CHAPTER II

CONCERNING THE SECTIONS OF SURFACES MADE BY ANY PLANES

26. Just as the intersections of lines are points, thus the intersections of surfaces are either right or curved lines . The intersection of two planes is a right line, as may be agreed from the Elements. But the section of a globe with a plane is a circle. But much more aid is available towards the recognition of a surface, if we may know the lines, by which the surface may be intersected by given planes. For in this manner at the same time the points of an infinitude of surfaces become known, but only if in the preceding method individual points of the surface may be produced from the individual values of the variable *z*.

27. Therefore since we may refer a surface to three intersecting planes normal to each other, before everything to be investigated the intersections of these planes with the surface will be agreed upon. Therefore in the first place with the plane (Fig. 121) *APQ* taken, so that it may be determined from the variables $AP = x,\ AQ = y$ (because the third variable *z* assigns the distance of each point of the surface from this plane), it is evident, if there may be put $z = 0$, those points of the surface will be going to be found, which shall be situated in the plane itself *APQ*, and on that account the equation remaining between *x* and *y* will show the line, by which the surface may be intersected by the plane *APQ*. In a similar manner it there may be put $y = 0$, the equation between *x* and *z* may express the intersection of the surface made by the plane *APR*, and on putting $x = 0$ the equation between *y* and *z* will give the intersection of the surface and of the plane *AQR*.

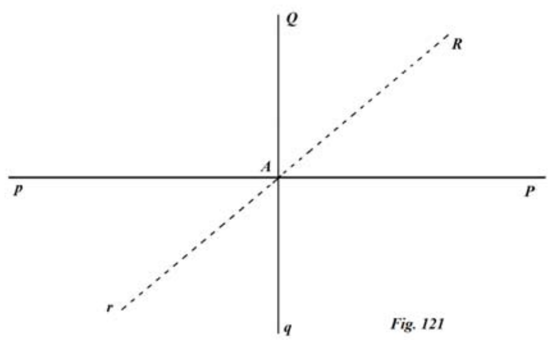

Fig. 121

28. Now we have hinted above that the surface of a globe having the centre at the point *A*, of which the radius $= a$, is expressed by this equation $xx + yy + zz = aa$; therefore I may use this by way of an example as an illustration of these. Therefore there shall be $z = 0$, and the equation $xx + yy = aa$ will show the intersection of the globe made with the plane *APQ*, which therefore appears to be a circle having the centre *A* and radius $= a$. In a similar manner on making $y = 0$ the intersection of the globe made with the plane *APR* will be the circle contained

in the equation $xx+zz=aa$. And in the same manner, if there may be put $x=0$, the equation $yy+zz=aa$ indicates an equal circle from the intersection of the globe with the plane *AQR*. Indeed these are well-known, since all the sections of a globe made by planes passing through its centre shall be great circles, or having a common radius with the globe.

29. The sections of a surface by other planes made parallel to one of these principal planes. A plane may be considered parallel to the plane *APQ* distant by an interval *h* from that, therefore all the points of the surface, of which the distance from the same plane *APQ*, which may be indicated by the variable *z*, is $=h$, likewise will be situated in this parallel plane and thus will form the intersection. Therefore an equation will be had for this intersection, if in the equation for the surface there may be put $z=h$; for then the equation will be had between the two orthogonal variables *x* and *y* expressing the nature of the surface. Moreover in the same manner the sections may be defined, which shall be made parallel either to the plane *APR* or *AQR*, from which it becomes superfluous, for what has been said for one, is repeated in the rest.

30. Therefore if in the equation for a surface between the three coordinates *x*, *y* and *z* with one these *z* being put constant$=h$, then a section of the surface formed by a plane parallel to the plane *APQ* and distant from that by an interval *h* arises. So that if therefore successively to this all the values of the letter *h* possible, both of the positive as well as negative, may be attributed, then all the sections of the surfaces will be obtained, which may be formed by a plane parallel to the plane *APQ*, and since all the surfaces of this kind shall be able to be dissected from parallel planes of this kind into an infinitude of parts and in this manner all the sections may become known, from these the whole surface made from all these sections will become known. Clearly all these sections will be expressed by a single equation between the coordinates *x* and *y* by involving a single indeterminate constant *h* ; from which all these sections will be lines either similar or at least contained by a single related equation.

31. Therefore all the sections of the surface parallel to the plane *APQ* will be equal to each other and are cut equally in the same manner by the planes *APR*, *AQR*, if an equation between *x* and *y* were to be prepared thus, so that it may remain the same, in whatever manner the value of *h* may be attributed. But this cannot come about, unless the variable *z*, of which the value *h* has been put in its place, shall be completely absent from the equation for the surface. Concerning which, if the third variable *z* shall not be present in the equation of the surface generally, then all the sections parallel to the plane *APQ* will be equal to each other, of which the nature itself will be expressed by the equation of the surface, certainly which involves the two variables *x* and *y* only. Truly in a similar manner, if in the equation for the surface either the variable *x* or *y* shall be absent, then all

the sections either parallel to the plane *AQR* or to the plane *APR* will be congruent among themselves.

32. Not only is a surface of this kind easily formed in the mind, but also it may be constructed and formed on a given material. For we may put the variable *z* to be missing in the given equation, thus so that the equation shall be only between the coordinates (Fig. 122) $AP = x$ and $AQ = PM = y$; from this the curved line *BMD* may be described in the plane *APQ*. With which done an infinite right line may be considered normal to the plane always following this line carried around the curve *BMD*, and this right line by its motion will produce or form the surface indicated by that equation. From which it is evident, if the line *BMD* were a circle, then the surface arising from that to be of a right cylinder; but if the line *BMD* were an ellipsis, then the surface to be generated will be of a scalene cylinder. But if the line *BMD* were not continued, but constructed from several right lines showing a rectilinear figure, then a prismatic surface will result.

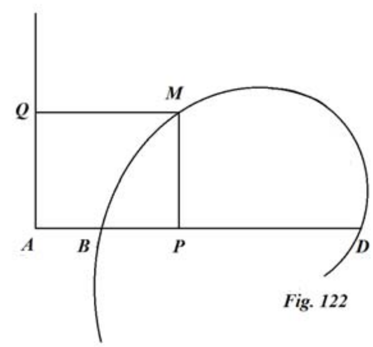

Fig. 122

33. Because this kind of surface includes cylinders and all prisms, this whole kind of surfaces may be agreed to be called *cylindrical* or *prismatic* ; moreover the individual kinds contained within this kind will be determined by the plane figure *BMD*, from which they shall arise in the manner described before ; and this figure *BMD* will be called the base. Therefore as many times in the equation for a surface with one of the three variables *x, y, z* absent, then the surface contained by this equation will be cylindrical or prismatic. But if moreover the two variables *y* and *x* were absent together, then on account of $x =$ a constant, the line *BMD* will be changed into a right line normal to the axis *AD* and therefore a plane surface is made normal to the plane *APQ.*

34. After this kind of surfaces, that deserves to become known arises from the equation between the three homogeneous variables *x, y* and *z,* or in which these three variables everywhere constitute a number of the same dimensions, of such a kind is $zz = mxz + xx + yy$. Hence indeed all the sections, which arise from planes parallel to one of the three principal planes, will be figures similar to each other. In as much as if the constant value *h* may be attributed to *z,* it is evident the equation $hh = mhx + xx + yy$, if for *h* successively all the other values may be attributed, infinitely many figures are contained similar to each other, the parameters of which shall be equal or proportionals of *h.* Therefore since these sections shall not only be similar, but also may increase in the ratio of the distances from the plane *APQ,* the lines, which may be drawn from the point *A* through the points of the individual sections, will be right.

