CHAPTER II

CONCERNING THE CHANGING OF COORDINATES

23. A given curve is described in some manner upon the axis *RS* by an equation between the coordinates *x* and *y*, of which that abscissa [*i.e. x*] specifies this applied line [*i.e. y*] (see Fig. 2), and by assuming the start of the abscissas as it pleases, at some point *A* ; thus in turn, if now the curved line were described, its nature will be expressed by an equation between the coordinates. But here, whatever curve shall be given, still two things are left to our choice : evidently the position of the axis *RS* and the starting point *A* of the abscissas. Which since they [*i.e.* the origin and the axis] may be varied in an indefinite number of ways, even for the same curved line innumerable equations are able to be shown, and on this account the diversity of equations does not always follow from the diversity of curved lines, even if diverse curves will always provide diverse equations.

24. Therefore since, with the variation of the axis as well as of the origin of the abscissas, innumerable equations expressing the nature of the same curve, all these will be compared amongst themselves, so that from a single given equation all the rest may be able to be found. Indeed from a given equation between the coordinates the curved line itself can be determined, but with this known, if we assume some right line for the axis and on that some point for the beginning of the abscissas, an equation between the orthogonal coordinates will be defined. Therefore in this chapter we will discuss a method, with the aid of which, if an equation were given for the curve, the equation may be found between the coordinates, according to some other axis and some start of the abscissas, which expresses the nature of this same curve. And in this manner generally all the equations will be found, which express the nature of the same curve, and thus the diversity of curved lines will be able to be judged from the diversity of the equations.

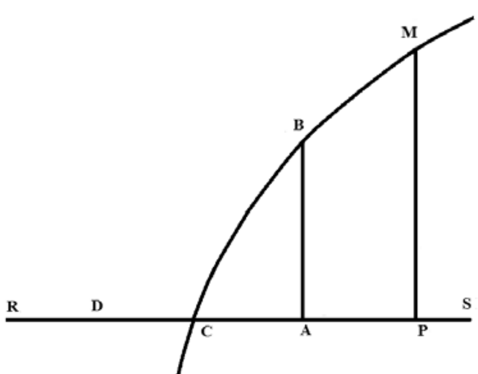


Fig. 7

25. Therefore some equation shall be given between *x* and *y*, from which with the right line *RS* taken (Fig. 7) for the axis and the point *A* for the start of the abscissas, thus so that *x* may denote the abscissa *AP* and *y* the applied line *PM*, the curved line *CBM* may be produced, the nature of which therefore is expressed by the given equation. Now at first we may retain the same axes *RS*, but we may assume another point on that, *D*, for the start of the abscissas, thus so that now the abscissa *DP* may correspond to another point of the curve *M*, which is put $= t$, truly the applied line MP will remain the same $= y$, as

before : therefore we may seek the equation between t and y, by which the nature of the same curve *CBM* is expressed. The interval may be put in place $AD = f$, so that from A to the left it lies in the region of negative abscissas, and there will be $DP = t = f + x$ and thus $x = t - f$. Whereby, if $t - f$ may be substituted in place of x in the equation given everywhere between x and y, an equation between t and y will be produced, which will show the same curved line CBM. Therefore since the magnitude $AD = f$ may depend on our choice, now we are able to adapt innumerable different equations, which all may express the same curved line.

26. If the curve may cut the axis somewhere *RS*, as in *C*, then with this point *C* taken for the origin of abscissas an equation will be obtained of this kind, which can be pout in place with the abscissa $CP = 0$ likewise the vanishing applied line *PM* shall be going to be allowed, if indeed it may correspond to the single point *C* of the axis. But the intersection *C*, if any or several may be given, will be found from the first equation proposed between x and y on putting $y = 0$ and by seeking from the equation the value or values of x. For where the curve falls on the axis, there it becomes $y = 0$, therefore in turn by making $y = 0$ all these abscissas or values of x will be elicited, where the curve falls on the axis.

27. Therefore the start of the abscissas will be changed, but with the axis retained, if the abscissa x may be increased or diminished by a given quantity, that is, if $t - f$ may be put in place of x ; where f will be a positive quantity, if the new beginning of the abscissas *D* were removed to the left from *A*; truly f will be a negative quantity, if the point *D* were placed to the right of *A*.

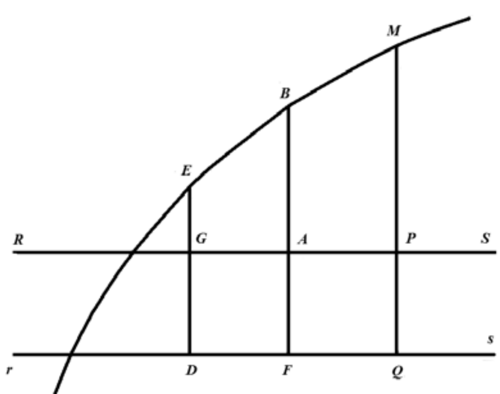


Fig. 8

Now we may put in place another axis *rs* parallel to be assumed for the curve described (Fig. 8) *LBM* with the beginning of abscissas *D* from the given equation between $AP = x$ and $PM = y$; but this axis may fall in the region of negative applied lines and its distance from the first axis shall be $AF = g$ and the interval may be $DF = AG = f$ put in place. Therefore if on this new axis the abscissa $DQ = t$ corresponds to the point of the curve *M* and the applied line $QM = u$, there will be

$$t = DF + FQ = f + x \text{ and } u = PM + PQ = g + y,$$

from which

$$x = t - f \text{ and } y = u - g.$$

Whereby, if in the given equation between *x* and *y* there may be substituted everywhere $t - f$ in place of x and $u - g$ in place of *y*, an equation will arise between *t* and *u* ; by which the nature of the same curved line may be expressed.

28. Therefore since the magnitudes f and *g* may depend on our choice and hence they may be defined in an indefinite number of way, there will be infinitely more diverse equations able to be formed as in the former case, which still all relate to the same equation. But if therefore two equations, the one between *x* and *y* and the other between *t* and *u*, may only differ in turn between each other in this respect, the one may be transformed into the other, if the coordinates of the one given may be increased or decreased by a given amount, then it is apparent both the equations although different will show the same curved line. Hence therefore innumerable diverse equations will be formed, which all express still the nature of the same curved line.

29. A new axis (Fig. 9) *rs* may be put in place normal to the first *RS* and cutting the same at the beginning of the abscissas *A*, thus so that for each axis there shall be the same start of the abscissas *A*. Because the equation is given for the axis *RS* the curve LM between the abscissa $AP = x$ and the applied line $PM = y$, from the point *M* of the curve a perpendicular *MQ* may be drawn to the new axis *rs* and it may be called the new perpendicular axis, and the new abscissa may be called $AQ = t$, the new applied line $QM = u$, and on that account there will be the rectangular parallelogram $APMQ$ $t = y$ and $u = x$. Hence from the given equation between *x* and *y* an equation will be formed between *t* and *u* by putting *u* in place of *x* and *t* in place of *y*. Therefore the first abscissa *x* now will change into the applied line $QM = u$ and the first applied line *y* now will be changed into the abscissa $AQ = t$, and thus for that new axis no other variation of the equation is induced, except that the coordinates *x* and *y* may be changed between themselves, and for this reason the abscissas and applied lines likewise are accustomed to be called the coordinates, with no distinction made, either may be taken for the abscissa or applied line. For indeed with an equation proposed between the two coordinates *x* and *y*, the same curve appears, whether *x* or *y* may be taken as indicating the abscissa.

