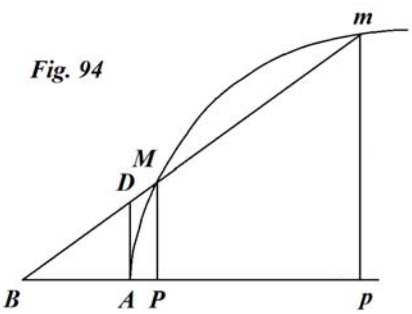
CHAPTER XIX

THE INTERSECTION OF CURVES

457. Just as curved lines will be intersected by right lines, as we have considered now more often in the preceding chapters, where we have shown lines of the second order are unable to be cut by right lines in more than two points, moreover lines of the third order by more than three intersections and of the fourth order by more than four cannot be admitted, and thus henceforth. Therefore since in this chapter I shall set out to define intersections, which any two curves may make between themselves, it will be necessary to increase this treatment with right lines and to investigate these points, in which a certain right line will cut across a given curve. For in this way it will be apparent from the mutual intersections of the curved lines being determined, how the argument shall be accustomed to be of the greatest use in the construction of equations of higher degree, concerning which I will treat further in the following chapter.

458. Therefore *AMm* shall be some proposed curve (Fig. 94), whose nature shall be given by an equation between the orthogonal coordinates $AP = x,\ PM = y$. Now some right line *BMm* may be drawn, of which it shall be required to determine how many and at what points it shall cut the curve *AMm*. Towards this end, an equation is sought for a right line equally between the orthogonal coordinates *x* and *y*, related to the same axis *AP* and likewise to the start of the abscissas *A*. Therefore the equation for this right line will be of this form $\alpha x + \beta y = \gamma$; from which the position $x = 0$ may be shown to be $y = AD = \frac{\gamma}{\beta}$,

Fig. 94



but the position $y = 0$ becomes $x = -AB = \frac{\gamma}{\alpha}$; from which the concurrence *B* of this right line with the axis will be known, and of which at *B* equally the tangent is $= \frac{AD}{AB} = -\frac{\alpha}{\beta}$. Therefore both the curve proposed as well as right line are expressed by equations between the coordinates *x* and *y*.

459. But if in each equation we may assume the abscissas *x* always to be equal, the applied lines *y* may be shown, if they shall be different, however far the corresponding points of the curve and of the right line may stand apart from each other for the same abscissa. If therefore from each equation an equal value may be produced for the applied line *y*, then the curve and the right line will have a common point there, and thus the intersection will be at this place. Therefore according to the intersections requiring to be found in each equation besides the abscissas *x* the equal applied lines *y* are to be put in place, and thus two equations will be had involving the two unknown quantities *x* and *y*,

from the resolution of which either the abscissas *x*, to which the intersections will correspond, or the applied lines *y* will be found. Clearly if from these two equations the unknown *y* may be eliminated, an equation will emerge involving the unknown *x* only, the values of which will show the abscissas *AP*, *Ap,* from which the applied lines *PM*, *pm* drawn will be crossing through the points of intersection *M* and *m*.

460. Since the equation for the right line *BMm* shall be $\alpha x+\beta y=\gamma$, from that there becomes $y=\frac{\gamma-\alpha x}{\beta}$, which value if it may be substituted into the equation for the curve in place of *y*, an equation will arise containing *x* only, the real roots of which will provide all the abscissas to which the intersections will correspond, and thus the number of intersections may be deduced from the number of real roots of *x,* which the equation found may provide. Because truly in the value of $y=\frac{\gamma-\alpha x}{\beta}$ the unknown *x* maintains a single dimension, after the substitution an equation will emerge, in which *x* will have no more dimensions, than were held jointly in the equation for both *x* and *y*. Therefore *x* will have just as many dimensions or less, if indeed the greatest powers of *x* may be removed by the substitution.

461. With the abscissas *AP*, *Ap* found in this manner, which correspond to the intersections, from these points of intersection themselves *M* and *m* will be found easily. Indeed since the applied lines erected through the points *P* and *p* shall pass through the intersections, only these points are required to be observed, where these applied lines cut the right line *BMm*. Also the points may be noted, at which these applied lines meet the curve *AMm* ; but since on many occasions a single applied line may cross the curve at several points, it may be unclear, which point of the curve likewise shall provide an intersection. But this inconvenience does not arise in use, if the intersections with the right line *BMm* may be considered, clearly in which a single applied line can be cut at a single point only. But if it arises that two values of *x* become equal to each other, then the two points of intersection *M* and *m* will merge into one ; in which case therefore either the right line *BM* is a tangent to the curve or it will cut that curve at a double point.

462. If the unknown *y* be eliminated, the resulting equation by which *x* is defined, may have no real root, then this is an indication that the curve is never cut by nor does it touch the right line *BMm* ; but the real roots (however many there should be,) of this equation will show just as many intersections, because one right applied line *BMm* will correspond to a single real abscissa ; since that cannot be equal to an applied line of the curve, if no intersection is present there. Therefore these are to be observed properly in this place, as the intersections of curved lines may not indicate always just as many individual roots ; the account of which will soon be made clear, since we will consider two curved lines and we shall investigate the intersections of these.

463. Therefore any two curves shall be described (Fig. 95) *MEm, MFm*, which mutually intersect each other ; towards defining the nature of which intersections each may be expressed by an equation between the orthogonal coordinates *x* et *y* for the same common axis *AB* and relative to the same origin of abscissas *A*. Therefore for each curves with equal abscissas *x*, where the intersections are given, in that place the applied lines *y* will come together. On which account, if from the two equations of the curves proposed on eliminating *y* a new equation will be formed involving the unknown *x* as the unknown, all the intersections *M, m, m,* however many there were, will be indicated by the real roots of this equation ; clearly the abscissas *AP, Ap, Ap* etc., which correspond to the intersections *M, m, m* etc., will be the values of *x* in agreement for that equation.

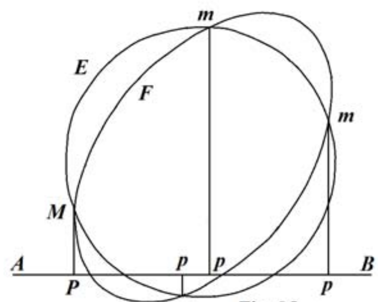


Fig. 95

464. But with these abscissas *AP, Ap* etc. found, which agree with the intersections, it will not be so easy to define their points of intersection. If indeed for each curve of the same abscissa *AP* several applied lines may correspond, which arises, if for each curve *y* were a multiform function of *x*, then it is required to select from this duplicity of the applied lines those which are equal to each other ; which investigation will be more troublesome with that, in which the applied line *y* may possess several values in each curve. Yet for this difficulty may be met easily, if, while from the two proposed equations, the applied line *y* may be eliminated, that equation may be called into help, by which *y* may be defined by *x* ; for from this equation for some value of *x* with the value found, the magnitude of the applied line from the point *P* may be known reaching as far as to the intersection; nor thus will there be a need to consider the nature of either one or the other or both curves at once.

465. One curve shall be a parabola, [not the curves shown in the above figure] the nature of which is expressed by this equation

$$yy - 2xy + xx - 2ax = 0;$$

truly the other shall be a circle expressed by the equation

$$yy + xx - cc = 0.$$

Towards eliminating *y* in the first place the first equation may be subtracted from the second and there will remain

$$2xy + 2ax - cc = 0, \text{ from which } y = \frac{cc - 2ax}{2x},$$

from which now it may be apparent, whatever values for *x* may result, always from these the real values of y are going to be found. Therefore this value found for *y* may be substituted into the other equation and it will produce

$$c^4 - 4accx + 4(aa - cc)xx + 4x^4 = 0,$$

thus the individual real roots of this equation will provide the true intersections.
We may put $c = 2a$ and thus

$$4a^4 - 4a^3x - 3aaxx + x^4 = 0,$$

one real root of which equation is $x = 2a$, with which extracted this equation will remain

$$x^3 + 2axx + aax - 2a^3 = 0,$$

which at this stage provides one real root ; but for each the agreeing applied line is found from this equation $y = \frac{2aa - ax}{x}$; in the first case, clearly $x = 2a$ will correspond to $y = 0$, thus so that is made on the axis itself.

466. Hence it is understood, as many times as both the equations between *x* and *y* should be prepared thus, so that in the exercise of the elimination of *y* a rational function of *x* may be found, which shall be equal to *y*, as long as the final equation will provide some single real root of *x* in the end, (after *y* has been duly eliminated), the true intersection will be shown. Truly if during the elimination no rational function of *x* may be found, which shall be equal to *y*, then it can arise, that not all the real roots may be able to be elicited from the final equation providing the real intersections. For one after another such a magnitude of the value of *x* can be produced, for which a real applied line may not correspond in either curve ; nor yet in this case can it be proved to be the calculation which is in error. For since abscissas of this kind may correspond to imaginary applied lines for each curve, but both in imaginary and in real equations both equalities and inequalities may have a place, nothing hinders these imaginary applied lines from not being equal to each other, and thus a false intersection may arise.

467. Towards showing this more clearly (Fig. 96) the parabola *EM* may be described on the same axis *BAE* $= 2a$ and beyond that the circle *AmB* of radius *c*, with the interval present $AE = b$, thus so it is certain that on no account shall there be given an intersection. *A* may be taken for the start of the abscissas, which shall be positive towards *E*, but may be put in place negative towards *B* backwards ; and this equation will

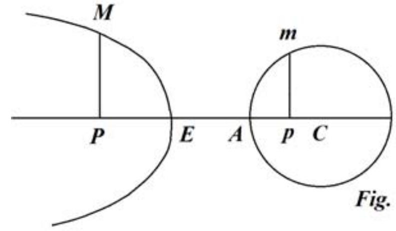


Fig. 96

be had for the parabola $yy = 2ax - 2ab$; truly there will be this equation for the circle $yy = -2cx - xx$. But if now, as if we might want to investigate the intersection, we may eliminate *y*, and at once we will have

$$xx + 2(a+c)x - 2ab = 0,$$

from which two real values for *x* are found, indeed

$$x = -a - c \pm \sqrt{((a+c)^2 + 2ab)},$$

the one positive and the other negative, as yet no intersection will emerge. Cleary for these two abscissas found, both the parabola as well as the circle will show imaginary applied lines, which, however imaginary, will yet be equal to each other ; moreover with these values of *x* substituted the equation becomes

$$y = \sqrt{(-2aa - 2ac - 2ab \pm 2ay\sqrt{(aa + 2ac + cc + 2ab)})},$$

which expression is certainly imaginary.