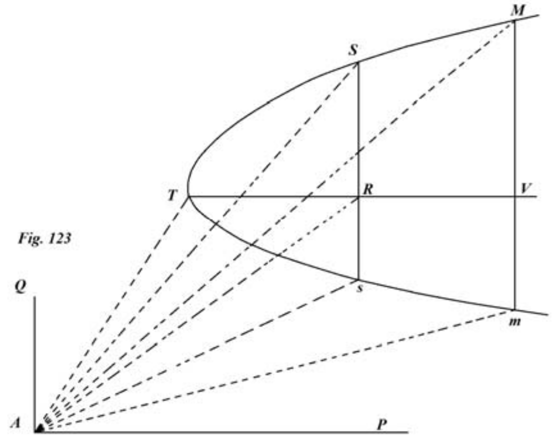

Fig. 123

35. Therefore with a proposed homogeneous equation of this kind between the three variables *x*, *y* and *z*, a given value of *z* may be attributed (Fig. 123) $AR = h$, and *TSsMm* shall be a figure in a plane parallel to *APQ* itself and drawn through the point *R*, as the equation between *x* and *y* will show, thus so that there shall be $RV = x$ and $VM = y$. But if this section *TSsMm* therefore were described once, about which it may be understood, an infinite number of right lines may be drawn around its perimeter always passing through the point *A* ; and this right line by its motion describes the surface contained in the proposed equation. Truly it is evident, if the figure *TSsMm* were a circle having the centre at *R* , then a right cone emerges ; if *R* shall not be the centre, a scalene cone emerges ; but if that figure were rectilinear, pyramids of each kind will arise. So that the surfaces on that account, which will be contained in this kind of equations, we will call here *conics* or *pyramids*.

36. From these it is clear, if the equation between the three variables *x*, *y* and *z* were homogeneous and thus the surface conical or pyramidal, then not only shall all the sections, parallel amongst themselves to a single principal plane *APQ*, be similar figures, the parameters of which shall be proportional to the distances of the sections from the vertex *A*, but on account of the same ratio it is understood also that all the sections, which shall be parallel either to the plane *APR* or to the plane *AQR*, to be provided with that same property, so that the figures shall be similar among themselves, of which the homologous sides may hold the ratio of the distances from *A*. Truly below all the sections of bodies of this kind will be shown generally, which are parallel between themselves, or which parallel to some plane drawn through the vertex *A*, are to be similar between themselves also and the parameters of these to be proportional to the distances from the vertex *A*.

37. A more general kind of surfaces appears, to which I shall now progress. Let *Z* be some function of *z* and some homogeneous equation shall be proposed between the three variables *x*, *y* and *Z*. Let $Z = H$ on putting $z = h$ and, since in this case an homogeneous equation may be produced between *x*, *y* and *H*, all the sections parallel to the plane *APQ* will be figures similar to each other ; but the parameters of which shall not be proportionals of the distances *h*, but the proportionals of these will be to the functions *H*. From which the lines drawn through homologous points of these sections will not be right lines, but curves depending on the nature

of the function *Z*. Then truly also it hence does not follow the sections, which shall be parallel to some other plane, to be similar between themselves.

38. Both the preceding are included in this kind. For if there were $Z = z$ or $Z = \alpha z$, on account of the homogeneous equation between *x*, *y* and *z* conic surfaces will arise . Likewise it comes about, if there were $Z = \alpha + \beta z$ this only indicates that the vertex of the cone may not fall on the point *A* itself ; clearly, if there were $Z = \frac{b-z}{b}$, the vertex of the cone will be distant from *A* by the interval *b*. But if now there may be put $b = \infty$, the conic figure will be changed into a cylinder and there becomes $Z = 1$. Hence the equation for the surface of a cylinder has been prepared thus, so that in that the variables *x* and *y* together with the constant 1 may contain a number of the same dimensions everywhere. But just as the equation between *x* and *y* were prepared, if the third variable *z* were not present in that, it is possible to fill the homogeneity by one ; from which, as above we have shown now, each equation will express a cylindrical surface with one variable missing.

39. Among these bodies, in which all the sections parallel to one principal plane *APQ* are similar figures, these are noteworthy mainly, these sections of which are circles having centres on the same right line *AR* normal to the plane *APQ*. Bodies of this kind are formed with a lathe and thence may be called *turned* bodies [*i.e.* surfaces of rotation]. Therefore the general equation for bodies of this kind will be $ZZ = xx + yy$: for whatever value may be attributed to the variable *z*, so that it becomes $Z = H$, the equation $HH = xx + yy$ will be produced for a plane section parallel to the plane *APQ*, which is the equation for a circle of radius $= H$ and by having the centre on the right line *AR*. If there were $ZZ = zz$, a right cone will be had ; if $ZZ = aa$, a cylinder and, if $ZZ = aa - zz$, a globe will be produced, which are all special kinds of bodies of rotation.

40. We may consider bodies of this kind (Fig.124), all the sections *PTV* of which shall be triangles normal to the axis *AP* and the apices of these *T* shall be placed on the right line *DT* parallel to the axis *AP*. Let *AVB* be the base of this body or its section made in the plane *APQ*, which shall be some curve. The distance of the right line *DT* from the axis *AB*, evidently *AD*, $= c$; and with the three variables $AP = x,\ PQ = y,\ QM = z$ put in place, so that at this stage, *PV* will be a certain function of *x;* that shall be $PV = P$, on account of the similar triangles *VQM*, *VPT* there shall be :

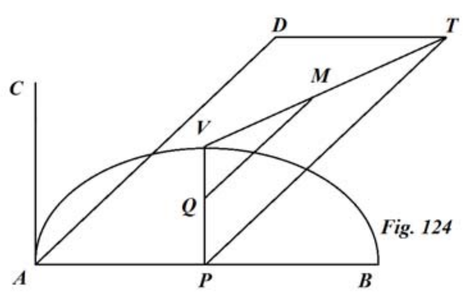

Fig. 124

$$P : c = P - y : z \text{ or } z = c - \frac{cy}{P}.$$

Therefore for bodies of this kind, $\frac{c-z}{y}$ will be equal to a certain function of *x*.

Therefore these bodies differ from conics, because they shall end in the sharp right line *DT*, with conics ending in sharp points. If the base *AVB* may be supposed to be a circle, the resulting body treated at length by WALLIS and called *a circular wedge*.
[*i.e.* one end is circular while the other end tapers to a sharp edge].

