CHAPTER V

CONCERNING LINES OF THE SECOND ORDER

85. Because only right lines may be contained in the first order of lines, the nature of which is now agreed upon well enough from geometry, we will carefully consider some *lines of the second order*, because these shall be the simplest among all the curved lines and may have the fullest use through all higher geometry. But these are the lines provided, which are also called conic sections, with many conspicuous properties, which were elicited by the ancient geometers, then enlarged on more recently. The necessary knowledge of these properties thus may be examined at once, as it is usually set out at once by many authors past the elementary stage. Truly all these properties cannot be derived from a single principle, but some an equation makes apparent, others by the generation of a conic section, and finally others are to be described in other ways, here we will investigate these properties only, which may be made available by an equation alone, without other help.

86. Therefore we will consider the general equation for lines of the second order, which is

$$0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy,$$

thus we have shown which equation to be prepared, so that, whatever the angle the inclination of the applied lines may put in place to the axis, yet always it includes these lines of the second order in itself. Now this form of the equation itself may be granted :

$$yy + \frac{(\varepsilon x + \gamma)y}{\zeta} + \frac{\delta xx + \beta x + \alpha}{\zeta} = 0,$$

from which it is apparent for each abscissa x there corresponds either two applied lines y or none, just as the two roots of y were either real or imaginary. But if moreover there were $\zeta = 0$, then certainly one of the applied lines will correspond to an individual abscissa, while the other goes off to infinity, on account of which this case will not disturb our investigation.

87. But as often as both the values of y should be real, which arises if the applied line *PMN* (see Fig. 19) may cut the curve in the two points *M* and *N*, the sum of the roots will be

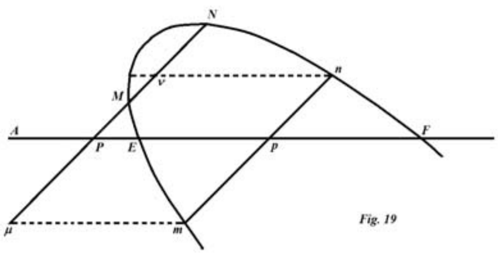


Fig. 19

$$PM + PN = \frac{-\varepsilon x - \gamma}{\zeta} = \frac{-\varepsilon \cdot AP - \gamma}{\zeta},$$

with the right line *AEF* taken for the axis, *A* for the beginning of the abscissas and the angle *APN,* by which the applied lines stand upon the axis, is placed obliquely as it pleases. But if therefore some other applied line *npm* may be drawn under the same angle, the value of which certainly *pm* is negative, in the same manner there will be

$$pn - pm = \frac{-\varepsilon \cdot Ap - \gamma}{\zeta}.$$

This equation may be taken away from the first, there will become

$$PM + pm + PN - pn = \frac{\varepsilon\left(Ap - AP\gamma\right)}{\zeta} = \frac{\varepsilon \cdot Pp}{\zeta}.$$

Right lines may be drawn from the points *m* and *n* parallel to the axis, then they cross the first applied lines at the points μ and v, and there will be

$$M\mu + Nv = \frac{\varepsilon \cdot Pp}{\zeta}$$

or the sum $M\mu + Nv$ to Pp or to $m\mu$ or to nv will have the constant ratio as ε to ζ. Clearly this ratio will be the same always, wherever the right lines *MN* and *mn* may be drawn on the curve*,* as long as they make the given angle with the axis and the right lines nv and $m\mu$ may be drawn parallel to the axis.

88. If there the applied line *PMN* may be advanced (see Fig. 20), so that the points *M* and *N* shall meet, then the applied line touches the curve ; for where the two intersections meet, there the cutting line will become a tangent. Therefore *KCI* shall be a tangent of this kind, to which some number of parallel right lines *MN, mn* may be drawn each crossing the curve, right lines of this kind are accustomed to be called *chords* and *ordinates.* Then from the points *M, N, m, n* the right lines *MI, NK* and *mi, nk* may be produced to the tangent parallel to

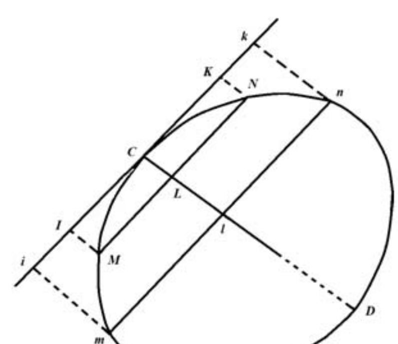


Fig. 20

the first axis assumed. Because now the intervals *OK, Ok* lie on opposite sides of the point *C*, they must be taken negative.
Hence there will be

$$CI - CK : MI = \varepsilon : \zeta \text{ and } Ci - Ck : mi = \varepsilon : \zeta$$

and thus

$$CI - CK : MI = Ci - Ck : mi$$

or

$$MI : mi = CI - CK : Ci - Ck .$$

89. Because the position of the axis with respect to the curve is arbitrary, the right lines *MI, NK, mi, nk* can be drawn as it pleases, as long as they shall be parallel to each other, and there will be always

$$MI : mi = CI - CK : Ci - Ck .$$

But if therefore the parallel right lines *MI* and *NK* may be drawn thus, so that there becomes $CI = CK$, so that it arises, if *MI* and *NK* may be put in place parallel to the right line *CL,* which may bisect the ordinate *MN* drawn from the point of contact *C* in *L*, then on account of $CI - CK = 0$ there becomes also

$$Ci - Ck = \frac{mi}{MI}\left(CI - CK\right) = 0.$$

Whereby the right line *CL* produced in *l,* because on account of *mi* and *nk* themselves equally parallel to *CL*, there is $ml = Ci$ and $nl = Ck$, there will be $ml = nl$. From which it follows the right line *CLl,* which drawn from the point of contact *C* may bisect the one ordinate *MN* parallel to the tangent, all the same ordinates *mn* parallel to the same tangent are to be cut into two equal parts.

90. Therefore since the right line *CLl* may cut all the ordinates parallel to the tangent *ICK* into two equal parts, this line *CLl* is accustomed to be called a *diameter of the second order* or *of a conic section.* Hence innumerable diameters are able to be drawn in any line of the second order, because the tangent is given at the individual points of the curve. For wherever the given tangent *ICK* should be*,* some one ordinate *MN* may be drawn hence parallel to this tangent, which bisected in *L*, the right line *CL* will be a diameter of a line of the second order cutting all the ordinates parallel to the tangent *IK* into two equal parts.

91. From these it follows also, if the right line *L1* may bisect any two lines parallel to the ordinates *MN* and *mn*, all the same remaining ordinates parallel to these are to be bisected ; for the right line *IK* will be given touching the curve somewhere parallel to these coordinates and thus will give a diameter. Hence a new method is had for finding the

innumerable diameters in a given line of the second order ; for two ordinates may be drawn as it pleases or the chords *MN* and *mn* parallel to each other, with which bisected in *L* and *l* right lines drawn through these points will bisect equally all the remaining ordinates parallel to these and on that account will be a diameter. And when the diameter produced may cut the curve in *C*, through that the line *IK* drawn parallel to the ordinates will touch the curve at the point *C*.

92. We have led to that property by consideration of the sum of the two roots of y from the equation

$$yy+\frac{(\varepsilon x+y)}{\zeta}y+\frac{\delta xx+\beta x+\alpha}{\zeta}=0.$$

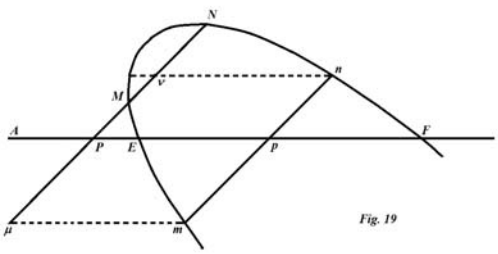


Fig. 19

Truly from the same equation it is agreed that the product of both roots to be (see Fig. 19) $PM\cdot PN=\frac{\delta xx+\beta x+\alpha}{\zeta}$, which expression $\frac{\delta xx+\beta x+\alpha}{\zeta}$ either has two simple real factors or otherwise. That arises, if the axis may cut the curve in two points *E* and *F*; because indeed with these in place, there becomes $y=0$, and the product $\frac{\delta xx+\beta x+\alpha}{\zeta}=0$ and hence the roots of x will be *AE* and *AF* and therefore the factors $(x-AE)\ (x-AF)$, thus so that there shall be

$$\frac{\delta xx+\beta x+\alpha}{\zeta}=\frac{\delta}{\zeta}(x-AE)\ (x-AF)=\frac{\delta}{\zeta}\cdot PE\cdot PF$$

on account of $x=AP$. Therefore because of this there will be

$$PM\cdot PN=\frac{\delta}{\zeta}PE\cdot PF,$$

or the rectangle $PM\cdot PN$ will be to $PE\cdot PF$ in a constant ratio as δ to ζ wherever the applied line *PMN* may be drawn, provided the angle *NPF* may be assumed equal, by which the applied lines may be inclined to the axis. Therefore in a similar manner, if the applied line *mn* may be drawn, on account of the negative *Ep* and *pm* :

$$pm\cdot pn=\frac{\delta}{\zeta}pE\cdot pF\,.$$

93. Therefore with some right line (see Fig. 21) *PEF* drawn cutting the second order line in the two points *E, F,* if some ordinates *NMP, npm* parallel to each other may be drawn, there will be always

$$PM \cdot PN : PE \cdot PF = qM \cdot qN : qe \cdot qf = pm \cdot pn : pE \cdot pF ;$$

Therefore by another way:

$$qe \cdot qf : pE \cdot pF = qM \cdot qN : pm \cdot pN .$$

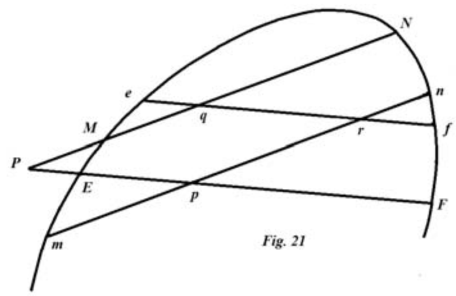


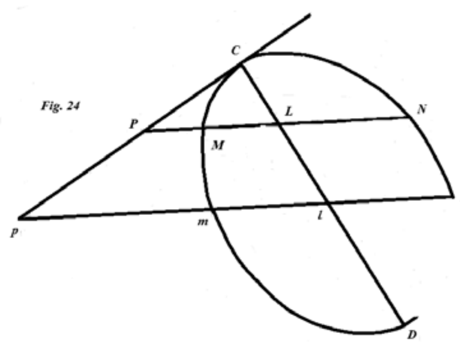
Fig. 21

Therefore with two parallel ordinates *ef* & *EF* given, if some other two ordinates parallel between themselves may be drawn *MN* & *mn*, cutting these in the points *P, p, q, r ,* all these ratios will be equal to each other :

$$PM \cdot PN : PE.PF = pm \cdot pn : pE \cdot pF = qM \cdot qN : qe \cdot qf = rm \cdot rn : re \cdot rf .$$

Which is another general property of lines of the second order.

94. Therefore if the two points of the curve *M* & *N* may coincide, the right line *PMN* becomes a tangent to the curve at the concurrence of these two points (Fig. 24), and the rectangle $PM \cdot PN$ will be changed into the square of *PM* or *PN,* from which a new property of the tangent will be obtained. Without doubt the right line *CPp* may touch the line of the second order at the point *C,* and some mutually parallel lines *PMN* , *pmn* may be drawn, which therefore all may make the same angle with the tangent. Therefore from the property found before there will be



$$PC^2 : PM \cdot PN = pC^2 : pm \cdot pn ,$$

or with some ordinate *MN* may be drawn to the tangent under a give angle, there will always be a constant ratio of the square of *CP* to $PM \times PN$.

95. It also follows from the same place, if some diameter *CD* may be drawn for a line of the second order, bisecting all the ordinates *MN*, *mn* [Fig. *20*] parallel to each other, and the diameter of the curve crosses at the two points *C* and *D*, that there shall be

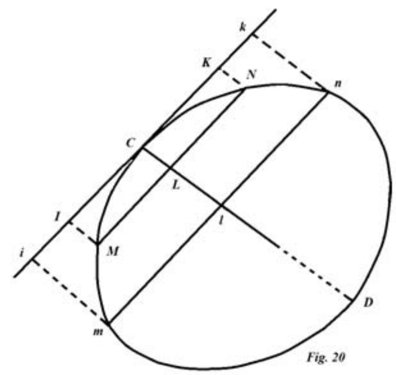
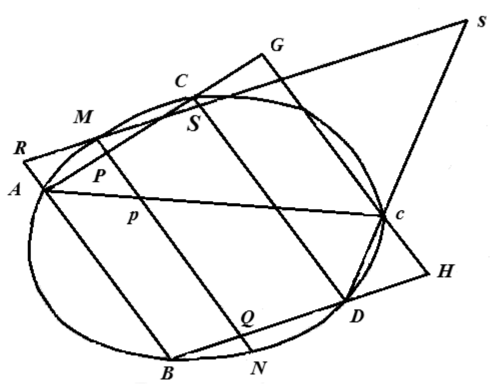


Fig. 20

$$CL \cdot LD : LM \cdot LN = Cl \cdot lD : lm \cdot ln.$$

For since there shall be $LM = LN$, & $lm = ln$, there will be $LM^2 : lm^2 = CL \cdot LD : Cl \cdot lD$, for always the square of the semi–ordinate *LM* shall be in a constant ratio to the rectangle $CL \cdot LD$. Hence with the diameter *CD* taken for the axe, and the semi–ordinate *LM* for the applied line, the equation of a line of the second order will be found. Indeed let the $CD = a$. The abscissa $CL = x$ and the applied line $LM = y$, on account of $LD = a - x$, y^2 will be in a constant ratio to $ax - xx$, which shall be as *h* to *k*, from which this equation arises for a line of the second order $yy = \frac{b}{k}(ax - xx)$.

96. Moreover from both the properties of lines of the second order now found taken together other properties will be able to be elicited. [See Fig. 22.] Two ordinates may be given on a line of the second order parallel to each other *AB* & *CD,* and the quadrilateral *ACDB* may be completed, so that if now through some point of the curve *M,* the ordinate *MN* may be drawn parallel to these *AB* & *CD* cutting the right lines *AC* & *BD* in the points *P* & *Q*, the parts *PM* & *QN* will be equal to each other. For a right line, which may bisect the two mutually parallel ordinates *AB* & *CD*, will bisect the ordinate *MN* too : but, by elementary geometry, the same right line bisecting the sides *AB* & *CD* also will bisect the part *PQ*. Therefore since the lines *MN* & *PQ* will be bisected at the same point, it is necessary that $MP = NQ$ & $MQ = NP$. Therefore besides the given four points of the line of the second order *A*, *B, C* & *D*, with the fifth point *M* given from that, a sixth point *N* may be found, on taking $NQ = MP$.

Fig. 22


97. Now since $MQ\cdot QN$ to $BQ\cdot DQ$ shall be in a constant ratio , on account of $QN=MP$, $MP\cdot MQ$ to $BQ\cdot DQ$ will also be in the same constant ratio. Evidently, if some other point of the curve may be taken, as *c*, and through that the right line *GcH* may be drawn parallel to *AB & CD* themselves, then it may meet the sides *AC, BD* in the points *G* & *H,* and also $cG.cH$ to $BH.DH$ will be in the same constant ratio, and thus $cG\cdot cH : BH\cdot DH = MP\cdot MQ : BQ\cdot DQ$. So that if moreover *RMS* may be drawn through *M* parallel to the base *BD* crossing the parallel ordinates *AB, CD* in *R & S ,* there will be, on account of $BQ=MR$ & $DQ=MS$, this constant ratio also $MP\cdot MQ : MR\cdot MS$.

98. If in place of the ordinate *CD*, which has been put in place parallel to *AB* itself, from the point *D* some other *Dc* is substituted in place of this, and the chord *Ac* is joined : thus so that now the right lines *MQ & RMS* drawn*,* as before, through *M* parallel to the sides *AB* & *BD*, may cut the sides of the quadrilateral *ABDc* in the points *p, Q, R* & *s* ; a similar property will be had in place. Indeed since there shall be $MP\cdot MQ : BQ\times DQ = cG\cdot cH : BH\cdot DH$ or $MP\cdot MQ : MR\cdot MS = cG\cdot cH : BH\cdot DH$, on account of the right line *RS* parallel and equal to *BD* itself. Truly the similar triangles *APp*, *AGc & DSs, cHD,* provide these proportions $Pp : AP = Gc : AG$, or, on account of $AP : AG = BQ : BH$ this, $Pp : BQ = Gc : BH$: the other similarity gives this $DS(MQ) : Ss = cH : DH$, with which taken together there becomes :

$$MQ\cdot Pp : MR\cdot Ss = cG\cdot c\text{H} : BH\cdot DH, \text{ as } BQ = MR.$$

This proportion taken with the above presents :

$$MP\cdot MQ : MR\cdot MS = Pp\cdot MQ : MR\cdot Ss,$$

from which with the preceding and the following being added becomes :

$$MP\cdot MQ : MR\cdot MS = Mp\cdot MQ : MR\cdot Ms,$$

therefore wherever the points *c* & *M* may be taken on the curve , the ratio $Mp\cdot MQ$ to $MR\cdot Ms$ will always be the same, as long as the right lines *MQ* & *Rs* are drawn through *M* parallel to the chords *AB* & *BD*. Indeed from the above proportion it follows that $MP : MS = Mp : Ms$. Therefore since, with the variation of the point *c,* only the points *p* & *s* may be changed, *Mp* to *Ms* will be in the same give ratio, however the point *c* may be changed, while the fixed point *M* may be used.

