CHAPTER TWO

THE RESTORATION OF FLOATING BODIES TO EQUILIBRIUM

PROPOSITION 14

THEOREM

110. *In order that a body may be floating securely on water, so that only a part of it shall be immersed in water, how great shall the part required be for equilibrium to be found, or if the right line joining the centre of gravity and the magnitude of the submerged part were not vertical, then the body will be moved, until it will arrive at a state of equilibrium.*

DEMONSTRATION

For indeed initially if a large enough great part is not submerged in the water, as is required for equilibrium, it is evident the forces of gravity and the pressure to be going to become unequal, and thus cannot mutually cancel each other out. Whereby the body then will be moving either by ascending or descending, until it may arrive at the position of equilibrium. But if the right line joining the centres of gravity and of the displaced magnitude were not vertical, the forces of gravity and pressure, even if they were equal, yet will not act on each other in opposite directions. Therefore since also in this case the forces, by which the body may be acted on do not cancel each other out, it is necessary that they will produce a rotational motion in the body, which will last for some time, until the body will be established in equilibrium. Q. E. D.

[It is interesting to note that Euler does not consider the possibility of an oscillatory motion to occur in the vertical case also, until equilibrium is established.]

COROLLARY 1

111. Therefore if the body were moved aside from the position of equilibrium by an external force, then with that force ceasing it will begin to move, until it will have reached the equilibrium position.

COROLLARY 2

112. Therefore if a body may be given, which may be prepared thus, so that no situation of equilibrium will be permitted then this body must be assigned to move continuously in the water, and shall show true perpetual motion.

COROLLARY 3

113. Therefore since perpetual motion of this kind shall involve a contradiction, it follows every body has at least one situation in which it shall be in equilibrium.

COROLLARIUM 4

114. Because a body must be able to be moving in water while it reconstitutes itself, it will be acted on also by the resistance of the water, which will be greater there, where the speed of the body would be greater against the water. Moreover, the resistance arising from the speed shall no longer affect the slowest motion.

COROLLARY 5

115. Yet meanwhile this is certain, a body cannot be restored as quickly, if the water were offering no resistance. Therefore it will be restored more slowly, where a greater resistance were requiring to be overcome.

SCHOLIUM

116. Therefore in this chapter it will be required for us to enquire, what kind of motion may be generated in a body, which is not going to be destroyed by the forces themselves acting. Moreover for this, before all else, there is a need for several propositions to be sought from mechanics, concerned with the effect of forces on finite bodies. Concerning which, since scarcely anything may be treated as determined, it is necessary that we may draw the principles from the fonts of mechanics themselves. Therefore we will investigate in the following lemmas, the forces which must be acting on the bodies to produce this kind of motion; and these principles, which we will establish here, are necessary not only in this chapter, but also are of so great necessity in all the following chapters, as in no way shall we be able to be without these.

LEMMA

117. *If there were several individual bodies A, B, C, D* (Fig. 20) *which may arrive at a, b, c, d moving forwards in parallel directions; then the common centre of gravity O of these also will be moving in the parallel direction Oo, and it shall arrive at o so that there shall become:*

$$Oo = \frac{A \cdot Aa + B \cdot Bb + C \cdot Cc + D \cdot Dd}{A + B + C + D}$$

with A , B, C, D denoting the respective masses of the bodies.

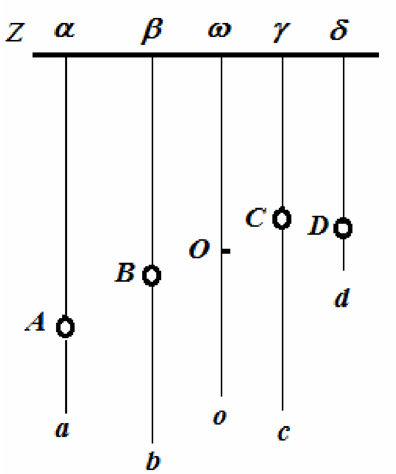


Fig. 20

DEMONSTRATION

The directed lines *Aa, Bb, Cc, Dd,* may be produced until they cross over some assumed right line *Zδ* at the points $\alpha, \beta, \gamma, \delta$, and the right line from the centre of gravity *O* at ω.
Therefore there will become, from the known property of the centre of gravity :

$$Z\omega = \frac{A \cdot Z\alpha + B \cdot Z\beta + C \cdot Z\gamma + D \cdot Z\delta}{A + B + C + D}$$

from which equation it is understood, the point ω to be the origin of the centre of gravity both while the bodies are at *A* , *B, C, D* as well as when they are situated at *a, b, c, d*, while the direction, in which the centre of gravity of the bodies is moved, will be parallel to the directions of the bodies.
Then truly with the bodies present at *A* , *B, C, D* there will be

$$O\omega = \frac{A \cdot A\alpha + B \cdot B\beta + C \cdot C\gamma + D \cdot D\delta}{A + B + C + D}.$$

Moreover with the bodies at *a, b, c, d,* translated, the centre of gravity shall arrive at *o,* and by the same centre of gravity property

$$o\omega = \frac{A \cdot a\alpha + B \cdot b\beta + C \cdot c\gamma + D \cdot d\delta}{A + B + C + D}.$$

Now, since there shall be $Oo = o\omega - O\omega$, there will become

$$oO = \frac{A \cdot Aa + B \cdot Bb + C \cdot Cc + D \cdot Dd}{A + B + C + D}.$$

Q.E.D.

COROLLARY

118. Therefore however great shall be the number of bodies the demonstration given prevails equally, as for the case of only four bodies. And it is understood also from the demonstration, that to retain its strength equally, whether the bodies shall be situated in the same plane or otherwise.

LEMMA

119. *If forces a, b, c, d were applied to the bodies A*, *B, C, D* (Fig. 20) *respectively*, *the directions of which shall be parallel to each other, and the bodies may be moved by these forces, then the common centre of gravity O of these will be moved in the same manner, and as if the sum of all these masses* $A + B + C + D$ *may be concentrated at the centre of gravity O, with the sum of the forces* $a + b + c + d$ *acting in the same direction.*

DEMONSTRATION

In the point of time *dt* the bodies *A, B, C, D* shall be moved by their respective forces *a, b, c, d* through the elements of the distances *Aa, Bb, Cc, Dd,* [as they stand, these equations are considered as corresponding, rather than being equal; thus, we consider *Aa* etc., to be the increment in the distance corresponding to the increment in the speed during an increment of the time , etc. Thus, in the following related propositions we will consider the equality sign to mean 'corresponding to', rather than being 'equal to'.], by the known principles of mechanics there will become:

$$Aa = \frac{a \cdot dt}{A},\ Bb = \frac{b \cdot dt}{B},\ Cc = \frac{c \cdot dt}{C},\ Dd = \frac{d \cdot dt}{D}.$$

Therefore, likewise, by the preceding lemma the centre of gravity *O* will arrive at *o,* so that [the differential quantity] *Oo* shall be parallel to the directions of the forces, and

$$Oo = \frac{dt\left(a + b + c + d\right)}{A + B + C + D}.$$

But if the sum of the bodies $A + B + C + D$ shall be concentrated at *O,* and that shall be acted on in the direction *Oo* by the sum of the forces $a + b + c + d$,then in the increment of the time *dt* the concentrated mass will also arrive at *o* so that there [the differential] shall arise

$$Oo = \frac{dt\left(a+b+c+d\right)}{A+B+C+D}.$$

Whereby since the individual bodies *A* , *B, C, D,* are acted on by their own respective forces *a, b, c, d* in the parallel directions amongst themselves, the centre of gravity *O* will be moved in the same manner as if the sum of the bodies were concentrated at *O,* and acted on by the sum of the forces $a+b+c+d$ in the same direction . Q. E. D.

COROLLARY

120. Therefore this demonstration follows equally, either if the bodies shall be put in the same plane, or if otherwise provided that the directions shall be parallel to each other, by which the individual forces are acting.

LEMMA

121. *If the bodies A, B, C, D* (Fig. 21) *shall be acted on by some forces, and by these the bodies shall be made to move, then the centre of gravity O will be moved in the same manner as if the masses of all the bodies shall be concentrated at that point, and the sum of these applied lines shall be whatever forces are acting in the direction parallel to these, on which they are acting.*

DEMONSTRATION

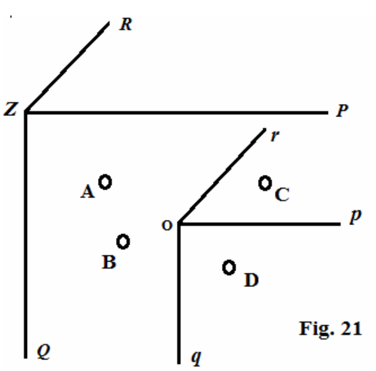


Fig. 21

For argument sake the three fixed directions shall be taken normal to each other *ZP, ZQ, ZR* normal to each other, and some forces, by which the bodies *A, B, C, D* may be acted on, may be resolved into three parts, the directions of which shall be parallel to the directions *ZP, ZQ, ZR* respectively. The sum of all the forces arising from this resolution, the directions of which are parallel to ZP itself $= P$, the sum of these which are parallel to *ZQ* $= Q$, and the sum of these which are parallel to *ZR* itself $= R$. Now if the individual bodies were acted on only by these forces, the directions of which shall be parallel to *ZP* itself, then the centre of gravity *O* will be moved in the same manner, as there the sum of the [masses of] the bodies $A+B+C+D$ is acted on by the force *P* in the direction *Op* parallel to *ZP* itself. In a similar manner, if the individual bodies were disturbed only by forces of such a size parallel to *ZQ* then the centre of gravity will be moved in the same way, as if there the sum of the bodies $A+B+C+D$ were acted on by the force *Q,* of which the direction *Oq* also shall be parallel to *ZQ*. Finally, in the same manner, if the individual bodies were acted on only by forces, the directions of which are parallel to *ZR*, then the centre of gravity *o* will be moved in the same manner, as if there the sum of the bodies $A+B+C+D$ were acted on by the force *R* in the direction *OR* parallel to *ZR*. Whereby the individual bodies *A, B, C, D* which may be acted on by all the forces applied at the same time, then the centre of gravity *O* will be moved in the same manner as if the sum of the bodies $A+B+C+D$ were concentrated there, and that to be disturbed by the three forces *P, Q, R* acting in the directions *Op, Oq, Or*. But if all the forces, by which the individual bodies may be disturbed, were applied in the parallel directions at the point *O*, then the three forces *P, Q, R* in the directions *Op, Oq, Or* will be equivalent to all these forces. On account of which the centre of gravity *O* will be moved in the same manner as if all the bodies were concentrated there , and all the forces were applied to these in the same directions.

Q.E.D.

LEMMA

122. *If some forces were applied to a rigid body, the parts of which next to each other bound together most firmly, whatever forces were applied, its centre of gravity will be moved in the same manner, as if the mass of the whole body were concentrated together, and all these forces applied jointly in their own directions.*

DEMONSTRATION

For the moment all the particles of the body may be freed from each other in turn, so that any of these may be moved freely by forces, then from the preceding lemma it is agreed the centre of gravity of the body to be moved in the same way, as if the whole mass of the body were concentrated in that, and that shall be acted on by the forces in the same respective directions. But since meanwhile the individual particles of the body shall be removed from their mutual place, in which they must be able to maintain the connections with each other, these themselves must be considered to restore the order, by forces with which they will mutually attract or repel each other. But by actions of this kind by which bodies mutually act between themselves, do not change the position of the centre of gravity. On account of which after the reconstitution the centre of gravity will remain at the same, and therefore it will be moved in the same manner as if the individual particles of the body themselves were to remain loose. Therefore the mutual binding, by means of which the particles cohere together, does not disturb the motion of the centre of gravity. Q.E.D.

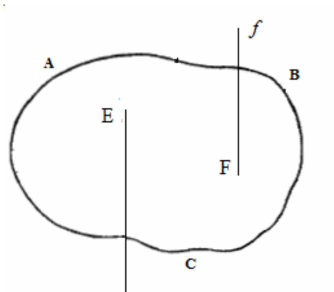


Fig. 22

COROLLARY 1

123. Therefore if the forces, by which the body may be acted on were prepared thus, so that, if all were applied to the same point, they would mutually cancel each other out and remain in equilibrium, then the centre of gravity of the body will remain at rest.

COROLLARY 2

124. But if the forces applied to the body were not prepared thus, so that if all were applied to the same point, they shall be in equilibrium, then the centre of gravity of the body will be moved about the middle of these forces.

COROLLARY 3

125. Therefore if two forces *Ee* and *Ff* may act on the body *ABC* (Fig. 22), of which the one shall act downwards and the other upwards were applied, then the centre of gravity will remain at rest if there were $Ee = Ff$. But if $Ee > Ff$ the centre of gravity will descend, and truly ascend if there were $Ee < Ff$.

COROLLARY 4

126. Therefore whatever forces will have been applied to the body, the motion of the centre of gravity will be able to be determined by this lemma.

SCHOLIUM.

127. But the motion, which the forces from the centre of gravity induce, is not the only effect which the forces produce within the body, nor does it suffice to know the motion of the centre of gravity for the effect of applied body forces. For meanwhile while the centre of gravity either may remain at rest, or it is moved in the manner described, it can happen, that the remaining parts of the body may be revolving around the centre of gravity, and they may adopt a rotational motion, whereby the motion of the centre is not disturbed. On account of which it remains, that we shall investigate any gyratory motion the forces produce applied to bodies moving around the centre of gravitation, indeed this whole effect which the forces are able to produce on rigid bodies, will become well-known,.

LEMMA

128. *If some forces were applied to a rigid body, and thus so that from these forces the motion of the centre of gravity may be obtained, that we have assigned to this, meanwhile the whole body will be moving equally about the centre of gravity, as if the centre of gravity were at rest, or it were fixed.*

DEMONSTRATION

We may consider a new force to be applied to the body at its centre of gravity equal and opposite to that which arises from the addition of all the forces if they were applied to the centre of gravity, therefore the centre of gravity then will be at rest. Now if the motion may be determined in this case, which the forces induce to the body around the centre of gravity, we consider a new force, which we consider applied to the body at the centre of gravity, this will not disturb the motion about the centre of mass. Whereby even if that again may be removed, the motion about the centre of mass will remain unchanged. On this account the body will be moved about the centre of mass in the same manner, as if it were fixed. Q. E. D.

COROLLARY

129. Therefore for the motion requiring to be determined about the centre of mass, it will be allowed to consider the centre of gravity as if it were fixed, since the motion about the centre of gravity shall agree with the rotational motion about a fixed centre of gravity.

SCHOLIUM

130. Therefore so that we may determine this motion about the centre of mass, we will begin by considering the motion requiring to be defined about some fixed point, and we will investigate the rotational motion, which the forces prevail to induce on a body suspended from some fixed point. Indeed for this determination it will be easy to transfer the fixed point to the centre of gravity itself and thus the motion to be assigned, which any forces applied to the body will generate about the centre of gravity.

LEMMA

131. *If a particle C moveable about the fixed point O may be acted on by the force CF* (Fig. 23), *the angle generated in a given time dt about O will vary directly as the product from the force CF by the sine of the angle FCO by the increment of the time dt and inversely as the mass of the particle C drawn from the distance CO.*

DEMONSTRATION

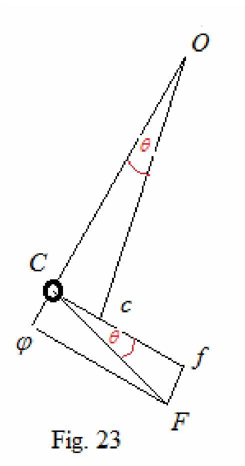


Fig. 23

With the force put in place $CF = p$ and sin ang.FCO $= m = \cos\theta$, the force CF shall be resolved into the two sides following : Cf normal to CO and $C\varphi$ along CO, of which the latter will be taken up completely in extending the thread CO or by leaving the distance of the particle from O unchanged, and thus provides nothing to the motion. Truly the other force in the direction Cf, which will be equal to $= mp$, is completely involved in producing the motion. Therefore in the increment of the time dt, the particle C shall be drawn through the increment of the distance Cc, and there will be the correspondence $Cc = \frac{mpdt}{C} = [(p/C)dt\cos\theta]$. Hence truly the increment Cc divided by the distance CO will give the angle COc [$=\theta$], generated in the time increment dt which consequently will become $=\frac{mpdt}{C\cdot CO}$ $[=\omega dt$ in modern terms$]$. Q.E.D.

COROLLARY 1

132. Therefore if the rotational force shall be made proportional to this angle, which is generated in the given time increment, the rotational force will be as the force *CF* by the sine of the angle *FCO* divided by [the mass of] the body *C* at the distance *CO.*

COROLLARY 2

133. Therefore if the rotational force may always be expressed by this ratio, then the angular motion will become known at once. For the expression of such a rotational force multiplied by the time *dt* shows at once the angle arising about *O* in this time.

$$=\frac{mp}{C\cdot CO}\left[\text{\& on setting } CO=r, \frac{(p/C)d\theta}{CO}=\frac{(acc.\ towards\ O).\omega dt}{r}, \text{ where } \theta=\omega t.\right]$$

COROLLARY 3

134. And if the rotational force found for each body of this kind may be expressed in this manner, then the angular motion is known at once, or what amounts to the same thing, a single body can be assigned so that a certain force may be had at *O* equal to the rotational motion.

