CHAPTER IV

CONCERNING CONCORDS

1. Several simple sounds being played together constitute a composite sound, which we will call here a concord [or consonance]. Indeed, the voice of the concord may be taken in a stricter sense than with other sounds, so that it may specify only composite sounds pleasing to be heard and having much charm in it, and musicians distinguish this from a discord [or dissonance], which is composed from sounds played together containing little or no charm. But because there is some difficulty in defining the boundaries between concords and discords, truly this distinction only agrees partially with our treatment, by which we are going to judge the degree of pleasantness sounds produce together, according to chapter II, which depends on several simple sounds being heard together [or in a sequence], to which we will attribute the name concord.

2. Therefore so that concords of this kind may please, it will be required that a ratio may be perceived, which the simple sounds constituting that hold between each other. But since here the duration of the sounds is not to be observed, only their variations, which is present in the high and low notes of the sounds, that same variation will hold the perceived charm or pleasing to the ear. On account of which, since the low and high sounds are produced by pulses in the same time, it is evident a number shall be required to be measured, which may comprise a mutual relation between these numbers, by which that same charm of the concords should be perceived.

3. But now above, we have represented these sound by the pulses which they make in a given time, and from this the numbers to be measured expressing the quantity of the sound or tenor of the piece of music, which may be present in the account of the high and low notes. And thus, so that the proposed concords may please, it is necessary that a ratio be perceived which the number of the simple sounds, or that the sounds themselves maintain among themselves (for we may consider the sounds as quantities). Therefore in this manner we may recall a perception of the concords for an understanding of the numbers, by which the precepts concerning this matter have been treated in the second chapter, from which it can be understood, how the charm shall be required to be judged from whatever concords in some way.

4. Therefore it will be easy to reduce concords of any perceived kind to certain orders of charm [*i.e.* orders of pleasurable listening], from which it will be apparent, whether it will be an easy or difficult thing to do, and in addition to which order the proposed concords may be had in mind. Besides truly several concords can be compared amongst themselves and from these it will be possible to judge, which shall be considered the easier or more difficult, and likewise it will be able to be defined, when some one easier than another may be able to be taken. Therefore from the given number of concords there must be found, which is the least common divisor of the numbers expressing the simple sounds, and that to be investigated, to

whatever order it may pertain. Indeed from this it will be evident, how many concords may be required to be considered.

5. Therefore since there shall be a need for the least common divisor of the simple sounds [*i.e.* the individual tones of the scale used], it will be required always to express these sounds by the smallest whole numbers, which maintain the same ratio between themselves ; concerning which this may be evident, if these whole numbers may have no common divisor apart from unity. Therefore without doubt the first operation then is the least common dividend requiring to be found according to the precepts treated in the second chapter. Finally it may become known from the same precepts, to which order of the charm this least common divisor may pertain, and it is required to be considered to pertain to the concords of the same. Indeed as long as this smallest common dividend itself does not exceed ten, there is no need for this last operation, since the table given above contains all these orders.

6. But in the following we will call this smallest common divisor of the simple sounds composing the concords the *exponent* of the concords ; indeed with this known, likewise the nature of its concords is seen. But how the degree of the charm must be found from this given exponent is explained in Ch. II, § 27, in this manner: here the exponent is resolved into its simple factors and of which the sum shall be *s*. Truly the number of these factors may be put $=n$; the degree of the charm, to which the proposed concords refer , will be $s-n+1$; and where this number is found to be smaller, thus the concords will be have more charm, or be easier to perceive.

7. Also not incongruously, the number of concords may be divided according to the simplicity of the sounds, from which they are composed ; and hence some will be bi-chords, others tri-chords and others multi-chords, just as the sound may consist of two, three or more sounds played together. In bi-chords therefore there shall be two sounds, consisting of *a* and *b*, or these numbers at least may maintain this ratio of the notes. Therefore *a* and *b* will have to be whole numbers relatively prime to each other. And on this account the smallest divisor of these will be *ab* and thus here the number *ab* will be the exponent of the proposed concords, to which the degree of the charm, to which it pertains, becomes known. Moreover we may review the concords of this kind according to the degree of the charm, so that from the order itself it may be apparent, as for each how easy or difficult it may be perceived to be.

8. Truly for an enumeration of this kind to be completed for this it is needed only, that the individual numbers from the table of Ch. II may be picked out and added according to the order, and any two factors of which may be resolved into two factors relatively prime, which often will be able to be done in several ways. With this done, two factors of this kind will give the sounds of a bi-chord, of which the exponent will be that number itself, from which these factors will be derived. For example in the fifth order 12 is had, which can be resolved into two relatively prime factors : 1, 12 and 3, 4. Therefore sounds of this kind constitute the concords pertaining to the order V, of which the exponent is 12.

9. Therefore for the first order, in which unity is present, there is no concern with bi-chords or several chords. For since the sounds constituting the concord ought to be different, of these unity alone will be able to be the least common divisor or the exponent. On this account the simplest concord will pertain to the second order and that they will constitute sounds maintaining the ratio 1 : 2, of which therefore the exponent is 2, which number is found alone in the second order. This concord is called a *diapason* or octave by musicians and it is considered by them to be the most simple and perfect concord ; for it may be discerned and distinguished the most easily by ear from the others concords.

10. For the third order we report on the two numbers 3 and 4, each of which is resolved into two factor prime between themselves or besides having no other common divisor apart from unity, clearly the first is resolved into 1 and 3, truly the second into 1 and 4. Therefore the two produce bi-chord concords pertaining to the third order, of which the one agrees with sounds having the ratio 1:3, the other truly from sounds in the ratio
1 : 4. The one is accustomed to be called the twelfth or the double fifth, truly the other the fifteenth or double octave, nor from these can there be any doubt, why the following may not be perceived more easily.

11. In this way the following table of bi-cord concords can be assembled, in which these have been set out following the second order of the charm established above, as far as to the tenth order :

Gr. II: 1:2.
Gr. III: 1 : 3, 1 : 4.
Gr. IV: 1 : 6, 2 : 3, 1 : 8
Gr. V: 1 : 5, 1 : 9, 1 : 12, 3 : 4, 1 : 16.
Gr. VI: 1 : 10, 2 : 5, 1 : 18, 2 : 9, 1 : 24, 3 : 8, 1 : 32.
Gr. VII: 1 : 7, 1 : 15, 3: 5, 1 : 20, 4 : 5, 1 : 27, 1 : 36, 4: 9, 1 : 48, 3 : 16, 1 : 64.
Gr. VIII: 1 : 14, 2: 7, 1: 30, 2: 15, 3: 10, 5: 6, 1 : 40, 5: 8, 1 :54, 2: 27, 1: 72,
8: 9, 1 : 96, 3: 32, 1 : 128.
Gr. IX: 1 : 21, 3 : 7, 1 : 25, 1 : 28, 4 : 7, 1 : 45, 5: 9, 1 : 60, 3: 20, 4 : 15, 5 : 12,
1 : 80, 5: 16, 1 : 81, 1: 108, 4: 27, 1 : 144, 9: 16, 1 : 192, 3: 64, 1 : 256.
Gr. X: 1 : 42, 3: 14, 6: 7, 1 :50, 2: 25, 1: 56, 7: 8, 1 : 90, 2: 45, 5: 18, 9: 10,
1 : 120, 3 : 40, 5 : 24, 8 : 15, 1 : 160, 5 : 32, 1 : 162, 2 : 81, 1 : 216, 8 : 27,
1 : 288, 9 : 32, 1 : 384, 3 : 128, 1 : 512.

12. From Ch. I, §11 it is understood how the two chords must be played, so that the sounds produced maintain the given ratio ; therefore in this manner it will be easy to play these concords with strings and for the whole piece itself to be expressed, however easier or more difficult it may be perceived to be; moreover it will be found much experience agrees with this theory. Truly I judge from the experience to be gained from listening to music carefully not only is it very useful, but also to be especially necessary ; for on this account distinct ideas of these simpler concords will be agreed on, and from these more suitable ideas emerge for the practice of performing music.