41. Just as (Fig. 125) all the sections shall be normal to the axis *AB* so that the triangles *PTV* shall be right angled at *P*, moreover the vertices *T* of which may constitute some curve *AT*, but the base shall be the figure *AVB*. With the three variables put in place $AP = x$, $PQ = y$ and $QM = z$ the right line *PV* will be a certain function of *x* in the curve *AVB*, which shall be $= P$ *:* then truly *PT* also will be a function of *x*, which shall be $= Q$; with which put in place there will be

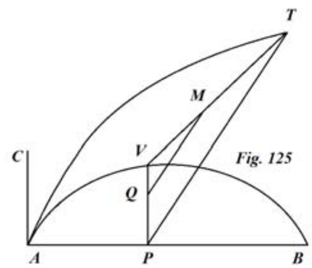

Fig. 125

$P : Q = P - y : z$ and thus $z = Q - \frac{Qy}{P}$ or $Pz + Qy = PQ$ or $\frac{z}{Q} + \frac{y}{P} = 1$, or equal to a constant. But therefore if nowhere in the equation can both the variables *y* and *z* be put in place with more than one dimension, then the body will belong to the kind that we have described here.

42. Because now we have considered these bodies, all the sections of which, parallel to one principal plane, are similar to each other, now we will consider these in which all the sections of this kind shall be figures only related to each other or , which with homologous abscissas taken may have the applied lines proportional to each other. Therefore let the three principal sections of a body of this kind (Fig.126) be *ABC, ACD* and *ABD*, all the parallel sections of which must be related to that figure *ACD*. Whereby in that figure the base may be put $AC = a$ and the altitude $AD = b$, and with the coordinates taken $Aq = p$ and $qm = q$, *q* shall be some function of *p*. Now some parallel section may be considered *PTV* with the interval in place $AP = x$ and the base *PV* will be equal to a function of *x*, which shall be $= P$, and with the altitude *PT* equal to a function of *x*, which shall be $= Q$. Now there may be called $PQ = y$ and $QM = z$ *;* and from the proportionalities the nature will be

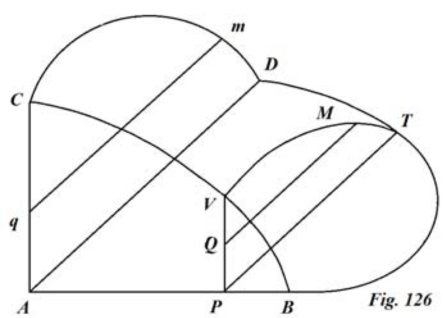

Fig. 126

$$a : p = P : y \text{ and } b : q = Q : z \text{ or } y = \frac{Pp}{a} \text{ and } z = \frac{Qq}{b}$$

43. But if therefore all three principal sections of the body were given, *ABC, ACD* and *ABD,* hence the nature of the body itself will be determined, so that all the sections parallel to *ACD* likewise shall have the same proportionality between each other. Indeed in the first place the functions *P* and *Q* may be given functions x ; then truly q is a function of p; from which from the two variables x and p both the variables y and z are defined. Truly, if we should wish for an equation between the three coordinates x, y and z, because q is a function of p or because an equation may be given between p and q, in this equation $p = \frac{ay}{P}$ and $q = \frac{bz}{Q}$ may be substituted and thus on account of the functions of x P and Q an equation will arise between the three coordinates x, y and z, from which the nature of the bodies belonging to this kind will be expressed. Moreover it is apparent on putting $x = 0$, that there is required to be $P = a$ and $Q = b$.

44. If in the equation for the surface the two variables y and z may constitute a number of the same dimensions everywhere, then all the sections normal to the axis *AP* will be rectilinear figures. For on putting some constant value for x, a homogeneous equation will be produced between y and z, which indicates one or more right lines. Therefore since the number of dimensions, which may be formed from the two y and z, shall be the same everywhere, it will be either even or odd ; and on account of this, as it has been shown above in § 20, bodies of this kind will have two parts equal to each other. Clearly the parts in the first and fifth regions will be similar to each other, then truly also in the second and third, and thus concerning the rest, as the small table given in that place indicates.

45. Now we have considered here more kinds of bodies, in which an infinity of rectilinear sections are given just as that finally treated, as well as both cylinders and conics. Truly these have been prepared thus, so that the sections made through the axis *AP* shall be rectilinear ; but this kind can be extended further. For let (Fig. 127) *AKMP* be the section of a body made through the axis *AP* for the angle $MPV = \varphi$; on putting $AP = x$, $PQ = y$ and $QM = z$, $\frac{z}{y}$ will be the tangent of the angle

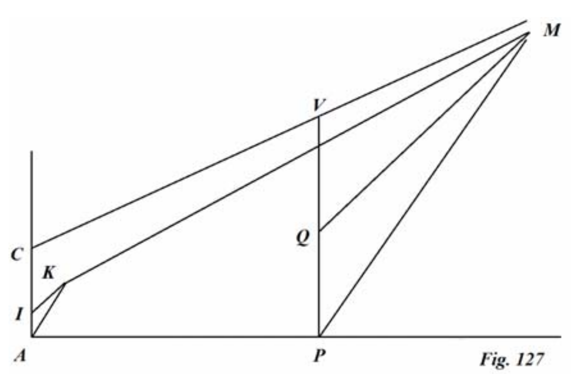

Fig. 127

φ and with the right line $PM = \frac{z}{sin.\varphi}$. But if now the line *KM* shall be right, $\frac{z}{sin.\varphi}$ will become $= \alpha x + \beta$; where α and β will be constants depending on the angle φ and thus they will be functions of zero dimensions of y and z themselves. *R* and *S* shall be functions of this kind and there will be $x = Rz + S$ or $x = Ry + S$. Or with *T* denoting a function of one dimension and *S* of no dimensions of y and z, all the bodies of this kind will be included in this general equation $x = T + S$.

46. But whatever the proposed surface should have been, whose nature may be defined by an equation between the three variables *x*, *y* and *z*, it will be easy to determine any section of that made along the axis *AP*. For let the angle *VPM*, by which that section *AKMP* may be inclined to the plane *ACVP*, $= \varphi$; and putting the right line $PM = v$, which will be the applied line of the section sought ; with which made there will be had $QM = z = v \cdot \sin.\varphi$ and $PQ = y = v \cdot \cos.\varphi$. But if therefore in the equation for the surface in place of the variables *y* and *z* these values $v \cdot \cos.\varphi$ and $v \cdot \sin.\varphi$ may be substituted, an equation will arise between the two variables *x* and *v*, by which the nature of the section *AKMP* will be expressed. Truly in a similar manner all the sections also, which may be found will be made along either of the two remaining principal axes (Fig. 121) *AQ* or AR. Indeed these three axes *AP*, *AQ* and *AR*, on which the three variables *x*, *y* and *z* depend, thus are permuted among themselves, so that always, whatever may be demonstrated by one of these may be transferred to the remaining two.

