CHAPTER V

CONCERNING THE MOTION OF WATER THROUGH TUBES DISTURBED BY A DIFFERENT DEGREE OF HEAT

PROBLEM 61

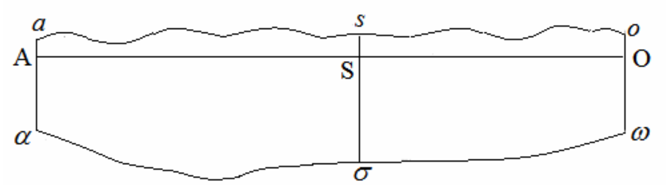


Fig. 63

124. *With a given degree of heat (Fig. 63) at the individual places of the tube, which we assume to be communicated at once with the water contained therein, to definite the motion, which the water will be able to receive in a tube of this kind.*

SOLUTION

AO shall be a tube, on account of the cross-section and curvature being variable in some manner, and on taking in that indefinite interval $AS = s$, where the cross-section shall be $= \omega$ and the height of the point *S* above a plane fixed to the horizontal $S\sigma = z$; but such a degree of heat, so that there a density $= q$ may be attributed to the water, which therefore by hypothesis is variable and to be a certain function of $AS = s$, since at the same place we assume the water always to bear the same degree of heat. But in the passage of the time *t*. the speed of the water flowing through the section *Ss* directly into the region *SO* shall be $= \mathfrak{T}$ and the pressure $= p$, which are functions of each of the variables *s* and *t*. With these in place , since $\left(\frac{dq}{dt}\right) = 0$, we continue with these two equations from problem 46:

$$\left(\frac{d\cdot q\omega\mathfrak{T}}{ds} \right) = 0 \quad \text{and} \quad \frac{2gdp}{q} = -2gdz - \mathfrak{T}d\mathfrak{T} - ds\left(\frac{d\mathfrak{T}}{dt} \right).$$

It follows from the first equation to become $q\mathfrak{T}\omega = \Gamma : t$, thus so that in the same time the quantity $q\mathfrak{T}\omega$ shall obtain the same value through the same tube. Therefore we may put the speed to become $= \upsilon$, at a certain place in the tube, where the cross-section $= ff$ and the density of water $= 1$, which therefore will be a function of the time *t* only; and the first condition produces $q\mathfrak{T}\omega = ff\upsilon$, thus so that there shall become $\mathfrak{T} = \frac{ff\upsilon}{q\omega}$, and hence, since the quantities *q* and ω will depend only on the variable *s*, there will become

$\left(\frac{d\mathfrak{T}}{dt}\right)=\frac{ff}{q\omega}\frac{d\upsilon}{dt}$, which value substituted into the other equation, where the time t may be considered constant, provides

$$2gdp=-2gqdz-q\mathfrak{T}d\mathfrak{T}-\frac{ffd\upsilon}{dt}\cdot\frac{ds}{\omega},$$

and from which by integrating we elicit :

$$2gp=\varDelta:t-2g\int qdz-\tfrac{1}{2}q\mathfrak{T}\mathfrak{T}+\tfrac{1}{2}\int\mathfrak{T}\mathfrak{T}dq-\frac{ffd\upsilon}{dt}\int\frac{ds}{\omega}$$

or with the value $\frac{f^4\upsilon\upsilon}{qq\omega\omega}$ substituted in place of $\mathfrak{T}\mathfrak{T}$

$$2gp=\varDelta:t-2g\int qdz-\frac{f^4\upsilon\upsilon}{2qq\omega\omega}+\tfrac{1}{2}f^4\upsilon\upsilon\int\frac{dq}{qq\omega\omega}-\frac{ffd\upsilon}{dt}\int\frac{ds}{\omega}$$

Therefore so that if the pressure were known by some means at two places, this equation can be applied at these two places, in the first place the function of the time $\varDelta:t$ to be removed, then truly the speed υ will be able to be determined for some time, from which thence everything will become known which is concerned with the motion .

[The left-hand term 2*gp* is essentially related to the familiar gravitational potential energy ρgh, where here 2*g* is numerically equal to the acceleration of gravity (being twice the distance fallen by an object from rest in one second), *p* is the water pressure measured in units of height, and the density of water *q* is nominally taken as 1; most of the terms following on the right-hand side correspond to the sum of the kinetic energies of the flowing liquid, changes in which can arise due to changes in the speed of flow occurring, due in turn to changes in the cross-section ω of the tube.]

COROLLARY 1

125. Since there shall become

$$\int\frac{dq}{qq\omega\omega}=-\frac{1}{q\omega\omega}-2\int\frac{d\omega}{q\omega^3},$$

the equation found also will be represented thus :

$$2gp=\varDelta:t-2g\int qdz-\frac{f^4\upsilon\upsilon}{2qq\omega\omega}+f^4\upsilon\upsilon\int\frac{d\omega}{q\omega^3}-\frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

in which here it happens conveniently, so that, as often as the tube everywhere is of the same width, the term $\int\frac{d\omega}{q\omega^3}$ vanishes and likewise there becomes $\int\frac{ds}{\omega}=\frac{ds}{\omega}$.

COROLLARY 2

126. Since υ specifies the speed in a given section of the tube, of which the cross-section $=f$, and where the density q becomes $=1$, this section will be allowed to be assumed as it pleases; since the density, as the water there has a certain degree of heat, so that the natural density can be considered, from which the pressures are defined.

SCHOLIUM

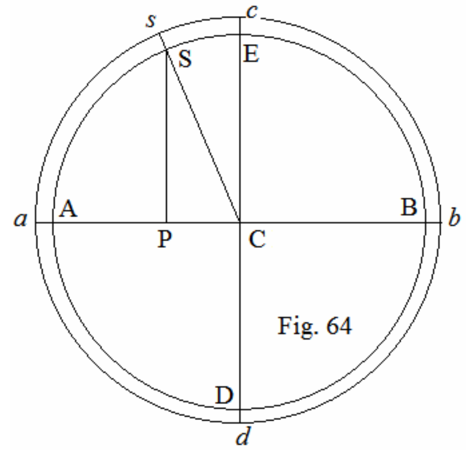


Fig. 64

127. As above we have seen that the mass of the heavy fluid cannot be in equilibrium, unless it may have the same density everywhere at equal heights, certainly there will be a need here to establish cases of this kind, where equilibrium will be unable to be sustained. And indeed in the first place a circular tube is presented (Fig. 64) placed vertical, which is maintained with one part warm and with another cold. Evidently *ASBD* shall be the circle placed in a vertical circle, of which *ACB* shall be the horizontal diameter ; we suppose this tube to become warm around *A*, so that the water contained there only may not boil, and in the region about *B* the tube shall be cold, with the degree of the heat from *A* to *B* successively decreasing on progressing either upwards or downwards, so that at *A* the heat shall be a maximum, at *B* truly a minimum. Therefore so that we may make the tube to be as narrow as possible, water by that motion at some point of time at some place of the tube will receive heat. But if now we may suppose the whole tube full of water, nothing at all can happen, as the water placed together itself comes to an equilibrium, therefore the motion of this kind shall be going to arise, we will investigate in the following problem.

PROBLEM 62

128. *If* (Fig. 64) *the circular tube ASBD shall be put in place vertically, hot at A and truly cold at B , then truly it may be filled with water, which shall receive the temperature* of the tube everywhere, to determine the motion of the water in the tube, since it is not given to be in equilibrium.*
[* There is no precise word in Latin for temperature, instead *temperamentum* is a measure of the mildness or otherwise of the weather, and may also applied to the state of ones feelings.]

SOLUTION

The radius of the circle shall be $CA = CB = c$, AB the horizontal diameter and the cross-section of the tube shall be the same everywhere $= ff$. Now since at *A* the heat is a maximum, truly a minimum at *B*, the density of the water at *A* will be a minimum, at *B* truly a maximum: we may set the middle density $=1$, to which clearly we will refer the evaluation of the pressure; then truly at *A* the density shall be $=1-\alpha$, at *B* truly $=1+\alpha$, on progressing from *A* to *B* the density thus shall increase, so that at some point *S* with the angle put to be $ACS=\varphi$ the density shall become $q=1-\alpha cos\varphi$, certainly which formula gives the density $1-\alpha$ for the point *A*, moreover for B, the density $1+\alpha$. Now again the height of the point *S* above the horizontal line *AB* is $PS=csin\varphi=z$ and the arc $AS=c\varphi=s$. From the beginning, when all the water at this stage was at rest, the time elapsed shall be $=t$, and the receding speed from the point *A* at the point *S* will be called $=\mathfrak{T}$, truly the pressure there $=p$. So that if now in that place, where the density is $=1$, the speed of the water may be put $=\upsilon$, on account of the cross-section everywhere the same $\omega=ff$ there will become $\mathfrak{T}=\frac{\upsilon}{q}=\frac{\upsilon}{1-\alpha\cos\varphi}$. Hence from the principles established previously we may define, before everything also, the pressure at an indefinite place *S*, and $z=csin\varphi$, $q=1-\alpha cos\varphi$ there will become :

$$\int qdz=c\int d\varphi cos\varphi\left(1-\alpha cos\varphi\right)=c\int d\varphi\left(cos\varphi-\tfrac{1}{2}\alpha-\tfrac{1}{2}\alpha cos2\varphi\right);$$

and thus

$$\int qdz=c\sin\varphi-\tfrac{1}{2}\alpha c\varphi-\tfrac{1}{4}\alpha c\sin2\varphi.$$

Thence on account of $\omega=ff$ there is $\int\frac{d\omega}{q\omega^3}=0$ and $\int\frac{ds}{\omega}=\frac{s}{ff}=\frac{c\varphi}{ff}$; with these substitutions made we follow with this equation:

$$2gp=\Delta:t-2gc\left(\sin\varphi-\tfrac{1}{2}\alpha\varphi-\tfrac{1}{4}\alpha\sin2\varphi\right)-\frac{\upsilon\upsilon}{1-\alpha\cos\varphi}-\frac{c\varphi d\upsilon}{dt}.$$

Hence initially at the point *A* this equation is produced :

$$2gp=\Delta:t-\frac{\upsilon\upsilon}{1-\alpha},$$

truly for the point *B* on putting $\varphi=\pi=180°$ there becomes:

$$2gp=\Delta:t+\alpha gc\pi-\frac{\upsilon\upsilon}{1+\alpha}-\frac{\pi cd\upsilon}{dt}.$$

We may run round the whole circle, so that we may return to the point *A*, and on putting $\varphi=2\pi$ for the point *A* also this equation is produced:

$$2gp=\Delta:t+\alpha\pi gc-\frac{\upsilon\upsilon}{1-\alpha}-\frac{2\pi cd\upsilon}{dt}.$$

Therefore it shall be necessary, that this pressure shall be equal to that for the same point *A*, hence we deduce this equation :

$$2\alpha\pi gc-\frac{2\pi cd\upsilon}{dt}=0\ \text{ or }\ d\upsilon=\alpha gdt,$$

which integrated gives $\upsilon = \alpha gt$, from which we learn, since the initial speed were zero, that to increase uniformly with the time, thus so that there shall become $\upsilon = \alpha gt$. Then truly on account of $\frac{d\upsilon}{dt} = \alpha g$ the pressure will be for whatever location *S* in the elapsed time *t*:

$$p = \Delta : t - c\left(\sin\varphi - \tfrac{1}{2}\alpha\varphi - \tfrac{1}{4}\alpha\sin 2\varphi\right) - \frac{\alpha\alpha gtt}{2(1-\alpha\cos\varphi)} - \tfrac{1}{2}\alpha c\varphi\,,$$

or

$$p = \Sigma : t - c\sin\varphi + \tfrac{1}{4}\alpha\sin 2\varphi - \frac{\alpha\alpha gtt}{2(1-\alpha\cos\varphi)}\,,$$

from which we concluded the pressures:

for *A*, where $\varphi = 0$, $\quad p = \Sigma : t - \frac{\alpha\alpha gtt}{2(1-\alpha)}$

for *E*, where $\varphi = 90°$, $\quad p = \Sigma : t - \frac{\alpha\alpha gtt}{2} - c$

for *B*, where $\varphi = 180°$, $\quad p = \Sigma : t - \frac{\alpha\alpha gtt}{2(1+\alpha)}$

for *D*, where $\varphi = 270°$, $\quad p = \Sigma : t - \frac{\alpha\alpha gtt}{2} + c$.

COROLLARY 1

129. Therefore since the water will have been at rest in the tube, thus at once it will begin to move, so that in the lower part *ADB* it may be carried from the colder to the hotter part, in the upper part AEB, on the other hand, from the hotter to the colder part, and a flow shall appear in the region *AEBD*, which shall be continually accelerated uniformly.

COROLLARY 2

130. This same accelerated motion will be faster, where the difference between the maximum heat at *A* and the minimum cold at *B* were greater. If the water were almost boiling at *A* , just about freezing at *B*, the fraction α is around $\frac{1}{30}$, and thus $\upsilon = \frac{1}{30}gt = \frac{1}{2}t$ ft. on account of $g = 15$ ft. and thus after one second the motion thus will now become rapid, so that in a second it will have traversed a distance of $\frac{1}{2}$ ft., moreover after the first minute a distance of 30 ft.

COROLLARY 3

131. So that it may attain the pressures, which the tube meanwhile may sustain, these indeed are not defined, since the pressure can be varied as it pleases even by being pressed on externally by water. Yet meanwhile at *B* the pressure always will be a maximum , indeed on taking

$$\Sigma : t = \frac{\alpha\alpha gtt}{2(1-\alpha)},$$

so that the pressure at *A* may vanish, at *B* the pressure will become $= \frac{\alpha^3 gtt}{1-\alpha\alpha}$, [*given as* $\frac{\alpha^3 gff}{1-\alpha\alpha}$ *in the original*] and thus will increase in a twofold manner in the account of the time.

SCHOLIUM 1

132. Moreover it is easily understood, if the matter may be investigated experimentally, the acceleration of the motion by no means will be going to become as rapid as we have found in the calculation, evidently the reason is due to this, so that also the water will have acquired a measurable speed, its degree of heat may not prevail at once to be adapted to the warmth of the tube, and for that reason, the tube at *B* to be conserving its former colder temperature for some longer time, while the water at *A* is going to become less warm. Therefore since the same may eventuate, if the fraction α may be rendered smaller, the acceleration of the motion also will be required to be reduced, yet which generally cannot be reduced to zero; and indeed likewise here if may occur that the time may not be sufficient for the heat to be received at some place in the tube, the motion again may be renewed as if from the beginning. From which, for this reason, it is evident the motion is going to be accelerated as far as to a certain level, at which thenceforth it shall be going to persist indefinitely, evidently as long as a difference of the temperature is present in the tube. Indeed since this moderation of the motion will depend mainly on this reason, by which the tube along with the water are to be considered, and since the tube itself in place in turn transfers a degree of heat, whereby likewise it will be required to consider the mass of each, here from the theory scarcely any modification will be required to be put in place. Just as if , to be revived with the aid of fire around *A*, here at this location in the tube, a significant degree of heat always will be impressed and if the tube shall be large enough, in order that so much heat may be transferred to a place other than the opposite side *B**, plainly there is no doubt, why water with a rapid enough motion may not be maintained in the region always .

