CHAPTER VI

THE SUBDIVISION OF LINES OF THE SECOND ORDER INTO KINDS

131. The properties relating to lines of the second order, which we have elicited in the preceding chapter, agree equally for all lines ; indeed nor have we made mention of any variation, by which these lines may be distinguished from each other. Although moreover all lines of the second order make use of these properties in common, yet these differ the most amongst themselves on account of the figure described ; on this account it may be agreed that the lines present in this order may be agreed to be distributed into kinds, by which the different figures which arise in this order, are distinguished and the properties may be set out, which finally are agreed upon for the individual kinds.

132. Moreover the general equation for lines of the second order, by changing only the axis and the starting point of the abscissas, we have reduced to this, so that all lines of the second order may be contained in this equation

$$yy = \alpha + \beta x + \gamma xx ,$$

in which x and y may denote orthogonal coordinates. Therefore since for any abscissa x the applied line y may adopt a two-fold value, the one positive and the other negative, that axis, in which the abscissas x are taken, will cut the curve into two similar and equal parts ; and thus this axis will be the orthogonal diameter of the curve, and any line of the second order will have an orthogonal diameter, upon which I have taken the abscissas as the axis.

133. Therefore three constant quantities α, β, and γ are present in this equation, which since they may be able to be varied in an infinite number of ways among themselves, will give rise to innumerable variations in the curved lines, but which more or less will differ in turn among themselves from the account of the figure. For in the first place the same figure will result in an infinite number of ways from the proposed equation $yy = \alpha + \beta x + \gamma xx$, surely from a variation of the starting point of the abscissas on the axis, which comes about, while the abscissa x may be increased or decreased by a given amount. Then also the same figure will be encountered under a difference in magnitude, thus so that infinitely many curved lines may be produced, which only differ on account of a magnitude between each other, just as circles described from different radii. From which it is evident not any variation of the letters α, β, and γ produces diverse species or kinds of lines of the second order.

134. But the nature of the coefficient γ, which is encountered in the equation

$$yy = \alpha + \beta x + \gamma xx,$$

suggests the maximum distinction between the curved lines according to whether it will have either a positive or negative value. For if γ should have a positive value, with the abscissa x put infinite, in which case the term γxx must emerge infinitely greater than the remaining terms $\alpha + \beta x$ and therefore the expression $yy = \alpha + \beta x + \gamma xx$ retains a positive value, and the applied line y equally will have an infinitely great value, the one positive and the other negative, which likewise arises, if there is put $x = -\infty$, in which case the expression $yy = \alpha + \beta x + \gamma xx$ will adopt an infinitely great positive value. On this account, with the positive quantity γ present, the curve will have four branches departing to infinity, corresponding to the two abscissas $x = +\infty$ and the two abscissas $x = -\infty$. Therefore these curves provided with the four branches running off to infinity are considered to constitute a single kind of line of the second order, and they are called *hyperbolas*.

135. But if the coefficient γ should have a negative value, then, on putting $x = +\infty$ or $x = -\infty$, the expression $yy = \alpha + \beta x + \gamma xx$ will retain the negative value and thus the applied line y becomes imaginary. Therefore at no time will the abscissa in these curves be able to become infinite and thus no part of the curve departs to infinity, but the whole curve will be retained in a finite and determined space. Therefore this second kind of lines of the second order has acquired the name *ellipse*, the nature of which therefore is contained in this equation $yy = \alpha + \beta x + \gamma xx$, if γ were a negative quantity.

136. Therefore since the value of γ itself, according as this was either positive or negative, so may produce the different character of lines of the second order, so that hence deservedly the two different kinds may be put in place: if there may be put $\gamma = 0$, which holds a mean place between the positives and negatives, a resulting curve also hence may be put in place, a certain mean kind between hyperbolas and ellipses which is called the *parabola*, the nature of which will be expressed by this equation $yy = \alpha + \beta x$. Here likewise it is the case, whether β were a positive or negative quantity, because the nature of the curve is not changed with the abscissa made negative. Therefore β shall be a positive quantity, and it is clear, with the abscissa x increased to infinity the applied line y also becomes infinite both positive and negative, from which the parabola will have two branches extending to infinity, but it cannot have more than two, because on putting $x = -\infty$ the value of the applied line y becomes imaginary.

137. Therefore we have three kinds of lines of the second order, the ellipse, the parabola, and the hyperbola, which differ so much from each other, that these generally will not be able to be confused. For essentially the difference consists of the number of branches extending to infinity ; for the ellipse has no part extending to infinity, but is enclosed

completely in a finite space, the parabola truly has two branches running off to infinity and the hyperbola has four. Whereby, since in general we have considered the properties of the conic sections in the preceding chapter, now, we may see with which properties whatever species will be endowed.

138. We may begin from the ellipse (Fig. 31), the equation of which is this :

$$yy = \alpha + \beta x + \gamma xx$$

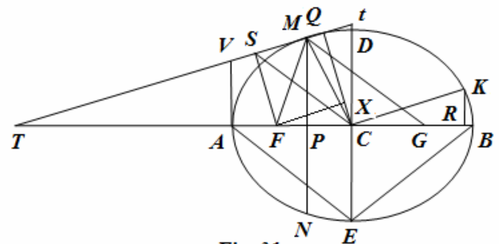

Fig. 31

with the abscissas taken on the orthogonal diameter. Truly because the beginning of the abscissas depends on our choice, if we may remove that amount $\frac{\beta}{2\gamma}$ from the interval, an equation of this form will arise : $yy = \alpha - \gamma xx$, in which the abscissas are taken from the centre of the figure.

[i.e. the origin is moved along the diameter : if we define $x' = x - \frac{\beta}{2\gamma}$, then the linear term vanishes.]

Therefore the centre shall be *C* and *AB* the orthogonal diameter, and the abscissa will be $CP = x$ and the applied line $PM = y$. Therefore on taking $x = \pm\sqrt{\frac{\alpha}{\gamma}}$, making $y = 0$ and if *x* may pass through these limits $+\sqrt{\frac{\alpha}{\gamma}}, -\sqrt{\frac{\alpha}{\gamma}}$, the applied line *y* becomes imaginary; which indicates the whole curve to be contained within those limits. Therefore there will be $CA = CB = \sqrt{\frac{\alpha}{\gamma}}$; then by making $x = 0$ there becomes $CD = CE = \sqrt{\alpha}$. Therefore the semidiameter or the semi principle axis may be put $CA = CB = a$ and with the conjugate axis taken $CD = CE = b$, there will be $\alpha = bb$ and $\gamma = \frac{bb}{aa}$. From which the equation for this ellipse will arise :

$$yy = bb - \frac{bbxx}{aa} = \frac{bb}{aa}(aa - xx).$$

139. When these conjugate semidiameters a and b become equal to each other, then the ellipse will change into a circle on account of $yy = aa - xx$ or $yy + xx = aa$; indeed there will be $CM = \sqrt{(xx + yy)} = a$ and thus all the points M of the curve will be equally distant from the centre C, which is the property of the circle. But if the semiaxes a and b were unequal to each other, then the curve will be oblong, evidently there will be either AB greater than DE or DE greater than AB. Because truly the conjugate axes AB and DE can be interchanged among themselves and likewise, in whatever axis we take, we may put AB to be the major axis, or a greater than b; and on this axis the foci of the ellipse F and G are present on taking $CF = CG = \sqrt{(aa - bb)}$, truly the semiparameter or semilatus rectum of the ellipse will be $= \frac{bb}{a}$, which expresses the magnitude of the applied line erected at either focus F or G.

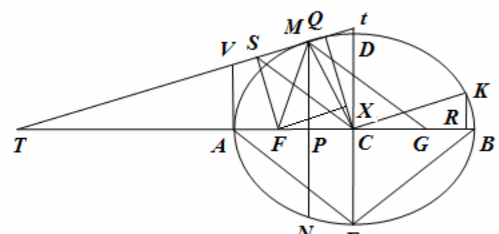

Fig. 31

140. The right lines FM and GM are drawn to the point M of the curve, and as we have seen there will be, [*c.f.* $DM = a - \frac{(a-d)x}{a}$ in § 130 and Fig. 29, previous chapter ;]

$$FM = AC - \frac{CF \cdot CP}{AC} = a - x\frac{\sqrt{(aa - bb)}}{a}$$

and

$$GM = a + x\frac{\sqrt{(aa - bb)}}{a},$$

from which there is made :

$$FM + GM = 2a.$$

Whereby, if the right lines FM and GM are drawn to any point of the curve M from both the foci, the sum of these will be equal always to the major axis $AB = 2a$; from which since it is seen from the characteristic property of the foci, then the manner is deduced easily, for an ellipse to be described mechanically.

141. The tangent *TMt* is drawn from the point *M*, which crosses with the axis at the points *T* and *t*, and there will be, as we have shown above, $CP : CA = CA : CT$; from which $CT = \frac{aa}{x}$, and in a similar manner with the coordinates interchanged, $Ct = \frac{bb}{y}$.
Therefore there will be

$$TP = \frac{aa}{x} - x, \quad TF = \frac{aa}{x} - \sqrt{(aa - bb)},$$

and $TA = \frac{aa}{x} - a.$

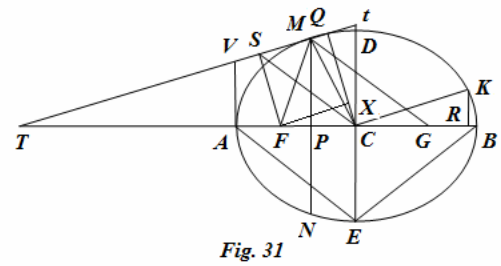

Fig. 31

And thus there becomes:

$$TP = \frac{aa - xx}{x} = \frac{aayy}{bbx}$$

$$\text{and } TM = \frac{y\sqrt{(b^4 xx + a^4 yy)}}{bbx},$$

and hence

$$\text{tang.}\, CTM = \frac{bbx}{aay}, \quad \sin.\, CTM = \frac{bbx}{\sqrt{(b^4 xx + a^4 yy)}}$$

and

$$\cos.\, CTM = \frac{aay}{\sqrt{(b^4 xx + a^4 yy)}}.$$

Whereby, if *AV* may be erected normal to the axis at *A*, which likewise touches the curve, there will be

$$AV = \frac{a(a-x)}{x} \cdot \frac{bbx}{aay} = \frac{bb(a-x)}{ay} = b\sqrt{\frac{a-x}{a+x}}$$

on account of $ay = b\sqrt{(aa - xx)}$.

142. Since there shall be

$$FT = \frac{aa - x\sqrt{(aa - bb)}}{x} \text{ and } FM = \frac{aa - x\sqrt{(aa - bb)}}{a},$$

there will be $FT : FM = a : x$. In a similar manner on account of

$$GT = \frac{aa + x\sqrt{(aa-bb)}}{x} \text{ and } GM = \frac{aa + x\sqrt{(aa-bb)}}{a}$$

there will be $GT : GM = a : x$; from which there becomes $FT : FM = GT : GM$. But there is

$$FT : FM = \sin.FMT : \sin.CTM \quad \text{and} \quad GT : GM = \sin.GMt : \sin.CTM ,$$

on account of which there will be
$\sin.FMT = \sin.GMt$ and thus

$$\text{angle } FMT = \text{angle } GMt .$$

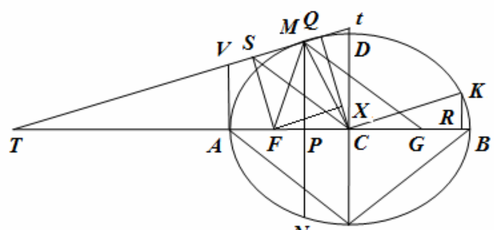

Fig. 31

Therefore both the right lines from the foci drawn to some point *M* of the curve are inclined equally to the tangent of the curve at that point *M*, which is the principal property of the foci.

