CHAPTER VII

ON THE INVESTIGATION OF INFINITE BRANCHES

166. If a curved line may have a branch or part extending to infinity and from a point of this set down at infinity some applied line may be sent normal to an axis, then either the abscissa x, or the applied line y, or each coordinate will be infinite. For unless either one or the other shall be infinite, then the distance of the point on the curve taken from the beginning of the abscissas may indeed become finite $=\sqrt{(xx+yy)}$, contrary to the hypothesis. On account of which, if the curve shall have a branch extending to infinity, either an actual infinite applied line will correspond to a certain finite abscissa, or an infinitely great abscissa will correspond to an actual applied line, either finite or infinite. Therefore from this beginning the branches of curves extending to infinity will be able to be investigated.

167. An algebraic equation shall be proposed between the coordinates x and y of any order, such as n, and the terms separately, in which the variables x and y may possess n dimensions, will be

$$\alpha y^n+\beta y^{n-1}x+\gamma y^{n-2}xx+\delta y^{n-3}x^3+\cdots+\zeta x^n,$$

which expression will be resolvable into simple factors of the form $Ay+Bx$, either real or imaginary. And, if it may have imaginary factors, the number of those will be even and a pair jointly will give a double real factor of the form

$$AAyy-2ABxy\cdot\cos.\varphi+BBxx.$$

But a factor of this kind, (either x or y or each may be put infinite, $=\infty$),
always will adopt the same infinite value $=\infty^2$, because the term $2AByx\cdot\cos.\varphi$ is less always than the two remaining $AAyy+BBxx$, for neither A nor B can be $=0$. Therefore a factor of this kind

$$AAyy-2ABxy\cdot\cos.\varphi+BBxx,$$

if either x or y or each may be put infinite, can neither become equal to zero, nor any quantity, either finite nor even infinite, since it itself becomes $=\infty^2$, which is infinitely greater than ∞.

168. But if therefore the greatest part of the equation

$$\alpha y^n + \beta y^{n-1}x + \gamma y^{n-2}xx + \cdots + \zeta x^n$$

may have no simple real factor, which indeed cannot happen, unless n shall be an even number, then it will be put together from a mixture of double factors of this form :

$$AAyy - 2ABxy \cdot \cos.\varphi + BBxx .$$

Whereby, if either x or y or each is made infinite, the value of that expression itself adopts the infinite value $= \infty^n$, therefore neither can it be equal to any quantity, either finite or infinite $= \infty^m$, the exponent m of which shall be less than n. Therefore the remaining terms of the equation, in which the variables x and y have fewer dimensions, because they present infinities with exponents less than n , are not able to become equal to that highest infinity ; and thus in this case the equation cannot exist, if either x or y or both may be made infinite.

169. Hence therefore a curved line, which is expressed by an equation between the coordinates x and y, of which the greatest member has no simple real factors, will have no branches extending to infinity, and thus the whole curve will be contained in a finite space, in the form of an ellipse or a circle. On account of which, if in the general equation of the second order

$$\alpha yy + \beta xy + \gamma xx + \delta y + \varepsilon x + \zeta = 0 ,$$

the greatest member, $\alpha yy + \beta xy + \gamma xx$, in which the variables x and y have the dimension two, may not have simple real factors, which comes about, if $\beta\beta$ shall be greater than $4\alpha\gamma$, then the curve will have no branch extending to infinite and thus it becomes an ellipse.

170. So that this may be enabled to be set out more distinctly, we may distinguish each equation proposed between the coordinates x and y thus into members, in order that we may refer all the terms of the equation to the greatest or first term, in which the variables x and y maintain the same maximum dimension, the exponent of which shall be n. Truly to the second member I refer all the terms, in which the variables both constitute $n-1$ dimensions. The third member will contain these terms, in which the number of dimensions of x and y themselves is $n-2$, and thus so forth, finally arriving at the last term, in which no dimension is present of x and y, and which therefore will be composed from a constant quantity only. Moreover P shall be the first or greatest member, Q the second member, R the third, S the fourth and so on.

171. Therefore because if the greatest member P has no simple real factor, with the equation of the curved line $P+Q+R+S+$ etc. $=0$ indicated has no branch extending to infinity, we may now put the greatest member P to have a single simple real factor $ay-bx$, thus so that there shall be $P=(ay-bx)M$, with M being a function of x and y of dimensions $n-1$, which may have no simple real factors. Therefore with either x or y or each placed infinite, M becomes $=\infty^{n-1}$; Q indeed likewise can be infinite, but R, S etc. become infinities of smaller orders. Consequently the equation

$$P+Q+R+\text{ etc.}=0$$

will be able to be terminated, if $ay-bx=$ some finite quantity or zero, and thus the curve may be extended to infinity.

172. Therefore let $ay-bx=p$, with p a finite quantity present, which must be prepared thus, so that the curve on going off to infinity becomes

$$pM+Q+R+S+\text{etc.}=0$$

or

$$p=\frac{-Q-R-S-\text{etc.}}{M}$$

But since M shall be an infinite quantity of a higher order than R and S etc., the fractions $\frac{R}{M}, \frac{S}{M}$ etc. will become $=0$ and thus $p=\frac{-Q}{M}$. Hence on account of this, the fraction $\frac{-Q}{M}$ will give the value of p, if the variables x and y become infinite. Therefore since there shall be $ay-bx=p$, there will be

$$y=\frac{bx+p}{a} \text{ and } \frac{y}{x}=\frac{b}{a}+\frac{p}{ax}=\frac{b}{a}, \text{ on account of } \frac{p}{ax}=0, \text{ if } x=\infty.$$

Therefore the curve in departing to infinity becomes $y=\frac{bx}{a}$.

173. Therefore since Q and M shall be homogeneous functions of dimensions $n-1$, $\frac{-Q}{M}$ will be a function of no dimensions and thus , if there may be put $y=\frac{bx}{a}$, a constant value for p will prevail. Or, because the function $\frac{-Q}{M}$ may be determined, only if the ratio between y and x will be determined, which is $b:a$, the value of p itself will be

obtained, if in the expression $\frac{-Q}{M}$, b may be written in place of y and a in place of x. Therefore with p found in this way there will be $ay - bx = p$, which equation will be contained in the proposed equation itself $P + Q + R + S + \text{ etc.} = 0$, if the curve may go off to infinity.

174. And thus the part of the curve itself extending to infinity will be expressed by this equation $ay - bx = p$; which since it shall be for a right line, this right line produced to infinity at last will merge with the curved line. Therefore the right line will be an asymptote with this curve, because the curved line extended to infinity agrees with the right line and thus continually approaches closer to that. And since the proposed equation $P + Q + R + S + \text{ etc.} = 0$ on putting either x or $y = \infty$ may change into the equation $ay - bx = p$, likewise it is understood this line produced in each direction finally agrees with the curve. On account of which the curved line will have two branches extending to infinity opposite to each other, since the one meets the right line produced indefinitely forwards, and the other meets the same produced indefinitely backwards.

175. Therefore since the curve, if the greatest member P of the equation $P + Q + R + S + \text{ etc.} = 0$ may have one simple real factor, it shall be provided with two branches extending to infinity and will converge to the same right line on both sides, which is called the asymptote of this line, now we may consider the greatest member P to have two simple real factors $ay - bx$ and $cy - dx$, thus so that there shall be $P = (ay - bx)\ (cy - dx)M$; M will be a homogeneous function of $n - 2$ dimensions. But two cases here come to be considered, just as these to factors themselves shall be equal or unequal to each other.

176. These factors shall be unequal to each other; and it is evident that the equation $(ay - bx)(cy - dx)M + Q + R + S + \text{etc.} = 0$ is able to be terminated in two ways either for infinite abscissas or applied lines, whether $ay - bx$ or $cy - dx$ will be equal to a finite quantity. Therefore let $ay - bx = p$; and, since p shall be a finite quantity, at infinity it will become $\frac{y}{x} = \frac{b}{a}$, and thus as before there becomes

$$p = \frac{-Q - R - S - \text{etc.}}{(cy - dx)M} = \frac{-Q}{(cy - dx)M},$$

which is a function of zero dimensions of x and y themselves ; whereby, if there is put $\frac{y}{x} = \frac{b}{a}$ or, what returns the same, if everywhere there may be written b in place of y and a in place of x, the true value of the constant sought p will be produced. Therefore there will be

$$p = \frac{-Q}{(bc-ad)M}$$

and on account of the unequal factors $bc-ad$ it will not be $=0$ nor also M, because no simple real factor is included entirely, that is able to go to zero ; from which a finite value for p, or also $=0$, arises, which comes about, either if the member Q therefore may be absent, or may have a factor $ay-bx$.