468. From this example, imaginary intersections of curves are understood to be given also, which, even if they shall be non existent, yet they may be indicated equally by the calculation just as with the real ones. And on that account from the number of real roots of *x*, which the final equation may contain, that may not include the correct number of intersections ; for it can come about that more real roots may be present than there are intersections, and generally also even no intersection may be present, still two or more real roots of *x* may result. Yet meanwhile any intersection will lead to one real root of *x* in the final equation and hence on that account the real roots of *x* will be always just as many, at least, as there shall be intersections, even if meanwhile more real roots might be present [than there are intersections]. But whether an actual intersection may correspond to each root of *x* may be seen easily, the corresponding value of *y* must be sought, which if it may arise real, then the intersection will be real, but if it shall be imaginary, then intersection also will be imaginary or not present at all.

469. Therefore with this exception or difference between the number of real roots of *x* and the number of intersections only has a place, if either in each equation the applied line *y* may have equal dimensions and thus the principal axis likewise shall be a diameter of each curve, or if both equations were prepared, so that, while *yy* may be eliminated, likewise *y* will depart from the calculation ; and thus *y* may not be able to be expressed by a retinal function of *x*. Thus, if either equation were

$$yy - xy = aa,$$

truly the other

$$y^4 - 2xy^3 + x^3y = bbxx,$$

since from the first

$$(yy - xy)^2 = a^4 \text{ or } y^4 - 2xy^3 = a^4 - xxyy,$$

this value may be substituted into the other, and there will be

$$a^4 - xxyy + x^3y = bbxx \text{ or } yy - xy = \frac{a^4 - bbxx}{xx} = aa;$$

from which there becomes

$$xx = \frac{a^4}{aa + bb}$$

and thus

$$x = \frac{\pm aa}{\sqrt{(aa + bb)}}.$$

Therefore a two-fold intersection may be seen to be given, but, whether each shall be a real intersection, must be deduced from the value of y, which is supplied by this equation $yy - xy = aa$. Therefore there will be

$$yy = \frac{\pm aay}{\sqrt{(aa + bb)}} + aa,$$

since all the roots of which are real, it is apparent four intersections are given, thus so that two real intersections correspond to each abscissa

$$x = \frac{\pm aa}{\sqrt{(aa + bb)}}.$$

470. But when neither the diameter of each curve exists nor this case has a place, so that, while higher powers of y may be eliminated, then, because a rational function of x may arise equal to y, just as many individual real roots of the final equation will indicate the true intersections, thus so that there shall be no need for caution in these cases. This arises, if either curve may become a right line, as we have seen before, or if its applied line may be expressed by a uniform function of x; for then no abscissas will correspond to imaginary applied lines and thus the individual roots of x will show true intersections. But furthermore, even if y may reach more dimensions in each equation, yet during the elimination of y it is customary to arrive at an equation, from which the value of y is expressed by a rational and thus a uniform function of x.

471. But as often as it happens that some number of intersections shall be imaginary, which the calculation shows, that is happens not only in these cases, when neither curve has a real applied line these corresponding abscissas found, which indeed has been done in the above example of the circle and parabola, but also cases of this kind can be shown, in which one curve provides real applied lines for all the abscissas, nor yet do the intersections of x correspond to individual real roots. A line of the third order provides an example of this kind expressed by this equation :

$$y^3 - 3ayy + 2aay - 6axx = 0,$$

which provided real applied lines for all the abscissas, and indeed three, if x were less than $\frac{a}{3}\sqrt[4]{\frac{1}{3}}$. But since if this curve may be combined with a parabola contained in the equation $yy - 2ax = 0$, no real applied line of this is given, if x shall be negative, and thus no intersection can arise from the negative x abscissas.

472. Now y may be eliminated and, since from the latter equation there shall be $yy = 2ax$, the first will be changed into this :

$$2axy - 6aax + 2aay - 6axx = 0,$$

from which there becomes

$$y = \frac{6aax + 6axx}{2aa + 2ax} = 3x.$$

Truly because that equation is divisible by $y - 3x$, if it may be divided, this equation $2aa + 2ax = 0$ will arise free from y, from which there arises $x = -a$. Therefore there must be an intersection of the curves corresponding to the abscissa $x = -a$, to which no real applied line will correspond in the parabola ; but in the line of the third order on putting $x = -a$ there becomes

$$y^3 - 3ayy + 2aay - 6a^3 = 0,$$

from which a single real applied line arises $y = 3a$; the remaining two values of y contained in the equation $yy + 2aa = 0$ are imaginary ; evidently in this place these imaginary applied lines become equal to the same applied lines of the parabola and thus in this place they will have two imaginary intersections. Truly also two real intersections will be had arising from the factor $y - 3x = 0$ of the above equation ; from which there becomes $9xx - 2ax = 0$. Therefore in the first place an intersection will exist at the start of the abscissas themselves, where $x = 0$ and likewise $y = 0$; the other will correspond to the abscissa $x = \frac{2a}{9}$, where there is $y = 3x = \frac{2a}{3}$.

473. Therefore in this case it has come to imaginary intersections, even if in the business of eliminating y itself the equation will have been produced:

$$2axy - 6aax + 2aay - 6axx = 0,$$

in which y obtains a single dimension only, thus so that thence y may be considered to be expressed by a rational function of x, which just as we have noted before as the criterion of no imaginary intersections. But actually, if this equation may have no divisors, no place may be relinquished for imaginary intersections ; because truly in this case by division an equation is elicited no further involving the applied line y, and it is the same, if y cannot be expressed by a rational function of x. Clearly as often as an equation of this kind is resolvable into factors, a judgment has to be made on each separate factor, from which it arises, so that while some factor involving imaginary intersections must be rejected completely, another same factor may be admitted.

474. From these considerations we may show a little more distinctly, how the intersections from any two curves must be defined ; and, since this inquiry may depend on the elimination of the other coordinate y, it will be by looking at the dimensions of this only, which it may reach in each equation. For the elimination may be resolved in the same manner, in whatever way the equation may be affected by the other coordinate x. Therefore P, Q, R, S, T etc. and likewise p, q, r, s, t etc. shall be some rational functions of x; and indeed in the first place we may consider both curves, the intersection of which is required, to be expressed by these equations

I.
$$P + Qy = 0,$$

II.
$$p + qy = 0.$$

The first equation is multiplied by p, the second by P; and these equations subtracted from each other in turn leave this equation completely free from y :

$$pQ - Pq = 0.$$

Therefore regarding this equation, in which only the unknown x is present besides the constants, all the real roots of x will provide points on the axis, from which the intersections arise. For any value of x found the real value of y will be had from either equation

$$y = -\frac{P}{Q} = -\frac{p}{q},$$

which will indicate the intersection ; from which, if the applied line *y* of each curve may be expressed by a rational or uniform function of *x*, no imaginary intersections will find a place.

475. Now the applied line *y* of a higher curve may be expressed by a uniform function of *x* as before, truly of a higher biform function, thus so that there shall be

I.
$$P+Qy=0,$$

II.
$$p+qy+ryy=0;$$

the first equation will be multiplied by *p*, the following by *P* and they may be subtracted from each other in turn, on division by *y* there will be

III.
$$pQ-Pq-Pry=0 \quad \text{or} \quad (Pq-pQ)+Pry=0 .$$

Now the first will be multiplied by *Pr* and the third by *Q*, and with the subtraction made this equation will emerge free from *y*

$$PPr-PQq+pQQ=0 .$$

Therefore the individual roots of this equation will provide the corresponding abscissas for the intersections, from which agreeing with the real applied lines

$$y=-\frac{P}{Q}=\frac{pQ-Pq}{Pr},$$

the intersections will be real.

476. As before there shall be the applied line of a higher [order] curve equal to a uniform function of *x*, truly the higher curve will be expressed by a cubic equation or it shall be a triform function of *x*, thus so that both equations shall be of this kind :

I.
$$P+Qy=0,$$

II.
$$p+qy+ryy+sy^3=0 .$$

The former may be multiplied by *p* and the latter by *P;* and with one taken from he other and by dividing by *y* there will be made

III.
$$(Pq - pQ) + Pry + Psyy = 0\,,$$

in which if in place of *y* the value from the first $y = -\frac{P}{Q}$ may be substituted, and it may be freed from fractions, this equation arises

$$PQQq - pQ^3 - PPQr + P^3 s = 0$$

or

$$Q^3 p - PQQq + PPQr - P^3 s = 0\,,$$

which the same will produce at once, if in the second equation in place *y* the value from the first $-\frac{P}{Q}$ may be substituted. Therefore all the real roots of *x* of this final equation, because the applied real lines correspond to each one by the first equation $y = -\frac{P}{Q}$, will show just as many true intersections.

477. In a similar manner, if the applied line y of a higher curve may be expressed by an equation of four or of more dimensions, while the applied line of the other remains a uniform or function of *x*, the unknown *y* may be eliminated. Indeed both the proposed equations

I.
$$p + Qy = 0\,,$$

II.
$$p + qy + ryy + sy^3 + ty^4 = 0\,;$$

and, since from the first there shall be $y = -\frac{P}{Q}$, this value substituted into the other will give this equation between *x* and known constants only :

$$Q^4 p - PQ^3 q + PPQQr - p^3 Qs + P^4 t = 0\,.$$

Therefore the individual real roots *x* of this equation will provide just as many true intersections, therefore so that one real applied line *y* can be assigned to each of the abscissas *x*, without doubt $y = -\frac{P}{Q}$.