SCHOLIUM

135. Since the oblique force pulling a small body *C* may be reduced so easily to a normal force *CO*, and other forces, whose direction shall lie on *CO* itself, shall not affect the rotational motion; in the following, where the rotational motion must be determined, we will consider only normal forces applied to each of the particles of the body. And indeed if *C* were a particle of the body moving around *O*, from this lemma it may now be understood, if only the particle *C* shall be present, with all the remaining parts of the body removed, such a rotational motion will be induced by the force *Cf* normal to *CO* acting on the particle *C*. Clearly the particle *C* will be acted on requiring to describe a circle, whose centre is at *O* and the radius *AO,* and this same circle will be placed in a plane, in where both the right line *CO* together with the direction of the force *Of* are placed. But the force by which this circle requiring to be described may be acting, or the rotational force, as we will call that, will be equal to the force *Of* divided by the product formed from the mass of the particle *C* and the distance *CO.*

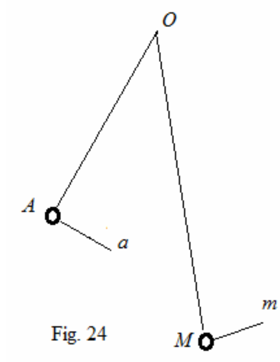


Fig. 24

LEMMA

136. *The body with the mass,*

$$M = \frac{A \cdot AO^2}{MO^2} ,$$

can be substituted at the distance MO, in place of the body A, for the rotational motion being excited around O by the force Aa (Fig. 24)*, which body is placed in the same plane as AO and Aa, so that it may be acted on by the force* $Mm = \frac{AO \cdot A}{MO}$ *for the rotary motion, the direction of which shall be situated in the same plane, and tending to move in the same plane and into the same region as Aa.*

DEMONSTRATION

Where in place of the force Aa applied at the distance AO , in the first place the force Mm shall be substituted to be applied at the distance MO, so that the forces MO and Mm shall be placed at AO and Aa in the same plane ; then truly it is required that the moments of the forces Aa and Mm shall be equal, from which there follows to become $Aa \cdot AO = Mm \cdot MO$, or

$$Mm = \frac{Aa \cdot AO}{MO}.$$

Indeed with this agreed on, the force Mm will establish the same effect about O, as the force Aa has about O. Besides truly the body at M need only be relocated so that the rotational force at M and A about O shall be equal, which will happen, if there were

$$\frac{Mm}{M \cdot MO} = \frac{Aa}{A \cdot AO}.$$

Whereby therefore so that the rotational force of each shall be the same, it is necessary that there shall be

$$M = \frac{A \cdot AO \cdot Mm}{MO \cdot Aa} = \frac{A \cdot AO^2}{MO^2}$$

on account of

$$Mm = \frac{Aa \cdot AO}{MO}.$$

Q.E.D.

COROLLARY 1

137. Likewise therefore is the direction assumed for the right line *MO,* only if that shall be placed in the same plane, in which the right lines *AO* and *Aa* have been placed. And in a similar manner the extent of the force *MO* may be taken arbitrarily.

COROLLARY 2

138. Moreover, with the length of the right line *MO* placed in the plane *OAa*, at the same time both the body being placed at *M,* as well as the force *Mm* being applied to that, are determined.

COROLLARY 3

139. Therefore in place of any particle *A* rotating about the fixed point *O,* another particle can be substituted at a given distance, so that at once the same effect may be established in the rotational motion generated about *O*, as that which would be produced by that particle *A* at the distance *AO.*

LEMMA

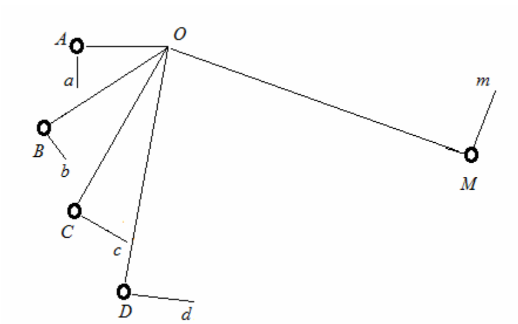


Fig. 25

140. *If there were several particles A, B, C, D* (Fig. 25), *firmly connected to each other, and situated in the same plane which shall be acted on by the forces Aa, Bb, Cc, Dd, the directions of which shall be placed in the same plane as well, so that the common rotational force around the fixed point O shall be placed in the same plane, the strength of the common rotational force will be*

$$=\frac{Aa\cdot AO+Bb\cdot BO+Cc\cdot CO+Dd\cdot DO}{A\cdot AO^2+B\cdot BO^2+\ C\cdot CO^2+D\cdot DO^2}.$$

[We would now call such a quantity the angular acceleration α, defined in familiar terms by the equation : $\dfrac{torque\ \ \tau}{moment\ of\ \ inertia\ I}=angular\ acceleration\ \alpha.$]

DEMONSTRATION

With an arbitrary right line *OM* taken in the plane, in which just as many particles are situated with the point *O,* as there are the directions by which they are disturbed, in place of the particle *A* with the force *Aa* acting there may be substitute for *M* the particle $\frac{A\cdot AO^2}{MO^2}$ with the exciting force $\frac{Aa\cdot AO}{MO}$ (§136), and in a similar manner in place of the particles *B, C, D* with the exciting forces *Bb, Cc, Dd,* the particles at *M* may be substituted

$$\frac{B\cdot BO^2}{MO^2},\ \frac{C\cdot CO^2}{MO^2},\ \frac{D\cdot DO^2}{MO^2}$$

with the forces

$$\frac{Bb\cdot BO}{MO},\frac{Cc\cdot CO}{MO},\frac{Dd\cdot DO}{MO}$$

acting. Therefore with this done in place of all the particles acted on by their own forces the body can be substituted a *M*

$$\frac{A\cdot AO^2+B\cdot BO^2+C\cdot CO^2+D\cdot DO^2}{MO^2},$$

which will be acted on in the direction *Mm* by the force which is

$$=\frac{Aa\cdot AO+Bb\cdot BO+Cc\cdot CO+Dd\cdot DO}{MO},$$

Therefore the rotary strength of this body will be

$$=\frac{Aa\cdot AO+Bb\cdot BO+Cc\cdot CO+Dd\cdot DO}{A\cdot AO^2+B\cdot BO^2+C\cdot CO^2+D\cdot DO^2}$$

to which the rotary strength is equal, which arises from the rotary strengths of all the bodies *A, B, C, D,* taken together. Q.E.D.

COROLLARY 1

141. Therefore the rotational force of several particles of this kind is equal to the sum of the moments of the individual forces about the fixed point *A*, divided by the sum of the products formed from the individual particles multiplied by the squares of these distances from the fixed point *O,* or from the axis about which the rotational motion occurs.

COROLLARY 2

142. Therefore the angle, through which the system of particles *A,* B, *C, D* will rotate through in the increment of time *dt* about the fixed point *O* will be equal to

$$= dt\frac{Aa\cdot AO+Bb\cdot BO+Cc\cdot CO+Dd\cdot DO}{A\cdot AO^2+B\cdot BO^2+C\cdot CO^2+D\cdot DO^2},$$

from which the rotational motion will become known.

COROLLARY 3

143. If some number of corpuscles may be drawn in the opposite direction, then these for which the moments of the forces acting on the with respect to the remainder will become negative, and thus must be subtracted in the expression found.

COROLLARY 4

144. Therefore with the same remaining forces acting about the fixed point *O*, thus the rotational force will be greater, for which the denominator of the fraction found will have been greater, that is where the individual particles of the body will have been situated closer to the point *O*.

COROLLARY 5

145. If the angles, which constitute the directions of the forces acting with the right lines drawn from the point *O*, were not right angles, then the individual products $Aa\cdot AO, Ab\cdot BO$ etc. in addition must be multiplied by the sines of the angles *OAa, OBb* etc. respectively ; with unity taken for the whole sine.

COROLLARY 6

146. Therefore the rotational force will vanish, if the moments of the forces acting on the body at a fixed point may cancel each other out; therefore in this case the body will remain at rest.

SCHOLIUM

147. Therefore these are the lemmas, for which there shall be a need for explaining the motion of bodies floating on water ; from which it is understood well enough, if the forces shall be applied to some rigid body, then the centre of gravity of this to be moved in the same manner, as if the whole mass of the body shall be concentrated at its centre of gravity, and for each of all these forces to be applied together in their own direction.

Truly in addition the whole mass of the body meanwhile will be moved about its centre of gravity in the same way, as if the centre of gravity were fixed, and here the rotational motion will become known from the moments of all the forces about the centre of gravity. Indeed if these moments may cancel each other out, then clearly the body will have no rotational motion about its centre of gravity ; therefore this rotational motion truly will be greater where the moment arising from all the forces together at the centre of gravity were greater. Indeed in the motion of bodies floating on water this same rotational motion is stopped at once, if which arises, and on this account in this case it will suffice to determine that situation of the body in which the rotational motion shall stop; for which the lemmas requiring to be adapted are already defined.

PROPOSITION 15

THEOREM

148. *If a body sitting on the surface of water were not in equilibrium were not in equilibrium, then it will be turning about its centre of gravity, and meanwhile the centre of gravity either will be at rest, or moving upwards or downwards.*

DEMONSTRATION

If the body may be immersed in water in some manner it will be acted on by two forces, of which one will be the equal to the weight of the body, and it acts vertically downwards through the centre of gravity, the other truly is equal to the weight of the water equal to the submerged part, and forces the body directly upwards in direction passing through the centre of the submerged part of the magnitude. Therefore since the centre of gravity may accept the same motion from these two same forces, and if the whole mass of the body shall be concentrated there and it may be acted on by each force jointly, it is evident the centre of gravity either remains at rest or it must it must move directly upwards or downwards. But unless the direction of the force, which act on the body, shall pass through the centre of gravity, then the body meanwhile will be turned around the centre of gravity. Q. E. D.

COROLLARY 1

149. Therefore it is understood, if an exceedingly large part of the body were immersed in water, then on that account the force urging upwards to be greater than the other, and the centre of gravity must rise ; truly on the other hand if the immersed part were exceedingly small, the centre of gravity must fall.

COROLLARY 2

150. Therefore if as great a part of the body were immersed in the water, as the amount required for equilibrium, then the centre of gravity remains at rest on account of the forces acting being equal and opposite to each other.

COROLLARY 3

151. Therefore if as great a part were submerged in the water, as the amount required for equilibrium, then the body itself on being restored to equilibrium will remain at rest about the centre of gravity.

SCHOLIUM

152. Hence moreover it does not follow that, if one part of such a size were submerged in the water, it is necessary for just as large a part to be produced for equilibrium, while the centre of gravity, which was turned for a long time, at last must be at rest. For indeed during the rotation certainly it can happen, that the part of the body present under the water, which indeed may be changing continually, may become just as large or smaller. Therefore since due to this it may happen, to be necessary that at once the centre of gravity may be raised or lowered. But the motion both of the centre of gravity as well as of the inversions will leave no record, when the body will have arrived at a state of equilibrium, but just as it happens in the motion of pendulums, sometimes it will be inclined contrariwise and it will revert, and this oscillatory motion will endure for a long time, until everything may be absorbed by the resistance.

PROPOSITION 16

PROBLEM

153. *For a plane figure resting vertically on water EAHBF* (Fig. 26) *to be dislodged from its state of equilibrium, to determine the motion by which it may be restored to its state of equilibrium.*

SOLUTION

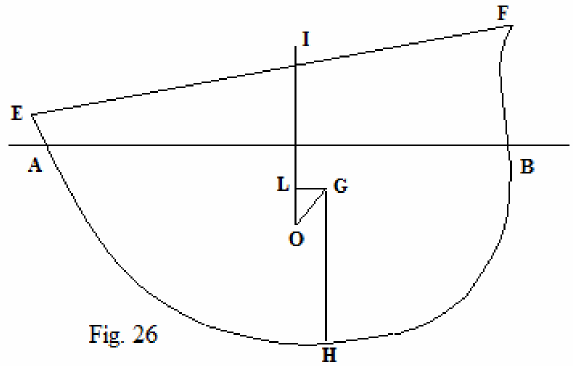


Fig. 26

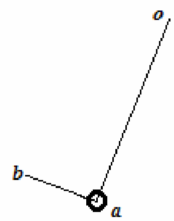


Fig. 27

AHB shall be the part immersed in the water, *O* its centre of magnitude, and *G* the centre of gravity of the whole figure, which will be placed in the centre of the plane *AHB*. Therefore this shape will be acted on by the pressure of the water upwards in the direction of the vertical line *OI* by a force equal to the weight of the water, occupying the volume *AHB*; truly it will be forced downwards in the direction *GH* by a force equal to the weight of the figure. So that now it may be concerned with the motion of the centre of gravity *G*, that will be propelled upwards, if the force *OI* were greater than the force *GH;* truly it will be pressed downwards if the force *GH* may overcome the force *OI*; finally in the case where these two forces are equal to each other, the centre of gravity *G* will remain at rest. Thence either the centre of gravity shall be at rest, or may rise or fall, and the whole figure will turn about that point, unless the directions *OI* and *GH* lie on the same vertical line ; which rotational motion is understood to arise from the moments of these forces about the centre of gravity. Indeed the force *GH,* which certainly passes through the centre of gravity itself, the moment is zero, but the moment of the other force *OI* is *OI.GL,* with *GL* drawn from *G* normal to the right line *OI*. Therefore this figure will be moved around the centre of gravity *G* by this force in the direction *BHA*. But the magnitude of this rotational force will depend above on the sum of the products from the individual particles of the figure by the square of their distances from the centre of gravity *G,* of which if the sum of the products may be called *S,* the rotational force will be $= \frac{OI \cdot GL}{S}$. Q.E.I.

COROLLARY 1

154. Therefore the accelerative thrust [i.e. α and the moment τ], by which the figure may be rotated around *G* thus will be greater, where both the force *OI* as well as the distance *GL* were greater, and where the sum of the products from the figures of the individual particles into the square of the distances from the centre of gravity *G* were smaller [*i.e.* the moments of inertia of the moving parts].

COROLLARY 2

155. Therefore with *OI* and *GL* remaining unchanged the strength of the figure requiring to be restored thus will be greater, where *S* were smaller, that is where all the particles of

the figure were closer to the centre of gravity *G.* Truly on the other hand the restoration there will become slower, where the greater particles shall be more removed from the centre of gravity.

COROLLARY 3

156. The rotational force, by which the figure may be rotated around the centre of the circle, can be reduced to the force by which a simple body *a* may be made to rotate about the fixed point *o* by the force *ab*. Clearly this body *a* will rotate with the same velocity, as that by the figure *EHBF,* if the *ab* force were $=\frac{a\cdot ao\cdot GL}{S}\cdot OI$. (Fig. 26, 27).

COROLLARY 4

157. Therefore from the given individual moments, while the figure is rotating, the position of the centres of gravity and of the changeable magnitude, the angular velocities can be will be able to be determined from some moments, and the time whereby the whole restitution is resolved.

SCHOLIUM I

158. If the body resting on the water were a cylinder, of which all the transverse sections shall be similar and equal to the figure *EAHBF,* it is evident this body is going to move in water in a similar manner to the figure *EAHBF* only. On account of which, if a cylindrical body of this kind thus were displaced from its state of equilibrium, so that all these sections were to remain permanently in place vertically, then the centres of gravity of the individual sections will rotate about the horizontal right line passing through the centre of gravity *G* of this body, until it arrives at a state of equilibrium. And just as the motion of this restoration will agree entirely with the motion of the figure *EAHBF* and may be resolved by the same speed. Therefore the force, by which this body, while equal and similar parts of the individual sections *AHB* themselves are submerged in the water, will be rotated around the axis passing through the centre of gravity *G* by all the sections, by a force equal to the weight of the water in the volume of the total part submerged to be exerted in the interval *LG ,* and to be divided by the sum of the masses of all the particles of the body each multiplied by the squares of the distances of the same from the axis of the rotation.

SCHOLIUM 2

159. Just as this cylindrical body, of which all the transverse sections not only are equal, but also the loads are similarly distributed by these, can rotate about a fixed axis passing through the centre of gravity, thus also the same can eventuate in other bodies of each kind, so that they may be able to rotate about a fixed axis. But generally this same motion of restitution to the state of equilibrium does not occur about an axis at rest, but equally about a centre of gravity accustomed to be moving, which motion of the axis itself may

be determined with difficulty. Yet meanwhile always, whatever were the motion of a body about the centre of gravity, just as if these may be observed from the motions made about two axes, which considered thus, so that while the body may be turning about a certain axis, at the same time also it may be considered to be turning also about another axis. Indeed why may it not be able to happen, that the motion about the centre of gravity may be composed from the motions about three axes. On account of which were a motion of this kind to be allowed, to be accomplished more carefully, before these laws of rotational motion about a fixed axes may be investigated by us, which then also for motion composed from other kinds of motion may be able to be accommodated.