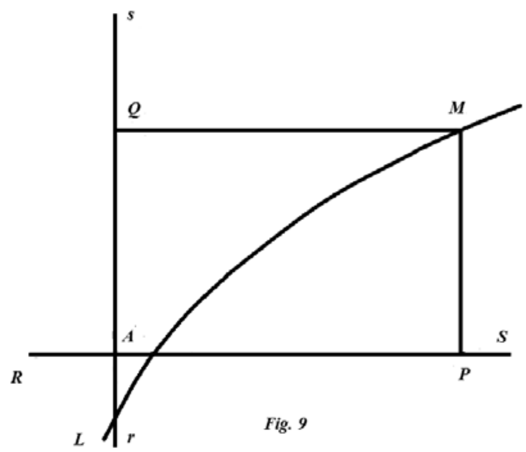


Fig. 9

30. Here we have put a part *As* of this new axis *rs* to show the positive abscissas and the region of the positive applied lines to the right of *rs*, which since this may depend on

choice, may be changed as you wish. Clearly, if the part *Ar* may be designated to the positive axis, certainly there will be $AQ = -t$ and thus in the equation between *x* and *y* in place of *y* there must be put $-t$. Then, if the region of negative applied lines may be put in place to the right of the axis *rs*, there becomes $QM = -u$, and $-u$ must be written for *x*. And hence the nature of the curved line is understood not to have changed, even if in the equation between the coordinates either one or both may be placed negative ; since that is preserved in all equations.

31. Now the new axis *rs* may cut the former axis *RS* at some angle *SAs* (see Fig. 10) and the intersection may be made at the beginning of the abscissas *A*, which point may constitute the beginning of each of the abscissas. Therefore some equation shall be given for the curve LM with the axis *RS* between the abscissa $AP = x$ and the applied line $PM = y$, from which the equation ought to be found for the same curve for the new axis *rs*, or by sending a perpendicular *MQ* from the point M of the curve to the new axis, between the new abscissa $AQ = t$ and the applied line $MQ = u$. Let the angle $SAs = q$, the sine of which $= m$ and the cosine $= n$, with unity taken for the whole sine, so that there shall be $mm + nn = 1$. From *P* the normals *Pp* and *Pq* shall be drawn to the new coordinates, and there will be on account of $AP = x$

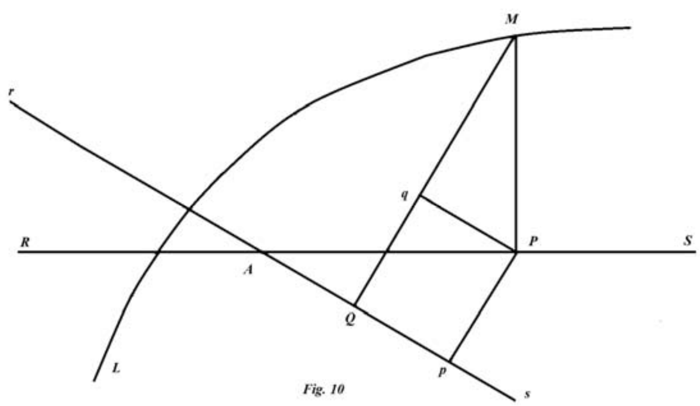


Fig. 10

$$Pp = x \cdot \sin. q, \quad Ap = x \cdot \cos. q$$

then, because the angle $PMQ = PAQ = q$, because $PM = y$

$$AQ = t = Ap - Qp = x \cdot \cos. q - y \cdot \sin. q$$

Therefore there arises from these

$$QM = u = Mq + Pp = x \cdot \sin. q + y \cdot \cos. q$$

and

$$Pq = Qp = y \cdot \sin. q, \quad Mq = y \cdot \cos. q .$$

32. But since there shall be $\sin. q = m, \quad \cos. q = n$, there will be

$$t = nx - my \text{ and } u = mx + ny,$$

hence there becomes

$$nt + mu = nnx + mmx = x \text{ and } nu - mt = nny + mmy = y.$$

Therefore the equation sought between t and u will be found, if in the equation proposed between x and y in place of x everywhere there may be written $mu + nt$ and $nu - mt$ in place of y, if indeed the portion *As* of the axis may contain positive abscissas, and positive applied lines may fall in the region QM . Here also we have put the angle *BAS* to lie in the region of negative applied lines ; because if moreover *AS* may lie above *AB*, in the calculation, the angle $BAS = q$ must be taken negative and therefore a negative sign m also must be taken.

33. Now some position *rs* may be granted to the new axis (see Fig. 11) and in which some point *D* may be taken for the start of the abscissas. Let *RS* be the former axis, for which an equation may be had between the $AP = x$ and the applied line $PM = y$, from which the nature of the curve *LM* is being expressed ; from which an equation between the other coordinates *t* and *u* related to the new axis *rs* must be able to be shown. Clearly with the perpendicular *MQ* sent from some point *M* of the curve to the new axis *rs* , the abscissa may be called $DQ = t$ and the applied line $QM = u$. So that the equation may be found between which, from the beginning of the new abscissas *D* a perpendicular *DG* may be drawn to the former axis *RS* and there may be put $AG = f$ and $DG = g$, then through *D* a line *DO* may be produced parallel to the former axis *RS,* to which the former applied line *PM* produced will cross at *O,* and there will be

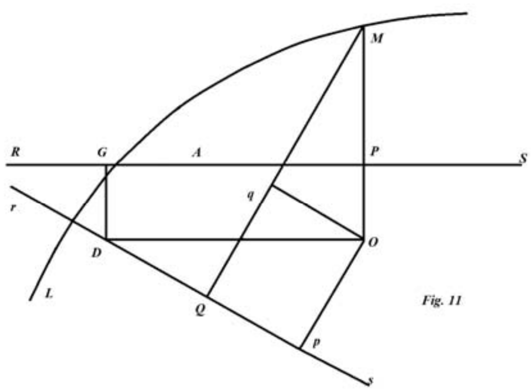


Fig. 11

$$MO = y + g \text{ and } DO = GP = x + f$$

And at last there may be put the angle $ODQ = q$, the sine of which shall be $= m$ and the cosine $= n$, always with the whole sine put $= 1$, so that there shall be $mm + nn = 1$.

34. Now from the point *O* the normals *Op* and *Oq* may be drawn both to the new axis *DQ* as well as to the applied line *MQ;* and on account of the angle $OMQ = ODQ$ and

$DO = x + f$ and $MO = y + g$, there will be :

$$Op = Qq = (x + f)\cdot \sin. q = mx + mf$$

and

$$Dp = (x + f)\cdot \cos. q = nx + nt.$$

And again,

$$Oq = Qp = (y + g)\cdot \sin. q = my + mg$$

and

$$Mq = (y + g)\cdot \cos. q = ny + ng\ .$$

Therefore from these it may be deduced :

$$DQ = t = nx + nf - my - mg$$

and

$$QM = u = mx + mf + ny + ng\ ,$$

and thus from the [old coordinates] *x* and *y* the new coordinates *t* and *u* may be defined. Hence truly there will be :

$$nt + mu = x + f \quad \text{and} \quad nu - mt = y + g$$

on account of $mm + nn = 1$, on account of which there will be had :

$$x = mu + nt - f \quad \text{and} \quad y = nu - mt - g\ ,$$

which values therefore if they may be substituted into the equation given between *x* and *y* in place of *x* and *y*, will produce an equation between *t* and *u,* from which the nature of the same curve *LM* will be expressed.

35. Because no axis *rs* can be imagined, which indeed shall be placed in the same plane with the curve, which may not be present in this latter determination, also for the same curve *LM,* no equation may exist between the orthogonal coordinates, which may not be contained in this equation between *t* and *u*. Therefore since the quantities *f* and *g* with the angle *q*, on which *m* and *n* may depend, may be able to be varied in an infinite number of ways, all the equations which have been found contained in an equation between *t* and *u* in this manner, will express the nature of the same curved line. On that account the equation between *t* and *u* is accustomed to be called the general equation for the curve

LM, because that includes within itself all the equations completely, which relate to the same curved line.

36. Now above we have indicated that it is with difficulty to judge from the diversity of some number of equations between the coordinates, whether these may refer to the same curved line, or to different ones : therefore now the way is apparent in which all questions of this kind are to be decided. Indeed let there be two equations proposed, the one between x and y and the other between t and u, putting in the former :

$$x = mu + nt - f \quad \text{et} \quad y = nu - mt - g\ ,$$

where m and n thus may depend on each other, so that there shall be $mm + nn = 1$; with which done it will be seen clearly, whether that other equation may be contained in this between t and u, but which has been elicited, or whether the quantities f, g thus shall be able to be defined from m and n, so that the other equation itself may result between t and u. Because if were to happen, both equations express the same curved line, but if otherwise, different curved lines.