177. Therefore on account of the simple real factor of the greatest member P, the curve $ay-bx$, as in the first case, will have a single asymptote, the position of which is indicated by the asymptote $ay-bx=p$. Truly in a similar manner on account of the other factor, $cy-dx$ also will give an asymptote, determined by this equation : $cy-dx=q$, with q being $=\frac{-Q}{(ay-bx)M}$, after which these determined values d and c will be able to be substituted everywhere in place of the values y and x. On account of which the curved line generally will have two asymptotes and therefore four branches extending to infinity, which finally agree with the these right lines. Moreover the case here is itself found in the place of the hyperbola ; whereby, if in the equation for lines of the second order $\alpha yy+\beta xy+\gamma xx+\delta y+\varepsilon x+\zeta=0$ the greatest member $\alpha yy+\beta xy+\gamma xx$ may have two simple real unequal factors, which will come about, if $\beta\beta$ may be greater than $4\alpha\gamma$, then the curve will be a hyperbola.

178. Both the factors $ay-bx$ and $cy-dx$ shall be equal to each other, thus so that there shall be $P=(ay-bx)^2M$. Therefore since there shall be $P+Q+R+S+\text{etc.}=0$, there will be

$$(ay-bx)^2 = \frac{-Q-R-S-\text{etc.}}{M}$$

But because Q is a function of $n-1$ dimensions, R of $n-2$, dimensions, and S of $n-3$ dimensions, on account of the M of $n-2$ dimensions, in the infinite case there will be , $\frac{S}{M}=0$ and thus

$$(ay-bx)^2 = -\frac{Q}{M}-\frac{R}{M} = -\frac{Q}{M(\mu y+\nu x)}(\mu y+\nu x)-\frac{R}{M}.$$

But $\frac{Q}{M(\mu y+\nu x)}$ and $\frac{R}{M}$ are functions of zero dimensions of x and y. Whereby, since at infinity there shall be $y:x=b:a$, if this ratio $\frac{b}{a}$ may be substituted for $\frac{y}{x}$ or b for y and a for x, each function from that will become a constant quantity.

179. Therefore with this substitution made

$$\frac{Q}{M(\mu y+\nu x)}=A \text{ and } \frac{R}{M}=B\ ;$$

[The editor of the *O.O.* edition acknowledges that this expression is incomplete, and that an extra term $C\sqrt{\mu y+\nu x}$ must be added; see p. 97 of Series I, Vol. 9 for details and an example ; the previous French and German translations have not made this comment.] and there will be

$$(ay-bx)^2=-A(\mu y+\nu x)-B,$$

which is the equation for the curved line, with which curved line expressed by the equation

$$P+Q+R+S+\text{etc.}=0,$$

will merge after it has proceeded to infinity. Truly, because the quantities μ and ν are arbitrary, there is taken $\mu=b$ and $\nu=a$, and, with the coordinates unchanged, there becomes

$$ay-bx=u\sqrt{(aa+bb)} \text{ and } by+ax=t\sqrt{(aa+bb)},$$

and this will be the equation for that same curve

$$uu+\frac{At}{\sqrt{(aa+bb)}}+\frac{B}{aa+bb}=0,$$

which appears to be the equation for a parabola. Therefore the curve sought will be prepared, so that it may be merged with a parabola on being extended to infinity. Therefore there will be only two branches extending to infinity, the asymptote of which will not be a straight line, but a parabola expressed by the above equation.

180. This arises, if there were not $A=0$; but if there were $A=0$, (which arises, if the second member Q either is absent or were divisible by $ay-bx$), then the equation ceases to be for a parabola and it becomes, three cases of which will be considered. Clearly in the first place, if B were a negative quantity, on putting $\frac{B}{aa+bb}=-ff$, the equation $uu-ff=0$ will include the two roots in itself $u-f=0$ and $u+f=0$, which will be for

two right lines parallel to each other, each of which will be an asymptotic curve, as in the first case ; and thus the curve will have four branches extending to infinity, which will merge with these two right lines.

181. The second case is that, so that B shall be a positive quantity, it may be for example $+ff$. Because truly in this case the equation $uu+ff=0$ is impossible, the curve will have no branch extending to infinity, but the whole will be contained in a finite space. Therefore not only the curve, which will be contained in this equation $P+Q+R+S+\text{etc.}=0$, will have no branch extending to infinity, if the greatest member P should have no simple real factors, but also the same can come about in use, whatever factors P may have, as we have just seen. But several cases of this kind occur at this stage.

182. The third case is, in which B also is made $=0$, in which each of the preceding is allowed to happen, but from which it is ambiguous in what manner the future curve shall be prepared. Hence towards defining the shape of the curve the following terms will have to be examined. Clearly, since there shall be $P+Q+R+S+\text{etc.}=0$ and $P=(ay-bx)^2 M$, at infinity there will be

$$\frac{y}{x}=\frac{b}{a} \text{ and } (ay-bx)^2+\frac{Q}{M}+\frac{R}{M}+\frac{S}{M}+\frac{T}{M}+\text{etc.}=0.$$

Therefore there becomes, with the substitution made, as before

$$\frac{y}{x}=\frac{b}{a},\ \frac{Q}{M}=A(by+ax),\ \frac{R}{M}=B,$$

then truly, since S, T, V etc. shall be functions of dimensions $(n-3),(n-4)$ etc., with the function M being of $(n-2)$ dimensions,

$$\frac{S(by+ax)}{M}=C,\quad \frac{T(by+ax)^2}{M}=D,\quad \frac{V(by+ax)^3}{M}=E \text{ etc.},$$

there will be

$$(ay-bx)^2+A(by+ax)+B+\frac{C}{by+ax}+\frac{D}{(by+ax)^2}+\frac{E}{(by+ax)^3}+\text{etc.}=0.$$

Therefore this equation will express the nature of the curved line, a part of which at an infinite distance, which it will produce, if $by+ax$ may be put infinite, since it will agree that the curve presented by the equation $P+Q+R+S+\text{etc.}=0$. For although, with the curve extending to infinity, $(ay-bx)^2$ may obtain either a finite or infinite value of order, but less than ∞^2, yet $by+ax$ will have an infinite value.

183. But we may change the axis, to which we refer that asymptote line, and taking the abscissa $\frac{ax+by}{\sqrt{(aa+bb)}}=t$ and the applied line $\frac{ay-bx}{\sqrt{(aa+bb)}}=u$

for that, and for brevity putting $\sqrt{(aa+bb)}=g$, there will be the equation :

$$uu+\frac{At}{g}+\frac{B}{gg}+\frac{C}{g^3t}+\frac{D}{g^4tt}+\frac{E}{g^5t^3}+\text{ etc.}=0\,.$$

Therefore since in the case that we want to examine, there shall be $A=0$ and $B=0$, there becomes

$$uu+\frac{C}{g^3t}+\frac{D}{g^4tt}+\frac{E}{g^5t^3}+\text{etc.}=0$$

But if now C were not $=0$, on putting t to become infinite, the terms $\frac{D}{g^4tt}+\frac{E}{g^5t^3}+\text{etc.}$ will vanish besides $\frac{C}{g^3t}$ and there will be

$$uu+\frac{C}{g^3t}=0\,;$$

from which equation the nature of the curved line may be expressed, which on putting $t=\infty$ will merge with the curve sought. Whereby, since hence there shall be $u=\ \pm\sqrt{\frac{-C}{g^3t}}$ the curve will have two branches each converging to the same part of the axis.

184. But if above there were $C=0$, then on taking that equation

$$uu+\frac{D}{g^4t}=0,$$

where again three cases occur, just as D should be a positive, negative, or zero quantity. In the first case, because of an impossible equation, the curve will have no branches extending to infinity, but it will be contained completely in a finite space. In the second case, if $\frac{D}{g^4} = -ff$, because $uu = \frac{ff}{tt}$, which on putting either $t = +\infty$ as well as $t = -\infty$, the applied line u will obtain a two fold vanishing value, positive or negative, and the curve will have four branches each converging at one side or the other, and at both ends of the axis. But in the third case, in which $D = 0$, on taking the equation $uu + \frac{E}{g^5 t^2} = 0$, of which the account is the same, as in the preceding paragraph; and thus the reasoning will have to be continued, so that the equation $P + Q + R + S + \text{etc.}$ will furnish the higher terms.