13. Nor truly is it necessary that anyone who contributes to music, should give attention to the enumerations of all the concords about which he may have formed ideas, for it is sufficient to bear in mind primarily those to be expressed properly, namely, only
1 : 2, 1 : 3, 2 : 3, 1 : 5, 2 : 5, or 4 : 5. For anyone who will know, not only how to distinguish these from all the others, but also how to produce these, either to be formed by the voice or to be heard with the aid of strings, this also to be put in place for all the remaining concords alone will be able to be heard, of which the exponents have no other divisors except 2, 3 and 5. And this will suffice for the day to day music and for the musical instruments requiring to be constructed. Truly I will be setting this out further in the following.

14. Now I have reminded myself here that under the name of concords both concords as well as discords thus commonly said to be included. Moreover, from the nearby table and from our method in a certain way boundaries are seen to be able to be defined. For the discords belong to higher orders, and the concords are obtained, from those which belong to the lower orders. Thus the tone, which agrees with sounds having the ratio 8 : 9 and is related to the VIIIth order, is counted amongst the discords, truly the double tone or the third major contained in the ratio 4 : 5, which belongs to the VIIth order, pertains to the concords. Yet neither can discords be put in place from the beginning of the VIIIth order; for the ratios 5 : 6 and 5 : 8 are present in the same, which are not considered to be discords.

15. Hence if the matter be considered more carefully it will be agreed not only an account of the discords and concords be required, but also an account of the whole composition must be considered. Which concords indeed in singing which are less able to be used conveniently, are called by the name of discords, even if perhaps they may be easier to be perceived than others which are referred to as concords. And this is the reason why the tone 8 : 9 may be counted with the discords and other much higher concords composed may be considered as concords. In a similar manner it is required to be explained from this, why the musical fourth formed from two sounds having the ratio
3 : 4 considered by musicians as discords rather than concords, since still there shall be no doubt why that may not be perceive very easily.

16. Indeed according to the ancient musicians this fourth was considered as very pleasant concord, as may be apparent from their writings. But uses were made by other methods for distinguishing between discords and concords, which were less well based on the nature of the matter and deduced from precarious principles. Indeed PYTHAGORAS did not judge other sounds suitable for producing concords, except these which could be considered from the ratio of two sounds consisting either a pure multiple, a non-integral multiple greater than one, or a multiple of this fraction less than one; truly those were thought to produce a discord whenever the ratio of these two sounds was a non-integral fraction or a multiple of this fraction.

17. PTOLEMY in his work *Libri Harmonicorum* [*Books of Harmonies*] refutes this belief of the Pythagoreans in the experimental testing alleging the agreement of the double fourth, which is a concord, although the ratio 3 : 8 shall be two and two thirds. Then he observed that the Pythagoreans themselves did not use this rule with care, while they did not make use of

other ratios apart from the double, triple, quadruple, one and a half, and one and a third, [*i.e.* 1 : 2, 1 : 3, 1 : 4, 2 : 3, 3 : 4, etc.] since yet enumerable others justly shall be able to follow the same rule. Truly I find nothing reprehensible in this refutation by Ptolemy; indeed not for the kinds of ratios, but for the simplicity and the ease of perception required to be seen.

18. Yet neither is this principle of Ptolemy firmer, of which use is made in this matter to denounce the other; for only two concords are allowed between the octave and double octave, which may be held in approximately equal non-integral fractional ratios and by producing together a double ratio. Moreover the ratios 2 : 3 and 3 : 4 are of this kind, which together give the ratio 1 : 2. From the first arises the said fifth concord, and from the latter truly the fourth concord. Then Ptolemy considers this second principle : any concord increased by an octave remains a concord and loses nothing of its charm, and in this manner a number of concords receive these ratios 1 : 2, 1 : 4, 2 : 3, 1 : 3, 3 : 4 and 3 : 8.

19. In spite of what we have just seen, nevertheless Ptolemy does not attribute a great prerogative to the large non-integer fractional ratios over the smaller purely fractional ratios; nor indeed does he call other sounds discords which maintain fractional ratios other than 2 : 3 and 3 : 4 , but he calls them by a name that holds a certain mean position between being concords or discords, namely *neat* [or *elegant*] chords. Truly the remaining fractional ratios besides 3 : 8 produce strong discords. But I do not judge it necessary to measure the degree of the charm of concords evidently be a precarious method, without refuting a superstructure based on firmer principles, since the truth of our principles is now abundantly clear and from that itself the nature to be derived. It remains for us now to make known the opinions set forth on this subject by another school of ancient musicians, of which the author was Aristoxenus, concerning which I may establish truly at once that these ratios of numbers were rejected, thus so that judgement of concords and discords was left entirely to the senses, in which this did not differ greatly from that of the Pythagoreans.

20. The enumeration of three-sound and multi-sound concords may be composed in a similar manner to concords of two sounds according to the degree of charm, thus so that thus it would be quite superfluous to set these out. Yet it is convenient to bear in mind the simplest of the three sound concord pertaining to the third order of the charm and to consist of the sounds 1 : 2 : 4, of which the exponent is 4. From which it is understood how several concord sounds shall be composed from that, and these pertain also to a higher order of charm, even if it shall be the simplest in its kind.

21. But I shall not pursue this division of the concords further here, since soon I shall be bringing forth another much more suitable and useful division, which shall be into *complete* and *incomplete* concords. Moreover I call a concord complete, to which no extra sound can be added, without likewise the concord itself shall be required to be referred to a higher order, or its exponent may be made more composite ; the concord of the sounds 1 : 2 : 3 : 6 consists of this kind, of which the exponent is 6. Indeed with any now sound added the exponent must be made greater. On the other hand a concord is *incomplete* for me, for which one or more sounds may be allowed before the multiplication of the exponent ; so that the exponent of this concord

1: 2: 3 shall not be greater, even if the sound 6 may be added, on account of which I call that incomplete.

22. Moreover from the preceding it is understood any simple number denoting a sound to be a divisor of the exponent of the concord. Whereby if all the divisors of the exponent may be taken and from them just as many simple sounds may be expressed, the complete concord of that exponent will be expressed; for besides these there will be no other number, which hence may divide the exponent. Thus the concord consisting of the sounds 1 : 2 : 3 : 4 : 6 : 12 will be complete, since only these numbers are divisors of the exponent of this concord, which is 12, and no other number besides divide the number 12.

23. Therefore as often as the exponent of the concord is a prime number, the complete concord is a bi-cord, as 1 : a, if a may denote a prime number. If the exponent were a^m, the complete concord will consist of $m+1$ sounds, namely

$$1:a:a^2:a^3:\cdots\cdots:a^m.$$

If the exponent may have this form ab, made from two prime numbers, the complete concord will be the will be the four sounds

$$1:a:b:ab$$

and with $a^m b^n$ being the exponent present, the complete concord will be expressed by

$$mn+m+n+1$$

sounds. And more generally if the exponent were $a^m b^n c^p$, the complete concord will contain

$$(m+1)(n+1)(p+1)$$

sounds and, following the rule given in § 6, will pertain to the order of the charm or agreement expressed by the number

$$ma+nb+pc-m-n-p+1;$$

for the sum of all the simple factors of the exponent is $ma+nb+pc$ and the number of the factors is $m+n+p$.

24. It is evident from the manner established of forming complete concords, if one or more sounds may be omitted from this, the remaining concords then become incomplete. In which it is required to be noted sounds of this kind should be omitted, so that the exponent of the remainder may not become simpler : as if from this concord 1: 2: 4, the exponent of which is 4, the sound 1 or 4 may be missing, the concord will be produced 1 : 2 or 2 : 4 agreeing with

that concord, of which the exponent no longer will become 4, but only 2. Truly the mean sound 2 can be omitted ; for the exponent of the concord 1 : 4 also now is 4, on account of which the complete concord is 1 : 2 : 4.

