CHAPTER V

SURFACES OF THE SECOND ORDER

101. Therefore with the establishment of surfaces with the order following the number of the dimensions, which the sum of the three coordinates *x, y, z* taken jointly shall fulfill, if an algebraic equation may be proposed for the surface, the order to which that surface must refer will be able to be assigned at once. Since therefore all the surfaces of the first order shall be shown to be planes, in this chapter I will make the subject to be examined surfaces of the second order. But in these [of higher order] a greater diversity will be apparent at once, than in lines of the second degree, which it will be apparent for each to be attended to easily. Therefore I shall provide the aid, so that I may explain these different kinds distinctly. Truly with higher orders the diversity will increase by so great a multitude of kinds, that we must abstain completely from these developments.

102. Because the nature of surfaces of the second order are expressed by an equation of the second order, in which the variables *x, y* and *z* rise to two dimensions, the cylinder and the cone both right as well as scalene and the sphere, the properties of which we have now described, will be contained in this second order. Truly all the surfaces belonging to this order are contained in this general equation

$$\alpha zz + \beta yz + \gamma xz + \delta yy + \varepsilon xy + \zeta xx + \eta z + \theta y + \iota x + \chi = 0.$$

For whatever three coordinates may be taken, the equation will always be contained in this form. Therefore the diverse kinds of related surfaces will depend here on the different mutual relations between the constants, which, even if the same surface may be expressed by an infinitude of equations, yet the equations will support an infinite number of variations in the surface.

103. Just as with plane curved lines we have chosen the first division thence, so that they may be extended to infinity or enclosed within a finite space, thus in a similar manner all surfaces belonging to some order may be divided into two classes, for the one of which we may refer to these surfaces which depart to infinity, and to the other truly, which will be enclosed in a finite space. Thus the cylinder and the cone will be counted in the first class, the sphere in the second. Indeed in the second class no surfaces will be given with odd orders ; for since any surface of an odd order shall have plane sections of the same order, but all curves of odd orders may be extended to infinity, and it is necessary that the surfaces of these orders are extended to infinity.

104. But as often as a certain surface is extended to infinity, it is necessary, that at least one of the three variables *x, y* and *z* shall go off to infinity. Whereby since it shall be likewise, whichever in this case shall be assumed to become infinite, we may make *z* to

become infinite, if indeed the surface may be extended to infinity. Therefore we may put $z=\infty$ to investigate the nature of this part going off to infinity, and now the first term αzz must be examined mainly, whether it shall be present or truly missing. Therefore this term shall be present in the equation and before that the terms ηz and χ vanish, this equation will be had for the part going off to infinity :

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx+\theta y+\iota x=0,$$

from which in turn all the terms vanish, which are not infinite or at least less than αzz.

105. We may put in place all the terms to be present, in which the variables maintain two dimensions ; indeed whatever surface it should be, all the terms of the sum of the dimensions will be present always in its most general equation, nor on account of this hypothesis, by which we may put all the terms of two dimensions to be present, is any strength of the general solution inferred. But when the terms *yz* and *xz* are present, before these the terms θy and ιx vanish, and this equation will be left :

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx=0,$$

from which there may be elicited :

$$z=\frac{-\beta y-\gamma x\pm\sqrt{((\beta\beta-4\alpha\delta)yy+(2\beta y-4\alpha\varepsilon)xy+(\gamma\gamma-4\alpha\zeta)xx)}}{2\alpha}.$$

Therefore the nature of the part extending to infinity is expressed by this equation.

106. Therefore if as the surface may have a part extended to infinity, that will agree with the infinite part of the surface, which is expressed by this equation :

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx=0,$$

thus so that this surface shall be as if the asymptote of the surface expressed by the general equation. Because truly in this equation the three variables everywhere have two dimensions, this will be the equation for the a conical surface, but having the vertex at the start of the coordinates, where all vanish at the same time ; therefore it will always be possible to show the conical surface, which will be the asymptote of the proposed surface, if indeed it may be extended to infinity, or of which the infinite part will agree with the proposed surface either completely, or only an interval from which the finite part has been removed. Therefore as we may distinguish the branches of curves going off to infinity by right lines, thus it will be allowed to distinguish the parts of surfaces extending to infinity by the asymptotic surfaces of cones.

107. Therefore as often as the asymptote conic surface shall be real, so also is the surface itself extended to infinity and thus indeed, so that the infinite parts of each shall be in agreement; and thus from the nature of the asymptote surface the nature of the proposed surface itself will be able to be deduced. But if the asymptote surface were made imaginary, that surface will have no part extended to infinity, but the whole will be enclosed in a finite space. Therefore for surfaces of the second order, which may be contained in a finite space, there is only a need for investigating, as we may consider, in which cases the asymptotic equation for the surface becomes imaginary ; which comes about, if this whole surface shall vanish in single point. And so that if any extension may be had or a point be placed beyond the vertex, by necessity it must be expanded to infinity, therefore as above we have shown a whole right cone, which may be drawn through the vertex and one point of the surface, is to be put on that surface itself.

108. Therefore when a conical surface expressed by this equation

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx=0,$$

will be changed into a single point, all its sections made through the vertex must vanish equally in the same point. Therefore in the first place by making $z=0$ the equation $\delta yy+\varepsilon xy+\zeta xx=0,$ must be impossible, unless both $x=0$ and $y=0$, which comes about, if $4\delta\zeta$ were greater than $\varepsilon\varepsilon$. Then likewise it must arise on putting either $x=0$ or $y=0$; therefore $4\alpha\delta$ will be greater than $\beta\beta$ and $4\alpha\zeta$ greater than $\gamma\gamma$. Therefore, unless in the equation for the surface of the second order

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx+\eta z+\theta y+\iota x+\chi=0,$$

$4\delta\zeta$ should be greater than $\varepsilon\varepsilon$, $4\alpha\delta$ greater than $\beta\beta$, $4\alpha\zeta$ and greater than $\gamma\gamma$, the surface certainly will have parts extending to infinity.

109. Indeed nor are these three conditions sufficient for the surface to be enclosed in a finite space; it is required in addition, that the value of *z* elicited from the above equation for the asymptote shall be made imaginary ; which will come about, if that expression

$$(\beta\beta-4\alpha\delta)yy+2\left(\beta y-2\alpha\varepsilon\right)xy+\left(\gamma\gamma-4\alpha\zeta\right)xx$$

may maintain a negative value always, if indeed some values other than 0 may be substituted for each of the variables *x* and *y*. Which comes about, since $\beta\beta-4\alpha\delta$ and $\gamma\gamma-4\alpha\zeta$ shall be negative quantities, if $(\beta\gamma-2\alpha\varepsilon)^2$ shall be made less than $(\beta\beta-4\alpha\delta)\left(\gamma\gamma-4\alpha\zeta\right)$; that is, if $\alpha\varepsilon^2+\delta\gamma^2+\zeta\beta^2$ were less than $\beta\gamma\varepsilon+4\alpha\delta\zeta$, if indeed α had a positive value, because we have divided that equation by α. But if indeed α may have a positive value, on account of the above equations ($4\alpha\zeta$ greater than $\gamma\gamma$, $4\alpha\delta$ greater than $\beta\beta$ and $4\delta\zeta$ greater than $\varepsilon\varepsilon$) the coefficients δ and ζ will be positive.

110. Therefore surfaces of the second order will be contained in a finite space, if in its equation the four following conditions may be put in place, clearly if there shall be :

$$4\alpha\zeta \text{ greater than } \gamma\gamma,\ 4\alpha\delta \text{ greater than } \beta\beta,\ 4\delta\zeta \text{ greater than } \varepsilon\varepsilon$$

and

$$\alpha\varepsilon^2 + \delta\gamma^2 + \zeta\beta^2 \text{ shall be less than } \beta\gamma\varepsilon + 4\alpha\delta\zeta.$$

And hence we have defined the first class of surfaces of the second order, to which all these kinds belong, which do not run off to infinity, but which may be enclosed in a finite space. Therefore the sphere belongs to this class, of which the equation is

$$zz + yy + xx = aa\ ,$$

for since here there shall be $\alpha = 1, \delta = 1, \zeta = 1, \beta = 0, \gamma = 0, \varepsilon = 0$, it will satisfy the four conditions found. Truly here this more general equation will pertain

$$\alpha zz + \delta yy + \zeta xx = aa\ ,$$

which, if $\alpha,\ \delta,\ \zeta$ were positive quantities, always is for a closed surface, unless one or two coefficients may vanish.