478. Now the applied line *y* of each curve may be expressed by a quadratic equation and indeed at first perfect, thus so that both the equations shall be of this kind

I.
$$P + Ryy = 0,$$

II.
$$p + ryy = 0,$$

from which by eliminating *yy* this equation will be obtained at once :

$$Pr - Rp = 0,$$

of which the individual real roots then show only true intersections, if the values of *x* found were prepared thus, so that $-\frac{P}{R}$ or $-\frac{p}{r}$ become positive quantities ; for then on account of $yy = -\frac{P}{R} = -\frac{p}{r}$ the applied line *y* will obtain two real values, the one positive and the other negative ; and thus for each value of the abscissa *x* found from the equation $Pr - Rp = 0$ two intersections will correspond equally distant from the axis on each side, so that, since the axis of each curve shall be present, the diameter cannot arise otherwise. But if nevertheless from which value of *x* found from the equation $Pr - Rp = 0$, may lead to a negative value in the expressions $-\frac{P}{R} = -\frac{p}{r}$, then on account of imaginary *y*, the intersections also will be imaginary.

479. Now in each equation with the proposed quadratic a term following containing *y* may be present also and both these proposed equations shall become

I.
$$P + Qy + Ryy = 0,$$

II.
$$p + qy + ryy = 0.$$

Towards eliminating the unknown *y* from these equations that first equation may be multiplied by *p*, truly the other by *P*, and with the subtraction made and on division by *y* there will be :

III.

$$(Pq - Qp) + (Pr - Rp)y = 0 .$$

Then the first equation may be multiplied by *r*, the latter truly by *R*, and with the one subtracted from the other there will be had

IV.

$$(Pr - Rp) + (Qr - Rq)y = 0 .$$

Therefore since from these two equations there shall be

$$y = \frac{Qp - Pq}{Pr - Rp} = \frac{Rp - Pr}{Qr - Rq}$$

there will be

$$(Qp - Pq)(Qr - Rq) + (Pr - Rp)^2 = 0$$

or

$$PPrr - 2PRpr + RRpp + QQpr - PQqr - QRpq + PRqq = 0 .$$

The individual real roots of this equation will show just as many true intersections, if indeed for each value of *x* a real value of *y* will agree from equations III or IV. Yet meanwhile it can arise, that the intersections shall be imaginary, which comes about, if the equations III and IV may have factors ; thus so that from these now by division an equation may be able to be elicited free from *y*. For then this equation to be substituted in place of the last, and for the values of *x* thence removed the corresponding values of y must be sought from the first equations ; which if they should be imaginary, this will be an indication the intersections to be imaginary.

480. Again in one curve the applied line *y* shall be a biform, but in the other a triform of *x*, or both the proposed equations of the curves shall be these :

I.

$$P + Qy + Ryy = 0 ,$$

II.

$$p + qy + ryy + sy^3 = 0 .$$

The first will be multiplied by *p*, the second by *P*, with either subtracted from the other there will remain

III.

$$(Pq-Qp)+(Pr-Rp)y+Psyy=0,$$

which conjointly with the first will show the case pertaining to the preceding chapter ; thus so that, which there were *p, q, r,* here shall be $Pq-Qp$, $Pr-Rp$ and Ps ; and thus here there will be found

$$y=\frac{PQq-QQp-PPr+PRp}{PPs-PRq+QRp}$$

and

$$y=\frac{PRq-QRp-PPs}{PQs-PRr+RRp};$$

from which there becomes

$$0=(PRq-QRp-PPs)^2+(PQs-PQr+RRp)(PQq-QQp-PPr+PRp),$$

which equation expanded out gives

$$\begin{aligned}P^4ss-2P^3Rqs&+3PPQRps-PQRRpq+QQRRpp\\ -P^3Qrs&+PPRRqq-PQ^3ps-QQRRpp\\ +P^3Rrr&+PPQQqs+PQQRpr\\ &-PPQRqr+PR^3pp\\ &-2PPRRpr=0\end{aligned}$$

which on account of the final terms vanishing is divisible by *P*, and thus this equation will be produced

$$\begin{aligned}&P^3ss-2PPRqs-PPQrs+PPRrr+3PQRps+PRRqq+PQQqs\\ &-PQRqr-2PRRpr-QRRpq-Q^3ps+QQRpr+R^3pp=0.\end{aligned}$$

From the real roots of which equation the intersections will become known, if indeed real values of *y* may be taken to correspond.

481. Now each equation may be expressed by a cubic equation and both these proposed equations shall be

I.
$$P + Qy + Ryy + sy^3 = 0,$$

II.
$$p + qy + ryy + sy^3 = 0.$$

Now the former may be multiplied by *p*, the latter by *P*, and with the subtraction made of one from the other there will remain

III.
$$(Pq - Qp) + (Pr - Rp)y + (Ps - Sp)yy = 0.$$

Then the former may be multiplied by *s* and the latter by *S*, and with the subtraction made there will remain

IV.
$$(Sp - Ps) + (Sq - Qs)y + (Sr - Rs)yy = 0.$$

These equations III and IV, if they may be compared with the two equations treated in §479, become as follows :

$$\begin{array}{r|l} P = Pp - Qp & p = Sp - Ps \\ Q = Pr - Rp & q = Sq - Qs \\ R = Ps - Sp & r = Sr - Rs. \end{array}$$

With which substituted into the final equation, there will emerge :

$$\begin{aligned} &+(Pq-Qp)^2(Sr-Rs)^2 - 2(Pq-Qp)(Ps-Sp)(Sp-Ps)(Sr-Rs) + (Ps-Sp)^2(Sp-Ps)^2 \\ &+(Pr-Rp)^2(Sp-Ps)(Sr-Rs) - (Pq-Qp)(Pr-Rp)(Sq-Qs)(Sr-Rs) \\ &-(Pr-Rp)(Ps-Sp)(Sp-Ps)(Sq-Qs) + (Pq-Qp)(Ps-Sp)\ (Sq-Qs)^2 = 0. \end{aligned}$$

In this equation there are seven terms, which all are divisible by $Sp - Ps$, except the first and fifth ; which nevertheless if they may be taken together, will have these two factors, the one $(Pq - Qp)(Sr - Rs)$, the other truly

$$(Pq - Qp)(Sr - Rs) - (Pr - Rp)(Sq - Qs),$$

[*i.e.* $(Pq-Qp)^2(Sr-Rs)^2-(Pq-Qp)(Pr-Rp)(Sq-Qs)(Sr-Rs)$
$=(Pq-Qp)(Sr-Rs)\big((Pq-Qp)(Sr-Rs)-(Pr-Rp)(Sq-Qs)\big)$.]

which latter factor resolved shall be

$$=PQrs+RSpq-PRqs-QSpr$$

and therefore

$$=(Sp-Ps)(Rq-Qr),$$

from which the terms I. and V. will merge into this form :

$$(Pq-Qp)(Sr-Rs)(Sp-Ps)(Rq-Qr),$$

also divisible by $Sp-Ps$. On which account this equation will arise :

$$0=(Pq-Qp)(Sr-Rs)(Rq-Qr)+2(Pq-Qp)(Sp-Ps)(Sr-Rs)+(Sp-Ps)^3$$
$$+(Pr-Rp)^2(Sr-Rs)+(Pr-Rp)(Sp-Ps)(Sq-Qs)-(Pq-Qp)(Sq-Qs)^2,$$

which expanded out will give

$$\begin{aligned}
&S^3p^3-3PSSpps+PPSr^3+2PRRprs-PPRrrs+PPQrss\\
&+PRSqqr-P^3s^3+3PPSpss-R^3pps-2PRSprr\\
&+RRSppr-RSSppq-QQRprs-PRRqqs-PQSqrr\\
&+PQRqrs+3PSSpqr-3PPSqrs+PQSprs+QQSprr\\
&+QRRpqs-QRSpqr-3PQRpss+3QRSpps-PRSpqs\\
&+2PPRqss+2PQSqqs-PSSq^3-PQQqss-2QSSppr-2QQSpqs\\
&+Q^3pss+QSSpqq=0.
\end{aligned}$$

482. So that this method of eliminating y from two equations of higher degree may be understood more clearly, we may put each proposed equation of the fourth order

I.
$$P+Qy+Ryy+Sy^3+Ty^4=0,$$

II.
$$p+qy+ryy+sy^3+ty^4=0;$$

the former equation may be multiplied by *p*, the latter by *P*, and after subtraction there will remain :

III.
$$(Pq-Qp)+(Pr-Rp)y+(Ps-Sp)yy+(Pt-Tp)y^3=0.$$

Then equation I may be multiplied by *t*, the latter II by *T*, and with the subtraction made there will remain :

IV.
$$(Pt-Tp)+(Qt-Tq)y+(Rt-Tr)yy+(St-Ts)y^3=0.$$

Now for brevity's sake, putting :

$Pq-Qp=A$	$Pt-Tp=a$	$Sq-Qs=\alpha$
$Pr-Rp=B$	$Qt-Tq=b$	$Rq-Qr=\beta$
$Ps-Sp=C$	$Rt-Tr=c$	
$Pt-Tp=D$	$St-Ts=d$	

where it is required to be noted that not only $a=D$, but also there are :

$$Ad-Cb=(Pt-Tp)(Sq-Qs)=D\alpha,$$
$$Ac-Bb=(Pt-Tp)(Rp-Qr)=D\beta.$$

Therefore with these substitutions the equations III and IV will be led to these forms

III.
$$A+By+Cyy+Dy^3=0,$$

IV.
$$a+by+cyy+dy^3=0.$$

Now again these equations may be multiplied respectively by *d* and *D* and subtracted from each other in turn, and the equation will be produced :