25. If the exponent is a prime number, it is apparent the concord has to be complete, because there it consists only of two-notes. But the remaining concords can all become incomplete and that with all the intermediate pairs of sounds [of increasing frequency] missing, except the lowest and the highest ; because indeed here from this exponent, truly that is expressed by unity, the exponent of this two-note concord will not be simpler than the complete concord: so that from the concord 1 : 2 : 3 : 6 with the notes 2 and 3 removed the exponent of the concord of that 1: 6 equally is 6 . Thence in concords, of which the exponent is of this form a^m, neither the deepest sound 1 nor the sharpest a^m can be rejected ; truly with all the remaining both the lowest as well as the uppermost, each and every one can be omitted, and even all these two at a time.

26. If a concord has been prepared thus, so that in that no sound may be omitted, whereby likewise a simpler concord may not emerge and it may not pertain to a lower order, here we will call that *pure*. All the bi-cords are of this kind, because with either of the sounds omitted they cease to be concords. In a similar manner the concords 3 : 4 : 5, 4 : 5 : 6 are pure, as well as 1 : 6 : 9, 2: 3 : 12, in which no sound can be omitted, so that they may not become simpler. Thus the use of these concords consists in this, so that the number of the sounds, may be diminished as far as possible, thus yet, so that the exponent may not be made smaller.

27. But any concords can be made simpler by removing any one or more of the sounds in two ways ; the first of which occurs when the smallest multiple of the remaining sounds, emerges smaller than that of all the sounds which compose the concord, as in the concord 2 : 3 : 5 : 6 with the sound 5 removed, the least common multiple of the remaining 2 : 3 : 6 is 6, which before was 30. The other way makes a simpler concord, when the remainder of the rest of the sounds have a common divisor ; for then they must be divided by that, as the smallest common multiple or exponent may be defined, so that in this concord 2: 3: 4: 6 with the sound 3 removed the rest divided by 2 constitute the concord 1 : 2: 3, of which the exponent is 6; truly which was 12 before.

28. Also in each way the adjoining concord can become simpler with one or more sounds removed, evidently when the number of the remaining simpler sounds have a least common multiple and also a common divisor : just as there shall become in this concord 3 : 6 : 8 : 9 : 12, of which the exponent is 72, if the sound 8 may be removed ; for the least common multiple of the rest 3: 6: 9: 12 is 36; or since these individual numbers can be divided by 3, the resulting concord is agreed to be 1: 2: 3:4, of which the exponent therefore will be 12. And thus with a much simpler concord emerging from the proposed concord with the single sound 8 omitted.

29. Moreover so that it may be understood more clearly, how any proposed concord can be made simpler, we will consider the complete concord, of which the exponent is a^mP, where P

is a quantity including prime numbers besides a. Therefore in this case, if all the sounds and its multiples corresponding to a^m and its multiples may be rejected, a simpler concord of the exponent P will remain, which reduction according to the first way has been made. But by the second way the concord may be made simpler, if all the sounds may be omitted, which are expressed by the sounds not containing a, and the rest divided by a, and of which the exponent is a^{m-1}. From which it is understood, how a simpler concord collection may be effected by each method.

30. The distinction, which the listener perceives between complete and incomplete concords, consists in this, as can be easily understood, because complete concords may be considered complete much more distinctly by the ear than incomplete concords. And even if all sounds affect the ear at the same time, it is necessary that the clearer of the individual sound offer themselves to the sense, as if the exponent must be deduced from fewer sounds. Thus from the concord 1 : 2 : 3 : 6 its exponent which is 6 is understood more distinctly than only from the two sounds 1 : 6. But for this it is required that all the sounds may correspond most precisely to the numbers, by which they are expressed.

31. Moreover of all the complete concords, which are contained in the twelve first orders, it is seen the following suitable table to be added, in which Roman numerals indicate the order, Arabic numbers relate these concords which relate each to its order.

I.	1.
II.	1:2.
III.	1:3,
	1:2:4.
IV.	1:2:3:6,
	1:2:4:8.
V.	1:5,
	1:3:9,
	1: 2:3:4:6:12,
	1:2:4:8:16.
VI.	1:2:5:10,
	1:2:3:6:9:18,
	1:2:3:4:6:8:12:24,
	1:2:4:8:16:32.
VII.	1:7,
	1:3:5:15,
	1:2:4:5:10:20,
	1:3:9:27,
	1:2:3:4:6:9:12:18:36,
	1:2:3:4:6:8:12:16:24:48,
	1:2:4:8:16:32:64.
VIII.	1:2:7:14,
	1:2:3:5:6:10:15:30,
	1:2:4:5:8:10:20:40,

1:2:3:6:9:18:27:54,
1:2:3:4:6:8:9:12:18:24:36:72,
1:2:3:4:6:8:12:16:24:32:48:96,
1:2:4:8:16:32:64:128.

IX. 1:3:7:21,
1:5:25,
1:2:4:7:14:28,
1:3:5:9:15:45,
1:2:3:4:5:6:10:12:15:20:30:60,
1:2:4:5:8:10:16:20:40:80,
1:3:9:27:81,
1:2:3:4:6:9:12:18:27:36:54:108,
1:2:3:4:6:8:9:12:16:18:24:36:48:72:144,
1:2:3:4:6:8:12:16:24:32:48:64:96:192,
1:2:4:8:16:32:64:128:256.

X. 1:2:3:6:7:14:21:42,
1:2:5:10:25:50
1:2:4:7:8:14:28:56,
1:2:3:5:6:9:10:15:18:30:45:90,
1:2:3:4:5:6:8:10:12:15:20:24:30:40:60:120,
1:2:4:5:8:10:16:20:32:40:80:160,
1:2:3:6:9:18:27:54:81:162,
1:2:3:4:6:8:9:12:18:24:27:36:54:72:108:216,
1:2:3:4:6:8:9:12:16:18:24:32:36:48:72:96:144:288,
1:2:3:4:6:8:12:16:24:32:48:64:96:128:192:384,
1:2:4:8:16:32:64:128:256:512.

XI. 1:11,
1:5:7:35,
1:3:7:9:21:63,
1:3:5:15:25:75,
1:2:3:4:6:7:12:14:21:28:42:84,
1:2:4:5:10:20:25:50:100,
1:2:4:7:8:14:16:28:56:112
1:3:5:9:15:27:45:135,
1:2:3:4:5:6:9:10:12:15:18:20:30:36:45:60:90:180,
1:2:3:4:5:6:8:10:12:15:16:20:24:30:40:48:60:80:120:240,
1:3:9:27:81:243,
1:2:4:5:8:10:16:20:32:40:64:80:160:320,
1:2:3:4:6:9:12:18:27:36:54:81:108:162:324,
1:2:3:4:6:8:9:12:16:18:24:27:36:48:54:72:108:144:216:432,
1:2:3:4:6:8:9:12:16:18:24:32:36:48:64:72:96:144:192:288:576,
1:2:3:4:6:8:12:16:24:32:48:64:96:128:192:256:384:768,
1:2:4:8:16:32:64:128:256:512:1024.

XII. 1:2:11:22,
1:2:5:7:10:14:35:70,
1:2:3:6:7:9:14:18:21:42:63:26,
1:2:3:5:6:10:15:25:30:50:75:150,
1:2:3:4:6:7:8:12:14:21:24:28:42:56:84:168,
1:2:4:5:8:10:20:25:40:50:100:200,
1:2:4:7:8:14:16:28:32:56:112:224,
1:2:3:5:6:9:10:15:18:27:30:45:54:90:135:270,
1:2:3:4:5:6:8:9:10:12:15:18:20:24:30:36:40:45:60:72:90:120:180:360,
1:2:3:4:5:6:8:10:12:15:16:20:24:30:32:40:48:60:80:96:120:160:240:480,
1:2:3:6:9:18:27:54:81:162:243:486,
1:2:4:5:8:10:16:20:32:40:64:80:128:160:320:640,
1:2:3:4:6:8:9:12:18:24:27:36:54:72:81:108:162:216:324:648,
1:2:3:4:6:8:9:12:16:18:24:27:32:36:48:54:72:96:108:144:216:288:432:864,
1:2:3:4:6:8:9:12:16:18:24:32:36:48:64:72:96:128:144:192:288:384:576:1152,
1:2:3:4:6:8:12:16:24:32:48:64:96:128:192:256:384:512:768:1536,
1:2:4:8:16:32:64:128:256:512:1024:2048.