LEMMA

160. *If some forces were applied to a body, which is transfixed by the axis EF* (Fig. 28)*, the rotational force about the axis EF will be equal to the sum of the moments of all the forces applied to the axis EF divided by the sums of all the [masses of the] particles of the body multiplied by the squares of the distances of the same from the axis EF.*

DEMONSTRATION

From the points *A, B, C, D* at which the forces are applied, some perpendiculars *Aa, Bb, Cc, Dd* may be dropped to the axis *EF*; but is evident the force *B* applied to have the same effect on the body by rotating around *EF*, as if the right line *Bb* were to be moved to some other point of the axis. On account of which all the points *a, b, c, d* may be considered to fall on the same point *a*, thus so that at the same time all the particles of the body in the plane normal to *EF* may be arranged passing through *a*. With which done the whole body to be reduced in a like manner will be acted on in this plane by the forces for rotation about the fixed point *a* as before. Whereupon the rotational force will be equal to the sum of the moments of all the forces acting on the axes *EF* divided by the sum of all the particles multiplied by the squares of their distances. Q.E. D.

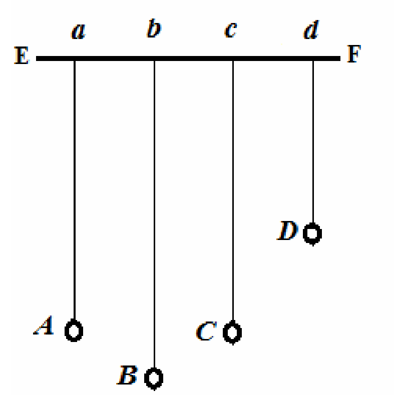


Fig. 28

COROLLARY 1

161. Therefore if the rotary force must be determined acting on the body, which is present around the fixed axis arising from some applied forces, in the first place the individual particle of the body are required to be multiplied by the squares of their distances from that fixed axis, and to be gathered into one sum.

COROLLARY 2

162. Truly with this sum found of all the products from the individual particles into the squares of their distances from the fixed axis, the moments of the forces with respect to this axis must be found, the sum of which divided by that sum will show the rotational force about the axis.

COROLLARY 3

163. Moreover with the rotational force known, if that may be multiplied by the element of the time, the product will give the element of the rotational speed generated in this time; from which the whole rotational motion will be able to be determined in the same manner where the motion of the body by a force acting in a given direction.

SCHOLIUM

164. From the demonstration the given rule for finding the rotational force, equally has a place, whether the axis may or may not pass through the centre of gravity of the body. Yet meanwhile from the reasons brought forth above, another axis to be at rest itself is not possible, unless which axis passes through the centre of gravity. Therefore this rule also will take care of the situation, whereby, if the body were mobile about another axis indeed not passing through the centre of gravity, truly which may be held in place by another fixed force.

DEFINITION

165. The sum of all the particles of the body multiplied by the squares of the distances respectively of these from this axis to be called the moment of inertia.

COROLLARY

166. Therefore if forces were applied to a body moveable about some fixed axis, the rotational force will be expressed by the sum of the moments of all the forces with respect to this axis, divided by the moment of inertia of this body with respect to this same axis.

COROLLARARY 2

167. Since the axes, about which a given body can be moved, shall be able to be infinite, any body also will have infinite moments. On this account if the discussion is concerned with the movement of the body, the axis must be defined likewise, with respect of which the moment of inertia can be taken, just as has been declared in the definition.

SCHOLIUM 1

168. It is observed to have introduced this quantity, which I shall call the moment of inertia, for the sake of brevity ; so that the fuller descriptions which shall be encountered most often may be avoided. Indeed, just as in rotational motion, the moment of the force may now be called the force applied for that effect; thus also with rotating bodies the sum of all these products is called the moment of inertia of the body, since in turn the masses or inertias of the bodies shall sustain the effect of the applied forces.

[Thus: Euler compares rotational motion with linear motion: forces are replaced by the moments of forces about an axis, masses by moments of inertia about the same axis, and the acceleration arising becomes the angular acceleration about this axis, which he calls simply the acceleration, which we now call the angular acceleration about this axis.]

SCHOLIUM 2

169. So that we may acquire more clearly the notion of moments of inertia of bodies, it will help to have investigated the moments of inertia of some bodies, which also will be able to be used in the following developments. Moreover I shall consider only the axes passing through the centre of gravity, and I will determine the moments with regard to these; since below I shall be going to show, how from the given moments of inertia with regard to the axis passing through the centre of gravity, the moments of inertia with regard to some other axis shall be able to be found.

EXAMPLE 1

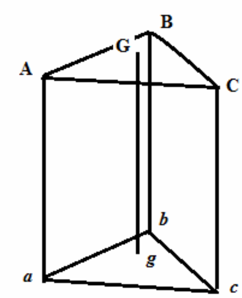


Fig. 29

170. *ABCabc* shall be a right triangular prism constructed from some uniform material, of which the mass shall be $= M$ (Fig. 29). The vertical axis *Gg* shall pass through its centre of gravity, which likewise will pass through the centres of gravity of the individual sections, the moment of inertia of this prism with respect of this axis will be :

$$= \frac{M\left(\mathrm{AB}^2 + AC^2 + BC^2\right)}{36}.$$

EXAMPLE 2

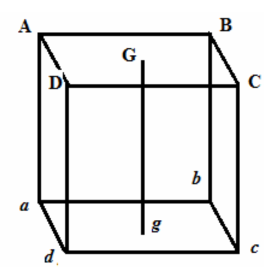


Fig. 30

171. *ABCDabcd* shall be a right angled or parallelepiped prism, of which the bases *ABCD* and *abcd* shall be parallelograms, constructed from a uniform material and having the mass equal to $= M$ (Fig. 30). The axis *Gg* shall be drawn through its centre of gravity, passing through the centres of gravity *G* and *g* of

each end, the moment of inertia of this parallelepiped with respect to this axis will be

$$= \frac{M\left(\mathrm{AB}^2 + BC^2\right)}{12}.$$

Therefore the bases shall be either rectangular or oblique angled , the moment of inertia is determined in an equal manner from the sides and mass only.

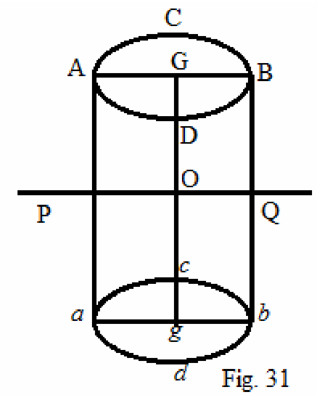


Fig. 31

EXAMPLE 3

172. *ABCDabcd* shall be a right cylinder, the bases *ABCD* and *abcd* of which shall be circles, constructed from a uniform material, and of which the mass shall be *M* (Fig. 31). The centres of the bases *G, g,* of the cylinder shall be joined by the axes of the cylinder, and the moment of inertia of this cylinder with respect to the axis $Gg = \frac{M \cdot AG^2}{2} = \frac{M \cdot AB^2}{8}$.

COROLLARY

173. If pyramids and right cones may be produced from the bases and altitudes of the preceding prisms and cylinders, the moments of inertia of these with respect to the same axis will be five times smaller than the moments of the prisms and cylinders. Indeed the masses will become three times the former, and the latter moments of inertia to the former shall maintain the ratio 3 : 5.

EXAMPLE 4

174. But if in the same cylinder the transverse axis *PQ* shall be drawn through the centre of gravity *O* normal to the first axis *Gg,* the moment of inertia of the cylinder with respect to this axis will be $PQ = M\left(\frac{Gg^2}{12} + \frac{AB^2}{6}\right)$.

EXAMPLE 5

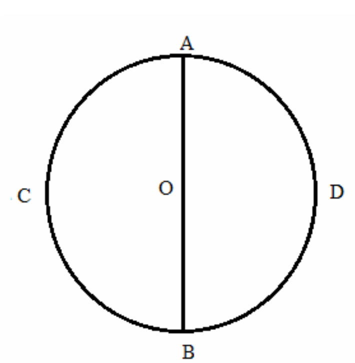


Fig. 32

175. If a globe were made from uniform material, of which the mass is *M* (Fig. 32), the moment of inertia shall be required with respect of the diameter *AB,* likewise this may be taken about some diameter, here the moment will be found

$= \frac{2}{5} M \cdot AG^2$ or $= \frac{M \cdot AB^2}{10}$. Namely to be the product from the mass into the square of the tenth part of the diameter gives the moment of inertia with respect to any axis passing through the centre.

EXAMPLE 6

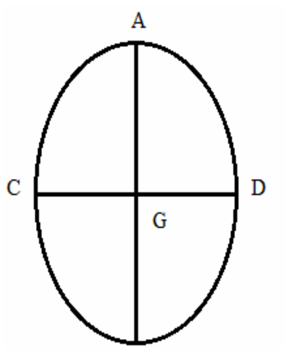


Fig. 33

176. If the body were an elliptic spheroid, arising from the rotation of the ellipse *ACBD* about the axis *AB,* constructed from a uniform material, and its mass $=M$ (Fig. 33), its moment of inertia with respect to the axis *AB,* which likewise is the axis of the spheroid, will be $=\frac{M\cdot CD^2}{10}$. Truly the moment of inertia of this same spheroid with respect to the axis *CD* normal to the first axis will be $=\frac{M\cdot\left(AB^2+CD^2\right)}{20}$. Truly the volume of the sphere of diameter *AB* to the volume of this sphere as AB^2 to CD^2.

SCHOLIUM 3

177. The manner, in which the moments of inertia of these bodies have been found, has been seen to be superfluous here, since it may be agreed to be purely an analytical matter, and for the others, who will have treated the calculation of the centre of the oscillation, the exposition shall be satisfactory enough. But where, if the bodies were more composite and irregular, or if the moments with respect of some other axes were required, which do not pass through the centre of mass of the body, so that the whole matter may be resolved without the greatest tedium, I have attached the following lemma.

PROBLEM

178. *With the moment of inertia given with respect of the axis AB passing through the centre of gravity G* (Fig. 34), *to find the moment of inertia with respect to any other axis CD you please, to be parallel to the first axis AB.*

SOLUTION

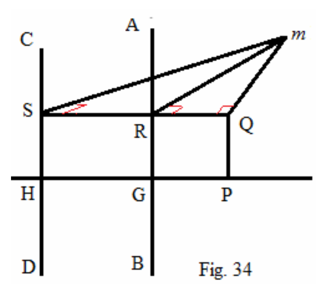


Fig. 34

The mass of the body shall be $=M$ and its moment of inertia with respect to its axis $AB=S$. The right line *HG* shall be drawn cutting the parallel axes normally, where the distance of the axis *GH* may be had, or the distance of the centre of gravity of the body from the axis *CD,* with respect to which the moment of inertia is sought. Some small part *m* of the body may be considered, and from that the perpendicular *mQ* may be dropped to the plane *ABDC,* in which the axes are situated. *QRS* shall be acting parallel to *GH,* and likewise *mR* and *mS,* which shall be normal to the

axes. Therefore the question will be satisfied if the sum of all the terms $m \cdot mS^2$ may be defined; as we have indicated by $\int m \cdot mS^2$.
[Note again the use of *m* as a point on the diagram, as well as *m*, which would now be written as the differential *dm* in the integral; however, we will keep the original notation.]
Indeed the moment of this force with respect to the axis *AB* is $= \int mR^2$, which since it may be given, will be $\int m \cdot mR^2 = S$; but $\int m$ or the sum of all the small parts of the body shall be equal to the whole mass *M.* Now since there shall become [there is a misprint in this formula in the original text used]

$$mS^2 = mR^2 + SR^2 + 2SR.\,RQ$$

there will become

$$\int m \cdot mS^2 = \int m \cdot mR^2 + \int m \cdot SR^2 + 2\int m \cdot SR \cdot RQ$$
$$= \int m \cdot mR^2 + SR^2 \cdot \int m + 2SR\int m \cdot RQ$$

on account of $SR = GH$, and thus to be constant. Moreover since *AB* passes through the centre of gravity *G* of the body, there will be by the known property of the centre of gravity, $\int m \cdot RQ = 0$.
Whereby since there shall become $\int m \cdot mR^2 = S$ and $\int m = M$ the moment of inertia sought of the proposed body with respect to the axis *CD* shall be $= S + M \cdot GH^2$. Q.E.I.

COROLLARY 1

179. Therefore the moment of inertia of the body with respect to the axis *GD* is equal to the moment of inertia of the same body with respect to the axis *AG* passing through the centre of gravity *G,* together with that made from the square of the distance of the centre of gravity *G* from the axis *GD.*

COROLLARY 2

180. Therefore from the given moment of inertia of the body with respect to a certain axis passing through the centre of gravity, the moment of inertia of the same body with respect to some other axis parallel to that axis will be determined easily.

COROLLARY 3

181. Therefore if the body, the moment of inertia of which is sought with respect to some axis, shall be composed from several parts, the individual moments of inertia of which may be given with respect to axes parallel to that axis passing through the centre of gravity of each part, the moment of inertia sought will be equal to the sum of all the moments of inertia of the parts together with the products from the individual parts multiplied by the squares of the distances of each centre of gravity from the axis.

COROLLARY 4

182. Hence the method therefore works more easily, where the moment of inertia of the body can be found for some axis, before doing an especially composite calculation.

SCHOLIUM

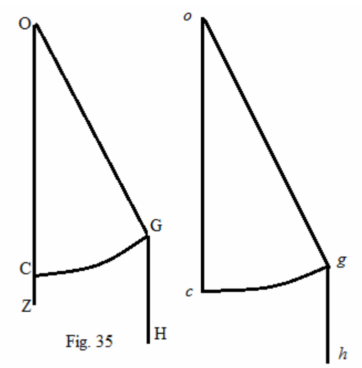


Fig. 35

183. From these an easy and especially natural way follows for determining the centre of oscillation for any kind of body oscillating about the axes, even though it may not seem to belong here, yet it will be appropriate to put this in place. There shall be some body, which may oscillate about a horizontal axis passing through *O* (Fig. 35), and *C* a point placed on the vertical line *OC*, at which the centre of gravity of the body will fall, thus so that *OC* shall be the distance of the centre of oscillation from the axis. Moreover we may consider another body thrust away out from the vertical position, and its centre of gravity at *G,* thus so that with that describing the angle *GOC*, until it may reach the position of equilibrium. But the force, which disturbs the body to produce this angular motion, is the weight of the body, by which it is pushed downwards in the direction *GH*. Now if the mass or weight of this body $= M$, and its moment of inertia, with respect to the axis passing through the centre of gravity *G,* and thus parallel to the axis of the other oscillating body $= S$, the moment of inertia of this [composite] body with respect to the axis of oscillation will be $= S + M \cdot OC^2$. But the moment of the weight requiring to generate the angular motion about *O* is $M \cdot GO \cdot \sin O$; and thus the rotational force [*i.e.* the angular acceleration] will be $= \frac{M \cdot GO \cdot \sin O}{S + M \cdot OC^2}$. Now we may consider a simple pendulum *og* placed at an angle equal to *goc* at a distance *oc* from the vertical, to which at *p* an infinitely small weight attached at *g* shall be swinging about *o*, the rotational force, by which the small weight *p* is going to resolve the angle *goc* to be acting $= \frac{\sin o}{oc}$. Therefore if this rotary force were equal to the former, the simple pendulum *og* and the composite pendulum *OG* will arrive at the same vertical position at the same time, since each traversing through the same angle.

Therefore we may put $\frac{M\cdot GO\cdot\sin O}{S+M\cdot OC^2}=\frac{\sin o}{oc}$
and there becomes

$$oc=\frac{S+M\cdot OC^2}{M\cdot CO},$$

which is the length of the simple isochronous pendulum, or the distance of the centre of oscillation for the composite pendulum *OC* from the axis of the oscillation, therefore the centre of oscillation shall be at *Z,* so that there shall become

$$OZ=CO+\frac{S}{M\cdot CO};$$

from which it is apparent the centre of oscillation always to fall below the centre of gravity *G*, and the extra distance to become

$$CZ=\frac{S}{M\cdot CO}.$$

HYPOTHESIS

184. *In all bodies floating in water* (Fig. 36) *truly for ships especially, it is allowed to consider three axes passing through the centre of gravity G normal to each other, the first evidently CGD, the second the horizontal RS parallel to the keel AGB, situated in the plane of the diameter ARSB and the third EGF equally horizontal, if indeed the ship were in a state of equilibrium, and normal to the previous AGB. Then it is allowed to put a body of this kind thus able to turn with some forces acting about each one of its axis, so that the rotary motion about one of these axes shall not be disturbed by the rotational motions about the remaining axes.*

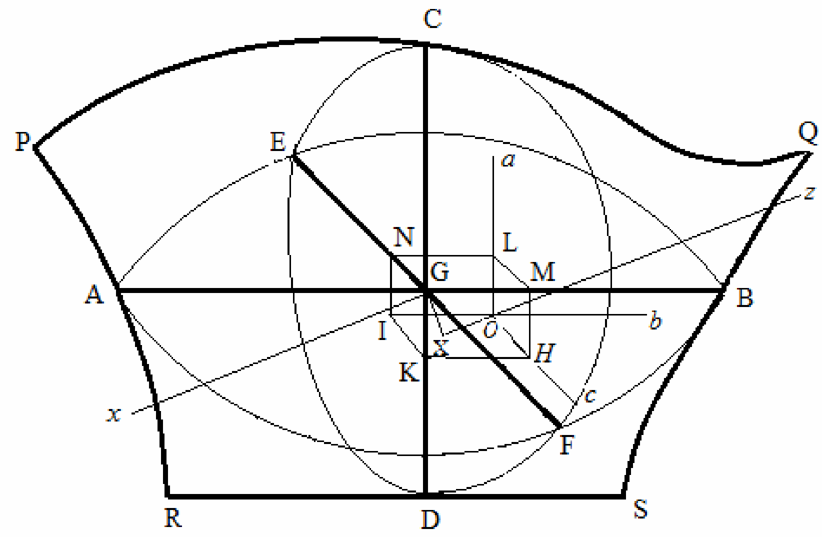


Fig. 36

SCHOLIUM 1

185. It is known well enough from above that a body is unable to rotate about another fixed axis , unless all the centrifugal forces shall cancel each other out. On account of which if forces acting may try to rotate the body about another axis, a motion will arise with the greatest irregularity, when also the axis may be inclined; which motion even now is defined with the greatest difficulty. Therefore a solution may be brought forth for this inconvenience of such an irregular motion may be able to be resolved in two or three rotational motions can be made around the fixed axis at the same time ; then indeed with a known motion about some separate axis, the whole motion then will be deduced easily. Although moreover this resolution may not succeed accurately, yet if in practice we may consider, with care it will be able to be used well enough, if these three axes were normal to each other; then certainly the motion around one will be disturbed minimally by the motion around the others. Besides truly if these axes shall be prepared thus so that a body may be able to rotate about any of these fixed axes, that resolution therefore will agree more with the truth. But in ships, for which this treatment to be put in place to be especially adapted, three axes of this kind either will be actually true, or usually approximately true. For any rotation of the ship about the vertical axis *CD* can remain unchanged, and also about the axis *AB,* certainly which is situated in the diametrical plane; but the third axis *EF* takes the precedence, just as experience prevails well enough to demonstrate.