V.
$$(Ad-Da)+(Bd-Db)y+(Cd-Dc)yy=0.$$

Then these same equations may be multiplied by *a* and *A*, and after subtraction there will remain :

VI.
$$(Ab-Ba)+(Ac-Ca)y+(Ad-Da)yy=0.$$

Now again for the sake of brevity :

$$\begin{array}{r|r|r} Ab-Ba=E & Ad-Da=e & \\ Ac-Ca=F & Bd-Db=f & Cb-Bc=\zeta \\ Ad-Da=G & Cd-Dc=g & \end{array}$$

and there will be $G=e$ and $Eg-Ff=G\zeta$; thus so that also $Eg-Ff$ shall be divisible by *G*.
Hence we will have the following equations

V.
$$E+Fy+Gyy=0,$$

VI.
$$e+fy+gyy=0.$$

From which by a similar operation these will be elicited :

VII.
$$(Ef-Fe)+(Eg-Ge)y=0,$$

VIII.
$$(Eg-Ge)+(Fg-Gf)y=0.$$

Then again for the sake of brevity

$$\begin{array}{r|r} Ef-Fe=H & Eg-Ge=h \\ Eg-Ge=I & Fg-Gf=i, \end{array}$$

thus so that there shall be $I=h$, and there will be had

VII.
$$H+Iy=0,$$

VIII.
$$h+iy=0$$

from which finally this equation free from *y* may be deduced :

$$Hi - Ih = 0.$$

In which if the preceding values may be restored successively, an equation will be obtained, that only the functions *P, Q, R* etc. *p, q, r* etc. of the first equations may enter. Truly the equation between *E, F, G, e, f, g* will be divisible by $G = e$; and, if it may be advanced to the letters *A, B, C, D, a, b, c, d*, the resulting equation will allow division by $D^2 = a^2$, thus so that in the final equation any term shall include only eight letters, four capital letters and four lower case letters. And thus in this manner in general, whatever the dimensions of *y* each proposed equation may contain. the unknown *y* will be able to be eliminated always and an equation can be found, which involves the unknown *x* alone.

483. And if the use of this method of eliminating one unknown from two equations may be most widely apparent, I may adjoin at this stage yet another method, which may not need just as many repeated substitutions. Two equations therefore may be proposed of some number of dimensions

I.

$$Py^m + Qy^{m-1} + Ry^{m-2} + Sy^{m-3} + \text{etc.} = 0,$$

II.

$$py^n + qy^{n-1} + ry^{n-2} + sy^{n-3} + \text{ etc.} = 0,$$

from which one equation must be put in place, in which *y* shall no longer be present. Towards this the latter equation may be multiplied by the quantity

$$Py^{k-n} + Ay^{k-n-1} + By^{k-n-2} + Cy^{k-n-3} + \text{etc.},$$

which includes $k - n$ arbitrary letters *A, B, C,* etc. Therefore the first equation may be multiplied by

$$py^{k-m} + ay^{k-m-1} + by^{k-m-2} + cy^{k-m-3} + \text{ etc.},$$

in which $k - m$ arbitrary letters *a, b, c* etc. are present. Then both products may be put equal to each other, so that all the terms, which contain the powers of *y*, mutually cancel and the final terms being without *y* itself may show the equation sought. But the greatest powers freely cancel each other, for in each product the greatest term will be Ppy^k; therefore at this stave $k-1$ terms remain, which must cancel each other, for which just as many arbitrary letters are to be determined. But the number of arbitrary letters introduced thus is $2k - m - n$, which since it must be equal to $k-1$, making $k = m + n - 1$.

484. On this account the first equation may be multiplied by this indeterminate quantity

$$py^{n-1} + ay^{n-2} + by^{n-3} + cy^{n-4} + \text{ etc.},$$

truly the second equation may be multiplied by this equation

$$Py^{m-1} + Ay^{m-2} + By^{m-3} + Cy^{m-4} + \text{ etc.}$$

And with the individual terms, in which similar powers of y occur, the following equations may arise equal to each other :

$$Pp = Pp$$
$$Pa + Qp = pA + qP$$
$$Pb + Qa + Rp = pB + qA + rP$$
$$Pc + Qb + Ra + Sp = pC + qB + rA + sP$$
$$\text{etc.}$$

Therefore a number of equations $m + n$ of this kind will be had, likewise computed with the first $Pp = Pp$, from which if the arbitrary letters *A, B, C* etc., *a, b, c* etc. may be determined, and the final equation thus will contain none but given letters *P, Q, R* etc., *p, q, r* etc., and thus the question will be satisfied.

485. But this determination of the arbitrary letters may be put in place easier, if the members of each equation are put equal to the new indeterminate quantities α, β, γ etc.; which will appear clearer from the following example.
These two equations shall be proposed :

I.
$$Pyy + Qy + R = 0,$$

II.
$$py^3 + qyy + ry + s = 0;$$

the first therefore may be multiplied by $pyy + ay + b$ and the other by $Py + A$ and these equations will be produced :

$$Pp = Pp,$$
$$Pa + Qp = pA + qP = \alpha,$$
$$Pb + Qa + Rp = qA + rP = \beta,$$
$$Qb + Ra = rA + sP,$$
$$Rb = sA.$$

With the first equation identically omitted, from the second there shall be

$$a = \frac{\alpha - Qp}{P},$$
$$A = \frac{\alpha - Qp}{p}.$$

Truly from the third there will be found

$$b = \frac{\beta}{P} - \frac{Qa}{P} - \frac{Rp}{P} = \frac{\beta}{P} - \frac{\alpha Q}{PP} + \frac{QQp}{PP} - \frac{Rp}{P}$$

and

$$\beta = \frac{\alpha q}{p} - \frac{qqP}{p} + rP,$$

from which with the value of β substituted there will be

$$b = \frac{\alpha q}{Pp} - \frac{qq}{p} + r - \frac{\alpha Q}{PP} + \frac{QQp}{PP} - \frac{Rp}{P}$$

or

$$b = \frac{\alpha(Pq - Qp)}{PPp} + \frac{(QQpp - PPqq)}{PPp} + \frac{(Pr - Rp)}{P},$$

which value substituted into the fourth equation will give

$$\frac{\alpha Q(Pq - Qp)}{PPp} - \frac{Q(Pq - Qp)(Qp + Pq)}{PPp} + \frac{Q(Pr - Rp)}{P} + \frac{\alpha R}{P} - \frac{RQp}{P}$$
$$= \frac{\alpha r}{p} - \frac{Prq}{p} + Ps,$$

or on multiplying by PPp

$$\alpha Q(Pq-Qp)+\alpha P\left(Rp-Pr\right)-Q\left(Pq-Qp\right)(Pq+Qp)$$
$$+PQp(Pr-2Rp)+P^3qr-P^3ps=0.$$

Hence there becomes

$$\alpha=\frac{PPQqq-Q^3pp-PPQpr+2PQRpp-P^3qr+P^3ps}{PQq-QQp+PRp-PPr}.$$

Truly the final equation will be

$$\frac{\alpha R\left(Pq-Qp\right)}{PPp}-\frac{R(PPqq-QQpp)}{PPp}+\frac{R(Pr-Rp)}{P}=\frac{\alpha S}{p}-\frac{Pqs}{p},$$

from which also there is elicited

$$\alpha=\frac{PPRqq-QQRPP-PPRpr+PRRpp-P^3qs}{PRq-QRp-PPs},$$

which twin values of α provide the equation sought, which will be reduced to the same form finally, we have found above § 480 for the same case.

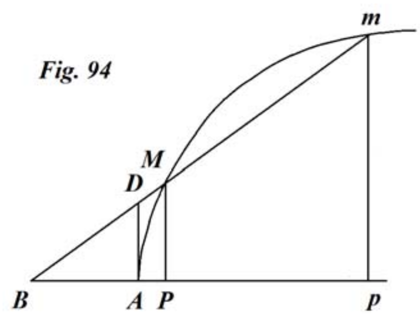
CAPUT XIX

DE INTERSECTIONE CURVARUM

457. Quemadmodum lineae curvae a rectis intersecentur, in praecedentibus capitibus iam saepius vidimus, ubi ostendimus lineas secundi ordinis a rectis in pluribus quam duobus punctis secari non posse, lineas autem tertii ordinis plures quam tres intersectiones et quarti ordinis plures quam quatuor et ita porro non admittere. Cum igitur in hoc capite constituerim intersectiones; quas duae quaevis curvae inter se faciunt, definire, oportebit hanc tractationem a lineis rectis inchoare atque ipsa illa puncta indagare, in quibus recta quaepiam data curvam datam traiicit. Hoc enim modo via parabitur ad intersectiones mutuas linearum curvarum determinandas, quod argumentum maximum usum habere solet in construendis aequationibus altiorum graduum, qua de re in sequenti capite fusius tractabo.

458. Sit igitur proposita (Fig. 94) curva quaecunque *AMm*, cuius natura data sit per aequationem inter coordinatas orthogonales $AP = x,\ PM = y$. Ducatur iam recta quaecunque *BMm*, quae quot et quibusque in punctis sectura sit curvam *AMm*, definiri oporteat. Ad hoc quaeratur aequatio pro linea recta pariter inter coordinatas orthogonales *x* et *y* ad eundem axem *AP* idemque abscissarum initium *A* relata. Aequatio ergo pro linea recta erit huiusmodi $\alpha x + \beta y = \gamma$; qua indicatur posito

Fig. 94



$x = 0$ fore $y = AD = \frac{\gamma}{\beta}$, posito autem $y = 0$ fore $x = -AB = \frac{\gamma}{\alpha}$; unde concursus *B* huius rectae cum axe pariterque ad *B*, cuius tangens est $= \frac{AD}{AB} = -\frac{\alpha}{\beta}$, innotescit. Sic igitur tam curva quam recta proposita per aequationes inter communes coordinatas *x* et *y* exprimuntur.