111.With these four conditions examined, by which the surface is restrained in a finite space, if some determined equation of the second order may be proposed, it will be able to be judged at once, whether the surface expressed by that equation may have parts extending to infinity or not. But if in fact one of these four conditions may be absent, the surface certainly is extended to infinity. But in this case some subdivisions are required to be made, in which the individual variety in the parts extended to infinity is introduced. The first subdivision therefore may be put in place, if

$$\alpha\varepsilon\varepsilon + \delta\gamma\gamma + \zeta\beta\beta \text{ were greater than } \beta\gamma\varepsilon + 4\alpha\delta\zeta\ ,$$

in which case the surface may be extended to infinity and it will have a conical surface for an asymptote, as we have now shown previously. And this case is the opposite direction from the preceding diameter, by which the whole surface is contained in a finite space.

112. But in addition certain intermediate cases are given, in which the surface will depart to infinity, yet in a similar manner maintains a place between the preceding two cases, in which a parabola may be contained between the ellipse and the hyperbola. This case arises, if there were

$$\alpha\varepsilon\varepsilon + \gamma\gamma + \zeta\beta\beta = \beta\gamma\varepsilon + 4\alpha\delta\zeta,$$

and therefore there will be

$$\alpha z = -\beta y - \gamma x + y\sqrt{(\beta\beta - 4\alpha\delta)} + x\sqrt{(\gamma\gamma - 4\alpha\zeta)}.$$

Therefore the asymptotic equation

$$\alpha zz + \beta yz + \gamma xz + \delta yy + \varepsilon xy + \zeta xx = 0$$

will have two simple factors, which will be either real or imaginary, or equal to each other. Therefore that threefold difference provides three kinds of surfaces extending to infinity, and thus altogether we have arrived at five kinds of surfaces of the second order, which now we will pursue more diligently.

113. Because with a change in the position of the three axes, to which the coordinates are parallel, it will be possible to reduce the general equation to a simpler form, we may use that reduction, so that we may reduce the general equation for surfaces of the second order to the most simple form, which still includes all the kinds equally and general among themselves. Therefore since the general equation for surfaces of the second order shall be

$$\alpha zz + \beta yz + \gamma xz + \delta yy + \varepsilon xy + \zeta xx + \eta z + \theta y + \iota x + \chi = 0,$$

we may seek an equation between three other coordinates p, q and r, which indeed will cross each other at the same point as the three prior coordinates. Towards this end from § 92 the equations may be put in place :

$$x = p(\cos.k\cdot\cos.m - \sin.k\cdot\sin.m\cdot\cos.n) + q(\cos.k\cdot\sin.m + \sin.k\cdot\cos.m\cdot\cos.n) - r\cdot\sin.k\cdot\sin.n,$$

$$y = -p(\sin.k\cdot\cos.m + \cos.k\cdot\sin.m\cdot\cos.n) - q\left(\sin.k\cdot\sin.m - \cos.k\cdot\cos.m\cdot\cos.n\right) - r\cdot\cos.k\cdot\sin.n,$$

and

$$z = -p\cdot\sin.m\cdot\sin.n + q\cdot\cos.m\cdot\sin.n + r\cdot\cos.n,$$

from which this equation will result :

$$App + Bqq + Crr + Dpq + Epr + Fqr + Gp + Hq + Ir + K = 0.$$

114. Now these arbitrary angles *k*, *m* and *n* can be defined thus, so that the three coefficients *D*, *E* and *F* may vanish. Nevertheless the calculation indeed may become exceedingly involved, in order that the determination of these angles may actually be shown, still, if perhaps this may be doubted, or that the elimination may lead to real values of these angles always, this certainly must be conceded that perhaps two of the coefficients *D* and *E* can be reduced to equal zero. But if this were effected, the position of the third axis, to which the ordinates *r* are parallel in the plane normal to the ordinates *p* , thus can be changed easily, so that the coefficient *F* shall vanish also. For there may be put in place

$$q = t \cdot \sin.i + u \cdot \cos.i \quad \text{and} \quad r = t \cdot \cos.i - u \cdot \sin.i ,$$

thus so that in place of the term *qr* the new term *tu* may be introduced, the coefficient of which with the aid of the angle *i* will become equal to zero. Therefore in this manner the general equation for surfaces of the second order may be produced according to this form :

$$App + Bqq + Crr + Gp + Hq + Ir + K = 0 .$$

115. Now in addition the coordinates *p*, *q*, *r* will be able to be increased or diminished by some given amounts, so that the coefficients *G*, *H* and *I* shall vanish ; which may be done only by changing that point, from which all the coordinates have their beginning. And in this way all surfaces of the second order will be contained in this equation

$$App + Bqq + Crr + K = 0 ,$$

from which it is understood each one of the three principal planes drawn through the start of the coordinates bisects the surface into two similar and equal parts.
Therefore all the surfaces of the second order not only have one plane through the diameter, but thus three, which will mutually intersect each other through the same point ; which point therefore will constitute the centre of the surface, even if in some cases this centre may be at an infinite distance, evidently in a like manner, by which all the conic sections are said to be provided with a centre, even if in the parabola the centre may be infinitely removed from the vertex.

116. Therefore on being led to an equation of the simplest form, in which all the surfaces of second order are contained, this equation will show the first kind of these surfaces :

$$App + Bqq + Crr = aa,$$

if indeed all three coefficients *A*, *B* and *C* may possess positive values. Therefore surfaces belonging to this first kind not only will be enclosed in a finite space, but will have a centre also, in which the three diametrical planes cross each other mutually at right angles. Let *C* be the centre of this figure (Fig. 143) and *CA*, *CB*, *CD* these principal axes normal between themselves, to which the coordinates *p*, *q*, *r* are parallel, the three diametrical planes will be *ABab*, *ADa* and *BDb*, by which this body will be cut into two equal similar parts.

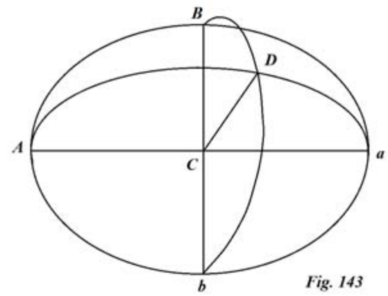

Fig. 143

117. On putting $r = 0$, the equation $App + Bqq = aa$ expresses the nature of the principal section *ABab;* which therefore will be an ellipse having the centre at *C*, of which the semiaxes will be

$$CA = Ca = \frac{a}{\sqrt{A}} \text{ and } CB = Cb = \frac{a}{\sqrt{B}}.$$

If there may be put $q = 0$, the equation $App + Crr = aa$ will be for the principal section *ADa*, which equally will be an ellipse having the centre at *C*, the semiaxis of which will be

$$CA = Ca = \frac{a}{\sqrt{A}} \text{ and } CD = \frac{a}{\sqrt{C}}$$

Moreover on putting $p = 0$, the equation for the third principal section *BDb* will be $Bqq + Crr = aa$, which also will be an ellipse having the centre at *C* and the semiaxis

$$CB = Cb = \frac{a}{\sqrt{B}} \text{ and } CD = \frac{a}{\sqrt{C}}.,$$

But with these three principal sections known or only with the semiaxes of these

$$CA = \frac{a}{\sqrt{A}}, \ CB = \frac{a}{\sqrt{B}} \text{ and } CD = \frac{a}{\sqrt{C}},$$

the nature of this body is determined and known. Hence it will be agreed to call this first genus [kind] of surfaces of the second order *ellipsoids*, because its three principal sections are ellipses.