EXAMPLE

It will be apparent in this manner that these two equations

$$yy - ax = 0$$

and

$$16uu - 24tu + 9tt - 55au + 10at = 0$$

to be referring to the same curved line, even if they may differ greatly themselves : for if we may put in the first equation :

$$x = mu + nt - f \quad \text{and} \quad y = nu - mt - g\ ,$$

that will be transformed into this :

$$\begin{aligned} nnuu - 2mntu + mmtt - 2ngu + 2mgt + gg& \\ - \ mau - nat \quad + af \quad &= 0 \end{aligned}$$

Whether therefore it may be contained in that other latter equation, we may multiply that by nn, that truly by 16, so that the first terms on each side may agree, and there will be had

$$16nnuu - 24nntu + 9nntt - 55nnau + 10nnat = 0$$

and

$$16nnuu - 32mntu + 16m^2tt - 32ngu + 32mgt + 16gg$$
$$-16mau - 16nat + 16af = 0.$$

Now it may be inquired, how many terms, requiring to be determined from the arbitrary *f, g, m* et *n*, can be returned equal, and indeed we will have in the first place $24nn = 32mn$ and $9nn = 16mm$, each of which gives $3n = 4m$, and because $mm = 1 - nn$ also there will be $25nn = 16$, hence

$$n = \frac{4}{5} \text{ and } m = \frac{3}{5}$$

and thus now three terms agree. The fourth and fifth give

$$55nna = 32ng + 16ma \text{ and } 10nna = 32mg - 16na,$$

from which, it being seen whether the same value may be elicited for g ; truly the first equation gives

$$g = \frac{55na}{32} - \frac{ma}{2n} = \frac{11a}{8} - \frac{3a}{8} = a$$

and the latter :

$$g = \frac{5na}{16} + \frac{na}{2m} = \frac{a}{3} + \frac{2a}{3} = a,$$

therefore each value is in agreement, and now the five terms agree. No other therefore can be present, unless there shall be $gg + af = 0$, because, since f shall not yet be determined, no difficulty is had, for there may be put $f = -a$. Therefore it has been shown that these two equations proposed represent the same curved line.

37. But although it can happen, that very diverse equations may represent the same curved line, yet on most numerous occasions it may be concluded from the diversity of the equations that they represent different curved lines. This arises, if the proposed equations may belong to diverse orders or in which the greatest dimensions are different, which constitute the coordinates x and y or t and u ; for in this case the curved lines, which may be indicated by these equations, certainly are different. For there may be an equation of any order between x and y, if there may be put

$$x = mu + nt - f \text{ and } y = nu - mt - g,$$

an equation will result between t and u of the same order ; whereby, if another equation proposed between t and u belonged to another order, it will indicate also a different curve.

38. Therefore unless the two equations, the one between x and y, the other between t and u , belong to the same order, it is to be concluded at once that the curved lines, which may be expressed by these equations, are different. Therefore there can only be room for doubt, if both equations were of the same order, and for these alone there will be a need for the cases to be examined as before, but which may eventuate to be laborious enough, if the equations may relate to some higher order; below more expedient rules will be treated, from which it will be possible at once to distinguish between the kinds of curves.

39. Which matters treated here are the precepts concerned with finding the general equation for some curved line, the same can be applied to straight lines. For in place of a curved line a right line LM shall be proposed (see Fig. 12), which we may put in place parallel to the axis RS; therefore the start of the abscissas may be taken at some place A,

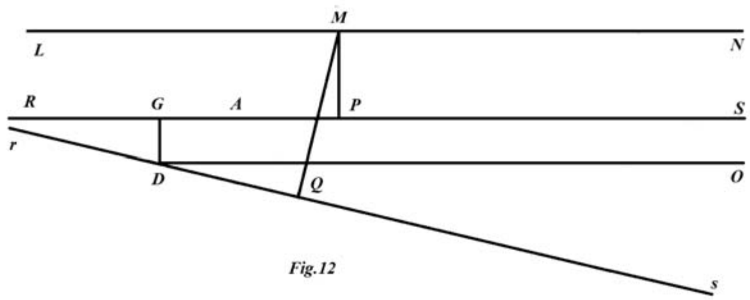


Fig.12

the applied line PM will always be of a constant magnitude, or $y = a$; which therefore is the equation for a right line parallel to the axis. Hence we may seek the general equation of a right line referring to some axis rs; therefore on putting $DG = g$, the sine of the angle $ODs = m$, the cosine $= n$ and with the abscissa called $DQ = t$ and the applied line $MQ = u$, on account of

$$y = nu - mt - g$$

there will be

$$nu - mt - g - a = 0,$$

which is the general equation for the right line. That may be multiplied by a constant k and there may be put $nk = \alpha$, $mk = -\beta$ and $(g + a)k = -b$, and the equation for the right line will be

$$\alpha u + \beta t + b = 0,$$

which since it shall be the general equation of the first order between t and u, it is apparent every equation of the first order between two coordinates does not show a curved line, but a right line.

40. Therefore such an equation

$$\alpha x+\beta y-a=0,$$

will be produced between the coordinates *x* and *y*, as often as that will produce a right line, the position of which thus will be determined with respect to the axis *RS* (see Fig. 13). In the first place there may be put $y=0$, and thus the point *G* on the axis is found, where this line cuts the axis, indeed it becomes $AC=\frac{a}{\alpha}$; then there may be put $x=0$, and there becomes $y=\frac{a}{\beta}$, which is the value of the applied line *AB* at the beginning of the abscissa. Therefore since the two point *B* and *C* may be had on the sought right line, that will be defined and thus the right line *LM* will satisfy the proposed equation. For some abscissa $AP=x$ and the corresponding right line $MP=y$, on account of the similar triangles *PM, CAB*, there will be $CP:PM=CA:AB$, that is

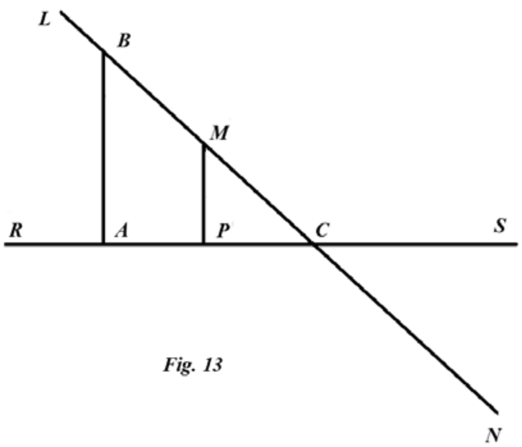


Fig. 13

$$\frac{a}{\alpha}-x:y=\frac{a}{\alpha}:\frac{\alpha}{\beta},$$

from which there becomes

$$\frac{ay}{\alpha}=\frac{aa}{\alpha\beta}-\frac{ax}{\beta}$$

or

$$\alpha x+\beta y=a,$$

which is the proposed equation itself.

41. If either α or $\beta=0$, then that construction cannot be used, but truly these cases by themselves are most easy. For let there be $\alpha=0$ and $y=a$, from which it is apparent that the satisfying line is a right line parallel to the axis and distant from that by the interval $=a$; but if there shall be $a=0$ or $y=0$, the satisfying line lies on the axis. But if indeed there were $\beta=0$ and $x=a$, it is seen that the satisfying line is a line normal to the axis, which will stand apart from the beginning of the abscissas by an interval $=a$. Clearly in this case all the applied lines will correspond to a single abscissa, thus so that the abscissa shall cease to be a variable quantity. From these, therefore, it is seen more clearly, how right lines can described by equations between orthogonal coordinates.