185. Now we may consider the greatest member P of the equation $P + Q + R + S + \text{etc.} = 0$ to have three simple real factors; and it is evident, if these factors should be unequal to each other, then with any single value whatever these have been explained above for a single real value ; therefore in which case the curve will have six infinite extensions, converging to the three right line asymptotes. If two factors were equal, then from the third unequal one the same will be set out, as before ; but from the two equal ones the same precepts are to be observed, which we have given before. Therefore only the third case remains to be established, in which all three factors are equal to each other. Therefore let $P = (ay - bx)^3 M$. And because the equation

$$P + Q + R + S + \text{etc.} = 0$$

cannot be stopped at infinity, unless $(ay - bx)^3$ may have a certain finite or infinite value, but of order less than ∞^3, so that the power of the infinite, into which the greatest member P may be changed, becomes less than ∞^n, certainly there will be at infinity $\frac{y}{x} = \frac{b}{a}$.

186. Towards setting out this case, first it is necessary to look at the second member Q, to see whether or not it shall have the same factor $ay - bx$; where it is to be observed, if it is absent entirely, then it will be present in the first, because it is recognized to be zero in some factor. And in the first place Q shall not be divisible by $ay - bx$. And, since Q shall be a function of $n-1$ dimensions, M truly will be of $n-3$ dimensions, and $\frac{Q}{(ax+by)^2 M}$ will be a function of zero dimensions and thus on putting $\frac{y}{x} = \frac{b}{a}$ it will be changed into a constant, which shall be $= A$, and there will be $\left(ay - bx\right)^3 + A\left(ax + by\right)^2 = 0$, for the following members will provide the terms which vanish at infinity before $A(ax + by)^2$.

187. Therefore the curved line, which is expressed by this equation, will be prepared thus, so that on being produced to infinity, it may agree with the equation

$$P+Q+R+S+\text{etc.}=0$$

expressed for the curved line. But towards that becoming known more closely, the above can be referred to by another axis, in which the abscissa shall be $t=\frac{ax+by}{g}$ and the applied line $u=\frac{ay-bx}{g}$ on putting $\sqrt{(aa+bb)}=g$, and there will be

$$u^3+\frac{Att}{g}=0,$$

which equation, if there is put $t=\infty$, will give the part of the curve sought

$$P+Q+R+\text{etc.}=0$$

present at infinity. Whereby, if the figure of the curve $u^3+\frac{Att}{g}=0$ were known,

likewise the figure of the infinity part of the curve $P+Q+R+\text{etc.}=0$ will be known. Moreover in the following chapter we will give special consideration to these curved line asymptotes.

188. But if the second member Q may have the factor $ay-bx$, whether likewise it will be divisible by $(ay-bx)^2$ or otherwise. We may consider it not to be divisible by $(ay-bx)^2$ and this function of zero dimension $\frac{Q}{(ay-bx)(ax+by)M}$ which on putting $\frac{y}{x}=\frac{b}{a}$ may give rise to that constant quantity A, and there will be

$$\left(ay-bx\right)^3+A\left(ay-bx\right)(ax+by)+\frac{R}{M}+\frac{S}{M}+\text{etc.}=0\,.$$

Here $\frac{R}{M}$ will become, on putting $\frac{y}{x}=\frac{b}{a}$, either $B(ay-bx)$ or $B(ax+by)$, provided R were divisible or not by $ay-bx$; indeed $\frac{S}{M}$ will become a constant quantity C. Hence as we have established before, that by an equation related to another axis between the coordinates t and u, or that will become

$$u^3+\frac{Atu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0$$

or

$$u^3+\frac{Atu}{g}+\frac{Bt}{gg}+\frac{C}{g^3}=0$$

Because moreover only this case here will be considered, since $t=\infty$, the final terms vanish.
Therefore in the first case there will be

$$u^3+\frac{Atu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0,$$

which gives rise to a double asymptote, surely both $u=0$ and $uu+\frac{At}{g}=0,$ the one a straight line, and the other a parabola. In the latter case too, with t being $=\infty$, or u will have a finite value and there will be, on account of the finite values vanishing before the infinite,

$$\frac{Atu}{g}+\frac{Bt}{gg}=0 \text{ and thus } u=\frac{-B}{Ag}$$

for a straight line. Therefore truly u will be able to have an infinite value ; and thus, with the third term vanishing, there becomes

$$uu+\frac{At}{g}=0$$

for a parabola. Whereby in each case a two fold asymptote will be produced, the one a right line, the other a parabola, from which there will be no need to distinguish between these cases.

189. Let Q also be divisible by $(ay-bx)^2$, as long as R were divisible by $(ay-bx)$ or otherwise, with the same the operations put in place from which before these equations will be produced between t and u :
either

$$u^3+\frac{Auu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0 \quad \text{or} \quad u^3+\frac{Auu}{g}+\frac{Bt}{gg}=0.$$

The first case is for three right lines parallel to each other, if indeed all the roots of the equation

$$u^3+\frac{Auu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0$$

were real, or for a single right line asymptote, if two of the roots were imaginary. Hence truly variations are produced, according as the three asymptotes are parallel to each other or two or all coincide. But in the latter case

$$u^3+\frac{Auu}{g}+\frac{Bt}{gg}=0$$

on putting $t=\infty$ in place cannot be had, unless u shall be infinite and thus the term $\frac{Auu}{g}$ will vanish before the first term u^3, and

$$u^3+\frac{Bt}{gg}=0$$

will be the equation for the curvilinear asymptote of the third order.

190. But if there were $A=0,\ B=0$ and $C=0$, then it is required to return to the following terms of the equation $P+Q+R+S+\text{etc.}=0$, which provide an equation of this kind

$$u^3+\frac{D}{g^4t}+\frac{E}{g^5tt}+\frac{F}{g^6t^3}+\text{ etc.}=0,$$

in which, unless there shall be $D=0$, the third vanishes with the following, so that there shall be $u^3+\frac{D}{g^4t}=0$; and if $D=0$, there will be $u^3+\frac{E}{g^5tt}=0$ and, if also $E=0$, there will be $u^3+\frac{F}{g^6t^3}=0$, etc., which equations denote curved lines, which on putting $t=\infty$ may agree with the curve contained in the equation $P+Q+R+S+\text{etc.}=0$. But these same equations, because the odd power u^3 is present, always are real and thus indicate for sure branches extending to infinity. Yet meanwhile for these same cases the right line expressed by the equation $u=0$ also will be an asymptote, because it is an asymptote of the curves

$$u^3+\frac{D}{g^4t}=0,\quad u^3+\frac{E}{g^5tt}=0\quad \text{etc.}$$

191. Therefore since the branches of the curvatures converging to right line asymptotes may be able to differ so much, it is agreed that this diversity be considered more carefully, which can be done, if the simplest curved line may be defined, which may merge the same right line asymptote with the proposed curve. Thus, even if the equation

$$u^3+\frac{Auu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0,$$

if it may have all the roots real, shows three right line asymptotes to be parallel to each other, yet it is not yet apparent, whether the legs of the curve extended to infinity shall be hyperbolic, that is, expressed by this equation $u=\frac{C}{t}$ or of another kind, just as expressed by the equation $u=\frac{C}{tt}$ or $u=\frac{C}{t^3}$ etc. Towards knowing this, the following nearest term, which the equation suggests, surely $\frac{D}{g^4t}$ or, if this be absent, $\frac{E}{g^5tt}$ or also with this deficient, $\frac{F}{g^6t^3}$. We may take, so that we may resolve the matter generally, the following term to be $\frac{K}{t^k}$; and from the nature of the equation $P+Q+R+S+\text{etc.}=0$, which is of n dimensions, it is apparent that k cannot be a number greater than $n-3$. The roots or factors of the equation

$$u^3+\frac{Auu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0$$

shall be $(u-\alpha)(n-\beta)(u-\gamma)$, and there will be

$$(u-\alpha)(u-\beta)(u-\gamma)-\frac{K}{t^k}=0.$$

Let $u-\alpha=\frac{I}{t^\mu}$ which equation will express the nature of one asymptote, and it will be

$$\frac{I}{t^\mu}\left(\alpha-\beta+\frac{I}{t^\mu}\right)(\alpha-\gamma+\frac{I}{t^\mu})=\frac{K}{t^k}$$