118. Within this kind three species are held more noteworthy from the others. The first is, if all three principal axes *CA*, *CB* and *CD* were equal to each other, in which case the three principal sections will change into circles and the body itself into a sphere, the equation of which, as we have observed above, will be

$$pp + qq + rr = aa\ .$$

The second species includes these cases, in which only two of the principal axes are equal to each other. Without doubt there shall be $CD = CB$ or $C = B$, and the section *BDb* will become a circle, moreover from the equation

$$App + B(qq + rr) = aa$$

it is understood that all the sections parallel to this equally will be circles ; so that the body will be a spheroid or an oblong, if AC shall be greater than *BC*, or shortened, if AC shall be less than *BC*. And finally the species includes these bodies, in which the coefficients *A*, *B*, *C* are unequal, which thus generally will retain the name of *ellipsoids*.

119. The following kinds of surfaces of the second order will be contained in this equation

$$App + Bqq + Crr = aa\ ,$$

and indeed in the first place, if none of the coefficients *A*, *B*, *C* in short may be absent, but one or two of these may have negative values. Let one only be negative and we will consider this equation

$$App + Bqq - Crr = aa\ ,$$

in which now we put *A*, *B*, and *C* to denote positive numbers. So that it is constrained to the centre of this body and to the diametrical plane, everything shall be prepared in the same way as before. Therefore it is apparent (Fig. 144) the first principal section *ABab* of the body is an ellipse, of which the semiaxis

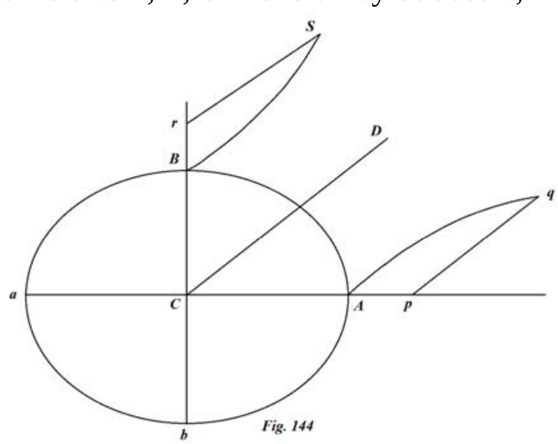

Fig. 144

$AC = \frac{a}{\sqrt{A}}$ and the other $BC = \frac{a}{\sqrt{B}}$. The two remaining principal sections *Aq* and *BS* will be hyperbolas having the centre at *C* and the conjugate semiaxes $= \frac{a}{\sqrt{C}}$.

120. Therefore this surface will represent a kind of funnel, diverging along hyperbolas upwards and downwards. So that this asymptotic surface will have a vertical cone expressed by the equation $App + Bqq - Crr = 0$, with the centre had at *C*, and the sides of which are the asymptotes of hyperbolas. Moreover this asymptote cone will stand between the surface and a right cone, if there were $A = B$, truly a scale cone, if *A* will not be equal to *B*. But the axis of the right cone will be the normal *CD* to the plane *ABa*. Moreover all the sections normal to the axis *CD* will be ellipses similar to the ellipse *ABab*, truly all the sections normal to the plane section *ABab* will be hyperbolas ; from which these surfaces may be agreed to be called *elliptic-hyperboloids*, circumscribed by their asymptotic cone. The surfaces of which for us therefore constitute the second kind.

121. Again three species [or sorts] can be noted in this genus, the first of which will be, if $a = 0$, in which case the ellipse *ABab* vanishes to a point and the hyperbolas will change into right lines, truly the surface will be merged completely with its asymptote, from which this first species will include all the cones, either right or scalene ; so that a new subdivision can be made. There will be another species if $A = B$, in which case the ellipse *ABab* is changed into a circle and the surface becomes rounded or turned [*i.e*. a surface of revolution]. Clearly this surface will arise, if some hyperbola may be turned about its conjugate axis. The third species will not be described from that kind.

122. We may define the third genus, if two coefficients of the terms *pp, qq* and *rr* become negative, the equation of which therefore shall be :

$$App - Bqq - Crr = aa .$$

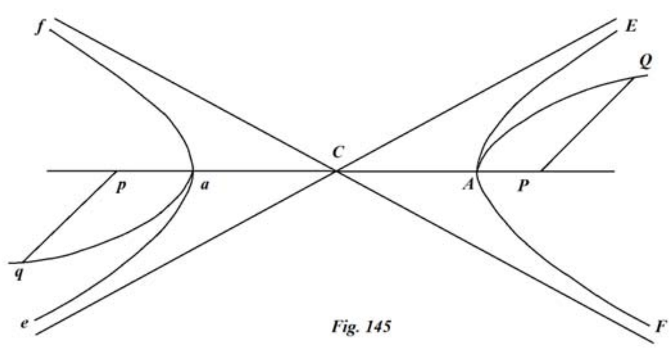

Fig. 145

Therefore putting $r = 0$ the first principal section (Fig. 145) will be the hyperbola *EAFeaf* having the centre at *C*, the transverse semiaxis of which will be $= \frac{a}{\sqrt{A}}$ and the conjugate semiaxis $=\frac{a}{\sqrt{B}}$. The other principal section on putting $q=0$ equally will be the hyperbola *AQ*, *aq* provided with the same transverse semiaxis, but the semiconjugate axis of which will be $=\frac{a}{\sqrt{C}}$; the third principal axis becomes imaginary. Finally the whole surface will be placed within the surface of the asymptotic cone, from which this genus can be called a *hyperbolic-hyperboloid* inscribed in the asymptotic cone. If there be made $B=C$, the surface will be round, arising from the conversion of the hyperbola about it transverse axis, in which case it shall constitute a particular species. But if there may be put $a=0$, a conical surface arises, which now we have considered as a specimen of the preceding kind.

123. Towards recognizing the following kinds we may put one of the coefficients *A*, *B*, and *C* to vanish. Therefore let $C=0$, and the general equation found § 114 will be

$$App+Bqq+Gp+Hq+Ir+K=0,$$

in which by increasing or diminishing the ordinates *p* and *q* it will be able to remove the terms *Gp* and *Hq*, but not indeed *Ir*. Therefore the term *Ir* will remain in the equation ; truly with the help of which the final term *K* will be able to be removed, from which we will have an equation of this kind :

$$App+Bqq=ar\ ,$$

two cases of which are to be considered. The former, if each coefficient *A* and *B* were positive, the latter, if either shall be negative. But in each case the centre of the surface will be placed on the axis *CD*, but removed to an infinite distance.

124. At first both the coefficients *A* and *B* shall be positive ; in which case the fourth genus may be put in place contained in this equation :

$$App+Bqq=ar\ .$$

Therefore the first principal section arising (Fig. 146), if there may be put $r=0$, will vanish to a point ; the second on putting $q=0$ and the third on putting $p=0$ each will be a parabola, truly *MAm* and *NAn*. Therefore since all the sections of this surface shall be ellipses normal to the axis *AD*, truly the sections made through this axis itself make parabolas, and we will call bodies of this kind *elliptic-paraboloids*. Two species of which are to be noted, the one if there shall be $A=B$, in which case a rounded body arises that we call a *parabolic-conoid* ;

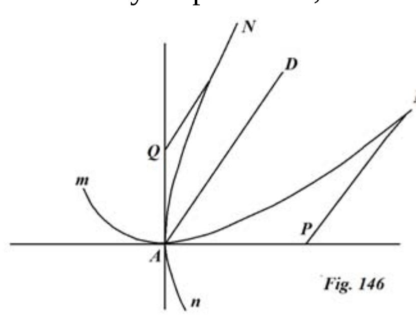

Fig. 146

and truly the other, if $a = 0$, and there becomes

$$App + Bqq = bb,$$

which give *cylinders* both *right,* if $A = B$, as well as *scalene,* if A and B were unequal.