32. Truly although complete concords may offer themselves to be heard more distinctly than incomplete ones, unless they shall be exceedingly simple, complete concords will not be used. For in the first place then a great number of sounds, if the musical instruments are not accurately tuned, because by no means can that be put into effect, for the ears rather will be confused by a loud noise than struck by a distinct harmony. Thence also several sounds indeed cannot be heard either on account of being exceedingly low or exceedingly high in pitch ; indeed in the first chapter it has now been shown no sound, which produces fewer than 30 beats per second or more than 7500 beats per second, is able to be heard by the ears. From which it is evident, whenever the concords of the sound may hold a ratio greater than 250 : 1, not even all its sounds are able to be heard.

33. It is convenient to attach to this principle concerning concords that which musicians are accustomed to deal with concerning the intervals between the sounds. Moreover the *interval* is called that distance, which is to be considered between two sounds, whether the one be lower or higher than the other. Therefore with that, the greater the interval is, by which the sounds may disagree in the ratio between the lowest and the highest, or where the greater is the ratio, which the sharper has to the deeper. Thus the interval of the sounds 1 : 3 is greater than that of the sounds 1 : 2; and of the equal sounds 1 : 1, since with no jump from one to the other arises, the interval is zero. From which the interval is understood to be defined thus, so that it shall be a measure of the difference between the higher and lower sounds.

34. There shall be three sounds $a : b : c$, of which c shall be the highest, a the lowest, b truly some intermediate ; it will be apparent from the preceding definition the interval of the sounds a and c to be the sum of the intervals between a and b and between b and c. Whereby if these two intervals between a and b and between b and c were equal, since that arises, when

$a:b=b:c$, the interval *a* : *c* will be twice as great as the interval between *a* : *b* or *b* : *c*. From which the interval 1 : 4 is seen to be twice as great as the interval 1: 2, and on this account, since this ratio 1: 2 may be put in place to constitute an octave interval, the ratio 1 : 4 will contain two octaves.

35. Anyone who may consider this more carefully, easily understands that the intervals must expressed from a measure of the ratios, which constitute the sounds. But the ratios may be measured from the logarithms of the fractions, the numerators of which denote the higher sounds, truly the denominators the lower ones. Accordingly the interval between the sounds $a:b$ will be expressed by the logarithm of the fraction $\frac{b}{a}$, which it is the custom to be designated by $\log.\frac{b}{a}$ or, since it returns the same, by $\log.b-\log.a$. Therefore the interval of the equal sounds *a* : *a* will be zero, as we have now observed, certainly because it is expressed by $\log.a-\log.a=0$.

36. And thus by having the interval, which is called the octave (in Greek *διαπασῶν*, *i.e. diapason*), since by having the double ratio of the sounds, it is expressed by the logarithm of two ; and the interval of the sounds 2 : 3, which is called the fifth or *diapente,* will be $\log.\frac{3}{2}$ or $\log.\,3-\log.\,2$. From which it is understood this interval generally to be incommensurable between themselves; for in no way can be ratio be assigned, what $\log.2$ has to $\log.\frac{3}{2}$, and on this account no interval can be given, however small, of which the octave and fifth may be aliquot parts. The ratio of all the other intervals is similar, which are expressed by the disparity of the logarithms, such as $\log.\frac{3}{2}$ and $\log.\frac{5}{4}$. On the other hand truly these intervals, which are expressed by the logarithms of numbers which shall be of powers of the same root, will be able to be compared between themselves; thus so that the interval of the sounds 27 : 8 will itself be had to the interval of the sounds 9 : 4 as 3 to 2; for
$\log.\frac{27}{8}=3\log.\frac{3}{2}$ and $\log.\frac{9}{4}=2\log.\frac{3}{2}$.

37. From which also it is readily apparent, for whichever intervals may arise from addition or subtraction or of several amongst themselves, with these being made from the same operations in terms of logarithms, which are measures of the intervals ;for with this done the resulting logarithm expresses the exponent of the interval arising. So that if the interval may be sought, which may remain with the musical fifth taken from the octave, $\log.\frac{3}{2}$ will be required, or $\log.3-\log.2$ to be taken from log.2 and the remainder will be $\log.2-\log.3+\log.2$, *i. e.* $2\log.2-\log.3$. But $2\log.2=\log.4$; from which the interval remaining will be $\log.4-\log.3$ or $\log.\frac{4}{3}$, which is called the musical fourth and which taken with the fifth completes the octave.

38. But any of the different logarithms of the divisors of the numbers cannot be compared between themselves, unless the numbers were powers of the same root, yet with the aid of tables

of logarithms an approximately true ratio of these can be defined and thus the different intervals, as far as it can be done, can be compared with each other. Therefore since the measure of the octave shall be log.2, which extracted from the table is $=0,3010300$, and of the fifth $\log.3-\log.2$, which difference is $=0,1760913$, the interval of the octave to the interval of the fifth will be approximately as 3010300 to 1760913. Which ratio, when it may be reduced to smaller numbers, is changed into this

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3}}}}$$

to 1, from which these simple ratios are derived 2 : 1, 3 : 2, 5 : 3, 7 : 4, 12 : 7 and 17 : 10, 29 : 17, 41 : 24, 53: 31, of which the final one is approximately true.

39. In a similar manner also the intervals can be divided into just as many parts equal parts of whatever size one would wish, and for the true sounds to be designated approximately, which the partial intervals of this kind may extend in turn between themselves. For the logarithm of the proposed interval is required to be divided into just as many parts and the corresponding number in the tables is taken, which will have the ratio sought to unity. For example an interval three times smaller than the octave may be sought ; its logarithm will be $=0,1003433$, certainly the third part of log.2, to which corresponds 126 : 100 or 63 :50, which less accurately is either 29: 23 or again 5:4, which is the least exact, by which finally the major third is indicated, which by people less well versed in music regard as the third of the octave.

CAPUT IV

DE CONSONANTIIS

1. Plures soni simplices simul sonantes constituunt sonum compositum, quem hic *consonantiam* appellabimus. Ab aliis quidem consonantiae vox strictiore sensu accipitur, ut tantum denotet sonum compositum auditui gratum multumque suavitatis in se habentem, hancque consonantiam distinguunta dissonantia, quae ipsis est sonus compositus parum vel nihil suavitatis complectens. At quia partim difficile est consonantiarum et dissonantiarum limites definire, partim vero haec distinctio cum nostro tractandi modo minus congruit, quo secundum suavitatis gradus capite II expositos sonos compositos sumus iudicaturi, omnibus sonitibus, qui ex pluribus sonis simplicibus simul sonantibus constant, consonantiae nomen tribuemus.

2. Quo igitur huiusmodi consonantia placeat, oportet, ut ratio, quam soni simplices eam constituentes inter se tenent, percipiatur. Quia autem hic duratio sonorum non spectatur, sola varietatis, quae in sonorum gravitate et acumine inest, perceptio istam suavitatem continebit. Quamobrem, cum gravita set acumen sonorum ex pulsuum eodem tempore editorum numero sint mensuranda, perspicuum est, qui horum numerorum mutuam relationem comprehendat, eandem suavitatem consonantiae sentire debere.

3. Supra autem iam constituimus ipsos sonos per pulsuum, quos dato tempore conficiunt, numeros exprimere ex hocque sonorum quantitatem seu tenorem, qui gravitatis et acuminis ratione continetur, metiri. Quo itaque proposita consonantia placeat, necesse est, ut ratio, quam sonorum simplicium quantitates seu ipsi soni (sonos enim tanquam quantitates consideramus) inter se tenent, percipiatur. Hoc igitur modo consonantiarum perceptionem ad numerorum contemplationem revocamus, qua de re in secundo capite praecepta sunt tradita, ex quibus intelligi potest, quomodo de cuiusvis consonantiae suavitate sit iudicandum.