99. But if any four points *A ,B ,C, D* are given on a line of the second order, and these may be joined by right lines, (see Fig. 23) so that the inscribed trapezium *ABDC* may be had*,* the most general property apparent of conic sections is deduced from the preceding. Evidently, if the right lines *MP, MQ*, *MR* & *MS* may be drawn from some point *M* of the

curve to the individual sides of the trapezium under the given angles, the rectangle of two of these lines always will be in a given ratio to each other to the opposite sides drawn: clearly $MP \cdot MQ$ to $MR \cdot MS$ will be in the same given ratio, wherever the point M may be taken on the curve, as long as the angles to P, Q, R, & S remain the same. To show this the two right lines Mq & rs may be drawn through the point M, that first one parallel to the side AB and this second one parallel to the side BD, and the points of intersection with the sides of the trapezium p, q, r, & s may be observed : and through the first $Mp \cdot Mq$ to $Mr \cdot Ms$ will be found in the given ratio. But because all the angles given will be in the given ratios $MP{:}Mp, MQ{:}Mq,\ MR{:}Mr,\ \&\ MS{:}Ms$, from which it follows that $MP \cdot MQ$ to $MR \cdot MS$ will be in some given ratio.

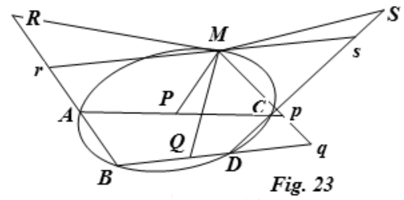


Fig. 23

100. Because we have seen above, if the parallel ordinates *MN, mn* (see Fig. 24) may be produced, as long as they may run to meet a certain tangent *CPp* in *P* & *r,* there shall be $PM \cdot PN{:}CP^2 = pm \cdot pn{:}Cp^2$. Whereby , if the points L & l may be noted, so that PL shall be the mean proportional between PM & PN, and equally pl the mean proportional between pm & pn, there will be $PL^2{:}CP^2 = pl^2{:}Cp^2$ and thus there will be $PL : CP = pl : Cp$ from which it is apparent all the points L ,l to be placed on a right line passing through the point of contact C. Whereby, if one applied line PMN thus may be cut at L so that there shall be $PL^2 = PM \cdot PN$, the right line CLD drawn through the points C & L thus will cut also all the remaining applied lines pmn at l so that pl shall be the mean proportional between pm & pn. Or, if two applied lines PN & pn thus may be cut at the points L & l, so that there shall be $PL^2 = PM \cdot PN$ & $pl^2 = pm \cdot pn$, then the right lines through L & l will be produced through the point of contact C, and all the remaining applied lines parallel to these will be cut in the same ratio.

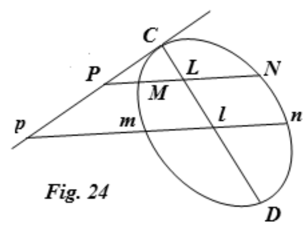


Fig. 24

101. With the properties of lines of the second order put in place, which follow at once from the form of the equation ; we may progress to investigating other more recondite properties. Therefore let the equation proposed for these lines of the general second order be

$$yy + \frac{(\varepsilon x + \gamma)}{\zeta} y + \frac{\delta xx + \beta x + \alpha}{\zeta} = 0,$$

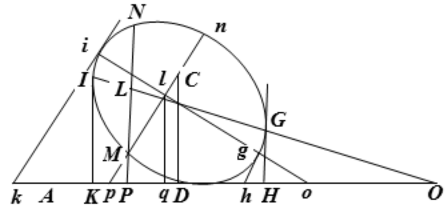


Fig. 25

from which since for some abscissa $AP = x$, the two fold applied line y certainly PM & PN may correspond , the position of the diameter cutting all the ordinates MN into two parts can be defined. For *IG* shall be a diameter, which will cut the ordinate *MN* at the mean point L, which point therefore is in the diameter. There may be put $PL = z$; and since there shall be $z = \frac{1}{2}PM + \frac{1}{2}PN$, there will be $z = \frac{-\varepsilon x - \gamma}{2\zeta}$, or $2\zeta z + \varepsilon x + \gamma = 0$, which is the equation determining the position of the diameter *IG*.

102. Hence again the length of the diameter *IG* can be defined , which gives two places on the curve, where the points *M* & *N* coincide, or where there becomes $PM = PN$. Truly from the equation there are given

$$PM + PN = \frac{-\varepsilon x - \gamma}{\zeta} \ \ \&\ \ PM.PN = \frac{\delta xx + \beta x + \alpha}{\zeta},$$

from which there becomes :

$$(PM - PN)^2 = (PM + PN)^2 - 4\,PM \cdot PN = \frac{(\varepsilon\varepsilon - 4\delta\zeta)xx + 2(\varepsilon\gamma - 2\beta\zeta)x + (\gamma\gamma - 4\alpha\zeta)}{\zeta\zeta} = 0,$$

or

$$xx - \frac{2(2\beta\zeta - \varepsilon\gamma)}{\varepsilon\varepsilon - 4\delta\zeta}x + \frac{\gamma\gamma - 4\alpha\zeta}{\varepsilon\varepsilon - 4\delta\zeta} = 0,$$

therefore the roots of which equation are *AK* & *AH* thus so that :

$$AK + AH = \frac{4\beta\zeta - 2\varepsilon\gamma}{\varepsilon\varepsilon - 4\delta\zeta} \ \ \&\ \ AK \cdot AH = \frac{\gamma\gamma - 4\alpha\zeta}{\varepsilon\varepsilon - 4\delta\zeta},$$

hence there shall be

$$(AK - AH)^2 = KH^2 = \frac{4(2\beta\zeta - \varepsilon\gamma)^2 - 4(\varepsilon\varepsilon - 4\delta\zeta)(\gamma\gamma - 4\alpha\zeta)}{(\varepsilon\varepsilon - 4\delta\zeta)^2}.$$

But $IG^2 = \frac{\varepsilon\varepsilon + 4\zeta\zeta}{4\zeta\zeta}KH^2$, if indeed applied lines normal to the axis may be put in place.

103. Without these applied lines which we have considered here normal to the axis *AH*; indeed now we may search for the equation of oblique angled applied lines. Therefore from some point *M* of the curve an applied line *Mp* may be drawn at an oblique angle to the axis, making the angle *MpH* with the axis, the sine of which shall be $= \mu$ and the cosine $= \nu$. Let the new abscissa $Ap = t$, the applied line $pM = u$, and there

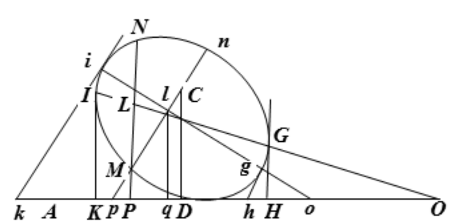


Fig. 25

will be $\frac{y}{u}=\mu$ & $\frac{Pp}{u}=\nu$; from which $y=\mu u$ & $x=t+\nu u$), which values substituted into the equation between x & y, which was $0=\alpha+\beta x+\gamma y+\delta xx+\varepsilon xy+\zeta yy$ provides

$$\begin{aligned} 0=\alpha+\beta t+\nu\beta u+\delta tt+2\nu\delta tu&+\nu\nu\delta uu\\ +\mu\gamma u\quad+\mu\varepsilon tu&+\mu\nu\varepsilon uu\\ &+\mu\mu\zeta uu \end{aligned}$$

or

$$uu+\frac{\big((\mu\varepsilon+2\nu\delta)t+u\gamma+\nu\beta\big)u+\delta tt+\beta t+\alpha}{\mu\mu\zeta+\mu\nu\varepsilon+\nu\nu\delta}=0.$$

104. Therefore here again some applied line will have a double value, certainly *pM* & *pn:* whereby the diameter *ilg* of the ordinates *Mn* will be defined in an equal manner as before. Evidently, with the ordinate *Mn* bisected in *l* there will be *l ,* a point on the diameter. Therefore putting $pl=\upsilon$, there will be

$$\upsilon=\frac{pM+pu}{2}=\frac{-(\mu\varepsilon+2\nu\delta)t-\mu\gamma-\nu\beta}{2(\mu\mu\zeta+\mu\nu\varepsilon+\nu\nu\delta)}.$$

From *l* the perpendicular *lq* may be sent to the axis *AH,* and putting in place $Aq=p$, $ql=q$, there will be $\mu=\frac{q}{\upsilon}$ & $\nu=\frac{pq}{\upsilon}=\frac{p-t}{\upsilon}$, from which there becomes $\upsilon=\frac{q}{\mu}$, & $t=p-\nu\upsilon=p-\frac{\nu q}{\mu}$. These values may be substituted into the equation between t & υ found before , and there will be produced

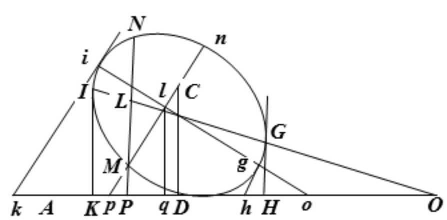


Fig. 25

$$\frac{q}{\mu}=\frac{-\mu\varepsilon p-2\nu\delta p+\nu\varepsilon q+2\nu\nu\delta q:\mu-\mu\gamma-\nu\beta}{2\mu\mu\zeta+2\mu\nu\varepsilon+2\nu\nu\delta}$$

or

$$(2\mu\mu\zeta+\mu\nu\varepsilon)q+(\mu\mu\varepsilon+2\mu\nu\delta)p+\mu\mu\gamma+\mu\nu\beta=0,$$

or

$$(2\mu\zeta+\nu\varepsilon)q+(u\varepsilon+2\nu\delta)p+\gamma\mu+\nu\beta=0,$$

from which equation, the position of the diameter *ig* is defined.

105 . The first diameter *IG* produced, the position of which was being determined by this equation $2\zeta z+\varepsilon x+\gamma=0$, may meet with the axis at *O*, and there will be $AO=\frac{-\gamma}{\varepsilon}$ and hence there becomes $PO=\frac{-\gamma}{\varepsilon}-x$, and the tangent of the angle will be $=\frac{z}{PO}=\frac{-\varepsilon z}{\varepsilon x+\gamma}=\frac{\varepsilon}{2\zeta}$, and the tangent of the angle *MLG,* under which the diameter *IG* may bisect the ordinate *MN* will be $=\frac{2\zeta}{\varepsilon}$. Truly the other diameter *ig* produced may run to meet the axis at *o* , and there will be $Ao=\frac{-\mu\gamma-\nu\beta}{\mu\varepsilon+2\nu\delta}$,
[from $(2\mu\zeta+\nu\varepsilon)q+(u\varepsilon+2\nu\delta)p+\gamma\mu+\nu\beta=0$, with $q=0$;]
and the tangent of the angle *Aol* will be $=\frac{\mu\varepsilon+2\nu\delta}{2\mu\zeta+\nu\varepsilon}$. Now since the tangent of the angle *AOL* shall become $=\frac{\varepsilon}{2\zeta}$, both diameters will intersect each other mutually at a certain point *C,* and making the angle $OCo=Aol-AOL$, therefore the tangent of which is

$$=\frac{4\nu\delta\zeta-\nu\varepsilon\varepsilon}{4\mu\zeta\zeta+2\nu\delta\varepsilon+2\nu\varepsilon\zeta+\mu\varepsilon\varepsilon}.$$

But the angle, which this other diameter includes may bisect its ordinate, is $Mlo=180^0-lpo-Aol$: therefore the tangent of this angle is

$$=\frac{2\mu\mu\zeta+2\mu\nu\varepsilon+2\nu\nu\delta}{\mu\mu\varepsilon+2\mu\nu\delta-2\mu\nu\zeta-\nu\nu\varepsilon}.$$

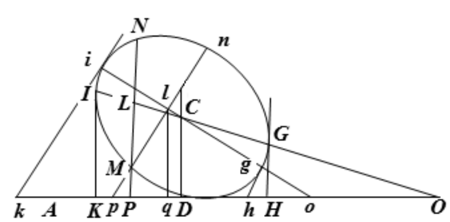


Fig. 25

106. Moreover we may inquire into the point *C*, where these two diameters mutually intersect each other : from which the perpendicular *CD* may be sent to the axis, and calling $AD=g$, $CD=h$; in the first place there will be $2\zeta h+\varepsilon g+\gamma=0$, because *C* stands on the diameter *IG.* Then, because *C* also is found on the diameter *ig*, there will be

$$(2\mu\zeta+\nu\varepsilon)h+(\mu\varepsilon+2\nu\delta)g+\mu\gamma+\nu\beta=0.$$

Hence the first equation multiplied by μ may be taken away from this equation, and there will remain

$$v\varepsilon h + 2v\delta g + v\beta = 0,\ \text{or}\ \ \varepsilon h + 2\delta g + \beta = 0.$$

From these arises $h = \frac{-\varepsilon g - \gamma}{2\zeta} = \frac{-2\delta g - \beta}{\varepsilon}$, and thus

$$\left(\varepsilon\varepsilon - 4\delta\zeta\right) g = 2\beta\zeta - \gamma\varepsilon,\ g = \frac{2\beta\zeta - \gamma\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}\ \ \&\ \ h = \frac{2\gamma\delta - \beta\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}.$$

In which determinations, since the quantities μ & v shall not be present on which the obliquity of the applied lines *pMn* depends, it is evident that the point *C* remains the same, in whatever manner the obliquity may be varied.

107. Therefore all the diameters *IG* & *ig* mutually cross each other at the same point *C* : because therefore if one were found, all the diameters will pass through that, and in turn all the diameters drawn passing through that will be diameters, which bisect all the ordinates drawn up to a certain angle. Therefore since this point shall be one of a kind in whatever line of the second order, and in that all the diameters will cross over each other, this point is accustomed to be called the CENTRE of the conic section . Therefore so that from the proposed equation between *x* and *y*

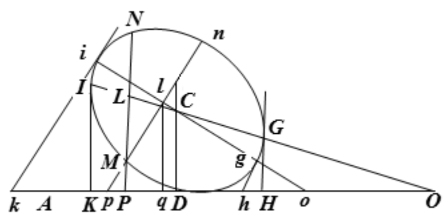


Fig. 25

$$0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy$$

thus it is found that by assuming $AD = \frac{2\beta\zeta - \gamma\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}$, there is found $CD = \frac{2\gamma\delta - \beta\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}$.

108 . Moreover above we have found that $AK + AH = \frac{4\beta\zeta - 2\gamma\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}$: But *IK* & *GH* are the perpendiculars from the ends of the diameter *IG* sent to the axis; from which there is seen to be $AD = \frac{AK + AH)}{2}$ and thus the point *D* will be the midpoint between the points *K* & *H.* On which account the centre *C* also will be placed in the mid-point of the diameter *IG*, because with whatever other diameter it may be prevail to equal, it follows that not only all the diameters mutually cross each other at the same point *C*, but also in turn they bisect each other.

109. Now we may assume any diameter *AI* for the axis to which the applied ordinates *MN* shall be up to the angle $APM = q$, the sine of which $= m$, & the cosine $= n$. Putting the abscissa $AP = x$ and the applied line $PM = y$, of which since there shall be two equal values, and the one or the other of these negative, the sum thus $= 0$, and the general equation for the line of the second order will change into this form $yy = \alpha + \beta x + \gamma xx$: which, if there is put $y = 0$, will give the points *G* & *I* on the axis, where this is crossed by the curve ; clearly the roots of the equation $xx + \frac{\beta}{\gamma}x + \frac{\alpha}{\gamma} = 0$ will be $x = AG$ & $x = AI$; and thus there will be had $AG + AI = \frac{-\beta}{\gamma}$ & $AG.AI = \frac{\alpha}{\gamma}$. Therefore since the centre *C* shall be placed in the middle of the diameter *GI*, the centre of the conic section *C* will be found easily. For there shall be $AC = \frac{AG + AI}{2} = \frac{-\beta}{2\gamma}$.

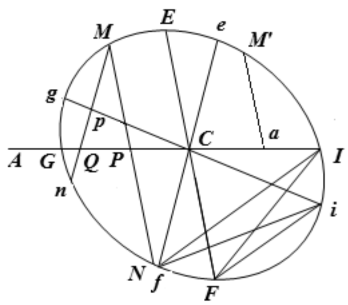


Fig. 26

110. Now with the centre of the conic section *C* known, it will be most convenient to take that for starting point of abscissas on the axis *AI*. Therefore $CP = t$ may be put in place, because $PM = y$ remains ; on account of $x = AC - CP = \frac{-\beta}{2\gamma} - t$, this equation will be produced between the coordinates *t* & *y* :

$$yy = \alpha - \frac{\beta\beta}{2\gamma} + \frac{\beta\beta}{4\gamma} - \beta t + \beta t + \gamma tt$$

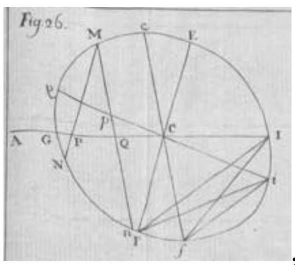
or

$$yy = \alpha - \frac{\beta\beta}{4\gamma} + \gamma tt.$$

Therefore by putting *x* in place of *t*, the general equation will be had for lines of the second order, with any line taken for the axis, and the centre for the beginning of the abscissas, which, with the form of the constants changed, will be $yy = \alpha - \beta xx$.

Therefore on putting $y = 0$ there becomes $CG = CI = \sqrt{\frac{\alpha}{\beta}}$; and thus the whole diameter *GI* will be $= 2\sqrt{\frac{\alpha}{\beta}}$.

111. Put $x=0$, and the ordinate passing through the centre *EF* will be found : evidently there becomes $CE=CF=\sqrt{\alpha}$; and thus the whole ordinate $EF=2\sqrt{\alpha}$, which, because it shall pass through the centre, equally will be a diameter, making the angle $ECG=q$ with that line *GI.* But this other diameter *EF* will cut equally all the prior parallel ordinates *GI*; for with the abscissa *AP* made negative, an applied line *aM* falling towards *I* will remain equal to the first line *PM*;
[This is not shown on the original Fig. 26 ; the French and German translations both show versions of this diagram differing from the original, which evidently is in error at this stage : I show the three diagrams taken from the original books, with the original Latin version first on the left, Labey's French translation (1796), and the German one by Michelson (1788) ; this translation uses Labey's version for Fig. 26 :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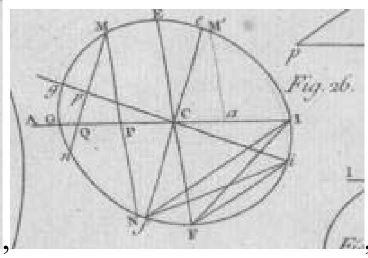
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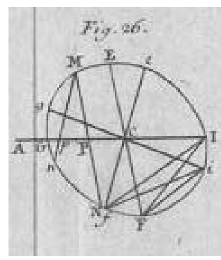

.]