and on putting t infinite it becomes

$$\frac{(\alpha-\beta)(\alpha-\gamma)I}{t^\mu}=\frac{K}{t^k}$$

192. This equation prevails, if the root α were unequal to the remaining roots β and γ, in which case there becomes

$$I=\frac{K}{(\alpha-\beta)(\alpha-\gamma)} \text{ and } \mu=k,$$

from which the root $u=\alpha$ will give the same curvilinear asymptote

$$u-\alpha=\frac{K}{(\alpha-\beta)(\alpha-\gamma)t^{k}}.$$

Therefore if all three roots were unequal to each other, they will provide individual asymptotes of this kind. But if two roots were equal, such as $\beta=\alpha$, two asymptotes will merge into one, and there becomes

$$\frac{I\cdot I(\alpha-\gamma)}{t^{2\mu}}=\frac{K}{t^{k}},$$

from which there becomes

$$I\cdot I=\frac{K}{\alpha-\gamma}\ \text{ and } 2\mu=k.$$

Whereby the nature of this two fold asymptote will be expressed by this equation

$$(u-\alpha)^{2}=\frac{K}{(\alpha-\gamma)t^{2k}}.$$

If all three roots were equal and thus the three asymptotes come together into one, the nature of this asymptote will be expressed by this equation

$$(u-\alpha)^{3}=\frac{K}{t^{k}}.$$

193. But if the greatest member P of the equation $P+Q+R+S+$ etc. may have four real factors, if these were either all unequal to each other or two or also three were equal, from the preceding the nature of the branches of the infinite extensions may be deduced together with the asymptotes. Therefore the single case, in which all the roots are equal to each other, needs an explanation. Therefore let $P=(ay-bx)^{4}M$, so that M shall be a function of $n-4$ dimensions; and, if with functions of zero dimensions, as we consider above $\frac{y}{x}=\frac{b}{a}$, so that they have constant quantities, and likewise with the axes changed, on putting $t=\frac{ax+by}{g}$ and $u=\frac{ay-bx}{g}$, with $g=\sqrt{(aa+bb)}$ present, the following equations will arise between t and u themselves for the asymptotes. Clearly in the first place, if Q were not divisible by $ay-bx$, there will be had

$$u^{4}+\frac{At^{3}}{g}=0$$

194. Then, if Q were indeed divisible by $ay-bx$ but not by $(ay-bx)^2$,

$$u^4+\frac{Attu}{g}+\frac{Btt}{gg}=0,$$

will be produced, in which on putting $t=\infty$ the applied line u can be either a finite or infinite quantity ; therefore it will produce two fold asymptotes, clearly the right line $u+\frac{B}{gA}=0$ and the curve $u^3+\frac{Att}{g}=0$. So that the right line may prevail, the nearest following term may be taken for a closer understanding, which shall be $\frac{K}{t^k}$, and there will be found

$$u+\frac{B}{gA}+\frac{gK}{At^{k+2}}=0,$$

which is the equation for the curve, of which the part corresponding to the abscissa $t=\infty$ will be merged with the curve sought.

195. Now Q shall be divisible by $(ay-bx)^2$ but not by $(ay-bx)^3$, it is required to be seen whether R shall be divisible by $ay-bx$ or otherwise. In the first case there will be produced

$$u^4+\frac{Attu}{g}+\frac{Btu}{gg}+\frac{Ct}{g^3}=0;$$

truly in the second case:

$$u^4+\frac{Attu}{g}+\frac{Btt}{gg}+\frac{Ct}{g^3}=0.$$

The first case gives a double equation, according as u is finite or infinite, and thus it is resolved into these two equations

$$uu+\frac{Bu}{gA}+\frac{C}{ggA}=0 \text{ and } uu+\frac{At}{g}=0;$$

of which the former, if it has both roots real and unequal, presents two parallel right lines, but if the roots shall be imaginary, it will show no branch extending to infinity ; truly this will give the parabolic asymptote $uu+\frac{At}{g}=0$.

The latter equation

$$u^4 + \frac{Attu}{g} + \frac{Btt}{gg} = 0.$$

(on account of $\frac{Ct}{g^3}$ vanishing before $\frac{Btt}{gg}$ on making $t = \infty$) holds two equations of the form $uu + \alpha t = 0$ and thus will produce two parabolic asymptotes, if AA were greater than $4B$, which unite into one, if $AA = 4B$, but become completely imaginary, if AA were less than $4B$, in which case no branch of the curve extending to infinity is designated.

196. Now Q shall be divisible by $(ay - bx)^3$; and, provided R and S shall be divisible or not by $ay - bx$, the following equations will be obtained:

$$u^4 + \frac{Au^3}{g} + \frac{Buu}{gg} + \frac{Cu}{g^3} + \frac{D}{g^4} = 0,$$

$$u^4 + \frac{Au^3}{g} + \frac{Buu}{gg} + \frac{Ct}{g^3} = 0,$$

$$u^4 + \frac{Au^3}{g} + \frac{But}{gg} + \frac{Ct}{g^3} = 0,$$

$$u^4 + \frac{Au^3}{g} + \frac{Btt}{gg} = 0.$$

The first of these equations is for four right lines parallel to each other, if indeed all the roots were real and unequal, but two or several more equal roots may be gathered into one. But indeed imaginary roots, either two or all are taken from the middle. In the second equation on account of $t = \infty$ the applied line u cannot avoid becoming infinite and therefore there will be $u^4 + \frac{Ct}{g^3} = 0$, an asymptote curve of the fourth order. From the third equation the finite value must give $u + \frac{C}{gB} = 0$, therefore indeed there will be had this equation $u^3 + \frac{Bt}{gg} = 0$, a line of the third order for the asymptote. And then the fourth equation on account of infinite u, if $t = \infty$, will change into $u^4 + \frac{Btt}{gg} = 0$, which equation, if B is a positive quantity, is impossible, but if negative, will designate two parabolas with opposite vertices, which produced to infinity will combine with the curve.

197. From these therefore a way is apparent, which one must follow to progress further, if several simple factors of the greatest member P were equal to each other. Because indeed concerning unequal factors, each of which is to be considered separately and the right line asymptote arising from that can be defined. But if two factors were equal, then the natures of the curves can be defined through that, which have been treated in §178 and in the following. And in a similar manner for three equal factors, the matters in §185 and in the following sections can be considered ; and the case, in which four factors are equal, that we have just set out, from which the equality of a great number factors can be treated at the same time. Moreover it is seen, however great the multiplicity and variety, curved lines are able to have a place only regarding branches extending to infinity; for we have not yet touched on the variation which happens in a finite space.

CAPUT VII

DE RAMORUM IN INFINITUM EXCURRENTIUM INVESTIGATIONE

166. Si curva linea quaecunque habeat ramum seu partem in infinitum excurrentem atque ex eius puncto infinite dissito ad axem quemcunque demittatur applicata normalis, tum vel abscissa x vel applicata y vel utraque coordinata erit infinita. Nisi enim vel alterutra vel utraque esset infinita, tum distantia puncti in curva assumti ab initio abscissarum foret finita nempe $=\sqrt{(xx+yy)}$, contra hypothesin. Quamobrem, si curva habeat ramum in infinitum excurrentem, vel abscissae cuipiam finitae conveniet applicata realis infinita, vel abscissae infinite magnae respondebit applicata realis, sive finita sive infinite magna. Ex hoc igitur fonte curvarum rami in infinitum excurrentes investigare poterunt.

167. Sit proposita aequatio algebraica inter coordinatas x et y cuiusvis ordinis, puta n, atque seorsim considerentur termini, in quibus variabiles x et y obtinent n dimensiones, qui erunt

$$\alpha y^n+\beta y^{n-1}x+\gamma y^{n-2}xx+\delta y^{n-3}x^3+\cdots+\zeta x^n,$$

quae expressio resolubilis erit in factores simplices formae $Ay+Bx$, sive reales sive imaginarios. Atque, si habeat factores imaginarios, eorum numerus erit par binique coniuncti dabunt factorem duplicem realem formae

$$AAyy-2ABxy\cdot\cos.\varphi+BBxx.$$

Huiusmodi autem factor, (sive x sive y sive utraque ponatur infinita $=\infty$), semper valorem induet infinitum $=\infty^2$, quia terminus $2AByx\cdot\cos.\varphi$ semper minor est quam duo reliqui $AAyy+BBxx$, neque enim A nec B potest esse $=0$. Huiusmodi ergo factor

$$AAyy-2ABxy\cdot\cos.\varphi+BBxx,$$

si vel x vel y vel utraque ponatur infinita, neque nihilo neque quantitati finitae neque etiam quantitati infinitae ∞ potest esse aequalis, cum ipsa fiat $=\infty^2$, quae infinities maior est quam ∞.

