CHAPTER XV

CONCERNING SERIES ARISING FROM THE EXPANSION OF FACTORS

264. Let a product be proposed, depending on either a finite or infinite number of factors of this kind,

$$(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)(1+\varepsilon z)(1+\zeta z) \text{ etc.},$$

so that, if it may be expanded by an actual multiplication, it may give

$$1+Az+Bz^2+Cz^3+Dz^4+Ez^5+Fz^6+\text{etc.},$$

and it is evident the coefficients *A, B, C, D, E* etc. thus are to be formed from the numbers

$\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ etc., so that there shall be

$A=\alpha+\beta+\gamma+\delta+\varepsilon+\zeta+\text{etc.}=$ sum of the individual terms,
B = sum of the factors from two different terms,
C = sum of the factors from three different terms,
D = sum of the factors from four different terms,
E = sum of the factors from five different terms
etc.,

finally the product from all the terms arises.

265. But if therefore there is put $z=1$, this product

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon) \text{ etc.}$$

will be equal to one, with the series of all the terms arising from these $\alpha, \beta, \gamma, \delta, \varepsilon$ etc., either with the individual terms taken, or with two or more different ones multiplied amongst themselves. And if likewise a number may be able to come about in two or more ways, likewise also it may arise in two or more ways in this series of numbers.

266. If there may be put $z=-1$, this product

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon) \text{ etc.}$$

will be equal to unity with a series of all the numbers, which arise from these $\alpha,\beta,\gamma,\delta,\varepsilon$ etc. either with the individual terms taken, or with two or more different terms taken multiplying each other ; so that indeed as before, truly with this distinction, so that these numbers, which either alone or in three's or fives or arising from any odd number shall be negative, and truly these, which result either from two, four, or from any even number, shall be positive.

267. For the numbers $\alpha,\beta,\gamma,\delta,\varepsilon$ etc. all the prime numbers may be written

$$2,\quad 3,\quad 5,\quad 7,\quad 11,\quad 13\quad \text{etc}.$$

and this product

$$(1+2)(1+3)(1+5)(1+7)(1+11)(1+13) \text{ etc.} = P$$

will be equal to unity with the series of all the prime numbers, either of the primes themselves , of from different primes arising through multiplication. Therefore there will be

$$P = 1+2+3+5+6+7+10+11+13+14+15+17+ \text{ etc.},$$

in which series all the natural numbers occur with the powers of these excepted, which are divisible by some power. Evidently the numbers 4, 8, 9,12, 16, 18 etc. are absent, because they are either powers such as 4, 8, 9, 16 etc., or divisible by powers such as 12, 18 etc.

268. The matter will be had in a similar manner, if some powers of the prime numbers $\alpha,\beta,\gamma,\delta,\varepsilon$ etc. may be substituted, evidently if we may put

$$P=\left(1+\tfrac{1}{2^n}\right)\left(1+\tfrac{1}{3^n}\right)\left(1+\tfrac{1}{5^n}\right)\left(1+\tfrac{1}{7^n}\right)\left(1+\tfrac{1}{11^n}\right) \text{ etc.}$$

For by putting a multiplication in place

$$P = 1+\tfrac{1}{2^n}+\tfrac{1}{3^n}+\tfrac{1}{5^n}+\tfrac{1}{6^n}+\tfrac{1}{7^n}+\tfrac{1}{10^n}+\tfrac{1}{11^n}+ \text{ etc.},$$

in which all the numbers from the fractions occur besides those which either are powers themselves, or divisible by some power. Indeed since all the whole numbers shall be either prime or composed from primes by multiplication, here only these numbers will be excluded, in the formation of which the same prime number enters two or more times.