47. Therefore with the plane *APQ* taken for the standard, to which all the sections of the surface may be referred, whatever section made in the plane will be either parallel to this plane or it will be inclined to that ; and in this case the plane of the section continued will intersect the plane *APQ* somewhere and the intersection of these planes will be a right line. Indeed in the first case, in which the plane of the section is parallel to the plane *APQ*, the nature of the intersection will be known by attributing a constant value to the quantity *z*. Truly in the latter case, in which the plane of the intersection may be inclined to the plane *APQ*, the nature of the intersection at this stage is only allowed to be defined, if either the right line *AP* or the right line *AQ* were the intersection of the plane cut with the plane *APQ*. Therefore it remains for all these sections requiring to be elicited entirely, that any other intersections of these two planes may be considered.

48. Let *ES* (Fig. 128) be a right line parallel to the axis *AP*, the intersection of the plane cut by the plane *APQ* and the angle of inclination *QSM*, by which the cutting plane *ESM* may be inclined to the plane *APQ* , is put $=\varphi$ and the distance *AE* may be called =*f*. Since now there shall be $AP = x,\ PQ = y$ and $QM = z$, there will be $ES = x$ and $QS = y + f$. But if the section therefore may be referred to the right line *ES* as the axis, the abscissa will be $ES = x$, truly the applied line *SM* may be put $= v$; from which on account of the angle,

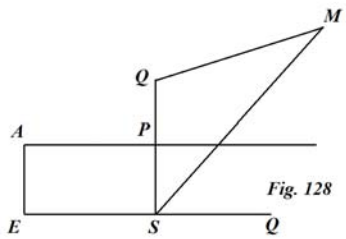

Fig. 128

$QSM = \varphi,\ QM = z = v \cdot \sin.\varphi$ will be obtained and $SQ = y + f = v \cdot \cos.\varphi$ and hence $y = v \cdot \cos.\varphi - f$. Whereby, if $y = v \cdot \cos.\varphi - f$ and $z = v \cdot \sin.\varphi$ may be substituted into the equation for the surface between *x*, *y* and *z* , an equation will arise between the coordinates *x* and *v* of the section sought *ESM*. If the intersection *ES* were normal to the axis *AP*, then, because it may be parallel to the other principal axis present in the plane *APQ*, with the variables *x* and *y* interchanged, the section may be found in the same way.

49. Now the intersection *ES* may be had (Fig. 129) placed in the plane *APQ* , to which the right line *AE* may cross the normal to the axis *AP* at the point *E*. Then *ETX* may be drawn parallel to the axis *AP*, putting $AE = f$ and the angle $TES = \theta$. Again with the three variables taken $AP = x,\ PQ = y$ and $QM = z$, with the normal *QS* drawn from *Q* to *ES* and the right line *MS* may be joined, the inclination of the cutting plane to the plane *APQ* will be the angle QSM , which may be put $=\varphi$. Then truly the coordinates of the section sought shall be $ES = t$ and $SM = v$. The perpendiculars *ST* and *SV* may be drawn from *S* to *EX* and *QP* produced ; and there will be

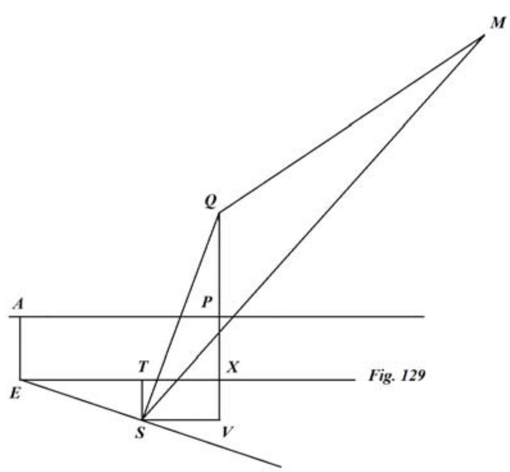

Fig. 129

$$QM = z = v \cdot \sin.\varphi,\ QS = v \cdot \cos.\varphi,\ SV = v \cdot \cos.\varphi.\sin.\theta$$

$$\text{and } QV = v \cdot \cos.\varphi \cdot \cos.\theta.$$

Indeed afterwards there will be

$$ST = VX = t \cdot \sin.\theta \text{ and } ET = t \cdot \cos.\theta .$$

From these finally it is gathered

$$AP = x = t \cdot \cos.\theta + v \cdot \cos.\phi \cdot \sin.\theta,$$

and

$$PQ = y = v \cdot \cos.\phi \cdot \cos.\theta - t \cdot \sin.\theta - f \ ;$$

which values, if they may be substituted in place of *x*, y and *z*, will give the equation sought for the section.

50. Therefore with the equation given for any solid [i.e. its surface in three dimensions], from that the equation will be able to be elicited easily for the section of that by some given plane. And indeed in the first place it is clear, if the equation for the solid between the three coordinates *x, y* and *z* were algebraic, then also all the sections of that become algebraic curves. Then truly, since the equation between the coordinates of the section *t* and *v* may arise by depending on the equation for the solid

$$y = v \cdot \sin.\varphi, \quad x = t \cdot \cos.\theta + v \cdot \cos.\varphi \cdot \sin.\theta$$

and $y = v \cdot \cos.\varphi \cdot \cos.\theta - t \cdot \sin.\theta - f$,

it is evident in the equation for some section the coordinates *t* and *v* cannot obtained more dimensions, than the three coordinates in the equation for the solid *x*, *y* and *z* may constitute. Yet it may happen whenever possible, that the equation for the section may be referred to a lower order, with the terms of higher degree themselves removed after the substitution.

51. Therefore if in the equation for the surface the three variables *x*, *y* and *z* may constitute a single dimension only, thus so that the equation shall be of this kind

$$\alpha x + \beta y + \gamma z = a \ ,$$

then all the sections of this surface shall be right lines. But in this case there will be a plane surface, as it will appear to be attended to easily as will be shown more clearly below ; and from the Elements it is known that the section of two planes is required to be a straight line. Hence it is understood in a similar manner of all solids, the nature of which may be contained in this general equation :

$$\alpha xx + \beta yy + \gamma zz + \delta xy + \varepsilon xz + \zeta yz + ax + by + cz + ee = 0 \, ,$$

the individual sections, unless they shall be right lines, must be lines of the second order, and it is not possible for any section to exist, the nature of which cannot be expressed by an equation of the second order.

CAPUT II

DE SECTIONIBUS SUPERFICIORUM A PLANIS QUIBUSCUNQUE FACTIS

26. Quemadmodum intersectiones linearum sunt puncta, ita superficierum intersectiones sunt lineae vel rectae vel curvae. Intersectio duorum planorum est linea recta, uti ex elementis constat. Globi autem plano secti figura est circulus. Plurimum autem ad cognitionem superficiei affertur subsidii, si lineas, quibus superficies a datis planis intersecatur, noverimus. Hoc enim modo simul infinita superficiei puncta innotescunt, cum modo praecedente singuli variabilis unius z valores singula tantum superficiei puncta praebeant.

27. Cum igitur superficies ad tria plana inter se normalia referamus, ante omnia investigari conveniet intersectiones superficiei et horum planorum. Sumto ergo (Fig. 121) primo plano *APQ*, quod variabilibus $AP = x,\ AQ = y$ determinatur (quoniam tertia variabilis z designat distantiam cuiusque superficiei puncti ab hoc plano), perspicuum est, si ponatur $z = 0$, ea superficiei puncta inventum iri, quae in ipso plano *APQ* sint sita, atque idcirco aequatio residua inter x et y exhibebit lineam, qua superficies a plano *APQ* intersecatur. Simili modo, si ponatur $y = 0$, aequatio inter x et z exprimet intersectionem superficiei a plano *APR* factam atque posito $x = 0$ aequatio inter y et z dabit intersectionem superficiei et plani *AQR*.