143. Since there shall be $GT : GM = a : x$, on account of $CT = \frac{aa}{x}$ there will be also $CT : CA = a : x$; from which $GT : GM = CT : CA$, whereby, if from the centre *C*, the right line *CS* is drawn parallel to the right line *GM*, crossing the tangent at *S*, there will be $CS = CA = a$; but in the same manner, if the right line *FM* may be drawn parallel to the tangent from *C*, that equally will be $= CA = a$. But since there shall be

$$TM = \frac{y}{bbx}\sqrt{(b^4xx + a^4yy)},$$

there will be, on account of $aayy = aabb - bbxx$,

$$TM = \frac{y}{bx}\sqrt{\left(a^4 - xx\left(aa-bb\right)\right)};$$

but

$$FT \cdot GT = \frac{a^4 - xx(aa-bb)}{xx},$$

from which

$$TM = \frac{y}{b}\sqrt{FT \cdot GT}.$$

Whereby, on account of $TG : TC = TM : TS$ there will be

$$TS = \frac{TM \cdot CT}{TG}$$

and thus

$$TS = \frac{y \cdot CT}{b}\sqrt{\frac{FT}{GT}} = \frac{y \cdot CT \cdot FT}{b\sqrt{FT \cdot GT}} = \frac{yy \cdot CT \cdot FT}{bb \cdot TM}.$$

Then there becomes

$$PT = \frac{aayy}{bbx} = \frac{CT \cdot yy}{bb}$$

therefore

$$TS = \frac{PT \cdot FT}{TM}$$

and thus

$$TM : PT = FT : TS;$$

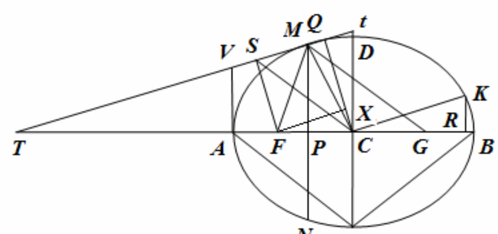

Fig. 31

from which it is understood that the triangles *TMP* and *TFS* are similar and thus and thus the right line *FS* from the focus *F* to be normal to the tangent. Indeed there will be $SV = \frac{AF \cdot MV}{GM}$, which may be elicited from these expressions.

144. But if therefore from the other focus *F* the perpendicular *FS* may be drawn to the tangent and the right line *CS* may be joined from the centre *C* to the point *S*, this right line *CS* is always equal to the major semiaxis $AC = a$. Truly there will be, on account of $TM : y = TF : FS$,

$$FS = \frac{y \cdot TF}{TM} = \frac{b \cdot TF}{\sqrt{FT} \cdot GT} = b\sqrt{\frac{FT}{GT}},$$

therefore

$$GT : FT = GM : FM = CD^2 : FS^2;$$

truly the perpendicular from the other focus sent to the tangent will be $= b\sqrt{\frac{GT}{FT}}$,

whereby the minor semiaxis $CD = b$ will be the mean proportional between these perpendiculars. Now also the perpendicular *CQ* may be sent from the centre *C* to the tangent, and there will be $TF : FS = CT : CQ$, therefore

$$CQ = \frac{b \cdot CT}{\sqrt{FT \cdot GT}} = \frac{bx \cdot CT}{a\sqrt{FM \cdot GM}} = \frac{ab}{\sqrt{FM \cdot GM}},$$

from which

$$CQ - FS = \frac{b \cdot CF}{\sqrt{FT \cdot GT}} = CX,$$

with FX parallel to the tangent. Hence there will be

$$CQ - CX = \frac{b \cdot TF}{\sqrt{FT \cdot GT}} \quad \text{and} \quad CQ + CX = \frac{b \cdot TG}{\sqrt{FT \cdot GT}},$$

from which

$$CQ^2 - CX^2 = bb \quad \text{and} \quad CX = \sqrt{\left(CQ^2 - bb\right)};$$

therefore with the minor axis given, a point X may be found on the perpendicular CQ, from which a normal drawn will pass through the focus F.

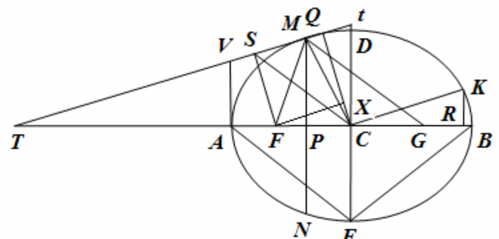

Fig. 31

145. With these properties of the foci established we may consider any two conjugate diameters. Moreover CM will be a semidiameter, the conjugate of which will be found, if CK may be drawn from the centre parallel to the tangent TM. Putting $CM = p$, $CK = q$ and the angle $MCK = CMT = s$, in the first place there will be $pp + qq = aa + bb$ and following this $pq \cdot \sin. s = ab$, as we have seen above. But truly there will be

$$pp = xx + yy = bb + \frac{(aa - bb)xx}{aa}$$

and

$$qq = aa + bb - pp = aa - \frac{\left(aa - bb\right)xx}{aa} = FM \cdot GM,$$

and in the same manner $pp = FK \cdot GK$. Then, since there shall be $CQ = \dfrac{ab}{\sqrt{FM \cdot GM}}$, there will be

$$\sin.CMQ = \sin.s = \frac{ab}{p\sqrt{FM \cdot GM}}.$$

And then there will be

$$TM : TP = \frac{y}{b}\sqrt{FT \cdot GT} : \frac{aayy}{bbx} = \sqrt{FM \cdot GM} : \frac{ay}{b} = CK : CR,$$

from which

$$CR = \frac{ay}{b} \quad \text{and} \quad KR = \frac{bx}{a}$$

and thus

$$CR.KR = CP \cdot PM.$$

Then there will be

$$\sin.FMS = \frac{b}{\sqrt{GM \cdot FM}} = \frac{b}{q};$$

because again there will be

$$x = CP = \frac{a\sqrt{(pp - bb)}}{\sqrt{(aa - bb)}} \quad \text{and} \quad y = \frac{b\sqrt{(pp - pp)}}{\sqrt{(aa - bb)}} = PM$$

and

$$CR = \frac{a\sqrt{(aa - pp)}}{\sqrt{(aa - bb)}} \quad \text{and} \quad KR = \frac{b\sqrt{(pp - pp)}}{\sqrt{(aa - bb)}},$$

there will be

$$\text{tang}.ACM = \frac{y}{x} \quad \text{and} \quad \text{tang}.2ACM = \frac{2yx}{xx - yy} = \frac{2ab\sqrt{(aa - pp)(pp - bb)}}{(aa + bb)pp - 2aabb}.$$

But there is

$$ab = pq \cdot \sin.s, \quad aa + bb = pp + qq$$

and

$$\sqrt{(aa - pp)(pp - bb)} = -pq \cdot \cos.s,$$

from which there becomes

$$\tan.2ACM = \frac{-qq \cdot \sin.2s}{pp + qq \cdot \cos.2s},$$

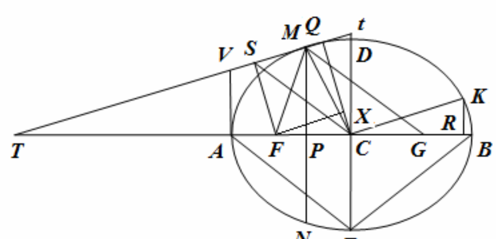

Fig. 31

because cos.*s* is negative. Finally there is $CK^2 = MT \cdot Mt$; truly from the above there is elicited

$$MV = q\sqrt{\frac{AP}{BP}} \quad \text{and} \quad AV = b\sqrt{\frac{AP}{BP}};$$

from which there will be $AV : MV = b : q = CE : CK$. Therefore the right lines *AM* and *EK*, if they may be drawn, will be parallel to each other.

146. Because $pq \cdot \sin.s = ab$, pq will be greater than *ab;* and, since there shall be $pp + qq = aa + bb$, the quantities *p* and *q* approach more to the ratio of equality than *a*

and *b*, from which all the conjugate diameters among these, which are orthogonal, disagree the most between themselves in turn. Therefore two conjugate diameters will be equal to each other, according to which it shall be required to find $q = p$, and there shall be

$$2pp = aa + bb \text{ and } p = q = \sqrt{\frac{aa+bb}{2}}$$

and

$$\sin.s = \frac{2ab}{aa+bb} \text{ and } \cos.s = \frac{-aa+bb}{aa+bb} \;;$$

from which there shall be

$$\sin.\tfrac{1}{2}s = \sqrt{\frac{aa}{aa+bb}} \text{ and } \cos.\tfrac{1}{2}s = \sqrt{\frac{bb}{aa+bb}} \;;$$

therefore

$$\text{tang}.\tfrac{1}{2}s = \frac{a}{b} = \text{tang}.CEB \text{ and } MCK = 2CEB = AEB.$$

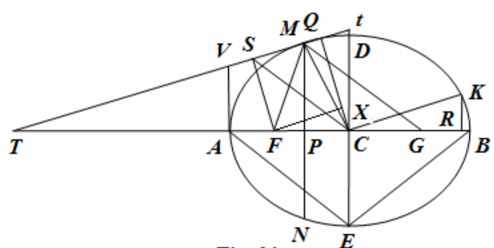

Fig. 31

Again,

$$CP = \frac{a}{\sqrt{2}}, \quad CM = \frac{b}{\sqrt{2}},$$

whereby the conjugate semidiameters equal to each other *CM, CK* will be parallel to the chords *AE* and *BE.*

147. If the abscissas will be computed from the vertex *A* and there is put $AP = x$, $PM = y$, since now there shall be $a - x$, which before was x, this equation will be had

$$yy = \frac{bb}{aa}(2ax - xx) = \frac{2bb}{a}x - \frac{bb}{aa}xx,$$

where it is apparent that $\frac{2bb}{a}$ is the parameter or latus rectum of the ellipse. The semilatus rectum or the applied line at the focus may be put $= c$ and the distance of the focus from the vertex $AF = d$, there will be

$$\frac{bb}{a} = c \text{ and } a - \sqrt{(aa-bb)} = d = a - \sqrt{(aa-ac)},$$

from which there becomes

$$2ad - dd = ac \text{ and } a = \frac{dd}{2d-c}.$$

Hence there will be

$$yy = 2cx - \frac{c(2d-c)xx}{dd},$$

which is the equation for the ellipse between the orthogonal coordinates *x* and *y*, with the abscissas *x* on the principal axis *AB* computed from the vertex *A*, which will be obtained from the given distance of the focus from the vertex $AF = d$ and the semilatus rectum $= c$; where always it is to be observed that *2d* must be greater than *c*, because

$$AC = a = \frac{dd}{2d-c} \text{ and } CD = b = d\sqrt{\frac{c}{2d-c}}.$$

148. But if there were $2d = c$, there will be $yy = 2cx$, as we have seen the above equation to be for a parabola (Fig. 32): for the equation above $yy = \alpha + \beta x$ is reduced to that form, with the beginning interval of the abscissas $= \frac{\alpha}{\beta}$ changed . Therefore there shall be the parabola *MAN*, the nature of which is expressed by this equation $yy = 2cx$

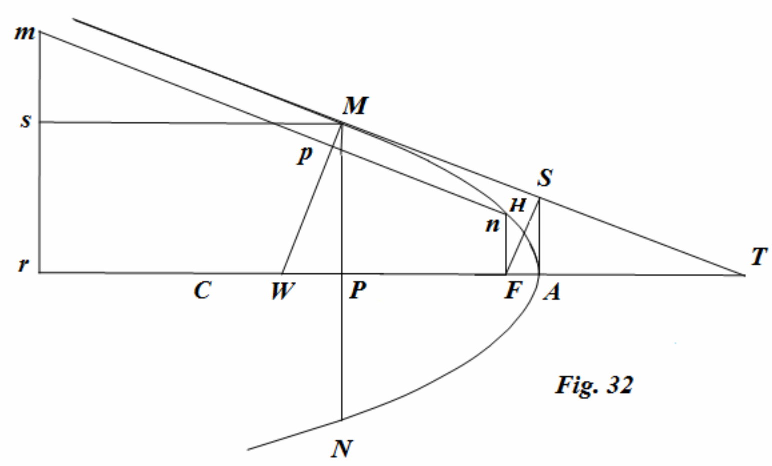

Fig. 32

between the abscissa $AP = x$ and the applied line $PM = y$. Therefore the distance of the focus from the vertex $AF = d = \frac{1}{2}c$ and the semiparameter $FH = c$, and everywhere $PM^2 = 2FH \cdot AP$; from which, with the abscissa *AP* put infinite, likewise the applied lines *PM* and *PN* increase to infinity and thus the curve at each part of the axis

AP is extended to infinity. But with the abscissa *x* put negative the applied line becomes imaginary, and hence no part of the curve corresponds to the axis beyond *A* towards *T*.

149. Since the equation for the ellipse shall change into a parabola by making $2d = c$, it is evident that the parabola is no other than the ellipse, the semiaxis of which $a = \frac{dd}{2d-c}$ becomes infinite; on account of which all the properties which we have found for the ellipse, are transferred to the parabola, with the axis *a* made infinite. But first, since there shall be $AF = \frac{1}{2}c$, there will be $FP = x - \frac{1}{2}c$, and thus hence with the right line *FM* drawn from the focus *F* to the point *M* of the curve there will be

$$FM^2 = xx - cx + \frac{1}{4}cc + yy = xx + cx + \frac{1}{4}cc$$

$$FM = x + \frac{1}{2}c = AP + AF,$$

which is a particular property of the focus for the parabola.

