubi existente $\omega = ff$.
His positis prima aequatio ibi inventa praebet $q\left(\frac{ds}{dS}\right) = Q$; deinde, quia ob tubum horizontaliter positum gravitas motum non afficit, altera aequatio ibi inventa hanc induet formam $\frac{2gdp}{q} = -ds\left(\frac{dds}{dt^2}\right)$, quae cum tempus hic ut constans spectetur, ita repraesentari potest

$$\frac{2g}{q}\left(\frac{dp}{dS}\right) + \left(\frac{ds}{dS}\right)\left(\frac{dds}{dt^2}\right) = 0;$$

ubi meminisse iuvabit quantitates *p, q, s* functiones esse duarum variabilium *S* et *t*, quantitatem *Q* vero tantum functionem ipsius *S*. Aëris autem natura hic praeterea introducatur, qua novimus pressionem *p* perpetuo densitati *q* esse proportionalem; unde si densitati datae *b* conveniat pressio $= a$, erit $p = \frac{aq}{b}$, ex quo posterior aequatio fit

$$\frac{2ga}{bq}\left(\frac{dq}{dS}\right) + \left(\frac{ds}{dS}\right)\left(\frac{dds}{dt^2}\right) = 0,$$

prima existante $q\left(\frac{ds}{dS}\right)=Q$. Hinc igitur densitatem $q$ elidere licet, cum sit

$$\left(\frac{dq}{dS}\right)=\frac{dQ}{dS\left(\frac{ds}{dS}\right)}-\frac{Q\left(\frac{dds}{dt^2}\right)}{\left(\frac{ds}{dS}\right)^2} \quad \text{ob } q=\frac{Q}{\left(\frac{ds}{dS}\right)}$$

ideoque

$$\frac{1}{q}\left(\frac{dq}{dS}\right)=\frac{dQ}{QdS}-\frac{\left(\frac{dds}{dS^2}\right)}{\left(\frac{ds}{dS}\right)}.$$

Quocirca habebimus hanc aequationem, qua motus determinatio continetur

$$\frac{2gadQ}{bQdS}\left(\frac{dq}{dS}\right)-\frac{2ga}{b}\left(\frac{dds}{dS^2}\right)+\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right)=0,$$

quam aequationem ita resolvi oportet, ut posito $t=0$ fiat $s=S$ , tum autem fiet $\left(\frac{ds}{dS}\right)=1$ ideoque $q=Q$ , ut rei natura postulat.

## COROLLARIUM 1

2. Si ergo hanc aequationem ita resolvere liceret, ut, qualis futura sit functio $Q$, quantitas $s$ binarum variabilium $S$ et $t$ assignari posset, tum omnes motus, qui quidem in aërem tubo contentum cadere queant, definiri possent. Sicque totum negotium ad resolutionem huius aequationis est perductum.

## COROLLARIUM 2

3. Cum illa aequatio differentialia secundi gradus involvat, eius integrale completum duas functiones indefinitas continere debet, quas deinceps ex statu initiali determinari oportet, et quoniam non solum cuiusque particulae locus initio sumitur datus, sed etiam motus, ad hoc efficiendum utique binis functionibus indeterminatis opus est.

## SCHOLION 1

4. Determinatio ergo motus aëris opus est longe difficillimum, cum casus simplicissimus, quo aërem eodem caloris gradu praeditum in tubo aequaliter amplo et gravitatis effectu remoto moveri ponimus, ad eiusmodi aequationem sit perductus, cuius resolutio nullo adhuc artificio cognito expediri potest: quod eo minus est mirandum, quod ea Analysis infinitorum pars, quorsum haec aequatio est referenda, nuper demum excoli est coepta, neque in ea ultra prima elementa vix quicquam adhuc est praestitum. Maxime ergo arduae sunt iudicandae omnes quaestiones, quae circa motum aëris instituuntur, etiamsi forte primo intuitu facillimae videantur, veluti si aër in tubo ope emboli vel condensetur vel rarefiat, cuius certe motus determinatio frustra susciperetur. Quando enim ex vi aëris condensati vulgo motus globuli in sclopeto pneumatico definiri solet, nullo modo ad motum ipsius aëris respicitur, sed is potius quovis momento, quasi esset

inquiete, spectatur, ex quo neglectu etiamsi in motu globuli vix ullus error gigni videatur, minime tamen hoc exemplum proferre licet, in quo circa motum aëris quicquam fuerit definitum: quin potius confiteri cogimur nos circa hanc Theoriae motus fluidorum partem etiamnunc in maxima ignoratione versari.

## SCHOLION 2

5. Quoniam solutionem problematis methodo posteriori supra exposita sum aggressus, ne quis suspicetur methodum priorem feliciori successu forte adhiberi, solutionem inde petitam hic apponam. Secundum problema 44 igitur sine respectu ad statum initialem habito ad tempus $=t$ consideremus aëris particulam in $s$ versantem vocato spatio $As=s$ sitque ibi densitas $=q$, pressio $=p$ et celeritas $=\mathfrak{T}$ in directione $sB$, et quoniam amplitudo tubi est constans seu $\omega = ff$ et nullae admittuntur vires sollicitantes, habebuntur hae duae aequationes:

$$\left(\frac{d\cdot q\mathfrak{T}}{ds}\right)+\left(\frac{dq}{dt}\right)=0 \quad \text{et} \quad \frac{2gdp}{q}=-\mathfrak{T}d\mathfrak{T}-ds\left(\frac{d\mathfrak{T}}{dt}\right).$$

Cum autem ex natura aëris sit $p=\frac{aq}{b}$ posterior aequatio fiet

$$\frac{2gadq}{bq}+\mathfrak{T}d\mathfrak{T}+ds\left(\frac{d\mathfrak{T}}{dt}\right)=0,$$

quae, cum tempus $t$ ponatur constans, reducitur ad hanc formam:

$$\frac{2ga}{bq}\left(\frac{dq}{ds}\right)+\mathfrak{T}\left(\frac{d\mathfrak{T}}{ds}\right)+\left(\frac{d\mathfrak{T}}{dt}\right)=0.$$