&, since it shall be parallel to the same, both the points joined to *M* will give a line parallel to the diameter *GI*, and thus bisected by the diameter *EF*. Therefore both these diameters *GI* & *EF* thus are disposed between themselves, so that the one will bisect all the parallel ordinates of the other, which on account of the reciprocal property these two diameters are called CONJUGATE between themselves. Therefore if other right lines are drawn from the ends *G* & *I* of the *GI* parallel to the other diameter *EF*, these are tangents to the curved line, and in a like manner right lines drawn parallel through *E* & *F* parallel to the diameter *GI,* these will touch the curve at the points *E* & *F.*

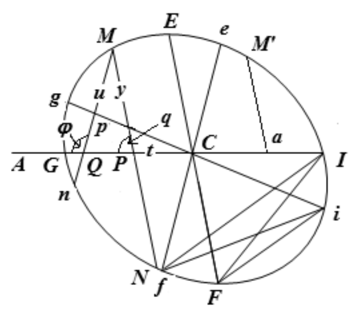


Fig. 26

112 . Now some oblique angled applied line *MQ* may be drawn ; and the angle $AQM=\phi$, the sine of which $=\mu$ and the cosine $=v$. Putting the abscissa $CQ=t$, and the applied line $MQ=u$, and in the triangle *PMQ* on account of the angle $PMQ=\phi-q$, therefore $\sin .PMQ.=\mu n-vm$, $y:u:PQ=\mu:m \ : \ \mu n-vm$, and hence $y=\frac{\mu u}{m}$ & $PQ=\frac{(\mu n-vm)u}{m}$, from which $x=t-PQ=t-\frac{(\mu n-vm)u}{m}$. These values may be substituted into the above equations, $yy=\alpha-\beta xx$ or $yy+\beta xx-\alpha=0$, and there will arise

$$\left(\mu\mu+\beta(\mu n-\nu m)^2\right)uu-2\beta m(\mu n-\nu m)tu+\beta m^2tt-\alpha m^2=0,$$

from which the applied line u obtains these two values QM & $-Qn$ and there will be

$$QM-Qn=\frac{2\beta m(\mu n-\nu m)t}{\mu\mu+\beta(\mu n-\nu m)^2}.$$

The ordinate Mn may be bisected at p, and the line Cpg will be a new diameter cutting all the ordinates parallel to Mn itself into two parts, and there will be

$$Qp=\frac{\beta m(\mu n-\nu m)t}{\mu\mu+\beta(\mu n-\nu m)^2}.$$

113. Hence moreover the tangent of this angle GCg may be found

$$=\frac{\mu.Qp}{CQ+\nu.Qp},\quad \text{or}\quad \text{tang.}GCg=\frac{\beta m(\mu n-\nu m)}{\mu+n\beta(\mu n-\nu m)}$$

$$\&\ \text{tang. } Mpg=\frac{\mu.CQ}{pQ+\nu.CQ}=\frac{\mu\mu+\beta(\mu n-\nu m)^2}{\mu\nu+\beta(\mu n-\nu m)(\nu n+\mu m)},$$

which is the angle included by which the new ordinate Mn may be bisected by a diameter.
Again truly there will be

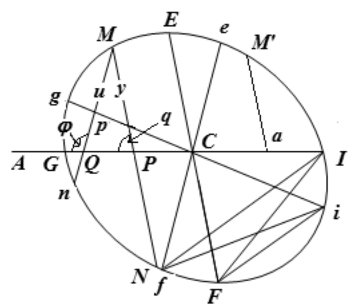


Fig. 26'

$$Cp^2=CQ^2+Qp^2+2\nu\cdot CQ\cdot Qp$$

$$=\frac{\mu^4+2\beta\mu^3n(\mu n-\nu m)+\beta\beta\mu\mu(\mu n-\nu m)^2}{\left(\mu\mu+\beta(\mu n-\nu m)^2\right)^2}tt:$$

and thus

$$Cp=\frac{\mu t\sqrt{\left(\mu^2+2\beta\mu n(\mu n-\nu m)+\beta\beta(\mu n-\nu m)^2\right)}}{\mu\mu+\beta(\mu n-\nu m)^2}.$$

Putting $Cp=r$, & $pM=s$, and there becomes

$$t=\frac{\left(\mu\mu+\beta\left(\mu n-\nu m\right)^{2}\right)r}{\mu\sqrt{\left(\mu^{2}+2\beta\mu n\left(\mu n-\nu m\right)+\beta\beta\left(\mu n-\nu m\right)^{2}\right)}},$$

$$\&\ u=s+Qp=s+\frac{\beta m\left(\mu n-\nu m\right)r}{\mu\sqrt{\left(\mu^{2}+2\beta\mu n\left(\mu n-\nu m\right)+\beta\beta\left(\mu n-\nu m\right)^{2}\right)}},$$

which values give again ,

$$y=\frac{\mu s}{m}+\frac{\beta\left(\mu n-\nu m\right)r}{\sqrt{(\cdots)}}$$

$$x=-\frac{\left(\mu n-\nu m\right)s}{m}+\frac{\mu r}{\sqrt{(\cdots)}},$$

from which equation $yy+\beta xx-\alpha$ there will arise

$$\frac{\mu\mu+\beta\left(\mu n-\nu m\right)^{2}ss}{mm}+\frac{\beta\left(\mu\mu+\beta\left(\mu n-\nu m\right)^{2}\right)rr}{\mu\mu+2\beta\mu n\left(\mu n-\nu m\right)+\beta\beta\left(\mu n-\nu m\right)^{2}}-\alpha=0.$$

114. We may call now the semidiameter $CG=f$ and the semiconjugate $CE=CF=g$, then there will be

$f=\sqrt{\frac{\alpha}{\beta}}$ & $g=\sqrt{\alpha}$, or

$\alpha=gg$ & $\beta=\frac{gg}{ff}$: from which there becomes

$yy+\frac{ggxx}{ff}=gg.$ Again we may put the angle $GCg=p$, and there will be

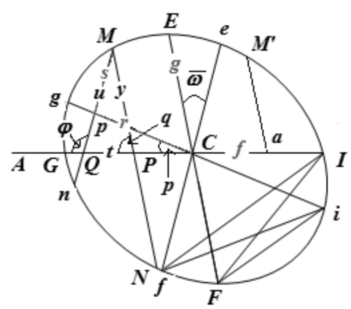


Fig. 26'

$$\text{tang.}p=\frac{\beta m\left(\mu n-\nu m\right)}{\mu+n\beta\left(\mu n-\nu m\right)}.$$

But, on account of the angle $GCE=q$, putting the angle $ECe=\varpi$, there comes about $AQM=\phi=q+\varpi$; and thus

$$\mu=\sin.\left(q+\varpi\right);\ \nu=\cos.\left(q+\varpi\right),\ m=\sin.q\ \ \&\ \ n=\cos.q.$$

[Clearly this book was produced in the days before π became sacrosanct as we know it; thus, in this translation, the symbol ϖ is used for the angle ; this symbol has also become confused in the printing with π in a random manner in the early editions of the book, and π is used in the O.O. edition.]

Therefore

$$\text{tang}.p = \frac{\beta\sin.q\cdot\sin.\varpi}{\sin.(q+\varpi)+\beta\cos.q\cdot\sin.\varpi} = \frac{\beta\text{tang}.q\cdot\text{tang}.\varpi}{\text{tang}.q+\text{tang}.\varpi+\beta\text{tang}.\varpi},\ \&$$

$$\sin.p = \frac{\beta\sin.q\cdot\sin.\varpi}{\sqrt{\left(\mu^2+2\beta\mu n(\mu n-vm)+\beta\beta(\mu n-vm)^2\right)}},$$

and

$$\mu\mu+\beta(\mu n-vm)^2 = \left(\sin.(q+\varpi)\right)^2+\beta(\sin.\varpi)^2,$$

with which values called in aid, this equation will be produced between *r* & *s* [in §113],

$$\frac{\left(\left(\sin.(q+\varpi)\right)^2+\beta(\sin.\varpi)^2\right)ss}{(\sin.q)^2}+\frac{\beta\left(\left(\sin.(q+\varpi)\right)^2+\beta(\sin.\varpi)^2\right)rr(\sin.p)^2}{\beta\beta(\sin.q)^2(\sin.\varpi)^2}-\alpha=0\ ;$$

But there is

$$\beta = \frac{\text{tang}.p\sin.(q+\varpi)}{(\sin.q-\cos.q\cdot\text{tang}.p)\cdot\sin.\varpi} = \frac{\text{tang}.p(\text{tang}.q+\text{tang}.\varpi)}{\text{tang}.\varpi(\text{tang}.q-\text{tang}.p)}$$

$$=\frac{gg}{ff}=\frac{\cot.\varpi\cdot\text{tang}.q+1}{\cot.p\cdot\text{tang}.q-1},$$

or,

$$\text{tang}.q = \frac{ff+gg}{gg\cdot\cot.p-f\cdot f\cot.\varpi},$$

from which many corollaries may be deduced. Indeed there will be

$$\frac{gg}{ff}=\frac{\sin.p\cdot\sin.(q+\varpi)}{\sin.\varpi\cdot\sin.(q-p)}.$$

[There is considerable confusion between the use of ϖ & π in naming the angle ECe in the equations derived in the versions of the book considered; here we have used ϖ exclusively.]

115. Let the semidiameter be $Cg = a$, of which the conjugate semidiameter shall be $Ce = b$; from the equation found before there will be :

$$a = \frac{\sin.q \cdot \sin.\varpi \cdot \sqrt{\alpha}\beta}{\sin.p \cdot \left(\left(\sin.\overline{q+\varpi} \right)^2 + \beta \left(\sin.\varpi \right)^2 \right)} = \frac{gg \cdot \sin.q \cdot \sin.\varpi}{\sin.p \cdot \left(ff \left(\sin.\overline{q+\varpi} \right)^2 + g^2 \left(\sin.\varpi \right)^2 \right)}$$

and

$$b = \frac{fg \cdot \sin.q}{\sqrt{\left(ff \left(\sin.\overline{q+\varpi} \right)^2 + gg \left(\sin.\varpi \right)^2 \right)}},$$

hence there will be $a : b = g \cdot \sin.\varpi : f \cdot \sin.p.$ Again indeed there is :

$$\left(\sin.(q+\varpi) \right)^2 + \frac{gg}{ff} \left(\sin.\varpi \right)^2 = \frac{\sin.(q+\varpi)}{\sin.(q-p)} \left(\sin.(q-p) \cdot \sin.(q+\varpi) + \sin.p \cdot \sin\varpi \right)$$

$$= \frac{\sin.q \cdot \sin.(q+\pi) \cdot \sin.(q+\pi-p)}{\sin.(q-p)},$$

from which there becomes

$$a = \frac{gg \cdot \sin.\varpi}{f \cdot \sin.pf} \sqrt{\frac{\sin.q \cdot \sin.(q-p)}{\sin.(q+\varpi) \sin.(q+\varpi-p)}};$$

or, on account of

$$\frac{gg}{ff} = \frac{\sin.p \cdot \sin.(q+\varpi)}{\sin.\varpi \cdot \sin.(q-p)},$$

there will be

$$a = f \sqrt{\frac{\sin.q \cdot \sin.(q+\varpi)}{\sin.(q-p) \cdot \sin.(q+\varpi-p)}} \;\&\; b = g \sqrt{\frac{\sin.q \cdot \sin.(q-p)}{\sin.(q+\varpi) \cdot \sin.(q+\varpi-p)}},$$

hence there will be

$$a:b=f\cdot\sin.(q+\varpi):g\cdot\sin.(q-p)\ \ \&\ \ ab=\frac{fg\cdot\sin.q}{\sin.(q+\varpi-p)}.$$

116. Therefore if in a conic section two pairs of conjugate diameters may be had : *GI, EF* & *gi* , *ef* ; initially there will be [from above, $a:b=g\cdot\sin.\varpi:f\cdot\sin.p.$] :

$$Cg:Ce=CG\cdot\sin.\ ECe:CG\cdot\sin.GCg.$$

Therefore

$$\sin.GCg:\sin.\ ECe=CE\cdot Ce:CG\cdot Cg.$$

and if the chords *Ee* & *Gg* may be drawn , hence there becomes triangle *CGg* = triangle *CEe*. Then there will be

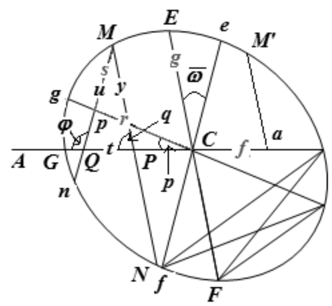


Fig. 26'

$$Cg:Ce=CG\cdot\sin.GCe:CE\cdot\sin.gCE,$$

or

$$Ce\cdot CG\cdot\sin.GCe=CE\cdot Cg\cdot\sin.gCE:$$

from which, if the chords *Ge* & *gE* may be drawn, the triangle *GCe* & g*CE* will be equal to each other [*i.e.* in area], or from the neighbouring triangles, the triangle *ICf* = Triangle *iCF*. Finally indeed the equation

$$ab\cdot sin.(\ q+\varpi-p\ )=fg\cdot\sin.q$$

will give

$$Cg\cdot Ce\cdot\sin.gCe=CG\cdot CE\cdot\sin.GCE.$$

Because if the chords *EG* & *eg* therefore may be drawn, or equally from the region *FI* & *fi* likewise will be equal triangles *ICF* & *iCf* : from which it follows that all the parallelograms, which may be described around the two conjugate diameters, are equal to each other.

117. Clearly there will be found three pairs of triangle equal to each other:

I. Triangle *FCf* is equal to triangle *ICi.*
II. Triangle *fCI* is equal to triangle *FCi.*
III.Triangle *FCI* is equal to triangle *fCi.*

From which it follows that the trapeziums *FfCI* & *iICf* are equal to each other ; from which if the same triangle *fCI* may be taken away, there will be triangle Fif = triangle Ifi: which since above they shall have the same base in place fI, it is necessary that the chord *Fi* be parallel to the chord *fI*. And thus again there will be triangle FIi = triangle ifF, to which if the equal triangle *FCI* & *fCi* may be added, there will also be these equal trapeziums $FCIi = iCfF$.

118. Hence also the method is deduced for drawing a tangent *MT* to whatever point M of a line of the second order. For with the diameter *GI* taken for the axis, [Fig. 27] for which *EC* shall be the half conjugate, from the point *M, MP* may be drawn parallel to the axis C*E*, which will be the semi-ordinate, and $PN = PM$. Draw *CM,* which will be a semidiameter, of which the conjugate semidiameter *CK* is sought, to which the tangent *MT* will be parallel. Let the angle *GCE*= *q;* $GCM = p$ & $ECK = \varpi$; there will be, as we have seen; [as in § 115 :

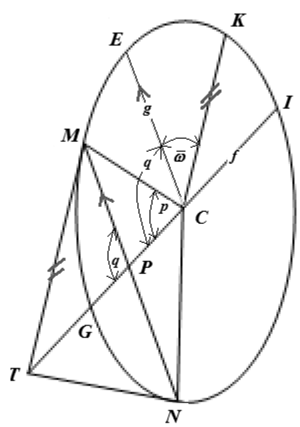


Fig. 27

$$\frac{gg}{ff} = \frac{\sin.p \cdot \sin.(q+\varpi)}{\sin.\varpi \cdot \sin.(q-p)},]$$

$$\frac{EC^2}{GC^2} = \frac{\sin.p \cdot \sin.(q+\varpi)}{\sin.\varpi \cdot \sin.(q-p)}$$

and

$$MC = CG\sqrt{\frac{\sin.p \cdot \sin.(q+\varpi)}{\sin.(q-p) \cdot \sin.(q+\varpi-p)}}$$

But in triangle *CMP* there is

$$MC^2 = CP^2 + MP^2 + 2PM \cdot CP \cdot cos.q$$

and

$$MP : MC = sin.p : sin.q$$

and also

$$MP : CP = sin.p : sin.(q-p)$$

Then in triangle CMT, on account of the given angles, there will be

$$CM : CT : MT = sin.(q+\varpi) : \sin.(q+\varpi - p) : sin.p\,.$$

Hence, with the angles eliminated, there will be $MC = CG\sqrt{\frac{MC \cdot CM}{CP \cdot CT}}$, or $CG^2 = CP.CT$. Hence there will be $CP : CG = CG : CT$, from which the position of the tangent is quickly found. But from this *on taking* , $CP : PG = CG : TG$; and on account of $CG = CI$ *on adding,* $CP : IP = CG : TI$.