150. Because the parabola arises from the ellipse with the greater axis increased to infinity, we may consider the parabola, to be as it were an ellipse, and its semi major axis $AO = a$, with the quantity *a* become infinite, thus so that the centre *C* may be infinitely removed from the vertex *A*. The tangent of the curve *MT* may be drawn to *M* crossing the axis at *T*; because there was

$$CP : CA = CA : CT, \quad \text{there will be } CT = \frac{aa}{a-x},$$

on account of $CP = a - x$; and hence $AT = \frac{ax}{a-x}$. But, since *a* shall be an infinite quantity, the abscissa *x* will vanish before that and there will be $a - x = a$, and thus $AT = x = AP$; which likewise can be shown in the same manner : since there shall be $AT = \frac{ax}{a-x}$, there will be $AT = x + \frac{xx}{a-x}$, but because the denominator of the fraction $\frac{ax}{a-x}$ is infinite, with a finite numerator present, the value of the fraction will be vanishing and thus $AT = AP = x$.

151. But if therefore the line *MC* may be drawn from the point *M* to the centre of the parabola infinitely distant *C*, which will be parallel to the axis *AC* , that also will be a diameter of the curve bisecting all the parallel chords of the tangent *MT*. Evidently, if a chord or ordinate *mn* may be drawn parallel to the tangent *MT*, that will be bisected by the diameter *Mp* at *p*. Therefore each right line *AP* drawn parallel to the axis will be an

oblique angle diameter. Towards eliciting the nature of diameters of this kind there shall be $Mp = t$, $pm = u$, *msr* may be drawn from *m* normal to the axis*;* there will be, on account of $PT = 2x$ and

$$MT = \sqrt{(4xx+2cx)},\ \sqrt{(4xx+2cx)} : 2x : \sqrt{2}cx = pm : ps : ms ,$$

from which there will be found:

$$ps = \frac{2xu}{\sqrt{(4xx+2cx)}} = u\sqrt{\frac{2x}{2x+c}} \text{ and } ms = u\sqrt{\frac{c}{2x+c}};$$

hence there will be

$$Ar = x+t+u\sqrt{\frac{2x}{2x+c}} \text{ and } mr = \sqrt{2}cx+u\sqrt{\frac{c}{2x+c}}.$$

Indeed because there is $mr^2 = 2c\cdot Ar$, there becomes

$$2cx+2cu\sqrt{\frac{2x}{2x+c}}+\frac{cuu}{2x+c} = 2cx+2ct+2cu\sqrt{\frac{2x}{2x+c}}$$

and hence

$$uu = 2t(2x+c) = 4FM\cdot t \text{ or } pm^2 = 4FM\cdot Mp.$$

But of the angle of obliquity *mps* there will be

$$\text{the sine} = \sqrt{\frac{c}{2x+c}} = \sqrt{\frac{AF}{FM}}, \quad \text{the cosine} = \sqrt{\frac{2x}{2x+c}} = \sqrt{\frac{AP}{FM}},$$

and thus

$$\sin.2mps = \frac{2\sqrt{2}cx}{2x+c} = \frac{y}{FM} = \sin.MFp,$$

therefore there will be

$$\text{the angle } mps = MTP = \tfrac{1}{2}MFr.$$

152. Because there is $MF = AP + AF$, on account of $AP = AT$ there will be $FM = FT$; and thus the triangle *MFT* will be isosceles, and the angle $MFr = 2MTA$, as we have just found. Then since there shall be $MT = 2\sqrt{x(x+\frac{1}{2}c)}$, there will be $MT = 2\sqrt{AP\cdot FM}$, hence with a perpendicular sent from the focus *F* to the tangent , there will be

$$MS = TS = \sqrt{AP} \cdot FM = \sqrt{AT} \cdot TF \ ,$$

from which there shall be $AT : TS = TS : TF$. From which analogy it is seen that the point *S* is on the right line *AS* of the normal to the axis at the vertex *A*. Indeed there will be

$$AS = \tfrac{1}{2} PM \quad \text{and} \quad AS : TS = AF : FS \ ,$$

therefore $FS = \sqrt{AF \cdot FM}$, and *FS* will be the mean proportional between *AF* and *FM*. Besides truly there will be

$$AS : MS = AS : TS = FS : FM = \sqrt{AF} : \sqrt{FM} \ .$$

But if the normal *MW* may be drawn to the tangent at *M* cutting the axis at *W*, there will be

$$PT : PM = PM : PW \quad \text{or} \quad 2x : \sqrt{2cx} = \sqrt{2cx} : PW \ ;$$

from which there becomes $PW = c$ *;* therefore generally the interval *PW*, which is intercepted on the axis between the applied line *PM* and the normal *WM*, has a constant magnitude and is equal to half of the latus rectum or the applied line *FH*. Moreover there will be

$$FW = FT = FM \quad \text{and} \quad MW = 2\sqrt{AF \cdot FM} \ .$$

153. Now we come to the hyperbola, the nature of which is expressed by this equation :

$$yy = \alpha + \beta x + \gamma xx$$

with the abscissas taken on the orthogonal diameter. But if the starting point of the abscissas may be transferred by the interval $\frac{\beta}{2\gamma}$, an equation of this kind may arise $yy = \alpha + \gamma xx$, in which the abscissas are computed from the centre. But γ must be a positive quantity ; because truly regarding α , likewise it shall be either a positive or negative quantity ; for with the coordinates *x* interchanged *y* a positive quantity α is changed into a negative and reciprocally. On account of which α shall be a negative quantity and $yy = \alpha + \gamma xx$, and it is apparent the applied line *y* vanishes twice, clearly if there were

$$x = +\sqrt{\frac{\alpha}{\gamma}} \quad \text{and} \quad x = -\sqrt{\frac{\alpha}{\gamma}} \ .$$

Therefore with *C* denoting the centre (Fig. 33), *A* and *B* shall be the places, were the axis is cut by the curve ; and, on putting the semiaxes $CA = CB = a$, there will be

$a = \sqrt{\frac{\alpha}{\gamma}}$ and $\alpha = \gamma aa$, from which there becomes

$$yy = \gamma xx - \gamma aa.$$

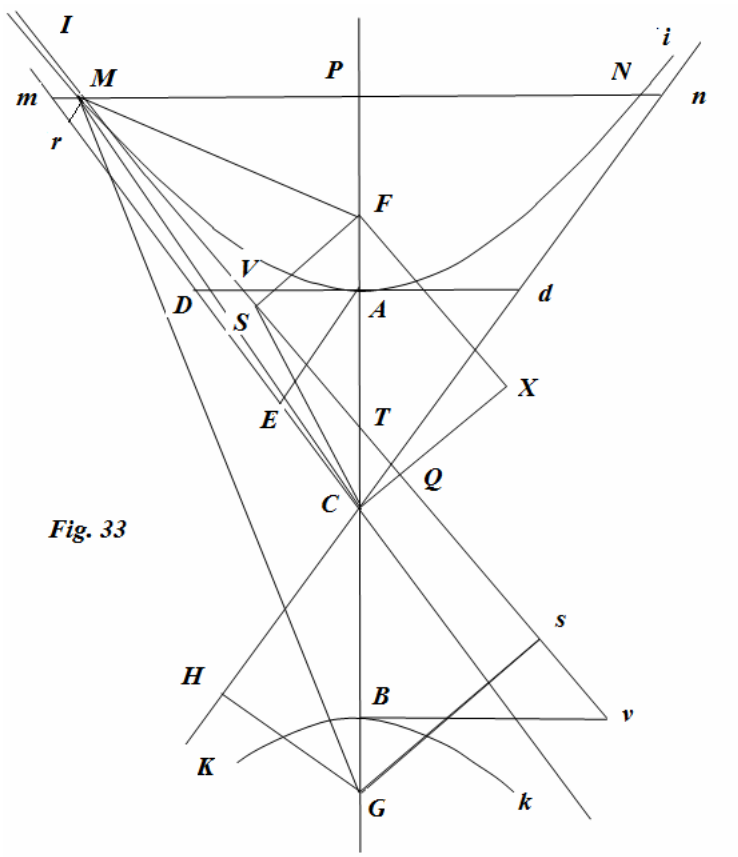

Fig. 33

Therefore as long as x^2 is less than a^2, the applied line will be imaginary, from which no part of the curve corresponds to the whole axis *AB*. Truly with *xx* greater than *aa*, the applied lines continually increase and finally depart to infinity ; therefore the hyperbola will have the four branches *AI*, *Ai*, *BK*, *Bk* running off to infinity and among themselves they are similar and equal, which is the principal property of hyperbolas.

154. Because on putting $x = 0$ there becomes $yy = -\gamma aa$, the hyperbola will not have a conjugate axis like the ellipse, because at the centre *C* the applied line is imaginary. Therefore this conjugate axis will be imaginary, which, so that we may preserve some similarity to the ellipse, we may put $= b\sqrt{-1}$, thus so that there shall be

$\gamma aa = bb$ and $\gamma = \frac{bb}{aa}$. Therefore calling the abscissa $CP = x$ and the applied line $PM = y$ there will be

$$yy = \frac{bb}{aa}(xx - aa),$$

and thus the equation treated before for the ellipse

$$yy = \frac{bb}{aa}(aa - xx),$$

is changed into the equation for the hyperbola by putting $-bb$ in place of bb. Therefore on this account, the similar properties of the ellipse found before are transferred readily to the hyperbola. And indeed in the first place, since for the ellipse the distance of the foci from the centre was $= \sqrt{(aa - bb)}$, for the hyperbola there will be $CF = CG = \sqrt{(aa + bb)}$. Hence there will be

$$FP = x - \sqrt{(aa + bb)} \text{ and } GP = x + \sqrt{(aa + bb)}\,;$$

so that, on account of $yy = -bb + \frac{bbxx}{aa}$, there becomes

$$FM = \sqrt{\left(aa + xx + \frac{bbxx}{aa} - 2x\sqrt{(aa + bb)}\right)} = \frac{x\sqrt{(aa + bb)}}{a} - a$$

and

$$GM = \sqrt{\left(aa + xx + \frac{bbxx}{aa} + 2x\sqrt{(aa + bb)}\right)} = \frac{x\sqrt{(aa + bb)}}{a} + a.$$

Therefore with the right lines *FM, GM* drawn from each focus to the point *M* of the curve there will be

$$FM + AC = \frac{CP \cdot CF}{CA} \text{ et } GM - AC = \frac{CP \cdot CF}{CA},$$

therefore the difference of these right lines $GM - GM$ is equal to $2AC$. Therefore just as for the ellipse, the sum of these two lines is equal to the principal axis *AB*, thus for the hyperbola the difference is equal to the principal axis *AB*.

155. Hence also the position of the tangent *MT* can be defined, for there is always $CP : CA = CA : CT$ for lines of the second order, from which there becomes

$$CT = \frac{aa}{x} \text{ and } PT = \frac{xx - aa}{x} = \frac{aayy}{bbx};$$

and hence

$$MT = \frac{y}{bbx}\sqrt{(b^4x^2 + a^4y^2)} = \frac{y}{bx}\sqrt{(aaxx + bbxx - a^4)}$$

But there is

$$FM \cdot GM = \frac{aaxx + bbxx - a^4}{aa},$$

therefore $MT = \frac{ay}{bx}\sqrt{FM \cdot GM}$. Then there is

$$FT = \sqrt{(aa + bb)} - \frac{aa}{x} \text{ and } GT = \sqrt{(aa + bb)} + \frac{aa}{x},$$

therefore

$$FT : FM = a : x \text{ and } GT : GM = a : x,$$

from which it follows that $FT : GT = FM : GM$, which proportion shows the angle *FMG* to be bisected by the tangent *MT* and there shall be $FMT = GMT$. But the right line *CM* produced will be an oblique angled diameter bisecting all the ordinates parallel to the tangent *MT*.