Quia igitur prior evoluta praebet

$$\mathfrak{T}\left(\frac{dq}{ds}\right)+q\left(\frac{d\mathfrak{T}}{ds}\right)+\left(\frac{dq}{dt}\right)=0,$$

ponamus $q=e^y$, inde fiet:

$$\mathfrak{T}\left(\frac{dy}{ds}\right)+\left(\frac{d\mathfrak{T}}{ds}\right)+\left(\frac{dy}{dt}\right)=0,$$

ex altera vero

$$\frac{2ga}{b}\left(\frac{dy}{ds}\right)+\mathfrak{T}\left(\frac{d\mathfrak{T}}{ds}\right)+\left(\frac{d\mathfrak{T}}{dt}\right)=0.$$

Hinc elicitur

$$\left(\frac{dy}{ds}\right)=-\frac{b\mathfrak{T}}{2ga}\left(\frac{d\mathfrak{T}}{ds}\right)-\frac{b}{2ga}\left(\frac{d\mathfrak{T}}{dt}\right),$$

ex illa vero

$$\left(\frac{dy}{dt}\right)=-\left(\frac{d\mathfrak{T}}{ds}\right)+\frac{b\mathfrak{T}\mathfrak{T}}{2ga}\left(\frac{d\mathfrak{T}}{ds}\right)+\frac{b\mathfrak{T}}{2ga}\left(\frac{d\mathfrak{T}}{dt}\right).$$

Quare, cum sit $\left(\frac{ddy}{dsdt}\right)=\left(\frac{ddy}{dtds}\right)$, concludimus

$$\left(\mathfrak{T}\mathfrak{T}-\frac{2ga}{b}\right)\left(\frac{dd\mathfrak{T}}{ds^2}\right)+2\mathfrak{T}\left(\frac{d\mathfrak{T}}{ds}\right)^2+2\left(\frac{d\mathfrak{T}}{ds}\right)\left(\frac{d\mathfrak{T}}{dt}\right)+2\mathfrak{T}\left(\frac{dd\mathfrak{T}}{dtds}\right)+\left(\frac{dd\mathfrak{T}}{dt^2}\right)=0,$$

unde nunc investigari oportet, qualis fit $\mathfrak{T}$ functio binarum variabilium *t* et *s*. Haec autem aequatio ea, quam in solutione problematis dedimus, non solum non est tractatu facilior, sed illa etiam hoc commodo est praedita, ut felici successu ad aëris agitationes minimas accommodari possit, quemadmodum in sequente problemate docebimus.

## PROBLEMA 68

6. *In casu praecedentis problematis, si motum aëris ita comparatum esse noverimus, ut singulae particulae non nisi quam minime a loco initiali recedant, istas aëris agitationes minimas determinare.*

## SOLUTIO

Praecedentis problematis solutio ad hunc casum accommodabitur, si spatium *Ss* (Fig. 70), quo aëris particulam *s* a situ suo initiali *S* remotam ponimus, in calculo tanquam minimum tractemus. In hunc finem ponamus $s=S+z$, ita ut *z* spectanda sit ut quantitas minima, atque aequatio motum determinans hanc induet formam:

$$\frac{2gadQ}{bQdS}\left(1+\left(\frac{dz}{dS}\right)\right)-\frac{2ga}{b}\left(\frac{ddz}{dS^2}\right)+\left(1+\left(\frac{dz}{ds}\right)\right)^2\left(\frac{ddz}{dt^2}\right)=0,$$

quae, cum formula $\left(\frac{dz}{dS}\right)$ prae unitate quasi evanescat, contrahitur in hanc:

$$\frac{2gadQ}{bQdS}-\frac{2ga}{b}\left(\frac{ddz}{dS^2}\right)+\left(\frac{ddz}{dt^2}\right)=0,$$

cuius simodo primus terminus abesset, integrale ex iis, quae iam in hoc novo calculi genere sunt comperta, dari posset; foret enim

$$z=\Gamma:\left(S+t\sqrt{\frac{2ga}{b}}\right)+\Delta:\left(S-t\sqrt{\frac{2ga}{b}}\right).$$

Cum autem primus terminus solam variabilem *S* contineat, cuius *Q* est functio data, integratio eo non turbatur eritque aequationis nostrae integrale completum:

$$z=\int dSl\,\frac{Q}{B}+\Gamma:\left(S+t\sqrt{\frac{2ga}{b}}\right)+\Delta:\left(S-t\sqrt{\frac{2ga}{b}}\right),$$

quo invento ut reliquae motus conditiones eliciantur, ob $s=S+z$ erit

$$\left(\frac{ds}{dS}\right)=1+l\,\frac{Q}{B}+\varGamma':\left(S+t\sqrt{\frac{2ga}{b}}\right)+\varDelta':\left(S-t\sqrt{\frac{2ga}{b}}\right)$$

et

$$\left(\frac{ds}{dt}\right)=\sqrt{\frac{2ga}{b}}\varGamma':\left(S+t\sqrt{\frac{2ga}{b}}\right)-\varDelta':\left(S-t\sqrt{\frac{2ga}{b}}\right),$$

ex quarum formarum illa colligitur densitas aëris in *s* elapso tempore *t*, quae est $q=\frac{Q}{\left(\frac{ds}{dS}\right)}$; hincque porro pressio $p=\frac{aq}{b}$, ex hac vero celeritas in eodem loco $\mathfrak{T}=\left(\frac{ds}{dt}\right)$, sicque ad quodvis tempus status, in quo aër versabitur, perfecte assignari poterit, simodo binae functiones indefinitae $\varGamma$ et $\varDelta$ ex statu initiali dato debite determinentur: id quod sequenti modo fieri debet. Status initialis duabus conditionibus continetur, quarum altera pro singulis locis *S* datur aëris densitas *Q*, altera vero motus ei in initio impressus; ponamus ergo tum particulae in *S* existentis celeritatem in plagam *SB* fuisse $\varUpsilon$, ita ut *Q* et $\varUpsilon$ sint functiones ipsius *S* datae. Hinc in formulis generalibus inventis ponendo tempus $t=0$, primo fieri necesse est $s=S$ seu $z=0$, unde habetur