4. Facile igitur erit consonantiae cuiusvis perceptionem ad certum suavitatis gradum reducere, ex quo apparebit, utrum facile an difficile et insuper quo gradu proposita consonantia mente comprehendatur. Praeterea vero etiam plures consonantiae inter se poterunt comparari de iisque iudicare licebit, quae sit perceptu facilior quaeve difficilior, simulque definiri poterit, quanto alia facilius quam alia possit comprehendi. Data ergo consonantia numerus debet inveniri, qui est minimus communis dividuus numerorum simplices sonos exponentium, isque investigari, ad quemnam gradum pertineat. Ex hoc enim manifestum erit, quantum ad consonantiam percipiendam requiratur.

5. Cum igitur opus sit minima communi dividuo sonorum simplicium, oportebit semper hos sonos numeris integris exponere iisque minimis, qui eandem inter se tenent rationem; cuius rei hoc habetur indicium, si isti numeri integri nullum habeant communem divisorem praeter unitatem. Hac ergo quasi prima operatione absoluta deinceps inveniendus est minimus

communis dividuus secundum praecepta capite secundo tradita. Denique per eadem praecepta innotescet, ad quem minimus hic communis dividuus gradum suavitatis pertineat, atque ad eundem ipsius consonantiae perceptio pertinere est censenda. Quoties quidem iste minimus communis dividuus non gradum sedecimum excedit, hac postrema operatione non est opus, quia tabula supra data hos omnes gradus continet.

6. Vocabimus autem in posterum minimum hunc communem dividuum sonorum simplicium consonantiam componentium *exponentem* consonantiae; hoc enim cognito simul ipsius consonantiae natura perspicitur. Quomodo antem ex dato hoc exponente gradus suavitatis inveniri debeat, § 27 cap. II docetur hoc modo: Exponens hic resolvatur in factores suos simplices omnes horumque summa sumatur, quae sit *s*. Factorum vero horum numerus ponatur $= n$; erit suavitatis gradus, ad quem proposita consonantia refertur, $s-n+1$; quo itaque minor reperitur hic numerus, eo erit consonantia suavior seu perceptu facilior.

7. Non incongrue etiam consonantiae dividuntur secundum sonorum simplicium, ex quibus sunt compositae, numerum; atque hinc aliae erunt bisonae, aliae trisonae aliaeque multisonae, prout duobus vel tribus vel pluribus constant sonis. In bisonis igitur sint duo soni, ex quibus constant, *a* et *b*, seu isti numeri rationem saltem teneant ipsorum sonorum. Debebunt ergo a et *b* esse numeri integri et primi inter se. Atque hanc ob rem minimus eorum dividuus erit *ab* ideoque hic ipse numerus *ab* erit exponens consonantiae propositae, ex quo suavitatis gradus, ad quem pertinet, innotescit. Recenseamus autem huiusmodi consonantias secundum suavitatis gradus, ut ex ipso ordine appareat, quam quaeque facilis vel difficilis sit perceptu.

8. Ad huiusmodi vero enumerationem perficiendam hoc tantum opus est, ut singuli numeri ex tabula capiti II adiecta iuxta ordinem excerpantur eorumque quilibet in duos factores inter se primos resolvatur, id quod saepe pluribus modis fieri poterit. Hoc facto dabunt huiusmodi bini factores sonos consonantiae bisonae, cuius exponens erit ille ipse numerus, ex quo hi factores erant derivati. Exempli gratia in quinto gradu habetur 12, qui duplici modo in factores inter se primos resolvi potest: 1, 12 et 3, 4. Huiusmodi soni igitur constituent consonantias ad gradum V pertinentes, quarum exponens est 12.

9. Ad primum igitur gradum, in quo habetur unitas, nulla refertur consonantia neque bisona neque plurium sonorum. Cum enim soni consonantianl constituentes debeant esse diversi, unitas eorum nunquam esse poterit minimus communis dividuus sive exponens. Hanc ob rem simplicissima consonantia pertinebit ad gradum secundum eamque constituent soni rationem 1 : 2 tenentes, cuius ergo exponens est 2, qui numerus solus in gradu secundo reperitur. Consonantia haec a Musicis *diapason* sive *octava* appellatur ab iisque pro simplicissima et perfectissima habetur; facillime enim auditu percipitur ab aliisque dignoscitur.

10. Ad tertium gradum retulimus duos numeros 3 et 4, quorum uterque in duos factores inter se primos seu praeter unitatem nullum alium communem habentes divisorem resolvitur, ille scilicet in 1 et 3, iste vero in 1 et 4. Duae igitur prodeunt consonantiae bisonae ad tertium gradum pertinentes, quarum altera constat ex sonis rationem 1:3 habentibus, altera vero ex

sonis 1 : 4. Illa vocari solet diapason cum diapente, haec vero disdiapason, neque de his dubium esse potest, quin sequentibus facilius percipiantur.

11. Hoc modo sequentem confeci tabulam consonantiarum bisonarum, in qua eae sunt secundum suavitatis gradus supra expositos dispositae, ad decimum usque gradum:

Gr. II: 1:2.
Gr. III: 1 : 3, 1 : 4.
Gr. IV: 1 : 6, 2 : 3, 1 : 8
Gr. V: 1 : 5, 1 : 9, 1 : 12, 3 : 4, 1 : 16.
Gr. VI: 1 : 10, 2 : 5, 1 : 18, 2 : 9, 1 : 24, 3 : 8, 1 : 32.
Gr. VII: 1 : 7, 1 : 15, 3: 5, 1 : 20, 4 : 5, 1 : 27, 1 : 36, 4: 9, 1 : 48, 3 : 16, 1 : 64.
Gr. VIII: 1 : 14, 2: 7, 1: 30, 2: 15, 3: 10, 5: 6, 1 : 40, 5: 8, 1 :54, 2: 27, 1: 72,
8: 9, 1 : 96, 3: 32, 1 : 128.
Gr. IX: 1 : 21, 3 : 7, 1 : 25, 1 : 28, 4 : 7, 1 : 45, 5: 9, 1 : 60, 3: 20, 4 : 15, 5 : 12,
1 : 80, 5: 16, 1 : 81, 1: 108, 4: 27, 1 : 144, 9: 16, 1 : 192, 3: 64, 1 : 256.
Gr. X: 1 : 42, 3: 14, 6: 7, 1 :50, 2: 25, 1: 56, 7: 8, 1 : 90, 2: 45, 5: 18, 9: 10,
1 : 120, 3 : 40, 5 : 24, 8 : 15, 1 : 160, 5 : 32, 1 : 162, 2 : 81, 1 : 216, 8 : 27,
1 : 288, 9 : 32, 1 : 384, 3 : 128, 1 : 512.

12. Ex cap. I § 11 intelligitur, quomodo duae chordae debeant intendi, ut sonos datam tenantes rationem edant; hoc ergo modo facile erit istas consonantias chordis exprimere atque re ipsa experiri, quae sit perceptu facilior quaeve difficilior; reperietur autem experientia egregie cum hac theoria conspirare. Huiusmodi vero experimentis auditum musicae studiosi exerceri non solum perutile iudico, sed etiam maxime necessarium; hac enim ratione sibi distinctas comparabit ideas harum simpliciorum consonantiarum magisque idoneus evadit ad musicam ipsa praxi tractandam.

13. Neque vero necesse est, ut, qui musicae operam dat, omnium enumeratarum consonantiarum distinctas habeat ideas, sed sufficit primarias tantum animo probe imprimere, quae sunt 1 : 2, 1 : 3 vel 2 : 3, 1 : 5 vel 2 : 5 vel 4 : 5. Has enim qui noverit non solum ab aliis distinguera, sed etiam ipse vel voce formare vel chordis auditus ope producere, is quoque omnes reliquas consonantias, quarum exponentes alios non habent divisores nisi 2, 3 et 5, solo auditu poterit efficere. Atque hoc sufficiet ad musicam hodiernam et ad instrumenta musica attemperanda. In sequentibus vero pluribus haec sum expositurus.