156. The perpendicular *CQ* may be sent from the centre *C* to the tangent, and there will be

$$TM : PT : PM = CT : TQ : CQ$$

or

$$\frac{ay}{bx}\sqrt{FM \cdot GM} : \frac{aayy}{bbx} : y = \frac{aa}{x} : TQ : CQ;$$

from which there becomes:

$$TQ = \frac{a^3y}{bx\sqrt{FM \cdot GM}} \text{ and } CQ = \frac{ab}{\sqrt{FM \cdot GM}}.$$

In a like manner the perpendicular *FS* may be sent from the focus *F* to the tangent, and there will be

$$TM : PT : PM = FT : TS : FS,$$

or

$$\frac{ay}{bx}\sqrt{FM\cdot GM}:\frac{aayy}{bbx}:y=\frac{a\cdot FM}{x}:TS:FS\,;$$

from which there becomes

$$TS=\frac{aay\cdot FM}{bx\sqrt{FM\cdot GM}}\ \text{ and }\ FS=\frac{b\cdot FM}{\sqrt{FM\cdot GM}}\,;$$

and equally, if the perpendicular Gs may be drawn from the other focus G to the tangent, there will be

$$Ts=\frac{aay\cdot FM}{bx\sqrt{FM\cdot GM}}\ \text{ and }\ Gs=\frac{b\cdot FM}{\sqrt{FM\cdot GM}}.$$

Hence therefore there will be had

$$TS\cdot Ts=\frac{a^4yy}{bbxx}=\frac{aa(xx-aa)}{xx}=CT\cdot PT\ \text{ and } TS{:}CT=PT:Ts.$$

Then there becomes $FS\cdot Gs=bb$. Because again there is $QS=Qs$, there will be

$$QS=\frac{TS+Ts}{2}=\frac{aay(FM+GM)}{2bx\sqrt{FM\cdot GM}}=\frac{ay\sqrt{(aa+bb)}}{b\sqrt{FM\cdot GM}}=Qs\,,$$

from which it follows :

$$CS^2=CQ^2+QS^2=\frac{aab^4+a^4yy+aabbyy}{bb\cdot FM\cdot GM}=\frac{aab^4+(aa+bb)\ (bbxx-aabb)}{bb\cdot FM\cdot GM}=\frac{(aa+bb)xx-a^4}{FM\cdot GM}=aa.$$

Therefore there will be, as in the ellipse, the right lines $CS=a=CA$. Then there is

$$CQ+FS=\frac{bx\sqrt{(aa+bb)}}{a\sqrt{FM\cdot GM}}$$

and thus

$$(CQ+FS)^2-CQ^2=\frac{bbxx(aa+bb)-a^4bb}{aa\cdot FM\cdot GM}=bb.$$

Whereby, if FX may be drawn from the focus F parallel to the tangent cutting the perpendicular CQ produced at X, there will be $CX = \sqrt{(bb + CQ^2)}$, for which a similar property has been found for the ellipse.

157. If perpendiculars may be erected to the axes from the vertices A and B, then they may cross the tangents in V and v, on account of

$$AT = \frac{a(x-a)}{x} \text{ and } BT = \frac{a(x+a)}{x},$$

$$PT : PM = AT : AV = BT : Bv,$$

hence there becomes

$$AV = \frac{bb(x-a)}{ay} \text{ and } Bv = \frac{bb(x+a)}{ay};$$

therefore

$$AV \cdot Bv = \frac{b^4(xx-aa)}{aayy} = bb$$

or

$$AV \cdot Bv = FS \cdot Gs$$

Then $PT : TM = AT : TV = BT : Tv$; therefore

$$TV = \frac{b(x-a)}{xy}\sqrt{FM \cdot GM} \text{ and } Tv = \frac{b(x+a)}{xy}\sqrt{FM \cdot GM};$$

from which there becomes

$$TV \cdot Tv = \frac{aa}{xx} FM \cdot GM = FT \cdot GT.$$

Moreover hence in a similar manner many other conclusions can be deduced.

158. Because $CT = \frac{aa}{x}$, it is apparent, by how much greater the abscissa $CP = x$ may be taken, so much less than that the interval CT will become ; and thus the tangent, which will touch the curve produced to infinity, will pass through the centre C itself and there becomes $CT = 0$.
But since there shall be

$$\text{tang. } PPM = \frac{PM}{PT} = \frac{bbx}{aay},$$

with the point *M* going off to infinity, or putting $x = \infty$, there becomes

$$y = \frac{b}{a}\sqrt{(xx - aa)} = \frac{bx}{a}.$$

Therefore the tangent of the curve produced to infinity both will pass through the centre *C* and constitute an angle to the axis *ACD,* the tangent of which $= \frac{b}{a}$. Therefore on putting $AD = b$ at the vertex *A* normal to the axis, then indeed the right line *CD* produced at no time touches each curve, but the curve will approach continually more to that, then at infinity it will be combined altogether with the right line *CI*. This will prevail the same with the part *Ck*, which will merge together at last with the branch *Bk*. And if the right line *KCi* may be drawn to the other part at the same angle , since that will come together with the branches *BK* and *Bi* produced to infinity. But right lines of this kind, to which a certain curved line approaches closer continually, running off to touch finally at infinity, are called *asymptotes*, so that the right lines *ICk, KCi* are the two asymptotes of the hyperbola.

159. Therefore the asymptotes mutually cross each other at the centre *C* of the hyperbola and are inclined to the axis at an angle $ACD = ACd$, the tangent of which $= \frac{b}{a}$, and the tangent of double the angle $DCd = \frac{2ab}{aa - bb}$, from which it is apparent, if there were $b = a$, the angle DCd within which the asymptotes intersect each other, becomes equal to a right angle ; in which case the hyperbola is said to be *equilateral*. But since there shall be $AC = a$, $AD = b$, there will be $CD = Cd = \sqrt{(aa + bb)}$; whereby, if a perpendicular *CR* may be sent from the focus *G* to whatever asymptote, on account of $CG = \sqrt{(aa + bb)} = CD$, there will be $CH = AC = BC = a$ and $GH = b$.

160. The ordinate $MPN = 2y$ may be produced on both sides, then it will cut the asymptotes at *m* and *n;* there will be

$$Pm = Pn = \frac{bx}{a} \text{ and } Cm = Cn = \frac{x\sqrt{(aa + bb)}}{a} = FM + AC = GM - AC.$$

Then indeed there will be

$$Mm = Nn = \frac{bx - ay}{a} \quad \text{and} \quad Nm = Mn = \frac{bx + ay}{a},$$

from which there becomes

$$Mm \cdot Nm = Mm \cdot Mn = \frac{bbxx - aayy}{aa} = bb\,,$$

on account of $aayy = bbxx - aabb$; therefore there will be everywhere

$$Mm \cdot Nm = Mm \cdot Mn = Nn \cdot Nm = Nn \cdot Mn = bb = AD^2\,.$$

Mr may be drawn from *M* parallel to the asymptote *Cd;* there will be

$$2b\sqrt{(aa+bb)} = Mm : mr(Mr)\,,$$

from which there becomes

$$mr = Mr = \frac{(bx-ay)\sqrt{(aa+bb)}}{2ab}$$

and

$$Cm - mr = Cr = \frac{(bx+ay)\sqrt{(aa+bb)}}{2ab}.$$

Hence therefore there may be put in place

$$Mr \cdot Cr = \frac{(bbxx-aayy)\ (aa+bb)}{4aabb} = \frac{aa+bb}{4}.$$

Or draw *AE* from *A* parallel to the asymptote *Cd* , there will be $AE = CE = \frac{1}{2}\sqrt{(aa+bb)}$ and thus there will be $Mr \cdot Cr = AE \cdot CE$; which is the primary property of the hyperbola related to the asymptotes.

161. But if therefore (see Fig. 34) the abscissas $CP = x$ may be taken on one asymptote from the centre and the applied lines $PM = y$ may be put in place parallel to the other asymptote, there will be

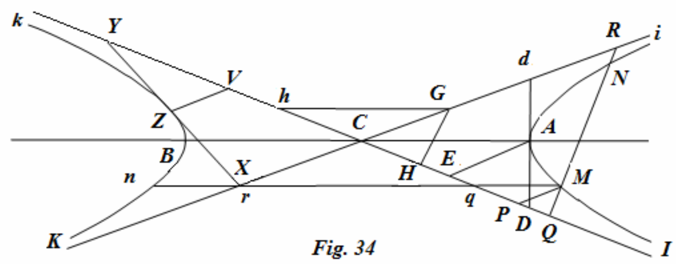

Fig. 34

$yx = \frac{aa+bb}{4}$, with $AC = BC = a$ present and $AD = Ad = b$; or, if there may be put $AE = CE = h$, there will be $yx = hh$ and $y = \frac{hh}{x}$. Therefore on putting $x = 0$ there becomes $y = \infty$, and in turn by making $x = \infty$ there becomes $y = 0$. Now some right line *QMNR* may be drawn through a point *M* on the curve, which shall be parallel as it pleases to the right line *GH*, and there may be put $CQ = t$, $QM = u$, there will be

$$GH : CH : CG = u : PQ : PM,$$

therefore

$$PQ = \frac{CH}{GH}u, \quad PM = \frac{CG}{GH}u;$$

from which

$$y = \frac{CH}{GH}u \text{ and } x = t - \frac{CG}{GH}u;$$

with which values substituted, there will be

$$\frac{CG}{CH}tu - \frac{CG \cdot GH}{GH^2} \cdot uu = hh,$$

or

$$uu - \frac{GH}{CH}tu + \frac{GH^2}{CH \cdot CG}hh = 0.$$

Therefore the applied line *u* will have a twofold value, surely *QM* and *QN*, the sum of which will be $= \frac{GH}{CH}t = QR$ and the rectangle $QM \cdot QN = \frac{GH^2}{CH.CG}hh$.

162. Therefore since there shall be $QM + QN = QR$, there will be $QM = RN$ and $QN = RM$. Whereby, if the points *M* and *N* may come together, in which case the right line *QR* touches the curve, then that will be bisected in the point of contact itself. Evidently, if the right line *XY* may touch the hyperbola, the point of contact *Z* will be placed in the middle of the line *XY*. From which, if *ZV* may be drawn parallel to the other asymptote from *Z*, there will be $CV = VY$, and hence the tangent may be drawn readily to any point *Z* of the hyperbola. Evidently there may be taken $VY = CV$, and the right line drawn through *Y* and a point *Z* of the curve touches the hyperbola at this point *Z*. Therefore since there shall be $CV \cdot ZV = hh = \frac{aa+bb}{4}$, there will be

$$CX \cdot CY = aa + bb = CD^2 = CD \cdot Cd ;$$

on account of which, if the right lines *DX* and *dY* may be drawn, these will be parallel to each other ; from which a method arises of drawing any number of tangents to the curve.

163. Then because the rectangle $QM \cdot QN = \frac{GH^2}{CH \cdot CG} \cdot hh$, it is apparent, wherever the right line *QR* may be drawn parallel to *HG*, $QM \cdot QN$ always shall be a rectangle of the same magnitude. Therefore there will be always

$$QM \cdot QN = QM \cdot MR = QN \cdot NR = \frac{CH^2}{CH.CG} hh .$$

But if therefore the tangent be drawn parallel to *QR* itself, because that will be bisected at the point of contact between the asymptotes, and if the half tangent may be called $= q$, there will be always

$$QM \cdot QN = QM \cdot MR = RN \cdot RM = RN \cdot NQ = qq ,$$

which is the distinguishing property of hyperbolas described between asymptotes.

164. Because the hyperbola may be made from two diametrically opposite parts *IAi* and *KBk*, these properties not only relate to right lines drawn within the asymptotes, which intersect the same part of the curve in two points, but also to these, which relate to opposite parts. Certainly the right line *Mqrn* may be drawn through the point *M* to the opposite part, to which the parallel right line *Gh* may be drawn, and there may be called $Cq = t$ and $qM = u$; there will be, on account of the similar triangles *CGh* and *PMq*,

$$PM = y = \frac{CG}{Gh} u \text{ and } qP = x - t = \frac{Ch}{Gh} u ;$$

from which there becomes $x = t + \frac{Ch}{Gh} u$. But since there shall be $xy = hh$, there arises

$$\frac{CG}{Gh}tu+\frac{CG\cdot Ch}{Gh^2}uu=hh$$

or

$$uu+\frac{Gh}{Ch}tu-\frac{Gh^2}{CG\cdot Ch}hh=0.$$

165. Therefore the applied line *u* will have a twofold value, certainly *qM* and − *qn,* with this *qn* being negative, because it is turned towards the other part of the asymptote *CP,* for the axis assumed. Therefore the sum of the two roots of these

$$qM-qn \text{ will be } =-\frac{Gh}{Ch}t=-qr,$$

and thus $qn-qM=qr$, from which there becomes $qM=rn$ and $qn=rM$. But then from the equation found the product of the roots is understood to be

$$-qM\cdot qn=-\frac{Gh^2}{CG.Ch}hh$$

or

$$qM\cdot qn=qM\cdot rM=rn\cdot qn=rn\cdot rM=\frac{Gh^2}{CG.Ch}hh.$$

Therefore these rectangles, however many right lines may be drawn *Mn* parallel to *Gh* itself, will always be of the same magnitude. Moreover these are the outstanding properties of the individual species of lines of the second order, which since if they may be combined with the general properties, will prepare an almost infinite multitude of significant properties.