$$0=\int dS\,l\,\frac{Q}{B}+\varGamma:S+\varDelta:S$$

hocque modo simul illi conditioni satisfit, qua posito $t=0$ prodire debet $q=Q$. Nam generatim, si *z* eiusmodi fuerit functio ipsorum *S* et *t*, ut posito $t=0$ fiat $z=0$, tum etiam fieri $\left(\frac{dz}{dS}\right)=0$, ideoque $\left(\frac{ds}{dS}\right)=1$ necesse est. Deinde pro motu, qui aëri ab initio fuerit impressus, haec obtinebitur aequatio

$$\varUpsilon=\sqrt{\frac{2ga}{b}}\varGamma':S-\sqrt{\frac{2ga}{b}}\varDelta':S,$$

ex quibus duabus ergo conditionibus indoles utriusque functionis $\varGamma$ et $\varDelta$ determinatur. Quo facto deinceps ad quodvis tempus status motusque aëris in tubo definiri poterit.

## COROLLARIUM l

7. Quoniam per hypothesin quantitas *z* minima esse debet, ut status initialis ad hunc casum sit accommodatus, densitas aëris *Q* ubique quam minime a densitate ad aequilibrium requisita, quam posui *B*, discrepare debet; si enim densitas nimium alteraretur, quantitas *z* inde tantum valorem accipere posset, qui hypothesi adversaretur.

## COROLLARIUM 2

8. Deinde ne functiones $\varGamma$ et $\varDelta$ ipsi *z* nimis magnum valorem inducant, eas per fractionem quandam minimam $\alpha$ multiplicari convenit, ut statuatur

$$z = s - S = \int dS\, l\, \tfrac{Q}{B} + \alpha\Gamma : \left(S + t\sqrt{\tfrac{2ga}{b}}\right) + \Delta : \left(S - t\sqrt{\tfrac{2ga}{b}}\right),$$

hoc modo nil impedit, quominus ipsae functiones valores quantumvis magnos adipiscantur.

COROLLARIUM 3

9. Hoc multiplicatore minimo introducto pro statu initiali his duabus aequationibus satisfieri oportebit:

$$0 = \int dS\, l\, \tfrac{Q}{B} + \alpha\Gamma : S + \alpha\Delta : S$$

et

$$\Upsilon = \alpha\sqrt{\tfrac{2ga}{b}}\Gamma' : S - \alpha\sqrt{\tfrac{2ga}{b}}\Delta' : S,$$

unde patet etiam in statu initiali non nisi celeritates minimas admitti posse.

COROLLARIUM 4

10. Quodsi tum ad quodvis tempus $t$ status aëris definiri debeat, primo pro densitate $q = \frac{Q}{\left(\frac{ds}{dS}\right)}$ habebitur :

$$\left(\tfrac{ds}{dS}\right) = 1 + l\, \tfrac{Q}{B} + \alpha\Gamma' : \left(S + t\sqrt{\tfrac{2ga}{b}}\right) + \alpha\Delta' : \left(S - t\sqrt{\tfrac{2ga}{b}}\right),$$

tum vero pro celeritate $\mathfrak{T} = \left(\frac{ds}{dt}\right)$ erit

$$\mathfrak{T} = \alpha\sqrt{\tfrac{2ga}{b}}\Gamma' : \left(S + t\sqrt{\tfrac{2ga}{b}}\right) - \alpha\sqrt{\tfrac{2ga}{b}}\Delta' : \left(S - t\sqrt{\tfrac{2ga}{b}}\right),$$

unde et haec celeritas semper erit minima.

SCHOLION 1

11. Si rem accuratius perpendamus, pro motus determinatione non absolute necessarium est, ut ipsa quantitas $z$ posito $s = S + z$ sit minima, dummodo ea ita sit comparata, ut formula inde orta $\left(\frac{dz}{dS}\right)$ fiat valde parva, id quod evenit, si ad valorem ipsius $z$ ante datum insuper adiiciamus terminum $\beta t$, existante $\beta$ quantitate quantumvis magna. Quoniam enim hinc neque formulae $\left(\frac{dz}{dS}\right)$ neque huius $\left(\frac{dds}{dt^2}\right)$ valor immutatur, aequationi propositae perinde satisfit. Inde autem tantum celeritates $\Upsilon$ et $\mathfrak{T}$ quantitate constante $\beta$ augebuntur totumque negotium eo redibit, ac si totus tubus cum aëre incluso motu uniformi deferreretur vel si toti massae aëreae in tubo motus quidam uniformis ab initio esset impressus, qui deinceps perpetuo conservaretur. Cum autem phaenomena hinc

oriunda per se sint perspicua, operae non est pretium huiusmodi casus seorsim pertractare.

SCHOLION 2

12. Quodsi ratio functionum $\Gamma$ et $\Delta$ pro lubitu assumatur, inde status initialis facile definietur, difficilius autem erit vicissim ex statu initiali dato indolem earum functionum elicere. Quoniam autem satisfieri oportet huic conditioni

$$0=\int dS\, l\tfrac{Q}{B}+\alpha\Gamma:S+\alpha\Delta:S\,,$$

statim atque altera functio $\Gamma:S$ fuerit assumta, hinc simul altera definitur, id quod ex illa ratione, qua functiones per applicatas curvarum repraesentari solent, clarissime hoc modo ostenditur. Referat (Fig. 71) recta *AB* tubi longitudinem, in qua capiatur abscissa $AS=S$, super ea construatur linea curva *DQE*, cuius sit applicata $SQ=\int dS\, l\tfrac{Q}{B}$, tum vero pro lubitu alia curva