459. Quodsi in utraque aequatione abscissas *x* perpetuo aequales assumamus, applicatae *y*, si sint diversae, ostendent, quantum curvae et rectae puncta eidem abscissae respondentia a se invicem distent. Si igitur ex utraque aequatione aequalis prodeat valor applicatae *y*, tum ibi curva et recta commune habebunt punctum, ideoque eo in loco dabitur intersectio. Ad intersectiones ergo inveniendas in utraque aequatione praeter abscissas *x* quoque applicatae *y* aequales sunt constituendae, sicque habebuntur duae aequationes duas quantitates incognitas *x* et *y* evolventes, ex quarum resolutione vel abscissae *x*, quibus intersectiones respondent, vel applicatae *y* reperientur. Scilicet si ex

istis duabus aequationibus eliminetur incognita *y*, aequatio nascetur solam incognitam *x* complectens, cuius valores exhibebunt abscissas *AP*, *Ap*, unde applicatae *PM*, *pm* eductae per intersectionum puncta *M* et *m* transibunt.

460. Cum aequatio pro recta *BM m* sit $\alpha x+\beta y=\gamma$, ex ea fiet $y=\frac{\gamma-\alpha x}{\beta}$ qui valor si in aequatione pro curva loco *y* substituatur, orietur aequatio tantum *x* continens, cuius radices reales praebebunt omnes abscissas, quibus intersectiones respondent, ideoque intersectionum numerus colligetur ex numero radicum realium ipsius *x*, quas aequatio inventa suppeditat. Quoniam vero in valore ipsius $y=\frac{\gamma-\alpha x}{\beta}$ incognita *x* unicam tenet dimensionem, post substitutionem emerget aequatio, in qua *x* non plures habebit dimensiones, quam antea in aequatione pro curva ambae *x* et *y* coniunctim tenuerant. Habebit ergo *x* vel totidem dimensiones vel pauciores, siquidem per substitutionem summae ipsius *x* potestates tollantur.

461. Inventis hoc modo abscissis *AP*, *Ap*, quae intersectionibus respondent, ex iis ipsa intersectionum puncta *M* et *m* facile definientur. Cum enim applicatae in punctis *P* et *p* erectae per intersectiones transeant, ea tantum puncta erunt notanda, ubi hae applicatae rectam *BMm* secant. Notari quoque possent puncta, quibus istae applicatae curvae *AMm* occurrunt; cum autem saepenumero una applicata curvae in pluribus punctis occurrat, incertum foret, quodnam curvae punctum simul intersectionem sit praebiturum. Hoc autem incommodum usu non venit, si intersectiones ex recta *BMm* aestimentur, quippe a qua unaquaeque applicata non nisi in unico puncto secari potest. Quodsi autem eveniat, ut duo ipsius *x* valores fiant inter se aequales, tum duo intersectionum puncta *M* et *m* in unum coalescent; quo ergo casu vel recta *BM* curvam tanget vel eam in puncto duplici secabit.

462. Si eliminata incognita *y* aequatio resultans, qua *x* definitur, nullam habeat radicem realem, tum hoc erit indicium curvam nusquam a recta *BMm* secari vel tangi ; radices autem reales (quotquot fuerint), illius aequationis ostendent totidem intersectiones, quia unicuique abscissae reali una rectae *BMm* applicata realis respondet; cui cum sit aequalis applicata curvae, fieri non potest, quin ibi nulla existat intersectio. Haec ideo isto loco probe sunt notanda, quod in intersectione linearum curvarum non semper singulae radices totidem intersectiones indicent; cuius ratio mox fiet manifesta, cum duas lineas curvas contemplabimur earumque intersectiones investigabimus.

463. Sint igitur (Fig. 95) descriptae duae curvae quaecunque *MEm*, *MFm*, quae se mutuo intersecent; ad quarum intersectiones definiendas natura utriusque exprimatur per aequationem inter coordinatas orthogonales *x* et *y* ad eundem communem axem *AB* idemque abscissarum initium *A* relatas. Sumtis ergo pro utraque curva abscissis *x* aequalibus, ubi dantur intersectiones, ibidem applicatae

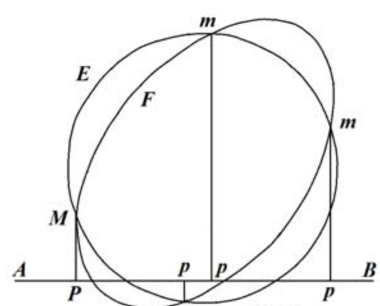


Fig. 95

y convenient. Quocirca, si ex duabus curvarum aequationibus propositis eliminando *y* formetur nova aequatio solam *x* tanquam incognitam involvens, intersectiones omnes *M, m, m,* quotquot fuerint, indicabuntur per radices reales istius aequationis; scilicet abscissae *AP, Ap, Ap* etc., quae intersectionibus *M, m, m* etc. respondent, erunt valores ipsius *x* convenientes pro illa aequatione.

464. Inventis autem abscissis his *AP, Ap* etc., quae intersectionibus conveniunt, non tam facile erit ipsa intersectionum puncta definire. Si enim pro utraque curva eidem abscissae *AP* plures applicatae respondeant, quod evenit, si pro utraque curva fuerit *y* functio multiformis ipsius *x*, tum ex hac duplici applicatarum multitudine eas, quae sint inter se aequales, eligi oportet; quae investigatio eo erit molestior, quo plures valores applicata *y* in utraque curva obtineat. Huic tamen difficultati facile occurretur, si, dum ex binis aequationibus propositis applicata *y* eliminatur, ea aequatio in subsidium vocetur, qua *y* per *x* definitur; ex hac enim aequatione pro quovis ipsius *x* valore invento cognoscetur magnitudo applicatae ex puncto *P* ad intersectionem usque pertingentis; neque ad hoc opus erit naturam alterutrius vel adeo utriusque curvae perpendisse.

465. Sit una curva parabola, cuius natura hac exprimatur aequatione

$$yy-2xy+xx-2ax=0;$$

altera vero sit circulus aequatione

$$yy+xx-cc=0$$

expressus.

Ad *y* eliminandum subtrahatur primum prior aequatio a posteriori ac remanebit

$$2xy+2ax-cc=0,\quad \text{unde fit}\quad y=\frac{cc-2ax}{2x}$$

ex qua iam patet, quicunque valores pro *x* resultent, iis semper valores ipsius *y* reales repertum iri. Substituatur ergo iste valor pro *y* inventus in altera aequatione ac prodibit

$$c^4-4accx+4(aa-cc)xx+4x^4=0,$$

cuius adeo aequationis singulae radices reales praebebunt intersectiones veras.
Ponamus esse $c=2a$ ideoque

$$4a^4-4a^3x-3aaxx+x^4=0,$$

cuius aequationis una radix est $x=2a$, qua extracta remanebit haec aequatio

$$x^3+2axx+aax-2a^3=0,$$

quae unam adhuc praebet radicem realem; utrique autem applicata conveniens invenitur ex hac aequatione $y=\frac{2aa-ax}{x}$; priori, scilicet $x=2a$, respondebit $y=0$, ita ut intersectio in ipso fiat axe.

466. Hinc intelligitur, quoties ambae aequationes inter *x* et *y* ita fuerint comparatae, ut in negotio eliminationis ipsius *y* inveniatur functio rationalis ipsius *x*, quae aequalis sit ipsi *y*, tum unamquamque radicem realem ipsius *x*, quam ultima aequatio (postquam *y* penitus est eliminata), praebebit, exhibituram esse intersectionem veram. Verum si inter eliminandum nulla inveniatur functio rationalis ipsius *x*, quae aequalis sit ipsi *y*, tum evenire potest, ut non omnes radices reales ex ultima aequatione erutae praebeant intersectiones veras. Tantus enim subinde valor pro *x* prodire potest, cui in neutra curva applicata realis respondeat; neque tamen hoc casu calculus erroris est arguendus. Cum enim huismodi abscissae pro utraque curva applicata imaginaria respondeat, in imaginariis autem aequalitas et inaequalitas aeque locum habeat atque in realibus, nihil impedit, quominus applicatae illae imaginariae inter se sint aequales ideoque intersectionem mentiantur.

467. Ad hoc clarius ostendendum (Fig. 96) describantur super eodem axe *BAE* parabola *EM* parametri $=2a$ et extra eam circulus *AmB* radii *c*, existente intervallo $AE=b$, ita ut certum sit nullam prorsus dari intersectionem. Sumatur *A* pro abscissarum initio, quae versus *E* affirmativae, retro autem versus *B* negativae statuantur; atque pro parabola habebitur haec aequatio $yy=2ax-2ab$; pro circulo vero haec $yy=-2cx-xx$. Quodsi iam, quasi intersectionem indagare velimus, eliminemus *y*, statim habebimus

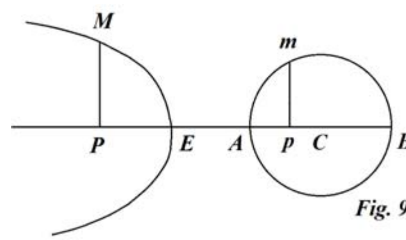


Fig. 96

$$xx+2(a+c)x-2ab=0,$$

ex qua duo pro *x* valores reales reperiuntur, nempe

$$x=-a-c\pm\sqrt{((a+c)^2+2ab)},$$

alter affirmativus alter negativus, cum tamen nulla existat intersectio. Pro his scilicet duabus abscissis tam parabola quam circulus exhibebit applicatas imaginarias, quae, utut imaginariae, tamen inter se erunt aequales; fiet autem hoc ipsius *x* valore substitute

$$y=\sqrt{(-2aa-2ac-2ab\pm 2ay\sqrt{(aa+2ac+cc+2ab)})},$$

quae expressio utique est imaginaria.