*This happens around 4.2 ^{o}C, due to the anomalous expansion of water.

SCHOLIUM 2

133. I have assumed in the tube problem at the one horizontal end *A* the maximum degree of heat to be introduced, truly at the other end *B* the minimum, which put in place is adapted for the maximum motion requiring to be produced. For if the maximum heat may be introduced either at the top *E* or even at the bottom *D*, and the minimum heat source shall be present either straight down or up, then plainly no motion will arise, for the water once put at rest will remain in that same state indefinitely. Whereby even if initially the tube around *A* will accept the maximum heat, unless this may be maintained by an external source, water passing through *A* there receives the heat and in turn will communicate the cold with the upper positions of the tube *S* and *E*, where that will be

imbued on passing through *B*, into the lower part of the tube *E*, where at last it will be brought about, so that it will be moved with the maximum heat in the part of the tube *E* and with the minimum at the lowest part *D*, then all motion shall cease, and the water shall arrive at a state of equilibrium, in which it may wish to remain at rest. Concerning the rest in the solution of the problem I have established a certain law, following which the density of the fluid shall be increased on progressing from *A* towards *B*, so that the increase from these distances taken on the horizontal right line *AB* to be put in place in proportion, thus so that the increase of the density at *S* above the density at *A* shall be proportional to the distance *AP*; which hypothesis is seen to agree with the truth well enough, if near *A* a fire or some other heat generating matter may be considered to be put in place, indeed since the heat providing force at any place *S* may be estimated to be proportional to the square of the distance *AS*, this square is proportional to the versed sine *AP* itself. Yet meanwhile I have made use of this hypothesis mainly in deliberating on the calculation, and below I will try to express the matter more generally.

PROBLEM 63

134. *A circular tube shall be used* (Fig. 64) *as in the preceding problem placed in the vertical plane and with that hot at A, truly cold at B; truly a different cross-section may be attributed to this tube; with this in place, to define its motion if it were filled with water.*

SOLUTION

As before, the radius of the circular shall be $CA = CB = c$, the density of the water at $A = 1-\alpha$, at $B = 1+\alpha$, but at E and $D = 1$, truly with some indefinite position *S* on putting the angle $ACS = \varphi$ the density shall become $q = 1-\alpha cos\varphi$. Then truly at *E* and *D* the cross-section shall be $= ff$, while at *A* it may be put $= ff(1-\beta)$, in $B = ff(1+\beta)$, while at *S* it becomes $\omega = ff(1-\beta\cos\varphi)$. Now in the elapsed time t at *E* or *D*, where the cross-section is ff and the density $=1$, the speed of the water shall be $=\upsilon$, from which at the indefinite place *S* it will be

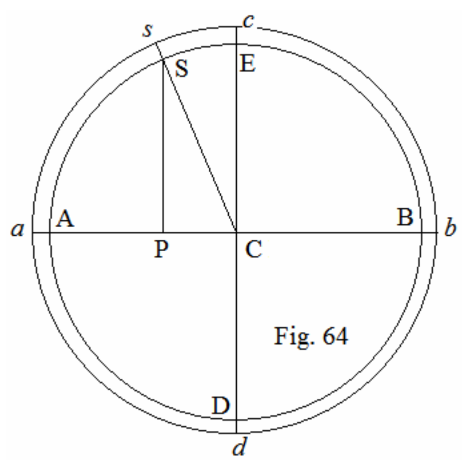


Fig. 64

$$\mathfrak{T} = \frac{\upsilon}{(1-\alpha\cos\varphi)(1-\beta\cos\varphi)}$$

as the first equation of motion supplies the condition. Again truly on putting the pressure at $f = p$ thus so that there shall become :

$$2gp = \varDelta : t - 2g\int qdz - \tfrac{1}{2}q\mathfrak{T}\mathfrak{T} + \tfrac{1}{2}\int \mathfrak{T}\mathfrak{T}dq - \tfrac{ffd\upsilon}{dt}\int \tfrac{ds}{\omega},$$

for each angle of which on account of $z = csin\varphi$ and $q = 1 - \alpha cos\varphi$ there shall be as before :

$$\int qdz = csin\varphi - \tfrac{1}{2}\alpha c\varphi - \tfrac{1}{4}\alpha csin2\varphi.$$

Then on account of $s = c\varphi$ and $\omega = ff\left(1 - \beta cos\varphi\right)$ there is

$$\int \frac{ds}{\omega} = \frac{c}{ff}\int \frac{d\varphi}{1-\beta\cos\varphi} = \frac{c}{ff\sqrt{(1-\beta\beta)}}\,\text{Ang sin}\,\frac{\sin\varphi\sqrt{(1-\beta\beta)}}{1-\beta\cos\varphi}.$$

Finally on account of $dq = \alpha d\varphi sin\varphi$ there becomes

$$\int \mathfrak{T}\mathfrak{T}dq = \alpha\upsilon\upsilon\int \frac{d\varphi\sin\varphi}{(1-\alpha cos\varphi)^2(1-\beta cos\varphi)^2},$$

from which on integrating there becomes:

$$\int \mathfrak{T}\mathfrak{T}dq = \frac{\alpha\upsilon\upsilon}{(\alpha-\beta)^2}\left(\frac{-(\alpha+\beta)+2\alpha\beta\cos\varphi}{(1-\alpha cos\varphi)(1-\beta cos\varphi)} + \frac{2\alpha\beta}{\alpha-\beta}\,l\,\frac{1-\beta cos\varphi}{1-\alpha cos\varphi}\right).$$

Now there may be put $\varphi = 0$, so that we may obtain the pressure at the point A

$$2gp = \varDelta : t - \frac{\upsilon\upsilon}{2(1-\alpha)(1-\beta)^2} + \frac{\alpha\upsilon\upsilon}{2(\alpha-\beta)^2}\left(\frac{-\alpha-\beta+2\alpha\beta}{(1-\alpha)(1-\beta)} + \frac{2\alpha\beta}{\alpha-\beta}\,l\,\frac{1-\beta}{1-\alpha}\right),$$

then truly for the same point there shall be $\varphi = 2\pi$, there will become

$$2gp = \varDelta : t + 2\pi\alpha gc - \frac{\upsilon\upsilon}{2(1-\alpha)(1-\beta)^2} + \frac{\alpha\upsilon\upsilon}{2(\alpha-\beta)^2}\left(\frac{-\alpha-\beta+2\alpha\beta}{(1-\alpha)(1-\beta)} + \frac{2\alpha\beta}{\alpha-\beta}\,l\left(\frac{1-\beta}{1-\alpha}\right)\right) - \frac{2\pi cd\upsilon}{dt\sqrt{(1-\beta\beta)}},$$

from the value of which equality there is elicited $d\upsilon = \alpha gdt\sqrt{\left(1-\beta\beta\right)}$ and hence $\upsilon = \alpha gt\sqrt{\left(1-\beta\beta\right)}$.

COROLLARY 1

135. The diverse cross-section of the tube has the effect, if indeed the law put in place in the solution may be followed, that the speed will be produces a little smaller, and that likewise, the greatest cross-section may be put in place either at *B* or at *A*. And if there were $\beta = 1$, in which case the cross-section will vanish at *A* or *B* , plainly no motion will arise, as is evident by itself.

COROLLARY 2

136. If there were $\beta=\alpha$, or the density everywhere were proportional to the cross-section of the tube, there will become

$$\int \mathfrak{T}\mathfrak{T}dq=\alpha\upsilon\upsilon\int\frac{d\varphi\sin\varphi}{(1-\alpha\cos\varphi)^4}=\frac{-\upsilon\upsilon}{3(1-\alpha\cos\varphi)^3}\text{ and }\mathfrak{T}\mathfrak{T}q=\frac{\upsilon\upsilon}{(1-\alpha\cos\varphi)^3}$$

and thus

$$-\tfrac{1}{2}q\mathfrak{T}\mathfrak{T}+\tfrac{1}{2}\int\mathfrak{T}\mathfrak{T}dq=\frac{-2\upsilon\upsilon}{3(1-\alpha\cos\varphi)^3},$$

it is apparent by using which formulas for finding the pressure .

SCHOLIUM

137. Therefore since we have seen, how great the inequality in the cross-section of the tube may be directed to the motion of the water, now we may inquire also, how great a motion may be going to arise in the same circular tube, if the locations of the maximum and minimum heat may not be placed horizontally, but at some other oblique position, where indeed again we may put in place everywhere the same cross-section of the tube.

PROBLEM 64

138. *As so far* (Fig. 65) *the circular tube shall be placed in a vertical plane and it is filled with water everywhere; in truth the maximum heat may be found at A, the minimum at B, so that here the diameter AB shall be inclined to the horizontal HI at the angle* $ACH=\zeta$ *; and since the tube will be filled with water, to define its motion.*

SOLUTION

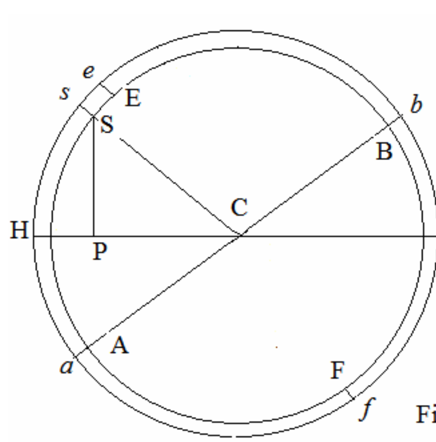


Fig. 65

The radius of the circle $CA=CB=c$, with the constant cross-section of the tube $=ff$, so that there shall be $\omega=ff$; and for some point S with the angle $ACS=\varphi$; the density of the water shall be $q=1-\alpha cos\varphi$, thus so that at the points E and F that may become $=1$, where the speed of the water may be put $=\upsilon$ after the elapsed time t, which therefore at the same time at S will be $\mathfrak{T}=\frac{\upsilon}{1-\alpha cos\varphi}$, of which the height of the point S above the horizon , since there shall be $SP=csin(\varphi-\zeta)=z$,
if the pressure at S may be called $=p$, on account of the arc $AS=cq$; there will become

:

$$2gp = \Delta : t - 2gc\int\left(1-\alpha cos\varphi\right)d\varphi\ cos\left(\varphi-\zeta\right) - \tfrac{\upsilon\upsilon}{1-\alpha cos\varphi} - \tfrac{c\varphi d\upsilon}{dt}.$$

But there becomes:

$$\int d\varphi cos\varphi cos\left(\varphi-\zeta\right) = \tfrac{1}{2}\int d\varphi\left(cos\zeta + cos\left(2\varphi-\zeta\right)\right) = \tfrac{1}{2}\varphi cos\zeta + \tfrac{1}{4}sin\left(2\varphi-\zeta\right)$$

and thus there will be had:

$$2gp = \Delta : t - 2gcsin\left(\varphi-\zeta\right) + \alpha gc\varphi\cos\zeta + \tfrac{1}{2}\alpha gcsin\left(2\varphi-\zeta\right) - \tfrac{\upsilon\upsilon}{1-\alpha cos\varphi} - \tfrac{c\varphi d\upsilon}{dt}.$$

Hence for the position *A* we will be able to express the pressure in a twofold manner, just as we may put either $\varphi = 0$ or $\varphi = 2\pi$; the first position gives

$$2gp = \Delta : t + 2gcsin\zeta - \tfrac{1}{2}\alpha gcsin\zeta - \tfrac{\upsilon\upsilon}{1-\alpha},$$

the other truly

$$2gp = \Delta : t + 2gcsin\zeta + 2\alpha\pi gc\cos\zeta - \tfrac{1}{2}\alpha gcsin\zeta - \tfrac{\upsilon\upsilon}{1-\alpha} - \tfrac{2\pi cd\upsilon}{dt},$$

which two expressions, since they must be equal to each other, there becomes

$$\alpha g\cos\zeta = \tfrac{d\upsilon}{dt} \quad \text{and hence} \quad \upsilon = \alpha gt\cos\zeta,$$

from which for any time at any location the speed becomes known, indeed the direction of which precedes into the region $AEBF$. Then truly the pressure at any place *S* will become :

$$p = \Sigma : t - csin\left(\varphi-\zeta\right) + \tfrac{1}{4}\alpha csin\left(2\varphi-\zeta\right) - \tfrac{\alpha\alpha gtt\cos^2\zeta}{2\left(1-\alpha\cos\varphi\right)}.$$

COROLLARY 1

139. Hence therefore it is clear, if the diameter AB passing through the places of the maximum and minimum heat were vertical, thus so that the maximum heat shall be either at the top or bottom place of the circle, on account of $\cos\zeta = 0$, no motion is going to be generated from the difference of the degrees of heat, but the water in this case is able to remain in equilibrium, therefore so that the same degree of heat is found for equally high positions in the tube.

COROLLARY 2

140. If the position of the maximum degree of heat *Aa* may differ less than a quadrant from the horizontal *H* , either above or below, the motion of the water will become in the direction *AEBF*; but if that distance *HA* may exceed the quadrant, since then $\cos\zeta$ becomes negative, the motion will be directed in the opposite region *AFBE*.

COROLLARY 3

141. Therefore the motion will be done in lower places from a colder region to a warmer region, truly on the other hand in higher places from warmer to colder places, generally as now above has been observed concerning equilibrium, even if the motion itself will not be allowed to be defined.

SCHOLIUM

142. Therefore since now not only shall the water prevail in the circular tube, in which for equal heights the degree of the warmth shall differ, not able to remain in equilibrium, but also we will have determined the motion itself thence arising, this being maintained properly to happen only, if the whole tube shall be filled with water; for if a smaller supply of water shall be poured into that, there a situation of this kind will be able to be had, in which it may remain permanently at rest. In which certainly a remarkable paradox must become known, because, while in a tube of this kind some empty space may be allowed, always it shall be able to give an equilibrium, that with the whole empty space removed suddenly and by necessity motion must arise; but this puzzle will become much greater, as I will show also with the empty space above the water admitted, as long as it shall be very small, to exclude equilibrium, thus so that, as often as that vacuum were smaller than some certain amount, then by necessity motion will be produced, in whatever manner water shall be disposed in the tube, but if that vacuum itself were a greater size, then always it must be attributed to the water, that it may remain permanently at rest. Therefore there will be the greatest need so that we may resolve this significant puzzle with the greatest care.