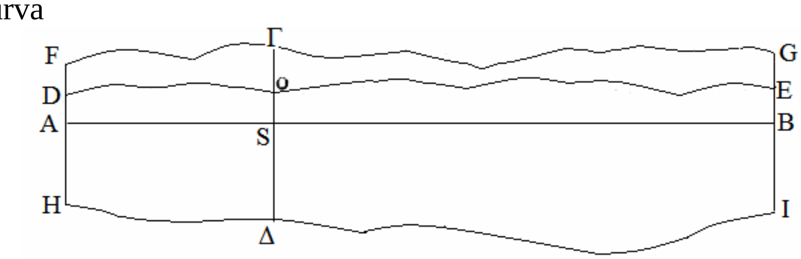


Fig. 71

$F\Gamma G$, cuius applicata sit $S\Gamma=\alpha\Gamma:S$. Iam infra axem *AB* construatur nova curva $H\Delta I$ hac lege, ut ubique sit eius applicata $S\Delta=SQ+S\Gamma$, eritque $S\Delta=-\alpha\Delta:S$; sicque hoc modo ex binis prioribus curvis haec tertia functionem $\Delta:S$ referens construetur. Hinc autem porro celeritas aëris in statu initiali pro quovis loco *S* facile innotescet ex hac aequatione:

$$\Upsilon\sqrt{\tfrac{b}{2ga}}=\alpha\Gamma':S-\alpha\Delta':S,$$

quandoquidem est

$$\alpha\Gamma':S=\tfrac{d\cdot S\Gamma}{d\cdot AS} \text{ et } \alpha\Delta':S=\tfrac{-d\cdot S\Delta}{d\cdot AS},$$

ita ut sit

$$\Upsilon\sqrt{\tfrac{b}{2ga}}=\tfrac{d(S\Gamma+S\Delta)}{d\cdot AS}=\tfrac{d\cdot\Gamma\Delta}{d\cdot AS},$$

sicque tangentibus ducendis facile assignatur. Quando autem vicissim celeritas $\Upsilon$ in singulis locis *S* pro statu initiali datur, quomodo inde vicissim ambas curvas $F\Gamma G$ et $H\Delta I$ definiri oporteat, in sequente problemate investigabimus.

## PROBLEMA 69

13. *Datis in statu initiali pro quovis tubi loco S tam aëris densitate Q quam celeritate* $\Upsilon$ *in plagam SB directa, elapso inde tempore t definire statum et motum aëris in tubo, siquidem status initialis valde parum a statu aequilibrii fuerit diversus.*

## SOLUTIO

Totum negotium iam eo est perductum, ut indoles functionum $\Gamma$ et $\Delta$ ex his duabus aequationibus determinetur:

$$0=\int dS\, l\tfrac{Q}{B}+\alpha\Gamma:S+\alpha\Delta:S$$

et

$$\Upsilon\sqrt{\tfrac{b}{2ga}}=\alpha\Gamma':S-\alpha\Delta':S.$$

Iam prior differentiata praebet

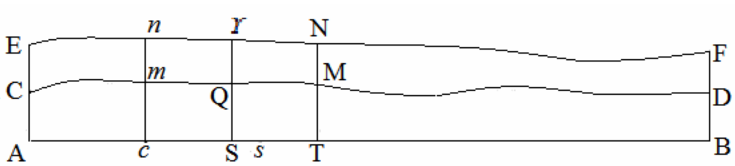
$$-l\tfrac{Q}{B}=\alpha\Gamma':S+\alpha\Delta':S,$$

ex qua cum altera combinata deducimus:

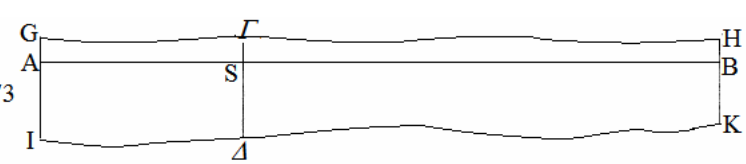
$$\alpha\Gamma':S=\tfrac{1}{2}\Upsilon\sqrt{\tfrac{b}{2ga}}-\tfrac{1}{2}l\tfrac{Q}{B}$$

et

$$\alpha\Delta':S=-\tfrac{1}{2}\Upsilon\sqrt{\tfrac{b}{2ga}}-\tfrac{1}{2}l\tfrac{Q}{B},$$

Fig. 72



quos valores sequenti modo (Fig. 72) construere licebit. Ex data in puncto *S* densitate *Q*

Fig. 73



valde parum a densitate *B*, quae in aequilibrio subsistit, discrepante, colligatur $l\,\frac{Q}{B}$ , pro quo sumsisse sufficiet $\frac{Q-B}{B}$ , hincque describatur super axe *AB* linea curva *CQD*, et

cuique abscissae $AS = S$ conveniat applicata $SQ = l\frac{Q}{B} = \frac{Q-B}{B}$ . Deinde cum etiam detur celeritas in *S*, qua uno minuto secundo conficiatur spatium $= \Upsilon$ , constituatur in *S* applicata $S\Upsilon = \Upsilon\sqrt{\frac{b}{2ga}}$ sicque describatur curva $E\Upsilon F$ . His iam duabus lineis constructis *CQD* et $E\Upsilon F$ super eodem axe *AB* (Fig. 73) duae novae lineae formentur, primo scilicet linea $G\Upsilon H$ supra axem *AB* sumendo applicatas $S\Gamma = \frac{1}{2}S\Upsilon - \frac{1}{2}SQ,$ tum vero linea $I\Delta K$ infra axem sumendo applicatas $S\Delta = \frac{1}{2}S\Upsilon + \frac{1}{2}SQ$ ; quo facto erit
$\alpha\Gamma' : S = S\Gamma$ et $\alpha\Delta' : S = -S\Delta,$ , hincque integralia per areas exhibendo:

$$\alpha\Gamma : S = AGS\Gamma \text{ et } \alpha\Delta : S = -AIS\Delta.$$

Per binas autem solas superiores curvas (Fig. 72) habebimus:

$$\alpha\Gamma' : S = \tfrac{1}{2}\big(S\Upsilon - SQ\big), \quad \alpha\Delta' : S = -\tfrac{1}{2}(S\Upsilon + SQ),$$
$$\alpha\Gamma : S = \tfrac{1}{2}\big(AES\Upsilon - ACSQ\big), \ \alpha A : S = -\tfrac{1}{2}(AES\Upsilon + ACSQ).$$