125. The fifth genus will be contained in this equation

$$App - Bqq = ar,$$

the first principal section of which (Fig. 147) on making $r = 0$ will be two right lines *Ee, Ff* mutually crossing each other at the point *A*. Truly all the sections parallel to this will be hyperbolas having their centres on the axis *AD* and within the right asymptotes *Ee* and *Ff* put in place . Therefore the two planes, which stand normally to the plane *ABC* in the lines *Ee* and *Ff*, will be congruent with the proposed surface at infinity and thus this surface will have two planes mutually crossing each other for an asymptote. The remaining principal sections made in the planes *ACD* and *ABd* will be parabolas, from which we will call the surfaces pertaining to this genus *parabolic-hyperboloids* having two planes for asymptotes, the species of which (if $a = 0$, so that there shall be $App - Bqq = bb$), will be the *cylindric- hyperboloid,* all the sections of which normal to the axis *AD* will be hyperbolas equal to each other ; if in addition there shall be $b = 0$, these two asymptotic planes themselves arise.

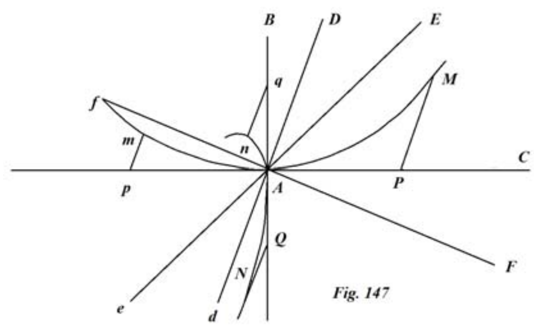

Fig. 147

126. Finally the sixth genus of surfaces of the second order will include this equation

$$App = aq,$$

which provides the *cylindrical paraboloid,* all the sections of which normal to the axis *AD* will be similar and equal parabolas, thus so that the vertices of the individual parabolas lie on the right line *AD*, and the axes shall be parallel to each other. Therefore all six kinds of surfaces of the second order can be reduced to these, thus so that none may be shown, which may not be contained in one of these kinds. Moreover, if in the final kind there is made $a = 0$, so that there shall be $App = bb$, this equation will provide two planes parallel to each other, which will constitute as if a specimen of this kind. Clearly the similarity used here applies to lines of the second order, where we have seen two right lines crossing each other constituting a kind of hyperbola, but two parallel lines constitute a kind of parabola

127. Nevertheless we have formed these six kinds from the simplest equation, to which a surface of the second order is allowed to be reduced, yet now, if some equation of the second order shall be proposed, it will be easy to assign the genus to which the surface may belong. So that if indeed this equation were proposed

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx+\eta z+\theta y+\iota x+\chi=0.$$

a judgment will be desired from the upper terms, in which the dimensions of two variables occur ; clearly these terms are to be examined :

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx,$$

in which if $4\alpha\zeta$ were greater than $\gamma\gamma$, $4\alpha\delta$ were greater than $\beta\beta$, $4\delta\zeta$ were greater than $\varepsilon\varepsilon$ and $\alpha\varepsilon\varepsilon+\delta\gamma\gamma+\zeta\beta\beta$ were less than $\beta\gamma\varepsilon+4\alpha\delta\zeta$, the surface will be closed and to belong to the first kind, which we have called ellipsoids.

128. If one or more of these conditions may be absent and nor yet shall there be $\alpha\varepsilon\varepsilon+\delta\gamma\gamma+\zeta\beta\beta=\beta\gamma\varepsilon+4\alpha\delta\zeta$, the surface will belong to either the second or the third genus and there a hyperbolic body will be provided with an asymptotic cone and that will be either circumscribed in the second genus, or inscribed in the third genus.
But, of there were

$$\alpha\varepsilon\varepsilon+\delta\gamma\gamma+\zeta\beta\beta=\beta\gamma\varepsilon+4\alpha\delta\zeta,$$

in which case the expression

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx$$

can be resolved into two simple factors, either imaginary or real, in the former case the surface will belong to the fourth genus, in the latter truly to the fifth. But if finally this expression may have two equal factors or it shall be a square, then the sixth genus will arise. And thus at once it will be easy to decide, to which genus some equation proposed may belong ; yet it will be more difficult to decide between the second genus and third, which both thus may be put together into one.

129. Surfaces of the third and of the following orders will be treated and divided into kinds in a similar manner. Clearly so many terms of the general equation must be considered and consequently for surfaces of the third order these, in which the coordinates possess three dimensions, which will be

$$\alpha z^3+\beta yzz+\gamma yyz+\delta xxz+\varepsilon xzz+\zeta xyz+\text{etc.}$$

Therefore initially it is required to be seen clearly, whether or not these terms be taken jointly, or the greatest member [*i.e.* the terms of the greatest power]of the equation may be able to be resolved into simple factors. If the resolution into factors may be rejected, the surface will have a cone of the third order for an asymptote. But because the nature of this cone is expressed by the uppermost member to be equal to zero, several cones of the third order of this kind will be given, from the diversity of which hence several kinds of surfaces will be put in place. For just as cones of the second order all refer to the same genus, because they are either right or scalene, in the third order much greater varieties still will find a place.

130. Therefore from these kinds put in place, the cases are required to be considered, in which the greatest member may be resolved into simple factors, whether they shall be real or imaginary. In the first place it may have one simple factor, which will be real ; from that the surface will have a plane asymptote. The other factor put equal to zero either will give a possible equation or not : if the equation were impossible, unless all the coordinates may vanish, there will be a single plane asymptote, but if that shall be impossible, the surface will have two asymptotes, the first a plane and the second a cone of the second order. But if it may have three simple factors, because one is real always, if the two remaining shall be either imaginary or real, two new kinds will arise. Finally if all three simple factors shall be real, according as two or all three shall be equal to each other, now two kinds of surfaces will be able to be put in place. But none is given in this order of surfaces, which does not extend to infinity.

CAPUT V

DE SUPERFICIEBUS SECUNDI ORDINIS

101. Constitutis ergo superficierum ordinibus secundum numerum dimensionum, quas summae trium coordinatarum x, y, z potestates in aequatione iunctim sumtae adimplent, si proponatur pro superficie aequatio algebraica, statim assignari potest ordo, ad quem illa superficies referri debet. Cum igitur omnis superficies primi ordinis ostensa sit esse plana, in hoc capite superficies secundi ordinis examini subiiciam. In iis autem maior statim deprehenditur diversitas, quam in lineis secundi gradus, quod quidem cuique attendenti facile patebit. Operam igitur dabo, ut haec diversa genera distincte exponam. In ordinibus vero altioribus tantopere multitudo generum increscit, ut ab iis evolvendis prorsus abstinere debeamus.

102. Quoniam natura superficierum secundi ordinis exprimitur aequatione, in qua variabiles x, y et z ad duas dimensiones assurgunt, cylindrus et conus tam rectus quam scalenus et globus, quorum proprietates iam descripsimus, in hoc secundo ordine continentur. Omnes vero superficies ad hunc ordinem pertinentes comprehenduntur in hac aequatione generali

$$\alpha zz + \beta yz + \gamma xz + \delta yy + \varepsilon xy + \zeta xx + \eta z + \theta y + \iota x + \chi = 0.$$

Utcunque enim tres coordinatae acciplantur, aequatio semper in hac forma continebitur. Varia ergo superficierum huc pertinentium genera a diversa coefficientium relatione mutua pendebunt, qui, etsi eadem superficies infinitis aequationibus exprimatur, tamen infinitam variarum superficierum multitudinem suppeditabunt.