PROBLEM 65

143. *A circular tube* (Fig. 66) *of the same cross-section everywhere placed in the vertical plane may be kept turning, truly with the maximum heat at A, the minimum at B, so that the right line AB shall be the horizontal diameter. So that if now only a part MN of this tube may contain water, to investigate its motion.*

SOLUTION

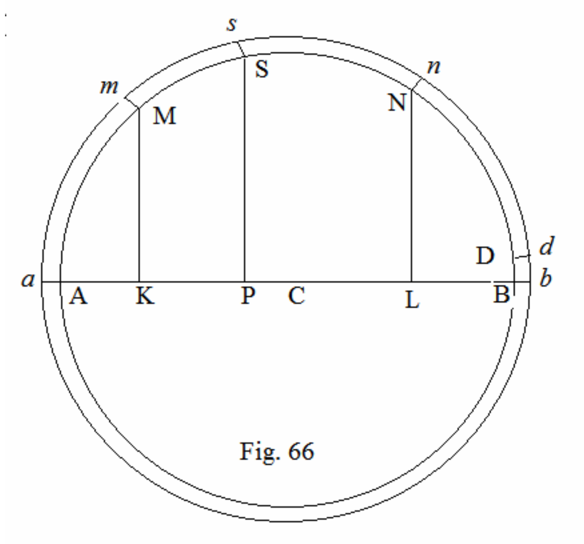


Fig. 66

The radius of the circle shall be $CA = CB = 1$, the cross-section of the tube shall be the same everywhere $\omega = ff$; but the heat will be prepared thus, so that at some place S with the arc $AS = s$, the density of the water shall become $q = 1 - \alpha\cos s$. Now after a time $= t$ it will happen with water poured into the tube volume and we may call the arc AM of the volume $MN = m$, and the arc $AN = n$; and indeed at first it is required to be considered carefully the mass of the water to remain the same always, therefore since at S the density shall be $q = 1 - \alpha\cos s$, the mass of the water, which will be going to be poured into the portion of the tube AS, will be $\int qds = s - \alpha\sin s$, from which we gather the mass of the water of the filled up part MN to become $= n - m - \alpha(sin\, n - sin\, m)$, which, since it shall be constant, it may be put $= 2e$. To this end there may be put :

$$m - \alpha sin\, m = u - e \text{ and } n - \alpha sin\, n = u + e,$$

and since from this minimal fraction, we will have approximately

$$m = u - e + \alpha sin(u - e) \text{ and } n = u + e + x sin(u + e),$$

where I observe, if the whole tube may be filled with water, to become

$$n = m + 2\pi \text{ and thus } 2e = 2\pi \text{ or } e = \pi,$$

thus so that so far the tube shall not be filled completely with water, in as much as the minor arc *e* is the semi perimeter π of the circle with the radius $=1$. Now since there shall be $z=\sin s$, there will become

$$\int qdz = sin\, s - \tfrac{1}{2}\alpha s - \tfrac{1}{4}\alpha sin\, 2s,$$

and by putting the pressure at $S = p$ there will be had

$$2gp = \Delta : t - 2g\left(sin\ s - \tfrac{1}{2}\alpha s - \tfrac{1}{4}\alpha sin\, 2s\right) - \tfrac{\upsilon\upsilon}{1-\alpha\cos s} - \tfrac{sd\upsilon}{dt},$$

from which the pressure for each end *M* and *N* can be deduced; but if besides water a vacuum or air may be present, the pressures at *M* and *N* by necessity shall be equal; from which there may become :

$$\left.\begin{aligned} &+2g\left(sin\, n - \tfrac{1}{2}\alpha n - \tfrac{1}{4}\alpha sin\, 2n\right) + \tfrac{\upsilon\upsilon}{1-\alpha\cos n} + \tfrac{nd\upsilon}{dt} \\ &-2g\left(sin\, m - \tfrac{1}{2}\alpha m - \tfrac{1}{4}\alpha sin\, 2m\right) - \tfrac{\upsilon\upsilon}{1-\alpha\cos m} - \tfrac{md\upsilon}{dt} \end{aligned}\right\} = 0.$$

From this equation at first it will be allowed to deduce, under which conditions an equilibrium will be able to be found. For if equilibrium may be present in this state, there will be required to be both $\upsilon = 0$ as well as $\tfrac{d\upsilon}{dt} = 0,$ which cannot happen, unless there shall be:

$$\sin n - \sin m - \tfrac{\alpha}{2}\left(n-m\right) - \tfrac{1}{4}\alpha\left(\sin 2n - \sin 2m\right) = 0.$$

Whereby, as often as it will be able for this equation to be satisfied, the equilibrium will be given, truly on the other hand by necessity motion will emerge. Moreover it is evident at once, if there shall become $n = 2\pi + m$, this equation by no means exists nor therefore can equilibrium be found. Therefore we may put $n = 2\pi + m - \delta$, and equilibrium demands this equation:

$$\sin\left(m-\delta\right) - \sin m - \tfrac{1}{2}\alpha\left(2\pi - \delta\right) - \tfrac{1}{4}\alpha\left(\sin\left(2m-\delta\right) - \sin 2m\right) = 0.$$

We may assume δ to be very small, and there will become:

$$-\delta\cos m - \tfrac{1}{2}\alpha\left(2\pi - \delta\right) + \tfrac{1}{4}\alpha\delta\cos 2m = 0,$$

from which there is deduced approximately $\cos m = \frac{-\alpha\left(2\pi-\delta\right)}{2\delta}$; therefore unless there shall be

$$\alpha\left(2\pi - \delta\right) < 2\delta \ \text{ or } \ \delta > \tfrac{2\alpha\pi}{2+\alpha},$$

clearly equilibrium cannot be found.

Moreover if equilibrium shall be excluded, or if the water may have been disturbed from another cause, the motion will be able to be defined from the above equation. Certainly with the new variable u introduced, so that there shall become

$$m = u - e + \alpha \sin(u-e) \text{ and } n = u + e + \alpha sin(u+e),$$

there will become:

$$\sin m = \sin(u-e) + \tfrac{1}{2}\alpha \sin 2(u-e),\ \cos m = \cos(u-e) - \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha \cos 2(u-e),$$
$$\sin n = \sin(u+e) + \tfrac{1}{2}\alpha \sin 2(u+e),\ \cos n = \cos(u+e) - \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha \cos 2(u+e),$$

$$\sin 2m = \sin 2(u-e) - \alpha \sin(u-e) + \alpha \sin 3(u-e),$$
$$\sin 2n = \sin 2(u+e) - \alpha \sin(u+e) + \alpha \sin 3(u+e).$$

Again since the speed at M shall be $= \frac{\upsilon}{1-\alpha\cos m}$, from the momentary advancement it is concluded the element of time

$$dt = \frac{dm(1-\alpha\cos m)}{\upsilon}$$

and with the substitution made there becomes $dt = \frac{du}{\upsilon}$. Then since there is approximately

$$\frac{1}{1-\alpha\cos n} = 1 + \alpha \cos n,$$

our equation adopts this form

$$2g\left(\sin n - \sin m - \tfrac{1}{2}\alpha(n-m) - \tfrac{1}{4}\alpha(\sin 2n - \sin 2m)\right) + \alpha\upsilon\upsilon(\cos n - \cos m) + \frac{\upsilon d\upsilon}{du}(n-m) = 0.$$

Now truly from the above forms there is deduced :

$$\sin n - \sin m = 2\sin e \cos u + \alpha \sin 2e \cos 2u,\ n - m = 2e + 2\alpha \sin e \cos u,$$
$$\cos n - \cos m = -2\sin e \sin u - \alpha \sin 2e \sin 2u,\ \sin 2n - \sin 2m = 2\sin 2e \cos 2u,$$

therefore with the substitution made, there will be produced:

$$2\upsilon d\upsilon(e + \alpha sin\, e\, cos\, u) - 2\alpha\upsilon\upsilon du\, sin\, e\, sin\, u$$
$$+ 2gdu\left(2sin\, e\, cos\, u - \alpha e + \tfrac{1}{2}\alpha sin\, 2e\, cos\, 2u\right) = 0,$$

which multiplied by $e + \alpha \sin e \cos u$ the integral is returned:

$$\upsilon\upsilon(e + \alpha \sin e \cos u)^2 + 2g\int du(2e \sin e \cos u - \alpha ee + \tfrac{1}{2}\alpha e \sin 2e \cos 2u + 2\alpha \sin^2 e \cos^2 u) = C,$$

thus hence so that there may be produced :

$$\upsilon\upsilon = \frac{C - 4ge\sin e\sin u + 2\alpha geeu - 2\alpha gu\sin^2 e - \frac{1}{2}\alpha ge\sin 2e\sin 2u - \alpha g\sin^2 e\sin 2u}{(e+\alpha\sin e\cos u)^2},$$

or

$$\upsilon\upsilon = \text{Const.} - \frac{4g}{e}\sin e\sin u + \frac{8\alpha g}{ee}\sin e\sin u + 2\alpha gu$$
$$- \frac{2\alpha gu}{ee}\sin^2 e - \frac{\alpha g\sin 2e}{2e}\sin 2u - \frac{\alpha g}{ee}\sin^2 e\sin 2u.$$

COROLLARY 1

144. If we may put $e = \pi$, so that the tube may become full, indeed which case we have now made clear above, the equation found there will be changed into this form $\upsilon = C + 2\alpha gu$, from which there becomes

$$dt = \frac{du}{\sqrt{(C+2\alpha gu)}} \text{ and } t = \frac{1}{\alpha g}\sqrt{(C+2\alpha gu)} = \frac{\upsilon}{\alpha g},$$

thus so that there shall become $\upsilon = \alpha gt$, as we have found above.

COROLLARY 2

145. For the limit, as far as to which the place of equilibrium can be had, we have found $\delta = \frac{2\alpha\pi}{2+\alpha}$, from which there becomes

$$m = \pi \text{ or } m = -\pi, \text{ and } n = \pi - \frac{2\alpha\pi}{2+\alpha} = \frac{2-\alpha}{2+\alpha}\pi.$$

In this case the lower semicircle is completely full of water, but the upper will contain water as far as to *Dd* with there being $BD = \frac{2\alpha}{2+\alpha}\pi;$ which is the extreme equilibrium state.

COROLLARY 3

146. Hence it follows, if the part of the tube without water were greater than $\frac{2\alpha}{2+\alpha}\pi;$ then equilibrium can be shown always, but if that part shall be smaller than $\frac{2\alpha}{2+\alpha}\pi;$ then clearly no place is left for equilibrium, but water may begin to move at once.

SCHOLIUM

147. Therefore the paradox mentioned above thus is resolved, so that, when the tube is not completely filled with water a vacuum is left in that space, indeed an equilibrium may be found always, provided this vacuum space were not very small. Indeed a certain very small boundary is given depending on the difference between the maximum and minimum density of the water, so that if that vacuum space were smaller, equilibrium may be excluded altogether and the water contained in the tube, whatever place it may maintain, by necessity will be stirred into motion. Since knowledge of this maximum

boundary shall be of importance, we may define that more carefully from the differential equation between υ and u , and since then we know the tube to be almost full, we may put for this boundary to become $e=\pi-\varepsilon$, with ε being the minimum arc; and so that both υ as well as $\frac{d\upsilon}{dt}$ may vanish, there shall be required to become

$$2\varepsilon cos\, u-\alpha\pi+\alpha\varepsilon-\alpha\varepsilon cos 2u=0 \text{ or } \varepsilon=\frac{\alpha\pi}{2cos\, u+\alpha-\alpha cos 2u},$$

which expression must be rendered a minimum, so that we may obtain the minimum value for ε and now also allowing equilibrium. Therefore there must be taken $u=0$, from which there becomes $\varepsilon=\frac{1}{2}\alpha\pi$, but then the outer extremity of this equilibrium state is found :

$$m=-\pi+\tfrac{1}{2}\alpha\pi=-\left(1-\tfrac{1}{2}\alpha\right)\pi \text{ and } n=\left(1-\tfrac{1}{2}\alpha\right)\pi,$$

so that the length of this vein of water contained in the tube shall become

$$MN=n-m=2\left(1-\tfrac{1}{2}\alpha\right)\pi=2\pi-\alpha\pi$$

and thus the vacuum volume $=\alpha\pi$; so that thus in equilibrium the location *B*, where the density is a maximum, will be occupied, so that the one extremity lies below the point *B*, the other falls above that with the interval between $\frac{1}{2}\alpha\pi$; which determination is more accurate with that , which is given in corollary 2 about the interval *BD*, even if an equilibrium may be able to be present equally in some nearby situation.

PROBLEM 66

148. *If* (Fig. 67) *a closed tube will have had in some manner a different degree of heat in there, so that the water, with which we assume that to be completely filled, cannot remain in equilibrium, to determine the motion generated in that.*

SOLUTION

The heat shall be a maximum at *A* and thus the density a minimum, which may be put $=1$, and there the cross-section of the tube $=ff$; at the location *B* truly the heat shall be a minimum and thus the maximum density $=1+\alpha$. Now we will consider some location of the tube *S* and the length along its directrix may be put $AS=s$, the cross-section $Ss=\omega$ and the height above a fixed horizontal plane $=z$; so that a plane will be allowed to be drawn through the same point *A*, truly the density at that place shall be $=q$. Now in the elapsed time $=t$ it will have acquired a motion of this kind, so that the speed at *A* in the region *AS* shall be $=\upsilon$ and thus at *S* there shall be

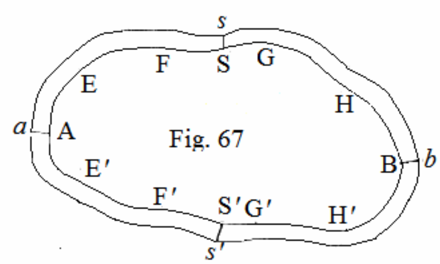


Fig. 67

going to become $\mathfrak{T} = \frac{ff\upsilon}{q\omega}$. Therefore so that if we may put the pressure at $S = p$, hence above we have elicited the equation:

$$2gp = \Delta : t - 2g\int qdz - \frac{f^4\upsilon\upsilon}{q\omega\omega} - f^4\upsilon\upsilon\int\frac{d\omega}{q\omega^2} - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

which on account of $\int qdz = qz - \int zdq$ will be transformed into this equation

$$2gp = \Delta : t - 2gqz + 2g\int zdq - \frac{f^4\upsilon\upsilon}{q\omega\omega} - f^4\upsilon\upsilon\int\frac{d\omega}{q\omega^3} - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

where I have assumed the integration to be taken along the whole length of the tube AS, thus so that on putting $s = 0$ that also will vanish. Therefore for the point A itself, where we have assumed also to become $z = 0$, there will become

$$2gp = \Delta : t - \upsilon\upsilon \text{ on account of } q = 1 \text{ and } w = ff,$$

but it will be necessary to increase the pressure also, if we may increase the arc s to that point, so that the whole length of the tube at the point S may be transferred to A ; then therefore it will be required to investigate such a form as our equation will adopt. And indeed I note in the first place, if the cross-section of the tube everywhere may be the same $w = ff$, then we are going to be able to express the formula for the whole length of the tube itself reverting to $ff\int\frac{ds}{\omega}$; therefore as far as the cross-section of the variable ω were either greater or smaller than *ff*, with that in place the value of the integral will be smaller or greater for that whole length. Therefore by putting this whole length $= a$, there may be put in place for the whole expanse of the tube :

$$ff\int\frac{ds}{\omega} = \lambda a.$$

Then the integral $\int\frac{d\omega}{q\omega^3}$ extended through the whole extent of the tube again either will vanish or adopt some certain value, just as the two variables q and ω were themselves prepared; therefore we may put the value of the integral $\int\frac{d\omega}{q\omega^3}$ extended through the whole tube $= \frac{\mu}{f^4}$. But the value of the integral $\int zdq$ demands a more careful investigation; we may take another location S' in the tube, where the water shall have the same density q as in the position S, but where the height above the fixed horizontal plane shall be $= z'$. Truly since by progressing from a to S the quantity q will be increased, but progressing further, with the course put in place through B as far as to S', the quantity q decreases, for some point S' we must write $-dq$ in place of dq, thus so that with the two elements of the tube at S et S' taken together we will have $(z-z')dq$, and now it will be required to have extended the integral $\int(z-z')dq$ from A only as far as to B. With this boundary taken with the right line (Fig. 68) $CA =$ of the minimum density 1, and CB = of the maximum $1+\alpha$, however many middle densities