Iam elapso tempore quocunque *t* minutorum secundorum in axe *AB* (Fig. 72) utrinque a puncto *S* capiatur spatium $ST = St = t\sqrt{\frac{2ga}{b}}$ , ut sit

$$AT = S+t\sqrt{\tfrac{2ga}{b}} \quad \text{et} \quad At = S - t\sqrt{\tfrac{2ga}{b}},$$

hincque evidens est fore

$$\alpha\Gamma' : \left(S+t\sqrt{\tfrac{2ga}{b}}\right) = \tfrac{1}{2}\big(TN - TM\big)$$
$$\alpha\Delta' : \left(S-t\sqrt{\tfrac{2ga}{b}}\right) = -\tfrac{1}{2}\big(tn + tm\big)$$
$$\alpha\Gamma : \left(S+t\sqrt{\tfrac{2ga}{b}}\right) = \tfrac{1}{2}\big(AETN - ACTM\big)$$
$$\alpha\Delta : \left(S-t\sqrt{\tfrac{2ga}{b}}\right) = -\tfrac{1}{2}\big(AEtn + ACtm\big).$$

His inventis post hoc tempus $= t$ particula aëris, quae initio erat in *S*, nunc erit translata in tubi locum *s*, ut sit

$$Ss = ACSQ + \tfrac{1}{2}AETN - \tfrac{1}{2}ACTM - \tfrac{1}{2}AEtn - \tfrac{1}{2}ACtm,$$

quae spatia rediguntur ad hanc formam:

$$Ss = \tfrac{1}{2}TNtn - \tfrac{1}{2}SQTM + \tfrac{1}{2}SQtm.$$

Deinde aëris particulae, quae nunc in *s* versatur, densitas inventa $q = \frac{Q}{\left(\frac{ds}{dS}\right)}$ formulis autem supra erutis ad figuram relatis fit

$$\left(\tfrac{ds}{dS}\right)=1+sQ+\tfrac{1}{2}TN-\tfrac{1}{2}TM-\tfrac{1}{2}tn-\tfrac{1}{2}tm,$$

cuius expressionis membra posteriora cum prae unitate sint minima, erit proxime

$$q=Q\left(1-SQ-\tfrac{1}{2}TN+\tfrac{1}{2}TM+\tfrac{1}{2}tn+\tfrac{1}{2}tm\right).$$

Denique pro motu, quo haec particula iam agitabitur, eius celeritas secundum directionem *sB* erit

$$\mathfrak{T}=\tfrac{1}{2}(TN-TM+tn+tm)\sqrt{\tfrac{2ga}{b}}.$$

COROLLARIUM 1

14. Data ergo agitatione, qua aër in tubo contentus ab initio de statu aequilibrii fuerit perturbatus et quae per binas lineas *CQD* et $E\Upsilon F$ repraesentatur, deinceps ad quodvis tempus translatio, densitas et motus cuiusque aëris particulae in tubo assignari poterit.

COROLLARIUM 2

15. Si initio aëris aequilibrium prorsus non fuerit turbatum, ut ubique fuerit $Q=B$ et motus nullus seu $\Upsilon=0$, tum binae lineae *CQD* et $E\Upsilon F$ fient rectae et in ipsum axem *AB* incident. Cum igitur tam omnes areae quam applicatae evanescant, nulla mutatio neque in loco neque densitate singularum particularum orietur ideoque aequilibrium per severabit.

COROLLARIUM 3

16. Si initio densitati naturali tantum mutatio quaedam sine ullo motu fuerit inducta, linea $E\Upsilon F$ in rectam *AB* incidet, indeque fiet:

$$Ss=-\tfrac{1}{2}SQTM+\tfrac{1}{2}SQtm;$$

tum vero

$$q=Q\left(1-SQ+\tfrac{1}{2}TM+\tfrac{1}{2}tm\right) \text{ et } \mathfrak{T}=\tfrac{1}{2}(tm-TM)\sqrt{\tfrac{2ga}{b}}.$$

COROLLARIUM 4

17. Si initio aëri in tubo motus quicunque fuerit impressus neque simul densitas primo saltem instanti ullam mutationem fuerit passa, linea *CQD* cum axe congruet, et pro agitatione sequente ad quodvis tempus *t* habebitur:

$$Ss=\tfrac{1}{2}TNtn,\ q=Q\left(1-\tfrac{1}{2}TN+\tfrac{1}{2}tn\right) \text{ et } \mathfrak{T}=\tfrac{1}{2}\left(TN+tn\right)\sqrt{\tfrac{2ga}{b}}.$$

## SCHOLION 1

18. Observari hic oportet binarum nostrarum curvarum *CQD* et $E\Upsilon F$ applicatas non quantitatibus linearibus, sed numeris absolutis exponi, ex quo earum constructio postulat, ut linea quaedam recta ad libitum assumta pro unitate accipiatur; ex qua deinceps quantitas singularum applicatarum debite determinetur; eam ergo rectam tantam statui conveniet, ut mutationes etiam minimae aëri inductae satis sensibiliter in figura referantur. Hinc meminisse iuvabit, quaenam litterae in calculum introductae numeros absolutos et quaenam quantitates lineares significent. Primum autem tempus *t* utpote in minutis secundis exprimendum numerum denotat absolutum, tum vero etiam litterae *b, B, Q* et *q*, quibus densitates indicamus, quoniam referuntur ad certam quandam densitatem unitate signatam, qua in pressionibus definiendis utor, sunt numeri absoluti. Reliquae litterae in calculum ingredientes sunt quantitates lineares; primo namque littera *g* denotat altitudinem, qua gravia uno minuto secundo delabuntur, quae aestimatur 15,625 ped. Rhen. Litterae vero $\Upsilon$ et $\mathfrak{T}$ pro celeritatibus usurpatae spatia denotant linearia, quae his celeritatibus uno minuto secundo percurrerentur; litterae denique pro pressionibus introductae *a* et *p* altitudines sicque etiam quantitates lineares significant. Iis enim denotatur altitudo columnae materia uniformi, cuius densitas ponitur 1, constantis, cuius pondus aequale est pressioni parem basin urgenti. Cum igitur nostrarum curvarum, quarum alteram *CQD* scalam densitatum, alteram $E\Upsilon F$ scalam celeritatum appellare licet, applicatae sint numeri absoluti, abscissae vero quantitates lineares, areae iis comprehensae quoque erunt quantitates lineares. His animadversis perspicuum est formulam ex tempore formatam $t\sqrt{\frac{2ga}{b}}$ esse quantitatem linearem, ut abscissae $AS = S$ addi ab eaque subtrahi possit; tum vero expressionem pro translatione *Ss* inventam esse quantitatem linearem perinde atque eam, quae pro celeritate $\mathfrak{T}$ est exhibita; rationem denique densitatum *q* et *Q* numero absoluto exprimi, uti rei natura postulat.
Postremo probe teneatur utramque scalam ita construi debere, ut, si initio ad *s* fuerit densitas $= Q$ et celeritas $= \Upsilon$, pro scala densitatum capi debeat applicata