CAPUT VI

DE LINEARUM SECUNDI ORDINIS SUBDIVISIONE IN GENERA

131. Proprietates, quas in capite praecedente elicuimus, in omnes lineas, quae ad ordinem secundum pertinent, aeque competunt; neque enim ullius varietatis, qua istae lineae aliae ab aliis distinguuntur, fecimus mentionem. Quanquam autem omnes lineae secundi ordinis his expositis proprietatibus communiter gaudent, tamen eae inter se ratione figurae plurimum differunt; quamobrem lineas in hoc ordine contentas distribui convenit in genera, quo facilius diversae figurae, quae in hoc ordine occurrunt, distingui atque proprietates, quae tantum in singula genera competunt, evolvi queant.

132. Aequationem autem generalem pro lineis secundi ordinis, mutando tantum axem et abscissarum initium, eo reduximus, ut omnes lineae secundi ordinis contineantur in hac aequatione

$$yy = \alpha + \beta x + \gamma xx ,$$

in qua x et y denotant coordinatas orthogonales. Cum igitur pro qualibet abscissa x applicata y duplicem induat valorem, alterum affirmativum alterum negativum, iste axis, in quo abscissae x capiuntur, curvam secabit in duas partes similes et aequales; eritque adeo iste axis diameter curvae orthogonalis atque omnis linea secundi ordinis habebit diametrum orthogonalem, super qua, tanquam axe, abscissas hic assumo.

133. Tres igitur ingrediuntur in hanc aequationem quantitates constantes α, β, et γ, quae cum infinitis modis inter se variari possint, innumerabiles varietates in lineis curvis orientur, quae autem vel magis vel minus a se invicem ratione figurae discrepabunt. Primum enim eadem figura infinities ex proposita aequatione $yy = \alpha + \beta x + \gamma xx$ resultat, variato nempe abscissarum initio in axe, quod fit, dum abscissa x data quantitate vel augetur vel minuitur. Deinde eadem quoque figura sub diversa magnitudine in aequatione continetur, ita ut infinitae lineae curvae prodeant, quae tantum ratione quantitatis a se invicem differant, uti circuli diversis radiis descripti Ex quibus manifestum est non omnem litterarum α, β, et γ variationem diversas linearum secundi ordinis species vel genera producere.

134. Maximum autem discrimen in lineis curvis, quae in aequatione

$$yy = \alpha + \beta x + \gamma xx$$

continentur, suggerit natura coefficientis γ, prout is vel affirmativum habuerit valorem vel negativum. Si enim γ habeat valorem affirmativum, posita abscissa x infinita, quo casu terminus γxx infinities maior evadet quam reliqui $\alpha + \beta x$ ac propterea expressio $yy = \alpha + \beta x + \gamma xx$ affirmativum obtinet valorem, applicata y pariter duplicem habebit valorem infinite magnum, alterum affirmativum alterum negativum, quod idem evenit, si ponatur $x = -\infty$, quo casu nihilominus expressio $yy = \alpha + \beta x + \gamma xx$ induet valorem infinite magnum affirmativum. Hanc ob rem, existente γ quantitate affirmativa, curva quatuor habebit ramos in infinitum excurrentes, binos abscissae $x = +\infty$ et binos abscissae $x = -\infty$ respondentes. Hae igitur curvae quatuor ramis in infinitum excurrentibus praeditae unum linearum secundi ordinis genus constituere censentur atque nomine *hyperbolarum* appellantur.

135. Sin autem coefficiens γ negativum habuerit valorem, tum, posito sive $x = +\infty$ sive $x = -\infty$, expressio $yy = \alpha + \beta x + \gamma xx$ negativum valorem tenebit ideoque applicata y imaginaria fiet. Neque igitur usquam in his curvis abscissa neque applicata poterit esse infinita ideoque nulla dabitur curvae portio in infinitum excurrens, sed tota curva in spatio finito ac determinato continebitur. Haec igitur linearum secundi ordinis species nomen *ellipsium* obtinuit, quarum propterea natura continetur in hac aequatione $yy = \alpha + \beta x + \gamma xx$, si γ fuerit quantitas negativa.

136. Cum igitur valor ipsius γ, prout is fuerit vel affirmativus vel negativus, tam diversam linearum secundi ordinis indolem producat, ut hinc merito duo diversa genera constituantur: si ponatur $\gamma = 0$, qui valor inter affirmativos et negativos medium tenet locum, curva quoque hinc resultans mediam quandam speciem inter hyperbolas atque ellipses constituet, quae *parabola* vocatur, cuius ergo natura hac exprimetur aequatione $yy = \alpha + \beta x.$ Hic perinde est, sive β fuerit quantitas affirmativa sive negativa, quoniam indoles curvae non mutatur sumta abscissa x negativa. Sit igitur β quantitas affirmativa, atque manifestum est, crescente abscissa x in infinitum applicatam y quoque infinitam fore tam affirmativam quam negativam, ex quo parabola duos habebit ramos in infinitum excurrentes, plures autem duobus habere non poterit, quia posito $x = -\infty$ applicatae y valor fit imaginarius.

137. Habemus ergo tres linearum secundi ordinis species, ellipsin, parabolam et hyperbolam, quae a se invicem tantopere discrepant, ut eas inter se confundere omnino non liceat. Discrimen enim essentiale in numero ramorum in infinitum excurrentium consistit; ellipsis enim nullam portionem habet in infinitum abeuntem, sed tota in spatio finito includitur, parabola vero duos habet ramos in infinitum excurrentes et hyperbola quatuor. Quare, cum in capite praecedente proprietates sectionum conicarum in genere simus contemplati, nunc, quibus proprietatibus quaeque species sit praedita, videamus.

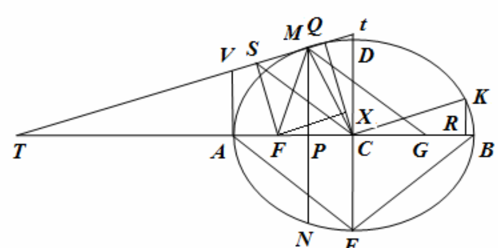

Fig. 31

138. Incipiamus ab ellipsi (Fig. 31), cuius aequatio est haec

$$yy = \alpha + \beta x + \gamma xx$$

sumtis abscissis in diametro orthogonali. Quoniam vero initium abscissarum ab arbitrio nostro pendet, si id removeamus intervallo $\frac{\beta}{2\gamma}$ orietur aequatio huius formae

$yy = \alpha - \gamma xx$,

in qua abscissae a centro figurae capiuntur. Sit igitur *C* centrum et *AB* diameter orthogonalis, atque erit abscissa $CP = x$ et applicata $PM = y$. Fiet ergo $y = 0$ sumta $x = \pm\sqrt{\frac{\alpha}{\gamma}}$, et, si *x* limites hos $+\sqrt{\frac{\alpha}{\gamma}}, -\sqrt{\frac{\alpha}{\gamma}}$ transgrediatur, applicata *y* fiet imaginaria; quod indicio est totam curvam intra istos limites contineri. Erit ergo $CA = CB = \sqrt{\frac{\alpha}{\gamma}}$; tum facto $x = 0$ fiet $CD = CE = \sqrt{\alpha}$. Ponatur ergo semidiameter seu semiaxis principalis $CA = CB = a$ et semiaxis coniugatus $CD = CE = b$, erit $\alpha = bb$ et $\gamma = \frac{bb}{aa}$. Unde pro ellipsi ista orietur aequatio

$$yy = bb - \frac{bbxx}{aa} = \frac{bb}{aa}(aa - xx).$$

139. Quando isti semiaxes coniugati *a* et *b* fiunt inter se aequales, tum ellipsis abibit in circulum ob $yy = aa - xx$ seu $yy + xx = aa$; erit enim $CM = \sqrt{(xx + yy)} = a$ ideoque omnia curvae puncta *M* aequaliter a centro *C* erunt remota, quae est proprietas circuli. Sin autem semiaxes *a* et *b* inter se fuerint inaequales, tum curva erit oblonga, nempe erit vel *AB* maior quam *DE* vel *DE* maior quam *AB.* Quia vero axes coniugati *AB* et *DE* inter se commutari possunt atque perinde est, in utro abscissas capiamus, ponamus *AB* esse axem maiorem, seu *a* maiorem quam *b;* atque in hoc axe existent foci ellipsis *F* et *G* sumendo $CF = CG = \sqrt{(aa - bb)}$, semiparameter vero seu semilatus rectum ellipsis erit $= \frac{bb}{a}$, quae exprimit magnitudinem applicatae in alterutro foco *F* vel *G* erectae.

140. Ad curvae punctum *M* ducantur ex utroque foco rectae *FM* et G*M,* eritque, uti supra vidimus,

$$FM = AC - \frac{CF \cdot CP}{AC} = a - \frac{x\sqrt{(aa - bb)}}{a}$$

et

$$GM = a + \frac{x\sqrt{(aa-bb)}}{a},$$

unde fit

$$FM + GM = 2a.$$

Quare, si ad quodvis curvae punctum *M* ex ambobus focis ducantur rectae *FM* et *GM,* earum summa semper aequabitur axi maiori $AB = 2a$*;* ex quo cum insignis focorum proprietas perspicitur, tum modus facilis ellipsin mechanice describendi colligitur.

141. In puncto *M* ducatur tangens *TMt,* quae axibus occurrat in punctis *T* et *t,* eritque, ut supra demonstravimus, $CP : CA = CA : GT$; unde $CT = \frac{aa}{x}$ similique modo, permutatis coordinatis, $Ct = \frac{bb}{y}$. Erit ergo

$$TP = \frac{aa}{x} - x, \quad TF = \frac{aa}{x} - \sqrt{(aa-bb)}, \text{ and } TA = \frac{aa}{x} - a.$$

Fiet itaque

$$TP = \frac{aa-xx}{x} = \frac{aayy}{bbx} \text{ et } TM = \frac{y\sqrt{(b^4xx + a^4yy)}}{bbx},$$

hincque

$$\text{tang.}\, CTM = \frac{bbx}{aay}, \quad \sin.\, CTM = \frac{bbx}{\sqrt{(b^4xx + a^4yy)}}$$

et

$$\cos.\, CTM = \frac{aay}{\sqrt{(b^4xx + a^4yy)}}.$$

Quare, si ad axem in *A* normalis erigatur *AV,* quae curvam simul tanget, erit

$$AV = \frac{a(a-x)}{x} \cdot \frac{bbx}{aay} = \frac{bb(a-x)}{ay} = b\sqrt{\frac{a-x}{a+x}}$$

ob $ay = b\sqrt{(aa-xx)}$.

142. Cum sit

$$FT = \frac{aa - x\sqrt{(aa-bb)}}{x} \text{ et } FM = \frac{aa - x\sqrt{(aa-bb)}}{a},$$

erit $FT : FM = a : x$. Simili vero modo ob

$$GT = \frac{aa + x\sqrt{(aa-bb)}}{x} \text{ et } GM = \frac{aa + x\sqrt{(aa-bb)}}{a}$$

erit $GT : GM = a : x$; unde erit $FT : FM = GT : GM$. At est

$$FT : FM = \sin.FMT : \sin.CTM \quad \text{et} \quad GT : GM = \sin.GMt : \sin.CTM,$$

quamobrem erit $\sin.FMT = \sin.GMt$ ideoque

$$\text{angulus } FMT = \text{angulo } GMt.$$

Ambae ergo rectae ex focis ad punctum curvae quodvis *M* ductae aequaliter inclinantur ad tangentem curvae in illo puncto *M*, quae est maxime principalis focorum proprietas.