468. Ex hoc exemplo intelligitur dari etiam curvarum intersectiones imaginarias, quae, etiamsi sint nullae, tamen per calculum aeque indicentur ac reales. Atque hanc ob rem ex numero radicum realium ipsius x, quas ultima aequatio continet, non semper intersectionum numerus recte concludetur; fieri enim potest, ut plures radices reales adsint quam intersectiones atque etiam nulla omnino existat intersectio, cum tamen duae pluresve radices reales ipsius x resultent. Interim tamen quaelibet intersectio semper unam inducet radicem realem ipsius x in aequationem ultimam et hanc ob rem semper tot, ad minimum,
erunt radices reales ipsius x, quot sunt intersectiones, etiamsi interdum plures radices reales affuerint. Utrum autem unicuique radici reali ipsius x intersectio realis respondeat, facile perspicietur, si valor ipsius y respondens quaeratur, qui si prodeat realis, intersectio erit realis, sin sit imaginarius, intersectio quoque erit imaginaria vel nulla.

469. Haec igitur exceptio seu differentia inter radicum realium ipsius x et intersectionum numerum tantum locum habet, si vel in utraque aequatione applicata y pares tantum ubique habeat dimensiones atque adeo axis principalis simul sit utriusque curvae diameter, vel si ambae aequationes ita fuerint comparatae, ut, dum eliminatur yy, simul y ex calculo excedat; sicque y per functionem rationalem ipsius x exprimi nequeat. Sic, si altera aequatio fuerit

$$yy-xy=aa\,,$$

altera vero

$$y^4-2xy^3+x^3y=bbxx\,,$$

cum ex priori sit

$$(yy-xy)^2=a^4 \quad \text{seu} \quad y^4-2xy^3=a^4-xxyy\,,$$

substituatur hic valor in altera, eritque

$$a^4-xxyy+x^3y=bbxx \quad \text{seu} \quad yy-xy=\frac{a^4-bbxx}{xx}=aa\,;$$

unde fit

$$xx=\frac{a^4}{aa+bb}$$

ideoque

$$x = \frac{\pm aa}{\sqrt{(aa+bb)}}.$$

Videtur ergo dari duplex intersectio, sed, an utraque sit realis, ex valore ipsius *y* colligi debet, quem haec aequatio $yy - xy = aa$ suppeditat. Erit ergo

$$yy = \frac{\pm aay}{\sqrt{(aa+bb)}} + aa,$$

cuius cum omnes radices sint reales, patet quatuor dari intersectiones, ita ut utrique abscissae

$$x = \frac{\pm aa}{\sqrt{(aa+bb)}}$$

binae intersectiones reales respondeant.

470. Quando autem neque axis utriusque curvae diameter existit neque iste casus locum habet, ut, dum altiores ipsius *y* potestates eliminantur, simul *y* prorsus eliminetur, tum, quia ad functionem rationalem ipsius *x* pervenietur ipsi *y* aequalem, singulae radices reales ultimae aequationis totidem indicabunt intersectiones veras, ita ut his casibus nulla cautione sit opus. Evenit hoc, si altera curva abeat in rectam, uti ante vidimus, vel si eius applicata exprimatur per functionem uniformem ipsius *x* ; tum enim nulli abscissae respondebit applicata imaginaria ideoque singulae radices ipsius *x* exhibebunt intersectiones veras. Plerumque autem, etiamsi *y* in utraque aequatione plures obtineat dimensiones, tamen inter eliminationem ipsius *y* perveniri solet ad aequationem, qua valor ipsius *y* per functionem rationalem ideoque uniformem ipsius *x* exprimitur.

471. Quoties autem accidit, ut aliquot intersectiones, quas calculus exhibet, sint imaginariae, id non solum iis evenit casibus, quando neutra curva habet applicatam realem illi abscissae inventae respondentem, quod quidem factum est in superiori circuli et parabolae exemplo, sed etiam eiusmodi casus exhiberi possunt, quibus una curva pro omnibus abscissis praebet applicatas reales, neque tamen singulis radicibus realibus ipsius *x* intersectiones respondeant. Huiusmodi exemplum praebet linea tertii ordinis hac aequatione expressa

$$y^3 - 3ayy + 2aay - 6axx = 0,$$

quae pro omnibus abscissis reales praebet applicatas, et quidem ternas, si fuerit *x* minor quam $\frac{a}{3}\sqrt[4]{\frac{1}{3}}$. Quodsi cum hac curva combinetur parabola aequatione $yy - 2ax = 0$

contenta, cuius nulla datur applicata realis, si x sit negativum, ideoque abscissis x negativis nulla intersectio convenire potest.

472. Eliminetur iam y et, cum sit ex aequatione posteriori $yy = 2ax$, prior aequatio abibit in hanc

$$2axy - 6aax + 2aay - 6axx = 0,$$

unde fit

$$y = \frac{6aax + 6axx}{2aa + 2ax} = 3x.$$

Quoniam vero illa aequatio est divisibilis per $y - 3x$, si dividatur, orietur aequatio ab y libera haec $2aa + 2ax = 0$, unde oritur $x = -a$. Deberet ergo esse intersectio curvarum respondens abscissae $x = -a$, cui in parabola nulla applicata realis respondet; in linea autem altera tertii ordinis posito $x = -a$ fit

$$y^3 - 3ayy + 2aay - 6a^3 = 0,$$

ex qua una nascitur applicata realis $y = 3a$; reliqui duo ipsius y valores in aequatione $yy + 2aa = 0$ contenti sunt imaginarii; hoc scilicet loco applicatae istae imaginariae aequales fiunt applicatis parabolae imaginariis eodem hoc loco sicque habebuntur duae intersectiones imaginariae. Habebuntur vero etiam duae intersectiones reales ex superioris aequationis factore $y - 3x = 0$ oriundae; ex qua fit $9xx - 2ax = 0$. Primum ergo in ipso abscissarum initio, ubi $x = 0$ simulque $y = 0$, existit intersectio; altera respondet abscissae $x = \frac{2a}{9}$, ubi est $y = 3x = \frac{2a}{3}$.

473. Hoc igitur casu perventum est ad intersectiones imaginarias, etiamsi in negotio eliminationis ipsius y prodierit aequatio

$$2axy - 6aax + 2aay - 6axx = 0,$$

in qua y unicam tantum obtinet dimensionem, ita ut inde y per functionem rationalem ipsius x exprimi posse videatur, quod ante tanquam criterium nullarum intersectionum imaginariarum annotavimus. Atque revera, si haec aequatio nullos haberet divisores, intersectionibus imaginariis nullus locus relinqueretur; quoniam vero hoc casu per divisionem elicitur aequatio applicatam y non amplius involvens, perinde est, ac si y per functionem rationalem ipsius x exprimi non posset. Quoties scilicet huiusmodi aequatio in factores est resolubilis, pro unoquoque factore seorsim iudicium ist ferendum, unde fit, ut, dum alter factor intersectiones imaginarias penitus respuit, alter easdem admittat.

474. His perpensis ostendamus aliquanto distinctius, quemadmodum duabus quibusvis curvis propositis earum intersectiones definiri debeant; atque, cum haec investigatio ab

eliminatione alterius coordinatae *y* pendeat, ad huius tantum dimensiones, quas in utraque aequatione obtinet, erit respiciendum. Eliminatio enim eodem modo absolvetur, utcunque aJtera coordinata *x* utramque aequationem afficiat. Sint igitur *P, Q, R, S, T* etc. itemque *p, q, r, s, t* etc. functiones quaecunque rationales ipsius *x;* ac primo quidem ponamus ambas curvas, quarum intersectiones requiruntur, exprimi his aequationibus

I.
$$P+Qy=0,$$

II.
$$p+qy=0.$$

Multiplicetur prior aequatio per *p*, posterior per *P;* atque hae aequationes a se invicem subductae relinquent hanc aequationem ab *y* prorsus liberam:

$$pQ-Pq=0.$$

Huius igitur aequationis, in qua sola incognita *x* praeter constantes inest, omnes radices reales ipsius *x* praebebunt puncta in axe, quibus intersectiones imminent. Pro quocunque valore ipsius *x* invento habebitur valor ipsius *y* realis ex alterutra aequatione

$$y=-\frac{P}{Q}=-\frac{p}{q},$$

qui intersectionem indicabit; unde, si utriusque curvae applicata *y* exprimatur per functionem rationalem seu uniformem ipsius *x*, nullae intersectiones imaginariae locum inveniunt.

475. Exprimatur iam alterius curvae applicata *y* per functionem uniformem ipsius *x* ut ante, alterius vero per functionem biformem, ita ut sit

I.
$$P+Qy=0,$$

II.
$$p+qy+ryy=0;$$

multiplicetur prior aequatio per *p*, posterior per *P* et a se invicem subtrahantur, factaque divisione per *y* erit

III.
$$pQ-Pq-Pry=0 \quad \text{seu} \quad (Pq-pQ)+Pry=0.$$

Nunc multiplicetur prima per *Pr* et tertia per *Q*, atque facta subtractione emerget haec aequatio ab *y* libera

$$PPr - PQq + pQQ = 0 .$$

Huius aequationis ergo singulae radices praebebunt abscissas intersectionibus respondentes, quibus cum applicatae reales

$$y = -\frac{P}{Q} = \frac{pQ - Pq}{Pr}$$

conveniant, intersectiones erunt reales.

476. Sit ut ante alterius curvae applicata aequalis functioni uniformi ipsius *x*, alterius vero curvae applicata exprimatur per aequationem cubicam seu sit functio triformis ipsius *x*, ita ut binae aequationes propositae sint huiusmodi:

I.
$$P + Qy = 0 ,$$

II.
$$p + qy + ryy + sy^3 = 0 .$$

Multiplicetur prior per *p* et posterior per *P;* alteraque ab altera subducta ac divisione per *y* facta erit

III.
$$(Pq - pQ) + Pry + Psyy = 0 ,$$

in qua si loco *y* valor ex prima $y = -\frac{P}{Q}$ substituatur et a fractionibus liberetur, proveniet ista aequatio

$$PQQq - pQ^3 - PPQr + P^3s = 0$$

seu

$$Q^3p - PQQq + PPQr - P^3s = 0 ,$$

quae eadem statim prodit, si in secunda aequatione loco *y* eius valor ex prima $-\frac{P}{Q}$ substituatur. Huius ergo ultimae aequationis omnes radices reales ipsius *x*, quoniam singulis per primam aequationem $y=-\frac{P}{Q}$ applicatae reales respondent, totidem intersectiones veras monstrabunt.