42. Up to this point we have assumed the coordinates, from which the nature of the curve is defined, to be normal to each other, truly in a like manner also from a given equation a curved line may be defined, if the applied lines may be inclined to the axis at some angle. Therefore in turn the nature of curved lines will be able to be expressed by an equation between two coordinates at oblique angles, and equations of this kind also are able to be varied by the axis as well as by the beginning of the abscissas in innumerable ways, with the curve remaining the same. And thus the general equation for the curve will be able to be shown for some obliquity of the coordinates. But if also now more and more oblique axes is put in place, the equation for the curve will be elicited much more broadly, that we will call the most general equation, because the nature of the curve not only will be expressed by an equation to some axis or another and relative to some starting point of the abscissas, but also for some obliquity of the coordinates. And therefore this most general equation will turn into the general equation, if the angle that the coordinates make between themselves may be set up as a right angle.

43. Let the equation be given for the curve *LM* between rectangular coordinates (see Fig. 14), truly between $AP = x$ and $PM = y$, and the equation is sought between the coordinates, with the axis *RS* and the start of the abscissas *A* kept the same, which take on a given angle which shall be $= \varphi$. Therefore the right line *MQ* is drawn from the point *M* to the axis *RS* at that given angle *MQA*, of which the sine shall be $= \mu$ and the cosine $= v$. Therefore the new abscissa will be *AQ* and *M Q* the new applied line ; therefore by putting $AQ = t$ and $QM = u$ in the right−angled triangle *PMQ* there will be :

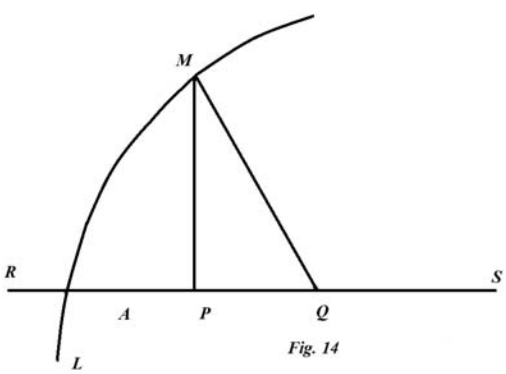


Fig. 14

$$\frac{y}{u} = \mu \text{ and } \frac{PQ}{u} = v = \frac{t-x}{u}.$$

On account of which there becomes :

$$u = \frac{y}{\mu} \text{ and } t = vu + x = \frac{vy}{\mu} + x$$

and in turn

$$y = \mu u \text{ and } x = t - vu.$$

Consequently, if in the proposed equation between *x* and *y* there may be put $x = t - vu$ and $y = \mu u$, an equation will be produced between the oblique angled coordinates *t* and u, which make the given angle φ between themselves.

44. But if moreover an equation were given for the curve *LM* between the oblique coordinates *AQ* and *QM*, from these in turn the equation may be found for the same curve between the orthogonal coordinates *AP* and *PM*. For let φ be the angle, which the applied lines *MQ* make with the abscissas *AQ*, the sine of which $= \mu l$ and the cosine $= \nu$, and the equation shall be given between $AQ = t$ and $QM = u$. From *M* the applied line *MP* may be drawn normal to the axis and, by putting the abscissa $AP = x$ and the applied line $MP = y$, because there is

$$u = \frac{y}{\mu} \text{ and } t = \frac{\nu y}{\mu} + x$$

if these values may be substituted into the proposed equation between *t* and *u*, the equation which is sought between *x* and *y* will be produced.

45. Now in the given equation between the orthogonal coordinates $AP = x$ and $PM = y$ for the curve *LM* (see Fig. 15), in this manner the most general equation for the same curved line will be able to be found. Some right line *rs* may be taken for the axis and at that point *D* the start of the abscissas, truly the applied lines *MT* drawn to this axis make an angle $DTM = \varphi$, the sine of which shall be $= \mu$ and the cosine $= \nu$; therefore there will be a new abscissa *DT* and applied line *TM*, between which the equation is sought. From *D* to the first axis *RS* the perpendicular *DG* may be drawn and there shall be $AG = f$, $DG = g$, and with *DO* drawn parallel to the axis *RS*, the sine of the angle *ODs* shall be $= m$, the cosine $= n$. From *M* to the new axis *rs* there may be drawn a new normal *MQ*, as we have done before, and there may be put $DQ = t$, $QM = u$, and the oblique angled coordinates moreover shall be $DT = r$, $TM = s$.

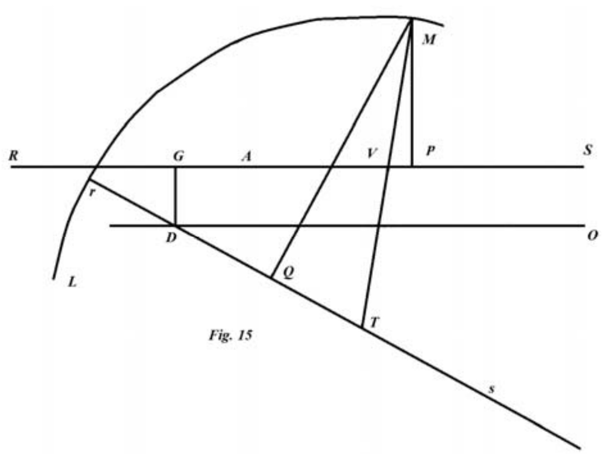


Fig. 15

Therefore there will be in the first place

$t = r - \nu z$ and $u = \mu s$ (from 43);

then truly there shall be

$$x = mu + nt - f \text{ and } y = nu - mt - y \text{ (from 36).}$$

Hence there becomes

$$x = nr - (n\nu - m\mu)s - f \text{ and } y = -mr + (\mu n + \nu m)s - g,$$

where $n\nu - m\mu$ is the cosine of the angle *AVM,* which the new applied lines set up with the former axis *RS*, and $\mu n + \nu m$ is the sine of this angle *AVM.* But if therefore in the equation between x and y in place of these values x and y these values found may substituted, an equation will be produced between the oblique angled coordinates r and $s,$ which will be the most general equation for the curve *LM*.

46. Because the dimension of the new variables r and $s,$ which are substituted in place of x and y, is present with a single dimension, it is evident the most general equation will be of this same order, as the proposed equation was between x et y. Therefore in whatever manner the equation for the same curve will be transformed, with both the axes changed in some manner and the starting point of the abscissas as well as with the angle of inclination of the coordinates, yet the equation will be always of the same order. Therefore although the equation between the coordinates, either orthogonal or oblique −angled, can be varied in infinite ways, so that it may relate to the same curve, yet neither will it be able to be carried to a higher order nor lowered to a lesser one. And because of this reason equations of different orders, in whatever way they were related to others, still show the same different curves.

CAPUT II

DE COORDINATARUM PERMUTATIONE

23. Quemadmodum ex aequatione inter coordinatas *x* et *y,* quarum illa abscissam, haec applicatam denotat, data curva (Fig. 2) describitur super axe *RS,* initio abscissarum *A* alicubi pro lubitu assumto, ita vicissim, si iam descripta fuerit linea curva, eius natura exprimi poterit per aequationem inter coordinatas. Hic autem, quamvis curva sit data, duae tamen res in arbitrio nostro relinquuntur, positio scilicet axis *RS* et principium abscissarum *A.* Quae cum infinitis modis variari queant, etiam pro eadem linea curva innumerabiles aequationes exhiberi poterunt, hancque ob causam ex aequationum diversitate non semper ad diversitatem linearum curvarum, quae illis aequationibus exprimantur, concludere licet, etiamsi diversae curvae perpetuo diversas praebeant aequationes.

24. Cum igitur, variato tam axe quam abscissarum initio, innumerabiles oriantur aequationes eiusdem curvae naturam exprimentes, hae omnes ita inter se erunt comparatae, ut ex data aequatione una reliquae omnes inveniri queant. Ex data enim aequatione inter coordinatas ipsa linea curva determinatur, hac autem cognita, si quaecunque linea recta pro axe et in ea punctum pro abscissarum principio assumatur, aequatio inter coordinatas orthogonales definietur. Hoc igitur capite methodum trademus, cuius ope, si aequatio pro curva fuerit data, ad alium axem quemcunque et abscissarum initium quodcunque aequatio inter coordinatas inveniri queat, quae eiusdem curvae naturam exprimat. Atque hoc modo reperientur omnes omnino aequationes, quae eiusdem curvae naturam comprehendant, sicque facilius diversitas linearum curvarum ex aequationum diversitate diiudicari poterit.