Fig. 68

CE, CF, CG, CH etc. may be observed and in the tube the two joined places *EE', FF', GG', HH'*, may be observed in which these densities shall be present, then for each excess, where the height of the points *E, F, G, H* exceeds the height of the points *E', F', G', H'*, the equal applied lines *Ee, Ff, Gg, Hh* may be put in place and the area of the curve drawn through the points *e, f, g, h* the area *AefghBA* will give the true value of the integral $\int zdq$, as far as it may be extended through the whole length of the tube. We may put this value $\int zdq = h$ and with the integration made for the circuit for the pressure at A we will have

$$2gp = \varDelta : t + 2gh - \upsilon\upsilon - \mu\upsilon\upsilon - \tfrac{\lambda a d\upsilon}{dt},$$

which value must be equal with that found before $2gp = \varDelta : t - \upsilon\upsilon$; for the motion requiring to be determined this equation will arise:

$$\lambda a d\upsilon + \mu\upsilon\upsilon dt = 2ghdt \quad \text{or} \quad dt = \tfrac{\lambda a d\upsilon}{2gh - \mu\upsilon\upsilon},$$

which provides three cases requiring to be considered :

I. If $\mu = 0$, there will become $t = \frac{\lambda a\upsilon}{2gh}$ and thus $\upsilon = \frac{2gh}{\lambda a}t$.

II. If $\mu > 0$; there may be put $\mu = \frac{2gh}{cc}$, there becomes $dt = \frac{\lambda accd\upsilon}{2gh(cc-\upsilon\upsilon)}$, and hence by integrating $t = \frac{\lambda ac}{4gh} l \frac{c+\upsilon}{c-\upsilon}$, if indeed on putting $t = 0$ there must become $\upsilon = 0$, we may make $\frac{4gh}{\lambda ac} = \gamma$ and there will become $\upsilon = \frac{e^{\gamma t}-1}{e^{\gamma t}+1}c$.

Therefore in this case the speed υ certainly increases, but not beyond the limit c, which it reaches at last after an infinite time. Hence the first case arises, if $c = \infty$.

III. If $\mu < 0$, there may be put $\mu = \frac{-2gh}{cc}$, so that there may become $dt = \frac{\lambda accd\upsilon}{2gh(cc+\upsilon\upsilon)}$ and hence on integrating :

$$t = \tfrac{\lambda ac}{2gh} \text{Ang tang} \tfrac{\upsilon}{c},$$

from which we deduce $\upsilon = c\,\mathrm{tang}\frac{2gh}{\lambda ac}t.$ Therefore in this case in the finite elapsed time $t = \frac{\pi\lambda ac}{4gh}$ the speed υ now becomes infinite.

EXAMPLE

149. A circular tube shall be placed in the vertical plane (Fig. 64) full of water and with the radius being $CA = CB = c$. But with the angle taken $ACS = \varphi$, the density of the water at S $q = 1 - \alpha\cos\varphi$; and the cross-section of the tube $\omega = ff\left(1 - \beta\sin\varphi\right)$, truly the height above the horizontal plane $z = c\sin\varphi$; with the arc being $AS = c\varphi = s$. Now since for the motion in the region *AECD*, situated in the section, where the density would be $=1$, and the cross-section $= ff$, with the speed $= \upsilon$, this equation shall be found:

$$0 = 2g\int zdq - f^4\upsilon\upsilon\int \frac{d\omega}{q\omega^3} - \frac{ffd\upsilon}{dt}\int \frac{ds}{\omega},$$

we have arranged for these individual integrations themselves to be extended through the whole circle. And at first, on account of $z = c\sin\varphi$; and $dq = \alpha d\varphi\sin\varphi$; there will be:

$$\int zdq = \alpha c\int d\varphi\sin^2\varphi = \tfrac{1}{2}\alpha c\int d\varphi\left(1 - \cos 2\varphi\right) = \tfrac{1}{2}\alpha c\left(\varphi - \tfrac{1}{2}\sin 2\varphi\right),$$

the value of which established through the whole circle by putting $\varphi = 2\pi$ provides $\int zdq = \pi\alpha c$.

Then on account of $ds = cd\varphi$ and $\omega = ff\left(1 - \beta\sin\varphi\right)$; there becomes

$$ff\int \frac{ds}{\omega} = c\int \frac{d\varphi}{1 - \beta\sin\varphi} = \frac{c}{\sqrt{(1-\beta\beta)}}\left(\mathrm{Ang}\sin\sqrt{\left(1 - \beta\beta\right)} - \mathrm{Ang}\sin\ \frac{\cos\varphi\sqrt{(1-\beta\beta)}}{1 - \beta\sin\varphi}\right).$$

With this same angle ψ, the sine of which is $\frac{\cos\varphi\sqrt{(1-\beta\beta)}}{1-\beta\sin\varphi}$, and on putting $\varphi = 90°$ there becomes $\psi = 0$, but on putting $\varphi = \pi$ there becomes

$$\psi = -\mathrm{Ang}\sin\sqrt{\left(1 - \beta\beta\right)}\ ,$$

again on putting $\varphi = 270°$ there becomes $\psi = -\pi$, finally by putting $\varphi = 2\pi$ it is deduced that

$$\psi = -2\pi + \mathrm{Ang}\sin\sqrt{\left(1 - \beta\beta\right)},$$

from which for the whole circle there becomes

$$ff\int\frac{ds}{\omega}=\frac{2\pi c}{\sqrt{(1-\beta\beta)}},$$

so that the same may become clearer, if we may consider β as very small, then indeed there will become

$$\int\frac{d\varphi}{1-\beta\sin\varphi}=\int d\varphi\left(1+\beta\sin\varphi\right)=\varphi-\beta\cos\varphi+\beta,$$

the value of which on putting $\varphi=2\pi$ becomes $=2\pi$. For the third formula of the integral on account of

$$d\omega=-\beta ffd\varphi\cos\varphi \text{ and } q=1-\alpha\cos\varphi$$

there will become

$$f^4\int\frac{d\omega}{q\omega^3}=-\beta\int\frac{d\varphi\cos\varphi}{(1-\alpha\cos\varphi)(1-\beta\sin\varphi)^3}.$$

Likewise again we will consider β as very small, so that the denominator may be able to be considered $=1-\alpha\cos\varphi-3\beta\sin\varphi$, and hence there shall be had

$$f^4\int\frac{d\omega}{q\omega^3}=-\beta\int d\varphi\cos\varphi\left(1+\alpha\cos\varphi+3\beta\sin\varphi\right)$$

or

$$f^4\int\frac{d\omega}{q\omega^3}=-\beta\left(\sin\varphi+\tfrac{1}{2}\alpha\varphi+\tfrac{1}{4}\alpha\sin2\varphi-\tfrac{3}{4}\beta\cos2\varphi+\tfrac{3}{4}\beta\right),$$

the value of which on putting $\varphi=2\pi$ becomes $=-\pi\alpha\beta$. On account of which our differential equation thus itself will be had:

$$0=2\pi\alpha gc+\pi\alpha\beta\upsilon\upsilon-\frac{2\pi c}{\sqrt{(1-\beta\beta)}}\cdot\frac{d\upsilon}{dt},$$

where, since we ignore higher dimensions of β, it will be allowed to write 1 for $\sqrt{(1-\beta\beta)}$, thus so that there shall become

$$dt=\frac{cd\upsilon}{\alpha\left(gc+\frac{1}{2}\beta\upsilon\upsilon\right)}.$$

Therefore with a comparison made with the form shown above [*i.e.* $dt=\frac{\lambda ad\upsilon}{2gh-\mu\upsilon\upsilon}$] there shall become $\lambda a=c$, $2gh=\alpha gc$ and $\mu=-\frac{1}{2}\alpha\beta$, if the number β shall be positive, from the third case there becomes $cc=\frac{2gh}{-\mu}=\frac{2gc}{\beta}$ and $c=\sqrt{\frac{2gc}{\beta}}$, from which there is deduced :

$$\upsilon=\sqrt{\frac{2gc}{\beta}}\ \text{tang}\,\alpha g\sqrt{\frac{\beta}{2gc}}\,t,$$

thus so that after the time $t=\frac{\pi}{2\alpha g}\sqrt{\frac{2gc}{\beta}}$ sec. the speed now may become infinite. But if β shall be a negative number, or the cross-section of the tube at *S* generally $\omega = ff\left(1+\beta sin\varphi\right)$, a comparison must be put in place with the second case ; from which on account of $\lambda a=c,\ 2gh=\ \alpha gc$ and $\mu=\frac{1}{2}\alpha\beta$ there becomes

$$cc=\frac{2gc}{\beta}\ \text{and}\ c=\sqrt{\frac{2gc}{\beta}}.$$

Therefore the number γ may be taken $=2\alpha g\sqrt{\frac{\beta}{2gc}}$ and from the given time *t* there will become

$$\upsilon=\sqrt{\frac{2gc}{\beta}}\cdot\frac{e^{\gamma t}-1}{e^{\gamma t}+1},$$

which speed therefore becomes $=\sqrt{\frac{2gc}{\beta}}$ at last after an infinite time.

COROLLARY 1

150. From the case $\omega=ff(1-\beta\sin\varphi)$ we learn in general, if the upper part of the tube *AEB* shall be narrower than the lower part *ADB*, then the motion of the water to be accelerated so much, so that now in a finite time the speed may become infinite. Truly from the other case $\omega=ff(1+\beta\sin\varphi)$ we gather in general, if the upper part of the tube *AEB* were wider than the lower part *ADB*, then the motion to be accelerated much less, so that in so far in an infinite time the speed shall not be going to exceed a certain limit.

COROLLARY 2

151. For the integral formula $f^4\int\frac{d\omega}{q\omega^3}$, if α may considered only to be a very small fraction, the value of β itself will require to be left at least less than unity, from the calculation performed its value is found through the whole extended circle

$$=\frac{-\pi\alpha\beta}{(1-\beta\beta)\sqrt{(1-\beta\beta)}}$$

and hence the motion may be expressed by this equation:

$$dt=\frac{2(1-\beta\beta)cd\upsilon}{2\alpha gc(1-\beta\beta)^{\frac{3}{2}}+\alpha\beta\upsilon\upsilon}.$$

COROLLARY 3

152. Hence with the calculation performed further for the cross-section $\omega = ff(1-\beta\sin\varphi)$, there is found

$$\upsilon = (1-\beta\beta)^{\frac{3}{4}}\sqrt{\frac{2gc}{\beta}}\ \text{tang}\ \frac{\alpha g}{(1-\beta\beta)^{\frac{1}{4}}}\sqrt{\frac{\beta}{2gc}}t,$$

truly for the other case with the cross-section $\omega = ff(1+\beta\sin\varphi)$ on taking

$$\gamma = \frac{2\alpha g}{(1-\beta\beta)^{\frac{1}{4}}}\sqrt{\frac{\beta}{2gc}}$$

there will become

$$\upsilon = (1-\beta\beta)^{\frac{3}{4}}\sqrt{\frac{2gc}{\beta}}\cdot\frac{e^{\gamma t}-1}{e^{\gamma t}+1}.$$

SCHOLIUM

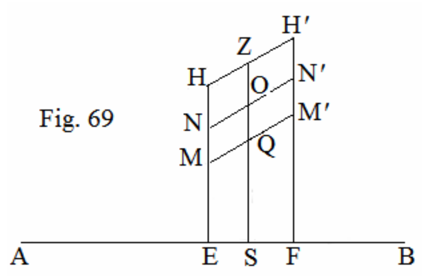


Fig. 69

153. When, as it may be happen in an example arising, the variable quantities *q, z* and ω are certain continuous functions of *s*, an investigation following the customary precepts can be put in place. Indeed if the tube may correspond to several parts with no law adjoining these together, then for the individual parts the values of the integral formulas, which are introduced in the determination of the motion, are to be investigated separately and then will be required to be brought together. With the directrix of the tube (Fig.69) extended in the direction *AB* for its part *EF*, there will be given at *E* the height $EH = h$, the cross-section of the tube $EN = ffn$ and the density of the water $EM = m$, at *F* truly the same elements shall be $EH' = h'$, $EN' = ffn'$ and $EM' = m'$, which we may assume to be changed uniformly from *E* and *F*, so that the scales representing these *HZH', NON'* and *MQM'* may be able to be had by right lines. Hence $EF = e$ and $ES = x$, so that there shall become $ds = dx$, there will become:

$$SZ = z = h + \frac{(h'-h)x}{e},$$

$$SO = \omega = ff\left(n + \frac{(n'-n)x}{e}\right) \text{ and } SQ = q = m + \frac{(m'-m)x}{e}.$$

On account of which, if we may consider the differences $h'-h,\ n'-n$ and $m'-m$ as very small, in the first place we will find

$$ff\int\frac{ds}{\omega} = \int\frac{edx}{en+(n'-n)x},$$

so that the expansion becomes through the interval $EF = e$

$$= \frac{e}{n'-n} l \frac{n'}{n} = e\left(\frac{1}{n} - \frac{(n'-n)}{2nn} + \frac{(n'-n)^2}{3n^3} - \text{etc.}\right).$$

Then there becomes:

$$\int zdq = \frac{m'-m}{e}\int dx\left(h + \frac{(h'-h)x}{e}\right),$$

with the integral equally expanded through the whole distance $EF = e$ provides

$$\int zdq = \tfrac{1}{2}(h'+h)(m'-m).$$

Finally the formula $f^4\int\frac{d\omega}{q\omega^3}$ will be changed into this form

$$\frac{n'-n}{en^2}\left(\frac{x}{mn} - \frac{3(n'-n)}{2emnn}xx - \frac{(m'-m)}{2emmn} - \text{etc.}\right)$$

and thus the value of this formula extended through the distance *EF* will become:

$$\frac{n'-n}{en^3} - \frac{3(n'-n)^2}{2mn^4} - \frac{(m'-m)(n'-n)}{2mmn^3},$$

of which it suffices to have taken the first part $\frac{n'-n}{en^3}$. I add no examples, since the precepts of the phenomena now have become evident well enough from the circular figure.