119. Since there shall be

$$\frac{CE^2}{CG^2} = \frac{\sin.p \cdot \sin.(q+\varpi)}{\sin.\varpi \cdot \sin.(q-p)}; \quad \frac{CK^2}{CM^2} = \frac{\sin.p \cdot \sin.(q-p)}{\sin.\varpi \cdot \sin.(q+\varpi)};$$

likewise

$$\frac{CM^2}{CG^2} = \frac{\sin.q \cdot \sin.(q+\varpi)}{\sin.(q-p)\cdot \sin.(q+\varpi-p)} \;\&\; \frac{CK^2}{CE^2} = \frac{\sin.q \cdot \sin.(q-p)}{\sin.(q+\varpi)\cdot \sin.(q+\varpi-p)},$$

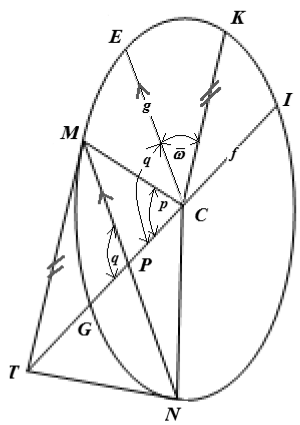


Fig. 27

there will be

$$\frac{CE^2 + CG^2}{CG^2} = \frac{\sin.p \cdot \sin.(q+\varpi) + \sin.\varpi \cdot \sin.(q-p)}{\sin.\varpi \cdot \sin.(q-p)}, \;\&$$

$$\frac{CK^2 + CM^2}{CM^2} = \frac{\sin.p \cdot \sin.(q-p) + \sin.\pi \cdot \sin.(q+\varpi)}{\sin.\varpi \cdot \sin.(q+\pi)}.$$

But there is :

$$\sin.A \cdot \sin.B = \frac{1}{2}\cos.(A-B) - \frac{1}{2}\cos.(A+B),$$

& in turn

$$\frac{1}{2}\cos.A - \frac{1}{2}\cos.B = \sin.\frac{A+B}{2} \cdot \sin.\frac{B-A}{2}.$$

From which there becomes

$$\sin.p \cdot \sin.(q+\varpi) + \sin.\varpi \cdot \sin.(q-p) = \frac{1}{2}\cos.(q+\varpi-p) - \frac{1}{2}\cos.(q+\varpi+p)$$
$$+\frac{1}{2}\cos.(q-\varpi-p) - \frac{1}{2}\cos.(q+\varpi-p)$$
$$= \frac{1}{2}\cos.(q-\varpi-p) - \frac{1}{2}\cos.(q+\varpi+p) = \sin.q \cdot \sin.(p+\varpi).$$

And

$$\sin.p\cdot\sin.(q-p)+\sin.\varpi\cdot\sin.(q+\varpi)=\frac{1}{2}\cos.(q-2p)-\frac{1}{2}\cos.q$$
$$+\frac{1}{2}\cos.q-\frac{1}{2}\cos.(q+2\varpi)$$
$$=\frac{1}{2}\cos.(q-2p)-\frac{1}{2}\cos.(q+2\varpi)=\sin.(q+\varpi-p)\cdot\sin.(p+\varpi).$$

Hence therefore there will be

$$\frac{CE^2+CG^2}{CG^2}=\frac{\sin.q\cdot\sin.(p+\varpi)}{\sin.\varpi\cdot\sin.(q-p)},\ \&$$

$$\frac{CK^2+CM^2}{CM^2}=\frac{\sin.(q+\varpi-p)\cdot\sin.(p+\varpi)}{\sin.\varpi\cdot\sin.(q+\varpi)}.$$

from which there is made

$$\frac{CE^2+CG^2}{CK^2+CM^2}=\frac{CG^2}{CM^2}\cdot\frac{\sin.q\cdot\sin.(q+\varpi)}{\sin.(q-p)\cdot\sin.(q+\varpi-p)}=\frac{CG^2}{CM^2}\cdot\frac{CM^2}{CG^2}.$$

Whereby there will be $CE^2+CG^2=CK^2+CM^2$, and thus the sum of the squares of the two conjugate diameters is always constant in the same line of the second order.

120 . Therefore since two conjugate semidiameters *CG & CE* may be given, with a semidiameter *CM* assumed as it pleases, the conjugate semidiameter *CK* is found at once, by taking $CK=\sqrt{(CE^2+CG^2-CM^2)}$. Therefore, from the above properties of conic sections, there will be

$$TG\cdot TI:TM^2=CG\cdot CI:CK^2=CG^2:CK^2=CG^2:CE^2+CG^2-CM^2;$$

and thus

$$TM^2=CG\sqrt{\left(\frac{CE^2+CG^2-CM^2}{TG\cdot TI}\right)}$$

In a similar manner, if the tangent *NT* may be drawn with the ordinate *MN* produced, both the tangents *MT* and *NT* will pass through the same point *T* for the axis *TI*. Indeed for each there will be $CP:CG=CG:CT$. But truly with the right line *CN* drawn there will be

$$TN = \frac{1}{CG}\sqrt{\left(TG \cdot TI\left(CE^2 + CG^2 - CN^2\right)\right)}$$

and thus

$$TM^2 : TN^2 = CE^2 + CG^2 - CM^2 : CE^2 + CG^2 - CN^2$$

Truly on this account *MN* will be bisected in *P*

$$\sin.CTM : \sin.CTN = TN : TM = \sqrt{\left(CE^2 + CG^2 - CN^2\right)} : \sqrt{\left(CE^2 + CG^2 - CM^2\right)}$$

121. The tangents *AK* and *BL* may be drawn (Fig. 28) from the ends of the diameter *A* and *B* and the tangent may be produced touching some tangent *MT,* then it will cut each tangent in the points *K* and *L.* Let *ECF* be the conjugate diameter, to which both the applied line *MP* as well as the tangents *AK* and *BL* are parallel. Now since from the nature of the tangent, there shall be :

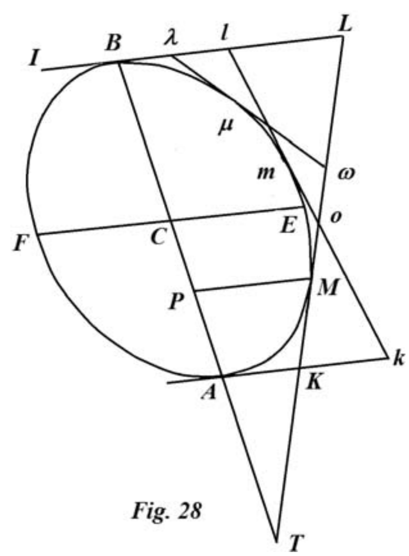


Fig. 28

$$CP : CA = CA : CT,$$

on account of $CB = CA$ there will be

$$CP : AP = CA : AT \text{ and } CP : BP = CA : BT,$$

$$[\text{for } CP : CA \pm CP = CA : CT \pm CA;]$$

therefore

$$CP : CA = CA : CT = AP : AT = BP : BT$$

and hence $AT : BT = AP : BP$. But there is $AT : BT = AK : BL$, therefore

$$AK : BL = AP : BP.$$

Then there is

$$AT = \frac{CA \cdot AP}{CP}, \quad BT = \frac{CA \cdot BP}{CP}$$

and

$$PT = \frac{CA \cdot AP}{CP} + AP = \frac{AP \cdot BP}{CP},$$

therefore

$$AT : PT = CA : BP = AK : PM \;;$$

and in a similar manner there will be

$$BT : PT = CA : AP = BL : PM \;;$$

from which there becomes

$$AK = \frac{CA \cdot PM}{BP}, \quad BL = \frac{CA \cdot PM}{AP}$$

and

$$AK \cdot BL = \frac{CA^2 \cdot PM^2}{AP \cdot BP}.$$

But there is $AP \cdot BP : PM^2 = AC^2 : CE^2$, from which that singular property

$$AK \cdot BL = CE^2$$

follows, and from which again there becomes

$$AK = CE\sqrt{\frac{AP}{BP}} \quad \text{and} \quad BL = CE\sqrt{\frac{BP}{AP}},$$

$$AP : BP = AK^2 : CE^2 = CE^2 : BL^2 = KM : ML,$$

and

$$AK : BL = KM : LM .$$

122. Therefore at any point *M* of a curve the tangent may be drawn crossing the parallel tangents *AK* and *BL* at *K* and *L,* and the semidiameter *CE* will always be the mean proportional between *AK* and *BL,* or there shall be $CE^2 = AK.BL$. But if therefore at some other point *m* of the curve in a similar manner the tangent *kml* may be drawn*,* there will be also $CE^2 = Ak.Bl$ and

$$AK : Ak = Bl : BL$$

and hence there will be also $AK : Kk = Bl : Ll$. The tangents *KL* and *kl* cross each other at *o,* and there will be

$$AK : Bl = Ak : BL = Kk : Ll = ko : lo = Ko : Lo .$$

And these are the principle properties of conic sections, by which Newton solved the most outstanding problems in the *Principia*. [Lemma 24, Cor. 1.]

123. Since there shall be $AK : Bl = Ko : Lo$, if the tangent *LB* may be produced to *I,* so that there shall be $BI = AK$, *I* will be the point, where the tangent on the other side parallel to *KL* shall itself cut this tangent *LB*, just as *K* is to a point on *LK*, where that may be cut by the tangent *AK* parallel to *BL* itself. Therefore the right line *IK* will be cut into two parts by the centre *C*. But if therefore any two tangents *BL, ML* may be produced to *I* and *K* in the manner prescribed and these may be cut by a third tangent *lmo* at the points *l* and *o*, there will be $BI : Bl = Ko : Lo$ and *by adding* [within the ratios] $IB : Il = Ko : KL$; therefore wherever the third tangent *lmo* may be drawn, there will be always $IB \cdot KL = Il.Ko$. Therefore with some fourth tangent drawn $\lambda\mu\omega$ on cutting the two first assumed *IL* and *KL* in λ and ω, equally there will be

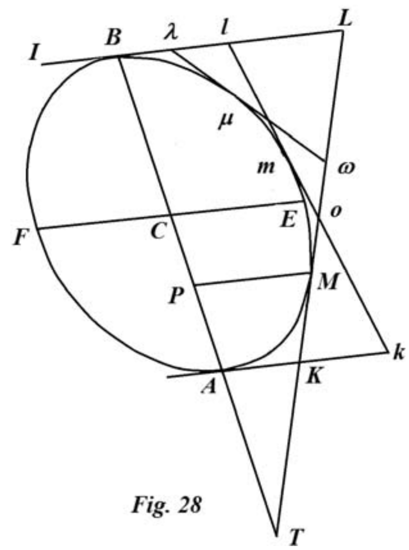


Fig. 28

$$IB \cdot KL = I\lambda \cdot K\omega$$

and thus $Il \cdot Ko = I\lambda.K\omega$ or $Il : I\lambda = K\omega : Ko$. Therefore with the right lines drawn $l\omega$, λo, in the ratio in which these may be cut, the right line passing through the point of division, will cut the right line *IK* in the same ratio. Whereby, if the right lines $l\omega$ and λo may be bisected, the right line through the point of bisection will bisect the right line *IK* also and thus will pass through the centre of the conic section *C* .

124. It will be shown from geometry in this manner, that if the right line *nmH* (see Fig. 30), which the right lines $l\omega$, λo cut in a given ratio, if indeed it were $Il : I\lambda = K\omega : Ko$ or $I\lambda : \lambda l = Ko : o\omega$, must cut the right line *KI* in the same ratio. The right line *mn* cuts each line $l\omega$ and λo in the given ratio $m : n$, or there shall be $\lambda m : mo = ln : n\omega = m : n$,

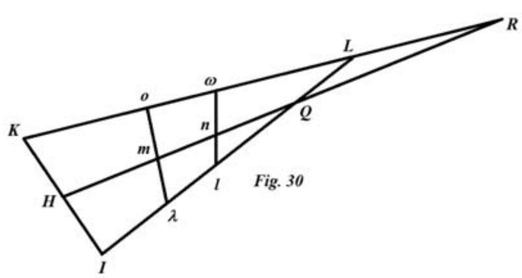


Fig. 30

and that right line produced cuts the tangents *IL* and *KL* in *Q* and *R* ; and there will be

$$\sin.Q:\sin.R=\frac{ln}{Ql}:\frac{n\omega}{R\omega}=\frac{\lambda m}{Ql}:\frac{mo}{Ro}=\frac{m}{Ql}:\frac{n}{R\omega},$$

therefore $Ql:R\omega=Q\lambda:Ro$ and on taking from the ratio [*dividendo*]

$$l\lambda:o\omega=Q\lambda:Ro=Ql:R\omega.$$

Indeed since there shall be $l\lambda:o\omega=I\lambda:Ko$, there will be also

$$QI:RK=l\lambda:o\omega \text{ and } \sin.Q:\sin.R=\frac{m}{l\lambda}:\frac{n}{o\omega}.$$

But also there is :

$$\sin.Q:\sin.R=\frac{HI}{QI}:\frac{HK}{KR}=\frac{HI}{l\lambda}:\frac{HK}{o\omega},$$

from which there becomes

$$HI:HK=m:n=\lambda m:mo=ln:n\omega.$$

125. With the two conjugate axes given (Fig. 27) *CG* and *CE,* which contain the oblique angle *GCE* = *q* between each other, two other conjugate semi-diameters *CM* and *CK* can be found always, which may constitute a right angle *MCK*. Let the angle $GCM=p$, and on putting $ECK=\varpi$ there will be $q+\varpi-p=90°$ and thus

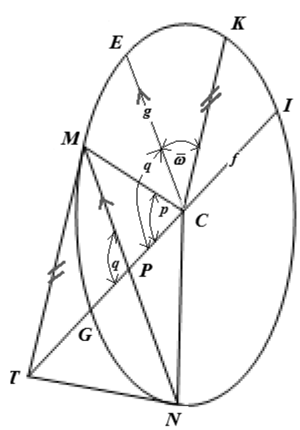


Fig. 27

$$\sin.\varpi=\cos.(q-p) \text{ and } \sin.(q+\varpi)=\cos.p.$$

From which (from § 119) there will be

$$\frac{CE^2}{CG^2}=\frac{\sin.p\cdot\cos.p}{\sin.(q-p)\cdot\cos.(q-p)}=\frac{\sin.2p}{\sin.2(q-p)}=\frac{\sin.2p}{\sin.2q\cdot\cos.2p-\cos.2q\cdot\sin.2p};$$

therefore

$$\frac{CG^2}{CE^2}=\sin.2q\cdot\cot.2p-\cos.2q,$$

and from which there becomes

$$\cot.2GCM = \cot.2q + \frac{CG^2}{CE^2 \cdot \sin.2q},$$

which equation always presents a possible solution. Indeed there will be

$$\frac{CM^2}{CG^2} = \frac{\sin.q \cdot \cos.p}{\sin.(q-p)} \text{ and } \frac{CG^2}{CM^2} = 1 - \frac{\text{tang}.p}{\text{tang}.q}$$

from which

$$\text{tang}.p = \text{tang}.q - \frac{CG^2}{CM^2}\text{tang}.q.$$

But since there shall be

$$CM^2 + CK^2 = CG^2 + CE^2 \text{ and } CK \cdot CM = CG \cdot CE \cdot \sin.q,$$

there will be

$$CM + CK = \sqrt{\left(CG^2 + 2CG \cdot CE \cdot \sin.q + CE^2\right)}$$

and

$$CM - CK = \sqrt{\left(CG^2 - 2CG \cdot CE \cdot \sin.q + CE^2\right)},$$

from which the orthogonal conjugate diameters themselves are found.

126. Therefore (Fig. 29) *CA* and *CE* shall be both the orthogonal conjugate semidiameters of a conic section, which are accustomed to be called the *principal diameters,* crossing each other normally at the centre *C*. Let the abscissa $CP = x$, the applied line $PM = y$, and there will be, as we have seen, $yy = \alpha - \beta xx$, but with the principal semidiameters called $AC = a$, $CE = b$

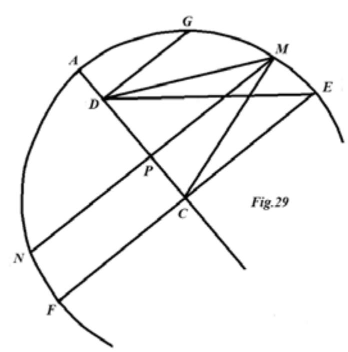


Fig.29

there will be $\alpha = bb$ and $\beta = \frac{bb}{aa}$, from which there becomes $yy = bb - \frac{bbxx}{aa}$.

[Thus eventually, after 27 at times grueling pages, Euler has given us the standard formula for the ellipse.]

From which equation it is understood, since it will not be changed, x and y may be taken either positive or negative, and the curve to have four similar and equal parts each placed around the diameters *AC* and *EF*. Clearly the quadrant *ACE* is similar and equal to the quadrant *ACF,* and with these, two like parts are put in place according to the other part of the diameter *EF*.

127. If from the centre C, which we have taken for the start of the abscissas, we may draw the right line CM, that will be

$$=\sqrt{(xx+yy)}=\sqrt{\left(bb-\frac{bbxx}{aa}+xx\right)},$$

from which it is understood, if there were $b=a$ or $CE=CA$, to become $CM=\sqrt{bb}=b=a$. Therefore in this case all the right lines produced from the centre C to the curve are equal to each other ; which since it shall be a property of the circle, it is evident that the conic section, of which the two conjugate principal diameters are equal to each other, is a circle, thus its equation between the orthogonal coordinates, putting $CP=x$ and $PM=y$, will be $yy=aa-xx$, with the radius of the circle being $CA=a$.

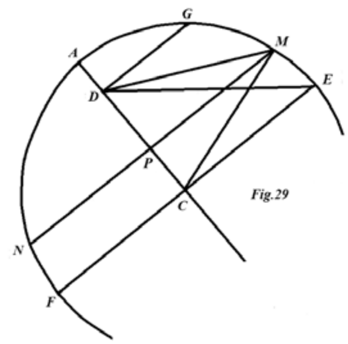


Fig.29

128. But if it was not the case that $b=a$, the right line CM on no account can be expressed rationally by x. But there will be given another point D on the axis, from which all the right lines drawn to the curve DM can be expressed rationally ; towards which arising there may be put $OD=f$, and on account of $DP=f-x$ there will be

$$DM^2=ff-2fx+xx+bb-\frac{bbxx}{aa}=bb+ff-2fx+\frac{(aa-bb)xx}{aa},$$

from which expression a square arises, if there were

$$ff=\frac{(aa-bb)\ (bb+ff)}{aa} \quad \text{or} \quad 0=aa-bb-ff\ ,$$

from which there becomes

$$f=\pm\sqrt{(aa-bb)},$$

therefore a point of this kind will give twin points on the axis AC, evidently each at a distance $CD=\pm\sqrt{(aa-bb)}$ from the centre. But then there will be

$$DM^2=aa-2x\sqrt{(aa-bb)}+\frac{(aa-bb)xx}{aa},$$

and hence

$$DM = a - \frac{x\sqrt{(aa-bb)}}{a} = AC - \frac{CD \cdot CP}{AC}.$$

Making $CP = 0$ there becomes $DM = DE = a = AC$, but with the abscissas $CP = CD$ or $x = \sqrt{(aa-bb)}$, the right line *DM* will be changed into the applied line *DG* and therefore there will be

$$DG = \frac{bb}{a} = \frac{CE^2}{AC}$$

or *DG* becomes the third proportional to *AC* and *CE*.