$$SQ = l\,\tfrac{Q}{B} = \tfrac{Q-B}{B},$$

pro scala celeritatum vero applicata

$$S\Upsilon = \Upsilon\sqrt{\tfrac{b}{2ga}}.$$

## SCHOLION 2

19. Commodissime hic usu venisse mirandum est, quod ex datis binis scalis densitatum et celeritatum ad statum initialem relatis tam expedite ad quodvis tempus elapsum aëris agitatio inde orta assignari possit, cum tamen quaestio haec primo intuitu vires analyseos superare sit visa, atque etiam certo superaret, nisi agitationes quam minimae essent assumtae; praeterea vero etiam hypothesis, qua tubo ubique eandem amplitudinem

tribuimus, plurimum ad hanc commodam solutionem contulisse est censenda, quandoquidem pro tubis inaequaliter amplis gravissima adhuc obstacula occurrunt. Quanquam autem haec solutio ad motus speciem maxime specialem restringitur, in investigationibus tamen physicis amplissimum praestat usum, indeque iam felicissimo successu duo phaenomena, quae adhuc maxime fuerunt abscondita frustraque a naturae scrutatoribus tractata, si solum Geometram acutissimum Taurinensem LUDOVICUM LA GRANGE excipiamus, explicari possunt. Alterum phaenomenon consistit in propagatione soni, ubi explicari oportet, quomodo, dum aëri uno in loco quaedam agitatio minima inducitur, inde similes agitationes successive ad maximas distantias proferantur. Alterum vero phaenomenon, in quo multo adhuc minus ab auctoribus est praestitum, versatur in explicatione soni, quem tibiae edunt infliatae, cuius quidem olim pulcra a me similitudo cum cordis vibrantibus est observata; nullo autem modo ipsam aëris agitationem, qua hi soni producuntur, definire licuit. Utrumque igitur phaenomenon, quatenus in tubis aequaliter amplis producitur, deinceps omni cura sum per secuturus.

## SCHOLION 3

20. Antequam autem hoc opus aggrediar, circa ambas scalas densitatum et celeritatum earumque continuationem quaedam circumstantiae maximi momenti sunt evolvendae. Primo quidem observo, si tubus utrinque in infinitum extendatur, solutionis nostrae applicationem nulli difficultati esse subiectam; quia enim tum pro statu initiali ambae scalae per se utrinque in infinitum continuantur; elapso quantumvis magno tempore $t$, si a quovis axis puncto $s$ utrinque abscindantur intervalla $ST = St = t\sqrt{\frac{2ga}{b}}$, his punctis $T$ et $t$ in utraque scala determinatae semper respondebunt applicatae, ex quibus status aëris ubique in tubo ad hoc tempus definiri poterit. Sin autem tubus vel utrinque vel ex altera saltem parte fuerit terminatus, ibique sive clausus sive apertus, ibidem quoque ambae scalae ad statum aëris initialem extractae terminentur necesse est; hinc necessario eveniet, ut tempore labente intervalla $ST$ et $st$ vel alterum saltem ultra scalarum terminum cadat, ita ut tum ipsae scalae nullas plane suppeditent applicatas, ex quibus aëris status ad haec tempora definiri queat. Cum igitur solutionis datae natura semper postulet, ut scalae utrinque in infinitum sint continuatae, etiamsi tubus finitam habeat longitudinem, maximum solutionis momentum in eo versatur, ut definiamus, qua lege his casibus utramque scalam continuari oporteat, ut inde vera solutio eliciatur. Atque idem quoque praestari debebit, quando tubus habet figuram in se redeuntem; quanquam enim hic tubi directricem ut lineam rectam repraesento, tamen iam satis est ostensum curvaturam ideo non excludi, quoniam hinc motus non alteratur.

## PROBLEMA 70

21. *Si tubus aequaliter amplus in puncto B terminetur ibique sit apertus, utramque scalam tum densitatum quam celeritatum, quae ex statu initiali super eo fuerit extructa, ultra punctum B super axe AB producto continuare.*

SOLUTIO

Quia tubus in *B* est apertus ideoque aër in tubo contentus cum aëre externo communicatur, in ipsa tubi extremitate *B* pressio aëris interni a pressione externi diversa esse nequit, quamobrem etiam densitas aëris interni in hoc loco convenire debet cum densitate externi, quae si vocetur $=B$, erit non solum initio, sed etiam perpetuo pro hac