168. Quodsi ergo aequationis pars summa

$$\alpha y^n+\beta y^{n-1}x+\gamma y^{n-2}xx+\cdots+\zeta x^n$$

nullum habeat factorem simplicem realem, quod quidem evenire non potest, nisi n sit numerus par, tum ex meris factoribus duplicibus huius formae

$$AAyy - 2ABxy \cdot \cos.\varphi + BBxx$$

constabit. Quare, si vel x vel y vel utraque ponatur infinita, ipsa ilia expressio valorem induet infinitum $= \infty^n$, neque igitur quantitati finitae neque ulli quantitati infinitae $= \infty^m$, cuius exponens m minor sit quam n, aequalis esse potest. Reliqua igitur aequationis membra, in quibus variabiles x et y pauciores habent dimensiones, quoniam infinita praebent minoris exponentis quam n, illud supremum infinitum adaequare non possunt; ideoque aequatio consistere non potest, si vel x vel y utraque statuatur infinita.

169. Hinc ergo linea curva, quae exprimitur aequatione inter coordinatas x et y, cuius supremum membrum nullos habet factores simplices reales, nullos habebit ramos in infinitum excurrentes ideoque tota curva continebitur in spatia finito, instar ellipsis seu circuli. Quamobrem, si in aequatione generali secundi ordinis

$$\alpha yy + \beta xy + \gamma xx + \delta y + \varepsilon x + \zeta = 0$$

membrum supremum, $\alpha yy + \beta xy + \gamma xx$, in quo variabiles x et y duas obtinent dimensiones, non habeat factores simplices reales, quod evenit, si $\beta\beta$ sit maior quam $4\alpha\gamma$, tum curva nullum habebit ramum in infinitum excurrentem eritque adeo ellipsis.

170. Quo haec distinctius evolvere liceat, omnem aequationem inter coordinatas x et y propositam ita in membra distinguamus, ut ad supremum seu primum referamus omnes aequationis terminos, in quibus variabiles x et y eandem summam dimensionem, cuius exponens sit n, teneant. Ad secundum vero membrum refero omnes terminos, in quibus variabiles ambae $n-1$ dimensiones constituunt. Tertium membrum continebit eos terminos, in quibus ipsorum x et y numerus dimensionum est $n-2$, et ita porro, donec perveniatur ad membrum ultimum, in quo nulla inest dimensio ipsarum x et y, et quod propterea sola quantitate constante constabit. Sit autem P membrum primum seu supremum, Q membrum secundum, R membrum tertium, S quartum et ita porro.

171. Quoniam igitur, si membrum supremum P nullum habet factorem simplicem realem, linea curva aequatione $P + Q + R + S +$ etc. $= 0$ indicata nullum habet ramum in infinitum excurrentem, ponamus iam membrum supremum P unicum habere factorem simplicem realem $ay - bx$, ita ut sit $P = (ay - bx)M$, existente M functione ipsarum x et y dimensionum $n-1$, quae nullos habeat factores simplices reales. Posita ergo vel x vel y vel utraque infinita, fiet $M = \infty^{n-1}$; Q vero simile poterit esse infinitum, at R, S etc. fient infinita minorum graduum. Consequenter aequatio

$$P + Q + R + \text{ etc.} = 0$$

poterit subsistere, si fuerit $ay-bx=$ quantitati finitae vel nihilo, ideoque curva in infinitum porrigetur.

172. Sit ergo $ay-bx=p$, existente p quantitate finita, quae ita debet esse comparata, ut curva in infinitum abeunte fiat

$$pM+Q+R+S+\text{etc.}=0$$

seu

$$p=\frac{-Q-R-S-\text{etc.}}{M}$$

At cum M sit quantitas infinita superioris ordinis quam R et S etc., erunt fractiones $\frac{R}{M}, \frac{S}{M}$ etc. $=0$ ideoque $p=\frac{-Q}{M}$. Hanc ob rem fractio $\frac{-Q}{M}$ dabit valorem ipsius p, si variabiles x et y fiant infinitae. Cum autem sit $ay-bx=p$, erit

$$y=\frac{bx+p}{a} \text{ et } \frac{y}{x}=\frac{b}{a}+\frac{p}{ax}=\frac{b}{a}, \text{ ob } \frac{p}{ax}=0, \text{ si } x=\infty.$$

Curva ergo in infinitum abeunte fit $y=\frac{bx}{a}$.

173. Cum igitur Q et M sint functiones homogenae $n-1$ dimensionum, erit $\frac{-Q}{M}$ functio nullius dimensionis ideoque, si ponatur $y=bax$, praebebit valorem constantem pro p. Vel, quia functio $\frac{-Q}{M}$ determinatur, si tantum ratio inter y et x determinetur, quae est $b:a$, valor ipsius p obtinebitur, si in expressione $\frac{-Q}{M}$ ubique b loco y et a loco x scribatur. Invento ergo hoc modo p erit $ay-bx=p$, quae aequatio in ipsa aequatione proposita $P+Q+R+S+$ etc. $=0$ continetur, si curva abeat in infinitum.

174. Portio itaque curvae in infinitum extensa ipsa exprimetur per hanc aequationem $ay-bx=p$; quae cum sit pro linea recta, haec linea recta in infinitum producta tandem cum linea curva confundetur. Erit ergo linea recta haec curvae asymptota, quoniam linea curva in infinitum porrecta cum recta congruet ideoque continuo propius ad eam accedet. Atque cum aequatio proposita $P+Q+R+S+$ etc. $=0$ posito x vel $y=\infty$ abeat in aequationem $ay-bx=p$, simul intelligitur hanc lineam rectam utrinque in infinitum productam tandem cum curva congruere. Quamobrem linea curva duos habebit ramos in infinitum excurrentes inter se oppositos, quorum alter cum ista linea recta antrorsum, alter cum eadem retrorsum infinite producta conveniet.

175. Cum igitur curva, si aequationis $P+Q+R+S+$ etc. $=0$ membrum supremum P unicum habeat factorem simplicem realem, praedita sit duobus ramis in infinitum extensis atque ad eandem lineam rectam utrinque convergentibus, quae linea recta eius asymptota vocatur, nunc ponamus supremum membrum P duos habere factores simplices reales $ay-bx$ et $cy-dx$, ita ut sit $P=(ay-bx)\,(cy-dx)M$; erit M functio homogenea $n-2$ dimensionum. Duo autem casus hic perpendendi veniunt, prout isti bini factores fuerint inter se aequales vel inaequales.

176. Sint hi factores inter se inaequales; atque manifestum est aequationem $\left(ay-bx\right)\left(cy-dx\right)M+Q+R+S+\text{etc.}=0$ duplici modo subsistere posse pro abscissis vel applicatis infinitis, vel si $ay-bx$ vel si $cy-dx$ aequetur quantitati finitae. Sit igitur $ay-bx=p$; et, cum p sit quantitas finita, in infinito erit $\frac{y}{x}=\frac{b}{a}$, atque ut ante fiet

$$p=\frac{-Q-R-S-\text{etc.}}{(cy-dx)M}=\frac{-Q}{(cy-dx)M},$$

quae est functio nullius dimensionis ipsarum x et y ; quare, si ponatur $\frac{y}{x}=\frac{b}{a}$ vel, quod eodem redit, si ubique scribatur b loco y et a loco x, verus prodibit valor constantis quaesitae p. Erit ergo

$$p=\frac{-Q}{\left(bc-ad\right)M}$$

et ob factores inaequales $bc-ad$ non erit $=0$ neque etiam M, quia nullum omnino factorem realem simplicem complectitur, in nihilum abire potest; unde valor pro p oritur finitus vel etiam $=0$, quod evenit, si vel membrum Q prorsus desit, vel factorem habeat $ay-bx$.