143. Cum sit $GT : GM = a : x$, ob $CT = \frac{aa}{x}$ erit quoque $CT : CA = a : x$;
unde $GT : GM = CT : CA$, quare, si ex centro *C* rectae *GM* parallela ducatur *CS*, tangenti in *S* occurrens, erit $CS = CA = a$; eodem autem modo, si ex *C* rectae *FM* parallela ducatur ad tangentem, erit ea pariter $= CA = a$.
Cum autem sit

$$TM = \frac{y}{bbx}\sqrt{(b^4xx + a^4yy)},$$

erit, ob $aayy = aabb - bbxx$,

$$TM = \frac{y}{bx}\sqrt{\left(a^4 - xx\left(aa-bb\right)\right)};$$

at est

$$FT \cdot GT = \frac{a^4 - xx(aa-bb)}{xx},$$

unde

$$TM = \frac{y}{b}\sqrt{FT \cdot GT}.$$

Quare, ob $TG : TC = TM : TS$ erit

$$TS = \frac{TM \cdot CT}{TG}$$

ideoque

$$TS = \frac{y \cdot CT}{b}\sqrt{\frac{FT}{GT}} = \frac{y \cdot CT \cdot FT}{b\sqrt{F \cdot GTT}} = \frac{yy \cdot CT \cdot FT}{bb \cdot TM}.$$

Deinde est

$$PT = \frac{aayy}{bbx} = \frac{CT \cdot yy}{bb}$$

ergo

$$TS = \frac{PT \cdot FT}{TM}$$

ideoque

$$TM : PT = FT : TS;$$

unde intelligitur triangula *TMP* et *TFS* esse similia ideoque rectam *FS* ad tangentem ex foco *F* esse normalem. Erit vero $SV = \frac{AF \cdot MV}{GM}$, quod ex his expressionibus eruere licet.

144. Quodsi ergo ex alterutro foco *F* in tangentem ducatur perpendiculum *FS* et ad punctum *S* ex centro *C* recta *CS* iungatur, erit haec *CS* perpetuo semiaxi maiori $AC = a$ aequalis. Erit vero, ob $TM : y = TF : FS$,

$$FS = \frac{y \cdot TF}{TM} = \frac{b \cdot TF}{\sqrt{FT \cdot GT}} = b\sqrt{\frac{FT}{GT}},$$

ergo

$$GT : FT = GM : FM = CD^2 : FS^2;$$

perpendiculum vero ex altero foco in tangentem demissum erit $= b\sqrt{\frac{GT}{FT}}$ quare inter haec perpendicula erit semiaxis minor $CD = b$ media proportionalis. Demittatur nunc quoque ex centro *C* in tangentem perpendiculum *CQ*, erit $TF : FS = CT : CQ$, ergo

$$CQ = \frac{b \cdot CT}{\sqrt{FT \cdot GT}} = \frac{bx \cdot CT}{a\sqrt{FM \cdot GM}} = \frac{ab}{\sqrt{FM \cdot GM}},$$

unde

$$CQ - FS = \frac{b \cdot CF}{\sqrt{FT \cdot GT}} = CX,$$

ducta *FX* tangenti parallela. Hinc erit

$$CQ - CX = \frac{b \cdot TF}{\sqrt{FT \cdot GT}} \quad \text{et} \quad CQ + CX = \frac{b \cdot TG}{\sqrt{FT \cdot GT}},$$

unde

$$CQ^2 - CX^2 = bb \quad \text{et} \quad CX = \sqrt{\left(CQ^2 - bb\right)};$$

ex dato ergo axe minori in perpendiculo *CQ* reperitur punctum *X*, unde normalis educta per focum *F* transibit.

145. His focorum proprietatibus expositis consideremus duas quasvis diametros coniugatas. Erit autem *CM* semidiameter, cuius coniugata reperietur, si tangenti *TM* ex centro parallela ducatur *CK*. Ponatur $CM = p$, $CK = q$ et angulus $MCK = CMT = s$, erit primo $pp + qq = aa + bb$ et secundo $pq \cdot \sin. s = ab$, uti supra vidimus. At vero erit

$$pp = xx + yy = bb + \frac{(aa - bb)xx}{aa}$$

et

$$qq = aa + bb - pp = aa - \frac{\left(aa - bb\right)xx}{aa} = FM \cdot GM,$$

eodemque modo $pp = FK \cdot GK$. Deinde, cum sit $CQ = \frac{ab}{\sqrt{FM \cdot GM}}$, erit

$$\sin.CMQ = \sin.s = \frac{ab}{p\sqrt{FM \cdot GM}}.$$

Denique erit

$$TM : TP = \frac{y}{b}\sqrt{FT \cdot GT} : \frac{aayy}{bbx} = \sqrt{FM \cdot GM} : \frac{ay}{b} = CK : CR,$$

unde

$$CR = \frac{ay}{b} \quad \text{et} \quad KR = \frac{bx}{a}$$

ideoque

$$CR.KR = CP \cdot PM.$$

Denique erit

$$\sin.FMS = \frac{b}{\sqrt{GM \cdot FM}} = \frac{b}{q};$$

quia porro est

$$x = CP = \frac{a\sqrt{(pp-bb)}}{\sqrt{(aa-bb)}} \quad \text{et} \quad y = \frac{b\sqrt{(pp-pp)}}{\sqrt{(aa-bb)}} = PM$$

atque

$$CR = \frac{a\sqrt{(aa-pp)}}{\sqrt{(aa-bb)}} \quad \text{et} \quad KR = \frac{b\sqrt{(pp-pp)}}{\sqrt{(aa-bb)}},$$

erit

$$\text{tang.}ACM = \frac{y}{x} \ \text{ et } \ \text{tang.}2ACM = \frac{2yx}{xx-yy} = \frac{2ab\sqrt{(aa-pp)(pp-bb)}}{(aa+bb)pp-2aabb}.$$

At est

$$ab = pq\cdot\sin.s\ , \quad aa+bb = pp+qq$$

et

$$\sqrt{(aa-pp)(pp-bb)} = -pq\cdot\cos.s\,,$$

unde fit

$$\tan.2ACM = \frac{-qq\cdot\sin.2s}{pp+qq\cdot\cos.2s},$$

quia cos.*s* est negativus. Tandem est $CK^2 = MT\cdot Mt$; ex superioribus vero eruitur

$$MV = q\sqrt{\frac{AP}{BP}} \ \text{ et } \ AV = b\sqrt{\frac{AP}{BP}} \ ;$$

unde erit $AV : MV = b : q = CE : CK$. Ergo rectae, si ducantur, *AM* et *EK* inter se erunt parallelae.

146. Quia est $pq\cdot\sin.s = ab$, erit *pq* maior quam *ab;* et, cum sit $pp+qq = aa+bb$, quantitates *p* et *q* magis ad rationem aequalitatis accedunt, quam *a* et *b,* unde inter omnes diametros coniugatas illae, quae sunt orthogonales, maxime a se invicem discrepant. Dabuntur ergo duae diametri conjugatae inter se aequales, ad quas inveniendas sit $q = p$, eritque

$$2pp = aa+bb \ \text{ et } p = q = \sqrt{\frac{aa+bb}{2}}$$

et

$$\sin.s = \frac{2ab}{aa+bb} \ \text{ atque } \cos.s = \frac{-aa+bb}{aa+bb} \ ;$$

unde fit

$$\sin.\tfrac{1}{2}s = \sqrt{\frac{aa}{aa+bb}} \text{ atque } \cos.\tfrac{1}{2}s = \sqrt{\frac{bb}{aa+bb}} \;;$$

ergo

$$\text{tang}.\tfrac{1}{2}s = \frac{a}{b} = \text{tang}.CEB \text{ et } MCK = 2CEB = AEB.$$

Porro

$$CP = \frac{a}{\sqrt{2}}, \quad CM = \frac{b}{\sqrt{2}},$$

quare semidiametri coniugatae inter se aequales *CM, CK* erunt parallelae cordis *AE* et *BE.*

147. Si abscissae a vertice *A* computentur ponaturque $AP = x$, $PM = y$, cum nunc sit $a - x$, quod ante erat *x,* habebitur ista aequatio

$$yy = \frac{bb}{aa}(2ax - xx) = \frac{2bb}{a}x - \frac{bb}{aa}xx,$$

ubi patet esse *2bb* parametrum seu latus rectum ellipsis. Ponatur semilatus rectum seu applicata in foco $= c$ et distantia foci a vertice $AF = d$, erit

$$\frac{bb}{a} = c \text{ et } a - \sqrt{(aa-bb)} = d = a - \sqrt{(aa-ac)},$$

unde fit

$$2ad - dd = ac \text{ et } a = \frac{dd}{2d-c}.$$

Hinc erit

$$yy = 2cx - \frac{c(2d-c)xx}{dd},$$

quae est aequatio pro ellipsi inter coordinatas orthogonales *x* et *y*, abscissis *x* in axe principali *AB* a vertice *A* computatis, quae obtinetur ex datis distantia foci a vertice $AF = d$ et semilatere recto $= c$ *;* ubi notandum est semper esse debere *2d* maiorem quam *c*, quia est

$$AC = a = \frac{dd}{2d-c} \text{ et } CD = b = d\sqrt{\frac{c}{2d-c}}.$$

148. Quodsi ergo fuerit $2d = c$, erit $yy = 2cx$, quam aequationem supra vidimus esse pro parabola (Fig. 32): aequatio enim superior $yy = \alpha + \beta x$ ad hanc formam reducitur, initio abscissarum intervallo $= \frac{\alpha}{\beta}$ mutato. Sit igitur *MAN* parabola, cuius

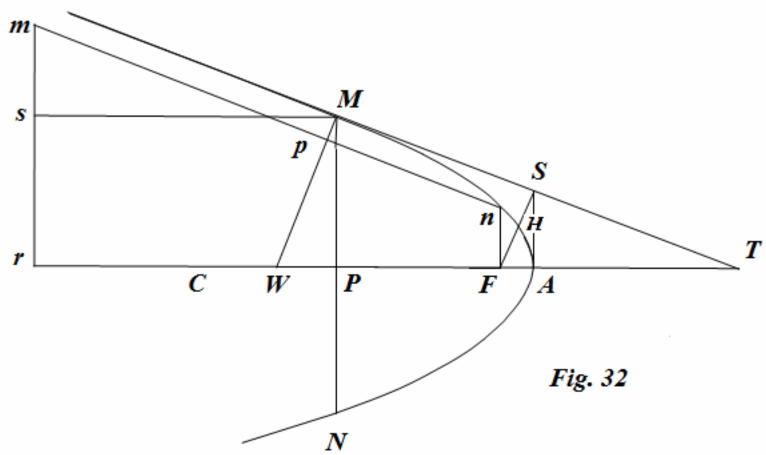

Fig. 32

natura inter abscissam $AP = x$ et applicatam $PM = y$ hac aequatione exprimatur $yy = 2cx$. Erit ergo distantia foci a vertice $AF = d = \frac{1}{2}c$ et semiparameter $FH = c$, atque ubique $PM^2 = 2FH \cdot AP$; unde, posita abscissa *AP* infinita, simul applicatae *PM* et *PN* in infinitum excrescunt ideoque curva ad utramque axis *AP* partem in infinitum extenditur. Posita autem abscissa *x* negativa applicata fit imaginaria, hincque axi ultra *A* versus *T* nulla curvae portio respondet.

149. Cum aequatio pro ellipsi abeat in parabolam facto $2d = c$, manifestum est parabolam nil aliud esse praeter ellipsin, cuius semiaxis $a = \frac{dd}{2d - c}$ fit infinitus; quamobrem proprietates omnes, quas pro ellipsi invenimus, ad parabolam transferentur, posito axe *a* infinito. Primum autem, cum sit $AF = \frac{1}{2}c$, erit $FP = x - \frac{1}{2}c$, ideo que hinc ducta ex foco *F* ad curvae punctum *M* recta *FM* erit

$$FM^2 = xx - cx + \frac{1}{4}cc + yy = xx + cx + \frac{1}{4}cc$$

$$FM = x + \frac{1}{2}c = AP + AF ,$$

quae est praecipua proprietas foci in parabola.

150. Quoniam parabola nascitur ex ellipsi axe maiore in infinitum aucto, consideremus parabolam, tanquam esset ellipsis, sitque eius semiaxis $AO = a$, existente *a* quantitate infinita, ita ut centrum *C* infinite distet a vertice *A*. Ad *M* ducatur tangens curvae *MT* axi occurrens in *T*; quia erat

$$CP : CA = CA : CT, \quad \text{erit } CT = \frac{aa}{a-x},$$

ob $CP = a - x$; hincque $AT = \frac{ax}{a-x}$. At, cum sit *a* quantitas infinita, abscissa *x* prae ea evanescet eritque $a - x = a$, ideoque $AT = x = AP$; quod idem hoc modo ostendi potest: cum sit $AT = \frac{ax}{a-x}$, erit $AT = x + \frac{xx}{a-x}$, at quia fractionis $\frac{ax}{a-x}$ denominator est infinitus, numeratore existente finito, valor fractionis erit evanescens ideoque $AT = AP = x$.