COROLLARY 1

186. Therefore if one or more forces were applied, the effect both in moving the body forwards as well as by rotating about the centre of gravity will be determined easily from the precepts treated thus far. Indeed in the first place all the forces may be considered to be applied in directions parallel to the centre of gravity itself, and from these the progressive motion of the centre of gravity can be concluded. Then the moments of the individual forces on these three axes may be sought, from which the rotational motion about any axis will become known. And finally from these motions gathered together from the rotational motions between themselves, truly the motion around the centre of gravity will be collected together accurately enough.

COROLLARY 2

187. Therefore three moments of inertia are required to be determined for this motion, clearly with regard to the three axis around which the rotation is considered to be made.

COROLLARY 3

188. Therefore if three axes of this kind may be given normal to each other in the body, and the moment of the body with regard to each axis were assigned, then some forces may disturb this body, truly the motion produced by these which will be able to be determined approximately: moreover, towards this end I have added the following lemma.

LEMMA

189. *If the body PARSBQ* (Fig. 36) *may be acted on by some forces, to find the motion which may be generated in the body.*

SOLUTION

It is agreed from the above from any forces acting on the body two kinds of progressive motion to be generated in the body : evidently of the centre of gravity, and of rotations about the centre of gravity ; of which the first motion is defined if the whole body may be considered to be concentrated at the centre of gravity, and all the forces may be put in place applied to that along parallel directions. Truly for the rotational motion requiring to be determined, *AB, CD* and *EF* shall be those axes drawn through the centre of gravity G and normal to each other, around any of which as if immobile, a body may be able to rotate. Now the moment of inertia of the body with respect of the axis $CD = P$, the moment with respect of the axis $AB = Q$, and the moment with respect of the axis $EF = R$. Now we will consider some one of the forces acting *OZ,* which shall be applied at the point *O*, or its direction shall pass through O . From this point *O* in the plane *ADBC* the normal *OH* may be drawn, and likewise the normal *OL* in the plane *AEBF,* and the normal *OI* in the plane *CEDF,* and with the lines joined *LN, LM, MH, HK, KI* and *NI* there will be had the rectangular parallelepiped *NGMLOHKI.* Then the force *OZ* at the point *O* is resolved into the three forces *Oa, Ob, Oc,* the directions of which shall be normal to each other and parallel to the axes *GD, AB* and *EF.* Now it is evident the moment of the force *Oa* with respect to the axis *AB* to become $Oa.\ LM = Oa \cdot GN$; likewise the moment of the force with respect to the axis *EF* to become $= Oa \cdot LN = Oa \cdot MG.$ In a similar manner the moment of the force *Ob* with respect to the axis *CD* will be $= Ob \cdot IK = Ob \cdot GN$, and the moment with respect to the $EF = Ob \cdot IN = Ob \cdot GK.$ Finally the moment of the force *Oc* with respect to the axis *CD* will be $= Oc \cdot HK = Oc \cdot GM$ and the moment with respect to the axis $AB = Oc \cdot HM = Oc \cdot GK$. Therefore the body will be acted on by two forces around each axis which either agree or oppose each other. Therefore with the like or unlike natures of the moments of the forces, the moment of the force *OZ* proposed for the body around the axis *CD* requiring to be changed to become $= Ob \cdot GN + Oc \cdot GM$. Truly the moment with respect to the axis *AB* will be

$$= Oa \cdot GN - Oc \cdot GK.$$

And the moment with respect to the axis *EF* will be $= Oa \cdot GM + Ob \cdot GK$. In a similar manner the remaining body forces acting are required to be resolved, and the moments of these to be sought on the individual axes, which according to whether these moments either favor or repel each other, for the sign + or - being required to be added to these. Therefore *p* may be put for the moment of the forces with respect to the axis *CD; q* for the moment with respect to the axis *AB;* and *r* for the moment with respect to the axis *EF.* Therefore with these found the rotational force around the axes $CD = \frac{p}{P}$; the rotational force with respect to the axis $AB = \frac{q}{Q}$; and the rotational force with respect to the axis $CD = \frac{r}{R}$; which forces since they arc equally jointly, and separately, will become known as the true rotary force. Q. E. I.

COROLLARY 1

190. If the direction of the force acting *OZ* shall pass through the centre of gravity of the body *G*, and the point *O* may be taken at *G* then the parallelepiped *GNLMHOIK* will vanish, and therefore all the rotational forces will depart into zero, so that indeed it agrees elsewhere.

COROLLARY 2

191. If the direction of forces acting were parallel to one axis *OZ,* then the body will not be able to turn about this axis, but only about the two remaining axes.

SCHOLIUM

192. Since the direction of the force acting is a right line, it would be pleasing to take some point O on that, at which the resolution can be put in place: from which doubt may arise, whether the same forces, whether the same rotational forces shall be going to be produces around the individual axes, with the point O changed, or truly otherwise. But what has to be given more attention, it will be easily understood, at whatever location of the right line OZ the point O may be taken, the same moments with respect to the axis must be found.

PROPOSITIO 17

PROBLEM

193. *To determine the motion of a body of any kind resting in water, displaced from its equilibrium state, by which it shall be restored to the state of equilibrium*.

SOLUTION

PARSBQ shall be the body (Fig. 36), the restoration of which we seek to its equilibrium state that it maintains in water, when displaced from that state of equilibrium ; G shall be the centre of gravity, and CD, AB and EF its three axes normal to each other and unmoved, around which some body may be able to rotate freely. The mass or weight of this body shall be $= M$ and its moment of inertia with respect to the axis $CD = P$; the moment with respect to the axis $AB = Q$ and the moment with respect to the axis $EF = R$. Now we may put this body thus to float on water, so that the centre of magnitude of the submerged part shall be at O, and the line OZ shall be normal to the surface of the water, or vertical ; N shall be the weight of water equal to the submerged part; therefore this body will be pressed upward from the pressure of the water in the direction OZ by a force $= N$. Likewise truly it will be forced downwards by the weight M in the direction GX on passing through the centre of gravity G. Therefore the centre of gravity G will be forced upwards or downwards by these two forces, just as N will have been either more or less than M, that is by itself apparent. But just as meanwhile the body shall be turning about the centre of gravity G, it may be defined in the following manner from the force OZ only; since the other force GX shall pass through the centre of gravity G. From O the parallel right lines OLa, bOI, OHc may be drawn to the individual axes, and there shall be $\cos \text{ang.} ZOa = a$; $\cos \text{ang.} ZOb = b$ and $\cos \text{ang.} ZOc = c$. With these in place if the force N pulling in the direction OZ may be resolved into the three forces pulling in the nearby directions Oa, Ob, et Oc, there will become $Oa = Na$; $Ob = Nb$ and $Oc = Nc$. From these compared with the preceding lemma, the rotational force will be found around the axes:

$$CD = \frac{N\left(b \cdot GN + c \cdot G\mathrm{M}\right)}{P}$$

acting in the sense *AEBF.* Truly around the axis *AB* the rotational force will be

$$= \frac{N\left(a \cdot GN - c \cdot GK\right)}{Q},$$

acting in the sense *ECFD.* Finally the rotational force around the axis *EF* will be

$$= \frac{N(a \cdot GM + b \cdot GK)}{R}$$

acting in the sense *BCA.* Which rotational forces, if they may be considered at the same time, the true rotational motion about the centre of gravity will be obtained. Q. E. I.

COROLLARY 1

194. If the perpendicular *GY* may be dropped from the centre of gravity *G* in the direction of the force acting *OZ* without the resolution of the force *OZ,* the moments of this will be able to be determined with respect to each axis.

COROLLARY 2

195. Indeed if the sine of the inclination of the axis *CD* to the plane *GYZ* were *k,* the moment of the force $OZ = N$ with respect to the axis $CD = N \cdot GY \cdot k$.

COROLLARY 3

196. Similarly if the sine of the inclination of the axis *AB* to the plane *GYZ* were *m,* the moment of the force *N* with respect to the axis $AB = N \cdot GY \cdot m$.

COROLLARY 4

197. And if *n* may be put for the sine of the angle of inclination of the axis *EF* to the plane *GYZ* , the moment of the force *N* with respect to the axis $FE = N \cdot GY \cdot n$.

COROLLARY 5

198. Hence therefore it appears easier to produce the same moments, the point *O* may be taken at any location on the line *OZ,* since its position shall not enter into these formulas.

SCHOLION

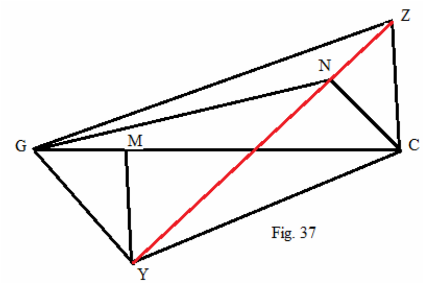


Fig. 37

199. Certainly the account of these moments assigned follows from the forms of the moments proposed, but yet can be returned more easily in the following way from the principles of statics. *G* shall be the centre of gravity of some body (Fig. 37), and *GC* the axis passing through that, with respect of which the moment of some force acting on the body must be determined. Clearly the body may be acted on by a force at *N* [and magnitude *N*] in the direction *YZ* [which we have made red in the diagram], on which the perpendicular *GY* from the centre of gravity *G* may fall. Now the perpendicular *ZC* may be dropped from *Z* onto the plane *GYC*, and *CY* may be joined, the plane *ZYC* being normal to the plane *GYC*, and *CY* normal to *GY*. The force *YZ* may be resolved into two lateral forces: *YC* and another force the direction of which is parallel to *ZC* itself; and this latter force $\frac{N\cdot ZC}{YZ}$ alone will have a moment with respect to the axis *GC*. Therefore with the perpendicular *YM* dropped from *Y* to *GC*, the moment of the force with respect to the axis *GC*

$$=\frac{N\cdot ZC\cdot YM}{YZ}=\frac{N\cdot ZC\cdot GY\cdot YC}{YZ\cdot GC},\ \left[\text{for } \frac{YM}{YC}=\frac{GY}{GC}\right].$$

Now again, with the perpendicular *CN* dropped from *C* to *YZ*, there will become $YZ:CZ=YC:CN$, from which that moment will be produced $=N\cdot GY\cdot\frac{CN}{GC}$. Truly *CN* dropped from *C* into the plane *GYZ* is perpendicular on account of the normals *CY* to *GY*, *ZY* to *GY* and *CN* to *YZ*. On account of which $\frac{CN}{GC}$ will express the sine of the angle, which the axis *GC* makes with the plane *GZY*. If which sine may be called k, the moment will be $=N\cdot GY\cdot k$, as asserted in the corollaries.

PROPOSITION 18

PROBLEM

200. *If the body resting on the water were disturbed from its state of equilibrium in some manner, to define the axis passing through the centre of gravity, about which the body will begin to rotate.*

SOLUTION

MPNQXP shall be the immersed part of the body (Fig. 38), and its centre of magnitude O and the weight of water equal to the submerged part $= N$. *PMQN* shall be the section of the water situated in a plane to the horizontal, and *OZ* the right vertical, therefore the water will be pressed upwards by the pressure of the water by the force *N* in the direction *OZ*. The centre of gravity of the body may be put at *G,* and the perpendicular *GY* may be sent from *G* to *OZ.* Then the normal *DC* may be drawn through *G* to the plane *GYZ* which hence will be a horizontal line. Moreover it will begin to turn about this horizontal line *DO* as the body begins to rotate about this axis restoring itself to the state of equilibrium. For since the axis *DO* shall be normal to the plane *GYZ*, the force *N* will have no moment about these other axes normal to the axis *DC*. Therefore the whole force will expend its strength turning the body about the axis *DC*, and its moment shall be $= N \cdot GY$. Q.E.I.

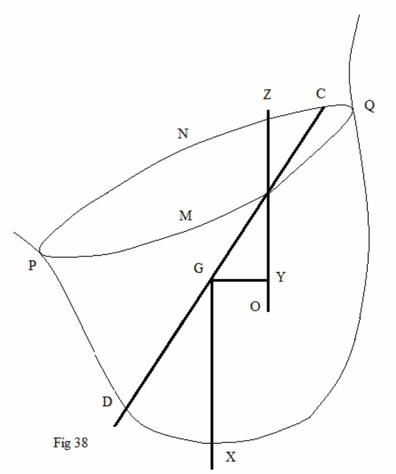


Fig 38

COROLLARY 1

201. Therefore if the axis *DC* were prepared thus, so that the centrifugal forces of the body around that axis may cancel each other out, then the body will continue to rotate about this fixed axis.

COROLLARY 2

202. Therefore a body floating on the water disturbed from its state of equilibrium, will have that restored by a rotational motion about a certain horizontal axis: while meanwhile the centre of gravity only is carried up or down.

SCHOLIUM

203. Moreover even if a body shall be able to rotate freely about a fixed axis *CD*, then yet it does not follow, the whole restoration to be made about this axis. For the direction *OZ* can be changed within this motion, from which also a variation of the axis *DO* arises. But yet from these, enough of the restitution motion is considered to be present.

CAPUT SECUNDUM

DE CORPORUM AQUAE INNATANTIUM

RESTITUTIONE IN AEQUILIBRIUM

PROPOSITIO 14

THEOREMA

110. *Ut corpus aquae ita insidat, ut vel non tanta eius pars aquae sit immersa, quanta ad aequilibrium requiritur, vel recta iungens centra gravitatis et magnitudinis non fuerit verticalis, tum corpus movebitur, donec in statum aequilibrii pervenerit.*

DEMONSTRATIO

Sit enim primo non tanta pars aquae submersa, quanta ad aequilibrium requiritur, manifestum est vires gravitatis et pressionum aquae fore inaequales, ideoque sese mutuo non destruere. Quare corpus tum movebitur vel ascendendo vel descendendo, donec in aequilibrii situm perveniat. At si recta iungens centra gravitatis et magnitudinis non sit verticalis, vires gravitatis et pressionum aquae, etiamsi fuerint aequales, tamen non directe sibi erunt oppositae. Cum igitur etiam hoc casu vires, quibus corpus sollicitatur se non destruant, necesse est ut motum gyratorium in corpore producant, qui tam diu durabit, donec corpus in aequilibrium fuerit constitutum. Q. E. D.

COROLLARIUM 1

111. Si igitur corpus vi externa ex situ aequilibrii fuerit declinatum, tum cessante hac vi moveri incipiet, donec in situm aequilibrii pervenerit.

COROLLARIUM 2

112. Si ergo daretur corpus, quod ita esset comparatum, ut nullum situm aequilibrii admitteret, tum hoc corpus aquae impositum perpetuo moveri deberet, atque perpetuum mobile verum repraesentaret.

COROLLARIUM 3

113. Cum igitur huiusmodi mobile perpetuum contradictionem involvat, sequitur omne corpus saltem unicum habere debere situm in quo in aequilibrio esse queat.

COROLLARIUM 4

114. Quia corpus dum se restituit in aqua moveri debet, patietur quoque ab aqua resistentiam, quae eo maior erit, quo maior fuerit celeritas corporis contra aquam. Motum autem resistentia prorsus impedire non potest cum motus tardissimos non amplius afficiat.

COROLLARIUM 5

115. Interim tamen hoc certum est corpus se non tam cito restituera posse, quam si aqua nulla prorsus resistentia reluctaretur. Eo tardius igitur corpus se restituet, quo maior superanda fuerit resistentia.