129. On account of this remarkable property, which the points *D* defined in this manner are able to use, generally such points of principal diameters are worthy of attention ; moreover these same aforementioned points have other outstanding properties, on account of which they have been given special names. Truly these points are called the *foci* or *umbilici* of a conic section ; and since they shall be placed on the greater diameter *a*, that diameter thus is distinguished from the conjugate *b*, so that the former may be called the *principal* or *transverse axis,* while the other *b* is called its conjugate axis. Indeed the applied line *DG* erected through either focus has been called the *semiparameter*, for the whole *parameter* is the ordinate at *D,* or *DG* taken twice, which also is called the *latus rectum*. Therefore the conjugate semiaxis *CE* is the mean proportional between the semiparameter *DG* and the transverse semiaxes *AC.* Again the transverse ends of the axis, where it is intersected by the curve, are called the *vertices,* as *A* ; and they have that property, so that in these places the tangent to the curve shall be normal to the principal axis *AC*.

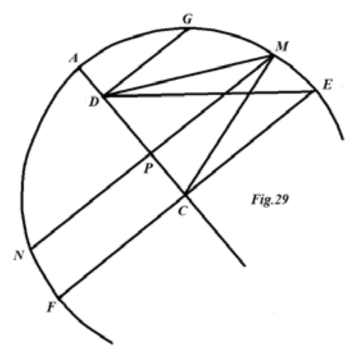


Fig.29

130. Putting the semiparameter $DG = c$ and the distance of the focus from the vertex $AD = d$, there will be

$$CD = a - d = \sqrt{(aa-bb)} \text{ and } DG = \frac{bb}{a} = c,$$

from which there becomes

$$bb = ac \text{ and } a - d = \sqrt{(aa-ac)};$$

therefore

$$ac = 2ad - dd, \quad a = \frac{dd}{2d-c} \text{ and, } b = d\sqrt{\frac{c}{2d-c}}.$$

Therefore with the given distance of the focus from the vertex $AD = d$ and with the semi-latus rectum $DG = c$, the conic section may be determined. Now on putting $CP = x$ there will be

$$DM = a - \frac{(a-d)x}{a} = \frac{dd}{2d-c} - \frac{(c-d)x}{d}.$$

Let $DP = t$, there will be

$$x = CD - t = \frac{(c-d)d}{2d-c} - t;$$

from which there becomes

$$DM = c + \frac{(c-d)t}{d}.$$

The angle may be called $ADM = \upsilon$, then there will be

$$\frac{t}{DM} = -\cos.\upsilon$$

and thus

$$d \cdot DM = cd + (d-c)DM \cdot \cos.\upsilon$$

and

$$DM = \frac{cd}{d-(d-c)\cdot\cos.\upsilon}, \text{ and at last } \cos.\upsilon = \frac{d(DM - DG)}{(d-c)DM}.$$

CAPUT V

DE LINEIS SECUNDI ORDINIS

85. Quia in linearum ordine primo sola linea recta continetur, cuius indoles iam satis ex Geometria elementari constat, *lineae secundi ordinis* aliquanto diligentius contemplemur, quod eae inter omnes lineas curvas sint simplicissimae atque per totam Geometriam sublimiorem usum habeant amplissimum. Praeditae autem sunt istae lineae, quae etiam *sectiones conicae* vocantur, plurimis insignibus proprietatibus, quas cum antiquissimi Geometrae eruerunt, tum recentiores amplificarunt. Harumque proprietatum cognitio adeo necessaria iudicatur, ut a plerisque auctoribus statim post Geometriam elementaremm
explicari soleant. Quoniam vero istae proprietates omnes non ex uno principio derivari possunt, sed alias aequatio patefecit, alias generatio ex sectione coni, alias denique alii describendi modi, hic tantum eas proprietates investigabimus, quas aequatio sola sine aliis subsidiis suppeditat.

86. Consideremus ergo aequationem generalem pro lineis secundi ordinis, quae est

$$0=\alpha+\beta x+\gamma y+\delta xx+\varepsilon xy+\zeta yy\,,$$

quam aequationem ita comparatam esse ostendimus, ut, quocunque angulo applicatae ad axem inclinatae statuantur, ea tamen semper omnes lineas secundi ordinis in se complectatur. Tribuatur iam isti aequationi haec forma

$$yy+\frac{(\varepsilon x+\gamma)y}{\zeta}+\frac{\delta xx+\beta x+\alpha}{\zeta}=0,$$

ex qua patet cuique abscissae x respondere vel duas applicatas y vel nullam, prout binae radices ipsius y fuerint vel reales vel imaginariae. Quodsi autem fuerit $\zeta=0$, tum unica quidem applicata singulis abscissis respondebit, altera abeunte in infinitum, quamobrem iste casus nostram indagationem non turbabit.

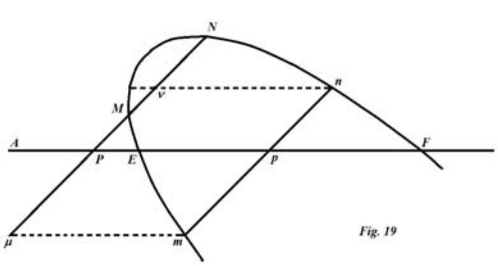


Fig. 19

87. Quoties autem ambo ipsius y valores fuerint reales, id quod evenit, si (Fig. 19) applicata PMN curvam in duobus punctis M et N intersecat, erit summa radicum

$$PM+PN=\frac{-\varepsilon x-\gamma}{\zeta}=\frac{-\varepsilon\cdot AP-\gamma}{\zeta},$$

sumta recta *AEF* pro axe, *A* pro initio abscissarum et angulo *APN,* quo applicatae axi insistunt, posito obliquo pro lubitu. Quodsi ergo sub eodem angulo ducatur quaevis alia applicata *npm,* cuius quidem valor

pm est negativus, erit eodem modo $pn - pm = \frac{-\varepsilon \cdot Ap - \gamma}{\zeta}$.

Subtrahatur haec aequatio a priori, erit

$$PM + pm + PN - pn = \frac{\varepsilon\left(Ap - AP\gamma\right)}{\zeta} = \frac{\varepsilon \cdot Pp}{\zeta}.$$

Ducantur ex punctis *m* et *n* rectae axi parallelae, donec priori applicatae occurrant in punctis μ et ν, eritque

$$M\mu + N\nu = \frac{\varepsilon \cdot Pp}{\zeta}$$

seu summa $M\mu + N\nu$ ad *Pp* seu $m\mu$ seu $n\nu$ rationem habebit constantem ut ε ad ζ. Ratio scilicet haec perpetuo erit eadem, ubicunque in curva ducantur rectae *MN* et *mn,* dummodo cum axe datum faciant angulum atque rectae $n\nu$ et $m\mu$ axi parallelae ducantur.

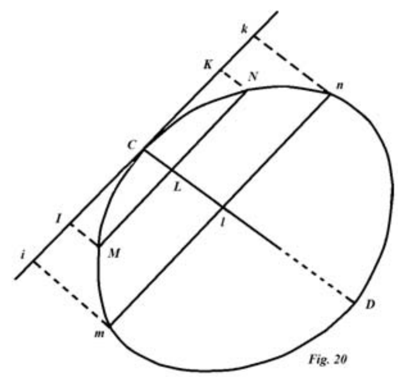


Fig. 20

88. Si (Fig. 20) applicata *PMN* eo promoveatur, quo puncta *M* et *N* coincidant, tum applicata tanget curvam; ubi enim duae intersectiones conveniunt, ibi linea secans abit in tangentem. Sit igitur *KCI* eiusmodi tangens, cui ducantur parallelae quotcunque rectae *MN, mn* curvae utrinque occurrentes, cuiusmodi rectae vocari solent *chordae* et *ordinatae.* Tum ex punctis *M, N, m, n* ad tangentem producantur rectae *MI, NK* et *mi, nk* axi prius assumto parallelae. Quia nunc intervalla *OK, Ok* ad contrariam puncti *C* partem cadunt, negative capi debebunt.
Hinc erit

$$CI - CK : MI = \varepsilon : \zeta \quad \text{et} \quad Ci - Ck : mi = \varepsilon : \zeta$$

ideoque

$$CI - CK : MI = Ci - Ck : mi$$

seu

$$MI : mi = CI - CK : Ci - Ck .$$

89. Quia positio axis respectu curvae est arbitraria, rectae *MI, NK, mi, nk* pro lubitu duci poterunt, dummodo inter se fuerint parallelae, eritque semper

$$MI : mi = CI - CK : Ci - Ck.$$

Quodsi ergo rectae parallelae *MI* et *NK* ita ducantur, ut fiat $CI = CK$, quod evenit, si parallelae *MI* et *NK* statuantur rectae *CL,* quae ex contactu *C* ducta ordinatam *MN* in *L* bisecat, tum ob $CI - CK = 0$ fiet quoque

$$Ci - Ck = \frac{mi}{MI}\left(CI - CK\right) = 0.$$

Quare producta recta *CL* in *l,* quia ob *mi* et *nk* pariter ipsi *CL* parallelas est $ml = Ci$ et $nl = Ck$, erit $ml = nl$. Unde sequitur rectam *CLl,* quae ex puncto contactus *C* ducta unam ordinatam *MN* tangenti parallelam bisecat, eandem omnes ordinatas *mn* eidem tangenti parallelas bifariam secare.

90. Cum igitur recta *CLl* omnes ordinatas tangenti *ICK* parallelas in duas partes aequales secet, haec linea *CLl* vocari solet *diameter lineae secundi ordinis* seu *sectionis conicae.* Hinc innumerabiles in unaquaque linea secundi ordinis duci possunt diametri, quia in singulis punctis curvae datur tangens. Ubicunque enim data fuerit tangens *ICK,* ducatur una quaevis ordinata *MN* hinc huic tangenti parallela, qua in *L* bisecata, erit recta *CL* diameter lineae secundi ordinis omnes ordinatas tangenti *IK* parallelas bifariam secans.

91. Ex his etiam sequitur, si recta *L1* duas quasvis parallelas ordinates *MN* et *mn* bisecet, eandem esse omnes reliquas ordinatas illis parallelas bisecturam; dabitur enim alicubi recta curvam tangens *IK* his ordinatis parallela ideoque dabitur diameter. Hinc nova habetur methodus in data linea secundi ordinis innumerabiles diametros inveniendi; ducantur enim pro lubitu duae ordinatae seu chordae *MN* et *mn* inter se parallelae, quibus bisectis in *L* et *l* recta per haec puncta ducta omnes reliquas ordinatas illis parallelas pariter bisecabit eritque propterea diameter. Atque ubi diameter producta curvam secat in *C*, per id recta *IK* ordinatis parallela ducta curvam in puncto *C* tanget.

92. Ad hanc proprietatem nos manuduxit consideratio summae binarum radicum ipsius *y* ex aequatione

$$yy + \frac{(\varepsilon x + y)}{\zeta}y + \frac{\delta xx + \beta x + \alpha}{\zeta} = 0.$$

Ex eadem vero aequatione constat fore productum ambarum radicum (Fig. 19) $PM \cdot PN = \frac{\delta xx + \beta x + \alpha}{\zeta}$, quae expressio $\frac{\delta xx + \beta x + \alpha}{\zeta}$ vel duos factores habet simplices reales vel secus. lllud evenit, si axis curvam in duobus punctis *E* et *F* secet; quia enim his in locis fit $y = 0$, erit $\frac{\delta xx + \beta x + \alpha}{\zeta} = 0$ hincque radices ipsius *x* erunt *AE* et *AF* atque adeo factores $(x - AE)\ (x - AF)$, ita ut sit

$$\frac{\delta xx+\beta x+\alpha}{\zeta}=\frac{\delta}{\zeta}(x-AE)\ (x-AF)=\frac{\delta}{\zeta}\cdot PE\cdot PF$$

ob $x=AP$. Hanc ob rem ergo erit

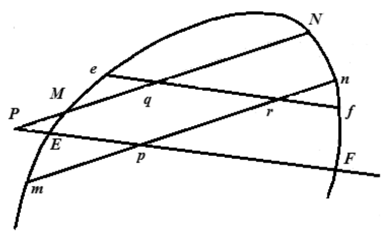
$$PM\cdot PN=\frac{\delta}{\zeta}PE\cdot PF,$$

seu rectangulum $PM\cdot PN$ ad rectangulum $PE\cdot PF$ constantem habebit rationem ut δ ad ζ ubicunque applicata *PMN* ducatur, dummodo sit angulus *NPF* assumto, quo applicatae ad axem inclinari ponuntur, aequalis. Erit ergo simili modo, si ducatur applicata *mn,* ob *Ep* et *pm* negativas

$$pm\cdot pn=\frac{\delta}{\zeta}pE\cdot pF\,.$$

93. Ducta ergo (Fig. 21) recta quacunque *PEF* lineam secundi ordinis secante in duobus punctis *E, F,* si ad eam parallelae ducantur ordinatae quotcunque *NMP, npm,* erit semper

Fig. 21



$$PM\cdot PN : PE\cdot PF = pm\cdot pn : pE\cdot pF\,;$$

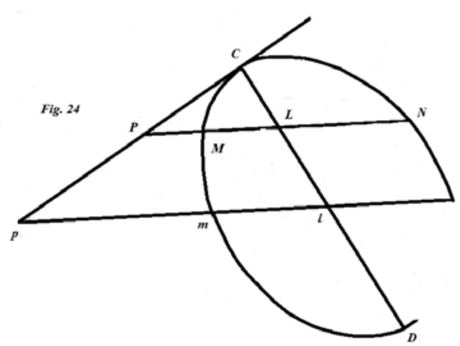
ideoque

Ergo alternando $qe\cdot qf : pE\cdot pF = qM\cdot qN : pm\cdot pN$. Datis igitur duabus ordinatis parallelis *ef* & *EF*, si aliae quaecunque duae ordinatae inter se parallelae *MN* & *mn* ducantur , illas secantes in punctis *P, p, q*, *r* , erunt hae rationes omnes inter se aequales,

$$PM\cdot PN : PE.PF = pm\cdot pn : pE\cdot pF = qM\cdot qN : qe\cdot qf = rm\cdot rn : re\cdot rf\,.$$

Quae est altera proprietas generalis linearum secundi ordinis.

Fig. 24



94. Si igitur duo curvae puncta *M* & *N* coincidant , recta *PMN* fiet curvae tangens in concursu illorum duorum punctorum (Fig. 24), abibitque rectangulum $PM\cdot PN$ in quadratum ipsius *PM* vel *PN,* unde nova tangentium proprietas obtinebitur.
Tangat nimirum recta *CPp* lineam secundi ordinis in puncto *C,* & ducantur linea quotvis *PMN* , *pmn* inter parallalae, quae ergo omnes cum tangente eundem angulum constituant. Ex proprietate igitur ante inventa erit

$$PC^2 : PM \cdot PN = pC^2 : pm \cdot pn,$$

seu quaecunque ordinate *MN* ad tangentem sub angulo dato ducatur, erit semper quadratum rectae *CP* ad ad angulem $PM \times PN$ in ratione constante.

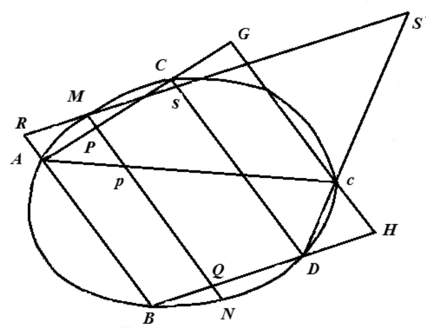
95. Indidem etiam sequitur , si linea secundi ordinis ductatur diameter quaecunque *CD,* omnes ordinatas *MN , mn* [Fig. *20*] inter se parallelas bifariam secans, atque ipsa diameter curvae occurrat in punctis duobus *C* & *D ,* fore

$$CL \cdot LD : LM \cdot LN = Cl \cdot lD : lm \cdot ln .$$

Cum autem set $LM = LN$, & $lm = ln$, erit $LM^2 : lm^2 = CL \cdot LD : Cl \cdot lD$, seu perpetuo crit quadratum semiordinatae *LM* ad rectangulum $CL \cdot LD$ in ratione constante. Hinc sumpta diametro *CD* pro axe, & semiordinatis *LM* pro applicatis, reperietur aequatio pro Lineis secundi ordinis. Sit enim diameter $CD = a$. Abscissa $CL = x$ & applicata $LM = y$, ob $LD = a - x$ erit, y^2 ad $ax - xx$ in ratione constante, quae sit ut *h* ad *k*, unde orietur ista pro lineis secundi ordinis aequatio $yy = \frac{b}{k}(ax - xx)$.

96. Ex ambabus autem jam inventis linearum secundi ordinis proprietatisbus conjunctim aliae erui poterunt proprietates. Dentur in linea secundi ordinis duae ordinatae inter se parallelae *AB* & *CD,* & compleatur quadrilaterum *ACDB ,* quod si jam per punctum quodcunque curvae *M* ducatur ordinata *MN* illis *AB* & *CD* parallela secans rectas *AC & BD* in punctis *P* & *Q* , erunt partes *PM* & *QN* inter se aequales. Nam recta , quae bisecat ordinatas duas *AB* & *CD* inter se parallelas , bisecabit quoque ordinatam *MN* : at, per Geometriam elementarem , eadem recta bisecans latera *AB & CD* quoque bisecabit portionem *PQ*. Cum igitur linae *MN & PQ* in eadem puncto bisecentur, necesse est ut sit, $MP = NQ$ & $MQ = NP$. Dato ergo praetcr quatuor linea secundi ordinis puncta *A , B. C* & *D,* quinto *M* ex eo reperietur sextum *N,* sumto $NQ = MP$.

Fig. 22



97. Cum jam sit $MQ \cdot QN$ ad $BQ \cdot DQ$ in ratione constante , ob $QN =$ MP erit quoque $MP \cdot MQ$ ad $BQ \cdot DQ$ in eadem ratione constante. Scilicet, si aliud quodcunque curvae punctum, uti *c ,* sumatur, & per id recta *GcH* ipsis *AB, & CD* parallela ducatur donec lateribus *AC, BD* occurrat in punctis *G* & *H,* erit quoque *cG.cH* ad *BH.DH* in eadem

ratione constante , ideoque $cG\cdot cH:BH\cdot DH=MP\cdot MQ:BQ\cdot DQ$. Quod si autem per *M* basi *BD* parallelae ducatur *RMS* ordinis parallelis *AB, CD* occurrens in *R & S* , erit , ob $BQ=MR$ & $DQ=MS$, haec quoque ratio $MP\cdot MQ:MR\cdot MS$ constans. Si igitur per quodvis curvae punctum *M* duae ducantur rectae, altera *MPQ* lateribus oppositis *AB, CD* parallela , altera vero *RMS* basi *BD* parallela, intersectiones *P, Q, R,* & *S* ita erunt comparatae, ut sit sit $MP\cdot MQ:MR\cdot MS$ in ratio constante.