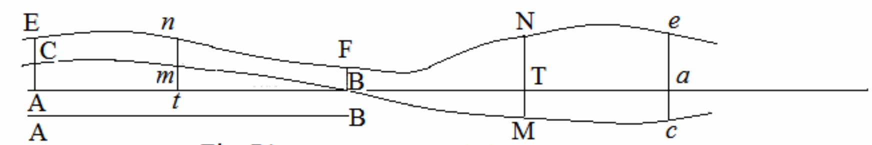


Fig. 74

tubi extremitate *B* tam densitas initialis $Q=B$ quam elapso tempore quocunque $q=B$. Pro scala ergo densitatum (Fig. 74) *CQD* ex statu initiali formata in extremitate *B* ob $Q=B$ fit applicata $BD=0$. Sit igitur *AB* tubus propositus in *B* terminatus et apertus, ultra *A* autem utcunque extensus, atque ex statu initiali formata sit scala densitatum *CmB* in *B* cum axe *AB* concurrens; scala autem celeritatum sit *EnF*. Quia iam elapso tempore quocunque $=t$ densitas aëris ad *B* perpetuo debet esse eadem $=B$, primum observo hanc conditionem perinde locum habere debere, quaecunque fuerit scala celeritatum; hoc est sive aëri in tubo impressus fuerit initio quispiam motus, sive tota agitatio tantum in perturbatione celeritatis substiterit. Quanquam autem haec proprietas illi aëris elemento, quod quovis tempore in orificio *BB* versatur, convenit, tamen, quia translatio singulorum elementorum est quam minima, etiam illi elemento, quod initio orificium *BB* occupaverat, constanter tribui potest. Effluxerit ergo tempus quodcunque $=t$, at a puncto *B* sumto utrinque intervallo

$$Bt=BT=t\sqrt{\tfrac{2ag}{b}},$$

continuatio scalarum ita esse debet comparata, ut fiat $q=Q=B$, quare ob $SQ=0$ hoc casu fieri oportet

$$-\tfrac{1}{2}TN+\tfrac{1}{2}TM+\tfrac{1}{2}tn+\tfrac{1}{2}tm=0$$

seu

$$TM+tm-TN+tn=0,$$

quod cum evenire necesse sit, quaecunque fuerit scala celeritatum, seorsim debet esse

$$\text{et } TM+tm=0 \text{ et } TN-tn=0\ .$$

Ex priori conditione patet scalam densitatum *CmB* ita ultra *B* continuari debere, ut curva *BMc* similis fiat et aequalis curvae *BmC*, sed ad partem axis contrariam disponatur. Deinde pro scala celeritatum continuatio *FNe* pariter similis et aequalis statuatur curvae *FnE* simulque ad eandem axis partem posita, hacque lege utramque scalam densitatum et celeritatum ultra terminum *B* continuari convenit.

## COROLLARIUM 1

22. Si tubus ex parte *A* in infinitum extendatur, hac constructione utraque scala *B* in infinitum continuabitur. At si tubus in *A* quoque sit terminatus, tum hoc modo scalas non ultra terminum *a* existente $Ba = AB$ continuari licebit. Quia vero similis continuatio ultra *A* institui debet, hac ratione utrinque in infinitum progredi poterimus.

## COROLLARIUM 2

23. In hac ergo scalarum continuatione neutiquam earum indoles interna spectatur, neque si eae fuerint curvae algebraicae vel aequatione quadam comprehensae, continuatio ex hac aequatione est petenda, sed sola figura externa eiusve ductus continuationem, qua hic opus est, suppeditat. Atque regula hic data perinde est observanda, sive scalae illae sint in se lineae continuae, sive irregulares, cuiusmodi libero manus ductu describuntur.

## SCHOLlON

24. Haec circumstantia eo maiore est attentione digna, quod vulgo in analysi nullae aliae lineae curvae, nisi quae certa quadam aequatione earum naturam exprimente contineantur, admitti lineaeque discontinuae nulla huiusmodi lege comprehensae inde penitus excludi solent. Analysis enim tam finitorum, quam ea pars infinitorum, quae adhuc potissimum est exculta, utique ad nullas alias lineas curvas, nisi quae certa continuitatis lege sint complexae, applicari potest, quandoquidem his casibus aequatio earum naturam exprimens semper in calculum introduci debet; neque ante lineis discontinuis, nulla certa lege ductis, locus in Analysi concedi potuit, quam sublimior Analyseos infinitorum pars, quae circa functiones duarum pluriumve variabilium versatur, excoli est coepta, cuius equidem naturam primus ita comparatam esse observavi, ut huiusmodi lineae discontinuae ad eam aeque referri debeant, atque lineae curvae regulares certa quadam aequatione expressae, neque adeo his posterioribus ulla praerogativa sit tribuenda, cum hic earum quasi interna natura neutiquam spectetur. Quare, si forte linea *CmB* fuerit arcus circuli, in nostro instituto ad circuli naturam plane non respicitur, sed in continuatione illi alius arcus aequalis *BMc* inverso situ adiungitur, prorsus uti fieri deberet, si linea *CmB* nulla certa ratione esset ducta, quod etiam de altera scala *EnF* est tenendum, cuius continuatio *FNe* illi semper similis et aequalis statui debet, etiamsi forte illius natura longe aliam continuationem involvat. Quoniam haec linearum curvarum consideratio prorsus est nova iisque, qui huic calculi generi nondum sunt assueti, a receptis Analyseos principiis maxime abhorrere videri solet, hanc circumstantiam saepius inculcasse minime superfluum est iudicandum.

## PROBLEMA 71

25. *Si* (Fig. 75) *tubus aequaliter amplus in B terminetur ibique sit clausus, utramque scalam tam densitatum quam celeritatum, ad statum initialem constructum, ultra terminum B super axe A B producto continuare.*

## SOLUTIO

Quia tubus *AB* in *B* est clausus, in hac ipsa extremitate aëri in tubo contento nullus plane motus inesse potest aërisque ultimum stratum operculo *BB* adiacens perpetuo in quiete perseverare debet, ex quo non solum in initio, sed etiam omni tempore celeritas aëris in puncto *B* nihilo debet esse aequalis. Hinc scalae celeritatum $E\Upsilon F$ (Fig. 72) extremum punctum *F* in axis punctum *B* incidat, necesse est; habeatque propterea haec scala figuram *EnB*, scala vero densitatum sit *CmD.* Cum nunc elapso tempore quocunque *t* formula pro celeritate supra inventa, si ad punctum *B* applicetur, semper evanescere debeat idque pro omni scala densitatum, continuationem utriusque scalae ad