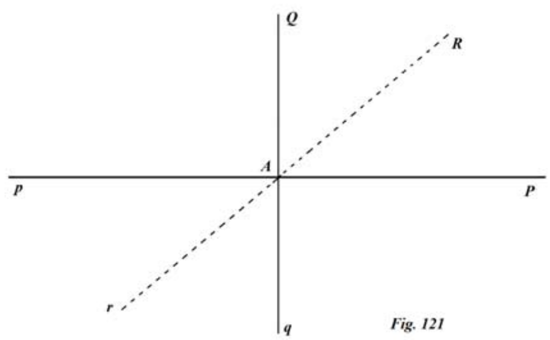

Fig. 121

28. Supra iam innuimus superficiem globi centrum in puncto *A* habentis, cuius radius $= a$, exprimi hac aequatione $xx + yy + zz = aa$; hoc ergo exemplo ad illustrationem harum intersectionum utar. Sit igitur $z = 0$, atque aequatio $xx + yy = aa$ exhibebit intersectionem globi a plano *APQ* factam, quam ergo patet esse circulum centrum *A* et radium $= a$ habentem. Simili modo facto $y = 0$ intersectio globi a plano *APR* facta erit circulus aequatione $xx + zz = aa$ contentus. Eodemque modo, si ponatur $x = 0$, aequatio $yy + zz = aa$ parem circulum pro intersectione plani *AQR* indicat. Haec quidem sunt satis nota, cum sectiones globi planis per eius centrum transeuntibus factae omnes sint circuli maximi seu cum globo radium communem habentes.

29. Haud difficilius erit sectiones superficiei per plana alia uni istorum planorum principalium parallela factas determinare. Concipiatur planum plano *APQ* parallelum ab eoque distans intervallo *h*, omnia ergo superficiei puncta, quorum ab eodem plano *APQ* distantia, quae per variabilem *z* indicatur, est $=h$, simul in isto plano parallelo sita erunt ideoque intersectionem formabunt. Pro hac ergo intersectione aequatio habebitur, si in aequatione pro superficie ponatur $z=h$; tum enim habebitur aequatio inter binas coordinatas orthogonales *x* et *y* naturam sectionis exprimens. Eodem autem modo sectiones, quae per plana vel ipsi *APR* vel *AQR* parallela fiunt, definientur, unde superfluum foret, quae de uno dicta sunt, in reliquis repetere.

30. Si ergo in aequatione pro superficie inter tres coordinatas *x*, *y* et *z* una earum *z* ponitur constans $=h$, tum sectio superficiei per planum plano *APQ* parallelum ab eoque intervallo *h* distans formata oritur. Quodsi ergo successive huic litterae *h* omnes valores . possibiles tam affirmativi quam negativi tribuantur, tum omnes sectiones superficiei, quae a planis plano *APQ* parallelis formantur, obtinentur atque, cum tota superficies huiusmodi planis parallelis in partes infinitas dissecari possit hocque modo omnes sectiones cognoscantur, ex istis omnibus sectionibus tota superficies innotescet. Omnes scilicet istae sectiones unica aequatione inter coordinatas *x* et *y* constantem indeterminatam *h* involvente exprimentur; ex quo omnes istae sectiones erunt lineae vel similes vel saltem affines una aequatione contentae.

31. Omnes ergo sectiones superficiei plano *APQ* parallelae erunt inter se aequales atque a planis *APR*, *AQR* aequali modo traiicientur, si aequatio inter *x* et *y* ita fuerit comparata, ut eadem maneat, quicunque valor ipsi *h* tribuatur. Hoc autem evenire nequit, nisi variabilis *z*, cuius loco *h* est posita, prorsus desit in aequatione pro superficie. Quo circa, si variabilis tertia *z* in aequationem superficiei omnino non ingrediatur, tum omnes sectiones plano *APQ* parallelae erunt inter se aequales, quarum natura exprim etur ipsa superficiei aequatione, quippe quae duas tantum variabiles *x* et *y* involvit. Simili vero modo, si in aequatione pro superficie vel variabilis *x* vel *y* desit, tum omnes sectiones vel plano *AQR* vel plano *APR* parallelae inter se congruent.

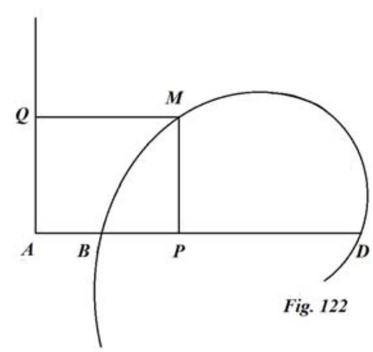

Fig. 122

32. Huiusmodi ergo superficies non solum animo facile concipitur, sed etiam construitur atque in data materia efformatur. Ponamus enim in aequatione deesse variabilem *z*, ita ut aequatio tantum sit (Fig. 122) inter coordinatas $AP=x$ et $AQ=PM=y$; ex hac in plano *APQ* describatur linea curva *BMD.* Quo facto concipiatur linea

recta infinita ad planum hoc perpetuo normalis secundum lineam hanc curvam *BMD* circumferri, atque haec recta motu suo producet seu efformabit superficiem per eam aequationem indicatam. Unde perspicuum est, si linea *BMD* fuerit circulus, tum superficiem ex eo ortam fore cylindri recti; sin autem linea *BMD* fuerit ellipsis, tum superficiem cylindri scaleni generari. Quodsi linea *BMD* non fuerit continua, sed ex pluribus rectis conflata figuram exhibens rectilineam, tum superficies resultabit prismatica.

33. Quod hoc superficierum genus cylindros et omnia prismata in se complectitur, universum hoc genus superficierum appellari conveniet *cylindricum* seu *prismaticum;* singulae autem species sub hoc genere contentae determinabuntur per figuram planam *BMD*, ex qua modo ante descripto sint ortae; atque ista figura *BMD basis* appellabitur. Quoties ergo in aequatione pro superficie una trium variabilium x, y, z deest, tum superficies hac aequatione contenta erit cylindrica seu prismatica. Quodsi autem duae variabiles y et x simul desint, tum ob $x =$ constanti linea *BMD* abibit in rectam ad axem *AD* normalem atque propterea superficies fiet plana normalis ad planum *APQ.*

34. Post hoc superficierum genus maxime notari meretur id, quod oritur ex aequatione inter tres variabiles x, y et z homogenea, seu in qua tres istae variabiles ubique eundem dimensionum numerum constituunt, cuiusmodi est $zz = mxz + xx + yy$. Hinc enim omnes sectiones, quae fiunt per plana uni ex tribus principalibus parallela, erunt figurae inter se similes. Namque si tribuatur ipsi z valor constans h, manifestum est aequationem $hh = mhx + xx + yy$, si pro h successive alii aliique valores tribuantur, infinitas continere figuras inter se similes, quarum parametri sint aequales seu proportionales ipsi h. Cum igitur hae sectiones non solum sint similes, sed etiam crescant in ratione distantiarum a plano *APQ*, lineae, quae ex puncto *A* per singularum sectionum puncta homologa ducuntur, erunt rectae.