98. Si loco ordinateae *CD*, quae posita est ipsi *AB* parallela, ex puncto *D* alia quaecunque *Dc* in ejus locum substitutur, & chorda *Ac* jungatur : ita ut nunc rectae *MQ* & *RMS,* ductae, ut ante, per *M* lateribus *AB* & *BD* parallelae, latera quadrilateri *ABDc* secent in punctis *p, Q, R* & *s* ; similis proprietas locum habebit. Cum enim sit $MP\cdot MQ:BQ\times DQ=cG\cdot cH:BH\cdot DH$ seu $MP\cdot MQ:MR\cdot MS=cG\cdot cH:BH\cdot DH$, ob rectam *RS* ipsi *BD* parallelam & aequalem. Triangula vero similia *APp*, *AGc* & *DSs, cHD,* praebent has proportiones $Pp:AP=Gc:AG$, seu, ob $AP:AG=BQ:BH$ hanc, $Pp:BQ=Gc:BH$:altera similitudo dat hanc $DS(MQ):Ss=cH:DH$, quibus coniunctis fit

$$MQ\cdot Pp:MR\cdot Ss=cG\cdot c\mathrm{H}:BH\cdot DH,\ \text{ob } BQ=MR.$$

Haec proportio cum superiori collata praebet

$$MP\cdot MQ:MR\cdot MS=Pp\cdot MQ:MR\cdot Ss,$$

unde addenda antecedentes & consequentcs fit

$$MP\cdot MQ:MR\cdot MS=Mp\cdot MQ:MR\cdot Ms,$$

ubicunque ergo sumantur puncta *c* & *M* in Curva , erit semper ratio $Mp\cdot MQ$ ad $MR\cdot Ms$ eadem , dummodo rectae *MQ* & *Rs* per *M* ducantur chordis *AB* & *BD* parallelae. Ex superiore vero proportone sequitur fore $MP:MS=Mp:Ms$.
Cum igitur , variato puncto *c* , tantum puncta *p* & *s* mutentur, erit *Mp* ad *Ms* in data ratione , utcunque punctum *c* varietur, dum punctum *M* fixum servatur.

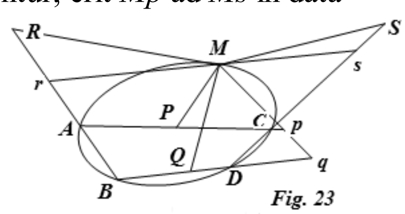


Fig. 23

99. Quod si quatuor quaecunque puncta *A ,B ,C, D* in linea secunda ordinis fuerit data, eaque jungantur rectis, (Fig. 23) ut habeatur trapezium inscriptum *ABDC,* proprietas Sectionum conicarum latissime patens ex praecedenti deducitur. Scilicet, si ex curvae puncto quovis *M* ad singula trapezii latera sub datis angulis ducantur rectae *MP, MQ*, *MR* & *M S*, erunt sempcr rectangula binarum harum linearum ad opposita latera ductarum inter se in data ratione, nempe erit $MP\cdot MQ$ ad $MR\cdot MS$ in data ratione eadem ,

ubicunque punctum *M* in curva capiatur, dummodo anguli ad *P, Q, R,* & *S* iidem serventur. Ad hoc ostendendum ducantur per *M* duae rectae *Mq* & *rs,* illa lateri *AB* haec lateri *BD* parallela , ac notentur intersectionum cum lateribus trapezii puncta *p, q, r ,* & *s :* eritque per prius inventum $Mp \cdot Mq$ ad $Mr \cdot Ms$ in data ratione. Propter omnes autem angulos datos datae erunt rationes $MP{:}Mp, MQ{:}Mq,\ MR{:}Mr,\ \&\ MS{:}Ms$, ex quibus sequitor fore $MP \cdot MQ$ ad $MR \cdot MS$ in data quoque ratione.

100. Quoniam supra vidimus, si ordinatae parallelae *MN, mn* (Fig. 24) producantur, quoad tangenti cuipiam *CPp*, occurrant in *P* & *r,* fore $PM \cdot PN{:}CP^2 = pm \cdot pn{:}Cp^2$. Quare , si puncta *L* & *l* notentur , ut sit *PL* media proportionalis inter *PM* & *PN ,* pariterque *pl* media proportionalis inter, *pm* & *pn,* erit $PL^2{:}CP^2 = pl^2{:}Cp^2$ ideoque erit $PL : CP = pl : Cp$ unde patet omnia puncta *L ,l* in linea recta per punctum contactus *C* transeunte esse sita. Quare, si una applicata *PMN* ita secetur in *L* ut sit $PL^2 = PM \cdot PN$, recta *CLD* per puncta *C* & *L* ducta omnes reliquas applicatas *pmn* ita quoque secabit in *l* ut sit *pl* media proportionalis inter *pm* & *pn.* Vel, si duae applicatae *PN* & *pn* ita in punctis *L* & *l*, secentur , ut sit $PL^2 = PM \cdot PN$ & $pl^2 = pm \cdot pn$ recta per *L* & *l*, producta per punctum contactus *C* transibit, atque omnes reliquas applicatas illis parallelas in eadem ratione secabit.

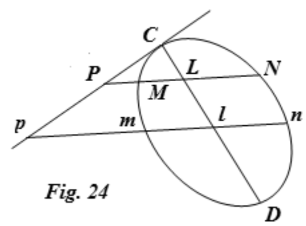


Fig. 24

101. His linearum secundi ordinis proprietatibus , quae ex forma aequationis immediate consequuntur, expositis ; progrediamur ad alias magis reconditas investigandas. Sit igitur proposita aequatio pro his lineis secundi ordinis generalis

$$yy + \frac{(\varepsilon x + \gamma)}{\zeta} y + \frac{\delta xx + \beta x + \alpha}{\zeta} = 0,$$

ex qua cum cuivis Abscissae $AP = x$, duplex applicata *y* nempe *PM* & *PN* respondeat , positio diametri omnes ordinatas *MN* bifariam secantis definiri potest. Sit enim *IG* ista diameter, quae ordinatam *MN* secabit in puncto medio *L*, quod ergo punctum est in diametro. Ponatur $PL = z$; &, cum sit $z = \frac{1}{2}PM + \frac{1}{2}PN$, erit $z = \frac{-\varepsilon x - \gamma}{2\zeta}$, seu $2\zeta z + \varepsilon x + \gamma = 0$, quae est aequatio positionem diametri *IG* praebens.

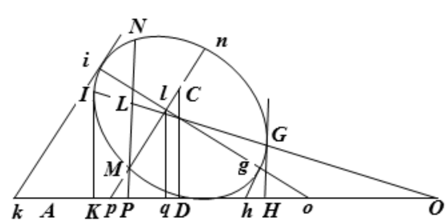


Fig. 25

102. Hinc porro longitudo diametri *IG* definiri poterit , quae dat loca bina in curva, ubi puncta *M & N* coincident, seu ubi fit $PM = PN$. Ex eaquatione vero dantur

$PM + PN = \frac{-\varepsilon x - \gamma}{\zeta}$ & $PM.PN = \frac{\delta xx + \beta x + \alpha}{\zeta}$, unde fit

$(PM - PN)^2 = \left(PM + PN \right)^2 - 4\, PM \cdot PN =$

$\frac{\left(\varepsilon\varepsilon - 4\delta\zeta\right)xx + 2\left(\varepsilon\gamma - 2\beta\zeta\right)x + \left(\gamma\gamma - 4\,\alpha\zeta\right)}{\zeta\zeta} = 0$, seu

$xx - \frac{2\left(2\beta\zeta - \varepsilon\gamma\right)}{\varepsilon\varepsilon - 4\delta\zeta}x + \frac{\left(\gamma\gamma - 4\,\alpha\zeta\right)}{\varepsilon\varepsilon - 4\delta\zeta} = 0$, cujus aequationis propterea radices sunt *AK* & *AH*

ita ut sit $AK + AH = \frac{4\beta\zeta - 2\varepsilon\gamma}{\varepsilon\varepsilon - 4\delta\zeta}$ & $AK \cdot AH = \frac{\gamma\gamma - 4\alpha\zeta}{\varepsilon\varepsilon - 4\delta\zeta}$, hinc sit

$$\left(AK - AH\right)^2 = KH^2 = \frac{4\left(2\beta\zeta - \varepsilon\gamma\right)^2 - 4\left(\varepsilon\varepsilon - 4\delta\zeta\right)\left(\gamma\gamma - 4\alpha\zeta\right)}{\left(\varepsilon\varepsilon - 4\delta\zeta\right)^2}.$$

At est $IG^2 = \frac{\varepsilon\varepsilon + 4\zeta\zeta}{4\zeta\zeta}KH^2$, si quidem applicatae ad axem normales statuantur.

103. Sine istae applicatae, quas hic sumus contemplati, normales ad axem *AH*; nunc vero hinc quaeram aequationem pro applicatis obliquangulis. Ducatur ergo ex quovis curvae puncto *M* ad axem applicata obliquangula *Mp,* faciens cum axe angulum *MpH,* cujus sinus sit $= \mu$ & cosinus $= \nu$. Sit nova abscissa $Ap = t$, applicata $pM = u$, erit $\frac{y}{u} = \mu$ & $\frac{Pp}{u} = \nu$; unde erit $y = \mu u$ & $x = t + \nu u$), qui valores in aequatione inter *x* & *y*, quae erat $0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy$ substituti praebent

$$\begin{aligned} 0 = \alpha + \beta t + \nu\beta u + \delta tt + 2\nu\delta tu + {} & \nu\nu\delta uu \\ + \mu\gamma u \qquad + \mu\varepsilon tu + {} & \mu\nu\varepsilon uu \\ + {} & \mu\mu\zeta uu \end{aligned}$$

seu

$$uu + \frac{\left(\left(\mu\varepsilon + 2\nu\delta\right)t + u\gamma + \nu\beta\right)u + \delta tt + \beta t + \alpha}{\mu\mu\zeta + \mu\nu\varepsilon + \nu\nu\delta} = 0.$$

104. Hic ergo iterum quaevis applicata duplicem habebit valorem, nempe *pM* & *pn:* quare ordinatatum *Mn* diameter *ilg* pari modo ut ante definietur. Scilicet, bisecta ordinate *Mn* in *l* erit *l ,* punctum in diametro. Ponatur ergo $pl = \upsilon$, erit

$$\upsilon = \frac{pM + pu}{2} = \frac{-(\mu\varepsilon + 2\nu\delta)t - \mu\gamma - \nu\beta}{2(\mu\mu\zeta + \mu\nu\varepsilon + \nu\nu\delta)}.$$

Demittatur ex l in axem AH perpendiculum lq, ac ponatur $Aq = p$, $ql = q$, erit $\mu = \frac{q}{\upsilon}$ & $\nu = \frac{pq}{\upsilon} = \frac{p-t}{\upsilon}$, unde fit $\upsilon = \frac{q}{\mu}$, & $t = p - \nu\upsilon = p - \frac{\nu q}{\mu}$. Substituantur hi valores in aequatione inter t & υ ante inventa , & prodibit

$$\frac{q}{\mu} = \frac{-\mu\varepsilon p - 2\nu\delta p + \nu\varepsilon q + 2\nu\nu\delta q : \mu - \mu\gamma - \nu\beta}{2\mu\mu\zeta + 2\mu\nu\varepsilon + 2\nu\nu\delta}$$

seu

$$(2\mu\mu\zeta + \mu\nu\varepsilon)q + (\mu\mu\varepsilon + 2\mu\nu\delta)p + \mu\mu\gamma + \mu\nu\beta = 0,$$

seu

$$(2\mu\zeta + \nu\varepsilon)q + (u\varepsilon + 2\nu\delta)p + \gamma\mu + \nu\beta = 0,$$

qua aequatione positio diametri ig definitur.

105 . Prior diameter IG, cujus positio per hanc aequationem determinabatur $2\zeta z + \varepsilon x + \gamma = 0$, producta cum axe concurrat in O, eritque $AO = \frac{-\gamma}{\varepsilon}$ hinc fit $PO = \frac{-\gamma}{\varepsilon} - x$, & anguli tangens erit $= \frac{z}{PO} = \frac{-\varepsilon z}{\varepsilon x + \gamma} = \frac{\varepsilon}{2\zeta}$, & tangens anguli MLG, sub quo diameter IG ordinatas MN bisecat erit $= \frac{2\zeta}{\varepsilon}$. Altera vero diameter ig producta axi occurrat in o , eritque $Ao = \frac{-\mu\gamma - \nu\beta}{\mu\varepsilon + 2\nu\delta}$, & anguli Aol tangens erit $= \frac{\mu\varepsilon + 2\nu\delta}{2\mu\zeta + \nu\varepsilon}$. Cum jam anguli AOL tangens fit $= \frac{\varepsilon}{2\zeta}$, ambae diametri fe mutua intersecabunt in puncto quodam C, facientque angulum $OCo = Aol - AOL$, cuius propterea tangens est

$$= \frac{4\nu\delta\zeta - \nu\varepsilon\varepsilon}{4\mu\zeta\zeta + 2\nu\delta\varepsilon + 2\nu\varepsilon\zeta + \mu\varepsilon\varepsilon}.$$

Angulus autem, sub quo hac altera diameter suas ordinatas bisecat, est $Mlo = 180^0 - lpo - Aol$: hujus propterea tangens est

$$= \frac{2\mu\mu\zeta + 2\mu\nu\varepsilon + 2\nu\nu\delta}{\mu\mu\varepsilon + 2\mu\nu\delta - 2\mu\nu\zeta - \nu\nu\varepsilon}$$

106. Inquiramus autem in punctum C, ubi hae duae diametri se mutua intersecant : ex quo ad axem perpendiculum *CD* demittatur , ac vocetur $AD = g$, $CD = h$; eritque primo, quod *C* in diametro *IG* extat, $2\zeta h + \varepsilon g + \gamma = 0.$ Deinde , quia *C* quoque in diametro *ig* reperitur , erit

$$(2\mu\zeta + \nu\varepsilon)h + (\mu\varepsilon + 2\nu\delta)g + \mu\gamma + \nu\beta = 0.$$

Subtrahatur hinc prior aequatio per μ multiplicata , ac remanebit

$$\nu\varepsilon h + 2\nu\delta g + \nu\beta = 0, \text{ seu } \varepsilon h + 2\delta g + \beta = 0.$$

Ex his fit $h = \frac{-\varepsilon g - \gamma}{2\zeta} = \frac{-2\delta g - \beta}{\varepsilon}$, ideoque

$\left(\varepsilon\varepsilon - 4\delta\zeta\right) g = 2\beta\zeta - \gamma\varepsilon$, & $g = \frac{2\beta\zeta - \gamma\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}$ & $h = \frac{2\gamma\delta - \beta\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}$. In quibus determinationibus cum non insint quantitates μ & ν a quibus obliquitas applicatarum *pMn* pendet, manifestum est punctum *C* idem manere , utcunque obliquitas varietur.

107. Omnes ergo diametri *IG* & *ig* se mutua in eadem puncto *C* decussant : quod ergo si semel fuerit inventum , omnes diametri per id transibunt , ac vicissim omnes rectae per id ductae erunt diametri, quae omnes ordinatas sub certo quodam angulo ductas bisecent. Cum igitur hoc punctum in quavis linea secundi ordinis sit unicum , in eoque omnes diametri se mutua decussent, hoc punctum vocari solet CENTRUM sectionis conicae . Quod ergo ex aequatione inter *x* & *y* proposita

$$0 = \alpha + \beta x + \gamma y + \delta xx + \varepsilon xy + \zeta yy$$

ita invenitur , ut sumta $AD = \frac{2\beta\zeta - \gamma\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}$, capiatur $CD = \frac{2\gamma\delta - \beta\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}$.

108 . Supra autem invenimus esse $AK + AH = \frac{4\beta\zeta - 2\gamma\varepsilon}{\varepsilon\varepsilon - 4\delta\zeta}$: Sunt autem *IK* & *GH* perpendicula ex terminis diametri *IG* in axem demissa ; unde perspicitur esse $AD = \frac{AK + AH)}{2}$ atque ideo punctum *D* erit medium inter puncta *K* & *H.* Quam ob rem centrum quoque *C* in medio diametri *IG* erit situm , quod cum de quavis alia diametro

aequae valeat, consequens est non solum omnes diametros se mutuo in eodem puncta C decussare,sed etiam se invicem bifariam secare.

109. Sumamus nunc quamcunque diametrum AI pro axe ad quem ordinatae MN applicatae sint sub angulo $APM = q$, cuius sinus $= m$, & cosinus $= n$. Ponatur abscissa $AP = x$

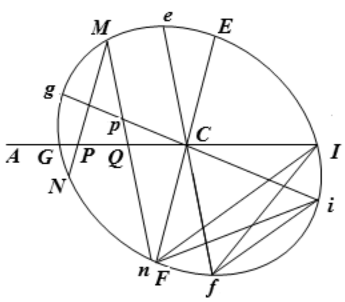


Fig. 26

& applicata $PM = y$, cuius cum duo sint valores aequales alter alterius negativus eorumque adeo summa $= 0$, aequatio generalis pro linea secundi ordinis abibit in hanc formam $yy = \alpha + \beta x + \gamma xx$: quae, si ponatur $y = 0$, dabit puncta G & I in axe, ubi is a curva trajicitur ; aequationis scilicet $xx + \frac{\beta}{\gamma} x + \frac{\alpha}{\gamma} = 0$ radices erunt $x = AG$ & $x = AI$; ideoque habibitur $AG + AI = \frac{-\beta}{\gamma}$ & $AG.AI = \frac{\alpha}{\gamma}$. Cum igitur centrum C in medio diametri GI sit positum, facile reperietur centrum sectionis conicae C. Erit enim $AC = \frac{AG + AI}{2} = \frac{-\beta}{2\gamma}$.