103. Quemadmodum in lineis curvis planis praecipuam divisionem inde desumsimus, quod vel in infinitum extendantur vel in spatio finito includantur, ita simili modo omnes superficies ad quemcunque ordinem pertinentes in duas classes dividentur, ad quarum alteram referemus eas, quae in infinitum abeunt, ad alteram vero, quae in spatio finito continentur. Ita cylindrus et conus priori classi, globus vero posteriori annumerabitur. Posterioris quidem classis nulla dabitur superficies in ordinibus imparibus; cum enim quaelibet superficies imparis ordinis habeat sectiones planas eiusdem ordinis, curvae autem imparium ordinum omnes in infinitum extendantur, necesse est, ut etiam ipsae superficies istorum ordinum in infinitum porrigantur.

104. Quoties autem quaepiam superficies in infinitum extenditur, necesse est, ut ad minimum una trium variabilium x, y et z in infinitum abeat. Quare cum perinde sit, quaenam hoc casu infinita fieri assumatur, ponamus z fieri infinitam, siquidem superficies in infinitum porrigatur. Naturam ergo huius partis in infinitum abeuntis investigaturi ponamus esse $z = \infty$, atque nunc potissimum spectari debet terminus primus αzz, utrum is adsit an vero deficiat. Adsit ergo primum iste terminus in aequatione atque

prae eo termini ηz et χ evanescent habiturque pro parte in infinitum excurrente haec aequatio

$$\alpha zz + \beta yz + \gamma xz + \delta yy + \varepsilon xy + \zeta xx + \theta y + \iota x = 0,$$

ex qua porro omnes termini, qui non sunt infiniti vel infinities minores saltem quam αzz, evanescunt.

105. Statuamus omnes terminos, in quibus variabiles duas tenent dimensiones, adesse; quaecunque enim fuerit superficies, in eius aequatione generalissima semper omnes inerunt termini summarum dimensionum neque idcirco hypothesis, qua omnes terminos duarum dimensionum adesse ponimus, universaliti solutionis ullam vim infert. Quando autem termini yz et xz adsunt, prae iis termini θy et ιx evanescunt relinqueturque haec aequatio

$$\alpha zz + \beta yz + \gamma xz + \delta yy + \varepsilon xy + \zeta xx = 0,$$

ex qua elicitor

$$z = \frac{-\beta y - \gamma x \pm \sqrt{((\beta\beta - 4\alpha\delta)yy + (2\beta y - 4\alpha\varepsilon)xy + (\gamma\gamma - 4\alpha\zeta)xx)}}{2\alpha}.$$

Hac igitur aequatione natura portionis in infinitum extensae exprimitur.

106. Si quam igitur superficies habeat portionem in infinitum extensam, ea congruet cum portione infinita superficiei, quae exprimitur hac aequatione

$$\alpha zz + \beta yz + \gamma xz + \delta yy + \varepsilon xy + \zeta xx = 0,$$

ita ut haec superficies sit quasi asymptota illius superficiei aequatione generali expressae. Quia vero in hac aequatione tres variabiles ubique duas habent dimensiones, erit ea pro superficie conica, verticem in initio coordinatarum, ubi omnes simul evanescunt, habente; semper ergo exhiberi potest superficies conica, quae erit asymptota superficiei propositae, siquidem in infinitum extenditur, seu cuius portio infinita cum superficie proposita vel penitus congruit, vel intervallo tantum finito ab eo est remota. Uti ergo ramos curvarum in infinitum abeuntes per lineas rectas asymptotas distinximus, ita superficierum partes in infinitum extensas per superficies conicas asymptotas distinguere licebit.

107. Quoties ergo superficies asymptota conica erit realis, toties superficies ipsa in infinitum extenditur atque ita quidem, ut utriusque partes infinitae congruant; sicque ex natura superficiei asymptotae natura ipsius superficiei propositae colligi poterit. Quodsi autem superficies asymptota fiat imaginaria, ipsa superficies nullam habebit partem in

infinitum extensam, sed tota spatio finito includetur. Ad superficies ergo secundi ordinis, quae in spatio finito contineantur, indagandas tantum opus est, ut videamus, quibus in casibus aequatio pro superficie asymptota fiat imaginaria; quod fit, si tota superficies haec in punctum unicum evanescit. Namque si ullam extensionem haberet vel punctum extra verticem situm, necessario in infinitum expandi deberet, propterea quod supra ostendimus totam rectam, quae per verticem et unum superficiei punctum ducitur, in ipsa superficie esse positum.

108. Quando ergo superficies conica asymptota hac aequatione expressa

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx=0,$$

in unicum punctum abit, omnes eius sectiones per verticem factae pariter in idem punctum evanescere debent. Primum ergo facto $z=0$ aequatio $\delta yy+\varepsilon xy+\zeta xx=0$, debet esse impossibilis, nisi sit $x=0$ et $y=0$, quod evenit, si fuerit $4\delta\zeta$ maior quam $\varepsilon\varepsilon$. Deinde idem evenire debet posito $x=0$ vel $y=0$; erit ergo $4\alpha\delta$ maior quam $\beta\beta$ et $4\alpha\zeta$ maior quam $\gamma\gamma$. Nisi ergo in aequatione pro superficie secundi ordinis

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx+\eta z+\theta y+\iota x+\chi=0.$$

fuerit $4\delta\zeta$ maior quam $\varepsilon\varepsilon$, $4\alpha\delta$ maior quam $\beta\beta$, $4\alpha\zeta$ maior quam $\gamma\gamma$, superficies certo habebit partes in infinitum extensas.

109. Neque vero hae tres conditiones sufficiunt ad superficiem in spatium finitum includendam; requiritur insuper, ut valor ipsius z ex aequatione asymptotica supra erutus fiat imaginarius; quod fit, si ista expression

$$(\beta\beta-4\alpha\delta)yy+2\left(\beta y-2\alpha\varepsilon\right)xy+\left(\gamma\gamma-4\alpha\zeta\right)xx$$

perpetuo obtineat valorem negativum, siquidem pro utraque variabili x et y valores quicunque praeter 0 substituantur. Quod, cum $\beta\beta-4\alpha\delta$ et $\gamma\gamma-4\alpha\zeta$ sint quantitates negativae, fiet, si $(\beta\gamma-2\alpha\varepsilon)^2$ minor quam $(\beta\beta-4\alpha\delta)\left(\gamma\gamma-4\alpha\zeta\right)$; hoc est, si fuerit $\alpha\varepsilon^2+\delta\gamma^2+\zeta\beta^2$ minor quam $\beta\gamma\varepsilon+4\alpha\delta\zeta$, siquidem α habuerit valorem affirmativum, quoniam illam aequationem per α divisimus. Quodsi vero α habeat valorem affirmativum, ob superiores aequationes ($4\alpha\zeta$ maior quam $\gamma\gamma$, $4\alpha\delta$ maior quam $\beta\beta$ et $4\delta\zeta$ maior quam $\varepsilon\varepsilon$) coefficientes δ et ζ erunt affirmativi.

110. Superficies ergo secundi ordinis in spatio finito continebitur, si in eius aequatione quatuor sequentes conditiones locum habeant, nempe si sit

$$4\alpha\zeta \text{ maior quam } \gamma\gamma,\ 4\alpha\delta \text{ maior quam } \beta\beta,\ 4\delta\zeta \text{ maior quam } \varepsilon\varepsilon$$

et

$$\alpha\varepsilon^2 + \delta\gamma^2 + \zeta\beta^2 \text{ minor quam } \beta\gamma\varepsilon + 4\alpha\delta\zeta.$$

Hincque genus primum superficierum secundi ordinis definimus, ad quod eae species omnes pertinent, quae non in infinitum excurrunt, sed in spatio finito includuntur. Ad hoc ergo genus pertinet globus, cuius aequatio est

$$zz + yy + xx = aa\,,$$

cum enim hic sit $\alpha = 1, \delta = 1, \zeta = 1, \beta = 0, \gamma = 0, \varepsilon = 0$, quatuor inventis conditionibus omnibus satisfit. Generalius vero hic pertinebit aequatio ista

$$\alpha zz + \delta yy + \zeta xx = aa\,,$$

quae, si $\alpha,\ \delta,\ \zeta$ fuerint quantitates affirmativae, semper est pro superficie clausa, nisi unus duove coefficientes evanescant.