35. Proposita ergo huiusmodi aequatione inter tres variabiles x, y et z homogenea tribuatur (Fig. 123) ipsi z valor datus $AR = h$ sitque *TSsMm* figura in plano ipsi *APQ* parallelo et per punctum *R* ducto, quam exhibebit aequatio inter x et y, ita ut sit $RV = x$ et $VM = y$. Quodsi ergo haec sectio una *TSsMm* fuerit descripta, concipiatur circa, eius perimetrum circumduci linea recta infinita perpetuo per punctum *A* transiens; atque haec recta motu suo describet superficiem in aequatione proposita contentam. Perspicuum vero est, si figura *TSsMm* fuerit circulus centrum in *R* habens, tum prodire conum rectum; sin *R* non sit centrum, conum scalenum; at si illa figura fuerit rectilinea,

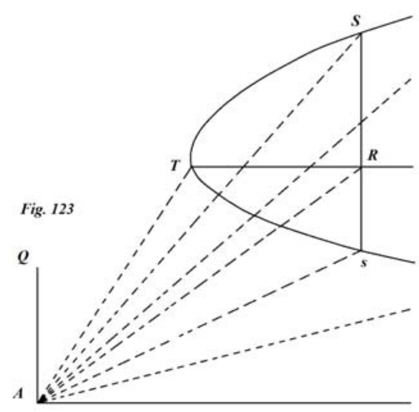

Fig. 123

orientur cuiusque generis pyramides. Quam ob rem superficies, quae in hoc aequationum genere continentur, hic *conicas* seu *pyramidales* vocabimus.

36. Ex his manifestum est, si aequatio inter tres variabiles x, y et z fuerit homogenea atque adeo superficies conica seu pyramidalis, tum non solum omnes sectiones uni plano principali APQ parallelas inter se esse figures similes, quarum parametri sint distantiis sectionum a vertice A proportionales, sed ob eandem rationem intelligitur quoque omnes sectiones, quae sint vel plano APR vel plano AQR parallelae, eadem illa proprietate esse praeditas, ut sint figurae inter se similes, quarum latera homologa teneant distantiarum ab A rationem. Infra vero ostendetur omnes omnino sectiones huiusmodi corporum, quae sunt inter se paralleiae seu quae sunt parallelae plano cuicunque per verticem A ducto, inter se quoque fore similes earumque parametros distantiis a vertice A esse proportionales.

37. Latius patet genus superficierum, ad quod nunc sum progressurus. Sit Z functio quaecunque ipsius z ac proponatur aequatio quaecunque homogenea inter tres variabiles x, y et Z. Fiat $Z = H$ posita $z = h$ et, cum hoc casu prodeat aequatio homogenea inter x, y et H, erunt omnes sectiones plano APQ parallelae figurae inter se similes; quarum parametri autem non distantiis h, sed earum functionibus H erunt proportionales. Ex quo lineae per harum sectionum puncta homologa ductae non erunt lineae rectae, sed curvae a functionis Z ratione pendentes. Tum vero etiam hinc non sequitur sectiones, quae alio cuipiam plano sint parallelae, fore inter se similes.

38. In hoc genere ambo praecedentia continentur. Si enim fuerit $Z = z$ seu $Z = \alpha z$, ob aequationem inter x, y et z homogeneam orientur superficies conicae . Idem evenit, si fuerit $Z = \alpha + \beta z$ hoc tantum discrimine, quod vertex coni non in ipsum punctum A cadat; scilicet, si fuerit $Z = \frac{b-z}{b}$, vertex coni ab A distabit intervallo b. Quodsi iam statuatur $b = \infty$, figura conica abibit in cylindricam fietque $Z = 1$. Hinc aequatio pro superficiebus cylindricis ita erit comparata, ut in ea variabiles x et y una cum constanti 1 ubique eundem dimensionum numerum adimpleant. Quomodocunque autem aequatio inter x et y fuerit comparata, si tertia variabilis z in eam non ingrediatur, semper per unitatem homogeneitas impleri potest; unde, uti supra iam ostendimus, omnis aequatio una variabili carens exprimit superficiem cylindricam.

39. Inter haec corpora, in quibus omnes sectiones uni plano principali APQ parallelae sunt figurae similes, maxime notatu sunt digna ea, quorum istae sectiones sunt circuli centra in eadem recta AR ad planum APQ normali habentes. Huiusmodi corpora torno efformantur indeque *tornata* appellantur. Pro huiusmodi ergo corporibus aequatio generalis erit $ZZ = xx + yy$: quicunque enim valor variabili z tribuatur, ut fiat $Z = H$, prodibit pro sectione

plano *APQ* parallela aequatio $HH = xx + yy$, quae est pro circulo radium $= H$ et centrum in recta *AR* habente. Si fuerit $ZZ = zz$, habebitur conus rectus; sin $ZZ = aa$, cylindrus et, si $ZZ = aa - zz$, prodibit globus, quae sunt species praecipuae corporum tomatorum.

40. Contemplemur eiusmodi corpora (Fig.124), quorum omnes sectiones *PTV* normales ad axem *AP* sint triangula horumque apices T in linea recta *DT* axi *AP* parallela sitae. Sit *AVB* basis huius corporis seu eius sectio in plano *APQ* facta, quae sit curva quaecunque. Sit distantia rectae *DT* ab axe *AB*, nempe $AD, = c$; positisque, ut hactenus, tribus variabilibus $AP = x,\ PQ = y,\ QM = z$

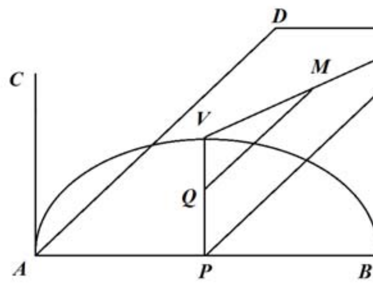

Fig. 124

erit *PV* functio quaepiam ipsius *x;* sit ea $PV = P$, erit ob triangula *VQM*, *VPT* similia

$$P : c = P - y : z \text{ seu } z = c - \frac{cy}{P}.$$

Pro huiusmodi ergo corporibus aequabitur $\dfrac{c - z}{y}$

functioni cuipiam ipsius *x.* Differunt igitur haec corpora a conicis, quod desinant in aciem rectam *DT*, cum conica desinant in cuspidem. Si basis *AVB* ponatur circulus, corpus resultans a WALLISIO fusius est pertractatum atque *cono-cuneus* appellatum.