110. Cognito iam centro sectionis conicae C, in axe AI, id convenientissime pro initio abscissarum accipietur. Statuatur ergo $CP = t$, quia manet $PM = y$, ob $x = AC - CP = \frac{-\beta}{2\gamma} - t$, prodibit haec equatio inter coordinatas t & y

$$yy = \alpha - \frac{\beta\beta}{2\gamma} + \frac{\beta\beta}{4\gamma} - \beta t + \beta t + \gamma tt$$

seu

$$yy = \alpha - \frac{\beta\beta}{4\gamma} + \gamma tt.$$

Posito igitur x loco t, habebitur aequatio generalis pro lineis secundi ordinis, sumta diametro quacunque pro axe, & centro pro abscissarum initio, quae, mutata constantium forma, erit $yy = \alpha - \beta xx$. Posito ergo $y = 0$ fiet $CG = CI = \sqrt{\frac{\alpha}{\beta}}$; ideoqae tota diameter GI erit $= 2\sqrt{\frac{\alpha}{\beta}}$.

111 . Ponatur $x = 0$, ac reperietur ordinata per centrum transcens EF : fiet scilicet $CE = CF = \sqrt{\alpha}$; ideoque tota ordinata $EF = 2\sqrt{\alpha}$, quae, quia per centrum transit , pariter

erit diameter, cum illa *GI* angulum faciens $ECG = q$. Haec autem altera diameter *EF* bisecabit omnes ordinatas priori *GI* parallelas ; facta enim abscissa *AP* negativa, applicata *aC* versus *I* cadens manebit priori *PM* aequalis ; &, cum eidem sit parallela, puncta ambo *M* juncta dabunt lineam diametro *GI* parallam , ideoque bisecandam a diametro *EF*. Hae igitur ambae diametri *GI* & *EF* ita inter se sunt affectae, ut altera bisecet omnes ordinatas alteri parallelas, quam ob reciprocam proprietatem hae duae diametri inter se CONJUGATAE appellantur. Si igitur in terminis *G* & *I* diametri *GI* ducantur rectae alterae diametro *EF* parallelae tangent hae lineam curvam, similique modo per *E* & *F* ducantur rectae diametro *GI* parallelae eae tangent curvam in punctis *E* & *F*.

112 . Ducatur nunc applicata quaevis *MQ* obliquangula ; sitque angulus $AQM = \phi$, eius sinus $= \mu$ & cos. $= \nu$. Ponatur abscissa $CQ = t$, & applicata $MQ = u$, eritque in triangulo *PMQ* ob ang. $PMQ = \phi - q$ ac propterea
$\sin.PMQ. = \mu n - \nu m$, $y : u : PQ = \mu : m \; : \; \mu n - \nu m$, hincque
$y = \frac{\mu u}{m}$ & $PQ = \frac{(\mu n - \nu m)u}{m}$, unde $x = t - \frac{(\mu n - \nu m)u}{m}$. Substituantur hi valores in aequationes superiori, $yy = \alpha - \beta xx$ seu $yy + \beta xx - \alpha = 0$, ac orietur

$$\left(\mu\mu + \beta(\mu n - \nu m)^2\right)uu - 2\beta m(\mu n - \nu m)tu + \beta m^2 tt - \alpha m^2 = 0,$$

ex qua applicata *u* duos obtinet valores *QM* & $-Qn$ eritque

$$QM - Qn = \frac{2\beta m(\mu n - \nu m)t}{\mu\mu + \beta(\mu n - \nu m)^2}.$$

Bisecetur ordinata *Mn* in *p*, eritque recta *Cpg* nova diameter secans omnes ordinatas ipsi *Mn* parallelas bifariam , eritque

$$Qp = \frac{\beta m(\mu n - \nu m)t}{\mu\mu + \beta(\mu n - \nu m)^2}.$$

113. Obtinetur autem hinc anguli *GCg* tangens

$$= \frac{\mu.Qp}{CQ + \nu.Qp}, \quad \text{vel} \quad \text{tang.}GCg = \frac{\beta m(\mu n - \nu m)}{\mu + n\beta(\mu n - \nu m)}$$

$$\text{\& tang. } Mpg = \frac{\mu.CQ}{pQ + \nu.CQ} = \frac{\mu\mu + \beta(\mu n - \nu m)^2}{\mu\nu + \beta(\mu n - \nu m)(\nu n + \mu m)},$$

qui est angulus sub quo novae ordinatae *Mn* a diametro bisecantur.

Porro vero erit

$$Cp^2 = CQ^2 + Qp^2 + 2\nu \cdot CQ \cdot Qp = \frac{\mu^4 + 2\beta\mu^3 n(\mu n - \nu m) + \beta\beta\mu\mu(\mu n - \nu m)^2}{\left(\mu\mu + \beta(\mu n - \nu m)^2\right)^2} tt:$$

ideoque

$$Cp = \frac{\mu t\sqrt{\left(\mu^2 + 2\beta\mu n(\mu n - \nu m) + \beta\beta(\mu n - \nu m)^2\right)}}{\mu\mu + \beta(\mu n - \nu m)^2}.$$

Ponatur $Cp = r$, & $pM = s$, eritque

$$t = \frac{\left(\mu\mu + \beta(\mu n - \nu m)^2\right) r}{\mu\sqrt{\left(\mu^2 + 2\beta\mu n(\mu n - \nu m) + \beta\beta(\mu n - \nu m)^2\right)}},$$

$$\& \ u = s + Qp = s + \frac{\beta m(\mu n - \nu m) r}{\mu\sqrt{\left(\mu^2 + 2\beta\mu n(\mu n - \nu m) + \beta\beta(\mu n - \nu m)^2\right)}},$$

qui valores porro dant ,

$$y = \frac{\mu s}{m} + \frac{\beta(\mu n - \nu m) r}{\sqrt{(\cdots)}}$$

$$x = -\frac{(\mu n - \nu m) s}{m} + \frac{\nu r}{\sqrt{(\cdots)}},$$

unde ex aequatione $yy + \beta xx - \alpha$ orietur

$$\frac{\mu\mu + \beta(\mu n - \nu m)^2 ss}{mm} + \frac{\beta\left(\mu\mu + \beta(\mu n - \nu m)^2\right) rr}{\mu\mu + 2\beta\mu n(\mu n - \nu m) + \beta\beta(\mu n - \nu m)^2} - \alpha = 0.$$

114. Vocemus jam semidiametrum $CG = f$ & semiconjugatum $CE = CF = g$, eritque, $f = \sqrt{\frac{\alpha}{\beta}}$ & $g = \sqrt{\alpha}$, seu

$\alpha = gg$ & $\beta = \frac{gg}{ff}$: unde fit $yy + \frac{ggxx}{ff} = gg.$ Ponamus porro angulum $GCg = p$, crit

$$\text{tang}.p = \frac{\beta m(\mu n - \nu m)}{\mu + n\beta(\mu n - \nu m)}.$$

At, ob angulum $GCE = q$, si ponatur angulus $ECe = \varpi$, erit $AQM = \phi = q + \varpi$; ideoque que $\mu = \sin.(q+\varpi)$; $\nu = \cos.(q+\varpi)$, $m = \sin.q$ & $n = \cos.q$.
Ergo

$$\text{tang}.p = \frac{\beta\sin.q \cdot \sin.\varpi}{\sin.(q+\varpi) + \beta\cos.q \cdot \sin.\varpi} = \frac{\beta\text{tang}.q \cdot \text{tang}.\varpi}{\text{tang}.q + \text{tang}.\varpi + \beta\text{tang}.\varpi}, \ \&$$

$$\sin.p = \frac{\beta\sin.q \cdot \sin.\varpi}{\sqrt{\left(\mu^2 + 2\beta\mu n(\mu n - \nu m) + \beta\beta(\mu n - \nu m)^2\right)}},$$

atque

$$\mu\mu + \beta(\mu n - \nu m)^2 = (\sin.(q+\varpi))^2 + \beta(\sin.\varpi)^2,$$

quibus valoribus in subsidium vocatis prodit ista aequatio inter *r* & *s,*

$$\frac{\left((\sin.(q+\varpi))^2 + \beta(\sin.\varpi)^2\right)ss}{(\sin.q)^2} + \frac{\beta\left((\sin.(q+\varpi))^2 + \beta(\sin.\varpi)^2\right)rr(\sin.p)^2}{\beta\beta(\sin.q)^2(\sin.\varpi)^2} - \alpha = 0 \ ;$$

At est

$$\beta = \frac{\text{tang}.p\sin.(q+\varpi)}{(\sin.q - \cos.q \cdot \text{tang}.p)\cdot\sin.\varpi} = \frac{\text{tang}.p(\text{tang}.q + \text{tang}.\varpi)}{\text{tang}.\varpi(\text{tang}.q - \text{tang}.p)}$$

$$= \frac{gg}{ff} = \frac{\cot.\varpi \cdot \text{tang}.q + 1}{\cot.p \cdot \text{tang}.q - 1}, \ \text{seu}$$

$$\text{tang}.q = \frac{ff + gg}{gg \cdot \cot.p - f \cdot f\cot.\varpi},$$

unde plurima consectaria deduci possunt. Erit vero

$$\frac{gg}{ff} = \frac{\sin.p \cdot \sin.(q+\varpi)}{\sin.\varpi \cdot \sin.(q-p)}.$$

115. Sit semidiameter $Cg = a$, eiusque semidiameter conjugata $Ce = b$; erit ex equatione ante inventa,

$$a = \frac{\sin.q\cdot\sin.\varpi\cdot\sqrt{\alpha\beta}}{\sin.p\cdot\left(\left(\sin.\overline{q+\varpi}\right)^2+\beta\left(\sin.\varpi\right)^2\right)} = \frac{gg\cdot\sin.q\cdot\sin.\varpi}{\sin.p\cdot\left(ff\left(\sin.\overline{q+\varpi}\right)^2+g^2\left(\sin.\varpi\right)^2\right)}$$

&

$$b = \frac{fg\cdot\sin.q}{\sqrt{\left(ff\left(\sin.\overline{q+\varpi}\right)^2+gg\left(\sin.\varpi\right)^2\right)}},$$

hinc erit $a:b = g\cdot\sin.\varpi : f\cdot\sin.p.$ Est vero porro

$$\left(\sin.(q+\varpi)\right)^2+\frac{gg}{ff}\left(\sin.\varpi\right)^2 = \frac{\sin.(q+\varpi)}{\sin.(q-p)}\left(\sin.(q-p)\cdot\sin.(q+\varpi)+\sin.p\cdot\sin\varpi\right)$$

$$= \frac{\sin.q\cdot\sin.(q+\pi)\cdot\sin.(q+\pi-p)}{\sin.(q-p)},$$

unde fiet

$$a = \frac{gg\cdot\sin.\varpi}{f\cdot\sin.pf}\sqrt{\frac{\sin.q\cdot\sin.(q-p)}{\sin.(q+\varpi)\sin.(q+\varpi-p)}};$$

seu, ob

$$\frac{gg}{ff} = \frac{\sin.p\cdot\sin.(q+\varpi)}{\sin.\varpi\cdot\sin.(q-p)},$$

erit

$$a = f\sqrt{\frac{\sin.q\cdot\sin.(q+\varpi)}{\sin.(q-p)\cdot\sin.(q+\varpi-p)}} \;\&\; b = g\sqrt{\frac{\sin.q\cdot\sin.(q-p)}{\sin.(q+\varpi)\cdot\sin.(q+\varpi-p)}},\text{ ergo erit}$$

$$a:b = f\cdot\sin.(q+\varpi):g\cdot\sin.(q-p) \;\;\&\;\; ab = \frac{fg\cdot\sin.q}{\sin.(q+\varpi-p)}.$$

116. Si ergo in sectione conica binae diametri conjugatae habeantur, *GI, EF* & *gi* , *ef* , erit primo

$$Cg : Ce = CG \cdot \sin.\ ECe : CG \cdot sin.GCg.$$

Ergo

$$sin.GCg : \sin.\ ECe = CE \cdot Ce : CG \cdot Cg.$$

& si chordae *Ee* & *Gg* ducantur , fiet hinc triangulum CGg = triangulo CEe. Deinde erit $Cg : Ce = CG \cdot \sin.GCe : CE \cdot \sin.gCE$, seu $Ce \cdot CG \cdot \sin.GCe = CE \cdot Cg \cdot \sin.gCE$: unde , si ducantur chordae *Ge, & gE,* erunt triangula *GCe* & g*CE* inter se aequalia , seu e regione erit triangulum *ICf* = Triangule *iCF.* Ultima vero aequatio $ab \cdot sin.(q + \varpi - p) = fg \cdot \sin .q$ dabit $Cg \cdot Ce \cdot \sin.gCE = CG \cdot CE \cdot \sin.GCe.$ Quod si ergo ducantur chordae *EG* & *eg ,* vel e regione *FI* & *fi* erunt pariter triangula *ICF* & *iCf* aequalia : unde sequitur omnia parallelogramma , quae circa binas diametros conjugatas dcscribuntur, inter se esse aequalia.

117. Habentur ergo tria triangulorum paria inter se aequalia, nempe ,
I. Triangulum *FCf* aequale Triangulo *ICi.*
II. Triangulum *fCI* aequalc Triangulo *FCi.*
III. Triangulum *FCI* aequale Triangulo *fCi.*
Unde sequitur fore trapezia *FfCI & iICf* inter se aequalia ;
a quibus si auferatur idem triangulum *fCI*, erit Triangulum Fif = Triangulo Ifi: quae cum super eadem basi *fI* sint constituta, necesse est ut sit chorda *Fi* chordae *fI* parallela.
Porro itaque erit Triangulum FIi = Triangulo ifF , ad quae si addantur triangula aequalia *FCI & fCi,* erunt quoque haec trapezia inter se aequlia $FCIi = iCfF$.

118. Hinc etiam deducitur methodus ad quodvis lineae secundi ordinis punctum *M* tangentem *MT* ducendi. Sumta enim diametro *GI* pro axe *,* [Fig. 27] cui conjugatae semissis sit *EC,* ex puncto *M* ipsi C*E* parallela ad axem ducatur *MP*, quae erit semiordinata, ac $PN = PM$. Ducta *CM,* quae erit semidiameter, quaeratur eius semidiameter conjugata *CK,* cui tangens *MT* quasita erit parallela. Sit angulus *GCE= q;* $GCM = p$ & $ECK = \varpi$; erit, uti vidimus;

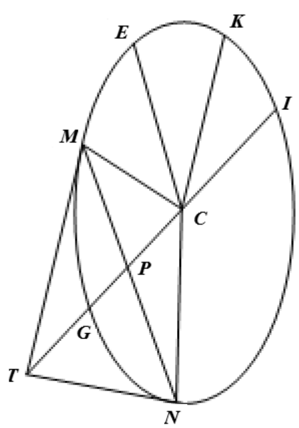


Fig. 27

$$\frac{EC^2}{CG^2} = \frac{\sin.p \cdot \sin.(q + \varpi)}{\sin.\varpi \cdot \sin.(q - p)}$$

&

$$MC = CG\sqrt{\frac{\sin.p \cdot \sin.(q + \varpi)}{\sin.(q - p) \cdot \sin.(q + \varpi - p)}}$$

At in triangulo *CMP* est

$$MC^2 = CP^2 + MP^2 + 2PM \times CP \cdot cos.q$$

&

$$MP : MC = sin.p : sin.q$$

&.

$$MP : CP = sin.q : sin.(q-p)$$

Deinde in triangulo CMT, ob angulos datos, erit

$$CM : CT : MT = sin.(q+\varpi) : \sin.(q+\varpi - p) : sin.p.$$

Hinc, angulis eliminatis, erit $MC = CG\sqrt{\frac{MC\cdot CM}{CP\cdot CT}}$,

seu $CG^2 = CP.CT$. Hinc erit $CP : CG = CG : CT$, unde positio tangentis expedite invenitur. Erit autem ex hac proportione *dividendo* $CP : PG = CG : TG$; & ob $CG = CI$ *componendo* $CP : IP = CG : TI$.

119. Cum sit

$$\frac{CE^2}{CG^2} = \frac{\sin.p\cdot\sin.(q+\varpi)}{\sin.\varpi\cdot\sin.(q-p)}; \frac{CK^2}{CM^2} = \frac{\sin.p\cdot\sin.(q-p)}{\sin.\varpi\cdot\sin.(q+\varpi)};$$

idemque

$$\frac{CM^2}{CG^2} = \frac{\sin.q\cdot\sin.(q+\varpi)}{\sin.(q-p)\cdot\sin.(q+\varpi-p)} \ \& \ \frac{CK^2}{CE^2} = \frac{\sin.q\cdot\sin.(q-p)}{\sin.(q+\varpi)\cdot\sin.(q+\varpi-p)},$$

erit

$$\frac{CE^2+CG^2}{CG^2} = \frac{\sin.p\cdot\sin.(q+\varpi)+\sin.\varpi\cdot\sin.(q-p)}{\sin.\varpi\cdot\sin.(q-p)}, \ \&$$

$$\frac{CK^2+CM^2}{CM^2} = \frac{\sin.p\cdot\sin.(q-p)+\sin.\pi\cdot\sin.(q+\varpi)}{\sin.\varpi\cdot\sin.(q+\pi)}.$$

At est

$$\sin.A\cdot\sin.B = \frac{1}{2}\cos.(A-B) - \frac{1}{2}\cos.(A+B),$$

& visissim

$$\frac{1}{2}\cos.A - \frac{1}{2}\cos.B = \sin.\frac{A+B}{2}\cdot\sin.\frac{B-A}{2}.$$

Unde erit

$$\sin.p\cdot\sin.(q+\varpi)+\sin.\varpi\cdot\sin.(q-p)=\frac{1}{2}\cos.(q+\varpi-p)-\frac{1}{2}\cos.(q+\varpi+p)$$
$$+\frac{1}{2}\cos.(q-\varpi-p)-\frac{1}{2}\cos.(q+\varpi-p)$$
$$=\frac{1}{2}\cos.(q-\varpi-p)-\frac{1}{2}\cos.(q+\varpi+p)=\sin.q\cdot\sin.(p+\varpi).$$

Atque

$$\sin.p\cdot\sin.(q-p)+\sin.\varpi\cdot\sin.(q+\varpi)=\frac{1}{2}\cos.(q-2p)-\frac{1}{2}\cos.q$$
$$+\frac{1}{2}\cos.q-\frac{1}{2}\cos.(q+2\varpi)$$
$$=\frac{1}{2}\cos.(q-2p)-\frac{1}{2}\cos.(q+2\varpi)=\sin.(q+\varpi-p)\cdot\sin.(p+\varpi).$$

Hinc ergo erit

$$\frac{CE^2+CG^2}{CG^2}=\frac{\sin.q\cdot\sin.(p+\varpi)}{\sin.\varpi\cdot\sin.(q-p)},\ \&$$

$$\frac{CK^2+CM^2}{CM^2}=\frac{\sin.(q+\varpi-p)\cdot\sin.(p+\varpi)}{\sin.\varpi\cdot\sin.(q+\varpi)}.$$

unde conficitur

$$\frac{CE^2+CG^2}{CK^2+CM^2}=\frac{CG^2}{CM^2}\cdot\frac{\sin.q\cdot\sin.(q+\varpi)}{\sin.(q-p)\cdot\sin.(q+\varpi-p)}=\frac{CG^2}{CM^2}\cdot\frac{CM^2}{CG^2}.$$

Quare erit $CE^2+CG^2=CK^2+CM^2$, ideoque in eadem linea secundi ordinis summa quadratorum binarum diametrorum coniugatarum semper est constans.