SCHOLION

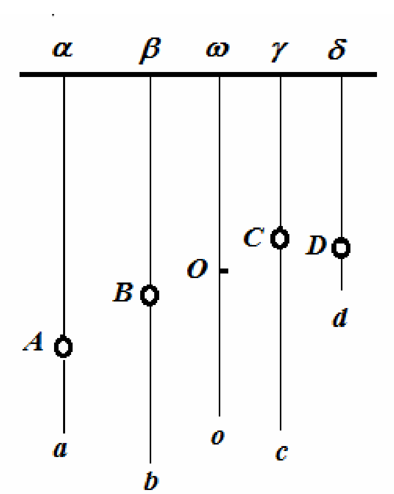


Fig. 20

116. In hoc ergo capite inquirendum nobis erit, cuiusmodi motus in corpore, quod a potentiis sese non destruentibus sollicitatur, generetur. Ad hoc autem praestandum pluribus opus est propositionibus ex mechanica petendis, circa effectum potentiarum in corpora finita. Qua dere cum vix quicquam adhuc certi sit traditum, necesse est ut principia ad hoc necessaria ex ipsis mechanicae fontibus hauriamus. In sequentibus igitur lemmatis investigabimus, cuiusmodi motum quaecunque potentiae corpus sollicitantes producere debeant; haecque principia, quae hic stabiliemus, non solum in hoc capite, sed etiam in omnibus sequentibus tantae sunt necessitatis, ut iis nullo modo carere queamus.

LEMMA

117. *Si plura fuerint corpora A, B, X, D* (Fig. 20) *quae singula in directionibus parallelis promota perveniant in a, b, c, d; tum eorum commune centrum gravitatis O in directione quoque parallela Oo movebitur, et perveniet in o ut sit*

$$Oo = \frac{A \cdot Aa + B \cdot Bb + C \cdot Cc + D \cdot Dd}{A + B + C + D}$$

denotantibus A , B, C, D massas corporum respective.

DEMONSTRATIO

Producantur directiones *Aa, Bb, Cc, Dd,* donec rectae $Z\delta$ pro lubitu assumptae occurrant in punctis $\alpha, \beta, \gamma, \delta$ et recta ex centro gravitatis *O* in ω.
Erit ergo ex nota centri gravitatis proprietate

$$Z\omega = \frac{A\cdot Z\alpha + B\cdot Z\beta + C\cdot Z\gamma + D\cdot Z\delta}{A+B+C+D}$$

ex qua aequatione intelligitur, punctum ω fore vestigium centri gravitatis tam dum corpora in *A* , *B, C, D* quam dum in *a, b, c, d* sunt sita, quare directio, in qua centrum gravitatis movetur, erit parallela directionibus corporum.
Deinde vero corporibus in *A* , *B, C, D* existentibus erit

$$O\omega = \frac{A\cdot A\alpha + B\cdot B\beta + C\cdot C\gamma + D\cdot D\delta}{A+B+C+D}.$$

Translatis autem corporibus in *a, b, c, d,* perveniat centrum gravitatis in *o,* erit per eandem centri gravitatis proprietatem

$$o\omega = \frac{A\cdot a\alpha + B\cdot b\beta + C\cdot c\gamma + D\cdot d\delta}{A+B+C+D}.$$

Cum iam sit $Oo = o\omega - O\omega$, , erit

$$oO = \frac{A\cdot Aa + B\cdot Bb + C\cdot Cc + D\cdot Dd}{A+B+C+D}.$$

Q.E.D.

COROLLARIUM

118. Quantuscunque ergo sit corporum numerus demonstratio data aeque valet, ac pro casu quatuor corporum tantum. Atque ex demonstratione quoque intelligitur, eam pariter vim suam retinere sive corpora sint in eodem plano sita sive secus.

LEMMA

119. *Si fuerint corporibus A*, *B, C, D potentiae a, b, c, d respective applicatae* (Fig. 20), *quarum directiones inter se sint parallelae, corporaque ab iis moveantur, tum eorum commune centrum gravitatis O eodem modo movebitur, ac si omnia corpora in ipso gravitatis centro O essent concentrata eorumque summa* $A+B+C+D$ *a summa potentiarum* $a+b+c+d$ *in eadem directione sollicitaretur.*

DEMONSTRATIO

Promoveantur puncto temporis *dt* corpora *A, B, C, D* a potentiis respectivis *a, b, c, d* per elementa *Aa, Bb, Cc, Dd,* erit per nota mechanicae principia

$$Aa = \frac{a \cdot dt}{A},\ Bb = \frac{b \cdot dt}{B},\ Cc = \frac{c \cdot dt}{C},\ Dd = \frac{d \cdot dt}{D}.$$

Interim igitur per lemma praecedens perveniet centrum gravitatis *O* in *o,* ut sit *Oo* parallela directionibus potentiarum, atque

$$Oo = \frac{dt\left(a+b+c+d\right)}{A+B+C+D}.$$

At si in *O* concentrata esset summa corporum $A+B+C+D$, eaque in directione *Oo* sollicitaretur a summa potentiarum $a+b+c+d$, tum tempusculo *dt* perveniret quoque in *o* ut esset

$$Oo = \frac{dt\left(a+b+c+d\right)}{A+B+C+D}.$$

Quare dum corpora singula *A* , *B, C, D,* a suis potentiis respectivis *a, b, c, d* in directionibus inter se parallelis urgentur, centrum gravitatis *O* eodem modo movebitur ac si summa corporum in *O* concentrata sollicitaretur a summa potentiarum $a+b+c+d$ in eadem directione. Q. E. D.

COROLLARIUM

120. Haec igitur demonstratio aeque succedit, sive corpora sint in eodem plano posita, sive secus dummodo potentiarum, quibus singula sollicitantur, directiones sint inter se parallelae.

LEMMA

121.*Si corpora A, B, C, D* (Fig. 21) *sollicitentur a quibuscunque potentiis, ab iisque moveantur, tum eorum centrum gravitatis O eodem modo movebitur ac si in eo omnium corporum massae essent concentratae, eorumque aggregato applicatae essent omnes potentiae quaeque in directione parallela ei, in qua singula corpora, sollicitantur.*

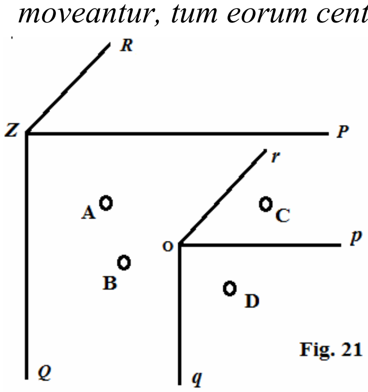


Fig. 21

DEMONSTRATIO

Sumantur pro lubito tres directiones fixae *Z P, ZQ, ZR* inter se normales, et quaelibet potentia, quibus corpora *A, B, C, D* sollicitantur, resolvatur in ternas, quarum directiones respective sint parallelae directionibus *ZP, ZQ,* *ZR.* Ponatur summa omnium potentiarum ex hac resolutione ortarum, quarum directiones sunt ipsi *Z P* parallelae $= P$, summa earum quae ipsi *ZQ* sunt parallelae $= Q$, earumque summa quae ipsi *ZR* sunt parallelae $= R$. si singula corpora tantum ab iis potentiis sollicitarentur, quarum directiones ipsi *ZP* sunt parallelae, tum in centrum gravitatis *O* eodem modo moveretur, ac si in eo summa corporum $A+B+C+D$ a potentia *P* in directionem *Op* ipsi *ZP* parallela sollicitaretur. Simili modo si singula corpora a potentiis tantum sollicitarentur quarum directiones sunt ipsi *ZQ* parallelae tum centrum gravitatis eodem modo moveretur, ac si in eo summa corporum $A+B+C+D$ a potentia *Q,* cuius directio *Oq* quoque ipsi *ZQ* est parallela, urgeretur. Denique eodem modo si singula corpora tantum a potentiis, quarum directiones ipsi *ZR* sunt parallelae, sollicitarentur, tum centrum gravitatis *o* eodem modo movebitur, ac si in eo summa corporum $A+B+C+D$ a potentia *R* in directione *Or* ipsi *ZR* parallela sollicitaretur. Quare si singula corpora *A, B, C, D* omnibus potentiis, quae ipsis sunt applicatae simul sollicitentur, tum centrum gravitatis *O* eodem modo movebitur ac si in eo summa corporum $A+B+C+D$ esset concentrata, eaque a tribus potentiis *P, Q, R* in directionibus *Op, Oq, Or* sollicitaretur. At si omnes potentiae, quibus singula corpora sollicitantur, in directionibus sibi parallelis in puncto *O* essent applicatae, tum iis omnibus tres potentiae *P, Q, R* in directionibus *Op, Oq, Or* aequivalerent. Quamobrem centrum gravitatis *O* eodem modo movebitur ac si in eo omnia corpora essent concentrata, iisque omnes potentiae in iisdem directionibus essent applicatae.

Q.E.D.

LEMMA

122. *Si corpori rigido, cuius partes firmissimo nexu inter se cohaerent,quaecunque potentiae fuerint applicatae, eius centrum gravitatis eodem modomovebitur, ac si in eo tota corporis massa esset concentrata, eique omnes potentiae in suis directionibus coniunctim applicatae.*

DEMONSTRATIO

Concipiantur ad momentum omnes corporis particulae a se invicem dissolutae, ut earum quaeque libere a potentiis moveantur, tum ex lemmate praecedente constat centrum gravitatis corporis eodem modo moveri, ac si in eo tota corporis massa esset concentrata, eaque ab omnibus potentiis in iisdem respective directionibus coniunctim urgeretur. Cum autem interea singulae corporis particulae ex mutuo situ, quem inter se tenere debent, sint dislocatae, concipiantur eae sese in debitum ordinem restituere, viribus, quibus se mutuo attrabant vel repellant. Huiusmodi autem actionibus quibus corpora in se mutuo agunt, locus centri gravitatis non mutatur. Quamobrem post restitutionem centrum gravitatis invariatum manebit, atque idcirca eodem modo movebitur ac si singulae corporis particulae a se invicem essent dissolutae. Mutuus igitur nexus quo corporis particulae inter se cohaerent, motum centri gravitatis assignatum non turbat. Q.E.D.

COROLLARIUM 1

123. Si igitur potentiae, quibus corpus sollicitatur ita fuerint comparatae, ut, si omnes eidem puncto applicentur, se mutuo destruant et in aequilibrio consistant, tum centrum gravitatis corporis quiescet.

COROLLARIUM 2

124. At si potentiae corpori applicatae non fuerint ita comparatae, ut, si omnes eidem puncto applicentur, sint in aequilibrio, tum centrum gravitatis corporis in media earum potentiarum directione movebitur.

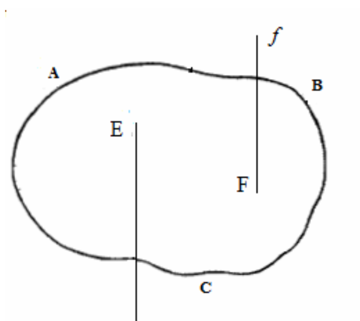


Fig. 22

COROLLARIUM 3

125. Si igitur corpori *ABC* (Fig. 22) duae potentiae *Ee* et *Ff,* quarum altera deorsum altera sursum urgeat, fuerint applicatae, tum centrum gravitatis quiescet si fuerit $Ee = Ff$. At si $Ee > Ff$ centrum gravitatis descendet, ascendet vero si fuerint $Ee < Ff$.

COROLLARIUM 4

126. Quaecunquae ergo potentiae corpori fuerint applicatae, motus centri gravitatis per hoc lemma poterit determinari.

SCHOLION

127. Motus autem, quem potentiae centro gravitatis inducunt, non est solus effectus quem potentiae in corpore producunt, neque ad effectum potentiarum corpori applicatarum sufficit motum centri gravitatis nosse. Interea enim dum centrum gravitatis vel quiescit, vel descripto modo movetur, fieri potest, ut reliquae corporis partes circa centrum gravitatis revolvantur, motumque gyratorium accipiant, quippe quo motu centri gravitatis motus non turbatur. Quamobrem superest, ut investigemus cuiusmodi motum gyratorium quaevis potentiae corpori applicatae circa centrum gravitatis producant, hoc enim cognito integer effectus, quem potentiae in corporibus rigidis edere possunt, innotescet.

LEMMA

128. *Si corpori rigido quaecunque potentiae fuerint applicatae, idque ab iis moveatur ita ut centrum gravitatis eum obtineat motum, quem ipsi assignavimus, totum corpus interea circa centrum gravitatis pariter movebitur, ac si centrum gravitatis quiesceret vel fixum esset.*

DEMONSTRATIO

Concipiatur corpori in ipso centro gravitatis potentia nova applicata aequalis et contraria ei quae oritur ex compositione omnium potentiarum si essent in centro gravitatis applicatae, tum igitur centrum gravitatis quiescet. Si nunc hoc casu determinetur motus, quem potentiae corpori circa centrum gravitatis inducunt, potentia nova, quam corpori in centro gravitatis applicatam concipimus, motum hunc circa centrum gravitatis non afficiet. Quare etiamsi, ea iterum auferatur, motus circa centrum gravitatis immutatus manebit. Hanc ob rem corpus, circa centrum gravitatis motum eodem modo movebitur, ac si esset fixum. Q. E. D.

COROLLARIUM

129. Ad motum igitur corporis circa centrum gravitatis determinandum, licebit centrum gravitatis tanquam fixum considerare, cum motus gyratorius circa centrum gravitatis motum conveniat cum motu gyratorio circa centrum gravitatis fixum.

SCHOLION

130. Quo ergo determinemus hunc circa centrum gravitatis motum, incipiemus a motu circa quodcumque punctum fixum definiendo, atque investigabimus motum gyratorium, quem quaecunque potentiae corpori ex quopiam puncto fixo suspenso inducere valent. Hoc enim determinato facile erit punctum fixum in ipsum centrum gravitatis transferre atque adeo motum, quem potentiae quaecunque corpori applicatae circa centrum gravitatis generant assignare.

LEMMA

131. *Si corpusculum C circa punctum fixum O mobile sollicitetur a potentia CF* (Fig. 23), *erit angulus dato tempusculo dt circa O genitus directe ut factum ex potentia CF in sin ang. FCO per dt multiplicatum et reciproce ut massa corpusculi O in distantiam CO ducta.*

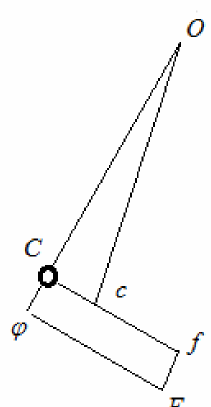


Fig. 23

DEMONSTRATIO

Positis potentia $CF = p$ et sin ang. $FCO = m$, resolvatur potentia CF in duas laterales secundum Cf normalem ad CO et $C\varphi$ ipsam CO, quarum posterior tota in tendendo filo CO vel immutanda corpusculi a C distantia consumetur, ideoque ad motum nihil praestat. Altera vero potentia in directione Cf, quae erit $= mp$, tota in motu producendo impendetur. Perducatur igitur corpusculum C tempusculo dt per spatiolum Cc, erit $Cc = \frac{mpdt}{C}$. Hoc vero spatiolum per distantiam CO divisum dabit angulum COc tempusculo dt genitum qui consequenter erit $= \frac{mpdt}{C \cdot CO}$. Q.E.D.

COROLLARIUM 1

132. Si ergo huic angulo, qui dato tempusculo generatur, vis gyratoria proportionalis ponatur, erit vis gyratoria ut potentia *CF* in sin ang. *FCO* divisa per corpus C in distantiam *CO.*

COROLLARIUM 2

133. Si igitur vis gyratoria perpetuo ista ratione exprimatur, tum statim cognoscetur motus angularis. Expressio enim talis vis gyratoriae per tempusculum *dt* multiplicata exhibet statim angulum hoc tempusculo genitum circa *O.*

COROLLARIUM 3

134. Atque si vis gyratoria compositi cuiusque corporis hoc modo expressa inveniatur, tum statim angularis motus cognoscetur, seu quod idem est, assignari potest corpus solitarium ut *O,* quod a certa potentia solicitatum aequalem motum gyratorium habeat.

SCHOLION

135. Cum potentia oblique corpusculum *C* trahens tam facile ad potentiam normalem ad *CO* reducatur, atque altera potentia, cuius directio in ipsam *CO* incidit, motum gyratorium non afficiat; in sequentibus, ubi motus gyratorius determinari debet, tantum potentias normales cuique corporis particulae applicatas concipiemus. Et quidem si *C* fuerit particula corporis circa *O* mobilis, ex hoc lemmate iam constat, si sola particula *C* adesset, reliquis corporis partibus omnibus annihilatis, qualem motum gyratorium potentia *Cf* normalis ad *CO* corpusculo *C* induceret. Scilicet corpusculum *C* ad circulum describendum incitabitur, cuius centrum est in *O* et radius *AO,* atque iste circulus situs erit in plano, in quo sitae sunt cum recta *CO* tum potentiae sollicitantis *Of* directio. Vis autem qua ad hunc circulum describendum incitatur, seu vis gyratoria, prout eam vocabimus, erit aequalis potentiae *Of* divisae per factum ex massa corpusculi *C* in distantiam *O O.*

LEMMA

136. *Loco corporis Aa potentia Aa circa O ad motum gyratorium incitati* (Fig. 24), *substitui potest in distantia MO, quae in eodem plano, in quo AO et Aa,*

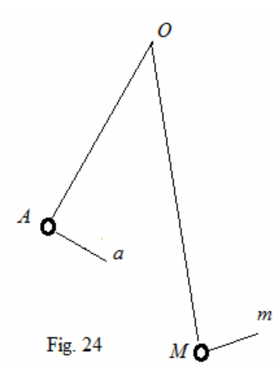


Fig. 24

sita est, corpus $M=\frac{A\cdot AO^2}{MO^2}$, *quod a potentia* $Mm=\frac{AO\cdot A}{MO}$,
cuius directio in eodem plano est sita et in eandem, in quam Aa plagam tendit, ad motum gyratorium circa O sollicitetur.