151. Quodsi ergo ex puncto *M* ad centrum parabolae *C* infinite distans ducatur linea *MC*, quae erit axi *AC* parallela, ea quoque erit diameter curvae omnes chordas tangenti *MT* parallelas bisecans. Scilicet, si ducatur chorda seu ordinata *mn* tangenti *MT* parallela, ea a diametro *Mp* bisecabitur in *p*. Omnis ergo recta axi *AP* parallela ducta in parabola erit diameter obliquangula. Ad huiusmodi diametrorum naturam eruendam sit $Mp = t$, $pm = u$, ducatur ex *m* ad axem normalis *msr;* erit, ob $PT = 2x$ et

$$MT = \sqrt{(4xx + 2cx)},\ \sqrt{(4xx + 2cx)} :\ 2x : \sqrt{2}cx = pm :\ ps : ms,$$

unde obtinetur

$$ps = \frac{2xu}{\sqrt{(4xx + 2cx)}} = u\sqrt{\frac{2x}{2x+c}} \ \text{ et }\ ms = u\sqrt{\frac{c}{2x+c}};$$

hinc erit

$$Ar = x + t + u\sqrt{\frac{2x}{2x+c}} \text{ et } mr = \sqrt{2}cx + u\sqrt{\frac{c}{2x+c}}.$$

Quia vero est $mr^2 = 2c \cdot Ar$, erit

$$2cx + 2cu\sqrt{\frac{2x}{2x+c}} + \frac{cuu}{2x+c} = 2cx + 2ct + 2cu\sqrt{\frac{2x}{2x+c}}$$

hincque

$$uu = 2t\left(2x + c\right) = 4FM \cdot t \ \text{ seu }\ pm^2 = 4FM \cdot Mp.$$

At anguli obliquitatis *mps* erit

$$\text{sinus} = \sqrt{\frac{c}{2x+c}} = \sqrt{\frac{AF}{FM}}, \quad \text{cosinus} = \sqrt{\frac{2x}{2x+c}} = \sqrt{\frac{AP}{FM}},$$

ideoque

$$\sin.2mps = \frac{2\sqrt{2cx}}{2x+c} = \frac{y}{FM} = \sin.MFp,$$

ergo erit

$$\text{angulus } mps = MTP = \frac{1}{2}MFr.$$

152. Quia est $MF = AP + AF$, ob $AP = AT$ erit $FM = FT$; ideoque triangulum *MFT* isosceles, et angulus $MFr = 2MTA$, ut modo invenimus. Cum deinde sit $MT = 2\sqrt{x(x+\frac{1}{2}c)}$, erit $MT = 2\sqrt{AP \cdot FM}$, hinc ex foco *F* in tangentem demisso perpendiculo erit

$$MS = TS = \sqrt{AP \cdot FM} = \sqrt{AT \cdot TF} ,$$

unde erit $AT : TS = TS : TF$. Ex qua analogia perspicitur punctum *S* fore in recta *AS* ad axem in vertice *A* normali. Erit vero

$$AS = \tfrac{1}{2}PM \quad \text{et} \quad AS : TS = AF : FS ,$$

ergo $FS = \sqrt{AF \cdot FM}$, et *FS* erit media proportionalis inter *AF* et *FM.* Praeterea vero erit

$$AS : MS = AS : TS = FS : FM = \sqrt{AF} : \sqrt{FM} .$$

Quodsi ducatur ad tangentem in *M* normalis *MW* axem secans in *W,* erit

$$PT : PM = PM : PW \quad \text{seu} \quad 2x : \sqrt{2cx} = \sqrt{2cx} : PW ;$$

unde fit $PW = c$; ubique igitur intervallum *PW,* quod in axe inter applicatam *PM* et normalem *WM* intercipitur, constantem habet magnitudinem atque aequale est semissi lateris recti seu applicatae *FH.* Erit autem

$$FW = FT = FM \quad \text{et} \quad MW = 2\sqrt{AF \cdot FM} .$$

153. Pervenimus iam ad hyperbolam, cuius natura exprimitur hac aequatione

abscissis super diametro orthogonali sumtis. Quodsi autem initium abscissarum transferatur intervallo $\frac{\beta}{2\gamma}$,orietur eiusmodi aequatio $yy = \alpha + \gamma xx$, in qua abscissae a centro computantur. Debet autem γ esse quantitas affirmativa ; quod vero ad α attinet, perinde est, sive ea sit quantitas affirmativa sive negativa; permutatis enim coordinatis x et y affirmatio quantitatis α in negationem mutatur et vicissim. Quamobrem sit α quantitas negativa et $yy = \alpha + \gamma xx$, atque apparet applicatam γ bis evanescere, scilicet si fuerit

$$x = +\sqrt{\frac{\alpha}{\gamma}} \text{ et } x = -\sqrt{\frac{\alpha}{\gamma}}.$$

Denotante ergo (Fig. 33) C centro, sint A et B loca, ubi axis a curva traiicitur; ac, posito semiaxe $CA = CB = a$, erit $a = \sqrt{\frac{\alpha}{\gamma}}$ et $\alpha = \gamma aa$, unde fit

$$yy = \gamma xx - \gamma aa.$$

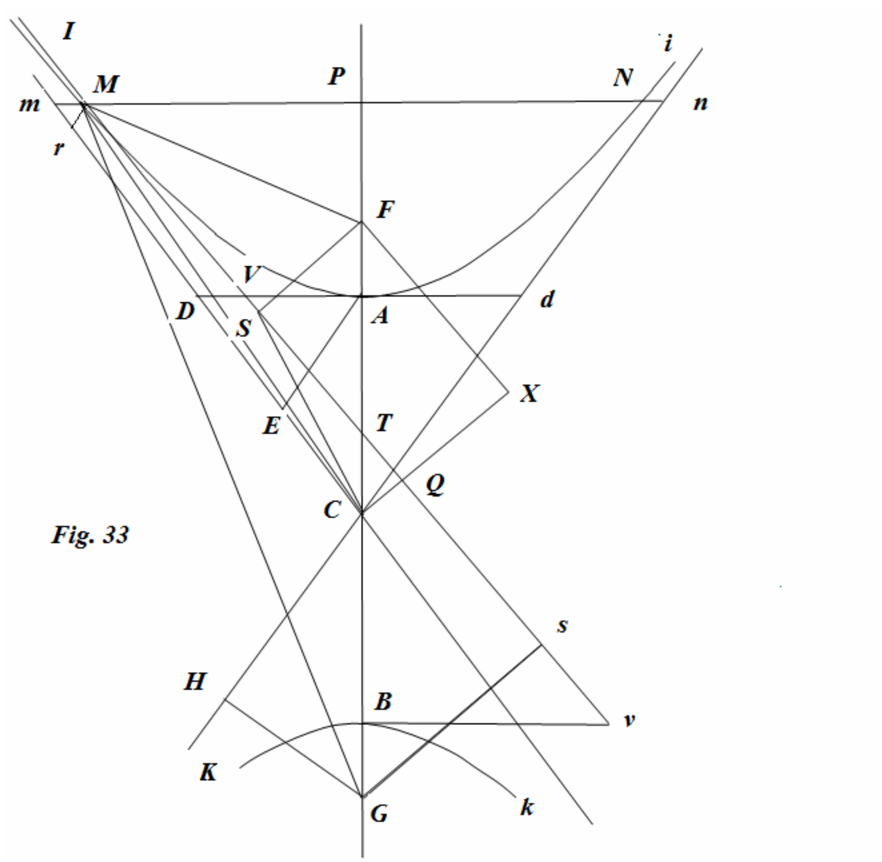

Fig. 33

Quamdiu ergo est x^2 minor quam a^2, applicata erit imaginaria, unde toti axi *AB* nulla curvae portio respondet. Sumto vero *xx* maiore quam *aa*, applicatae continuo crescunt atque tandem in infinitum abeunt; habebit ergo hyperbola quatuor ramos *AI, Ai, BK, Bk* in infinitum excurrentes et inter se similes atque aequales, quae est proprietas principalis hyperbolarum.

154. Quia posito $x = 0$ fit $yy = -\gamma aa$, hyperbola non instar ellipsis habebit axem coniugatum, quod in centro *C* applicata est imaginaria. Erit ergo ipse axis coniugatus imaginarius, quem, ut aliquam similitudinem ellipsis servemus, ponamus $= b\sqrt{-1}$, ita ut sit $\gamma aa = bb$ et $\gamma = \frac{bb}{aa}$. Vocata ergo abscissa $CP = x$ et applicata $PM = y$ erit

$$yy = \frac{bb}{aa}(xx - aa),$$

ideoque aequatio pro ellipsi ante tractata

$$yy = \frac{bb}{aa}(aa - xx),$$

transmutatur in aequationem pro hyperbola ponendo $-bb$ loco *bb*. Ob hanc ergo affinitatem proprietates ellipsis ante inventae facile ad hyperbolam transferuntur. Ac primo quidem, cum pro ellipsi distantia focorum a centro esset $= \sqrt{(aa - bb)}$, pro hyperbola erit $CF = CG = \sqrt{(aa + bb)}$. Hinc erit

$$FP = x - \sqrt{(aa + bb)} \text{ et } GP = x + \sqrt{(aa + bb)};$$

unde, ob $yy = -bb + \frac{bbxx}{aa}$, fiet

$$FM = \sqrt{\left(aa + xx + \frac{bbxx}{aa} - 2x\sqrt{(aa + bb)}\right)} = \frac{x\sqrt{(aa + bb)}}{a} - a$$

et

$$GM = \sqrt{\left(aa + xx + \frac{bbxx}{aa} + 2x\sqrt{(aa + bb)}\right)} = \frac{x\sqrt{(aa + bb)}}{a} + a.$$

Ductis ergo ex utroque foco ad curvae punctum *M* rectis *FM, GM* erit

$$FM + AC = \frac{CP \cdot CF}{CA} \text{ et } GM - AC = \frac{CP \cdot CF}{CA},$$

harum ergo rectarum differentia $GM - GM$ aequalis est $2AC$. Quemadmodum ergo in ellipsi summa harum duarum linearum aequatur axi principali *AB*, ita pro hyperbola differentia aequalis est axi principali *AB*.

155. Hinc etiam positio tangentis *MT* definiri potest, est enim perpetuo pro lineis secundi ordinis $CP : CA = CA : CT$, unde fit

$$CT = \frac{aa}{x} \text{ et } PT = \frac{xx - aa}{x} = \frac{aayy}{bbx};$$

hincque

$$MT = \frac{y}{bbx}\sqrt{(b^4x^2 + a^4y^2)} = \frac{y}{bx}\sqrt{(aaxx + bbxx - a^4)}$$

At est

$$FM \cdot GM = \frac{aaxx + bbxx - a^4}{aa},$$

ergo $MT = \frac{ay}{bx}\sqrt{FM} \cdot GM$. Deinde est

$$FT = \sqrt{(aa + bb)} - \frac{aa}{x} \text{ et } GT = \sqrt{(aa + bb)} + \frac{aa}{x},$$

ergo

$$FT : FM = a : x \text{ et } GT : GM = a : x,$$

unde sequitur $FT : GT = FM : GM$, quae proportio indicat angulum *FMG* per tangentem *MT* bisecari esseque $FMT = GMT$. Recta autem *CM* producta erit diameter obliquangula omnes ordinatas tangenti *MT* parallelas bisecans.