41. Sint (Fig. 125) ut modo omnes sectiones axi *AB* normales *PTV* triangula ad *P* rectangula, quorum vertices autem *T* constituant curvam quamcunque *AT*, basis autem sit figura *AVB.* Positis tribus variabilibus $AP = x,\ PQ = y$ et $QM = z$ erit in curva *AVB* recta *PV* functio quaedam ipsius *x*, quae sit $= P$ *:* tum vero erit *PT* quoque functio ipsius *x*, quae sit $= Q$; quibus positis erit

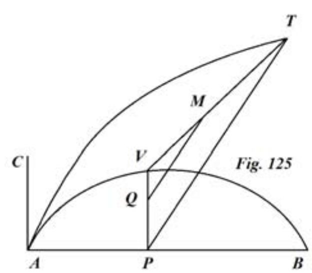

Fig. 125

$$P : Q = P - y : z \text{ ideoque } z = Q - \frac{Qy}{P} \text{ seu}$$

$$Pz + Qy = PQ \quad \text{vel} \quad \frac{z}{Q} + \frac{y}{P} = 1 \text{ vel constanti.}$$

Quodsi ergo in aequatione ambae variabiles *y* et *z* una plures dimensiones nusquam constituant, tum corpus ad hoc genus pertinebit, quod hic descripsimus.

42. Quoniam iam sumus contemplati ea corpora, quorum omnes sectiones, uni plano principali parallelae, sunt inter se similes, nunc ea consideremus, in quibus omnes istiusmodi sectiones sint figurae inter se saltem affines seu, quae sumtis abscissis homologis habeant applicatas inter se proportionales. Sint igitur (Fig.126) huiusmodi corporis tres sectiones principales *ABC, ACD* et *ABD,* quarum isti *ACD* omnes sectiones parallelae debeant esse figurae affines. Quare in ea ponatur basis $AC = a$ et altitude $AD = b$, sumtisque coordinatis $Aq = p$ et $qm = q$ sit *q* functio quaecunque ipsius *p.* Concipiatur nunc sectio quaecunque parallela *PTV* posito intervallo $AP = x$ eritque basis $PV =$ functioni ipsius *x*, quae sit $= P$, et altitudo $PT =$ functioni ipsius *x*, quae sit $= Q$. Vocetur iam $PQ = y$ et $QM = z$; atque ex affinitatis natura erit

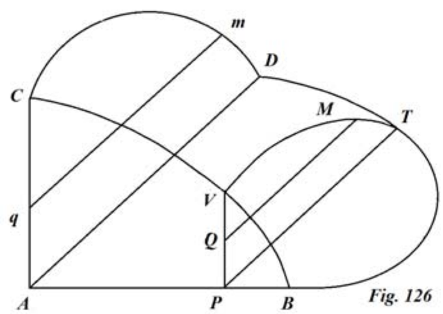

Fig. 126

$$a : p = P : y \text{ et } b : q = Q : z \text{ seu } y = \frac{Pp}{a} \text{ et } z = \frac{Qq}{b}$$

43. Quodsi ergo datae fuerint omnes tres sectiones principales corporis, *ABC, ACD* et *ABD* hinc natura ipsius corporis determinabitur, quod habeat omnes sectiones ipsi *ACD* parallelas simul eidem affines. Primum enim dantur *P* et *Q* functiones ipsius *x* ; tum vero est *q* functio ipsius *p;* unde ex binis variabilibus *x* et *p* definiuntur ambae variabiles y et *z.* Verum, si aequationem inter tres coordinatas *x, y* et *z* desideremus, quoniam *q* est functio ipsius *p* seu quia datur aequatio inter *p* et *q,* in hac aequatione substituatur $p = \frac{ay}{P}$ et $q = \frac{bz}{Q}$ sicque ob *P* et *Q* functiones ipsius *x* orietur aequatio inter tres coordinatas *x, y* et *z,* qua natura corporum ad hoc genus pertinentium exprimetur. Patet autem posito $x = 0$ fieri oportere $P = a$ et $Q = b$.

44. Si in aequatione pro superficie duae variabiles *y* et *z* ubique eundem dimensionum numerum constituant, tum omnes sectiones ad axem *AP* normales erunt figurae rectilineae. Posito enim pro *x* valore quocunque constante prodibit aequatio inter *y* et *z* homogenea, quae unam pluresve lineas rectas indicat. Cum igitur numerus dimensionum, qui a binis *y* et *z* constituitur, ubique sit idem, vel par erit vel impar; et hanc ob rem, uti supra § 20 ostensum est, huiusmodi corpora binas habebunt partes inter se aequales. Scilicet portiones in regionibus prima et quinta inter se erunt similes, tum vero etiam in regione secunda et tertia, et ita de ceteris, uti tabella loco citato indicat.

45. Iam plures hic contemplati sumus corporum species, in quibus dantur infinitae sectiones rectilineae veluti hanc ultimo pertractatam et cylindricas atque conicas. Hae vero ita sunt comparatae, ut sectiones per axem *AP* factae sint rectilineae; hoc autem genus latius patet. Sit enim (Fig. 127) *AKMP* sectio corporis per axem *AP* facta ad angulum $MPV = \varphi$; positis $AP = x,\ PQ = y$ et $QM = z$ erit $\frac{z}{y}$ tangens anguli φ et recta $PM = \frac{z}{sin.\varphi}$. Quodsi iam linea *KM* sit recta, debebit esse $\frac{z}{sin.\varphi} = \alpha x + \beta$; ubi α et β erunt constantes ab angulo φ pendentes ideoque erunt functiones nullius dimensionis ipsarum *y* et *z*. Sint *R* et *S* huiusmodi functiones eritque $x = Rz + S$ seu $x = Ry + S$. Vel denotante *T* functionem unius dimensionis et *S* nullius dimensionis ipsarum *y* et *z*, omnia huiusmodi corpora continebuntur in hac aequatione generali $x = T + S$.

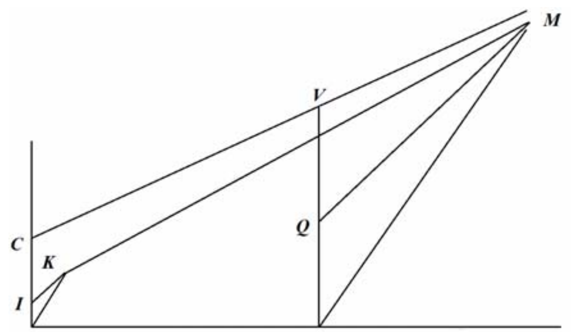

Fig. 127

46. Quaecunque autem fuerit proposita superficies, cuius natura per aequationem inter tres variabiles *x, y* et *z* definiatur, facile erit eius sectionem quamvis secundum axem *AP* factam determinare. Sit enim angulus *VPM*, quo ista sectio *AKMP* ad planum *ACVP* inclinatur, $= \varphi$; et ponatur recta $PM = v$, quae erit applicata sectionis quaesitae; quo facto habebitur $QM = z = v \cdot \sin.\varphi$ et $PQ = y = v \cdot \cos.\varphi$. Quodsi ergo in aequatione pro superficie loco variabilium *y* et *z* isti valores $v \cdot \cos.\varphi$ et $v \cdot \sin.\varphi$ substituantur, orietur aequatio inter duas variabiles *x* et *v*, qua natura sectionis *AKMP* exprimetur. Simili vero modo omnes quoque sectiones, quae fiunt secundum alterutrum binorum reliquorum axium principalium (Fig. 121) *AQ* vel AR, invenientur. Tres enim isti axes *AP, AQ* et *AR*, a quibus tres variabiles *x, y* et *z* pendent, ita inter se sunt permutabiles, ut perpetuo, quicquid de eorum uno docetur, ad binos reliquos transferatur.