111. Perspectis his quatuor conditionibus, quibus superficies in spatium finitum redigitur, si proponatur aequatio secundi ordinis quaecunque determinata, statim diiudicari poterit, utrum superficies ea aequatione expressa habeat partes in infinitum extensas an nullas. Quodsi enim unica illarum quatuor conditionum desit, superficies certo in infinitum extenditur. Hoc autem casu nonnullae subdivisiones sunt faciendae, quibus singularis varietas partibus in infinitum extensis inducitur. Prima subdivisio ergo constituatur, si fuerit

$$\alpha\varepsilon\varepsilon + \delta\gamma\gamma + \zeta\beta\beta \text{ maior quam } \beta\gamma\varepsilon + 4\alpha\delta\zeta\,,$$

quo casu superficies in infinitum extendetur atque superficiem conicam pro asymptota habebit, uti iam ante ostendimus. Hicque casus e diametro est oppositus praecedenti, quo tota superficies in spatio finito continetur.

112. Praeterea autem dantur casus quidam intermedii, quibus etsi superficies in infinitum abit, simili tamen modo inter duos praecedentes locum tenet, quo parabola inter ellipsin et hyperbolam continetur. Casus iste oritur, si fuerit

$$\alpha\varepsilon\varepsilon + \gamma\gamma + \zeta\beta\beta = \beta\gamma\varepsilon + 4\alpha\delta\zeta,$$

eritque propterea

$$\alpha z = -\beta y - \gamma x + y\sqrt{(\beta\beta - 4\alpha\delta)} + x\sqrt{(\gamma\gamma - 4\alpha\zeta)}.$$

Habebit ergo aequatio asymptotica

$$\alpha zz + \beta yz + \gamma xz + \delta yy + \varepsilon xy + \zeta xx = 0$$

duos factores simplices, qui erunt vel reales vel imaginarii vel inter se aequales. Triplex ista diversitas ergo tria genera superficierum in infinitum extensarum praebet, sicque omnino quinque genera superficierum secundi ordinis sumus adepti, quae nunc diligentius prosequemur.

113. Quia mutando positionem ternorum axium, quibus coordinatae sunt parallelae, aequatio generalis ad formam simpliciorem reduci potest, ista reductione ita utamur, ut aequationem generalem pro superficiebus secundi ordinis ad formam simplicissimam redigamus, quae tamen omnes species aeque ac generalis in se complectatur. Cum igitur aequatio generalis pro superficiebus secundi ordinis sit

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx+\eta z+\theta y+\iota x+\chi=0.$$

quaeramus aequationem inter alias ternas coordinatas *p, q* et *r,* quae quidem se mutuo in eodem puncto, quo ternae priores, se decussent. Ad hoc ex § 92 statuatur

$$x=p(\cos.k\cdot\cos.m-\sin.k\cdot\sin.m\cdot\cos.n)+q(\cos.k\cdot\sin.m+\sin.k\cdot\cos.m\cdot\cos.n)-r\cdot\sin.k\cdot\sin.n$$

et

$$y=-p(\sin.k\cdot\cos.m+\cos.k\cdot\sin.m\cdot\cos.n)-q\left(\sin.k\cdot\sin.m-\cos.k\cdot\cos.m\cdot\cos.n\right)-r\cdot\cos.k\cdot\sin.n$$

atque

$$z=-p\cdot\sin.m\cdot\sin.n+q\cdot\cos.m\cdot\sin.n+r\cdot\cos.n,$$

unde resultet ista aequatio

$$App+Bqq+Crr+Dpq+Epr+Fqr+Gp+Hq+Ir+K=0.$$

114. Iam anguli illi arbitrarii *k, m* et *n* ita definiri poterunt, ut tres coefficientes *D, E* et *F* evanescant. Quanquam enim calculus nimis fit prolixus, quam ut angulorum illorum determinatio actu ostendi possit, tamen, si quis forte dubitet, an semper ista eliminatio ad valores reales angulorum illorum perducat, is certe concedere debebit duos saltem coefficientes *D* et *E* nihilo aequales reddi posse. Hoc autem si fuerit effectum, positio tertii axis, cui ordinatae *r* sunt parallelae in plano ad ordinatas *p* normali, facile ita mutari potest, ut etiam coefficiens *F* evanescat. Statuatur enim

$$q=t\cdot\sin.i+u\cdot\cos.i\quad\text{et}\quad r=t\cdot\cos.i-u\cdot\sin.i\,,$$

ita ut loco termini *qr* novus terminus *tu* ingrediatur, cuius coefficiens ope anguli *i* nihilo aequalis fieri poterit. Hoc igitur modo aequatio generalis pro superficiebus secundi ordinis ad hanc formam perducetur

$$App + Bqq + Crr + Gp + Hq + Ir + K = 0 .$$

115. Nunc praeterea coordinatae *p, q, r* datis quantitatibus ita augeri diminuive poterunt, ut coefficientes *G, H* et *I* evanescent ; quod fiet mutato tantum puncto illo, unde omnes coordinatae initium habent. Atque hoc modo omnes superficies secundi ordinis in hac aequatione continebuntur

$$App + Bqq + Crr + K = 0 ,$$

ex qua intelligitur unumquodque trium planorum principalium per initium coordinatarum ductorum superficiem in duas partes similes et aequales bisecare. Omnis ergo superficies secundi ordinis non solum unum habet planum diametrale, sed adeo tria, quae se mutuo in eodem puncto normaliter intersecent; quod punctum propterea centrum superficiei constituet, etiamsi in nonnullis casibus hoc centrum in infinitum distet, simili scilicet modo, quo omnes sectiones conicae centro dicuntur praeditae, etiamsi in parabola centrum a vertice infinite removeatur.

116. Perducta ergo aequatione, qua omnes superficies secundi ordinis continentur, ad formam simplicissimam primum harum superficierum genus exhibebit ista aequatio

$$App + Bqq + Crr = aa,$$

siquidem omnes tres coefficientes *A, B* et *C* valores obtineant affirmativos. Superficies igitur ad hoc primum genus pertinentes non solum totae in finito spatio includentur, sed omnes quoque centrum habebunt, in quo tria plana diametralia se mutuo ad angulos rectos decussant. Sit (Fig. 143) *C* centrum huius figurae et *CA, CB, CD* axes illi principales inter se normales, quibus coordinatae *p, q, r* sunt parallelae, erunt tria plana diametralia *ABab, ADa* et *BDb*, quibus hoc corpus in binas portiones similes aequales secabitur.

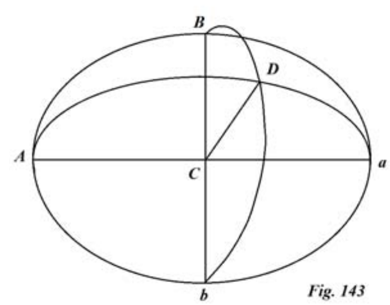

Fig. 143

117. Ponatur $r = 0$ et aequatio $App + Bqq = aa$ exprimet naturam sectionis principalis *ABab;* quae idcirco erit ellipsis centrum habens in *C,* cuius semiaxes erunt

$$CA = Ca = \frac{a}{\sqrt{A}} \quad \text{et} \quad CB = Cb = \frac{a}{\sqrt{B}} .$$

Si ponatur $q = 0$, aequatio $App + Crr = aa$ erit pro sectione principali *ADa*, quae pariter erit ellipsis centrum habens in *C*, cuius semiaxes erunt

$$CA = Ca = \frac{a}{\sqrt{A}} \text{ et } CD = \frac{a}{\sqrt{C}}$$

Posito autem $p = 0$ prodibit pro tertia sectione principali *BDb* aequatio $Bqq + Crr = aa$, quae etiam erit ellipsis centrum habens in *C* et semiaxes

$$CB = Cb = \frac{a}{\sqrt{B}} \text{ et } CD = \frac{a}{\sqrt{C}}.,$$

Cognitis autem his tribus sectionibus principalibus seu tantum earum semiaxibus

$$CA = \frac{a}{\sqrt{A}}, \ CB = \frac{a}{\sqrt{B}} \text{ et } CD = \frac{a}{\sqrt{C}},$$

natura huius corporis determinatur et cognoscitur. Hinc primum istud superficierum secundi ordinis genus *elliptoides* appellari conveniet, quia tres eius sectiones principales sunt ellipses.