CAPUT V

DE MOTU AQUAE PER TUBOS DIVERSO CALORIS GRADU INFECTOS

PROBLEMA 61

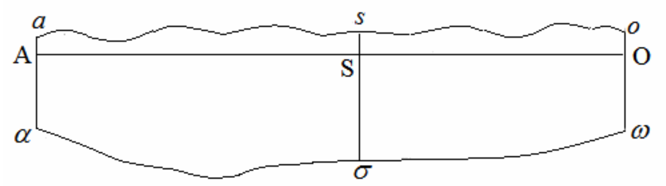


Fig. 63

124. *Dato caloris gradu (Fig. 63) in singulis tubi locis, quem statim cum aqua ibi contenta communicari assumimus, definire motum, quem aqua in huius modi tubo recipere poterit.*

SOLUTIO

Sit tubus *AO* ratione amplitudinis utcunque variabilis et incurvatus, sumtoque in eo intervallo indefinito $AS = s$, sit ibi amplitudo $= \omega$ et altitudo puncti *S* super plano horizontali fixo $S\sigma = z$; gradus autem caloris tantus, ut ibi aquae tribuatur densitas $= q$, quae ergo per hypothesin est variabilis et functio certa ipsius $AS = s$, quoniam in eodem loco aquam perpetuo eodem caloris gradu infectam assumimus. Elapso autem tempore *t* sit aquae per sectionem *Ss* transfluentis celeritas $= \mathfrak{T}$ in plagam *SO* directa et pressio $= p$, quae sunt functiones utriusque variabilis *s* et *t*. His positis, quia $\left(\frac{dq}{dt}\right) = 0$, ex problemate 46 has duas consequimur aequationes:

$$\left(\frac{d\cdot q\omega\mathfrak{T}}{ds}\right) = 0 \quad \text{et} \quad \frac{2gdp}{q} = -2gdz - \mathfrak{T}d\mathfrak{T} - ds\left(\frac{d\mathfrak{T}}{dt}\right).$$

Ex priori aequatione sequitur fore $q\mathfrak{T}\omega = \Gamma : t$, ita ut eodem tempore quantitas $q\mathfrak{T}\omega$ per totum tubum eundem obtineat valorem. Ponamus ergo in certo tubi loco, ubi amplitudo $= ff$ et densitas aquae $= 1$, celeritatem esse $= \upsilon$, quae ergo erit functio solius temporis *t*; ac prima conditio praebet $q\mathfrak{T}\omega = ff\upsilon$, ita ut sit $\mathfrak{T} = \frac{ff\upsilon}{q\omega}$, hincque, quia quantitates *q* et ω a sola variabili *s* pendent, erit $\left(\frac{d\mathfrak{T}}{dt}\right) = \frac{ff}{q\omega}\frac{d\upsilon}{dt}$, qui valor in altera aequatione, qua tempus *t* constans spectatur, substitutus praebet

$$2gdp = -2gqdz - q\mathfrak{T}d\mathfrak{T} - \frac{ffd\upsilon}{dt}\cdot\frac{ds}{\omega},$$

ex qua integrando elicimus:

$$2gp = \varDelta : t - 2g\int qdz - \tfrac{1}{2}q\mathfrak{T}\mathfrak{T} + \tfrac{1}{2}\int \mathfrak{T}\mathfrak{T}dq - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega}$$

seu loco $\mathfrak{TT}$ substituto valore $\frac{f^4\upsilon\upsilon}{qq\omega\omega}$

$$2gp = \varDelta:t - 2g\int qdz - \frac{f^4\upsilon\upsilon}{2qq\omega\omega} + \frac{1}{2}f^4\upsilon\upsilon\int\frac{dq}{qq\omega\omega} - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega}$$

Quodsi ergo in duobus locis pressio aliunde fuerit cognita, ad ea hanc aequationem applicando, primo functio temporis $\varDelta:t$ eliminari, tum vero celeritas υ pro quovis tempore determinari poterit, qua cognita deinceps omnia, quae ad motum spectant, innotescent.

COROLLARIUM 1

125. Cum sit

$$\int\frac{dq}{qq\omega\omega} = -\frac{1}{q\omega\omega} - 2\int\frac{d\omega}{q\omega^3},$$

aequatio inventa etiam ita repraesentabitur:

$$2gp = \varDelta:t - 2g\int qdz - \frac{f^4\upsilon\upsilon}{2qq\omega\omega} + f^4\upsilon\upsilon\int\frac{d\omega}{q\omega^3} - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

in qua hoc commodi occurrit, ut, quoties tubus ubique est aequaliter amplus, terminus $\int\frac{d\omega}{q\omega^3}$ evanescat simulque fiat $\int\frac{ds}{\omega} = \frac{ds}{\omega}$.

COROLLARIUM 2

126. Quia υ denotat celeritatem in data tubi sectione, cuius amplitudo $= f$, et ubi densitas q fit $=1$, hanc sectionem, ubi lubuerit, assumere licet; quoniam densitas, quam aqua ibi ob certum caloris gradum habet, ut densitas naturalis spectari potest, ex qua pressiones definiuntur.

SCHOLION

127. Quoniam supra vidimus massam fluidam gravem in aequilibrio esse non posse, nisi in aequalibus altitudinibus ubique eadem densitas locum habeat, operae omnino erit pretium hic eiusmodi casus evolvere, ubi aequilibrium prorsus subsistere nequit. Ac primo quidem se offert tubus (Fig. 64) circularis in situ verticali positus, qui ab una parte calidus, ab altera frigidus servatur. Sit scilicet *ASBD* tubus circularis in plano verticali positus, cuius *ACB* sit diameter

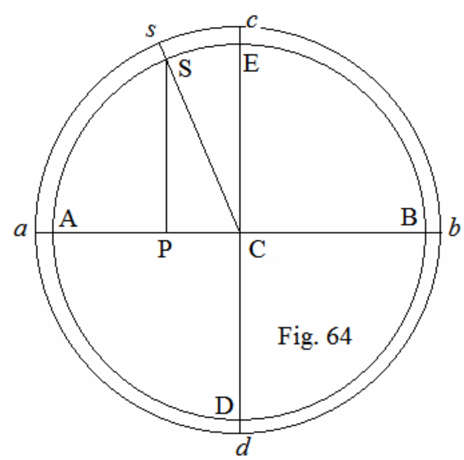


Fig. 64

horizontalis; hunc tubum circa a ita calefieri sumamus, ut aqua ibi contenta tantum non ebulliat, et regione vero in *B* tubus sit frigidus, gradu caloris ab *a* ad *B* sive sursum sive deorsum progrediendo successive decrescente, ut in *A* calor sit maximus, in *B* vero minimus. Quatenus ergo tubum vehementer angustum ponimus, aqua per eum mota quasi puncto temporis in quovis loco tubi calorem recipiet. Quodsi nunc totum tubum aqua plenum assumamus, fieri omnino nequit, ut aqua se ad aequilibrium componat, cuiusmodi igitur motum sit adeptura, in sequente problemate investigabimus.

PROBLEMA 62

128. *Si (Fig. 64) tubus circularis in situ verticali positus ASBD sit perpetuo in A calidus, in B vero frigidus, tum vero aqua repleatur, quae ubique tubi temperamentum statim recipiat, huius aquae motum in tubo, quia aequilibrium non datur, determinare.*

SOLUTIO

Sit radius circuli $CA = CB = c$, AB diameter horizontalis et amplitudo tubi ubique eadem $= ff$. Iam quia ad *A* calor est maximus, ad *B* vero minimus, densitas aquae ad *A* erit minima, ad *B* vero maxima: statuamus densitatem mediam $=1$, ad quam scilicet aestimationem pressionem referimus; tum vero in *A* sit densitas $=1-\alpha$, in *B* vero $=1+\alpha$, ab *A* vero ad *B* progrediendo densitas ita crescat, ut in puncto quovis *S* posito angulo $ACS = \varphi$ sit densitas $q = 1-\alpha cos\varphi$, quippe quae formula pro puncto *A* dat densitatem $1-\alpha$, pro B autem $1+\alpha$. Nunc porro altitudo puncti *S* super linea horizontali *AB* est $PS = csin\varphi = z$ et arcus $AS = c\varphi = s$. Ab initio, quo universa aqua adhuc erat in quiete, elapsum sit tempus $= t$, ac celeritas in puncto *S* vocetur $= \mathfrak{T}$ a termino *A* recedens, pressio vero ibidem $= p$. Quodsi nunc in eo loco, ubi densitas est $=1$, celeritas aquae ponatur $= \upsilon$, ob amplitudinem ubique eandem $\omega = ff$ erit $\mathfrak{T} = \frac{\upsilon}{q} = \frac{\upsilon}{1-\alpha\cos\varphi}$. Hinc ex principiis ante stabilitis definiamus ante omnia pressionem in loco indefinito *S*, ac primo ob $z = csin\varphi$, $q = 1-\alpha cos\varphi$ erit

$$\int qdz = c\int d\varphi cos\varphi(1-\alpha cos\varphi) = c\int d\varphi\left(cos\varphi - \tfrac{1}{2}\alpha - \tfrac{1}{2}\alpha cos2\varphi\right);$$

ideoque

$$\int qdz = c\sin\varphi - \tfrac{1}{2}\alpha c\varphi - \tfrac{1}{4}\alpha c\sin 2\varphi .$$

Deinde ob $\omega = ff$ est $\int \frac{d\omega}{q\omega^3} = 0$ et $\int \frac{ds}{\omega} = \frac{s}{ff} = \frac{c\varphi}{ff}$; his factis substitutionibus consequimur hanc aequationem:

$$2gp = \varDelta : t - 2gc\left(\sin\varphi - \tfrac{1}{2}\alpha\varphi - \tfrac{1}{4}\alpha\sin 2\varphi\right) - \frac{\upsilon\upsilon}{1-\alpha\cos\varphi} - \frac{c\varphi d\upsilon}{dt}.$$

Hinc pro initio in puncto *A* prodit haec aequatio:

$$2gp = \Delta : t - \tfrac{\upsilon\upsilon}{1-\alpha},$$

pro puncto *B* vero ponendo $\varphi = \pi = 180°$ haec

$$2gp = \Delta : t + \alpha g c\pi - \tfrac{\upsilon\upsilon}{1+\alpha} - \tfrac{\pi c d\upsilon}{dt}.$$

Percurramus totum circulum, ut revertamur in punctum *A*, et ponendo $\varphi = 2\pi$ pro puncto *A* prodit etiam haec aequatio:

$$2gp = \Delta : t + \alpha\pi g c - \tfrac{\upsilon\upsilon}{1-\alpha} - \tfrac{2\pi c d\upsilon}{dt}.$$

Cum igitur necesse sit, ut haec pressio illi pro eodem puncto *A* sit aequalis, hinc colligimus hanc aequationem

$$2\alpha\pi g c - \tfrac{2\pi c d\upsilon}{dt} = 0 \text{ seu } d\upsilon = \alpha g dt,$$

quae integrata dat $\upsilon = \alpha gt$, unde discimus, cum initio celeritas fuisset nulla, eam cum tempore uniformiter crescere, ita ut sit $\upsilon = \alpha gt$. Tum vero ob $\frac{d\upsilon}{dt} = \alpha g$ erit pro loco quocunque *S* elapso tempore *t* pressio

$$p = \Delta : t - c\left(\sin\varphi - \tfrac{1}{2}\alpha\varphi - \tfrac{1}{4}\alpha\sin 2\varphi\right) - \tfrac{\alpha\alpha gtt}{2(1-\alpha\cos\varphi)} - \tfrac{1}{2}\alpha c\varphi,$$

seu

$$p = \Sigma : t - c\sin\varphi + \tfrac{1}{4}\alpha\sin 2\varphi - \tfrac{\alpha\alpha gtt}{2(1-\alpha\cos\varphi)}$$

unde concludimus pressiones

pro *A*, ubi $\varphi = 0$, $\quad p = \Sigma : t - \frac{\alpha\alpha gtt}{2(1-\alpha)}$

pro *E*, ubi $\varphi = 90°$, $\quad p = \Sigma : t - \frac{\alpha\alpha gtt}{2} - c$

pro *B*, ubi $\varphi = 180°$, $\quad p = \Sigma : t - \frac{\alpha\alpha gtt}{2(1+\alpha)}$

pro *D*, ubi $\varphi = 270°$, $\quad p = \Sigma : t - \frac{\alpha\alpha gtt}{2} + c.$

COROLLARIUM 1

129. Cum igitur aqua primum in tubo quieverit, statim ita moveri incipiet, ut in parte inferiore *ADB* e locis frigidioribus in calidiora, in parte superiore AEB contra ex calidioribus in frigidiora feratur, fluxusque exoriatur in plagam *AEBD*, qui continuo uniformiter acceleretur.

COROLLARIUM 2

130. Ista motus acceleratio eo erit promtior, quo maius fuerit discrimen inter calorem maximum in *A* et minimum in *B*. Si in *A* aqua fere ebulliat, in *B* vero propemodum congelascat, fractio α est circiter $\frac{1}{30}$, ideoque $\upsilon = \frac{1}{30}gt = \frac{1}{2}t$ ped. ob $g = 15$ ped.

ped. ob g 15 ped. sicque post unum minutum secundum motus iam ita rapidus existeret, ut minuto secundo spatium $\frac{1}{2}$ pedis percurreret, post minutum primum autem spatium 30 pedum.

COROLLARIUM 3

131. Quod ad pressiones attinet, quas tubus interea sustinet, eae quidem non definiuntur, quia tubum vel aquam extrinsecus premendo ad quodvis tempus pressio pro lubitu variari potest. Interim tamen ad *B* pressio perpetua erit maxima, sumto enim

$$\Sigma : t = \frac{\alpha\alpha gtt}{2(1-\alpha)},$$

ut pressio in *A* evanescat, in *B* erit ea $= \frac{\alpha^3 gff}{1-\alpha\alpha}$, sicque in temporis ratione duplicata crescet.

SCHOLION 1

132. Facile autem intelligitur, si res experimentis exploretur, accelerationem motus neutiquam tam rapidam esse futuram, quam calculo invenimus, cuius ratio manifesta in eo est posita, quod statim atque aqua iam velocitatem notabilem acquisiverit, eius calor non subito se ad calorem tubi accommodare valeat, eaque proinde, pristinam temperaturam ad aliquod tempus conservans, in *B* magis calida quam tubus, in *A* vero minus sit futura. Cum igitur idem eveniat, ac si fractio α ex minor redderetur, motus quoque accelerationem relaxari oportebit, quae tamen omnino extingui nequit; simul enim atque hoc eveniret et aquae tempus suppeteret in quovis loco tubi calorem recipiendi, motus de novo uti ab initia instauraretur. Ex quo perspicuum est ob hanc causam motum tantum ad certum usque gradum acceleratum iri, in quo deinceps perpetua sit permansurus, quamdiu scilicet in ipso tubo discrimen caloris inest. Quoniam vero haec motus moderatio ab ea ratione potissimum pendet, qua tubus cum aqua, haecque vicissim cum tubo suum insitum caloris gradum communicat, ubi simul ad utriusque massam respici oportet, ex sola theoria hic vix quicquam statuere licebit. At si ope ignis circa *A* suscitati in hoc loco tubo perpetuo insignis caloris gradus imprimatur tubusque satis sit magnus, ut tantus calor non ad locum oppositum *B* transferri possit, nullum plane est dubium, quin aqua perpetuo motum satis velocem in plagam *AEBD* sit conservatura.