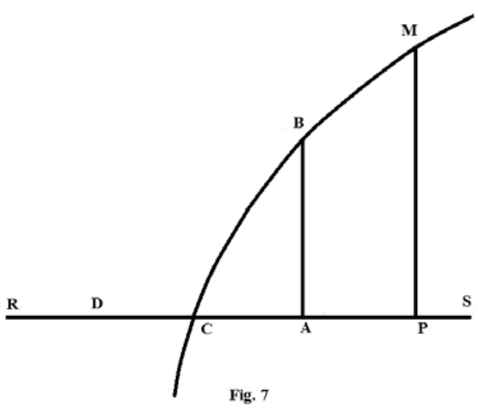


Fig. 7

25. Sit igitur data aequatio quaecunque inter *x* et *y,* ex qua sumta (Fig. 7) recta *RS* pro axe et *puncto A* pro initio abscissarum, ita ut *x* denotet abscissam *AP* et *y* applicatam *PM,* producatur linea curva *CBM,* cuius ergo natura per aequationem datam exprimitur. Retineamus iam primum eundem axem *RS*, at aliud punctum in eo *D* pro initio abscissarum assumamus, ita ut nunc puncto curvae *M* respondeat abscissa *DP*, quae ponatur $= t$, applicata vero MP manebit eadem $= y$, quae ante : quaeremus igitur aequationem inter *t* et *y*, qua eiusdem curvae *CBM* natura exprimatur. Ponatur intervallum $AD = f$, quod ab *A* sinistrorsum in regionem abscissarum negativarum cadat, eritque $DP = t = f + x$ ideoque $x = t - f$. Quare, si in aequatione inter *x* et *y* data

ubique loco *x* substituatur $t-f$, prodibit aequatio inter *t* et *y*, quae eandem lineam curvam CBM exhibebit. Cum igitur magnitudo $AD=f$ ab arbitrio nostro pendeat, iam innumerabiles diversas adepti sumus aequationes, quae omnes eandem lineam curvam exprimant.

26. Si curva alicubi axem *RS* traiiciat, uti in *C,* tum sumto hoc puncto *C* pro initio abscissarum eiusmodi obtinebitur aequatio, quae posita abscissa $CP=0$ simul applicatam *PM* evanescentem sit praebitura, siquidem unica tantum applicata puncto axis *C* respondeat. Intersectio autem *C*, si ulla pluresve dentur, invenietur ex aequatione primum proposita inter *x* et *y* ponendo $y=0$ et ex aequatione quaerendo valorem vel valores ipsius *x*. Ubi enim curva in axem incidit, ibi fit $y=0$, facto ergo vicissim $y=0$ omnes illae abscissae seu valores ipsius *x* elicientur, ubi curva in axem incidit.

27. Initium ergo abscissarum, retento axe, mutabitur, si abscissa *x* data quantitate sive augeatur sive minuatur, hoc est, si loco *x* ponatur $t-f$ *;* ubi *f* erit quantitas affirmativa, si novum abscissarum initium *D* sinistrorsum ab *A* fuerit remotum; erit vero *f* quantitas negativa, si punctum *D* ad dextram ab *A* fuerit situm.

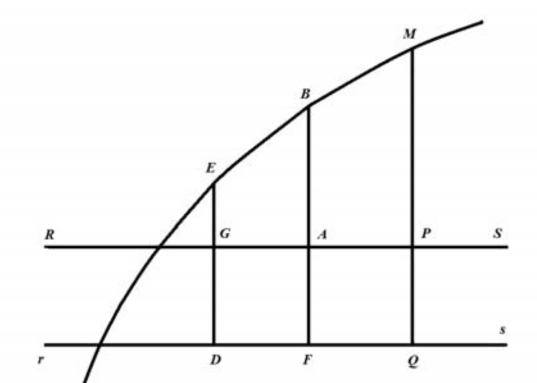


Fig. 8

Ponamus nunc (Fig. 8) descripta curva *LBM* ex data aequatione inter $AP=x$ et $PM=y$ alium assumi axem *rs* priori parallelum in eoque punctum *D* pro abscissarum initio; cadat autem iste axis in regionem applicatarum negativarum sitque eius a priori axe distantia $AF=g$ atque ponatur intervallum $DF=AG=f$. Sit igitur in hoc novo axe abscissa puncto curvae *M* respondens $DQ=t$ et applicata $QM=u$, eritque

$$t=DF+FQ=f+x \text{ et } u=PM+PQ=g+y,$$

unde fit

$$x=t-f \text{ et } y=u-g.$$

Quare, si in aequatione inter *x* et *y* data substituatur ubique $t-f$ loco *x* et $u-g$ loco *y,* orietur aequatio inter *t* et *u;* qua eiusdem lineae curvae natura exprimetur.

28. Cum igitur magnitudines *f* et *g* ab arbitrio nostro pendeant hincque infinitis modis definiri queant, infinities plures diversae formari poterunt aequationes quam priori casu, quae tamen omnes ad eandem lineam curvam pertineant. Quodsi ergo duae aequationes,

altera inter x et y et altera inter t et u, hoc tantum a se invicem discrepent, ut altera in alteram transformetur, si coordinatae unius datis quantitatibus sive augeantur sive minuantur, tum ambae aequationes licet diversae tamen eandem lineam curvam exhibebunt. Hinc igitur facile innumerabiles formabuntur aequationes diversae, quae tamen omnes eiusdem lineae curvae naturam exprimant.

29. Statuatur (Fig. 9) novus axis rs normalis ad priorem RS secansque ipsum in principio abscissarum A, ita ut pro utroque axe idem sit abscissarum initium A. Quoniam pro axe RS datur aequatio ad curvam LM inter abscissam $AP = x$ et applicatam $PM = y$, ducatur ex curvae puncto M in novum axem rs perpendicularis MQ et vocetur abscissa nova $AQ = t$, applicata nova $QM = u$, eritque ob $APMQ$ parallelogrammum rectangulum $t = y$ et $u = x$. Hinc ex aequatione inter x et y data formabitur aequatio inter t et u ponendo u loco x et t loco y. Prior ergo abscissa x nunc abit in applicatam $QM = u$ et prior applicata y nunc abit in abscissam $AQ = t$, pro isto itaque novo axe nulla alia aequationi variatio inducitur, nisi quod coordinatae x et y inter se commutentur, hancque ob rationem abscissa et applicata simul coordinatae vocari solent, nullo facto discrimine, utra pro abscissa applicatave accipiatur. Proposita enim aequatione inter duas coordinatas x et y, eadem curva emergit, sive x sive y ad abscissam indicandam accipiatur.

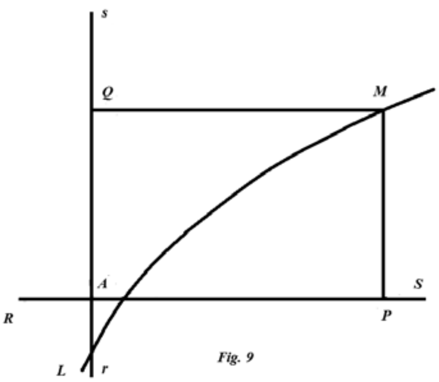
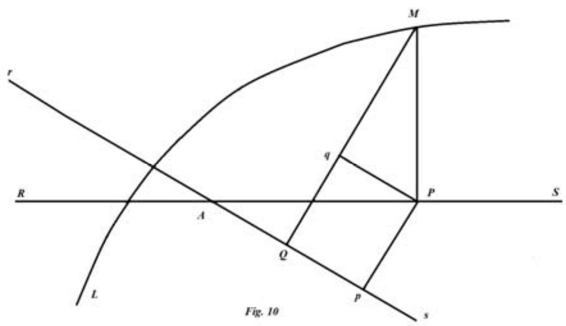


Fig. 9

30. Posuimus hic novi axis rs portionem As exhibere abscissas affirmativas atque ad dextram axis $r\,s$ statui regionem applicatarum affirmativarum, quae cum ab arbitrio pendeant, pro lubitu immutari poterunt. Scilicet, si axis portio Ar abscissis affirmativis destinetur, erit utique $AQ = -t$ sicque in aequatione inter x et y loco y poni debet $-t$. Deinde, si ad dextram axis rs regio applicatarum negativarum statuatur, fiet $QM = -u$, atque pro x scribi debebit $-u$. Atque hinc intelligitur naturam lineae curvae non mutari, etiamsi in aequatione inter coordinatas vel alterutra vel utraque negativa statuatur; id quod in omnibus aequationis transmutationibus est tenendum.