118. Sub hoc genere continentur tres species prae primis notatu dignae. Prima est, si omnes tres axes principales *CA*, *CB* et *CD* inter se fuerint aequales, quo casu tres sectiones principales abibunt in circulos ipsumque corpus in globum, cuius aequatio, uti supra vidimus, erit

$$pp + qq + rr = aa.$$

Secunda species eos complectitur casus, quibus duo tantum axes principales sunt inter se aequales. Sit nimirum $CD = CB$ seu $C = B$, atque sectio *BDb* fiet circulus, ex aequatione autem

$$App + B(qq + rr) = aa$$

intelligitur omnes sectiones huic parallelas pariter fore circulos; unde hoc corpus erit sphaeroides sive oblongum, si *AC* maior sit quam *BC*, sive compressum, si *AC* sit minor quam *BC*. Tertia denique species ea complectitur corpora, in quibus coefficientes *A*, *B*, *C* sunt inaequales, quae ideo nomen generale *elliptoidis* retinebunt.

119. Sequentia genera superficierum secundi ordinis hac continebuntur aequatione

$$App + Bqq + Crr = aa,$$

ac primo quidem, si nullus coefficientium *A*, *B*, *C* prorsus desit, eorum autem vel unus vel duo valores habeant negativos. Sit unus tantum negativus atque consideremus hanc aequationem

$$App + Bqq - Crr = aa,$$

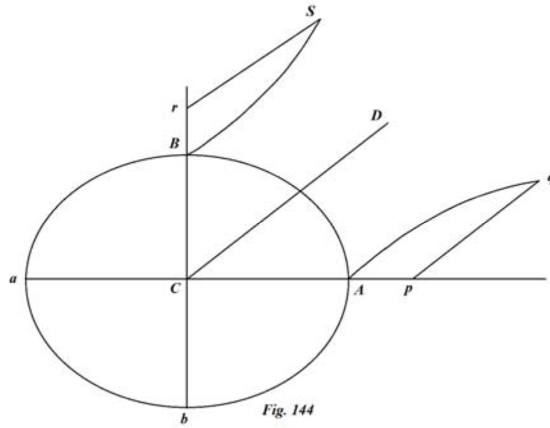

Fig. 144

in qua iam *A*, *B*, *C* numeros affirmativos denotare ponimus. Quod ad centrum huius corporis et plana diametralia attinet, omnia eodem modo sunt comparata ut ante. Patet igitur (Fig. 144) huius corporis sectionem principalem primam *ABab* esse ellipsin, cuius semiaxis $AC = \frac{a}{\sqrt{A}}$ alterque $BC = \frac{a}{\sqrt{B}}$. Binae reliquae sectiones principales *Aq*, *BS* erunt hyperbolae centrum in *C* et semiaxem coniugatum $= \frac{a}{\sqrt{C}}$ habentes.

120. Repraesentabit ergo haec superficies speciem infundibuli, sursum et deorsum secundum hyperbolas divergens. Unde ista superficies asymptoton habebit conum aequatione $App + Bqq - Crr = 0$ expressum, verticem, in centro *C* habentem, et cuius latera sunt asymptota hyperbolarum. Stabit autem iste conus asymptotos intra superficiem eritque conus rectus, si fuerit $A = B$, scalenus vero, si *A* non ipsi *B* aequabitur. Axis autem coni erit recta *CD* normalis ad planum *ABa*. Ceterum omnes sectiones axi *CD* normales erunt ellipses similes ellipsi *ABab*, sectiones vero plano *ABab* normales omnes erunt hyperbolae; unde istas superficies *elliptico-hyperbolicas* vocari conveniet, asymptoto suo conico circumscriptas. Huius igitur superficies nobis constituent genus secundum.

121. Species in hoc genere iterum tres notari poterunt, quarum prima erit, si $a = 0$, quo casu ellipsis *ABab* in punctum evanescit et hyperbolae in lineas rectas abibunt, superficies vero ipsa cum asymptota sua penitus confundetur, ex quo haec prima species complectetur omnes conos, sive rectos sive scalenos; unde nova subdivisio fieri posset. Altera species erit, si fiat $A = B$, quo casu ellipsis *ABab* in circulum mutatur et ipsa superficies fiet rotunda seu tornata. Orietur scilicet haec superficies, si hyperbola quaecunque circa axem coniugatum convertatur. Tertia species ab ipso genere non discrepabit.

122. Tertium genus definiamus, si duo coefficientes terminorum *pp*,*qq* et *rr* fiant negativi, cuius ergo aequatio sit

$$App - Bqq - Crr = aa.$$

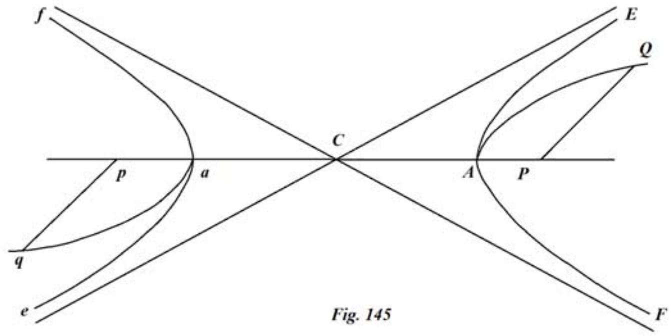

Fig. 145

Posito ergo $r = 0$ erit (Fig. 145) prima sectio principalis hyperbola *EAFeaf* centrum habens in *C*, cuius semiaxis transversus erit $= \frac{a}{\sqrt{A}}$ et semiaxis coniugatus $= \frac{a}{\sqrt{B}}$. Altera sectio principalis posito $q = 0$ pariter erit hyperbola *AQ*, *aq* eodem semiaxe transversa praedita, sed cuius axis semiconiugatus erit $= \frac{a}{\sqrt{C}}$; tertia sectio principalis fit imaginaria. Tota denique haec superficies intra superficiem conicam asymptotam erit sita, unde hoc genus vocari potest *hyperbolico-hyperbolicum* cono asymptoto inscriptum. Si fiat $B = C$, superficies erit rotunda, orta ex conversione hyperbolae circa suum axem transversum, quo casu species peculiaris constitui posset. Sin autem ponatur $a = 0$, oritur superficies conica, quam iam tanquam speciem generis praecedentis sumus contemplati.

123. Ad sequentia genera cognoscenda ponamus unum coefficientium *A*, *B*, *C* evanescere. Sit igitur $C = 0$, atque aequatio generalis § 114 inventa erit

$$App + Bqq + Gp + Hq + Ir + K = 0,$$

in qua augendo seu diminuendo ordinatas *p* et *q* termini *Gp* et *Hq*, non vero *Ir* tolli poterit. Relinquetur ergo terminus *Ir* in aequatione; eius vero ope tolli poterit terminus ultimus *K*, unde eiusmodi aequationem habebimus

$$App + Bqq = ar,$$

cuius duo casus sunt perpendendi. Prior, si uterque coefficiens *A* et *B* fuerit affirmativus, posterior, si alter sit negativus. Utroque autem casu centrum superficiei in axe *CD* erit situm, sed ad distantiam infinitam remotum.