14. Iam monui me hic sub consonantiae nomine tam consonantias quam dissonantias vulgo sic dictas complecti. Ex tabula autem apposita et methodo nostra limites quodammodo definiri posse videntur. Dissonantiae enim ad altiores pertinent gradus, et pro consonantiis habentur, quae ad inferiores gradus pertinent. Ita tonus, qui constat sonis rationem 8 : 9 habentibus et ad octavum gradum est relatus, dissonantiis annumeratur, ditonus vero seu tertia maior ratione 4 : 5 contentus, qui ad septimum gradum pertinet, consonantiis. Neque tamen ex his octavus gradus initium potest constitui dissonantiarum; nam in eodem continentur rationes 5 : 6 et 5 : 8, quae dissonantiis non accensentur.

15. Hanc rem autem attentius perpendenti constabit dissonantiarum et consonantiarum rationem non in sola perceptionis facilitate esse quaerendam, sed etiam ad totam componendi rationem spectari debere. Quae enim consonantiae in concentibus minus commode adhiberi possunt, eae dissonantiarum nomine sunt appellatae, etiamsi forte facilius percipiantur quam aliae, quae ad consonantias referuntur. Atque haec est ratio, cur tonus 8 : 9 dissonantiis annumeretur et aliae multo magis compositae consonantiae pro consonantiis habeantur. Simili modo ex hoc explicandum est, cur quarta seu diatessaron sonis rationem 3 : 4 habentibus constans a Musicis ad dissonantias potius quam ad consonantias referatur, cum tamen nullum sit dubium, quin ea admodum facile percipi queat.

16. Apud veteres quidem Musicos haec quarta tanquam valde suavis consonantia erat considerata, ut ex eorum scriptis liquet. At aliis prorsus usi sunt methodis dissonantias a consonantiis discernendi, quae in ipsa rei natura minus erant fundatae et ex precariis principiis deductae. PYTHAGOREI enim ad consonantias efficiendas alios sonos non iudicabant idoneos, nisi qui constarent ex duobus sonis rationem vel multiplicem vel superparticularem vel multiplicem superparticularem tenentes; dissonantiam vero prodire putarunt, quoties horum duorum sonorum ratio fuerit vel superpartiens vel multiplex superpartiens.

17. Hanc PYTHAGOREORUM sententiam refellit PTOLEMAEUS in *Libris Harmonicorum* experientiam testem allegans diapason diatessaron ratione 3 : 8 contentum esse consonantiam, quamvis haec ratio sit dupla superbipartiens tertias. Deinde notat hac regula ne ipsos quidem PYTHAGOREOS tuto uti esse ausos, dum praeter rationes duplam, triplam, quadruplam, sesquialteram et sesquitertiam alias ad consonantias efficiendas non adhibuissent, cum tamen praeterea innumerabiles alias eodem iure suam regulam sequentes adhibere potuissent. In hac vero PTOLEMAEI refutatione nihil reprehendendum reperio; non enim ad rationum genera, sed ad simplicitatem et percipiendi facilitatem respici oportet.

18. Neque tamen ipsius PTOLEMAEI principium, quo in hac re utitur, magis est firmum; consonantias enim post diapason et disdiapason duas tantum admittit, quae rationibus superparticularibus proxime aequalibus et coniunctis rationem duplam producentibus contineantur. Huiusmodi autem sunt rationes 2: 3 et 3:4, quae coniunctae dant rationem 1:2. Ex priore oritur consonantia diapente dicta, ex posteriore vero diatessaron. Deinde aliud insuper ponit principium hoc: consonantiam quamcunque octava auctam manere consonantiam nihilque de sua suavitate amittere, hocque modo in consonantiarum numerum recipit has rationes 1 : 2, 1 : 4, 2 : 3, 1 : 3, 3 : 4 et 3 : 8.

19. Nihilo tamen minus PTOLEMAEUS rationibus superparticularibus magnam tribuit praerogativam prae superpartientibus; neque enim sonos alias tenentes rationes superparticulares praeter 2 : 3 et 3 : 4 dissonos appellat, sed medio quodam inter consonos et dissonos nomine, scilicet concinnos. Reliquas vero rationes superpartientes praeter 3 : 8 dissonantias producere fortiter statuit. Non autem necesse esse iudico hanc consonantiarum suavitatem metiendi rationem utpote prorsus precariam nullisque principiis firmis superstructam refellere, cum veritas nostrorum principiorum abunde iam sit ob oculos posita et ex ipsa rei natura derivata. Restaret quidem, ut altertus sectae veterum Musicorum, cuius auctor ARISTOXENUS fuit, hac de re sententiam exponerem; verum uti hi numerorum rationes

prorsus reiecerunt, ita consonantiarum et dissonantiarum iudicium sensibus solis reliquerunt, in quo non multum a PYTHAGORAEIS dissenserunt.

20. Trisonarum et multisonarum consonantiarum secundum suavitatis gradus enumeratio simili modo perficietur, quo bisonarum, ita ut superfluum esset tam abunde de iis explicare. Id tantum animadverti convenit simplicissimam consonantiam trisonam ad gradum suavitatis tertium pertinere sonisque 1 : 2 : 4 constare, cuius exponens est 4. Ex quo intelligitur, ex quo pluribus sonis consonantia sit composita, eam ad eo altiorem quoque suavitatis gradum pertinere, etiamsi sit in suo genere simplicissima.

21. Eo autem magis hanc consonantiarum divisionem ulterius non persequor, cum aliam multo aptiorem et utiliorem divisionem sim allaturus, quae fit in *completas* et *incompletas* consonantias. Voco autem consonantiam *completam*, ad quam nullus sonus superaddi potest, quin simul ipsa consonantia ad altiorem gradum sit referenda seu eius exponens fiat magis compositus; huiusmodi est consonantia sonis 1 : 2 : 3 : 6 constans, cuius exponens est 6. Superaddito enim quocunque novo sono exponens fiet maior. Consonantia contra *incompleta* mihi est, ad quam unum vel plures sonos adiicere licet citra exponentis multiplicationem; ut huius consonantiae 1: 2: 3 exponens non fit maior, etiamsi sonus 6 addatur, quamobrem eam incompletam voco.

22. Ex praecedentibus autem intelligitur quemlibet numerum sonum simplicem denotantem esse divisorem exponentis consonantiae. Quare si exponentis omnes divisores accipiantur iisque totidem soni simplices exprimantur, habebitur consonantia completa illius exponentis; praeter hos enim numeros alius non erit, qui hunc exponentem dividat. Ita consonantia constans sonis 1 : 2 : 3 : 4 : 6 : 12 erit completa, quia hi soli numeri sunt divisores exponentis huius consonantiae, qui est 12, neque ullus alius praeter hos numerum 12 dividit.

23. Quoties igitur exponens consonantiae est numerus primus, completa consonantia erit bisona, ut 1 : a, si a denotet numerum primum. Si exponens fuerit a^m, constabit completa consonantia ex $m+1$ sonis, nempe

$$1:a:a^2:a^3:\cdots\cdots:a^m.$$

Si exponens habeat hanc formam ab, factum ex duobus numeris primis, erit completa consonantia quadrisona

$$1:a:b:ab$$

et existente exponente $a^m b^n$ habebit completa consonantia

$$mn+m+n+1$$

sonos. Atque generalius si exponens fuerit $a^m b^n c^p$, continebit consonantia completa

$$(m+1)(n+1)(p+1)$$

sonos ac secundum regulam § 6 datam pertinebit ad gradum

$$ma+nb+pc-m-n-p+1;$$

est enim summa omnium factorum simplicium exponentis $ma+nb+pc$ et numerus factorum est $m+n+p$.

24. Exposito modo consonantias completas formandi perspicuum est, si unus pluresve soni ex iis omittantur, consonantiam tum fieri incompletam. In quo est notandum huiusmodi sonos reiici oportere, ut reliquorum exponens non fiat simplicior: ut si ex hac consonantia 1: 2 : 4, cuius exponens est 4, sonus 1 vel 4 reiiceretur, consonantia prodiret 1 : 2 vel 2 : 4 congruens cum illa, cuius exponens non amplius foret 4, sed tantum 2. Verum medium sonum 2 reiicere licebit; consonantiae enim 1 : 4 exponens etiam nunc est 4, quemadmodum completae 1 : 2 : 4.