31. Secet nunc (Fig. 10) novus axis rs priorem RS sub angulo quocunque SAs fiatque intersectio in ipso abscissarum initio A, quod punctum in utroque axe initium abscissarum constituat. Data ergo sit pro axe RS aequatio quaecunque pro curva LM inter abscissam $AP = x$ et applicatam $PM = y$, ex qua reperiri debeat aequatio ad eandem curvam pro novo axe rs, seu ex curvae puncto M ad novum axem demisso perpendiculo MQ, inter abscissam novam $AQ = t$ et applicatam $MQ = u$. Sit angulus $SAs = q$, eius sinus $= m$ et cosinus $= n$, sumta unitate pro sinu toto, ut sit $mm + nn = 1$. Ex P ducantur normales Pp et Pq in novas coordinatas, eritque ob $AP = x$



$$Pp = x \cdot \sin. q, \quad Ap = x \cdot \cos. q,$$

deinde, quia angulus $PMQ = PAQ = q$, erit ob $PM = y$

$$AQ = t = Ap - Qp = x \cdot \cos. q - y \cdot \sin. q$$

Ex his ergo fiet

$$QM = u = Mq + Pp = x \cdot \sin. q + y \cdot \cos. q.$$

et

$$Pq = Qp = y \cdot \sin. q, \quad Mq = y \cdot \cos. q.$$

32. Cum autem sit $\sin. q = m, \quad \cos. q = n$, erit

$$t = nx - my \text{ et } u = mx + ny,$$

hinc fiet

$$nt + mu = nnx + mmx = x \text{ et } nu - mt = nny + mmy = y.$$

Aequatio ergo quaesita inter *t* et *u* reperietur, si in aequatione inter *x* et *y* proposita ubique loco *x* scribatur $mu + nt$ et $nu - mt$ loco *y*, siquidem axis portio *As* contineat abscissas affirmativas et applicatae affirmativae in regionem *QM* cadant. Posuimus hic etiam angulum *BAS* in regionem applicatarum negativarum cadere; quodsi autem *AS* supra *AB* caderet, in calculo angulus $BAS = q$ negativus ac propterea eius sinus *m* negative accipi deberet.

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33. Tribuatur nunc (Fig. 11) novo axi *rs* positio quaecunque in eoque sumatur punctum quodvis *D* pro abscissarum initio. Sit *RS* axis prior, pro quo habetur aequatio inter

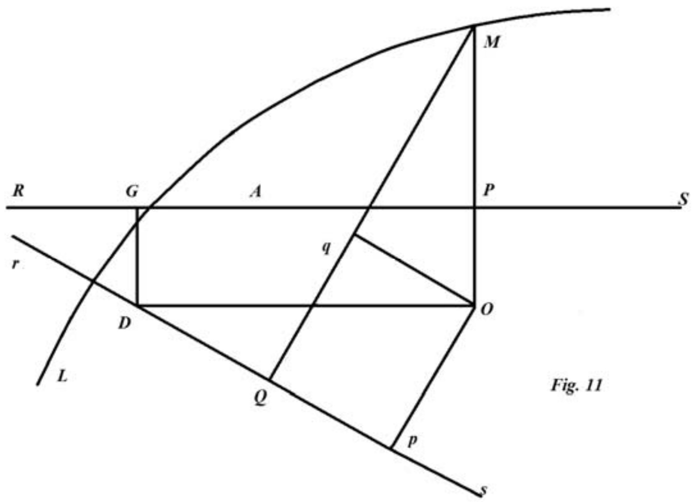

Fig. 11

abscissam $AP = x$ et applicatam $PM = y$, qua natura curvae *LM* exprimitur ; unde aequatio inter alias coordinatas *t* et *u* ad novum axem *rs* relatas exhiberi debet. Demisso scilicet ex quovis curvae puncto *M* in novum axem *rs* perpendiculo *MQ* vocetur abscissa $DQ = t$ et applicata $QM = u$. Inter quas ut aequatio inveniatur, ex novo abscissarum initio *D* in axem priorem *RS* ducatur perpendicularis *DG* ac ponatur $AG = f$ et $DG = g$, tum per *D* priori axi *RS* producatur parallela *DO,* cui prior applicata *PM* producta occurrat in *O,* eritque

$$MO = y + g \text{ et } DO = GP = x + f$$

Denique ponatur angulus $ODQ = q$, cuius sinus sit $= m$ et cosinus $= n$, posito semper sinu toto $= 1$, ut sit $mm + nn = 1$.

34. Iam ex puncto *O* ducantur tam in novum axem *DQ* quam in applicatam *MQ* normales *Op* et *Oq;* atque ob angulum $OMQ = ODQ$ et

$DO = x + f$ ac $MO = y + g$ erit

$$Op = Qq = (x + f)\cdot \sin. q = mx + mf$$

et

$$Dp = (x + f)\cdot \cos. q = nx + nt.$$

Porroque

$$Oq = Qp = (y + g)\cdot \sin. q = my + mg$$

et

$$Mq = (y+g)\cdot \cos. q = ny + ng \ .$$

Ex his igitur colligetur

$$DQ = t = nx + nf - my - mg$$

et

$$QM = u = mx + mf + ny + ng \ ,$$

sicque ex x et y definientur novae coordinatae t et u . Hinc vero erit

$$nt + mu = x + f \quad \text{et} \quad nu - mt = y + g$$

ob $mm + nn = 1$, quocirca habebitur

$$x = mu + nt - f \quad \text{et} \quad y = nu - mt - g \ ,$$

qui ergo valores si in aequatione inter x et y dato loco x et y substituantur, prodibit aequatio inter t et u, qua eiusdem curvae LM natura exprimetur.

35. Quoniam nullus excogitari potest axis rs, qui quidem in eodem plano cum curva sit situs, qui non in hac postrema determinatione contineatur, pro eadem quoque curva LM nulla existet aequatio inter coordinatas orthogonales, quae non in hac aequatione inter t et u inventa comprehendatur. Cum igitur quantitates f et g cum angulo q, unde m et n pendent, infinitis modis variari queant, omnes aequationes, quae in aequatione inter t et u hoc modo inventa continentur, eiusdem lineae curvae naturam exprimen t. Hanc ob rem ista aequatio inter t et u vocari solet aequatio generalis pro curva LM, quoniam ea in se complectitur omnes omnino aequationes, quae ad eandem lineam curvam pertinent.

36. Supra iam innuimus difficile esse ex diversitate aliquot aequationum inter coordinatas iudicare, utrum eae ad eandem lineam curvam, an ad diversas referantur: nunc igitur patet via omnes huiusmodi quaestiones diiudicandi. Sint enim duae propositae aequationes, altera inter x et y et altera inter t et u, ponatur in illa

$$x = mu + nt - f \quad \text{et} \quad y = nu - mt - g \ ,$$

ubi m et n ita a se invicem pendent, ut sit $mm + nn = 1$; quo facto dispiciendum erit, utrum altera illa aequatio inter t et u in hac, quae modo est eruta, contineatur, seu an quantitates f, g cum m et n ita definiri possint, ut ipsa altera aequatio inter t et u resultet. Quod si fieri possit, ambae aequationes eandem lineam curvam expriment, sin secus, diversas.