SCHOLION 2

133. Assumsi in problemate tubo in altera extremitate horizontali *A* maximum caloris gradum, in altera vero *B* minimum induci, quae dispositio ad motum generandum maxime est accommodata. Si enim maximus calor excitaretur in loco vel summo *E* vel imo *D*, et e regione minimus existeret, tum nullus plane motus oriretur, sed aqua semel in quiete posita perpetuo in eodem statu perseveraret. Quare etiamsi initio tubus circa *A* maximum calorem acceperit, nisi is a causa externa sustineatur, aqua per *A* transiens calorem

ibi receptum cum tubi locis superioribus *S* et *E* communicabit vicissimque frigus, quo per *B* transiens erat imbuta, in tubi regionem inferiorem *E* transferet, quo tandem efficietur, ut cum maximus calor in tubi locum summum *E* fuerit translatus minimusque in imum *D*, tum omnis motus sit cessaturus, et aqua in statum aequilibrii sit perventura, in quo acquiescera valeat. De cetero in solutione problematis certam legem stabilivi, secundum quam densitas fluidi ab *A* versus *B* progrediendo augeatur, quod augmentum ipsis distantiis in recta horizontali *AB* sumtis proportionale statui, ita ut excessus densitatis in *S* supra densitatem in *A* proportionalis esset spatio *AP*; quae hypothesis cum veritate satis consentire videtur, si prope *A* ignis aliave materia calorem gignens concipiatur constituta, cum enim vis calefaciendi in loco quovis *S* quadrato distantiae *AS* proportionalis aestimetur, hoc quadratum in circulo ipsi sinui verso *AP* est proportionale. Interim tamen hac hypothesi calculo potissimum consulens sum usus, et infra rem generalius expedire conabor.

PROBLEMA 63

134. *Sit* (Fig. 64) *uti in praecedente problemate tubus circularis in plano verticali positus isque in A calidus, in B vero frigidus; verum huic tubo diversa tribuatur amplitudo; hoc posito si tubus fuerit aqua repletus, eius motum definire*.

SOLUTIO

Sit ut ante radius circuli $CA = CB = c$, densitas aquae in $A = 1-\alpha$, in $B = 1+\alpha$, at in *E* et *D* $=1$, in loco vero quovis indefinito *S* posito angulo $ACS = \varphi$ sit densitas $q = 1-\alpha cos\varphi$. Tum vero in *E* et *D* sit amplitudo $= ff$, verum in *A* statuatur $= ff\left(1-\beta\right)$, in *B* $= ff\left(1+\beta\right)$, at in *S* fit $\omega = ff\left(1-\beta\cos\varphi\right)$. Elapso iam tempore *t* in *E* vel *D*, ubi amplitudo est *ff* et densitas $=1$, celeritas aquae sit $=\upsilon$, unde in loco indefinito *S* erit

$$\mathfrak{T} = \frac{\upsilon}{(1-\alpha\cos\varphi)(1-\beta\cos\varphi)}$$

quam aequationem prima motus conditio suppeditat. Altera vero posita pressione in $f = p$ ita se habet:

$$2gp = \Delta : t - 2g\int qdz - \tfrac{1}{2}q\mathfrak{T}\mathfrak{T} + \tfrac{1}{2}\int \mathfrak{T}\mathfrak{T}dq - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

pro cuius evolutione ob $z = csin\varphi$ et $q = 1-\alpha cos\varphi$ est ut ante

$$\int qdz = csin\varphi - \tfrac{1}{2}\alpha c\varphi - \tfrac{1}{4}\alpha csin2\varphi.$$

Deinde ob $s = c\varphi$ et $\omega = ff\left(1-\beta cos\varphi\right)$ est

$$\int \frac{ds}{\omega} = \frac{c}{ff}\int \frac{d\varphi}{1-\beta\cos\varphi} = \frac{c}{ff\sqrt{(1-\beta\beta)}}\,\text{Ang sin}\,\frac{\sin\varphi\sqrt{(1-\beta\beta)}}{1-\beta\cos\varphi}.$$

Denique ob $dq = \alpha d\varphi sin\varphi$ est

$$\int \mathfrak{T}\mathfrak{T}dq = \alpha\upsilon\upsilon\int \frac{d\varphi\sin\varphi}{(1-\alpha cos\varphi)^2(1-\beta cos\varphi)^2},$$

unde fit integrando:

$$\int \mathfrak{T}\mathfrak{T}dq = \frac{\alpha\upsilon\upsilon}{(\alpha-\beta)^2}\left(\frac{-(\alpha+\beta)+2\alpha\beta\cos\varphi}{(1-\alpha cos\varphi)(1-\beta cos\varphi)} + \frac{2\alpha\beta}{\alpha-\beta}\,l\,\frac{1-\beta cos\varphi}{1-\alpha cos\varphi}\right).$$

Ponatur nunc $\varphi = 0$, ut pressionem in puncto A obtineamus

$$2gp = \mathit{\Delta}:t - \frac{\upsilon\upsilon}{2(1-\alpha)(1-\beta)^2} + \frac{\alpha\upsilon\upsilon}{2(\alpha-\beta)^2}\left(\frac{-\alpha-\beta+2\alpha\beta}{(1-\alpha)(1-\beta)} + \frac{2\alpha\beta}{\alpha-\beta}\,l\,\frac{1-\beta}{1-\alpha}\right),$$

tum vero pro eodem puncto sit $\varphi = 2\pi$, erit

$$2gp = \mathit{\Delta}:t + 2\pi\alpha gc - \frac{\upsilon\upsilon}{2(1-\alpha)(1-\beta)^2} + \frac{\alpha\upsilon\upsilon}{2(\alpha-\beta)^2}\left(\frac{-\alpha-\beta+2\alpha\beta}{(1-\alpha)(1-\beta)} + \frac{2\alpha\beta}{\alpha-\beta}\,l\,\frac{1-\beta}{1-\alpha}\right) - \frac{2\pi cd\upsilon}{dt\sqrt{(1-\beta\beta)}},$$

ex quorum valorum aequalitate elicitur $d\upsilon = \alpha gdt\sqrt{(1-\beta\beta)}$ hincque $\upsilon = \alpha gt\sqrt{(1-\beta\beta)}$.

COROLLARIUM 1

135. Diversa ergo tubi amplitudo, siquidem legem in solutione positam sequitur, efficit, ut celeritas aliquanto minor generetur, idque perinde, sive maxima amplitudo statuatur in B sive in A. Ac si foret $\beta = 1$, quo casu amplitudo in A vel B evanesceret, motus plane nullus orietur, uti per se est manifestum.

COROLLARIUM 2

136. Si esset $\beta = \alpha$, seu densitas ubique tubi amplitudini esset proportionalis, foret

$$\int \mathfrak{T}\mathfrak{T}dq = \alpha\upsilon\upsilon\int \frac{d\varphi\sin\varphi}{(1-\alpha\cos\varphi)^4} = \frac{-\upsilon\upsilon}{3(1-\alpha\cos\varphi)^3} \quad \text{et} \quad \mathfrak{T}\mathfrak{T}q = \frac{\upsilon\upsilon}{(1-\alpha\cos\varphi)^3}$$

ideoque

$$-\tfrac{1}{2}q\mathfrak{T}\mathfrak{T} + \tfrac{1}{2}\int \mathfrak{T}\mathfrak{T}dq = \frac{-2\upsilon\upsilon}{3(1-\alpha\cos\varphi)^3},$$

quibus formulis pro pressione invenienda est utendum.

SCHOLION

137. Quoniam igitur vidimus, quantum inaequalitas in tubi amplitudine conferat ad motum aquae, inquiramus nunc etiam, qualis motus sit oriturus in eodem tubo circulari, si loca maximi et minimi caloris non in diametrum horizontalem, sed alium utcunque oblique positum incidant, ubi quidem amplitudinem tubi iterum ubique eandem statuamus.

PROBLEMA 64

138. *Sit* (Fig. 65) *ut hactenus tubus circularis in plano verticali positus isque ubique aeque amplus; verum maximus calor reperiatur in A, minimus in B, ut diameter AB sit ad horizontem HI inclinatus angulo* $ACH = \zeta$ *; atque cum hic tubus fuerit aqua plenus, eius motum definire.*

SOLUTIO

Sit radius circuli $CA = CB = c$, amplitudo tubi constans $= ff$, ut sit $\omega = ff$; ac pro puncto quovis S posito angulo $ACS = \varphi$; sit aquae densitas $q = 1 - \alpha cos\varphi$, ita ut in punctis E et F ea fiat $= 1$, ubi aquae celeritas elapso tempore t statuatur $= \upsilon$, quae ergo eodem tempore in S erit $\mathfrak{T} = \frac{\upsilon}{1-\alpha cos\varphi}$, cuius puncti S altitudo super horizonte cum sit $SP = csin(\varphi - \zeta) = z$, si pressio in S vocetur $= p$, ob arcum $AS = cq$; erit:

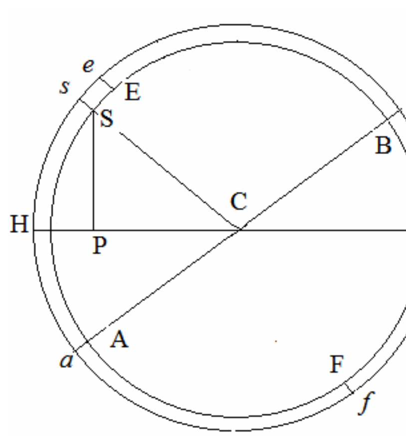


Fig. 65

$$2gp = \Delta : t - 2gc\int (1 - \alpha cos\varphi) d\varphi\ cos(\varphi - \zeta) - \frac{\upsilon\upsilon}{1 - \alpha cos\varphi} - \frac{c\varphi d\upsilon}{dt}.$$

At est

$$\int d\varphi cos\varphi cos(\varphi - \zeta) = \tfrac{1}{2}\int d\varphi\big(cos\zeta + cos(2\varphi - \zeta)\big) = \tfrac{1}{2}\varphi cos\zeta + \tfrac{1}{4}sin(2\varphi - \zeta)$$

ideoque habebitur:

$$2gp = \Delta : t - 2gcsin(\varphi - \zeta) + \alpha gc\varphi \cos\zeta + \tfrac{1}{2}\alpha gcsin(2\varphi - \zeta) - \frac{\upsilon\upsilon}{1 - \alpha cos\varphi} - \frac{c\varphi d\upsilon}{dt}.$$

Hinc pro loco A pressionem duplici modo exprimere poterimus, prout ponamus vel $\varphi = 0$ vel $\varphi = 2\pi$; prior positio dat

$$2gp = \Delta : t + 2gc\sin\zeta - \tfrac{1}{2}\alpha gc\sin\zeta - \tfrac{\upsilon\upsilon}{1-\alpha},$$

altero vero

$$2gp = \Delta : t + 2gc\sin\zeta + 2\alpha\pi gc\cos\zeta - \tfrac{1}{2}\alpha gc\sin\zeta - \tfrac{\upsilon\upsilon}{1-\alpha} - \tfrac{2\pi cd\upsilon}{dt},$$

quae duae expressiones cum inter se debeant esse aequales, est

$$\alpha g\cos\zeta = \tfrac{d\upsilon}{dt} \quad \text{hincque} \quad \upsilon = \alpha gt\cos\zeta,$$

unde ad quodvis tempus in quovis loco celeritas innotescit, cuius quidem directio in plagam *AEBF* tendit. Tum vero pressio in loco quocunque *S* erit:

$$p = \Sigma : t - c\sin(\varphi - \zeta) + \tfrac{1}{4}\alpha c\sin(2\varphi - \zeta) - \tfrac{\alpha\alpha gtt\cos^2\zeta}{2(1-\alpha\cos\varphi)}.$$

COROLLARIUM 1

139. Hinc ergo patet, si diameter AB per loca maximi minimique caloris transiens fuerit verticalis, ita ut maximus calor sit in circuli loco vel summo vel imo, ob $\cos\zeta = 0$ nullum motum a diversitate caloris generatum iri, sed aquam hoc casu in aequilibrio consistere posse, propterea quod in tubi locis aeque altis par caloris gradus reperitur.

COROLLARIUM 2

140. Si locus maximi caloris *Aa* puncto horizontali *H* minus quadrante distet, sive sursum sive deorsum, motus aquae fiet in directione *AEBF*; sin autem illa distantia *HA* quadrantem superet, quia tum $\cos\zeta$ fit negativus, motus in contrariam plagam *AFBE* erit directus.

COROLLARIUM 3

141. Semper ergo in locis inferioribus motus fiet a regione frigidiore in calidiorem, in superioribus vero contra a regione calidiore in frigidiorem, omnino uti iam supra circa aequilibrium est observatum, etiamsi ibi motum ipsum definire haud licuerit.

SCHOLION

142. Cum igitur iam non solum sit evictum aquam in tubo circulari, in quo ad pares altitudines gradus caloris sit diversus, in aequilibrio consistere non posse, sed etiam ipsum motum inde genitum determinaverimus, probe tenendum est hoc tantum evenire, si totus tubus sit aqua repletus; si enim minor aquae copia ei sit infusa, ea semper eiusmodi situm habere poterit, in quo perpetuo acquiescat. In quo certe ingens paradoxon agnosci

debet, quod, dum in eiusmodi tubo vacuum aliquod spatium admittitur, semper aequilibrium dari possit, id omni vacuo remoto subito tollatur ac necessario motus oriri debeat; multo maius autem hoc fiet paradoxon, cum ostendero etiam admisso spatio ab aqua vacuo, dummodo sit minimum, aequilibrium excludi, ita ut, quoties illud vacuum certa quadam quantitate fuerit minus, tum semper necessario motus generetur, quomodocunque aqua in tubo sit disposita, sin autem id vacuum ista quantitate fuerit maius, tum semper aquae eiusmodi situs tribui queat, in quo perpetuo acquiescat. Maxime igitur operae pretium erit, ut hoc insigne paradoxon accuratissime evolvamus.

PROBLEMA 65

143. *Sit* (Fig. 66) *tubus circularis in plano verticali positus ubique eiusdem amplitudinis, calor vero maximus in A, minimus in B versetur, ut recta AB sit diameter horizontalis. Quodsi iam huius tubi tantum portio MN aquam contineat, eius motum investigare.*

SOLUTIO

Sit radius circuli $\mathrm{CA} = \mathrm{CB} = 1$, amplitudo tubi ubique eadem $\omega = ff$; calor autem ita comparatus, ut in loco quocunque *S* posito arcu $AS = s$ sit densitas aquae $q = 1 - \alpha \cos s$.