177. Ob supremi ergo membri P factorem realem simplicem curva $ay-bx$, uti in priori casu, unam habebit asymptotam, cuius positio indicatur aequatione $ay-bx=p$. Simili vero modo ob alterum factorem $cy-dx$ quoque habebit asymptotam, quam praebebit aequatio haec: $cy-dx=q$, existente $q=\frac{-Q}{\left(ay-bx\right)M}$ postquam ubique loco y et x hi valores determinati d et c fuerint substituti. Quocirca linea curva omnino duas habebit asymptotas ideoque quatuor ramos in infinitum extensos, qui cum illis rectis tandem congruant. Hic ipse autem casus locum supra invenit in hyperbola ; quare, si in aequatione pro lineis secundi ordinis $\alpha yy+\beta xy+\gamma xx+\delta y+\varepsilon x+\zeta=0$ supremum membrum $\alpha yy+\beta xy+\gamma xx$ duos habeat factores simplices inaequales reales, quod evenit, si $\beta\beta$ superet $4\alpha\gamma$, tum curva erit hyperbola.

178. Sint ambo factores $ay-bx$ et $cy-dx$ inter se aequales, ita ut sit $P=(ay-bx)^2M$. Cum igitur sit $P+Q+R+S+\text{etc.}=0$, erit

$$(ay-bx)^2=\frac{-Q-R-S-\text{etc.}}{M}$$

Quia autem est Q functio $n-1$, R $n-2$ et S functio $n-3$ dimensionum, ob M functionem $n-2$ dimensionum erit, casu infiniti, $\frac{S}{M}=0$ ideoque

$$(ay-bx)^2=-\frac{Q}{M}-\frac{R}{M}=-\frac{Q}{M\left(\mu y+\nu x\right)}(\mu y+\nu x)-\frac{R}{M}.$$

At est $\frac{Q}{M(\mu y+\nu x)}$ et $\frac{R}{M}$ functio nullius dimensionis ipsarum x et y. Quare, cum in infinito sit $y:x=b:a$, si haec ratio $\frac{b}{a}$ pro $\frac{y}{x}$ seu b pro y et a pro x substituatur, utraque illa functio abibit in quantitatem constantem.

179. Fiat ergo facta hac substitutione

$$\frac{Q}{M(\mu y+\nu x)}=A \text{ et } \frac{R}{M}=B\ ;$$

eritque

$$(ay-bx)^2=-A(\mu y+\nu x)-B,$$

quae est aequatio pro linea curva, cum qua linea curva aequatione

$$P+Q+R+S+\text{etc.}=0$$

expressa, postquam in infinitum processerit, confundetur. Verum, quia quantitates μ et ν sunt arbitrariae, sumatur $\mu=b$ et $\nu=a$, ac, immutandis coordinatis, fiat

$$ay-bx=u\sqrt{(aa+bb)} \text{ et } by+ax=t\sqrt{(aa+bb)},$$

eritque pro eadem ilia curva ista aequatio

$$uu+\frac{At}{\sqrt{(aa+bb)}}+\frac{B}{aa+bb}=0,$$

quam patet esse pro parabola. Curva ergo quaesita ita erit comparata, ut in infinitum protensa cum parabola confundatur. Habebit ergo duos tantum ramos in infinitum excurrentes, quorum asymptota non erit linea recta, sed parabola superiore aequatione expressa.

180. Evenit hoc, si non fuerit $A=0$; at si sit $A=0$, (quod evenit, si membrum secundum Q vel desit vel divisibile fuerit per $ay-bx$), tum aequatio cessat esse pro parabola eritque $uu+\frac{B}{aa+bb}=0$, cuius tres casus erunt evolvendi. Primo scilicet, si B fuerit quantitas negativa, puta $\frac{B}{aa+bb}=-ff$, aequatio $uu-ff=0$ duas in se complectetur aequationes $u-f=0$ et $u+f=0$, quae erunt pro duabus lineis rectis inter se parallelis, quarum utraque erit curvae asymptota, uti casu primo; atque ideo curva quatuor habebit ramos in infinitum excurrentes, qui cum istis duabus rectis confundentur.

181. Secundus casus est, quod sit B quantitas affirmativa, puta $+ff$. Quia vero hoc casu aequatio $uu+ff=0$ est impossibilis, curva nullum habebit ramum in infinitum excurrentem, sed tota in spatio finito continebitur. Non solum igitur curva, quae hac aequatione $P+Q+R+S+\text{etc.}=0$ continetur, nullum habebit ramum in infinitum extensum, si membrum supremum P nullum habeat factorem simplicem realem, sed etiam idem usu venire potest, quamvis P habeat factores, uti modo vidimus. Plures autem huiusmodi casus adhuc occurrent.

182. Tertius casus est, quo fit etiam $B=0$, in quem uterque praecedentium incidere potest, ex quo ambiguum est, quomodo curva futura sit comparata. Hinc ad figuram curvae definiendam sequentes termini spectari debebunt. Scilicet, cum sit $P+Q+R+S+\text{etc.}=0$ atque $P=(ay-bx)^2M$, in infinito erit

$$\frac{y}{x}=\frac{b}{a}\ \text{ et }\ (ay-bx)^2+\frac{Q}{M}+\frac{R}{M}+\frac{S}{M}+\frac{T}{M}+\text{etc.}=0.$$

Ponatur ergo, ut ante, facta substitutione

$$\frac{y}{x}=\frac{b}{a},\ \frac{Q}{M}=A(by+ax),\ \frac{R}{M}=B,$$

tum vero, cum S, T, V etc. sint functiones $(n-3),(n-4)$ etc. dimensionum, existente M functione $(n-2)$ dimensionum,

$$\frac{S(by+ax)}{M}=C,\quad \frac{T(by+ax)^2}{M}=D,\quad \frac{V(by+ax)^3}{M}=E\ \text{etc.},$$

erit

$$(ay-bx)^2 + A(by+ax) + B + \frac{C}{by+ax} + \frac{D}{(by+ax)^2} + \frac{E}{(by+ax)^3} + \text{etc.} = 0 .$$

Haec ergo aequatio exprimit naturam curvae lineae, cuius portio in infinitum distans, quae prodit, si $by+ax$ ponatur infinitum, conveniet cum curva in aequatione $P+Q+R+S+\text{etc.}=0$ contenta. Quamvis enim, curva in infinitum excurrente, $(ay-bx)^2$ valorem obtineat vel finitum vel infinitum ordinis tamen inferioris quam ∞^2, tamen $by+ax$ valorem habebit infinitum.

183. Mutemus autem axem, ad quem lineam istam asymptotam inventam referamus, ac in eo ponamus

$$\text{abscissam } \frac{ax+by}{\sqrt{(aa+bb)}} = t \quad \text{et applicatam } \frac{ay-bx}{\sqrt{(aa+bb)}} = u$$

sitque brevitatis gratia $\sqrt{(aa+bb)} = g$, atque erit aequatio

$$uu + \frac{At}{g} + \frac{B}{gg} + \frac{C}{g^3 t} + \frac{D}{g^4 tt} + \frac{E}{g^5 t^3} + \text{etc.} = 0 .$$

Cum igitur in casu, quem evolvere debemus, sit $A=0$ et $B=0$, fiet

$$uu + \frac{C}{g^3 t} + \frac{D}{g^4 tt} + \frac{E}{g^5 t^3} + \text{etc.} = 0$$

Quodsi iam non fuerit $C=0$, posito t infinito termini $\frac{D}{g^4 tt} + \frac{E}{g^5 t^3} + \text{etc.}$ prae $\frac{C}{g^3 t}$ evanescent eritque

$$uu + \frac{C}{g^3 t} = 0 ;$$

qua aequatione natura linea curvae continetur, quae posito $t=\infty$ cum curva quaesita confundetur. Quare, cum hinc sit $u = \pm\sqrt{\frac{-C}{g^3 t}}$ curva duos habebit ramos ad eandem axis partem utrinque convergentes.

184. Quodsi insuper fuerit $C = 0$, tum sumenda est ista aequatio

$$uu + \frac{D}{g^4 t} = 0,$$

ubi iterum tres casus occurrunt, prout D fuerit quantitas affirmativa vel negativa vel nulla. Primo casu, ob aequationem impossibilem, curva nullum habebit ramum in infinitum excurrentem, sed tota continebitur in spatio finito. Secundo casu, si $\frac{D}{g^4} = -ff$, , ob $uu = \frac{ff}{tt}$, quia posito tam $t = +\infty$ quam $t = -\infty$ applicata u duplicem obtinet valorem evanescentem, affirmativum et negativum, curva habebit quatuor ramos ad axem utrinque ad utramque partem convergentes. Tertio autem casu, quo $D = 0$, sumenda est aequatio $uu + \frac{E}{g^5 t^2} = 0,$ cuius par est ratio, atque in paragrapho praecedente; sicque consideratio continuari debebit, quoad aequatio $P + Q + R + S +$ etc. terminos ulteriores suppeditat.