DEMONSTRATIO

Quo loco potentiae *Aa* in distantia *AO* applicatae substitui queat potentia *Mm* in distantia *MO* applicata, oportet ut primo *MO* et *Mm* in eodem plano sint sitae, in quo sunt *AO* et *Aa;* deinde vero requiritur ut momenta potentiarum *Aa* et *Mm* sint aequalia, unde sequitur fore $Aa\cdot AO=Mm\cdot MO$ seu

$$Mm=\frac{Aa\cdot AO}{MO}.$$

Hoc enim pacto potentia *Mm* eundem praestabit effectum circa *O,* quem habet potentia *Aa* circa *O.* Praeterea vero corpus in *M* collocandum tantum esse debet, ut vires gyratoriae in *M* et *A* circa *O* sint aequales, quod eveniet, si fuerit

$$\frac{Mm}{M\cdot MO}=\frac{Aa}{A\cdot AO}.$$

Quo ergo et vis gyratoria utrinque sit eadem, necesse est ut sit

$$M=\frac{A\cdot AO\cdot Mm}{MO\cdot Aa}=\frac{A\cdot AO^2}{MO^2}$$

propter

$$Mm=\frac{Aa\cdot AO}{MO}.$$

Q.E.D.

COROLLARIUM 1

137. Perinde igitur est in qua directione assumatur recta *MO,* modo ea in eodem plano sit sita, in quo positae sunt rectae *AO* et *Aa.* Atque simili modo arbitrarium est quantum vis longa *MO* capiatur.

COROLLARIUM 2

138. Definita autem longitudine rectae *MO* in plano *OAa* positae, simul tam corpus in *M* collocandum, quam potentia ei applicanda *Mm* determinantur.

COROLLARIUM 3

139. Loco ergo cuiusvis corpusculi *A* circa punctum fixum *O* gyrantis aliud in data distantia substitui potest, quod prorsus eundem praestet effectum in motu gyratorio circa *O* generando, quem produceret illud corpusculum *A* in distantia *AO*.

LEMMA

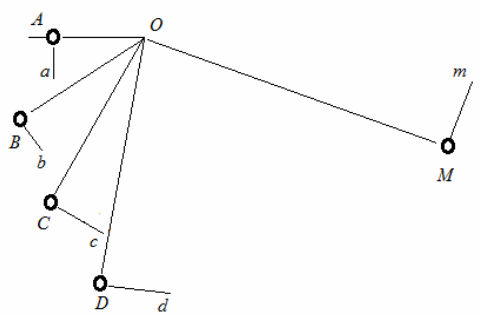


Fig. 25

140. *Si fuerint plura corpuscula A, B, C, D* (Fig. 25), *inter se firmiter connexa et in eodem plano sita quae a potentiis Aa, Bb, Cc, Dd, quarum directiones in eodem quoque plano sint positae, circa punctum fixum O in eodem plano situm ad motum gyratorium sollicitentur, erit vis gyratoria communis*

$$= \frac{Aa \cdot AO + Bb \cdot BO + Cc \cdot CO + Dd \cdot DO}{A \cdot AO^2 + B \cdot BO^2 + C \cdot CO^2 + D \cdot DO^2}.$$

DEMONSTRATIO

Sumta in plano, in quo sita sunt tam corpuscula cum puncto *O,* quam directiones, quibus sollicitantur, recta arbitraria *OM,* loco corpusculi *Aa* potentia *Aa* sollicitati substituatur in *M* corpusculum $\frac{A \cdot AO^2}{MO^2}$ a potentia $\frac{Aa \cdot AO}{MO}$ sollicitatum (§136), similique modo loco corpusculorum *B, C, D* a potentiis *Bb, Cc, Dd,* sollicitatorum, in *M* substituantur corpuscula

$$\frac{B \cdot BO^2}{MO^2}, \frac{C \cdot CO^2}{MO^2}, \frac{D \cdot DO^2}{MO^2}$$

a potentiis

$$\frac{Bb \cdot BO}{MO}, \frac{Cc \cdot CO}{MO}, \frac{Dd \cdot DO}{MO}$$

sollicitata. Hoc ergo facto loco omnium corpusculorum a suis potentiis sollicitatorum substitui potest in *M* corpus

$$\frac{A\cdot AO^2+B\cdot BO^2+C\cdot CO^2+D\cdot DO^2}{MO^2},$$

quod sollicitetur in directione *Mm* a potentia quae est

$$=\frac{Aa\cdot AO+Bb\cdot BO+Cc\cdot CO+Dd\cdot DO}{MO},$$

Huius igitur corporis vis gyratoria erit

$$=\frac{Aa\cdot AO+Bb\cdot BO+Cc\cdot CO+Dd\cdot DO}{A\cdot AO^2+B\cdot BO^2+C\cdot CO^2+D\cdot DO^2}$$

cui aequalis est vis gyratoria, quae ex viribus gyratoriis singulorum corporum, *A, B, C, D,* coniunctim oritur. Q.E.D.

COROLLARIUM 1

141. Vis ergo gyratoria plurium huiusmodi corpusculorum aequatur summae momentorum singularum potentiarum in punctum fixum *A* , divisae per summam factorum ex singulis corpusculis in quadrata eorum a puncto fixo *O* seu ab axe, circa quem fit motus gyratorius multiplicatis.

COROLLARIUM 2

142. Angulus igitur, per quem corpusculorum *A,* B, *C, D* systema tempusculo *dt* circa punctum fixum *O* gyrabitur erit

$$=dt\,\frac{Aa\cdot AO+Bb\cdot BO+Cc\cdot CO+Dd\cdot DO}{A\cdot AO^2+B\cdot BO^2+C\cdot CO^2+D\cdot DO^2}$$

unde ipse motus gyratorius cognoscetur.

COROLLARIUM 3

143. Si aliquot corpuscula in plagam oppositam trahantur, tum potentiarum, quibus ea sollicitantur momenta fient respectu reliquorum negativa, ideoque in expressione inventa subtrahi debebunt.

COROLLARIUM 4

144. Manentibus ergo potentiis corpus circa punctum fixum *O* sollicitantibus iisdem, vis gyratoria eo erit maior, quo minor fuerit denominator fractionis inventae, hoc est quo propius singulae corporis particulae puncto *O* fuerint sitae.

COROLLARIUM 5

145. Si anguli, quos potentiarum sollicitantium directiones cum rectis ad punctum *O* ductis constituunt, non fuerint recti, tum singula facta $Aa \cdot AO, Ab \cdot BO$ etc. insuper per sinus angulorum *OAa, OBb* etc. respective multiplicari debent; sumta unitate pro sinu toto.

COROLLARIUM 6

146. Vis ergo gyratoria evanescet, si momenta potentiarum corpus sollicitantium in punctum fixum se mutuo destruant; hoc ergo casu corpus quiescet.

SCHOLION

147. Haec igitur sunt lemmata, quibus ad motum corporum aquae innatantium explicandum opus erit; ex quibus satis intelligitur, si corpori cuicunque rigido applicatae sint potentiae quaecunque, tum corporis illius centrum gravitatis eodem modo moveri, ac si tota corporis massa in ipso centro gravitatis esset concentrata, eique omnes potentiae quaeque in sua directione coniunctim essent applicatae. Praeterea vero totum corpus interim circa centrum gravitatis eodem modo movebitur, ac si centrum gravitatis esset fixum, hicque motus gyratorius cognoscetur ex momentis omnium potentiarum in centrum gravitatis. Si enim haec momenta se mutuo destruant, tum corpus plane nullum habebit motum gyratorium circa centrum gravitatis; eo maior vero erit hic motus gyratorius quo maius fuerit momentum ex omnibus potentiis in centrum gravitatis coniunctim ortum. In motu quidem corporum aquae innatantium iste motus gyratorius si quis oritur statim sistitur, et hancobrem hoc casu sufficiet eum corporis situm determinare in quo motus gyratorius cessat; ad quem definiendum lemmata praemissa sunt accommodata.

PROPOSITIO 15

THEOREMA

148. *Si corpus aquae insidens non fuerit in aequilibrio, tum convertetur circa centrum gravitatis, atque centrum gravitatis interea vel quiescet vel directe sursum deorsumve movebitur.*

DEMONSTRATIO

Si corpus aquae quomodocunque immergatur id a duabus viribus urgebitur, quarum altera aequalis est ponderi corporis, et in verticali per centrum gravitatis transeunte deorsum tendit, altera vero aequalis est ponderi aquae volumine partem aquae submersam aequantis, et corpus directe sursum pellit in directione per centrum magnitudinis partis submersae transeunte. Cum igitur centrum gravitatis ab his potentiis eundem accipiat motum, ac si tota corporis massa in eo esset concentrata et ab utraque vi coniunctim sollicitaretur, manifestum est centrum gravitatis vel quiescere vel directe sursum deorsumue moveri debere. Nisi autem directio potentiae, qua corpus sursum urgetur, per centrum gravitatis transeat, tum corpus ab hac vi interea circa gravitatis centrum convertetur. Q. E. D.

COROLLARIUM 1

149. Intelligitur ergo, si nimis magna pars corporis aquae fuerit immersa, tum propter vim sursum urgentem altera maiorem, centrum gravitatis ascendere debere; contra vero si nimis exigua pars aquae immergatur, centrum gravitatis descendere debere.

COROLLARIUM 2

150. Si ergo tanta pars corporis aquae est immersa, quanta ad aequilibrium requiritur, tum centrum gravitatis quiescet propter vires sollicitantes inter se aequales et contrarias.

COROLLARIUM 3

151. Si igitur tanta corporis pars aquae fuerit submersa, quanta ad aequilibrium requiritur, tum corpus sese in aequilibrium restituendo circa centrum gravitatis quiescens convertetur.

SCHOLION

152. Hinc autem non sequitur, quod, si semel tanta corporis pars aquae fuerit submersa, quanta ad aequilibrium producendum est necessaria, tum centrum gravitatis, quam diu conversio fit, perpetuo quiescere debere. Inter convertendum enim utique evenire potest, ut pars corporis sub aqua existens, quippe quae continuo immutatur, fiat iusto vel maior vel minor. Hoc igitur cum accideret, necesse est, ut centrum gravitatis statim vel elevetur vel deprimatur. Ipse autem motus tum centri gravitatis tum conversionis none vestigio cessabit, cum corpus in statum aequilibrii pervenerit, sed prout in motu pendulorum evenit, in partem contrariam inclinabitur et revertetur, hicque motus oscillatorius tam diu durabit, quoad a resistentia omnino absorbeatur.

PROPOSITIO 16

PROBLEMA

153. *Figurae planae aquae verticaliter insistentis EAHBF* (Fig. 26) *ex situ aequilibrii depulsae, motum quo se in situm aequilibrii restituet, determinare.*

SOLUTIO

Sit *AHB* pars aquae immersa, *O* eius centrum magnitudinis et *G* centrum gravitatis totius figurae, quae centra in plano *AHB* erunt posita. Sollicitabitur

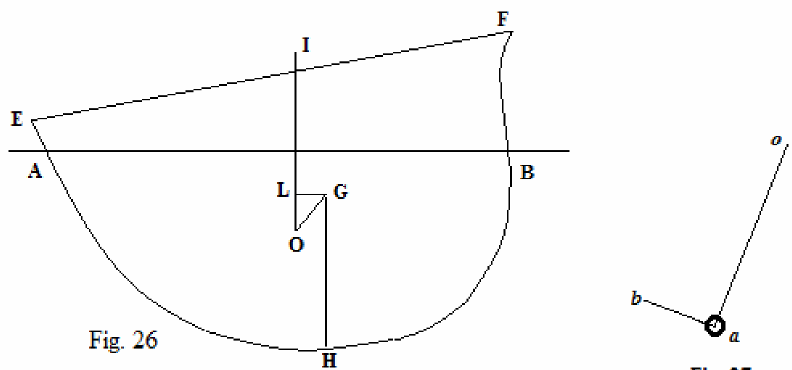


Fig. 26
Fig. 27

ergo haec figura a pressionibus aquae sursum in directione verticali *OI* vi aequali ponderi aquae spatium *AHB* occupantis; deorsum vero urgebitur in directione *GH* vi ponderi figurae aequali. Quod iam ad motum centri gravitatis *G* attinet, id sursum pelletur, si vis *OI* fuerit maior quam vis *GH;* deorsum vero nitetur si vis *GH* vim *OI* superet; casu denique quo hae vires sunt inter se aequales, centrum gravitatis *G* quiescet. Deinde sive centrum gravitatis quiescat sive ascendat sive descendat, tota figura circa id convertetur, nisi directiones *OI* et *GH* in eandem rectam verticalem incidant; qui motus conversionis ex momentis harum virium in centrum gravitatis cognoscetur. Vis quidem *GH,* quippe quae per centrum ipsum gravitatis transit, momentum est nullum, at alterius vis *OI* momentum est *OI.GL* ducta *GL* ex *G* in rectam *OI* normali. Hac igitur vi figura secundum directionem *BHA* circa centrum gravitatis *G* convertetur. Quantitas autem huius vis gyratoriae absoluta pendet insuper a summa productorum ex singulis figurae particulis in quadrata distantiarum a centro gravitatis *G,* quorum productorum summa si dicatur *S,* erit vis gyratoria $=\frac{OI\cdot GL}{S}$. Q.E.I.

COROLLARIUM 1

154. Impetus ergo, quo figura circa *G* convertetur eo erit maior, quo maiores fuerint tum vis *OI* tum distantia *GL,* atque quo minor fuerit summa productorum ex singulis figurae particulis in quadrata distantiarum a centro gravitatis *G.*

COROLLARIUM 2

155. Manentibus ergo *OI* et *GL* invariatis vis figurae se restituendi eo erit maior, quo minor fuerit *S,* hoc est quo propiores fuerint omnes figurae particulae centro gravitatis *G.* Contra vero restitutio eo tardius fiet, quo magis particulae a centro gravitatis removeantur.

COROLLARIUM 3

156. Vis gyratoria, qua figura circa centrum gravitatis convertitur, reduci potest ad vim qua simplex corpus *a* circa punctum fixum *o* a vi *ab* urgetur. Eadem scilicet velocitate hoc corpus *a* gyrabitur, qua figura *EHBF,* si fuerit vis $ab = \frac{a \cdot ao \cdot GL}{S} \cdot OI$. (Fig. 26, 27).

COROLLARIUM 4

157. Datis ergo singulis momentis, dum figura convertitur, situ centrorum gravitatis et magnitudinis mutuo, determinari poterit velocitas angularis quovis momento, atque tempus, quo tota restitutio absolvitur.

SCHOLION I

158. Si corpus aquae insidens fuerit cylindricum, cuius omnes sectiones transversales sint similes et aequales figurae *EAHBF,* manifestum est hoc corpus in aqua simili modo motum iri quo figura sola *EAHBF.* Quamobrem si huiusmodi corpus cylindricum ita fuerit ex statu aequilibrii depulsum, ut omnes illae sectiones in situ verticali permaneant, tum hoc corpus circa rectam horizontalem per singularum sectionum centra gravitatis *G* transeuntem tanquam circa axem convertetur, donec in statum aequilibri pervenerit. Atque praeterea motus huius restitutionis omnino congruet cum motu figurae *EAHBF* et eadem celeritate absolvetur. Vis ergo, qua corpus hoc, dum singularum sectionum partes similes et aequales ipsi *AHB* aquae sunt submersae, circa axem per omnium sectionum centra gravitatis *G* transeuntem convertetur, aequalis erit ponderi aquae volumine toti parti submersae aequali ducto in intervallum *LG* et diviso per aggregatum omnium corporis particularum per quadrata distantiarum earundem ab axe conversionis multiplicatarum.

SCHOLION 2

159. Quemadmodum hoc corpus cylindricum, cuius omnes sectiones transversales non solum sunt similes et aequales, sed etiam onera per eas similiter sunt digesta, circa axem immobilem per centrum gravitatis transeuntem converti potest, ita quoque idem in aliis

cuiusque formae corporibus evenire potest, ut circa axem immotum gyrari queant. Plerumque autem iste motus restitutionis in statum aequilibrii non circa axem quiescentem, sed pariter circa centrum gravitatis mobilem fieri solet, qui ipsius axis motus difficulter determinatur. Interim tamen semper qualiscunque fuerit corporis circa centrum gravitatis motus, is tanquam compositus spectari potest ex motibus circa duos axes factis, qui conceptus ita se habet, ut dum corpus circa axem quendam gyratur, eodem tempore quoque circa alium axem converti concipiendum sit. Quin etiam fieri potest, ut motus circa centrum gravitatis ex motibus circa tres axes sit compositus. Quamobrem quo huiusmodi motus curatius persequi liceat, ante nobis investigandae sunt leges motus gyratorii circa axem fixum, quae deinde etiam ad motum ex aliquibus huiusmodi motibus compositum accommodari queant.