EXEMPLUM

Hoc modo patebit has duas aequationes

$$yy - ax = 0$$

et

$$16uu - 24tu + 9tt - 55au + 10at = 0$$

ad eandem lineam curvam referri, etiamsi ipsae plurimum discrepent: si enim in priori aequatione ponamus

$$x = mu + nt - f \quad \text{et} \quad y = nu - mt - g \, ,$$

ea transformabitur in hanc

$$\begin{aligned} nnuu - 2mntu + mmtt - 2ngu + 2mgt + gg \\ - \; mau - nat + af = 0 \end{aligned}$$

Num igitur in hac altera illa aequatio contineatur, multiplicemus illam per *nn*, hanc vero per 16, ut termini primi utrinque congruant, habebiturque

$$16nnuu - 24nntu + 9nntt - 55nnau + 10nnat = 0$$

et

$$\begin{aligned} 16nnuu - 32mntu + 16m^2tt - 32ngu + 32mgt + 16gg \\ -16mau - 16nat + 16af = 0. \end{aligned}$$

Nunc inquiratur, quot termini, arbitrariis *f, g, m* et *n* determinandis, aequales reddi queant, ae primo quidem habebimus $24nn = 32mn$ et $9nn = 16mm$, quarum utraque dat $3n = 4m$, et ob $mm = 1 - nn$ erit quoque $25nn = 16$,
hinc

$$n = \frac{4}{5} \text{ et } m = \frac{3}{5}$$

sicque iam tres termini conveniunt. Quartus et quintus dant

$$55nna = 32ng + 16ma \quad \text{et} \quad 10nna = 32mg - 16na \, ,$$

unde, an idem pro *g* valor eruatur, videndum est; dat vero prior

$$g = \frac{55na}{32} - \frac{ma}{2n} = \frac{11a}{8} - \frac{3a}{8} = a$$

et posterior

$$g = \frac{5na}{16} + \frac{na}{2m} = \frac{a}{3} + \frac{2a}{3} = a,$$

uterque ergo valor congruit et iam quinque termini conveniunt. Nil aliud ergo superest, nisi ut sit $gg + af = 0$, quod, cum *f* nondum sit determinatum, nil habet difficultatis, fiet enim $f = -a$. Ostensum ergo est has duas aequationes propositas eandem lineam curvam exhibere.

37. Quanquam autem fieri potest, ut aequationes admodum diversae eandem lineam curvam repraesentent, tamen saepenumero ex aequationum diversitate tuto linearum curvarum diversitas concluditur. Evenit hoc, si aequationes propositae ad diversos ordines pertineant seu in quibus maximae dimensiones, quas coordinatae *x* et *y* seu *t* et *u* constituunt, sunt diversae ; hoc enim casu lineae curvae, quae per has aequationes indicantur, certo erunt diversae. Cuiuscunque enim ordinis fuerit aequatio inter *x* et *y*, si ponatur

$$x = mu + nt - f \quad \text{et} \quad y = nu - mt - g\ ,$$

resultabit aequatio inter *t* et *u* eiusdem ordinis; quare, si altera aequatio inter *t* et *u* proposita ad alium ordinem pertineat, curvam quoque diversam indicabit.

38. Nisi igitur duae aequationes, altera inter *x* et *y*, altera inter *t* et *u* , ad eundem ordinem pertineant, statim concludendum est lineas curvas, quae illis aequationibus exprimuntur, esse diversas. Dubitatio ergo tantum locum habere potest, si ambae aequationes fuerint eiusdem ordinis, hisque solum casibus investigatione ante tradita opus erit, quae autem cum satis operosa evadat, si aequationes ad altiorem quempiam ordinem pertineant, infra expeditiores regulae tradentur, ex quibus statim varietas curvarum dignosci poterit.

39. Quae hic de invenienda aequatione generali pro quavis linea curva sunt praecepta,

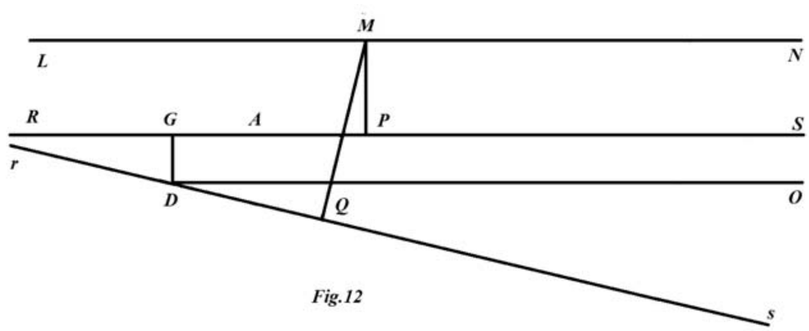

Fig.12

eadem ad lineam rectam accomodari possunt. Sit enim loco lineae curvae (Fig. 12) proposita linea recta *LM*, quam axi *RS* parallelam statuamus; ubicunque ergo initium abscissarum *A* capiatur, erit semper applicata *PM* constantis magnitudinis seu $y = a$; quae ergo est aequatio pro linea recta axi parallela. Quaeramus hinc aequationem generalem lineae rectae ad axem quemcunque *rs* relatam; posito ergo $DG = g$, anguli ODs sinu $= m$, cosinu $= n$ et vocata abscissa $DQ = t$ et applicata $MQ = u$, ob

$$y = nu - mt - g$$

erit

$$nu - mt - g - a = 0\,,$$

quae est aequatio generalis pro linea recta. Multiplicetur ea per constantem *k* et ponatur $nk = \alpha,\ mk = -\beta$ et $(g + a)k = -b$, eritque aequatio

$$\alpha u + \beta t + b = 0$$

pro linea recta, quae cum sit aequatio primi ordinis inter *t* et *u* generalis, patet omnem aequationem primi ordinis inter duas coordinatas nullam lineam curvam, sed rectam lineam exhibere.

40. Quoties. ergo inter coordinatas *x* et *y* talis prodit aequatio

$$\alpha x + \beta y - a = 0\,,$$

toties ea praebet lineam rectam, cuius (Fig. 13) positio respectu axis *RS* ita determinabitur. Ponatur primo $y = 0$, sicque in axe reperitur punctum *G*, ubi haec recta axem traiicit, fit enim $AC = \frac{a}{\alpha}$; tum ponatur $x = 0$, fietque $y = \frac{a}{\beta}$, qui est valor applicatae *AB* in initio abscissarum. Cum ergo habeantur duo puncta *B* et *C* in recta quaesita, ea erit definita ideoque aequationi propositae satisfaciet recta *LM*. Ponatur enim abscissa quaecunque $AP = x$ et respondens applicata $MP = y$, erit ob similitudinem triangulorum *CPM, CAB* $CP : PM = CA : AB$, hoc est

$$\frac{a}{\alpha} - x : y = \frac{a}{\alpha} : \frac{\alpha}{\beta}\,,$$

unde fit

$$\frac{ay}{\alpha} = \frac{aa}{\alpha\beta} - \frac{ax}{\beta}$$

seu

$$\alpha x + \beta y = a\,,$$

quae est ipsa aequatio proposita.

41. Si fuerit vel α vel $\beta = 0$, tum ista constructio usum habere non poterit, at vero isti casus per se sunt facillimi. Sit enim $\alpha = 0$ et $y = a$, unde patet lineam satisfacientem esse rectam axi parallelam ab eoque intervallo $= a$ remotam; sin sit $a = 0$ seu $y = 0$, linea satisfaciens in axem incidet. Quodsi vero fuerit $\beta = 0$ et $x = a$, perspicuum est lineam satisfacientem esse rectam ad axem normalem, quae ab initio abscissarum intervallo $= a$ distet. Hoc scilicet casu omnibus applicatis unica abscissa respondet, ita ut abscissa quantitas variabilis esse desinat. Ex his igitur luculenter perspicitur, quemadmodum lineae rectae per aequationes inter coordinatas orthogonales designari queant.