185. Ponamus nunc membrum supremum P aequationis $P + Q + R + S + \text{etc.} = 0$ tres habere factores simplices reales; atque manifestum est, si isti factores fuerint inter se inaequales, tum de unoquoque valere ea, quae supra de unico factore reali sunt exposita; quo ergo casu curva habebit sex ramos in infinitum excurrentes, ad tres lineas rectas asymptotas convergentes. Si bini factores fuerint aequales, tum de tertio inaequali idem erit tenendum, quod ante; at de duobus aequalibus eadem praecepta sunt notanda, quae ante dedimus. Tantum ergo superest casus tertius evolvendus, quo omnes tres factores sunt inter se aequales. Sit igitur $P = (ay - bx)^3 M$. Et quia aequatio $P + Q + R + S + \text{etc.} = 0$ subsistere non potest in infinito, nisi $(ay - bx)^3$ habeat valorem vel finitum vel infinitum quidem, at ordinis inferioris quam ∞^3, quo potestas infiniti, in quam membrum supremum P abit, fiat minor quam ∞^n, erit utique in infinito $\frac{y}{x} = \frac{b}{a}$.

186. Ad hunc casum exponendum primum spectari oportet membrum secundum Q, utrum id factorem habeat eundem $ay - bx$ an secus; ubi notandum est, si omnino desit, tum in priori contineri, quia nihilum quemcunque factorem agnoscit. Primum itaque non sit Q per $ay - bx$ divisibile. Et, cum Q sit functio $n-1$ dimensionum, M vero functio $n-3$ dimensionum, erit $\frac{Q}{(ax+by)^2 M}$ functio nullius dimensionis ideoque posito $\frac{y}{x} = \frac{b}{a}$ abibit in quantitatem constantem, quae sit $= A$, eritque $\left(ay - bx\right)^3 + A\left(ax + by\right)^2 = 0$, sequentia enim membra praebebunt terminos, qui in infinito prae $A(ax + by)^2$ evanescunt.

187. Linea igitur curva, quae hac aequatione exprimitur, ita erit comparata, ut in infinitum producta cum linea curva aequatione

$$P+Q+R+S+\text{etc.}=0$$

expressa congruat. Ad illam autem propius cognoscendam eam ad alium axem referamus, in quo sit abscissa $t=\frac{ax+by}{g}$ et applicata $u=\frac{ay-bx}{g}$ posito $\sqrt{(aa+bb)}=g$, eritque

$$u^3+\frac{Att}{g}=0,$$

quae aequatio, si ponatur $t=\infty$, dabit partem curvae quaesitae

$$P+Q+R+\text{etc.}=0$$

in infinito existentem. Quare, si figura curvae $u^3+\frac{Att}{g}=0$ cognita fuerit,

simul curvae $P+Q+R+\text{etc.}=0$ portionis infinitae figura erit cognita. In capite autem sequente has lineas curvas asymptotas data opera evolvemus.

188. Quodsi membrum secundum Q factorem habeat $ay-bx$, vel simul erit divisibile per $(ay-bx)^2$ vel secus. Ponamus non esse divisibile per $(ay-bx)^2$ ac sumatur ista functio nullius dimensionis $\frac{Q}{(ay-bx)(ax+by)M}$ quae posito $\frac{y}{x}=\frac{b}{a}$ praebeat istam quantitatem constantem A, eritque

$$\left(ay-bx\right)^3+A\left(ay-bx\right)(ax+by)+\frac{R}{M}+\frac{S}{M}+\text{etc.}=0\ .$$

Hic erit $\frac{R}{M}$, posito $\frac{y}{x}=\frac{b}{a}$, vel $B(ay-bx)$ vel $B(ax+by)$, prout R fuerit per $ay-bx$ divisibile vel minus; verum $\frac{S}{M}$ erit quantitas constans C. Hinc ista aequatione ad alium axem relata inter coordinatas t et u, ut ante fecimus, ea erit vel

$$u^3+\frac{Atu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0$$

vel

$$u^3+\frac{Atu}{g}+\frac{Bt}{gg}+\frac{C}{g^3}=0$$

Quia autem tantum casus hue spectat, cum $t=\infty$, termini ultimi evanescunt. Eritque ergo priori casu

$$u^3+\frac{Atu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0 \ ,$$

quae duplicem praebet asymptotam, nempe et $u=0$ et $uu+\frac{At}{g}=0$, alteram rectam, alteram parabolam. Posteriori casu quoque, existente $t=\infty$, vel u habebit valorem finitum eritque, ob finita prae infinitis evanescentia,

$$\frac{Atu}{g}+\frac{Bt}{gg}=0 \quad \text{ideoque } u=\frac{-B}{Ag}$$

pro linea recta. Praeterea vero u valorem infinitum habere poterit; sicque, evanescente termino tertio, fiet

$$uu+\frac{At}{g}=0$$

pro parabola. Quare utroque casu duplex prodit asymptota, altera recta altera parabola, ex quo hos casus a se distingui non opus est.

189. Sit Q etiam per $(ay-bx)^2$ divisibile atque, prout R per $(ay-bx)$ fuerit divisibile vel secus, iisdem, quibus ante, operationibus institutis prodibunt inter t et u hae aequationes: vel

$$u^3+\frac{Auu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0 \quad \text{vel} \quad u^3+\frac{Auu}{g}+\frac{Bt}{gg}=0.$$

Prior casus est pro tribus lineis rectis inter se parallelis, siquidem omnes aequationis

$$u^3+\frac{Auu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0$$

radices fuerint reales, vel pro unica recta asymptota, si duae radices fuerint imaginariae. Hinc vero varietates nascuntur, prout trium istarum asymptotarum inter se parallelarum vel binae vel omnes coincidunt. Posterior autem casus

$$u^3+\frac{Auu}{g}+\frac{Bt}{gg}=0$$

posito $t=\infty$ locum habere nequit, nisi u sit infinitum ideoque terminus $\frac{Auu}{g}$ prae primo u^3 evanescet, eritque

$$u^3 + \frac{Bt}{gg} = 0$$

aequatio pro asymptota curvilinea ordinis tertii.

190. Sin autem fuerit $A = 0$, $B = 0$ et $C = 0$, tum recurrendum est ad terminos aequationis $P + Q + R + S + \text{etc.} = 0$ sequentes, qui praebebunt huiusmodi aequationem

$$u^3 + \frac{D}{g^4 t} + \frac{E}{g^5 tt} + \frac{F}{g^6 t^3} + \text{ etc.} = 0,$$

in qua, nisi sit $D = 0$, tertius cum sequentibus evanescit, ut sit $u^3 + \frac{D}{g^4 t} = 0$;
sin et $D = 0$, erit $u^3 + \frac{E}{g^5 tt} = 0$ et, si etiam $E = 0$, erit $u^3 + \frac{F}{g^6 t^3} = 0$, etc.,
quae aequationes lineas curvas denotant, quae posito $t = \infty$ cum curva in aequatione $P + Q + R + S + \text{etc.} = 0$ contenta congruant. Istae autem aequationes, quia inest potestas impar u^3, semper sunt reales ideoque certo ramos in infinitum excurrentes declarant. Interim tamen pro his iisdem casibus linea recta aequatione $u = 0$ expressa quoque erit asymptota, quia est asymptota curvarum

$$u^3 + \frac{D}{g^4 t} = 0, \quad u^3 + \frac{E}{g^5 tt} = 0 \quad \text{etc.}$$

191. Cum igitur rami curvarum ad asymptotam rectam convergentes tantopere discrepare queant,convenit hanc diversitatem diligentius perpendere, quod fiet, si linea curva simplicissima definiatur, quae ad eandem asymptotam rectam relata cum curva proposita confundatur. Sic, etsi aequatio

$$u^3 + \frac{Auu}{g} + \frac{Bu}{gg} + \frac{C}{g^3} = 0,$$

si radices omnes habeat reales, tres ostendit asymptotas rectas inter se parallelas, tamen nondum patet, utrum crura curvae in infinitum extensa sint hyperbolica, hoc est aequatione $u = \frac{C}{t}$ expressa, an alius generis, veluti aequatione $u = \frac{C}{tt}$ vel $u = \frac{C}{t^3}$ etc. expressa. Ad hoc cognoscendum sumatur sequens proximus terminus, quem aequatio suggerit, nempe $\frac{D}{g^4 t}$ vel, si hic desit, $\frac{E}{g^5 tt}$ vel etiam hoc deficiente $\frac{F}{g^6 t^3}$. Sumamus, ut rem generaliter absolvamus, terminum sequentem esse $\frac{K}{t^k}$; atque ex natura aequationis