LEMMA

160. *Si corpori, quod axe fixo EF* (Fig. 28) *est traiectum, applicatae fuerint quaecunque potentiae, erit vis gyratoria circa axem EF aequalis summae momentorum omnium potentiarum corpori applicatarum in axem EF divisae per aggregatum omnium corporis particularum per quadrata distantiarum earundem ab axe EF multiplicatarum.*

DEMONSTRATIO

Ex punctis *A, B, C, D* in quibus vires sunt applicatae in axem *EF* demittantur perpendicula *Aa, Bb, Cc, Dd;* perspicuum autem est vim *B* applicatam eundem effectum in corpore circa *EF* convertendo habere, ac si recta *Bb* in quodlibet aliud axis punctum promoveatur. Quamobrem puncta omnia *a, b, c, d* in idem punctum *a* incidere concipiantur, ita ut simul omnes corporis particulae in planum ad *EF* normale et per *a* transiens collocentur. Quo facto totum corpus in hoc planum reductum pari modo circa punctum fixum *a* ad gyrandum incitabitur a potentiis, atque ante. Quocirca vis gyratoria aequalis erit summae momentorum potentiarum omnium in axem *EF* divisae per aggregatum omnium particularum per suarum ab axe *EF* distantiarum quadrata multiplicatarum. Q.E. D.

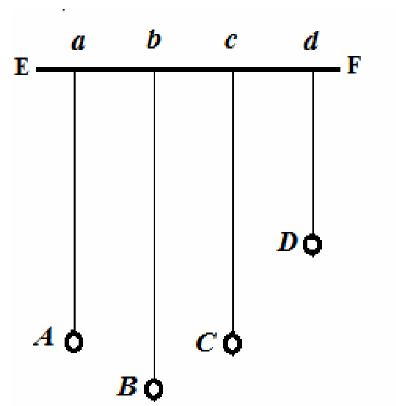


Fig. 28

COROLLARIUM 1

161. Si igitur corporis, quod circa axem fixum mobile existit, vis gyratoria orta a quibuscunque potentiis ipsi applicatis determinari debeat, primo singulae corporis particulae per quadrata distantiarum suarum ab axe illo fixo sunt multiplicandae et in unam summam coniiciendae.

COROLLARIUM 2

162. Inventa vero hac summa omnium productorum ex singulis particulis in quadrata distantiarum suarum ab axe fixo, quaeri debent momenta potentiarum respectu istius axis, quarum aggregatum per illam summam divisum exhibebit vim gyratoriam circa axem.

COROLLARIUM 3

163. Cognita autem vi gyratoria, si ea per elementum temporis multiplicetur, productum dabit elementum celeritatis gyratoriae hoc tempusculo genitum; unde totus motus gyratorius eodem modo determinari poterit quo motus corporis a potentia sollicitati in directum.

SCHOLION

164. Intelligitur ex demonstratione regulam datam pro invenienda vi gyratoria aeque locum habere, sive axis per centrum gravitatis corporis transeat sive secus. Interim tamen ob supra allatas causas alius axis per se immobilis esse non potest, nisi qui per centrum gravitatis transit. Inserviet igitur haec regula quoque, si corpus fuerit mobile circa axem non quidem per centrum gravitatis transeuntem, verum qui ab aliena vi fixus teneatur.

DEFINITIO

165. Momentum inertiae corporis respectu axis cuiusdam fixi voco aggregatum omnium corporis particularum per suarum respective ab hoc axe distantiarum quadrata multiplicatarum.

COROLLARIUM

166. Si ergo corpori circa fixum quempiam axem mobili potentiae fuerint applicatae, vis gyratoria exprimetur aggregato momentorum potentiarum respectu huius axis, diviso per momentum inertiae ipsius corporis respectu eiusdem axis.

COROLLARIUM 2

167. Cum infiniti esse possint axes, circa quos corpus datum moveri potest, infinita etiam quodlibet corpus habebit momenta. Hancobrem si de momento corporis sermo est, simul definiri debet axis, cuius respectu momentum inertiae accipitur prout in definitione est declaratum.

SCHOLION 1

168. Hanc momenti inertiae vocem brevitatis gratia introducere visum est, quo tam amplae descriptiones, quae saepissime occurrerent, evitentur. Quemadmodum enim in motu rotatorio momentum potentiae vocatur potentia ad effectum iam applicata, ita etiam commode in corporibus gyrantibus aggregatum omnium illorum productorum momentum inertiae corporis appellantur, cum vicem massae seu inertiae ad effectum applicatae sustineat.

SCHOLION 2

169. Quo de huiusmodi corporum momentis inertiae clariorem notitiam acquiramus, iuvabit corporum quorumdam momenta investigasse, quae etiam in sequentibus usum habere poterunt. Axes autem tantum considerabo per centrum gravitatis transeuntes, et eorum respectu momenta determinabo; cum infra sim ostensurus, quomodo ex datis momentis inertiae respectu axium per centrum gravitatis transeuntium momenta respectu aliorum quorumque axium inveniri queant.

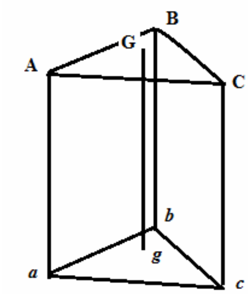


Fig. 29

EXEMPLUM 1

170. Sit prisma triangulare rectum *ABCabc* ex materia constans uniformi, cuius massa sit $= M$ (Fig. 29). Transeat per eius centrum gravitatis axis verticalis *Gg,* qui simul per singularum sectionum transversalium centra gravitatis transibit, erit momentum inertiae huius prismatis respectu huius axis $= \dfrac{M\left(\mathrm{AB}^2 + AC^2 + BC^2\right)}{36}$.

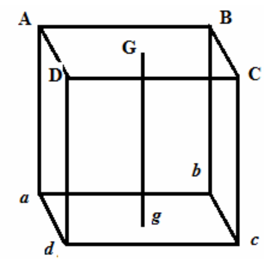


Fig. 30

EXEMPLUM 2

171. Sit prisma quadrangulare rectum seu parallelepipedum *ABCDabcd,* cuius bases *ABCD* et *abcd* sint parallelogramma, ex materia uniformi constans et massam habens $= M$ (Fig. 30). Ductus sit per eius centrum gravitatis axis *Gg,* per utriusque basis centra gravitatis *G* et *g* transiens, erit momentum inertiae huius parallelipiedi respectu huius axis $= \dfrac{M\left(\mathrm{AB}^2 + BC^2\right)}{12}$.

Sive ergo bases sint rectangulae sive obliquangulae, momentum inertiae pari modo ex solis lateribus et massa determinatur.

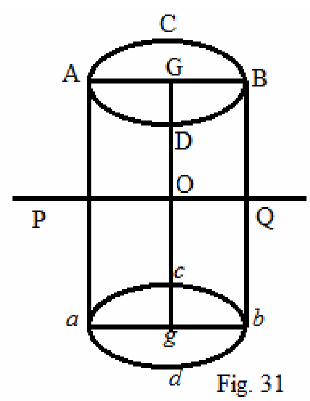


Fig. 31

EXEMPLUM 3

172. Sit corpus cylindrus rectus *ABCDabcd,* cuius bases *ABCD* et *abcd* sint circuli, ex materia uniformi constans eiusque massa *M* (Fig. 31). Traiiciatur iste cylindrus axe *G, g,* basium centra iungente, erit momentum inertiae huius cylindri respectu axis

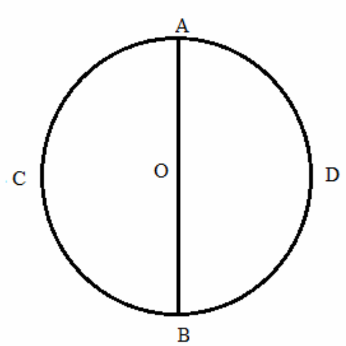
$$Gg = \frac{M \cdot AG^2}{2} = \frac{M \cdot AB^2}{8}.$$

COROLLARIUM

173. Si ex praecedentibus prismatibus et cylindris secentur pyramides et coni recti earumdem basium et altitudinum, erunt eorum momenta inertiae respectu eorundem axium quinquies minora, quam momenta prismatum et cylindrorum. Massae enim fiunt priorum trientes, alterique factores ad priores rationem 3 : 5 tenent.

EXEMPLUM 4

174. Si autem in eodem cylindro per centrum gravitatis *O* ducatur axis transversus *PQ* normalis ad priorem axem *Gg,* erit cylindri momentum inertiae respectu huius axis $PQ = M\left(\frac{Gg^2}{12} + \frac{AB^2}{6}\right)$.



Fig, 32

EXEMPLUM 5

175. Si globi ex materia uniformi constantis, cuius massa est *M* (Fig. 32), momentum inertiae requiratur respectu diametri *AB,* perinde enim est quaecunque diameter accipiatur, reperietur hoc momentum $= \frac{2}{5} M \cdot AG^2$ seu $= \frac{M \cdot AB^2}{10}$. Producti scilicet ex massa in quadratum diametri pars decima dat globi momentum inertiae respectu cuiusque axis per centrum transeuntis.

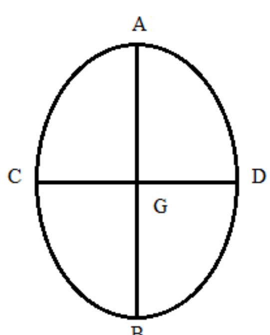


Fig. 33

EXEMPLUM 6

176. Si corpus fuerit sphaeroides ellipticum, genitum ex rotatione ellipsis *ACBD* circa axem *AB,* ex materia uniformi constans, eiusque massa $= M$ (Fig. 33), erit eius momentum inertiae respectu axis *AB,* qui simul est axis

sphaeroidis, $= \frac{M \cdot CD^2}{10}$. Momentum vero inertiae eiusdem sphaeroidis respectu axis *CD* ad priorem axis normalis erit $= \frac{M \cdot \left(AB^2 + CD^2\right)}{20}$. Est vero sphaerae diametri *AB* soliditas ad soliditatem huius sphaeroidis ut AB^2 ad CD^2.

SCHOLION 3

177. Modum, quo horum corporum momenta inertiae sunt inventa, superfluum visum est hic apponere, cum mera constet analysi, atque ab aliis, qui calculum centri oscillationis tradiderunt, iam satis sit expositus. Quo autem, si corpora fuerint magis composita et irregularia, vel si momenta respectu aliorum axium, qui per centrum gravitatis non transeunt, requirantur, totum negotium sine maxime taedioso calculo absolvit queat, sequens lemma adiunxi.

PROBLEMA

178. *Dato momento inertiae cuiusque corporis respectu axis AB per corporis centrum gravitatis G transeuntis* (Fig. 34), *invenire eiusdem corporis momentum inertiae respectu alius cuiusvis axis CD priori axi AB paralleli.*

SOLUTIO

Sit massa corporis $= M$ et momentum inertiae eius respectu axis $AB = S$. Ducatur recta *HG* parallelos axes normaliter secans, quo habeatur distantia axium *GH,* seu distantia centri gravitatis corporis, ab axe *CD* cuius respectu momentum inertiae quaeritur. Consideretur corporis quaecunque molecula *m,* ex eaque in planum *ABDC* in quo siti sunt axes, perpendiculum *mQ* demittatur. Agatur *QRS* parallela ipsi *GH,* itemque *mR* et *mS,* quae in axes erunt normales. Quaesito ergo satisfiet si summa omnium $m \cdot mS^2$ definiatur; quam per $\int m \cdot mS^2$ indicemus. Momentum vero huius corporis respectu axis *AB* est $= \int mR^2$, quod cum detur, erit $\int m \cdot mR^2 = S$; at $\int m$ seu summa omnium corporis molecularum aequatur massae toti *M.* Cum iam sit

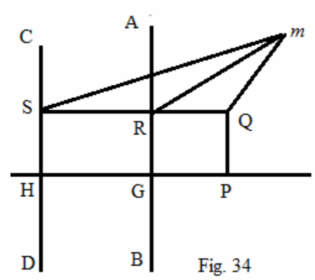


Fig. 34

$$MS^2 = mR^2 + SR^2 + 2SR.\ RQ$$

erit

$$\int m \cdot mS^2 = \int m \cdot mR^2 + \int m \cdot SR^2 + 2\int m \cdot SR \cdot RQ$$
$$= \int m \cdot mR^2 + SR^2 \cdot \int m + 2SR \int m \cdot RQ$$

ob $SR = GH$ ideoque constans. Quia autem *AB* per corporis centrum gravitatis

G transit, erit per notam centri gravitatis proprietatem $\int m \cdot RQ = 0$.
Quare cum sit $\int m \cdot mR^2 = S$ et $\int m = M$ erit quaesitum corporis propositi momentum inertiae respectu axis $CD = S + M \cdot GH^2$. Q.E.I.

COROLLARIUM 1

179. Momentum ergo inertiae corporis respectu axis *GD* aequale est momento inertiae eiusdem corporis respectu axis *AG* per centrum gravitatis *G* transeuntis, una cum facto ex massa in quadratum distantiae centri gravitatis *G* ab axe *GD.*

COROLLARIUM 2

180. Dato ergo momento inertiae corporis respectu axis cuiusdam per centrum eius gravitatis transeuntis, facile determinabitur eiusdem corporis momentum inertiae respectu alius cuiusque axis illi axi paralleli.

COROLLARIUM 3

181. Si igitur corpus, cuius momentum inertiae respectu axis cuiusvis quaeritur, ex pluribus compositum sit partibus, quarum singularum momenta respectu axium illi axi parallelorum et per cuiusque partis centrum gravitatis transeuntium dantur, erit momentum inertiae quaesitum aequale summae omnium momentorum partium una cum productis ex singulis partibus per quadrata distantiarum cuiusque centri gravitatis ab axe illo multiplicatis.

COROLLARIUM 4

182. Hinc igitur manat modus facilis quo citra calculum corporis maxime compositi momentum inertiae respectu cuiusvis axis invenire licet.

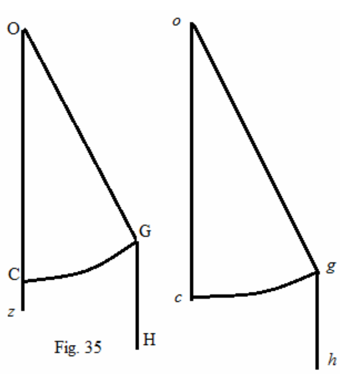


Fig. 35

SCHOLION

183. Ex his sequitur modus facilis et maxime naturalis determinandi centrum oscillationis in corporibus quibuscunque circa axem oscillantibus, *o* quem, etiamsi is huc non pertineat, tamen hic apponi conveniet. Sit corpus quodcunque, quod oscilletur circa axem horizontalem per *O* transeuntem (Fig. 35), atque *C* punctum in verticali *OC* situm, in quod corporis centrum gravitatis incidet, ita ut *OC* sit distantia centri oscillationis ab axe. Consideremus autem corpus extra situm verticalem detrusum, eiusque centrum gravitatis in *G,* ita ut ipsi angulus *GOC* sit describendus, donec in situm

aequilibrii pertingat. Vis autem, quae corpus ad hunc motum angularem sollicitat, est pondus corporis $= S$ quo in directione *GH* deorsum urgetur. Sit nunc massa seu pondus corporis $= M$ eiusque momentum inertiae respectu axis per centrum gravitatis *G* transeuntis et axi oscillationis paralleli, erit huius corporis momentum respectu axis oscillationis $= S + M \cdot OC^2$. Momentum autem gravitatis ad motum angularem circa *O* generandum est $M \cdot GO \cdot \sin O$;

eoque vis gyratoria erit $= \dfrac{M \cdot GO \cdot \sin O}{S + M \cdot OC^2}$. Contemplemur nunc pendumum simplex *og* aequali angulo *goc* a situ verticali *oc* distans cui in *g* ponduscculum infinite parvum *p* sit alligatum circa *o* oscillans, erit vis gyratoria, qua ponduscculum *p* ad angulum *goc* absolvendum animatur $= \dfrac{\sin O}{OC}$. Si ergo haec vis gyratoria aequalis fuerit priori, pendulum simplex *og* et compositum *OG* simul in situm verticalem pervenient, quia utrique aequalis angulus est percurrendus.