25. Si exponens est numet·us primus, patet consonantiam non posse esse non completam, eo quod duobus tantum constet sonis. At reliquae consonantiae omnes fieri possunt incompletae idque bisonae omittendis omnibus sonis praeter gravissimum et acutissimum; quia enim hic ipso exponente, ille vero unitate exprimitur, exponens huius consonantiae bisonae non erit simplieior quam completae: ut ex consonantia 1 : 2 : 3 : 6 reiectis sonis 2 et 3 consonantiae 1: 6 exponens est 6 pariter ac illius. Deinde in consonantiis, quarum exponens est huius formae a^m, neque sonus gravissimus 1 neque acutissimus a^m possunt reiici; in reliquis vero consonantiis omnibus tam infimus quam supremus, imo et uterque potest praetermitti.

26. Si qua consonantia ita est comparata, ut in ea nullus sonus omitti possit, quin simul ipsa consonantia simplicior evadat et ad gradum inferiorem quam ante pertineat, eam hic *puram* appellabimus. Huiusmodi sunt omnes consonantiae bisonae, quia praetermisso altero sono cessant esse consonantiae. Simili modo purae sunt consonantiae 3 : 4 : 5, 4 : 5 : 6 nec non 1 : 6 z 9, 2: 3 : 12, in quibus nullus sonus potest omitti, quin simul fiant simpliciores. Harum itaque consonantiarum usus in hoc consistit, quod sonorum numerus, quantum fieri potest, diminuatur, ita tamen, ut exponens non fiat minor.

27. Duplici autem modo consonantia quaecunque uno pluribusve sonis reiiciendis fieri potest simplicior; quorum prior est, quando residuorum sonorum seu numerorum vices eorum tenentium minimus communis dividuus minor evadit quam omnium, ut in consonantia 2 : 3 : 5 : 6 reiecto sono 5 reliquorum 2 : 3 : 6 minimus communis dividuus est 6, qui ante erat 30. Altero modo consonantia fiet simplicior, quando residui soni communem habent divisorem; tum enim per hunc ante debent dividi, quam minimus communis dividuus seu exponens definiatur, ut in hac consonantia 2: 3: 4: 6 reiecto sono 3 reliqui per 2 divisi constituunt consonantiam 1 : 2: 3, cuius exponens est 6; ante vero erat 12.

28. Utroque etiam modo coniunctim consonantia reiiciendis uno pluribusve sonis fieri potest simplicior, quando scilicet sonorum residuorum numeri et simpliciorem habent minimum communem dividuum et insuper communem divisorem: quemadmodum fit in hac consonantia 3 : 6 : 8 : 9 : 12, cuius exponens est 72, si reiiciatur sonus 8; reliquorum enim 3: 6: 9: 12 minimus communis dividuus est 36; at quia singuli hi numeri per 3 possunt dividi, consonantia resultans ex sonis 1: 2: 3:4 constare censenda est, cuius igitur exponens erit 12. Tanto itaque simplicior evadit proposita consonantia unico sono 8 reiecto.

29. Quo autem distinctius intelligatur, quomodo quaevis consonantia proposita effici possit simplicior, consideremus consonantiam completam, cuius exponens est $a^m P$, ubi P est quantitas quoscunque numeros primos praeter a complectens. In hac igitur, si omnes soni per a^m et huius multipla expositi reiiciantur, remanebit consonantia simplicior exponentis $a^{m-1}P$, quae reductio secundum primum modum est facta. Secundo modo autem consonantia fiet simplicior, si omnes soni, qui exprimuntur numeris a in se non continentibus, omittantur; tum enim reliqui soni omnes per a dividi poterunt eritque eorum exponens $a^{m-1}P$. Ex quo intelligitur, quomodo utraque methodo coniunctim consonantia efficiatur simplicior.

30. Discrimen, quod auditus inter consonantias completas et incompletas percipit, in hoc, ut facile intelligi potest, consistit, quod completas multo distinctius, incompletas vero minus distincte comprehendat. Etenim si omnes soni simul organum auditus afficiunt, clarius singulorum inter se relationes sese sensui offerant necesse est, quam si exponens ex paucioribus sonis deberet colligi. Ita ex consonantia 1 : 2 : 3 : 6 multo distinctius eius exponens qui est 6, cognoscitur quam ex duobus tantum sonis 1 : 6. Ad hoc autem requiritur, ut omnes soni quam exactissime numeris, quibus exprimuntur, respondeant.

31. Completarum autem consonantiarum omnium, quae in duodecim primis gradibus continentur, sequentem adiicere idoneum visum est tabulam, in qua numeri Romani gradus designant, Arabici autem ipsas consonantias quasque ad suum gradum relatas.

I.	1.
II.	1:2.
III.	1:3,
	1:2:4.
IV.	1:2:3:6,
	1:2:4:8.
V.	1:5,
	1:3:9,
	1: 2:3:4:6:12,
	1:2:4:8:16.
VI.	1:2:5:10,

1:2:3:6:9:18,
1:2:3:4:6:8:12:24,
1:2:4:8:16:32.

VII. 1:7,
1:3:5:15,
1:2:4:5:10:20,
1:3:9:27,
1:2:3:4:6:9:12:18:36,
1:2:3:4:6:8:12:16:24:48,
1:2:4:8:16:32:64.

VIII. 1:2:7:14,
1:2:3:5:6:10:15:30,
1:2:4:5:8:10:20:40,
1:2:3:6:9:18:27:54,
1:2:3:4:6:8:9:12:18:24:36:72,
1:2:3:4:6:8:12:16:24:32:48:96,
1:2:4:8:16:32:64:128.

IX. 1:3:7:21,
1:5:25,
1:2:4:7:14:28,
1:3:5:9:15:45,
1:2:3:4:5:6:10:12:15:20:30:60,
1:2:4:5:8:10:16:20:40:80,
1:3:9:27:81,
1:2:3:4:6:9:12:18:27:36:54:108,
1:2:3:4:6:8:9:12:16:18:24:36:48:72:144,
1:2:3:4:6:8:12:16:24:32:48:64:96:192,
1:2:4:8:16:32:64:128:256.

X. 1:2:3:6:7:14:21:42,
1:2:5:10:25:50
1:2:4:7:8:14:28:56,
1:2:3:5:6:9:10:15:18:30:45:90,
1:2:3:4:5:6:8:10:12:15:20:24:30:40:60:120,
1:2:4:5:8:10:16:20:32:40:80:160,
1:2:3:6:9:18:27:54:81:162,
1:2:3:4:6:8:9:12:18:24:27:36:54:72:108:216,
1:2:3:4:6:8:9:12:16:18:24:32:36:48:72:96:144:288,
1:2:3:4:6:8:12:16:24:32:48:64:96:128:192:384,
1:2:4:8:16:32:64:128:256:512.

XI. 1:11,
1:5:7:35,
1:3:7:9:21:63,
1:3:5:15:25:75,
1:2:3:4:6:7:12:14:21:28:42:84,
1:2:4:5:10:20:25:50:100,

1:2:4:7:8:14:16:28:56:112
1:3:5:9:15:27:45:135,
1:2:3:4:5:6:9:10:12:15:18:20:30:36:45:60:90:180,
1:2:3:4:5:6:8:10:12:15:16:20:24:30:40:48:60:80:120:240,
1:3:9:27:81:243,
1:2:4:5:8:10:16:20:32:40:64:80:160:320,
1:2:3:4:6:9:12:18:27:36:54:81:108:162:324,
1:2:3:4:6:8:9:12:16:18:24:27:36:48:54:72:108:144:216:432,
1:2:3:4:6:8:9:12:16:18:24:32:36:48:64:72:96:144:192:288:576,
1:2:3:4:6:8:12:16:24:32:48:64:96:128:192:256:384:768,
1:2:4:8:16:32:64:128:256:512:1024.