156. Demittatur ex centro *C* in tangentem perpendicularis *CQ*, erit

$$TM : PT : PM = CT : TQ : CQ$$

seu

$$\frac{ay}{bx}\sqrt{FM \cdot GM} : \frac{aayy}{bbx} : y = \frac{aa}{x} : TQ : CQ;$$

unde oritur

$$TQ = \frac{a^3y}{bx\sqrt{FM \cdot GM}} \text{ et } CQ = \frac{ab}{\sqrt{FM \cdot GM}}.$$

Demittatur simili modo ex foco F in tangentem perpendiculum FS, erit

$$TM : PT : PM = FT : TS : FS\,,$$

seu

$$\frac{ay}{bx}\sqrt{FM\cdot GM} : \frac{aayy}{bbx} : y = \frac{a\cdot FM}{x} : TS : FS\,;$$

unde oritur

$$TS = \frac{aay\cdot FM}{bx\sqrt{FM\cdot GM}} \text{ et } FS = \frac{b\cdot FM}{\sqrt{FM\cdot GM}}\,;$$

pariterque, si ex altero foco G in tangentem ducatur perpendicularis Gs, erit

$$Ts = \frac{aay\cdot FM}{bx\sqrt{FM\cdot GM}} \text{ et } Gs = \frac{b\cdot FM}{\sqrt{FM\cdot GM}}.$$

Hinc ergo habetur

$$TS\cdot Ts = \frac{a^4yy}{bbxx} = \frac{aa(xx-aa)}{xx} = CT\cdot PT \text{ et } TS{:}CT = PT : Ts.$$

Deinde fit $FS\cdot Gs = bb$. Quia porro est $QS = Qs$, erit

$$QS = \frac{TS+Ts}{2} = \frac{aay(FM+GM)}{2bx\sqrt{FM\cdot GM}} = \frac{ay\sqrt{(aa+bb)}}{b\sqrt{FM\cdot GM}} = Qs\,,$$

unde sequitur

$$CS^2 = CQ^2 + QS^2 = \frac{aab^4 + a^4yy + aabbyy}{bb\cdot FM\cdot GM} = \frac{aab^4 + (aa+bb)\ (bbxx-aabb)}{bb\cdot FM\cdot GM} = \frac{\left(aa+bb\right)xx - a^4}{FM\cdot GM} = aa.$$

Erit ergo, uti in ellipsi, recta $CS = a = CA$. Deinde est

$$CQ + FS = \frac{bx\sqrt{(aa+bb)}}{a\sqrt{FM\cdot GM}}$$

ideoque

$$(CQ+FS)^2 - CQ^2 = \frac{bbxx(aa+bb) - a^4bb}{aa\cdot FM\cdot GM} = bb.$$

Quare, si ducatur ex foco *F* tangenti parallela *FX* secans perpendiculum *CQ* productum in *X*, erit $CX=\sqrt{(bb+CQ^2)}$, cui similis proprietas pro ellipsi est inventa.

157. Si in verticibus *A* et *B* ad axem perpendiculares erigantur, donec tangenti occurrant in *V* et *v*, ob

$$AT=\frac{a(x-a)}{x} \text{ et } BT=\frac{a(x+a)}{x},$$

$$PT:PM=AT:AV=BT:Bv,$$

hinc fit

$$AV=\frac{bb(x-a)}{ay} \text{ et } Bv=\frac{bb(x+a)}{ay};$$

ergo

$$AV\cdot Bv=\frac{b^4(xx-aa)}{aayy}=bb$$

seu

$$AV\cdot Bv=FS\cdot Gs$$

Deinde $PT:TM=AT:TV=BT:Tv$; ergo

$$TV=\frac{b(x-a)}{xy}\sqrt{FM\cdot GM} \text{ et } Tv=\frac{b(x+a)}{xy}\sqrt{FM\cdot GM};$$

unde fit

$$TV\cdot Tv=\frac{aa}{xx}FM\cdot GM=FT\cdot GT.$$

Simili autem modo hinc plura alia consectaria deduci possunt.

158. Quia est $CT=\frac{aa}{x}$, patet, quo maior capiatur abscissa $CP=x$, eo minus futurum esse intervallum CT; atque adeo tangens, quae curvam in infinitum productam tangit, per ipsum centrum *C* transibit fietque $CT=0$.
Cum autem sit

$$\text{tang. } PPM=\frac{PM}{PT}=\frac{bbx}{aay},$$

puncto *M* in infinitum abeunte seu posito $x=\infty$, fit

$$y = \frac{b}{a}\sqrt{(xx - aa)} = \frac{bx}{a}.$$

Tangens ergo curvae in infinitum productae et per centrum *C* transibit et cum axe angulum constituet *ACD,* cuius tangens $= \frac{b}{a}$. Posita ergo in vertice *A* ad axem normali $AD = b$, tum recta *CD* in infinitum utrinque producta curvam nusquam quidem tanget, at curva continuo magis ad eam appropinquabit, donec in infinitum tota cum recta *CI* confundatur. Hoc idem valebit de parte *Ck,* quae tandem cum ramo *Bk* confundetur. Atque si ad alteram partem sub eodem angulo ducatur recta *KCi,* ea cum ramis *BK* et *Bi* in infinitum productis conveniet. Huiusmodi autem lineae rectae, ad quas linea quaepiam curva continuo propius accedit, in infinitum autem excurrens demum attingit, *asymptotae* vocantur, unde lineae rectae *ICk, KCi* sunt binae asymptotae hyperbolae.

159. Asymptotae ergo se mutuo in centro *C* hyperbolae decussant atque ad axem inclinantur angulo $ACD = ACd$, cuius tangens $= \frac{b}{a}$, angulique dupli *DCd* tangens $= \frac{2ab}{aa - bb}$, unde patet, si fuerit $b = a$, fore angulum, sub quo asymptotae se intersecant, $DCd =$ recto; quo casu hyperbola *aequilatera* dicitur. Cum autem sit $AC = a,\ AD = b,$ erit $CD = Cd = \sqrt{(aa + bb)}$; quare, si ex foco *G* in utramvis asymptotam perpendiculum *CR* demittatur, ob $CG = \sqrt{(aa + bb)} = CD$, erit $CH = AC = BC = a$ et $GH = b$.

160. Producatur ordinata $MPN = 2y$ utrinque, donec asymptotas secet in *m* et *n;* erit

$$Pm = Pn = \frac{bx}{a} \text{ et } Cm = Cn = \frac{x\sqrt{(aa + bb)}}{a} = FM + AC = GM - AC.$$

Tum vero erit

$$Mm = Nn = \frac{bx - ay}{a} \quad \text{et} \quad Nm = Mn = \frac{bx + ay}{a},$$

unde fit

$$Mm \cdot Nm = Mm \cdot Mn = \frac{bbxx - aayy}{aa} = bb,$$

ob $aayy = bbxx - aabb$; erit ergo ubique

$$Mm \cdot Nm = Mm \cdot Mn = Nn \cdot Nm = Nn \cdot Mn = bb = AD^2.$$

Ducatur ex *M* asymptotae *Cd* parallela *Mr;* erit

$$2b\sqrt{(aa+bb)} = Mm : mr(Mr),$$

unde fit

$$mr = Mr = \frac{(bx-ay)\sqrt{(aa+bb)}}{2ab}$$

et

$$Cm - mr = Cr = \frac{(bx+ay)\sqrt{(aa+bb)}}{2ab}.$$

Hinc ergo conficietur

$$Mr \cdot Cr = \frac{(bbxx-aayy)\ (aa+bb)}{4aabb} = \frac{aa+bb}{4}.$$

Vel ducta ex *A* asymptotae *Cd* parallela *AE* erit $AE = CE = \frac{1}{2}\sqrt{(aa+bb)}$ ideoque erit $Mr \cdot Cr = AE \cdot CE$; quae est proprietas primaria hyperbolae ad asymptotas relatae,

161. Quodsi ergo (Fig. 34) abscissae $CP = x$ in una asymptota a centro sumantur et applicatae $PM = y$ alteri asymptotae parallelae statuantur, erit

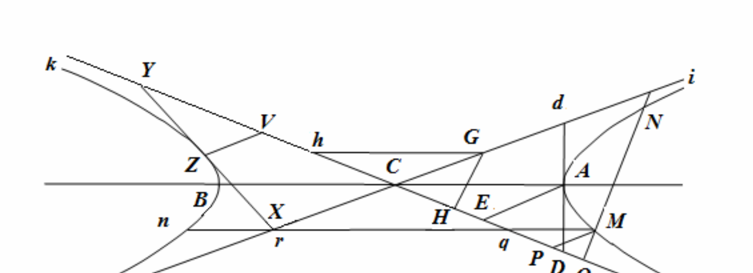

Fig. 34

$yx = \frac{aa+bb}{4}$. existente $AC = BC = a$ et $AD = Ad = b$; seu, si ponatur $AE = CE = h$, erit $yx = hh$ et $y = \frac{hh}{x}$. Posito ergo $x = 0$ fit $y = \infty$, ac vicissim facto $x = \infty$ fiet $y = 0$. Agatur iam per punctum curvae *M* recta quaecunque *QMNR*, quae parallela sit ductae pro libitu rectae *GH*, ac ponatur $CQ = t,\ QM = u$, erit

$$GH : CH : CG = u : PQ : PM ,$$

ergo

$$PQ=\frac{CH}{GH}u,\quad PM=\frac{CG}{GH}u\,;$$

unde

$$y=\frac{CH}{GH}u\ \text{ et }\ x=t-\frac{CG}{GH}u\,;$$

quibus valoribus substitutis, erit

$$\frac{CG}{CH}tu-\frac{CG\cdot GH}{GH^2}\cdot uu=hh,$$

seu

$$uu-\frac{GH}{CH}tu+\frac{GH^2}{CH\cdot CG}hh=0.$$

Habebit ergo applicata *u* duplicem valorem, nempe *QM* et *QN,* quarum summa erit $=\frac{GH}{CH}t=QR$ et rectangulum $QM\cdot QN=\frac{GH^2}{CH.CG}hh$.

162. Cum igitur sit $QM+QN=QR$, erit $QM=RN$ et $QN=RM$. Quare, si puncta *M* et *N* conveniant, quo casu recta *QR* curvam tanget, tum ea in ipso puncto contactus bisecabitur. Scilicet, si recta *XY* tangat hyperbolam, punctum contactus *Z* in medio rectae *XY* erit positum. Unde, si ex *Z* alteri asymptotae parallela ducatur *ZV,* erit $CV=VY$, hincque ad quodvis hyperbolae punctum *Z* expedite tangens ducetur. Sumatur scilicet $VY=CV$, ac recta per *Y* et curvae punctum *Z* ducta hyperbolam in hoc puncto *Z* tanget.

Cum ergo sit $CV\cdot ZV=hh=\frac{aa+bb}{4}$, erit

$$CX\cdot CY=aa+bb=CD^2=CD\cdot Cd\,;$$

quocirca, si rectae *DX* et *dY* ducerentur, eae inter se forent parallelae; unde facillimus oritur modus quotcunque curvae tangentes ducendi.

163. Quoniam deinde est rectangulum $QM\cdot QN=\frac{GH^2}{CH\cdot CG}\cdot hh$, patet, ubicunque recta *QR* ipsi *HG* parallela ducatur, fore semper rectangulum $QM\cdot QN$ eiusdem magnitudinis. Erit ergo etiam

$$QM\cdot QN=QM\cdot MR=QN\cdot NR=\frac{CH^2}{CH.CG}hh.$$

Quodsi ergo concipiatur ducta tangens ipsi *QR* parallela, quia ea intra asymptotas in puncto contactus bisecabitur, et si tangentis semissis vocetur $=q$, erit semper

$$QM \cdot QN = QM \cdot MR = RN \cdot RM = RN \cdot NQ = qq\ ,$$

quae est insignis proprietas hyperbolarum intra asymptotas descriptarum.

164. Quoniam hyperbola ex duabus partibus diametraliter oppositis *IAi* et *KBk* constat, istae proprietates non solum ad eas rectas intra asymptotas ductas pertinent, quae eandem curvae partem in duabus punctis intersecant, sed etiam ad eas, quae ad partes oppositas pertingunt. Ducatur nempe per punctum *M* recta *Mqrn* ad partem oppositam, cui parallela agatur *Gh,* ac vocetur $Cq = t$ et $qM = u$ *;* erit, ob triangula *CGh* et *PMq* similia,

$$PM = y = \frac{CG}{Gh}u \text{ et } qP = x - t = \frac{Ch}{Gh}u\ ;$$

unde fit $x = t + \frac{Ch}{Gh}u$. Cum autem sit $xy = hh$, fiet

$$\frac{CG}{Gh}tu + \frac{CG \cdot Ch}{Gh^2}uu = hh$$

seu

$$uu + \frac{Gh}{Ch}tu - \frac{Gh^2}{CG \cdot Ch}hh = 0\ .$$

165. Applicata ergo *u* habebit duplicem valorem, nempe *qM* et − *qn,* hoc *qn* existente negativo, quia ad alteram partem asymptotae *CP* pro axe assumtae vergit. Harum ergo binarum radicum summa

$$qM - qn \text{ erit} = -\frac{Gh}{Ch}t = -qr\ ,$$

ideoque $qn - qM = qr$, unde fit $qM = rn$ et $qn = rM$. Deinde autem ex aequatione inventa intelligitur fore radicum productum

$$-qM \cdot qn = -\frac{Gh^2}{CG.Ch}hh$$

seu

$$qM \cdot qn = qM \cdot rM = rn \cdot qn = rn \cdot rM = \frac{Gh^2}{CG.Ch}hh\ .$$

Haec ergo rectangula, quotcunque rectae *Mn* ipsi *Gh* parallelae ducantur, perpetuo eiusdem erunt magnitudinis. Hae autem sunt praecipuae singularum specierum linearum secundi ordinis proprietates, quae si cum proprietatibus generalibus conferantur, infinita fere insignium proprietatum multitudo conficitur.