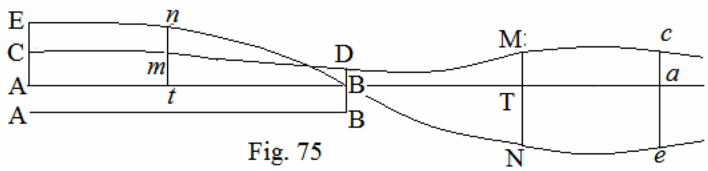


Fig. 75

hoc requisitum accommodari oportet. Super axe ergo utrinque a termino *B* capiantur intervalla aequalia

$$Bt = BT = t\sqrt{\tfrac{2ga}{b}},$$

et quia celeritas in puncto *B* supra ita exprimi est inventa, ut esset:

$$\mathfrak{T} = \tfrac{1}{2}(TN + tn - TM + tm)\sqrt{\tfrac{2ga}{b}},$$

hinc duplicem aequationem elicimus, alteram pro scala celeritatum

$$TN + tn = 0,$$

alteram pro scala densitatum *TM tm;* unde discimus scalam celeritatum *EnB* ita continuari debere, ut sit $TN = -tn$, ideoque partem continuatam *BNe* similem fore et aequalem scalae *BnE,* ad contrariam autem axis partem dispositam. Cum autem pro scala densitatum sit $TM = tm$, eius continuatio *DMc* omnino similis et aequalis erit lineae *DmC,* ad easdem axis partes disposita. Utraque scilicet similitudo refertur ad punctum *B,* ita ut inde sumtis utrinque abscissis aequalibus $BT = Bt$, in utraque scala etiam applicatae fiant aequales $TN = tn$ et $TM = tm$, illa quidem ad contrariam, haec vero eandem axis partem sita.

COROLLARIUM 1

26. Prout ergo tubus in *B* fuerit vel apertus vel clausus, utriusque scalae continuatio diverso quidem ratione axis, sed pari modo respectu ipsarum applicatarum institui debet. Illo scilicet casu scalae densitatum, hoc vero scalae celeritatum continuatio ad contrariam axis partem est disponenda.

## COROLLARIUM 2

27. Quoniam casu, quo tubus in *BB* est clausus, aëris ad *B* siti celeritas semper est nulla, hinc sponte sequitur eum nunquam de loco suo recedere: unde translationis spatium *Ss* supra in genere definitum hic quoque evanescere debet. Puncto autem *S* in *B* translato fit utique

$$Ss = \tfrac{1}{2}TNtn - \tfrac{1}{2}BDTM + \tfrac{1}{2}BDtm = \tfrac{1}{2}TNtn = 0,$$

quia est $TNtn = Btn - BTN = 0,$ area enim *BTN* in contrariam axis partem cadens negative capi debet.

## PROBLEMA 72

28. *Si tubus aequaliter amplus fuerit in se rediens, quamcunque habuerit figuram, utramque scalam densitatum et celeritatum ad statum initialem extructam utrinque in infinitum continuare.*

## SOLUTIO

Longitudo tubi (Fig. 76) in directum extensa in tabula repraesentetur linea recta *ASA'*, ita ut punctum *A'* tota perimetro percursa in punctum *A* recidere sit concipiendum. Exstructa ergo super hac linea *ASA'* tam scala

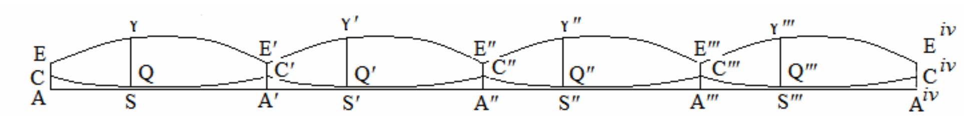


Fig. 76

densitatum *CQC'* quam scala celeritatum $E\Upsilon E'$, ita ut, si in *S* initio fuerit densitas $= Q$ et celeritas secundum directionem $SA' = \Upsilon$, sint applicatae

$SQ = l\frac{Q}{B} = \frac{Q-B}{B}$ et $S\Upsilon = \Upsilon\sqrt{\frac{b}{2ga}},$ evidens est in puncto *A'* esse debere

$A'C' = AC$ et $A'E' = AE,$ quandoquidem punctum *A'* cum puncto *A* convenit. Simili modo tubum pluries percurrendo pro singulis revolutionibus repetitis in axe capiantur intervalla *A'A''*, *A''A'''*, *A'''A''''* etc. tubi longitudini *AA'* aequalia, et quia singula puncta *A*, *A'*, *A''*, *A'''*, *A''''* etc. revera idem tubi punctum *A* repraesentant, continuatio ambarum scalarum ita debet esse comparata, ut applicatae in omnibus his punctis sint eaedem. Idem quoque tenendum est de omnibus punctis *S*, *S'*, *S''*, *S'''* etc. sumtis intervallis $AS = A'S' = A''S'' = A'''S'''$ etc., quae omnia cum unicum tubi punctum exhibere sint censenda, in omnibus quoque ambae applicatae congruere debent. Quare quaecunque fuerint ambae curvae *CQC'* et $E\Upsilon E'$, eaedem continuo super axe prolongato repetitae repraesententur, quod pariter super altera axis prolongatione in infinitum fieri est concipiendum. Cum autem hoc modo ambae scalae utrinque in infinitum fuerint continuatae, manifestum est, si pro quovis tempore elapso aëris elementi, quod initio fuerat in *S*, secundum praecepta supra data tam translatio *Ss* quam eius densitas et celeritas $\Upsilon$ definiatur, eosdem prodituros esse valores, ac si eadem investigatio pro punctis *S'*, *S''*, *S'''* etc. institueretur.

## COROLLARIUM 1

29. Neque ergo hoc casu ad naturalem utriusque scalae $CQC'$ et $E\Upsilon E'$ continuationem, quam ex sua propria indole essent habiturae, est respiciendum, sed utraque continuo eadem super axe $AA'$ utrinque producto construi debet.

## COROLLARIUM 2

30. Haec autem utriusque scalae continuatio utrinque in infinitum facta tam hoc casu quam praecedentibus ideo est necessaria, ut elapso tempore quantumvis magno $t$ a quovis puncto $S$ utrinque tanta spatia $t\sqrt{\frac{2ga}{b}}$ abscindi queant, in iisque punctis applicatae respondentes reperiantur.