Faciamus ergo $\dfrac{M \cdot GO \cdot \sin O}{S + M \cdot OC^2} = \dfrac{\sin o}{oc}$

prodibit

$$oc = \frac{S + M \cdot OC^2}{M \cdot CO}$$

quae est longitudo penduli simplicis isochroni, seu distantia centri oscillationis in pendulo composito *OC* ab axe oscillationis, erit ergo centrum oscillationis in *Z,* ut sit

$$OZ = CO + \frac{S}{M \cdot CO};$$

unde apparet centrum oscillationis perpetuo infra centrum gravitatis *G* cadere, esseque intervallum

$$CZ = \frac{S}{M \cdot CO}.$$

HYPOTHESIS

184. *In omnibus corporibus aquae innatantibus* (Fig. 36) *praecipue vero in navibus concipere licet tres axes per centrum gravitatis G transeuntes inter se normales, primum verticalem scilicet CGD, secundum horizontalem AGB spinae RS parallelum, in plano diametrali situm ARSB et tertium EGF pariter horizontalem, si quidem navis fuerit in statu aequilibrii, et ad priorem AGB normalem. Deinde ponere licet corpus huiusmodi a viribus sollicitantibus circa unumquemque horum axium ita converti posse, ut motus gyratorius circa unum horum axium non turbetur a motibus gyratoriis circa reliquos.*

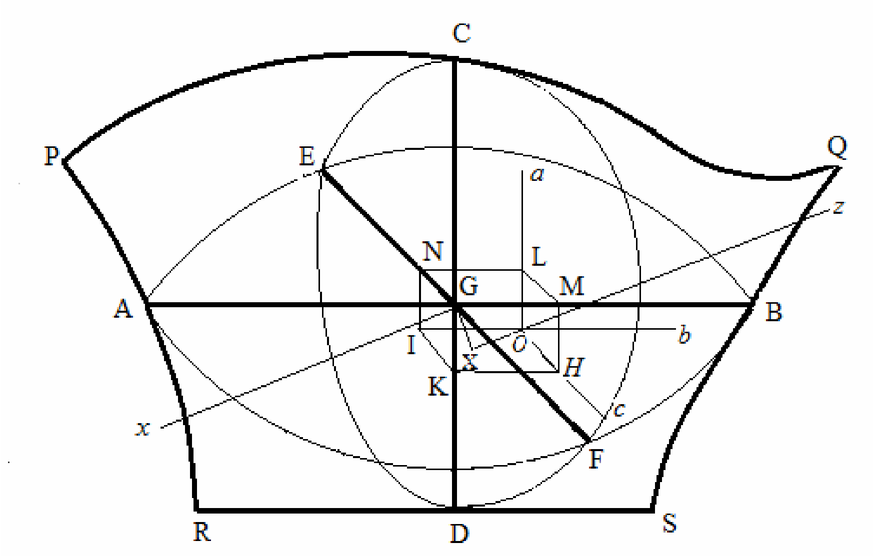


Fig. 36

SCHOLION 1

185. Ex superioribus satis intelligitur corpus circa alium axem liberum et immotum gyrari non posse, nisi circa quem omnes vires centrifugae se destruant. Quamobrem si vires sollicitantes corpus circa alium axem rotare conentur, motus orietur maxime irregularis, cum etiam axis inclinetur; quem motum definire difficillimum etiamnum est. Huic igitur incommodo medela afferetur, si talis motus irregularis resolvi posset in duos vel tres motus rotatorios circa axes fixos simul factos; tum enim cognito motu circa quemque axem seorsim, motus totus inde facile colligeretur. Quamvis autem haec resolutio accurate non succedat, tamen si ad praxin respiciamus, tuto satis adhiberi poterit, si tres illi axes inter se fuerint normales; tum enim motus circa unum minime a motibus circa reliquos turbabitur. Praeterea vero si hi axes ita sint comparati ut corpus circa quemlibet seorsim immotum gyrari queat, resolutio ista eo magis veritati erit consentanea. In navibus autem, ad quas hanc tractationem praecipue accomodare institui, huiusmodi tres axes vel revera vel proxime adesse solent. Quaelibet enim navis circa axem verticalem *CD* immotum gyrari potest, atque etiam circa axem *AB,* quippe qui in plano diametrali est situs; tertius autem axis *EF* pari praerogativa gaudet, prout experientia satis evincit.

COROLLARIUM 1

186. Si ergo huiusmodi corpori una pluresve potentiae fuerint applicatae, earum effectus tam in corpore promovendo, quam gyrando circa centrum gravitatis ex hactenus traditis praeceptis facile determinabitur. Primo enim omnes potentiae in directionibus sibi parallelis centro gravitatis concipiantur applicatae, ex iisque motus progressivus centri gravitatis concludatur. Deinde singularum potentiarum momenta in ternos illos axes

quaerantur, ex quibus motus rotatorius circa quemvis axem seorsim cognoscetur. Collatis denique his motibus rotatoriis inter se, verus motus circa centrum gravitatis satis accurate colligetur.

COROLLARIUM 2

187. Tria ergo requiruntur ad hos motus determinandos cuiusque corporis momenta inertiae respectu trium scilicet axium circa quos rotatio fieri concipitur.

COROLLARIUM 3

188. Si igitur in corpore tres huiusmodi axes inter se normales dentur, atque momentum corporis respectu cuiusque axis fuerit assignatum, tum quaecunque potentiae hoc corpus sollicitent, verus motus ab iis productus quam proxime poterit determinari: in hunc autem finem lemma sequens adieci.

LEMMA

189. *Si corpus PARSBQ* (Fig. 36) *urgeatur a quibuscunque potentiis, invenire motum qui in corpore generatur.*

SOLUTIO

Ex superioribus constat a potentiis quibuscunque in corpore duplicem generari motum progressivum scilicet centri gravitatis, et gyratorium circa centrum gravitatis; quorum motuum prior definitur si totum corpus in centro gravitatis concipiatur concentratum, eique omnes potentiae in directionibus parallelis ponantur applicatae. Ad motum gyratorium vero determinandum sint *AB, CD* ***et*** *EF* tres illi axes per centrum gravitatis *G* ducti et inter se normales, circa quorum quemlibet immotum corpus seorsim libere gyrari queat. Sit nunc momentum inertiae corporis respectu axis $CD = P$, momentum respectu axis $AB = Q$ et momentum respectu axis $EF = R$. Consideretur iam una quaecunque potentiarum sollicitantium *OZ,* quae in puncto *O* sit applicata, seu cuius directio per *O* transeat. Ex puncto hoc *O* in planum *ADBC* ducatur normalis *OH,* itemque in planum *AEBF* normalis *OL* atque in *CEDF* normalis *OI,* habebiturque iunctis *LN, LM, MH, HK, KI* et *NI* parallelepipedum rectangulum *NGMLOHKI.* Deinde potentia *OZ* in puncto *O* applicata resolvatur in tres potentias, *Oa, Ob, Oc,* quarum directiones sint inter se normales et parallelae axibus *GD, AB* et *EF.* Constat iam potentiae *Oa* momentum respectu axis *AB* fore $Oa.\ LM = Oa \cdot GN$; eiusdemque potentiae momentum respectu axis *EF* fore $= Oa \cdot LN = Oa \cdot MG$. Simili modo potentiae *Ob* momentum respectu axis *OD* erit $= Ob \cdot IK = Ob \cdot GN$, et momentum respectu axis $EF = Ob \cdot IN = Ob \cdot GK$. Denique potentiae *Oc* momentum respectu axis *CD* erit $= Oc \cdot HK = Oc \cdot GM$ atque momentum respectu axis $AB = Oc \cdot HM = Oc \cdot GK$. Corpus igitur circa unumquemque axem duobus momentis urgebitur quae inter se vel conspirant vel contrariantur. Spectata igitur congruentia vel repugnantia momentorum reperietur potentiae propositae *OZ* momentum ad corpus circa axem *CD* convertendum fore $= Ob \cdot GN + Oc \cdot GM$. Momentum vero respectu axis *AB* erit

$$= Oa \cdot GN - Oc \cdot GK.$$

Atque momentum respectu axis *EF* erit $= Oa \cdot GM + Ob \cdot GK$. Simili modo reliquae potentiae corpus sollicitantes sunt resolvendae, earumque momenta in singulos axes quaerenda, quae prout istis momentis vel favent vel repugnant, signo + vel - ipsis sunt adiiciendae. Ponatur igitur *p* pro momento potentiarum respectu axis *CD; q* pro momento respectu axis *AB;* et *r* pro momento respectu axis *EF.* His ergo inventis habebitur vis gyratoria circa axem $CD = \frac{p}{P}$; vis gyratoria respectu axis AB ~ ; atque vis gyratoria respectu axis $AB = \frac{q}{Q}$; quae vires cum coniunctim aeque agant, ac seorsim, verus motus gyratorius innotescet. Q. E. I.

COROLLARIUM

190. Si directio potentiae sollicitantis *OZ* per centrum gravitatis corporis *G* transeat, atque punctum *O* in *G* capiatur evanescet parallelepipedum *GNLMHOIK,* atque propterea vires gyratoriae omnes in nihilum abibunt, uti quidem alias constat.

COROLLARIUM 2

191. Si directio potentiae sollicitantis *OZ* parallela fuerit uni axium, tum corpus circa hunc axem non convertetur, sed tantum circa duos reliquos.

SCHOLION

192. Quia directio potentiae sollicitantis est linea recta, in ea ubicunque libuerit punctum *O,* in quo resolutio instituitur accipi potest: unde dubium oriri posset, utrum perpetuo eaedem vires gyratoriae circa singulos axes sint proditurae, mutato puncto *O,* an vero secus. Sed qui rem attentius perpendet, facile intelliget, in quocunque loco rectae *OZ* punctum *O* accipiatur, eadem momenta respectu axium reperiri debere.

PROPOSITIO 17

PROBLEMA

193. *Corporis cuiuscunque aquae insidentis et ex situ aequilibrii depulsi motum, quo se in situm aequilibrii restituet, determinare* (Fig. 36).

SOLUTIO

Sit *PARSBQ* corpus, cuius ex situ aequilibrii, quem in aqua tenet, depulsi restitutionem quaerimus; *G* eius centrum gravitatis, atque *CD, AB* et *EF* tres eius axes inter se normales, circa quorum quemvis immotum corpus libere rotari queat. Sit massa seu pondus huius corporis $= M$ atque momentum inertiae eius respectu axis *CDP;* momentum respectu axis $AB = Q$ et momentum respectu axis $EF = R$. Ponamus iam corpus hoc aquae ita insidere, ut partis submersae centrum magnitudinis sit in *O,* atque *O Z* linea ad superficiem aquae normalis, seu verticalis; sit *N* pondus aquae partem submersam volumine aequantis; urgebitur ergo hoc corpus a pressione aquae sursum in directione OZ vi $= N$. Simul vero deorsum urgebitur proprio pondere M in directione *GX* per centrum gravitatis *G* transeunte. Ab his ergo duabus viribus centrum gravitatis *G* sursum vel deorsum urgebitur prout *N* vel maius vel minus fuerit quam *M* id quod per se patet. Quemadmodum autem corpus interea circa centrum gravitatis *G* convertatur, sequenti modo ex sola vi *OZ* definietur; cum altera vis *GX* per centrum gravitatis ipsum *G* transeat. Ex *O* ducantur axibus singulis rectae parallelae *OLa, bOI, OHc* sitque $\cos \text{ang.} ZOa = a$; $\cos \text{ang.} ZOb = b$ et $\cos \text{ang.} ZOc = c$. His positis si potentia *N* in directione *OZ* trahens resolvatur in tres potentias iuxta directiones *Oa, Ob,* et *Oc* trahentes, erit $Oa = Na$; $Ob = Nb$ et $Oc = Nc$. Ex istis cum lemmate praecedente comparatis invenietur vis gyratoria circa axem

$$CD = \frac{N\left(b \cdot GN + c \cdot GM\right)}{P}$$

agens in sensum *AEBF.* Circa axem *AB* vero erit vis gyratoria

$$= \frac{N\left(a \cdot GN - c \cdot GK\right)}{Q},$$

agens in sensum *ECFD.* Vis denique gyratoria circa axem *EF* erit

$$= \frac{N(a \cdot GM + b \cdot GK)}{R}$$

agens in sensum *BCA.* Quae vires gyratoriae si simul considerentur, obtinebitur verus motus gyratorius circa centrum gravitatis. Q. E. I.

COROLLARIUM 1

194. Si ex centro gravitatis *G* in directionem potentiae sollicitantis *OZ* demittatur perpendiculum *GY* sine resolutione potentiae *OZ,* momenta eius respectu cuiusque axis poterunt determinari.

COROLLARIUM 2

195. Si enim sinus inclinationis axis *CD* ad planum *GYZ* fuerit *k* erit momentum potentiae $OZ = N$ respectu axis $CD = N \cdot GY \cdot k$.

COROLLARIUM 3

196. Similiter si sinus inclinationis axis *AB* ad planum *GYZ* fuerit *m,* erit momentum potentiae *N* respectu axis $AB = N \cdot GY \cdot m$.

COROLLARIUM 4

197. Atque si sinus anguli inclinationis axis *EF* ad planum *GYZ* ponatur *n* erit momentum potentiae *N* respectu axis $FE = N \cdot GY \cdot n$.

COROLLARIUM 5

198. Hinc ergo facilius patet momenta eadem prodire, in quocunque loco rectae *OZ* punctum *O* capiatur, cum eius positio in his formulis in calculum non ingrediatur.

SCHOLION

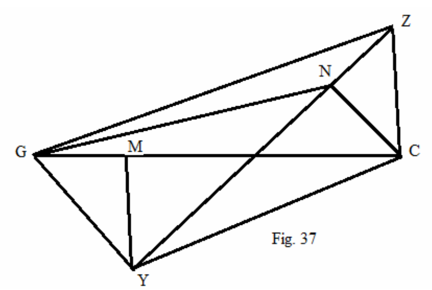


Fig. 37

199. Ratio horum momentorum in corollariis assignatorum fluit quidem ex forma momentorum propositionis, sed tamen facilius ex principiis staticis reddi potest sequenti modo. Sit (Fig. 37) corporis cuiusvis centrum gravitatis *G,* et axis per id transiens a :JC *GO,* cuius respectu momentum cuiuscunque potentiae corpus sollicitantis determinari oporteat. Urgeatur scilicet corpus a potentia *N* in directione *YZ,* in quam ex centro gravitatis *G* cadat perpendiculum *GY.* Iam ex *Z* in planum *GYC* demittatur perpendiculum *ZC,* iunctaque *CY,* erit planum *ZYC* normale ad planum *GYC,* atque *CY* normalis in *GY.* Resolvatur potentia *YZ* in binas laterales *YC* et alteram cuius directio est parallela ipsi *ZC;* habebitque haec posterior potentia sola, quae est $\frac{N \cdot ZC}{YZ}$ momentum respectu axis *GC.* Demisso igitur ex *Y* in *GC* perpendiculo *YM,* erit momentum potentiae respectu axis

$$GC = \frac{N \cdot ZC \cdot YM}{YZ} = \frac{N \cdot ZC \cdot GY \cdot YC}{YZ \cdot GC}.$$

Demisso nunc porro ex *C* in *Y Z* perpendiculo *CN,* erit $YZ : CZ = YC : CN$, unde prodit illud momentum $= N \cdot GY \cdot \frac{CN}{GC}$. Est vero *CN* perpendiculum ex *C* in planum *GYZ* demissum ob *CY* ad *G* et *ZY* ad *CY* atque *CN* ad *Y Z* normales. Quamobrem exprimet $\frac{CN}{GC}$ sinum anguli, quem axis *GC* cum plano *GZY* constituit. Qui sinus si dicatur *k* erit momentum $= N \cdot GY \cdot k$, uti in corollariis est assertum.

PROPOSITIO 18

PROBLEMA

200. *Si corpus aquae insidens utcunque fuerit ex situ aequilibrii deturbatum, definire axem per centrum gravitatis transeuntem, circa quem corpus gyrari incipiet.*

SOLUTIO

Sit corporis pars aquae immersa *MPNQXP* (Fig. 38), eiusque centrum magnitudinis *O* et pondus aquae volumine partem submersam aequantis $= N$. Sit *PMQN* sectio aquae in plano horizontali sita, et *OZ* recta verticalis, urgebitur ergo corpus a pressione aquae vi *N* in directione *OZ* sursum. Ponatur corporis centrum gravitatis in *G,* et ex *G* in *OZ* demittatur perpendicuium *GY.* Deinde per *G* ad planum *GY Z* ducatur normalis *DC* quae proin erit linea horizontalis. Circa hanc autem lineam horizontalem *DO* tanquam circa axem corpus in situm aequilibrii sese restituens, converti incipiet. Nam cum axis *DO* ad planum *GYZ* sit normalis, potentia *N* nullum habebit momentum in alios axes huic axi *DC* normales. Tota ergo potentia vim suam impendet ad corpus circa axem *DC* convertendum, eritque eius momentum $= N \cdot GY$. Q.E.I.

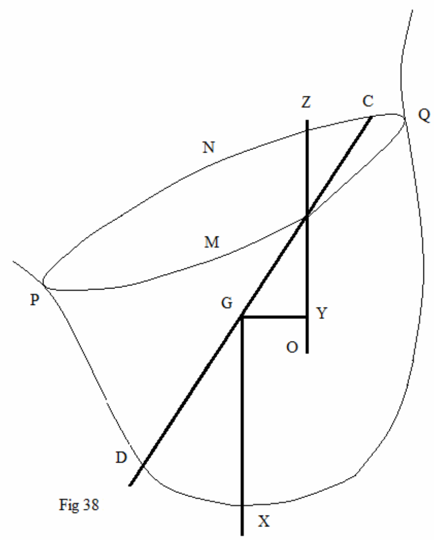


Fig 38

COROLLARIUM 1

201. Si ergo axis *DC* ita fuerit comparatus, ut corporis circa eum rotantis vires centrifugae se mutuo destruant, tum corpus perget circa hunc axem immotum gyrari.

COROLLARIUM 2

202. Corpus ergo aquae insidens ex situ aequilibrii deturbatum, in eum restituetur motu gyratorio circa axem quendam horizontalem: dum interea centrum gravitatis vel sursum vel deorsum tantum fertur.

SCHOLION

203. Etiamsi autem corpus circa axem *CD* immotum libere gyrari possit, tamen inde non sequitur, integram restitutionem fieri circa hunc axem. Nam inter hunc motum directio *OZ* mutari potest, unde quoque variatio axis *DO* oritur. Sed ex his tamen satis motus restitutionis perspicitur.