269. If the numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. may be taken negative, as we have made before (§ 266), and there may be put

$$P = \left(1 - \tfrac{1}{2^n}\right)\left(1 - \tfrac{1}{3^n}\right)\left(1 - \tfrac{1}{5^n}\right)\left(1 - \tfrac{1}{7^n}\right)\left(1 - \tfrac{1}{11^n}\right) \text{ etc.,}$$

there becomes

$$P = 1 - \tfrac{1}{2^n} - \tfrac{1}{3^n} - \tfrac{1}{5^n} + \tfrac{1}{6^n} - \tfrac{1}{7^n} + \tfrac{1}{10^n} - \tfrac{1}{11^n} + \tfrac{1}{14^n} + \tfrac{1}{15^n} - \text{ etc.,}$$

where again as before all the numbers occur besides the powers and divisible by the powers. Truly the prime numbers themselves and those which are themselves constructed from three, five, or from any odd number of terms, have a – sign prefixed, but those formed from two, four, six or from an even number of terms, have a + sign. Thus in this series the term $\frac{1}{30^n}$ occurs which is $30 = 2 \cdot 3 \cdot 5$ nor thus including a power ; truly here the term $\frac{1}{30^n}$ will have a – sign, because 30 has been made from three prime numbers.

270. Now we will consider this expression

$$\frac{1}{(1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z)(1-\varepsilon z) \text{ etc.}},$$

which actually expanded out by division provides this series :

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \text{etc.}$$

and is evident the coefficients *A*, *B*, *C*, *D*, *E* etc. are composed in the following manner from the numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ etc., so that there shall be

$$A = \text{sum of the individual terms},$$
$$B = \text{sum of the factors two at a time},$$
$$C = \text{sum of the factors three at a time},$$
$$D = \text{sum of the factors four at a time}$$
$$\text{etc.}$$

with the same factors not excluded.

271. Therefore on putting $z = 1$ this expression

$$\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon)\text{ etc.}}$$

will be equal to one with a series of all the numbers, which arise from these $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. either by being taken individually, or in two's or more multiplied by each other with equal numbers not excluded. Therefore this differs from that series of numbers, which was produced in § 265, because there finally the factors had to be taken different, but here the same factor may occur twice or several more times. Here clearly all the numbers are present, which are able to come about from these numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ etc.

272. Hence on account of this thing, the series will be constructed always from an infinite number of terms, whether the number of factors were infinite or finite. Thus there will be

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{etc.},$$

where all the numbers are present, which arise from multiples of two along, or all the powers of two. Then there will be

$$\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \text{etc.},$$

where all numbers do not occur, except those which are formed from the two numbers 2 and 3 by the original multiplication, or which have no other divisors besides 2 and 3.

273. Therefore if for $\alpha, \beta, \gamma, \delta, \varepsilon$ etc., one is written divided by all the prime numbers and there is put

$$P = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

it becomes

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \text{etc.},$$

where all the numbers occur, both primes as well as those which arise from the primes by multiplication. But since all the numbers either shall be themselves prime or to have arisen from the multiplication of primes, it is evident here all the whole numbers must be present entirely in the denominators.

274. The same comes about, if the powers of some number of prime numbers may be taken. Indeed if there may be put

$$P=\frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}},$$

there becomes

$$P=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{8^n}+\text{etc.},$$

where all the natural numbers without exception occur. But if moreover in the factors everywhere the + sign may be put in place, so that there shall be

$$P=\frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.}},$$

the expression becomes

$$P=1-\frac{1}{2^n}-\frac{1}{3^n}+\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}-\frac{1}{8^n}+\frac{1}{9^n}+\frac{1}{10^n}-\text{etc.},$$

where the prime numbers have a – sign ; those which have been produced from two primes, whether the same or different, have a + sign ; and generally, those of which the number of prime factors is even, have the + sign, but those which are made from an odd number of prime factors, have the – sign. Thus the term $\frac{1}{240^n}$ on account of $240=2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 5$ will have the sign +. The reasoning of this rule is understood from § 270, if there may be put $z=-1$.