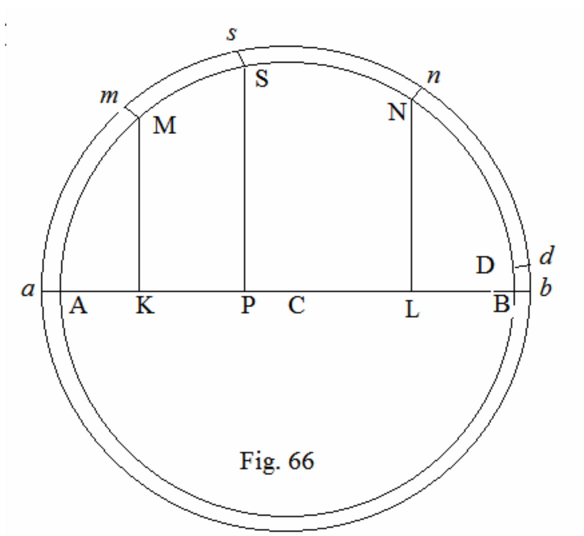


Fig. 66

Iam elapso tempore $= t$ occurret aqua tubo infusa spatium *MN* vocemusque arcus $AM = m$ et $AN = n$; ac primo quidem perpendendum est aquae massam perpetuo eandem manere, cum igitur in *S* sit densitas $q = 1 - \alpha \cos s$, massa aquae, quae tubi portionem *AS* esset impletura, erit $\int q ds = s - \alpha \sin s$, unde colligimus aquae portionem *MN* implentis massam fore $= n - m - \alpha(\sin n - \sin m)$, quae, cum sit constans, ponatur $= 2e$. In hunc finem statuatur:

$$m - \alpha \sin m = u - e \text{ et } n - \alpha \sin n = u + e,$$

et quia ex est fractio minima, habebimus proxime

$$m = u - e + \alpha \sin(u - e) \text{ et } n = u + e + x \sin(u + e),$$

ubi observo, si totus tubus esset aqua repletus, fore

$$n = m+2\pi \text{ ideoque } 2e = 2\pi \text{ seu } e = \pi,$$

ita ut tubus eatenus non sit totus aqua repletus, quatenus arcus e minor est semiperipheria circuli n radio existente $=1$. Cum nunc sit $z = \sin s$, erit

$$\int qdz = sin\, s - \tfrac{1}{2}\alpha s - \tfrac{1}{4}\alpha sin\, 2s,$$

et posita pressione in $S = p$ habebitur

$$2gp = \Delta : t - 2g\left(sin\ s - \tfrac{1}{2}\alpha s - \tfrac{1}{4}\alpha sin\, 2s\right) - \tfrac{\upsilon\upsilon}{1-\alpha\cos s} - \tfrac{sd\upsilon}{dt},$$

unde pressio pro utroque termino *M* et *N* colligi poterit; sive autem praeter aquam in tubo insit vacuum sive aër, semper pressiones in *M* et *N* aequales sint necesse est; ex quo fiet

$$\left.\begin{array}{l} +2g\left(sin\, n - \tfrac{1}{2}\alpha n - \tfrac{1}{4}\alpha sin\, 2n\right) + \tfrac{\upsilon\upsilon}{1-\alpha\cos n} + \tfrac{nd\upsilon}{dt} \\ -2g\left(sin\, m - \tfrac{1}{2}\alpha m - \tfrac{1}{4}\alpha sin\, 2m\right) - \tfrac{\upsilon\upsilon}{1-\alpha\cos m} - \tfrac{md\upsilon}{dt} \end{array}\right\} = 0.$$

Ex hac aequatione primum colligere licet, sub quibus conditionibus aequilibrium locum habere queat. Si enim hoc statu adsit aequilibrium, oportet sit
tam $\upsilon = 0$ quam $\frac{d\upsilon}{dt} = 0$, , quod fieri nequit, nisi sit:

$$\sin n - \sin m - \tfrac{\alpha}{2}(n-m) - \tfrac{1}{4}\alpha\left(\sin 2n - \sin 2m\right) = 0..$$

Quare, quoties huic aequationi satisfieri potest, aequilibrium dabitur, contra vero necessario motus exorietur. Statim autem patet, si sit $n = 2\pi+m$, hanc aequationem neutiquam subsistere neque propterea aequilibrium locum habere posse. Statuamus ergo $n = 2\pi+m-\delta$ atque aequilibrium postulat hanc aequationem:

$$\sin(m-\delta) - \sin m - \tfrac{1}{2}\alpha(2\pi-\delta) - \tfrac{1}{4}\alpha\left(\sin(2m-\delta) - \sin 2m\right) = 0.$$

Sumamus δ valde parvum, eritque

$$-\delta\cos m - \tfrac{1}{2}\alpha(2\pi-\delta) + \tfrac{1}{4}\alpha\delta\cos 2m = 0,$$

unde deducitur proxime $\cos m = \frac{-\alpha(2\pi-\delta)}{2\delta}$; nisi ergo sit

$$\alpha(2\pi-\delta) < 2\delta \text{ seu } \delta > \tfrac{2\alpha\pi}{2+\alpha},$$

aequilibrium plane locum habere nequit.

Sive autem aequilibrium excludatur, sive aqua ab alia causa fuerit agitata, motus ex superiori aequatione definiri poterit. Introducta nempe nova variabili u, ut sit

$$m = u - e + \alpha \sin(u-e) \quad \text{et} \quad n = u + e + \alpha \sin(u+e),$$

erit

$$\sin m = \sin(u-e) + \tfrac{1}{2}\alpha \sin 2(u-e),\ \cos m = \cos(u-e) - \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha \cos 2(u-e),$$
$$\sin n = \sin(u+e) + \tfrac{1}{2}\alpha \sin 2(u+e),\ \cos n = \cos(u+e) - \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha \cos 2(u+e),$$

$$\sin 2m = \sin 2(u-e) - \alpha \sin(u-e) + \alpha \sin 3(u-e),$$
$$\sin 2n = \sin 2(u+e) - \alpha \sin(u+e) + \alpha \sin 3(u+e).$$

Porro cum celeritas in M sit $= \frac{\upsilon}{1-\alpha\cos m}$, ex promotione momentanea concluditur temporis elementum

$$dt = \frac{dm(1-\alpha\cos m)}{\upsilon}$$

factaque substitutione fit $dt = \frac{du}{\upsilon}$. Quia deinde est proxime

$$\frac{1}{1-\alpha\cos n} = 1 + \alpha \cos n,$$

nostra aequatio induet hanc formam

$$2g\left(\sin n - \sin m - \tfrac{1}{2}\alpha(n-m) - \tfrac{1}{4}\alpha(\sin 2n - \sin 2m)\right) + \alpha\upsilon\upsilon(\cos n - \cos m) + \frac{\upsilon d\upsilon}{du}(n-m) = 0.$$

Iam vero ex superioribus formis elicitur

$$\sin n - \sin m = 2\sin e \cos u + \alpha \sin 2e \cos 2u,\ n - m = 2e + 2\alpha \sin e \cos u,$$
$$\cos n - \cos m = -2\sin e \sin u - \alpha \sin 2e \sin 2u,\ \sin 2n - \sin 2m = 2 \sin 2e \cos 2u,$$

facta ergo substitutione prodibit:

$$2\upsilon d\upsilon(e + \alpha \sin e \cos u) - 2\alpha\upsilon\upsilon du \sin e \sin u$$
$$+ 2g du\left(2\sin e \cos u - \alpha e + \tfrac{1}{2}\alpha \sin 2e \cos 2u\right) = 0,$$

quae per $e + \alpha \sin e \cos u$ multiplicata integrabilis redditur:

$$\upsilon\upsilon(e + \alpha \sin e \cos u)^2 + 2g \int du(2e \sin e \cos u - \alpha ee + \tfrac{1}{2}\alpha e \sin 2e \cos 2u + 2\alpha \sin^2 e \cos^2 u) = C,$$

ita ut hinc prodeat

$$\upsilon\upsilon = \frac{C - 4ge\sin e\sin u + 2\alpha geeu - 2\alpha gu\sin^2 e - \frac{1}{2}\alpha ge\sin 2e\sin 2u - \alpha g\sin^2 e\sin 2u}{(e+\alpha\sin e\cos u)^2},$$

seu

$$\upsilon\upsilon = \text{Const.} - \frac{4g}{e}\sin e\sin u + \frac{8\alpha g}{ee}\sin e\sin u + 2\alpha gu$$
$$-\frac{2\alpha gu}{ee}\sin^2 e - \frac{\alpha g\sin 2e}{2e}\sin 2u - \frac{\alpha g}{ee}\sin^2 e\sin 2u.$$

COROLLARIUM 1

144. Si ponamus $e = \pi$, ut tubus fiat aqua plenus, quem casum quidem iam supra enodavimus, aequatio hic inventa in hanc abit formam $\upsilon = C + 2\alpha gu$, unde fit

$$dt = \frac{du}{\sqrt{(C+2\alpha gu)}} \text{ et } t = \frac{1}{\alpha g}\sqrt{(C+2\alpha gu)} = \frac{\upsilon}{\alpha g},$$

ita ut sit $\upsilon = \alpha gt$, uti supra invenimus.

COROLLARIUM 2

145. Pro limite, ad quem usque aequilibrium locum habere potest, invenimus $\delta = \frac{2\alpha\pi}{2+\alpha}$, unde fit

$$m = \pi \text{ vel } m = -\pi, \text{ et } n = \pi - \frac{2\alpha\pi}{2+\alpha} = \frac{2-\alpha}{2+\alpha}\pi.$$

Hoc casu inferior semicirculus totus aqua plenus, superior vero aquam continebit usque ad *Dd* existente $BD = \frac{2\alpha}{2+\alpha}\pi$; qui est extremus status aequilibrii.

COROLLARIUM 3

146. Hinc sequitur, si portio tubi aqua destituta fuerit maior quam $\frac{2\alpha}{2+\alpha}\pi$; tum semper aequilibrium exhiberi posse, sin autem illa portio minor sit quam $\frac{2\alpha}{2+\alpha}\pi$; tum aequilibrio nullus plane locus relinquitur, sed aqua quasi sponte motum concipiet.

SCHOLION

147. Paradoxon ergo supra memoratum ita resolvitur, ut, quando tubus non omnino aqua est plenus in eoque spatium vacuum relinquitur, aequilibrium quidem semper locum habere possit, dummodo hoc spatium vacuum non fuerit valde parvum. Datur enim terminus quidam valde exiguus et a discrimine inter maximam minimamque aquae densitatem pendens, quo si spatium illud vacuum fuerit minus, aequilibrium penitus excludatur et aqua in tubo contenta, quemcunque situm tenuerit, necessario ad motum

concitetur. Cum cognitio huius termini maximi sit momenti, cum accuratius ex aequatione differentiali inter υ et *u* definiamus, et quia novimus tum tubum fere esse plenum, ponamus pro hoc termino esse $e=\pi-\varepsilon$, existente ε arcu minimo; atque ut tam υ quam $\frac{d\upsilon}{dt}$ evanescat, oportet sit

$$2\varepsilon\cos u-\alpha\pi+\alpha\varepsilon-\alpha\varepsilon\cos 2u=0 \text{ seu } \varepsilon=\frac{\alpha\pi}{2\cos u+\alpha-\alpha\cos 2u},$$

quae expressio minima reddi debet, ut valor pro ε minimus etiam nunc aequilibrium admittens obtineatur. Sumi igitur debet $u=0$, unde fit $\varepsilon=\frac{1}{2}\alpha\pi$, tum autem hoc aequilibrii statu extremo reperitur

$$m=-\pi+\tfrac{1}{2}\alpha\pi=-\left(1-\tfrac{1}{2}\alpha\right)\pi \text{ et } n=\left(1-\tfrac{1}{2}\alpha\right)\pi,$$

ut sit longitudo venae aqueae in tubo contentae

$$MN=n-m=2\left(1-\tfrac{1}{2}\alpha\right)\pi=2\pi-\alpha\pi$$

ideoque spatium vacuum $=\alpha\pi$; quod in aequilibrio ita locum *B*, ubi densitas est maxima, occupabit, ut altera extremitas infra punctum *B*, altera supra id cadat intervallo $\frac{1}{2}\alpha\pi$; quae determinatio accuratior est ea, quae in corollario 2 circa spatium *BD* est data, etiamsi aequilibrium in quovis situ proximo aeque subsistere queat.

PROBLEMA 66

148. *Si* (Fig. 67) *tubus in se rediens habuerit figuram quamcunque gradusque caloris in eo utcunque diversus, ut aqua, qua cum penitus repletum assumimus, in aequilibrio consistere nequeat, motum in ea genitum determinare.*

SOLUTIO

Sit in *A* calor maximus ideoque densitas minima, quae ponatur $=1$, ibique sit tubi amplitudo $=ff$; in loco *B* vero sit calor minimus ideoque densitas maxima $=1+\alpha$. Consideretur nunc locus tubi quicunque *S* et ponatur in eius directrice longitudo $AS=s$, amplitudo $Ss=\omega$ et altitudo supra planum horizontale fixum $=z$; quod planum per ipsum punctum *A* ducere licet, densitas vero ibidem sit $=q$. Iam elapso tempore $=t$ aqua eiusmodi motum acquisiverit, ut in *A* celeritas in plagam *AS* sit $=\upsilon$ ideoque in *S* futura sit $\mathfrak{T}=\frac{ff\upsilon}{q\omega}$. Quodsi ergo statuamus pressionem in $S=p$, hanc supra elicuimus aequationem:

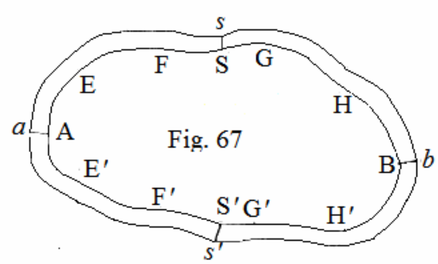


Fig. 67

$$2gp = \Delta : t - 2g\int qdz - \tfrac{f^4\upsilon\upsilon}{q\omega\omega} - f^4\upsilon\upsilon\int \tfrac{d\omega}{q\omega^2} - \tfrac{ffd\upsilon}{dt}\int \tfrac{ds}{\omega},$$

quae ob $\int qdz = qz - \int zdq$ transformetur in hanc

$$2gp = \Delta : t - 2gqz + 2g\int zdq - \tfrac{f^4\upsilon\upsilon}{q\omega\omega} - f^4\upsilon\upsilon\int \tfrac{d\omega}{q\omega^3} - \tfrac{ffd\upsilon}{dt}\int \tfrac{ds}{\omega},$$

ubi integralia per totam longitudinem tubi *AS* capi assumo, ita ut posito $s = 0$ ea quoque evanescant. Pro ipso ergo puncto *A*, ubi etiam fieri $z = 0$ sumimus, erit

$$2gp = \Delta : t - \upsilon\upsilon \text{ ob } q = 1 \text{ et } w = ff,$$

eandem autem pressionem prodire necesse est, si arcum *s* eousque augeamus, ut confecta tota tubi longitudine punctum *S* in *A* transferatur; qualem ergo formam tum nostra aequatio sit indutura, investigari oportet. Ac primo quidem observo, si amplitudo tubi ubique esset eadem $w = ff$, tum formulam $ff\int \frac{ds}{\omega}$ longitudinem totius tubi in se redeuntis esse expressuram; quatenus ergo amplitudo variabilis w fuerit vel maior vel minor quam *ff*, eatenus valor istius integralis vel minor erit vel maior illa longitudine tota. Posita ergo tota hac longitudine $= a$, statuatur integrale per totum tubum expansum

$$ff\int \tfrac{ds}{\omega} = \lambda a.$$

Deinde integrale $\int \frac{d\omega}{q\omega^3}$ per totum tubum extensum vel iterum evanescit vel certum quendam valorem induit, prout binae variabiles *q* et ω inter se fuerint comparatae; ponamus ergo valorem integralis $\int \frac{d\omega}{q\omega^3}$ per totum tubum extensi $= \frac{\mu}{f^4}$.
Integralis autem $\int zdq$ valor diligentiorem investigationem postulat; sumatur in tubo alius locus *S′*, ubi densitas aquae eadem sit *q* atque in loco *S*, ibi autem altitudo super plano horizontali fixo sit $= z'$. Quia vero ab *a* ad *S* progrediendo quantitas *q* augebatur, ulterius autem, cursu per *B* usque ad *S′* instituto, quantitas *q* decrescit, pro puncto *S′* loco *dq* scribere debemus $-dq$, ita ut binis tubi elementis in *S* et *S′* iunctim sumtis habeatur $(z - z')dq$, et nunc integrale $\int (z - z')dq$ ab *A* tantum usque *B* extendi oportet. Hunc in finem sumta (Fig. 68) recta $CA =$ densitati minimae 1, et CB = maximae $1+\alpha$, notentur quotcunque densitates mediae *CE, CF, CG, CH* etc. atque in tubo notentur bina loca coniugata *EE', FF', GG', HH'*, in quibus illae densitates insint, tum cuique excessui, quo altitudo punctorum *E, F, G, H* superat