$P+Q+R+S+\text{etc.}=0$, quae est n dimensionum, patet k non posse esse numerum maiorem quam $n-3$. Sint aequationis

$$u^3+\frac{Auu}{g}+\frac{Bu}{gg}+\frac{C}{g^3}=0$$

radices seu factores $(u-\alpha)(n-\beta)(u-\gamma)$, eritque

$$(u-\alpha)(u-\beta)(u-\gamma)-\frac{K}{t^k}=0.$$

Sit $u-\alpha=\frac{I}{t^\mu}$ quae aequatio exprimet naturam unius asymptotae, eritque

$$\frac{I}{t^\mu}\left(\alpha-\beta+\frac{I}{t^\mu}\right)(\alpha-\gamma+\frac{I}{t^\mu})=\frac{K}{t^k}$$

et posito t infinito fit

$$\frac{(\alpha-\beta)(\alpha-\gamma)I}{t^\mu}=\frac{K}{t^k}$$

192. Aequatio haec obtinet, si radix α fuerit inaequalis reliquis radicibus β et γ, hocque casu fiet

$$I=\frac{K}{(\alpha-\beta)(\alpha-\gamma)} \quad \text{et} \quad \mu=k,$$

unde radix $u=\alpha$ suppeditabit istam asymptotam curvilineam

$$u-\alpha=\frac{K}{(\alpha-\beta)(\alpha-\gamma)t^k}.$$

Si ergo omnes tres radices fuerint inter se inaequales, singulae huiusmodi asymptotas praebebunt. Sin autem duae radices sint aequales, puta $\beta=\alpha$, binae asymptotae coalescent in unam eritque

$$\frac{II(\alpha-\gamma)}{t^{2\mu}}=\frac{K}{t^k},$$

unde fit

$$II=\frac{K}{\alpha-\gamma} \quad \text{et} \quad 2\mu=k.$$

Quare huius duplicis asymptotae natura exprimetur hac aequatione

$$(u-\alpha)^2 = \frac{K}{(\alpha-\gamma)t^{2k}}.$$

Si omnes tres radices fuerint aequales ideoque tres asymptotae in unam concrescant, eius natura exprimetur hac aequatione

$$(u-\alpha)^3 = \frac{K}{t^k}.$$

193. Quodsi aequationis $P+Q+R+S+$ etc. supremum membrum P quatuor habeat factores simplices reales, si ii fuerint vel omnes inaequales inter se vel bini aequales vel etiam tres aequales, ex antecedentibus natura ramorum in infinitum excurrentium una cum asymptotis colligetur. Unicus ergo casus, quo omnes radices sunt inter se aequales, explanatione indiget. Sit igitur $P=(ay-bx)^4 M$, ut sit M functio $n-4$ dimensionum; atque, si in functionibus nullius dimensionis, uti supra, ponatur $\frac{y}{x}=\frac{b}{a}$, ut praebeant quantitates constantes, simulque ponatur mutato axe $t=\frac{ax+by}{g}$ et $u=\frac{ay-bx}{g}$, existente $g=\sqrt{(aa+bb)}$, pro lineis asymptotis sequentes inter t et u orientur aequationes. Primum scilicet, si Q non fuerit divisibile per $ay-bx$, habebitur

$$u^4 + \frac{At^3}{g} = 0$$

194. Deinde, si Q sit divisibile quidem per $ay-bx$ at non per $(ay-bx)^2$, prodibit

$$u^4 + \frac{Attu}{g} + \frac{Btt}{gg} = 0,$$

in qua posita $t=\infty$ applicata u potest esse vel quantitas finita vel infinita; ergo duplex prodit asymptota, recta scilicet $u+\frac{B}{gA}=0$ et curva $u^3+\frac{Att}{g}=0$. Quod ad rectam attinet, ad eam propius cognoscendarn sumatur terminus sequens proximus, qui sit $\frac{K}{t^k}$, ac reperietur

$$u + \frac{B}{gA} + \frac{gK}{At^{k+2}} = 0,$$

quae est aequatio pro curva, cuius pars respondens abscissae $t=\infty$ cum curva quaesita confundetur.

195. Sit nunc Q divisibile per $(ay-bx)^2$ at non per $(ay-bx)^3$, videndum est, utrum R sit divisibile per $ay-bx$ an secus. Priori casu prodibit

$$u^4+\frac{Attu}{g}+\frac{Btu}{gg}+\frac{Ct}{g^3}=0;$$

posteriori vero

$$u^4+\frac{Attu}{g}+\frac{Btt}{gg}+\frac{Ct}{g^3}=0.$$

Prior casus duplicem dat aequationem, prout u est finitum aut infinitum, ideoque resolvitur in has duas aequationes

$$uu+\frac{Bu}{gA}+\frac{C}{ggA}=0 \text{ et } uu+\frac{At}{g}=0;$$

quarum illa, si radices habet ambas reales et inaequales, praebet duas rectas parallelas, sin autem radices sint imaginariae, nullum ostendit ramum in infinitum excurrentem; haec vero $uu+\frac{At}{g}=0$ dat parabolam asymptotam.

Posterior aequatio

$$u^4+\frac{Attu}{g}+\frac{Btt}{gg}=0.$$

(ob evanescentem $\frac{Ct}{g^3}$ prae $\frac{Btt}{gg}$ facto $t=\infty$) duas continet aequationes formae

$uu+\alpha t=0$ ideo que duae prodeunt parabolae asymptotae, si fuerit AA maior quam $4B$, quae in unam coeunt, si $AA=4B$, at penitus imaginariae evadunt, si AA minor quam $4B$, quo casu nullus curvae ramus in infinitum excurrens designatur.

196. Sit iam Q divisibile, per $(ay-bx)^3$; atque, prout R et S fuerint divisibilia vel non per $ay-bx$, obtinebuntur sequentes aequationes:

$$u^4+\frac{Au^3}{g}+\frac{Buu}{gg}+\frac{Cu}{g^3}+\frac{D}{g^4}=0\,,$$

$$u^4+\frac{Au^3}{g}+\frac{Buu}{gg}+\frac{Ct}{g^3}=0\,,$$

$$u^4+\frac{Au^3}{g}+\frac{But}{gg}+\frac{Ct}{g^3}=0\,,$$

$$u^4+\frac{Au^3}{g}+\frac{Btt}{gg}=0\,.$$

Harum aequationum prima est pro quatuor rectis inter se parallelis, siquidem omnes radices fuerint reales et inaequales, radices autem aequales duas pluresve in unam colligent. At vero radices imaginariae penitus vel duas vel omnes e medio tollunt. In aequatione secunda ob $t=\infty$ applicata u non potest non esse infinita eritque ergo $u^4+\frac{Ct}{g^3}=0$, asymptota curva quarti ordinis. Ex aequatione tertia finitum valorem habere potest $u+\frac{C}{gB}=0$, praeterea vero habet hanc $u^3+\frac{Bt}{gg}=0$, lineam tertii ordinis, pro asymptota. Denique aequatio quarta ob u infinitam, si $t=\infty$, abit in $u^4+\frac{Btt}{gg}=0$, quae aequatio, si B est quantitas affirmativa, est impossibilis, sin negativa, designat duas parabolas ad verticem oppositas, quae in infinitum productae cum curva confundentur.

197. Et his igitur iam via patet, qua ulterius progredi licet, si plures factores simplices supremi membri P inter se fuerint aequales. Quod enim ad factores inaequales attinet, eorum quisque seorsim considerari atque linea recta asymptota ex eo nata definiri potest. Sin autem duo factores fuerint aequales, tum per ea, quae §§ 178 et sequentibus sunt tradita, indoles curvae definiri potest. Similique modo pro tribus factoribus aequalibus negotium conficient §§ 185 et sequentes; atque casum, quo quatuor factores sunt aequales, modo evolvimus, ex quo simul plurium factorum aequalitas tractari potest. Ceterum hinc perspicitur, quanta multiplicitas ac varietas in lineis curvis tantum ratione ramorum in infinitum excurrentium locum habere queat; varietatem enim, quae in spatia finito inesse potest, nondum attigimus.