275. If this expression may be brought together with the above, two series will arise, the product of which is equal to unity. For let

$$P=\frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}}$$

and

$$Q=\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.};$$

there will be

$$P=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{8^n}+\text{etc.},$$

$$Q=1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\frac{1}{11^n}-\text{etc.},$$

(§ 269) and it is evident that $PQ = 1$.

276. But if there may be put

$$P = \frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.}}$$

and

$$Q = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.},$$

there will be

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \text{ etc.},$$
$$Q = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{ etc.}$$

and in a like manner there will be had $PQ = 1$. Therefore with the sum of one of the series known, the other sum of the other series will become known.

277. Again in turn from the known sums of these series it will be possible to designate the values of the infinite factors. Let there be without doubt

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \text{ etc.},$$
$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \text{ etc.}$$

and there will be

$$M = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}}$$

$$N = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\left(1-\frac{1}{11^{2n}}\right)\text{ etc.}}$$

Hence there arises by division

$$\frac{M}{N} = \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{etc.},$$

and at last there will be

$$\frac{MM}{N}=\left(\frac{2^n+1}{2^n-1}\right)\left(\frac{3^n+1}{3^n-1}\right)\left(\frac{5^n+1}{5^n-1}\right)\left(\frac{7^n+1}{7^n-1}\right)\left(\frac{11^n+1}{11^n-1}\right)\cdot \text{etc.}$$

Therefore from the known M and N besides the values of these products the sums of these series will be found :

$$\frac{1}{M}=1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\frac{1}{11^n}-\text{etc.},$$
$$\frac{1}{N}=1-\frac{1}{2^{2n}}-\frac{1}{3^{2n}}-\frac{1}{5^{2n}}+\frac{1}{6^{2n}}-\frac{1}{7^{2n}}+\frac{1}{10^{2n}}-\frac{1}{11^{2n}}-\text{etc.},$$
$$\frac{M}{N}=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{10^n}+\frac{1}{11^n}+\text{etc.},$$
$$\frac{N}{M}=1-\frac{1}{2^n}-\frac{1}{3^n}+\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}-\frac{1}{8^n}+\frac{1}{9^n}+\frac{1}{10^n}-\text{etc.},$$

from the combination of which many other results can be deduced.

EXAMPLE 1

Let $n=1$, and because above we have shown [§ 123] to be

$$l\frac{1}{1-x}=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\text{etc.},$$

on putting $x=1$ there will be

$$l\frac{1}{1-1}=l\infty=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.},$$

But the logarithm of an infinitely large number ∞ itself is infinitely large, from which there will be

$$M=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\text{etc}=\infty.$$

Hence on account of $\frac{1}{M}=\frac{1}{\infty}=0$ there becomes

$$0=1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{10}-\frac{1}{11}-\frac{1}{13}+\frac{1}{14}+\frac{1}{15}-\text{etc.},$$

Then truly in the products [above] there will be had

$$M=\infty=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{ etc.}}$$

from which there becomes

$$\infty=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\text{etc.}$$

and

$$0=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{16}{17}\cdot\frac{18}{19}\cdot\text{etc.}$$

Then by the summation of the series above [see § 167]

$$N=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\text{etc.}=\frac{\pi\pi}{6}$$

will be examined. Hence these sums of the series will be obtained :

$$\frac{6}{\pi\pi}=1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}+\frac{1}{10^2}-\frac{1}{11^2}-\text{etc.},$$
$$\infty=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\frac{1}{11}+\text{etc.},$$
$$0=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\text{etc.}$$

And finally there will emerge from the factors

$$\frac{\pi\pi}{6}=\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdot\text{etc.}$$

or

$$\frac{\pi\pi}{6}=\frac{4}{3}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdot\frac{49}{48}\cdot\frac{121}{120}\cdot\frac{169}{168}\cdot\text{etc.}$$

and on account of $\frac{M}{N}=\infty$ or $\frac{N}{M}=0$ there will be had

$$\infty=\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{12}{11}\cdot\frac{14}{13}\cdot\frac{18}{17}\cdot\frac{20}{19}\cdot\text{etc.}$$

or

$$0=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{20}\cdot\text{etc.},$$

and

$$\infty=\frac{3}{1}\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{14}{12}\cdot\frac{18}{16}\cdot\frac{20}{18}\cdot\text{etc.},$$

or

$$0=\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{9}{10}\cdot\text{etc.},$$

the numerators of which fractions (with the first excepted) are less by one from the denominators, but the sum from the numerators and denominators of each fraction constantly provides the prime numbers 3, 5, 7, 11, 13, 17, 19 etc.