42. Assumsimus hactenus coordinatas, quibus natura curvae definitur, inter se esse normales, simili vero modo etiam ex data aequatione linea curva definietur, si applicatae ad axem sub angulo quocunque inclinentur. Vicissim ergo natura curvae exprimi poterit per aequationem inter duas coordinatas obliquangulas atque huiusmodi aequationes quoque variatis cum axe tum principio abscissarum innumerabilibus modis variari possunt, manente curva eadem. Sicque pro quavis obliquitate coordinatarum aequatio generalis ad curvam exhiberi potest. Quodsi vero etiam haec obliquitas alia atque alia statuatur, multo latius patens eruetur aequatio pro curva, quam aequationem generalissimam appellabimus, quoniam naturam curvae non solum exprimit per aequationem ad quemvis axem et quodcunque initium abscissarum relatam, sed etiam pro quacunque coordinatarum obliquitate. Haecque adeo aequatio generalissima abibit in aequationem generalem, si angulus, quem coordinatae inter se constituunt, rectus statuatur.

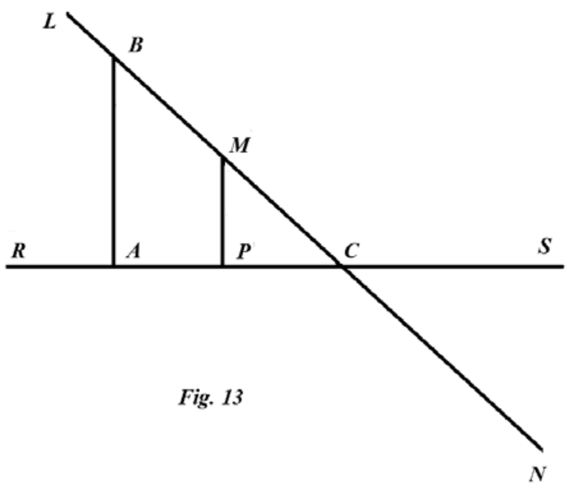

Fig. 13

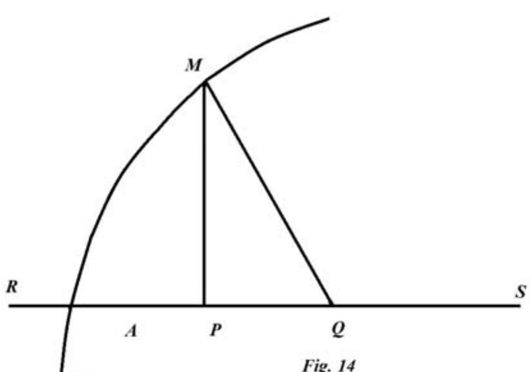

Fig. 14

43. Data sit (Fig. 14) pro curva *LM* aequatio inter coordinatas rectangulas, nempe inter $AP = x$ et $PM = y$, et quaeratur, retento axe *RS* et initio abscissarum *A* eodem, aequatio inter coordinatas, quae datum angulum comprehendant, qui sit $= \varphi$. Ex puncto ergo *M* ad axem *RS* ducatur recta *MQ* ad angulum illum datum *M QA*, cuius sinus sit $= \mu$ et cosinus

$=\nu$. Erit ergo *AQ* nova abscissa et *M Q* nova applicata; posito ergo $AQ = t$ et $QM = u$ erit intriangulo rectangulo *PMQ*

$$\frac{y}{u} = \mu \text{ et } \frac{PQ}{u} = \nu = \frac{t-x}{u}.$$

Quocirca fiet

$$u = \frac{y}{\mu} \text{ et } t = \nu u + x = \frac{\nu y}{\mu} + x$$

et vicissim

$$y = \mu u \text{ et } x = t - \nu u.$$

Consequenter, si in aequatione inter *x* et *y* proposita ponatur $x = t - \nu u$ et $y = \mu u$, prodibit aequatio inter coordinatas obliquangulas *t* et *u* , quae inter se datum angulum φ constituant.

44. Quodsi autem data fuerit pro curva *LM* aequatio inter coordinatas obliquangulas *AQ* et *QM,* ex ea vicissim reperietur aequatio pro eadem curva inter coordinatas orthogonales *AP* et *PM.* Sit enim φ angulus, quem applicatae *MQ* cum abscissis *AQ* constituunt, cuius sinus $= \mu l$ et cosinus $= \nu$, dataque sit aequatio inter $AQ = t$ et $QM = u$. Ex *M* ducatur ad axem applicata normalis *MP* et, posita abscissa $AP = x$ et applicata $MP = y$, quia est

$$u = \frac{y}{\mu} \text{ et } t = \frac{\nu y}{\mu} + x$$

si hi valores in aequatione inter *t* et *u* proposita substituantur, prodibit aequatio inter *x* et *y,* quae quaerebatur.

45. Data nunc aequatione inter coordinatas orthogonales (Fig. 15) $AP = x$ et $PM = y$ pro curva *LM,* hoc modo aequatio generalissima pro eadem linea curva inveniri poterit.
Sumatur recta quaecunque *rs* pro axe et in eo punctum *D* pro abscissarum initio, applicatae vero *MT* ad hunc axem ducetae faciant angulum $DTM = \varphi$, cuius sinus sit $= \mu$ et cosinus $= \nu$; erit ergo nova abscissa *DT* et applicata *TM,* inter quas aequatio
quaeritur. Ex *D* in axem priorem *RS* ducatur perpendicularis *DG* et sit

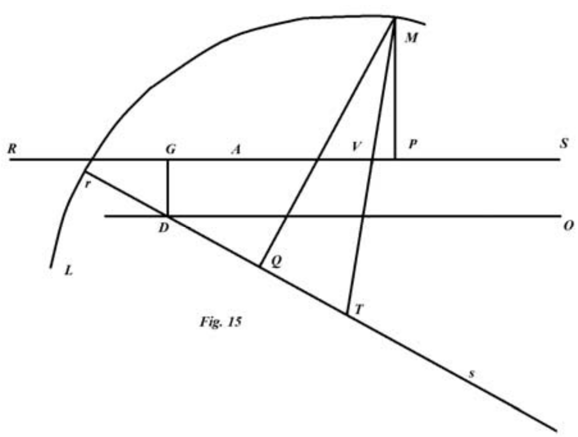

Fig. 15

$AG = f,\ DG = g$,
ductaque *DO* axi *RS* parallela sit anguli *ODs* sinus $= m$, cosinus $= n$. Ducatur, ut ante fecimus, ex *M* ad axem novum *rs* normalis *MQ* et ponatur $DQ = t,\ QM = u$, coordinatae autem obliquangulae sint $DT = r,\ TM = s$.
Erit ergo primo

$t = r - \nu z$ et $u = \mu s$ (43);

deinde vero est

$$x = mu + nt - f \quad \text{et} \quad y = nu - mt - y\,(36).$$

Hinc fiet

$$x = nr - (n\nu - m\mu)s - f \text{ et } y = -mr + (\mu n + \nu m)s - g,$$

ubi est $n\nu - m\mu$ cosinus anguli *AVM*, quem novae applicatae cum axe priori *RS* constituunt, et $\mu n + \nu m$ est sinus huius anguli *AVM*. Quodsi ergo in aequatione inter *x* et *y* loco *x* et *y* illi valores inventi substituantur, prodibit aequatio inter coordinatas obliquangulas *r* et *s*, quae erit aequatio generalissima pro curva *LM*.

46. Quoniam in valoribus, qui loco *x* et *y* substituuntur, novarum variabilium *r* et *s* unica inest dimensio, manifestum est aequationem generalissimam eiusdem esse ordinis, cuius erat aequatio proposita inter *x* et *y*. Quomodocunque ergo aequatio ad eandem curvam transformetur, mutatis utcunque tam axe et abscissarum initio quam inclinatione mutua coordinatarum, tamen perpetuo aequatio eiusdem erit ordinis. Quanquam ergo aequatio inter coordinatas sive orthogonales sive obliquangulas infinitis modis variari potest, ut ad eandem curvam pertineat, tamen neque ad ordinem altiorem evehi neque ad inferiorem deprimi poterit. Atque hanc ob causam aequationes diversi ordinis, utcunque alias fuerint affines, tamen semper curvas diversas exhibebunt.