124. Sint primo ambo coefficientes *A* et *B* affirmativi; quo casu constituatur genus quartum aequatione hac contentum

$$App + Bqq = ar.$$

Prima ergo sectio principalis (Fig. 146) oriunda, si ponatur $r=0$, in punctum evanescet ; altera posito $q=0$ et tertia posito $p=0$ utraque erit parabola, nempe *MAm* et *NAn.* Cum igitur huius superficiei omnes sectiones ad axem *AD* normales sint ellipses, sectiones vero per hunc ipsum axem factae parabolae, huius generis corpora *elliptico-parabolica* appellabimus. Cuius species sunt notandae duae, altera, si $A=B$, quo casu oritur corpus rotundum, *conoides parabolicum* vocatum; fitque altera vero, si $a=0$, fitque

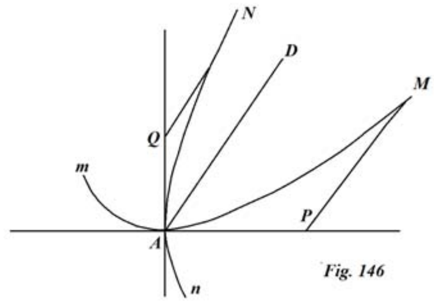

Fig. 146

$$App+Bqq=bb,$$

quae dat *cylindros* tam *rectos,* si $A=B$, quam *scalenos,* si *A* et *B* fuerint inaequales.

125. Quintum genus continebitur hac aequatione

$$App-Bqq=ar,$$

cuius (Fig. 147) sectio principalis prima facto $r=0$ erunt duae lineae rectae *Ee, Ff* se mutuo in puncto *A* decussantes. Omnes vero sectiones huic parallelae erunt hyperbolae sua centra in axe *AD* habentes et intra asymptotas rectas *Ee* et *Ff* constitutae. Duo igitur plana, quae plano *ABC* in lineis *Ee* et *Ff* normaliter insistunt, in infinitum cum superficie proposita congruent ideoque haec superficies pro asymptoto habebit duo plana se mutuo decussantia. Sectiones reliquae principales in planis *ACD* et *ABd* factae erunt parabolae, unde superficies ad hoc genus pertinentes vocabimus *parabolico-hyperbolicas* duo plana pro asymptotis habentes, cuius species (si $a=0$, ut sit $App-Bqq=bb$), erit *cylindrus hyperbolicus,* cuius omnes sectiones ad axem *AD* normales erunt hyperbolae inter se aequales; si insuper sit $b=0$, oriuntur duo illa ipsa plana asymptotica.

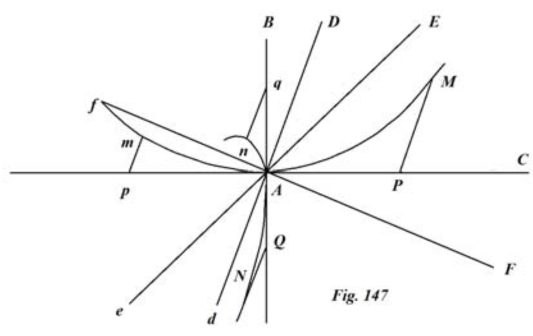

Fig. 147

126. Sextum denique genus superficierum secundi ordinis complectetur haec aequatio

$$App=aq,$$

quae praebet *cylindrum parabolicum,* cuius omnes sectiones axi *AD* normales erunt parabolae similes et aequales, ita ut singularum vertices in rectam *AD* incidant et axes inter se sint paralleli. Ad haec igitur sex genera omnes superficies secundi ordinis reduci poterunt, ita ut nulla exhiberi possit, quae non in uno horum generum contineatur. Ceterum, si in genere ultimo fiat $a=0$, ut sit $App=bb$, haec aequatio praebebit duo plana inter se parallela, quae quasi speciem huius generis constituent. Similitudo scilicet hic uti in lineis secundi ordinis obtinet, ubi vidimus duas rectas se decussantes hyperbolae speciem, duas autem lineas parallelas parabolae speciem constituere.

127. Quanquam haec sex genera ex aequatione simplicissima, ad quam superficies secundi ordinis reducere licet, formavimus, tamen nunc facile erit, si aequatio quaecunque secundi gradus sit proposita, genus assignare, ad quod superficies pertineat. Quodsi enim proposita fuerit haec aequatio

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx+\eta z+\theta y+\iota x+\chi=0.$$

iudicium ex supremis terminis, in quibus duae variabilium occurrunt dimensiones, petendum erit; spectari scilicet debebunt hi termini

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx,$$

in quibus si fuerit $4\alpha\zeta$ maior quam $\gamma\gamma$, $4\alpha\delta$ maior quam $\beta\beta$, $4\delta\zeta$ maior quam $\varepsilon\varepsilon$ et $\alpha\varepsilon\varepsilon+\delta\gamma\gamma+\zeta\beta\beta$ minor quam $\beta\gamma\varepsilon+4\alpha\delta\zeta$ superficies erit clausa et ad genus primum, quod elliptoides vocavimus, pertinebit.

128. Si una pluresve harum conditionum desint neque tamen sit $\alpha\varepsilon\varepsilon+\delta\gamma\gamma+\zeta\beta\beta=\beta\gamma\varepsilon+4\alpha\delta\zeta$, superficies vel ad secundum vel ad tertium genus pertinebit eritque corpus hyperbolicum cono asymptoto praeditum eique vel circumscriptum in genere secundo vel inscriptum in genere tertio. At, si fuerit

$$\alpha\varepsilon\varepsilon+\delta\gamma\gamma+\zeta\beta\beta=\beta\gamma\varepsilon+4\alpha\delta\zeta,$$

quo casu expression

$$\alpha zz+\beta yz+\gamma xz+\delta yy+\varepsilon xy+\zeta xx$$

resolvi poterit in duos factores simplices sive imaginarios sive reales, casu priori superficies pertinebit ad genus quartum, posteriori vero ad quintum. Quodsi denique ista expressio duos habeat factores aequales seu sit quadratum, tum orietur genus sextum. Sicque statim facile diiudicari poterit, ad quodnam genus quaevis aequatio proposita pertineat; difficilius tantum erit iudicium circa genus secundum et tertium, quae ambo ideo in unum conflari possent.

129. Simili modo superficies tertii et sequentium ordinum pertractari atque in genera dividi poterunt. Spectari scilicet tantum debebunt aequationis generalis termini supremi et consequenter pro superficiebus tertii ordinis ii, in quibus coordinatae tres obtinent dimensiones, qui erunt

$$\alpha z^3 + \beta yzz + \gamma yyz + \delta xxz + \varepsilon xzz + \zeta xyz + \text{etc.}$$

Primum igitur dispiciendum est, utrum hi termini coniunctim sumti seu aequationis membrum supremum resolvi possit in factores simplices an non.
Si resolutionem in factores respuat, habebit superficies conum tertii ordinis pro asymptoto. Quia autem natura huius coni exprimitur supremo membro nihilo aequali posito, plures huiusmodi coni tertii ordinis dabuntur, ex quorum diversitate hinc plura superficierum genera constituentur. Quamvis enim coni secundi ordinis omnes ad unum genus referuntur, quia sunt vel recti vel scaleni, tamen in tertio ordine multo maior varietas locum invenit.

130. Expositis ergo his generibus, considerandi sunt casus, quibus supremum membrum in factores simplices resolvi potest, sive sint reales sive imaginarii. Habeat primum unum factorem simplicem, qui erit realis; ex eo superficies habebit asymptotam planam. Alter factor nihilo aequalis positus vel dabit aequationem possibilem vel non: si aequatio fuerit impossibilis, nisi omnes coordinatae evanescant, unica erit asymptota plana, sin autem sit impossibilis, superficies duas habebit asymptotas, alteram planam alteram conum secundi ordinis. Quodsi habeat tres factores simplices, quia unus semper est realis, si bini reliqui sint vel imaginarii vel reales, duo nova genera oriuntur. Denique si omnes tres factores simplices sint reales, prout duo vel omnes sint inter se aequales, adhuc duo genera constitui poterunt. Nulla autem in hoc ordine datur superficies, quae non in infinitum extendatur.