Fig. 68


altitudinem punctorum *E', F', G', H'*, statuantur applicatae aequales *Ee, Ff, Gg, Hh*, et curvae per puncta *e, f, g, h* ductae area *AefghBA* dabit verum valorem integralis $\int zdq$, quatenus per totam tubi longitudinem extenditur. Statuamus hunc valorem $\int zdq = h$ et facto integro circuitu pro pressione in *A* habebimus

$$2gp = \Delta : t + 2gh - \upsilon\upsilon - \mu\upsilon\upsilon - \tfrac{\lambda ad\upsilon}{dt},$$

qui valor cum ante invento $2gp = \Delta : t - \upsilon\upsilon$ aequalis esse debeat; pro motu determinando nascetur haec aequatio:

$$\lambda ad\upsilon + \mu\upsilon\upsilon dt = 2ghdt \quad \text{seu} \quad dt = \tfrac{\lambda ad\upsilon}{2gh - \mu\upsilon\upsilon},$$

quae tres supeditat casus considerandos

I. Si $\mu = 0$, erit $t = \frac{\lambda a\upsilon}{2gh}$ ideoque $\upsilon = \frac{2gh}{\lambda a}t.$

II.Si $\mu > 0$; ponatur $\mu = \frac{2gh}{cc}$, fit $dt = \frac{\lambda accd\upsilon}{2gh(cc-\upsilon\upsilon)}$, hincque integrando $t = \frac{\lambda ac}{4gh} l \frac{c+\upsilon}{c-\upsilon}$,

siquidem posito $t = 0$ esse debet $\upsilon = 0$, faciamus $\frac{4gh}{\lambda ac} = \gamma$ eritque $\upsilon = \frac{e^{\gamma t}-1}{e^{\gamma t}+1}c.$
Hoc ergo casu celeritas υ quidem crescit, sed non ultra terminum *c*, quem demum elapso tempore infinito assequitur. Hinc casus primus nascitur, si $c = \infty$.

III. Si $\mu < 0$, ponatur $\mu = \frac{-2gh}{cc}$, ut fiat $dt = \frac{\lambda accd\upsilon}{2gh(cc+\upsilon\upsilon)}$ hincque integrando

$$t = \tfrac{\lambda ac}{2gh} \text{Ang tang} \tfrac{\upsilon}{c},$$

unde elicimus $\upsilon = c \tan \frac{2gh}{\lambda ac}t.$ Hoc ergo casu elapso tempore finito $t = \frac{\pi\lambda ac}{4gh}$ celeritas υ iam fit infinita.

EXEMPLUM

149. Sit (Fig. 64) tubus circularis in plano verticali positus aqua plenus radio existente $CA = CB = c$. Sumto autem angulo $ACS = \varphi$, sit in *S* densitas aquae $q = 1 - \alpha\cos\varphi$; et amplitudo tubi $\omega = ff(1 - \beta\sin\varphi)$, altitudo vero super plano horizontali $z = c\sin\varphi$; existente arcu $AS = c\varphi = s$. Cum iam pro motu in plagam *AECD*, posita in sectione, ubi foret densitas $= 1$ et amplitudo $= ff$, celeritate $= \upsilon$, haec inventa sit aequatio

$$0 = 2g\int zdq - f^4\upsilon\upsilon\int \tfrac{d\omega}{q\omega^3} - \tfrac{ffd\upsilon}{dt}\int \tfrac{ds}{\omega},$$

his integralibus per totum circulum extensis singula seorsim evolvamus. Ac primo quidem ob $z = c\sin\varphi$; et $dq = \alpha d\varphi \sin\varphi$; erit

$$\int zdq = \alpha c\int d\varphi \sin^2\varphi = \tfrac{1}{2}\alpha c\int d\varphi(1 - \cos 2\varphi) = \tfrac{1}{2}\alpha c\left(\varphi - \tfrac{1}{2}\sin 2\varphi\right),$$

cuius valor per totum circulum ponendo $\varphi = 2\pi$ expansum praebet $\int zdq = \pi\alpha c$.
Deinde ob $ds = cd\varphi$ et $\omega = ff(1-\beta\sin\varphi)$; fit

$$ff\int\frac{ds}{\omega} = c\int\frac{d\varphi}{1-\beta\sin\varphi} = \frac{c}{\sqrt{(1-\beta\beta)}}\left(\text{Ang}\sin\sqrt{(1-\beta\beta)} - \text{Ang}\sin\ \frac{\cos\varphi\sqrt{(1-\beta\beta)}}{1-\beta\sin\varphi}\right).$$

Sit ψ angulus iste, cuius sinus est $\frac{\cos\varphi\sqrt{(1-\beta\beta)}}{1-\beta\sin\varphi}$, et posito $\varphi = 90°$ fit $\psi = 0$, posito autem $\varphi = \pi$ fit

$$\psi = -\text{Ang}\sin\sqrt{(1-\beta\beta)}\ ,$$

posito porro $\varphi = 270°$ fit $\psi = -\pi$, posito denique $\varphi = 2\pi$ colligitur

$$\psi = -2\pi + \text{Ang}\sin\sqrt{(1-\beta\beta)},$$

ex quo pro toto circulo fit

$$ff\int\frac{ds}{\omega} = \frac{2\pi c}{\sqrt{(1-\beta\beta)}},$$

quod idem clarius fit, si β ut valde parvem spectemus, tum enim erit

$$\int\frac{d\varphi}{1-\beta\sin\varphi} = \int d\varphi(1+\beta\sin\varphi) = \varphi - \beta\cos\varphi + \beta,$$

cuius valor posito $\varphi = 2\pi$ fit $= 2\pi$. Pro tertia formula integrali ob

$$d\omega = -\beta ffd\varphi\cos\varphi \text{ et } q = 1-\alpha\cos\varphi$$

erit

$$f^4\int\frac{d\omega}{q\omega^3} = -\beta\int\frac{d\varphi\cos\varphi}{(1-\alpha\cos\varphi)(1-\beta\sin\varphi)^3}.$$

Consideremus iterum β perinde ac ex valde parvum, ut denominator censeri possit $= 1-\alpha\cos\varphi - 3\beta\sin\varphi$, hincque habeatur

$$f^4\int\frac{d\omega}{q\omega^3} = -\beta\int d\varphi\cos\varphi(1+\alpha\cos\varphi+3\beta\sin\varphi)$$

seu

$$f^4\int\frac{d\omega}{q\omega^3} = -\beta\left(\sin\varphi + \tfrac{1}{2}\alpha\varphi + \tfrac{1}{4}\alpha\sin2\varphi - \tfrac{3}{4}\beta\cos2\varphi + \tfrac{3}{4}\beta\right),$$

cuius valor posito $\varphi = 2\pi$ fit $= -\pi\alpha\beta$. Quocirca nostra aequatio differentialis ita se habebit:

$$0 = 2\pi\alpha gc + \pi\alpha\beta\upsilon\upsilon - \frac{2\pi c}{\sqrt{(1-\beta\beta)}}\cdot\frac{d\upsilon}{dt},$$

ubi, quia ipsius β altiores dimensiones negligimus, loco $\sqrt{(1-\beta\beta)}$ scribere licet 1, ita ut sit

$$dt = \frac{cd\upsilon}{\alpha\left(gc+\frac{1}{2}\beta\upsilon\upsilon\right)}.$$

Cum ergo facta comparatione cum forma supra exhibita sit
$\lambda a = c,\ 2gh = \alpha gc$ et $\mu = -\frac{1}{2}\alpha\beta$, si β sit numerus positivus, ex casu tertio fit
$cc = \frac{2gh}{-\mu} = \frac{2gc}{\beta}$ et $c = \sqrt{\frac{2gc}{\beta}}$, unde colligitur:

$$\upsilon = \sqrt{\frac{2gc}{\beta}}\ \mathrm{tang}\,\alpha g\sqrt{\frac{\beta}{2gc}}\,t,$$

ita ut post tempus $t = \frac{\pi}{2\alpha g}\sqrt{\frac{2gc}{\beta}}$ sec. celeritas iam fiat infinita. At si β sit numerus negativus seu amplitudo tubi in *S* generaliter $\omega = ff\left(1+\beta sin\varphi\right)$, comparatio cum casu secundo institui debet; ex quo ob $\lambda a = c,\ 2gh = \ \alpha gc$ et $\mu = \frac{1}{2}\alpha\beta$ fit

$$cc = \frac{2gc}{\beta} \text{ et } c = \sqrt{\frac{2gc}{\beta}}$$

Capiatur ergo numerus $\gamma = 2\alpha g\sqrt{\frac{\beta}{2gc}}$ et ad datum tempus *t* erit

$$\upsilon = \sqrt{\frac{2gc}{\beta}}\cdot\frac{e^{\gamma t}-1}{e^{\gamma t}+1},$$

quae ergo celeritas elapso demum tempore infinito fit $= \sqrt{\frac{2gc}{\beta}}$.

COROLLARIUM 1

150. From the case $\omega = ff(1-\beta\sin\varphi)$ discimus in genere, si tubi pars superior *AEB* angustior sit quam pars inferior *ADB*, tum motum aquae tantopere accelerari, ut iam tempore finito celeritas fiat infinita. Ex altero vero casu $\omega = ff(1+\beta\sin\varphi)$ colligimus in genere, si tubi pars superior *AEB* fuerit amplior inferiori *ADB*, tum motum multo minus accelerari, ut elapso adeo tempore infinito celeritas non sit certum limitem superatura.

COROLLARIUM 2

151. Pro formula integrali $f^4\int\frac{d\omega}{q\omega^3}$, si tantum α ut fractio valde parva spectetur, ipsi β valorem quemcunque unitate saltem minorem relinquendo, calculo subducto reperitur eius valor per totum circulum extensus

$$\frac{-\pi\alpha\beta}{(1-\beta\beta)\sqrt{(1-\beta\beta)}}$$

hincque motus hac aequatione exprimetur:

$$dt = \frac{2(1-\beta\beta)cd\upsilon}{2\alpha gc(1-\beta\beta)^{\frac{3}{2}}+\alpha\beta\upsilon\upsilon}.$$

COROLLARIUM 3

152. Calculo hinc ulteriori subducto pro amplitudine $\omega = ff(1-\beta\sin\varphi)$ reperitur

$$\upsilon = (1-\beta\beta)^{\frac{3}{4}}\sqrt{\frac{2gc}{\beta}}\ \text{tang}\ \frac{\alpha g}{(1-\beta\beta)^{\frac{1}{4}}}\sqrt{\frac{\beta}{2gc}}t,$$

$$dt = \frac{2(1-\beta\beta)cd\upsilon}{2\alpha gc+\alpha\beta\upsilon\upsilon}.$$

pro altero vero casu amplitudinis $\omega = ff(1+\beta\sin\varphi)$ sumto

$$\gamma = \frac{2\alpha g}{(1-\beta\beta)^{\frac{1}{4}}}\sqrt{\frac{\beta}{2gc}}$$

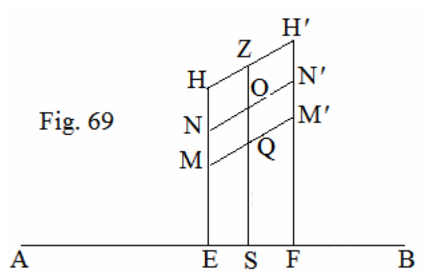
erit

$$\upsilon = (1-\beta\beta)^{\frac{3}{4}}\sqrt{\frac{2gc}{\beta}}\cdot\frac{e^{\gamma t}-1}{e^{\gamma t}+1}.$$

SCHOLION

153. Quando, uti in exemplo allato usu venit, quantitates variabiles *q, z* et ω sunt certae functiones continuae ipsius *s*, investigatio secundum praecepta analyseos consueta institui potest. Verum si tubus constet pluribus partibus nulla continuitatis lege inter se connexis, tum pro singulis partibus valores formularum integralium, quae in motus determinationem ingrediuntur,

Fig. 69



seorsim investigari ac deinceps colligi oportet. Directrice tubi (Fig.69) *AB* in directum extensa pro eius portione *EF* dentur in *E* altitudo $EH = h$, amplitudo tubi $EN = ffn$ et densitas aquae $EM = m$, in *F* vero sint eadem elementa $EH' = h'$, $EN' = ffn'$ et $EM' = m'$, quae ab E et F ita uniformiter mutari assumamus, ut scalae ea repraesentantes *HZH', NON'* et *MQM'* pro lineis rectis haberi possint. Hinc $EF = e$ et $ES = x$, ut sit $ds = dx$, erit

$$SZ = z = h+\frac{(h'-h)x}{e},$$

$$SO = \omega = ff\left(n+\frac{(n'-n)x}{e}\right) \text{ et } SQ = q = m+\frac{(m'-m)x}{e}.$$

Quamobrem, si differentias $h'-h$, $n'-n$ et $m'-m$ ut valde parvas spectemus, inveniemus primo

$$ff\int\frac{ds}{\omega}=\int\frac{edx}{en+(n'-n)x},$$

quod per spatium $EF=e$ expansum fit

$$=\frac{e}{n'-n}l\frac{n'}{n}=e\left(\frac{1}{n}-\frac{(n'-n)}{2nn}+\frac{(n'-n)^2}{3n^3}-\text{etc.}\right).$$

Deinde est

$$\int zdq=\frac{m'-m}{e}\int dx\left(h+\frac{(h'-h)x}{e}\right),$$

quod integrale pariter per totum spatium $EF=e$ expansum praebet

$$\int zdq=\tfrac{1}{2}(h'+h)(m'-m).$$

Denique formula $f^4\int\frac{d\omega}{q\omega^3}$ abit in hanc formam

$$\frac{n'-n}{en^2}\left(\frac{x}{mn}-\frac{3(n'-n)}{2emnn}xx-\frac{(m'-m)}{2emmn}-\text{etc.}\right)$$

sicque istius formulae valor per spatium *EF* extensus erit

$$\frac{n'-n}{en^3}-\frac{3(n'-n)^2}{2mn^4}-\frac{(m'-m)(n'-n)}{2mmn^3},$$

cuius sufficit partem sumsisse primam $\frac{n'-n}{en^3}$. Exempla non addo, quia praecipua phaenomena ex figura circulari satis iam sunt facta manifesta.