120 . Cum igitur dentur duae semidiametri conjugatae *CG & CE.* pro semidiametro *CM* ad lubitum assumta statim reperitur ut ejus semidiameter conjugata *CK* sumendo $CK=\sqrt{(CE^2+CG^2-CM^2)}$. Ex superioribus ergo sectionum conicarum proprietatibus erit $TG\cdot TI:TM^2=CG\cdot CI:CK^2=CG^2:CK^2=CG^2:CE^2+CG^2-CM^2$; ideoque

$$TM=\frac{1}{CG}\sqrt{\left(TG\cdot TI\left(CE^2+CG^2-CM^2\right)\right)}$$

Simili modo, si producta ordinata *MN* ducatur tangens *NT,* ambae tangentes *MT* et *NT* axi *TI* in eodem puncto *T* occurrent. Erit enim pro utraque $CP:CG=CG:CT$. At vero ducta recta *CN* erit

$$TN = \frac{1}{CG}\sqrt{\left(TG \cdot TI\left(CE^2 + CG^2 - CN^2\right)\right)}$$

adeoque

$$TM^2 : TN^2 = CE^2 + CG^2 - CM^2 : CE^2 + CG^2 - CN^2$$

Erit vero ob bisectam *MN* in *P*

$$\sin.CTM : \sin.CTN = TN : TM = \sqrt{\left(CE^2 + CG^2 - CN^2\right)} : \sqrt{\left(CE^2 + CG^2 - CM^2\right)}$$

121. Ducantur (Fig. 28) in terminis diametri *A* et *B* tangentes *AK, BL* ac producatur tangens quaecunque *MT,* donec utramque tangentem secet in punctis *K* et *L.* Sit *ECF* diameter coniugata, cui cum applicatae *MP* tum tangentes *AK* et *BL* erunt parallelae.

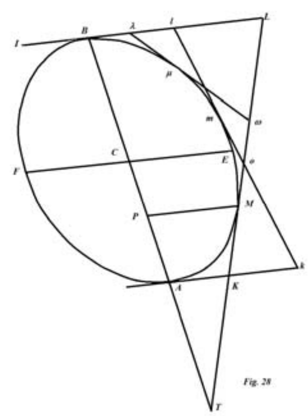


Fig. 28

Cum iam sit ex natura tangentis

$$CP : CA = CA : CT ,$$

ob $CB = CA$ erit

$$CP : AP = CA : AT \text{ et } CP : BP = CA : BT ,$$

ergo

$$CP : CA = CA : CT = AP : AT = BP : BT$$

hincque $AT : BT = AP : BP$. At est $AT : BT = AK : BL$, ergo

$$AK : BL = AP : BP .$$

Deinde est

$$AT = \frac{CA \cdot AP}{CP}, \quad BT = \frac{CA \cdot BP}{CP}$$

et

$$PT = \frac{CA \cdot AP}{CP} + AP = \frac{AP \cdot BP}{CP},$$

ergo

$$AT : PT = CA : BP = AK : PM ;$$

similique modo erit

$$BT : PT = CA : AP = BL : PM \;;$$

unde fit

$$AK = \frac{CA \cdot PM}{BP}, \quad BL = \frac{CA \cdot PM}{AP}$$

et

$$AK \cdot BL = \frac{CA^2 \cdot PM^2}{AP \cdot BP}.$$

At est $AP \cdot BP : PM^2 = AC^2 : CE^2$, unde consequitur ista egregia proprietas

$$AK \cdot BL = CE^2,$$

ex qua porro fit

$$AK = CE\sqrt{\frac{AP}{BP}} \quad \text{et} \quad BL = CE\sqrt{\frac{BP}{AP}}$$

et

$$AP : BP = AK^2 : CE^2 = CE^2 : BL^2 = KM : ML$$

atque

$$AK : BL = KM : LM.$$

122. In quocunque ergo curvae puncto *M* ducatur tangens occurrens tangentibus parallelis *AK, BL* in *K* et *L,* erit semper semidiameter *CE* tangentibus *AK* et *BL* parallela media proportionalis inter *AK* et *BL,* seu erit $CE^2 = AK.BL$. Quodsi ergo in alio quocunque curvae puncto *m* simili modo ducatur tangens *kml,* erit quoque $CE^2 = Ak.Bl$ ideoque

$$AK : Ak = Bl : BL$$

hincque erit quoque $AK : Kk = Bl : Ll$. Secent tangentes *KL* et *kl* se mutuo in *o,* eritque

$$AK : Bl = Ak : BL = Kk : Ll = ko : lo = Ko : Lo.$$

Atque hae sunt praecipuae sectionum conicarum proprietates, ex quibus Newtonus plurima insignia problemata resolvit in *principiis*.

123. Cum sit $AK : Bl = Ko : Lo$, si tangens *LB* producatur in *I,* ut sit $BI = AK$, erit *I* punctum, ubi tangens ex altera parte ipsi *KL* parallela hanc tangentem *LB* esset sectura, quemadmodum *K* in tangente *LK* est punctum, ubi ea a tangente *AX* ipsi *BL* parallela secatur. Transibit ergo recta *IK* per centrum *C* ibique bifariam secabitur. Quodsi igitur duae quaecunque tangentes *BL, ML* modo praescripto in *I* et *K* producantur eaeque

a tertia tangente *lmo* in punctis *l* et *o* secentur, erit $BI : Bl = Ko : Lo$ et *componendo* $IB : Il = Ko : KL$; ubicunque ergo tertia tangens *lmo* ducatur, erit perpetuo $IB \cdot KL = Il.Ko$. Ducta ergo quarta tangente quacunque $\lambda\mu\omega$ binas primum assumtas *IL* et *KL* secante in *A* et ω, erit pariter

$$IB \cdot KL = I\lambda \cdot K\omega$$

ideoque $Il \cdot Ko = I\lambda.K\omega$ seu $Il : I\lambda = K\omega : Ko$. Ductis ergo rectis $l\omega$, λo, in qua ratione hae secabuntur, recta per sectionum puncta transiens in eadem ratione secabit rectam *IK*. Quare, si rectae $l\omega$ et λo bisecentur, recta per bisectionis puncta transiens bisecabit quoque rectam *IK* ideo que per centrum sectionis conicae *C* transibit.

124. Quod (Fig. 30) recta *nmH*, quae rectas $l\omega$, λo in data ratione secat,

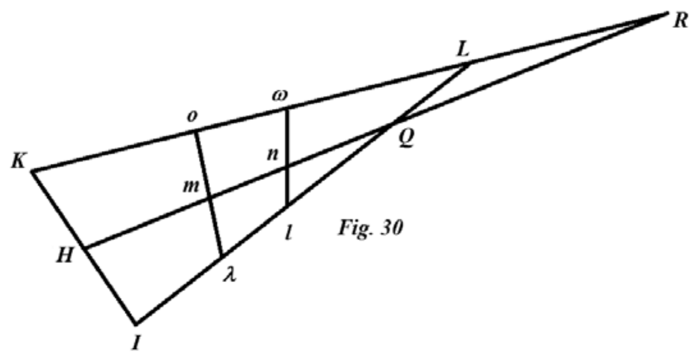


Fig. 30

in eadem ratione secare debeat rectam *KI*, siquidem fuerit $Il : I\lambda = K\omega : Ko$ seu $I\lambda : \lambda l = Ko : o\omega$, hoc modo ex Geometria ostendetur. Secet recta *mn* utramque $l\omega$ et λo in ratione $m : n$ seu sit $\lambda m : mo = ln : n\omega = m : n$ et ea producta traiiciat tangentes *IL* et *KL* in *Q* et *R* ; eritque

$$\sin.Q : \sin.R = \frac{ln}{Ql} : \frac{n\omega}{R\omega} = \frac{\lambda m}{Ql} : \frac{mo}{Ro} = \frac{m}{Ql} : \frac{n}{R\omega},$$

ergo $Ql : R\omega = Q\lambda : Ro$ et *dividendo*

$$l\lambda : o\omega = Q\lambda : Ro = Ql : R\omega .$$

Cum vero sit $l\lambda : o\omega = I\lambda : Ko$, erit quoque

$$QI : RK = l\lambda : o\omega \text{ et } \sin.Q : \sin.R = \frac{m}{l\lambda} : \frac{n}{o\omega}.$$

At est quoque

$$\sin.Q:\sin.R=\frac{HI}{QI}:\frac{HK}{KR}=\frac{HI}{l\lambda}:\frac{HK}{o\omega},$$

unde fit

$$HI:HK=m:n=\lambda m:mo=ln:n\omega.$$

125. Datis (Fig. 27) duabus semidiametris coniugatis *CG* et *CE,* quae angulum obliquum $GCE = q$ inter se comprehendant, semper reperiri poterunt duae aliae semidiametri coniugatae *CM* et *CK,* quae angulum *MCK* rectum constituant. Sit angulus $GCM = p$, et posito $ECK = \varpi$ erit $q+\varpi-p=90°$ ideoque

$$\sin.\varpi=\cos.(q-p) \text{ et } \sin.(q+\varpi)=\cos.p.$$

Unde (ex § 119) erit

$$\frac{CE^2}{CG^2}=\frac{\sin.p\cdot\cos.p}{\sin.(q-p)\cdot\cos.(q-p)}=\frac{\sin.2p}{\sin.2(q-p)}=\frac{\sin.2p}{\sin.2q\cdot\cos.2p-\cos.2q\cdot\sin.2p};$$

ergo

$$\frac{CG^2}{CE^2}=\sin.2q\cdot\cot.2p-\cos.2q,$$

ex quo fit

$$\cot.2GCM=\cot.2q+\frac{CG^2}{CE^2\cdot\sin.2q},$$

quae aequatio semper praebet solutionem possibilem. Erit vero

$$\frac{CM^2}{CG^2}=\frac{\sin.q\cdot\cos.p}{\sin.(q-p)} \text{ et } \frac{CG^2}{CM^2}=1-\frac{\text{tang.}p}{\text{tang.}q}$$

unde

$$\text{tang.}p=\text{tang.}q-\frac{CG^2}{CM^2}\text{tang.}q.$$

At cum sit

$$CM^2+CK^2=CG^2+CE^2 \text{ et } CK\cdot CM=CG\cdot CE\cdot\sin.q,$$

erit

$$CM+CK=\sqrt{\left(CG^2+2CG\cdot CE\cdot\sin.q+CE^2\right)}$$

et

$$CM - CK = \sqrt{\left(CG^2 - 2CG \cdot CE \cdot \sin.q + CE^2\right)} \;,$$

unde ipsae diametri coniugatae orthogonales reperiuntur.

126. Sint igitur (Fig. 29) *CA* et *CE* ambae semidiametri coniugatae sectionis conicae orthogonales, quae vocari solent *diametri principales,* sese in centro *C* normaliter decussantes. Sit abscissa $CP = x$, applicata $PM = y$, eritque, uti vidimus, $yy = \alpha - \beta xx$, vocatis autem semidiametris principalibus $AC = a$,

$CE = b$ erit $\alpha = bb$ et $\beta = \dfrac{bb}{aa}$, unde fit $yy = bb - \dfrac{bbxx}{aa}$.

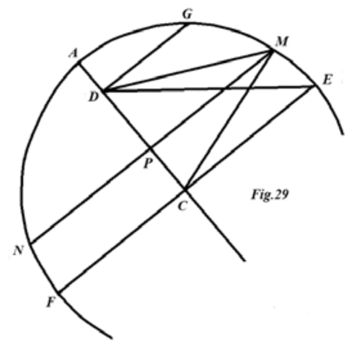


Fig.29

Ex qua aequatione intelligitur, cum non mutetur, sive *x* et *y* sumantur affirmativae sive negativae, curvam esse habituram quatuor partes similes et aequales utrinque circa diametros *AC* et *EF* sitas. Erit nempe quadrans *ACE* similis et aequalis quadranti *A CF,* hisque bini pares ad alteram partem diametri *EF* sunt positi.

127. Si ex centro *C*, quod pro initio abscissarum assumsimus, ducamus rectam *CM,* erit ea

$$= \sqrt{\left(xx + yy\right)} = \sqrt{\left(bb - \frac{bbxx}{aa} + xx\right)},$$

unde intelligitur, si fuerit $b = a$ seu $CE = CA$, fore $CM = \sqrt{bb} = b = a$. Hoc ergo casu omnes rectae ex centro *C* ad curvam productae inter se erunt aequales; quae cum sit proprietas circuli, manifestum est sectionem conicam, cuius binae diametri coniugatae principales sint inter se aequales, esse circulum, cuius adeo aequatio inter coordinatas orthogonales, positis $CP = x$ et $PM = y$, erit $yy = aa - xx$, existente radio circuli $CA = a$.

128. Sin autem non fuerit $b = a$, recta *CM* per *x* rationaliter nunquam exprimi poterit. Dabitur autem aliud punctum *D* in axe, a quo omnes rectae ad curvam ductae *DM* rationaliter exprimi possunt; ad quod inveniendum ponatur $OD = f$, atque ob $DP = f - x$ erit

$$DM^2 = ff - 2fx + xx + bb - \frac{bbxx}{aa} = bb + ff - 2fx + \frac{\left(aa - bb\right)xx}{aa},$$

quae expressio quadratum evadet, si fuerit

$$ff = \frac{(aa-bb)\ (bb+ff)}{aa} \quad \text{seu} \quad 0 = aa - bb - ff\ ,$$

unde fit

$$f = \pm\sqrt{(aa-bb)},$$

huiusmodi ergo punctum dabitur geminum in axe *AC,* utrinque scilicet a centro in distantia $CD = \pm\sqrt{(aa-bb)}$. Erit autem tum

$$DM^2 = aa - 2x\sqrt{(aa-bb)} + \frac{(aa-bb)xx}{aa},$$

hincque

$$DM = a - \frac{x\sqrt{(aa-bb)}}{a} = AC - \frac{CD\cdot CP}{AC}.$$

Facto $CP = 0$ fiet $DM = DE = a = AC$, sumta autem abscissa $CP = CD$ seu $x = \sqrt{(aa-bb)}$, recta *DM* abibit in applicatam *DG* eritque ergo

$$DG = \frac{bb}{a} = \frac{CE^2}{AC}$$

seu fiet *DG* tertia proportionalis ad *AC* et *CE.*

129. Ob singularem hanc proprietatem, qua puncta *D* hoc modo definita gaudent, ista diametri principalis puncta omnino attentione sunt digna; plurimis aliis autem haec eadem puncta praedita sunt eximiis proprietatibus, ob quas peculiaria nacta sunt nomina. Vocantur vero ista puncta *foci* seu *umbilici* sectionis conicae; et, cum in diametro maiori *a* sint posita, ista diameter a sua coniugata *b* ita distinguitur, ut ea vocetur *axis principalis et transversus,* dum altera *b* eius *axis coniugatus* appellatur. Applicata vero orthogonalis *DG* in ipso foco alterutro erecta nomen *semiparametri* obtinuit, tota enim *parameter* est ordinata in *D,* seu *DG* bis sumta, quae etiam *latus rectum* nuncupatur. Est ergo semiaxis coniugatus *CE* media proportionalis inter semiparametrum *DG* et semiaxem transversum *AC.* Termini porro axis transversi, ubi is a curva intersecatur, vocantur *vertices,* ut *A* ; atque hanc habent proprietatem, ut iis in locis tangens curvae sit ad axem principalem *AC* normalis

130. Ponatur semiparameter $DG = c$ et distantia foci a vertice $AD = d$, erit

$$CD = a - d = \sqrt{(aa - bb)} \text{ et } DG = \frac{bb}{a} = c,$$

unde fit

$$bb = ac \text{ et } a - d = \sqrt{(aa - ac)};$$

ergo

$$ac = 2ad - dd \text{ et } a = \frac{dd}{2d - c} \text{ et, } b = d\sqrt{\frac{c}{2d - c}}.$$

Ex datis ergo distantia foci a vertice $AD = d$ et semilatere recto $DG = c$ sectio conica determinatur. Posito nunc $CP = x$ erit

$$DM = a - \frac{(a - d)x}{a} = \frac{dd}{2d - c} - \frac{(c - d)x}{d}.$$

Sit $DP = t$, erit

$$x = CD - t = \frac{(c - d)d}{2d - c} - t;$$

unde fit

$$DM = c + \frac{(c - d)t}{d}.$$

Vocetur angulus $ADM = \upsilon$, erit

$$\frac{t}{DM} = -\cos.\upsilon$$

ideoque

$$d \cdot DM = cd + (d - c)DM \cdot \cos.\upsilon$$

et

$$DM = \frac{cd}{d - (d - c) \cdot \cos.\upsilon} \text{ et } \cos.\upsilon = \frac{d(DM - DG)}{(d - c)DM}.$$