47. Sumto ergo plano *APQ* pro norma, ad quod omnes sectiones superficiei referantur, sectio quaecunque plano facta vel erit parallela huic plano vel ad id erit inclinata; hocque casu planum sectionis continuatum alicubi intersecabit planum *APQ* atque intersectio istorum planorum erit linea recta. Priori quidem casu, quo planum sectionis parallelum est plano *APQ*, natura sectionis innotescet tribuendo quantitati *z* valorem constantem. Posteriori vero casu, quo planum sectionis ad planum *APQ* inclinatur, naturam sectionis adhuc tantum definire licet, si vel recta *AP* vel recta *AQ* fuerit intersectio plani secantis cum plano *APQ*. Ad omnes ergo

omnino sectiones eruendas superest, ut quascunque alias binorum illorum planorum intersectiones contemplemur.

48. Sit recta *ES* (Fig. 128) axi *AP* parallela intersectio plani secantis cum plano *APQ* angulusque inclinationis *QSM*, quo planum secans *ESM* ad planum *APQ* inclinatur, ponatur $=\varphi$ et distantia *AE* vocetur $=f$. Cum iam sit $AP=x,\ PQ=y$ et $QM=z$, erit $ES=x$ et $QS=y+f$. Quodsi ergo sectio ad rectam *ES* tanquam axem referatur, erit abscissa $ES=x$, applicata vero *SM* ponatur *v;* unde ob angulum $QSM=\varphi$ obtinebitur

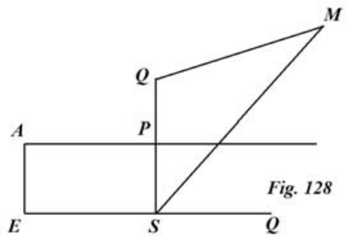

Fig. 128

$QM=z=v\cdot\sin.\varphi$ et $SQ=y+f=v\cdot\cos.\varphi$ hincque $y=v\cdot\cos.\varphi-f$.
Quare, si in aequatione pro superficie inter *x, y* et *z* substituatur

$$y=v\cdot\cos.\varphi-f \text{ et } z=v\cdot\sin.\varphi,$$

orietur aequatio inter coordinatas *x* et *v* sectionis *ESM* quaesitae. Si intersectio *ES* esset normalis ad axem *AP*, tum, quia foret parallela alteri axi principali in plano *APQ* existenti, permutandis variabilibus *x* et *y* sectio eodem modo invenietur.

49. Habeat iam (Fig. 129) inter sectio *ES* in plano *APQ* positionem quamcunque, cui recta *AE* ad axem *AP* normalis occurrat in puncto *E*. Tum ducatur. *ETX* axi *AP* parallela et ponatur $AE=f$ et angulus $TES=\theta$.
Sumtis porro tribus variabilibus $AP=x,\ PQ=y$ et $QM=z$ ex *Q* ad *ES* ducatur normalis *QS* et iungatur recta *MS*, erit angulus *QSM* inclinatio plani secantis ad planum *APQ*, qui ponatur $=\varphi$. Deinde vero sint sectionis quaesitae coordinatae $ES=t$ et $SM=v$. Ex *S* ad *EX* et *QP* productam ducantur perpendicula *ST* et *SV;* eritque

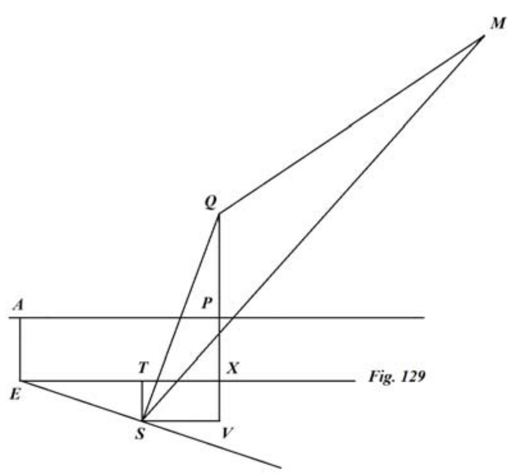

Fig. 129

$$QM = z = v \cdot \sin.\varphi,\ QS = v \cdot \cos.\varphi,\ SV = v \cdot \cos.\varphi.\sin.\theta$$
$$\text{et } QV = v \cdot \cos.\varphi \cdot \cos.\theta.$$

Postea vero erit

$$ST = VX = t \cdot \sin.\theta \text{ et } ET = t \cdot \cos.\theta .$$

Ex his colligitur tandem

$$AP = x = t \cdot \cos.\theta + v \cdot \cos.\phi \cdot \sin.\theta,$$
$$\text{et } PQ = y = v \cdot \cos.\phi \cdot \cos.\theta - t \cdot \sin.\theta - f \ ;$$

qui valores, si loco *x*, y et *z* substituantur, dabunt aequationem pro sectione quaesita.

50. Data ergo aequatione pro solido quocunque, ex ea facile elici potest aequatio pro sectione eius quacunque plana. Ac primo quidem perspicuum est, si aequatio pro solido inter tres coordinatas *x*, *y* et *z* fuerit algebraica, tum quoque omnes eius sectiones fore curvas algebraicas. Deinde vero, cum aequatio inter coordinatas sectionis *t* et *v* oriatur ponendo in aequatione pro solido

$$y = v \cdot \sin.\varphi, \quad x = t \cdot \cos.\theta + v \cdot \cos.\varphi \cdot \sin.\theta$$
$$\text{et } y = v \cdot \cos.\varphi \cdot \cos.\theta - t \cdot \sin.\theta - f ,$$

manifestum est in aequatione pro quavis sectione coordinatas *t* et *v* plures dimensiones obtinere non posse, quam in aequatione pro solido tres coordinatae *x*, *y* et *z* constituant. Fieri tamen quandoque potest, ut aequatio pro sectione ad ordinem inferiorem referatur supremis scilicet membris post substitutionem sese tollentibus.

51. Si igitur in aequatione pro superficie tres variabiles *x, y* et *z* unicam tantum constituant dimensionem, ita ut aequatio sit huiusmodi

$$\alpha x + \beta y + \gamma z = a ,$$

tum omnes huius superficiei sectiones erunt lineae rectae. Erit autem hoc casu superficies plana, uti cum attendenti facile patebit tum infra clarius ostendetur; atqui ex elementis notum est sectionem duorum planorum lineam rectam esse oportere. Simili modo hinc intelligitur omnium solidorum, quorum natura hac generali aequatione contineatur

$$\alpha xx + \beta yy + \gamma zz + \delta xy + \varepsilon xz + \zeta yz + ax + by + cz + ee = 0 ,$$

singulas sectiones, nisi sint lineae rectae, lineas secundi ordinis esse debere neque ullam dari sectionem, cuius natura per aequationem secundi gradus exprimi nequeat.