XII. 1:2:11:22,
1:2:5:7:10:14:35:70,
1:2:3:6:7:9:14:18:21:42:63:26,
1:2:3:5:6:10:15:25:30:50:75:150,
1:2:3:4:6:7:8:12:14:21:24:28:42:56:84:168,
1:2:4:5:8:10:20:25:40:50:100:200,
1:2:4:7:8:14:16:28:32:56:112:224,
1:2:3:5:6:9:10:15:18:27:30:45:54:90:135:270,
1:2:3:4:5:6:8:9:10:12:15:18:20:24:30:36:40:45:60:72:90:
120:180:360,
1:2:3:4:5:6:8:10:12:15:16:20:24:30:32:40:48:60:80:96:120:160:240:480,
1:2:3:6:9:18:27:54:81:162:243:486,
1:2:4:5:8:10:16:20:32:40:64:80:128:160:320:640,
1:2:3:4:6:8:9:12:18:24:27:36:54:72:81:108:162:216:324:648,
1:2:3:4:6:8:9:12:16:18:24:27:32:36:48:54:72:96:108:144:216:288:432:864,
1:2:3:4:6:8:9:12:16:18:24:32:36:48:64:72:96:128:144:192:288:384:576:1152,
1:2:3:4:6:8:12:16:24:32:48:64:96:128:192:256:384:512:768:1536,
1:2:4:8:16:32:64:128:256:512:1024:2048.

32. Quamvis vero completa consonantia se multo distinctius auditui offerat quam incompleta, tamen, nisi sint admodum simplices, completae consonantiae non adhibentur. Primo enim tam magnus sonorum numerus, si instrumenta musica non sunt accuratissime coaptata, id quod effici nequaquam potest, aures potius confuso strepitu quam distincta harmonia obtundit. Deinde etiam plures soni vel propter nimis profundam gravitatem vel propter nimis altum acumen ne quidem percipi possunt; primo enim capite iam est ostensum nullum sonum, qui minuto secundo vel pauciores quam 30 vel plures quam 7500 edat percussiones, auribus posse percipi. Ex quo perspicuum est, quoties consonantiae soni extremi maiorem teneant rationem quam 250 : 1, omnes eius sonos nequidem posse audiri.

33. Ad doctrinam de consonantiis referri convenit ea, quae Musici de intervallis sonorum tradere solent. Vocatur autem *intervallum* ea distantia, quae inter duos sonos, alterum graviorem alterum acutiorem, esse concipitur. Eo igitur maius est intervallum, quo magis soni

ratione gravis et acuti inter se discrepant, seu quo maior est ratio, quam acutior habet ad graviorem. Sic maius est intervallum sonorum 1 : 3 quam sonorum 1 : 2; et aequalium sonorum 1 : 1, quia nullo saltu ex altero ad alterum pervenitur, intervallum est nullum. Ex quo intelligitur intervallum ita esse definiendum, ut sit mensura discriminis inter sonum acutiorem et graviorem.

34. Sint tres soni $a : b : c$, quorum c sit acutissimus, a gravissimus, b vero intermedius quicunque; apparebit ex praecedente definitione intervallum sonorum a et c esse aggregatum intervallorum inter a et b atque inter b et c. Quare si haec duo intervalla inter a et b ac b et c fuerint aequalia, id quod evenit, quando est $a:b=b:c$, erit intervallum $a : c$ duplo maius quam intervallum $a : b$ seu $b : c$. Ex quo perspicitur intervallum 1 : 4 duplo esse maius intervallo 1: 2, et hanc ob rem, cum haec ratio 1: 2 octavam intervallum constituere ponatur, ratio 1 : 4 duas continebit octavas.

35. Qui haec attentius inspiciet, facile deprehendet intervalla exprimi debere mensuris rationum, quas soni constituunt. Rationes autem mensurantur logarithmis fractionum, quarum numeratores denotent sonos acutiores, denominatores vero graviores. Quocirca intervallum inter sonos $a:b$ exprimetur per logarithmum fractionis $\frac{b}{a}$, quem designari mos est per $\log.\frac{b}{a}$ seu, quod eodem redit, per $\log.b-\log.a$. Intervallum ergo sonorum aequalium $a : a$ erit nullum, ut iam notavimus, quippe quod exprimitur per $\log.a-\log.a=0$.

36. Intervallum itaque, quod octava (graece $\delta\iota\alpha\pi\alpha\sigma\widehat{\omega}\nu$) nuncupatur, quia continetur sonis rationem duplam habentibus, exprimetur logarithmo binarii; atque intervallum sonorum 2 : 3, quod quinta seu diapente appellatur, erit $\log.\frac{3}{2}$ seu $\log.\,3-\log.\,2$. Ex quo intelligitur haec intervalla omnino inter se esse incommensurabilla; nullo enim modo ratio, quam habet $\log.2$ ad $\log.\frac{3}{2}$, potest assignari et hanc ob rem nullum datur intervallum quantumvis exiguum, quod octavae simul et quintae esset pars aliquota. Similis est ratio omnium aliorum intervallorum, quae disparibus exprimuntur logarithmis, ut $\log.\frac{3}{2}$ et $\log.\frac{5}{4}$. Contra vero ea intervalla, quae logarithmis numerorum, qui sint potentiae eiusdem radicis, exponuntur, inter se poterunt comparari; ita intervallum sonorum 27 : 8 se habebit ad intervallum sonorum 9 : 4 ut 3 ad 2; est enim $\log.\frac{27}{8}=3\log.\frac{3}{2}$ et $\log.\frac{9}{4}=2\log.\frac{3}{2}$.

37. Ex his quoque facile liquet, quaenam intervalla ex additione vel subtractione plurium inter se oriantur, perficiendis his iisdem operationibus in logarithmis, qui mensurae sunt intervallorum; hoc enim facto logarithmus resultans exponet intervallum proveniens. Ut si quaeratur intervallum, quod restet diapente ab octava ablata, oportebit $\log.\frac{3}{2}$ sive $\log.3-\log.2$ auferre a $\log.2$ eritque residuum $\log.2-\log.3+\log.2$, i. e. $2\log.2-\log.3$. At est $2\log.2=\log.4$; ex quo residuum intervallum erit $\log.4-\log.3$ seu $\log.\frac{4}{3}$, id quod diatessaron seu quarta appellatur et cum quinta coniunctum integram octavam adimplet.

38. Quanquam autem diversorum numerorum logarithmi inter se non possunt comparari, nisi fuerint numeri potestates eiusdem radicis, tamen ope tabularum logarithmicarum verae proxima earum ratio potest definiri atque ita diversa intervalla, quantum fieri potest, exacte inter se conferri. Cum igitur octavae mensura sit log.2, qui ex tabulis excerptus est $=0,3010300$, et quintae $\log.3-\log.2$, quae differentia est $=0,1760913$, erit intervallum octavae ad intervallum quintae quam proxime ut 3010300 ad 1760913. Quae ratio, quo ad minores numeros reducatur, mutatur in hanc ad 1, ex qua istae simplices derivantur rationes 2: 1, 3: 2, 5: 3, 7 : 4, 12: 7 et 17 : 10, 29 : 17, 41 : 24, 53: 31, quarum postrema verae est proxima.

39. Simili quoque modo intervalla possunt dividi in tot, quot quis voluerit, partes aequales atque soni veris proximi assignari, qui huiusmodi intervalla partiali a se invicem distent. Logarithmus enim intervalli propositi in totidem partes est dividendus uniusque partis numerus in tabulis respondens accipiendus, qui ad unitatem quaesitam habebit rationem. Quaeratur verbi gratia intervallum ter minus quam octava; erit eius logarithmus = 0,1003433, tertia nimirum pars ipsius log.2, cui respondet ratio 126 : 100 seu 63 :50, quae minus accurata est vel 29: 23 vel 5:4, qua postrema tertia maior indicatur, quae etiam ab imperitioribus pro tertia parte unius octavae habetur.