477. Simili modo, si alterius curvae applicata *y* exprimatur per aequationem quatuor pluriumve dimensionum, dum alterius applicata manet functio uniformis seu rationalis ipsius *x*, facile incognita *y* eliminatur. Sint enim ambae aequationes propositae

I.
$$p+Qy=0,$$

II.
$$p+qy+ryy+sy^3+ty^4=0;$$

atque, cum ex priori sit $y=-\frac{P}{Q}$, hic valor in altera substitutus dabit aequationem inter *x* et cognitas tantum hanc

$$Q^4p-PQ^3q+PPQQr-p^3Qs+P^4t=0.$$

Huius ergo aequationis singulae radices ipsius *x* reales suppeditabunt totidem intersectiones veras, propterea quod unicuique abscissae *x* ex prima aequatione assignari potest una applicata *y* realis, nempe $y=-\frac{P}{Q}$.

478. Exprimatur iam utriusque curvae applicata *y* per aequationem quadraticam ac primo quidem puram, ita ut aequationes ambae sint huiusmodi

I.
$$P+Ryy=0,$$

II.
$$p+ryy=0,$$

ex quibus eliminando *yy* statim obtinetur haec aequatio

$$Pr-Rp=0,$$

cuius singulae radices reales tum solum demonstrant intersectiones veras, si valores ipsius *x* inventi ita fuerint comparati, ut $-\frac{P}{R}$ vel $-\frac{p}{r}$ fiat quantitas affirmativa; tum enim ob $yy=-\frac{P}{R}=-\frac{p}{r}$ applicata *y* duplicem nanciscetur valorem realem, alterum affirmativum alterum negativum; ideoque cuique abscissae *x* valori ex aequatione $Pr-Rp=0$ invento binae respondebunt intersectiones ab axe utrinque aequaliter distantes, quod, cum axis utriusque curvae diameter existat, aliter evenire non potest. Quodsi autem quis valor ipsius *x* ex aequatione $Pr-Rp=0$ inventus expressionibus $-\frac{P}{R}=-\frac{p}{r}$ inducat valorem negativum, tum ob *y* imaginarium intersectiones quoque erunt imaginariae.

479. Adsit nunc in utraque aequatione proposita quadratica secundus quoque terminus continens *y* sintque ambae aequationes propositae istae

I.
$$P+Qy+Ryy=0,$$

II.
$$p+qy+ryy=0.$$

Ad incognitam *y* ex his aequationibus eliminandam multiplicetur primum illa aequatio per *p*, haec vero per *P*, factaque subtractione et divisione per *y* erit

III.
$$(Pq-Qp)+(Pr-Rp)y=0.$$

Deinde multiplicetur prior aequatio per *r*, posterior vero per *R*, alteraque ab altera subtracta habebitur

IV.
$$(Pr-Rp)+(Qr-Rq)y=0.$$

Cum igitur ex his duabus aequationibus sit

$$y=\frac{Qp-Pq}{Pr-Rp}=\frac{Rp-Pr}{Qr-Rq}$$

erit

$$(Qp-Pq)(Qr-Rq)+(Pr-Rp)^2=0$$

seu

$$PPrr - 2PRpr + RRpp + QQpr - PQqr - QRpq + PRqq = 0 .$$

Cuius aequationis singulae radices reales ostendent totidem intersectiones veras, siquidem cuique valori ipsius x valor realis ipsius y convenit ex aequatione III. vel IV. Interim tamen fieri potest, ut intersectiones sint imaginariae, quod evenit, si aequationes III. et IV. habeant factores; ita ut ex iis iam per divisionem aequatio ab y libera elici queat. Tum enim haec aequatio in locum ultimae substitui atque ad valores ipsius x inde erutos ex primis aequationibus valores ipsius y respondentes quaeri debebunt; qui si fuerint imaginarii, hoc erit indicio intersectiones esse imaginarias.

480. Sit porro in una curva applicata y functio biformis, in altera autem triformis ipsius x, seu sint ambae curvarum aequationes propositae hae

I.
$$P + Qy + Ryy = 0 ,$$

II.
$$p + qy + ryy + sy^3 = 0 .$$

Multiplicetur prior per p, posterior per P, alteraque ab altera subtracta remanebit

III.
$$(Pq - Qp) + (Pr - Rp)\, y + Psyy = 0 ,$$

quae cum prima coniuncta exhibet casum in praecedente paragrapho pertractatum; ita ut, quae ibi erant p, q, r, hic sint $Pq - Qp$, $Pr - Rp$ et Ps ; ideoque reperietur hic

$$y = \frac{PQq - QQp - PPr + PRp}{PPs - PRq + QRp}$$

et

$$y = \frac{PRq - QRp - PPs}{PQs - PRr + RRp} ;$$

unde fit

$$0 = (PRq - QRp - PPs)^2 + (PQs - PQr + RRp)(PQq - QQp - PPr + PRp) ,$$

quae aequatio evoluta dat

$$\begin{aligned} &P^4ss-2P^3Rqs+3PPQRps-PQRRpq+QQRRpp\\ &-P^3Qrs\ +PPRRqq\ -PQ^3\ ps\ \ -QQRRpp\\ &+P^3Rrr\ +PPQQqs\ +PQQRpr\\ &\qquad -PPQRqr\ +PR^3pp\\ &\qquad -2PPRRpr=\ 0 \end{aligned}$$

quae ob ultimum terminum evanescentem divisibilis est per *P*, sicque prodibit haec aequatio

$$P^3ss-2PPRqs-PPQrs+PPRrr+3PQRps+PRRqq+PQQqs$$
$$-PQRqr-2PRRpr-QRRpq-Q^3ps+QQRpr+R^3pp=0.$$

Ex cuius aequationis radicibus realibus intersectiones cognoscentur, siquidem ipsis valores reales ipsius *y* respondere deprehendantur.

481. Exprimatur nunc utraque applicata per aequationem cubicam sintque ambae aequationes propositae hae

I.
$$P+Qy+Ryy+sy^3=0\,,$$

II.
$$p+qy+ryy+sy^3=0\,.$$

Multiplicetur prior per *p*, posterior per *P*, factaque subtractione alterius ab altera remanebit

III.
$$(Pq-Qp)+(Pr-Rp)\,y+(Ps-Sp)\,yy=0\,.$$

Deinde multiplicetur prior per *s* posteriorque per *S*, factaque subtractione remanebit

IV.
$$(Sp-Ps)+(Sq-Qs)\,y+(Sr-Rs)\,yy=0\,.$$

Hae aequationes III. et IV. si comparentur cum binis aequationibus § 479 tractatis, fiet, ut sequitur

$$\begin{array}{r|l} P=Pp-Qp & p=Sp-Ps \\ Q=Pr-Rp & q=Sq-Qs \\ R=Ps-Sp & r=Sr-Rs. \end{array}$$

Quibus in aequatione finali substitutis, emerget

$$\begin{aligned} &+(Pq-Qp)^2(Sr-Rs)^2-2(Pq-Qp)(Ps-Sp)(Sp-Ps)(Sr-Rs)+(Ps-Sp)^2(Sp-Ps)^2 \\ &+(Pr-Rp)^2(Sp-Ps)(Sr-Rs)-(Pq-Qp)(Pr-Rp)(Sq-Qs)(Sr-Rs) \\ &-(Pr-Rp)(Ps-Sp)(Sp-Ps)(Sq-Qs)+(Pq-Qp)(Ps-Sp)\ (Sq-Qs)^2=0. \end{aligned}$$

In hac aequatione septem sunt termini, qui omnes sunt divisibiles per $Sp-Ps$, praeter primum et quintum; qui autem si coniungantur, duos habebunt factores, alterum $(Pq-Qp)(Sr-Rs)$, alterum vero

$$(Pq-Qp)(Sr-Rs)-(Pr-Rp)(Sq-Qs),$$

qui posterior resolutus fit

$$=PQrs+RSpq-PRqs-QSpr$$

ideoque

$$=(Sp-Ps)(Rq-Qr),$$

unde termini I. et V. coalescent in hanc formam

$$(Pq-Qp)(Sr-Rs)(Sp-Ps)(Rq-Qr),$$

quoque per $Sp-Ps$ divisibilem. Quocirca orietur haec aequatio

$$\begin{aligned} 0=&(Pq-Qp)(Sr-Rs)(Rq-Qr)+2(Pq-Qp)(Sp-Ps)(Sr-Rs)+(Sp-Ps)^3 \\ &+(Pr-Rp)^2(Sr-Rs)+(Pr-Rp)(Sp-Ps)(Sq-Qs)-(Pq-Qp)(Sq-Qs)^2, \end{aligned}$$

quae evoluta dabit

$$\begin{aligned}&S^3p^3-3PSSpps+PPSr^3+2PRRprs-PPRrrs+PPQrss\\&+PRSqqr-P^3s^3+3PPSpss-R^3pps-2PRSprr\\&+RRSppr-RSSppq-QQRprs-PRRqqs-PQSqrr\\&+PQRqrs+3PSSpqr-3PPSqrs+PQSprs+QQSprr\\&+QRRpqs-QRSpqr-3PQRpss+3QRSpps-PRSpqs\\&+2PPRqss+2PQSqqs-PSSq^3-PQQqss-2QSSppr-2QQSpqs\\&+Q^3pss+QSSpqq=0.\end{aligned}$$

482. Quo methodus ista eliminandi y ex duabus aequationibus altiorum graduum clarius percipiatur, ponamus utramque aequationem propositam esse quarti ordinis

I.
$$P+Qy+Ryy+Sy^3+Ty^4=0,$$

II.
$$p+qy+ryy+sy^3+ty^4=0;$$

multiplicetur aequatio prior per p, posterior per P, atque post subtractionem relinquetur

III.
$$(Pq-Qp)+(Pr-Rp)y+(Ps-Sp)yy+(Pt-Tp)y^3=0.$$

Deinde multiplicetur aequatio I. per t, posterior II. per T, et facta subtractione remanebit

IV.
$$(Pt-Tp)+(Qt-Tq)y+(Rt-Tr)yy+(St-Ts)y^3=0.$$

Ponatur nunc brevitatis gratia

$Pq-Qp=A$	$Pt-Tp=a$	$Sq-Qs=\alpha$
$Pr-Rp=B$	$Qt-Tq=b$	$Rq-Qr=\beta$
$Ps-Sp=C$	$Rt-Tr=c$	
$Pt-Tp=D$	$St-Ts=d$	

ubi notandum est esse non solum $a=D$, sed esse quoque

$$Ad - Cb = (Pt - Tp)(Sq - Qs) = D\alpha,$$
$$Ac - Bb = (Pt - Tp)(Rp - Qr) = D\beta.$$

His ergo substitutionibus aequationes III. et IV. induent has formas

III.
$$A + By + Cyy + Dy^3 = 0,$$

IV.
$$a + by + cyy + dy^3 = 0.$$

Nunc porro aequationes hae multiplicentur respective per *d* et *D* et a se invicem subtrahantur, prodibitque

V.
$$(Ad - Da) + (Bd - Db)y + (Cd - Dc)yy = 0.$$

Tum eaedem illae aequationes multiplicentur per *a* et *A*, et post subtractionem relinquetur

VI.
$$(Ab - Ba) + (Ac - Ca)y + (Ad - Da)yy = 0.$$

Iam statuatur iterum brevitatis gratia

$Ab - Ba = E$	$Ad - Da = e$	
$Ac - Ca = F$	$Bd - Db = f$	$Cb - Bc = \zeta$
$Ad - Da = G$	$Cd - Dc = g$	

eritque $G = e$ et $Eg - Ff = G\zeta$; ita ut et $Eg - Ff$ sit divisibile per *G*.
Hinc sequentes habebimus aequationes

V.
$$E + Fy + Gyy = 0,$$

VI.
$$e + fy + gyy = 0.$$

Ex quibus per similem operationem eliciuntur istae

VII.

$$\left(Ef - Fe\right) + \left(Eg - Ge\right)y = 0,$$

VIII.

$$\left(Eg - Ge\right) + \left(Fg - Gf\right)y = 0.$$

Denique iterum ponatur brevitatis gratia

$Ef - Fe = H$	$Eg - Ge = h$
$Eg - Ge = I$	$Fg - Gf = i$,

ita ut sit $I = h$, habebiturque

VII.

$$H + Iy = 0,$$

VIII.

$$h + iy = 0$$

ex quibus tandem colligitur haec aequatio ab *y* libera

$$Hi - Ih = 0.$$

In qua si valores praecedentes successive restituantur, obtinebitur aequatio, quam solae functiones *P*, *Q*, *R* etc. *p*, *q*, *r* etc. primarum aequationum ingredientur. Aequatio vero inter *E*, *F*, *G*, *e*, *f*, *g* divisibilis erit per $G = e$; atque, si procedatur ad litteras *A*, *B*, *C*, *D*, *a*, *b*, *c*, *d*, aequatio resultans divisionem admittet per $D^2 = a^2$, ita ut in aequatione ultima quivis terminus octo tantum complexurus sit litteras, quatuor maiusculas totidemque minusculas. Hoc itaque modo in genere, quotcunque dimensiones ipsius *y* utraque aequatio proposita contineat, semper incognita *y* poterit eliminari atque aequatio, quae solam incognitam *x* involvat, inveniri.

483. Etsi huius methodi ex duabus aequationibus unam incognitam eliminandi usus latissime patet, tamen aliam adhuc methodum subiungam, quae tot repetitis substitutionibus non indigeat. Sint igitur propositae duae aequationes quotcunque dimensionum

I.

$$Py^m + Qy^{m-1} + Ry^{m-2} + Sy^{m-3} + \text{etc.} = 0,$$

II.

$$py^n + qy^{n-1} + ry^{n-2} + sy^{n-3} + \text{etc.} = 0,$$

ex quibus unam aequationem, in qua *y* amplius non insit, conflari oporteat. Ad hoc multiplicetur aequatio posterior per hanc quantitatem

$$Py^{k-n}+Ay^{k-n-1}+By^{k-n-2}+Cy^{k-n-3}+\text{etc.},$$

quae $k-n$ litteras arbitrarias *A*, *B*, *C* etc. continet. Aequatio vero prior multiplicetur per hanc quantitatem

$$py^{k-m}+ay^{k-m-1}+by^{k-m-2}+cy^{k-m-3}+\text{ etc.},$$

in qua $k-m$ litterae arbitrariae *a*, *b*, *c* etc. insunt. Tum ambo producta ita inter se aequalia ponantur, ut omnes termini, qui continent potestates ipsius *y*, se mutuo destruant terminique ultimi ipsa *y* carentes aequationem quaesitam exhibeant. Summae autem potestates iam sponte se destruunt, in utroque enim producto summus terminus erit Ppy^k; supersunt ergo adhuc $k-1$ termini, qui destrui debebunt, ad quod totidem litterae arbitrariae sunt determinandae. Numerus autem litterarum arbitrariarum sic introductarum est $2k-m-n$, qui cum aequalis esse debeat $k-1$, fiet $k=m+n-1$.

484. Hanc ob rem prima aequatio multiplicetur per hanc quantitatem indeterminatam

$$py^{n-1}+ay^{n-2}+by^{n-3}+cy^{n-4}+\text{ etc.},$$

secunda vero aequatio multiplicetur per hanc

$$Py^{m-1}+Ay^{m-2}+By^{m-3}+Cy^{m-4}+\text{ etc}.$$

Singulisque terminis, in quibus similes ipsius *y* occurrunt potestates, inter se coaequatis nascentur sequentes aequationes

$$Pp=Pp$$
$$Pa+Qp=pA+qP$$
$$Pb+Qa+Rp=pB+qA+rP$$
$$Pc+Qb+Ra+Sp=pC+qB+rA+sP$$
$$\text{etc.}$$

Huiusmodi ergo aequationes, prima $Pp = Pp$ simul computata, habebuntur numero $m+n$, ex quibus si litterae arbitrariae *A*, *B*, *C* etc., *a*, *b*, *c* etc. determinentur, ultima aequatio nonnisi litteras datas *P*, *Q*, *R* etc., *p*, *q*, *r* etc. continebit sicque quaesito satisfaciet.

485. Haec autem litterarum arbitrariarum determinatio facilius expedietur, si membra uniuscuiusque aequationis aequalia ponantur novis indeterminatis quantitatibus α, β, γ etc.; quod ex sequenti exemplo clarius apparebit.
Sint propositae hae aequationes duae

I.
$$Pyy + Qy + R = 0,$$

II.
$$py^3 + qyy + ry + s = 0;$$

multiplicetur ergo prima per $pyy + ay + b$ et altera per $Py + A$ prodibuntque hae aequalitates

$$Pp = Pp,$$
$$Pa + Qp = Pa + qP = \alpha,$$
$$Pb + Qa + Rp = qA + rP = \beta,$$
$$Qb + Ra = rA + sP,$$
$$Rb = sA.$$

Aequatione prima identica omissa, ex secunda fit

$$a = \frac{\alpha - Qp}{P},$$
$$A = \frac{\alpha - Qp}{p}.$$

Ex tertia vero obtinebitur
$$b = \frac{\beta}{P} - \frac{Qa}{P} - \frac{Rp}{P} = \frac{\beta}{P} - \frac{\alpha Q}{PP} + \frac{QQp}{PP} - \frac{Rp}{P}$$
et
$$\beta = \frac{\alpha q}{p} - \frac{qqP}{p} + rP,$$
quo valore ipsius β substituto erit

$$b = \frac{\alpha q}{Pp} - \frac{qq}{p} + r - \frac{\alpha Q}{PP} + \frac{QQp}{PP} - \frac{Rp}{P}$$

seu

$$b = \frac{\alpha(Pq - Qp)}{PPp} + \frac{(QQpp - PPqq)}{PPp} + \frac{(Pr - Rp)}{P},$$

qui valor in quarta aequatione substitutus dabit

$$\frac{\alpha Q(Pq-Qp)}{PPp}-\frac{Q(Pq-Qp)(Qp+Pq)}{PPp}+\frac{Q(Pr-Rp)}{P}+\frac{\alpha R}{P}-\frac{RQp}{P}$$
$$=\frac{\alpha r}{p}-\frac{Prq}{p}+Ps,$$

seu per *PPp* multiplicando

$$\alpha Q(Pq-Qp)+\alpha P\left(Rp-Pr\right)\ -Q\left(Pq-Qp\right)(Pq+Qp)$$
$$+PQp\ (Pr-2Rp)+P^3qr-P^3ps=0.$$

Ergo fiet

$$\alpha=\frac{PPQqq-Q^3pp-PPQpr+2PQRpp-P^3qr+P^3ps}{PQq-QQp+PRp-PPr}.$$

Ultima vero aequatio dabit

$$\frac{\alpha R\left(Pq-Qp\right)}{PPp}-\frac{R(PPqq-QQpp)}{PPp}+\frac{R(Pr-Rp)}{P}=\frac{\alpha S}{p}-\frac{Pqs}{p},$$

ex qua quoque elicitor

$$\alpha=\frac{PPRqq-QQRPP-PPRpr+PRRpp-P^3qs}{PRq-QRp-PPs},$$

qui gemini ipsius α valores praebebunt aequationem quaesitam, quae tandem reducetur ad eandem formam, quam supra § 480 pro eodem casu invenimus.