EXAMPLE 2

Let $n=2$ and from the above there will be

$$M=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\text{etc.}=\frac{\pi\pi}{6},$$
$$N=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4}+\text{etc.}=\frac{\pi^4}{90}.$$

Hence in the first place these series are summed :

$$\frac{6}{\pi\pi}=1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{10^2}-\frac{1}{11^2}-\text{etc.},$$
$$\frac{90}{\pi^4}=1-\frac{1}{2^4}-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{6^4}-\frac{1}{7^4}+\frac{1}{10^4}-\frac{1}{11^4}\text{etc.},$$
$$\frac{15}{\pi^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{10^2}+\frac{1}{11^2}+\text{etc.},$$
$$\frac{\pi\pi}{15}=1-\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}-\text{etc.}$$

From which the values of the following products become known :

$$\frac{\pi\pi}{6}=\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdot\text{etc.},$$

$$\frac{\pi^4}{90}=\frac{2^4}{2^4-1}\cdot\frac{3^4}{3^4-1}\cdot\frac{5^4}{5^4-1}\cdot\frac{7^4}{7^4-1}\cdot\frac{11^4}{11^4-1}\cdot\text{etc.},$$

$$\frac{15}{\pi\pi}=\frac{2^2+1}{2^2}\cdot\frac{3^2+1}{3^2}\cdot\frac{5^2+1}{5^2}\cdot\frac{7^2+1}{7^2}\cdot\frac{11^2+1}{11^2}\cdot\text{etc.},$$

or

$$\frac{\pi\pi}{15}=\frac{4}{5}\cdot\frac{9}{10}\cdot\frac{25}{26}\cdot\frac{49}{50}\cdot\frac{121}{122}\cdot\frac{169}{170}\cdot\text{etc.}$$

and

$$\frac{5}{2}=\frac{2^2+1}{2^2-1}\cdot\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\cdot\frac{7^2+1}{7^2-1}\cdot\frac{11^2+1}{11^2-1}\cdot\text{etc.}$$

or

$$\frac{5}{2}=\frac{5}{3}\cdot\frac{5}{4}\cdot\frac{13}{12}\cdot\frac{25}{24}\cdot\frac{61}{60}\cdot\frac{85}{84}\cdot\text{etc.}$$

or

$$\frac{3}{2}=\frac{5}{4}\cdot\frac{13}{12}\cdot\frac{25}{24}\cdot\frac{61}{60}\cdot\frac{85}{84}\cdot\text{etc.}$$

In these fractions the numerators are greater than the denominators by one, truly taken together they provide the squares of the prime numbers 3^2, 5^2, 7^2, 11^2 etc.

EXAMPLE 3

Because from the above values of M in §167, it is allowed for n to be assigned an even number only, if we may put $n = 4$, the value will be

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} = \frac{\pi^4}{90},$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \text{etc.} = \frac{\pi^8}{9450}.$$

Hence initially the following series are summed :

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{etc.},$$
$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8} - \text{etc.},$$
$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4} + \text{etc.},$$
$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} + \text{etc.}$$

Then also the values of the following products will be obtained :

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \text{etc.},$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8-1} \cdot \frac{3^8}{3^8-1} \cdot \frac{5^8}{5^8-1} \cdot \frac{7^8}{7^8-1} \cdot \frac{11^8}{11^8-1} \cdot \text{etc.},$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4} \cdot \frac{3^4+1}{3^4} \cdot \frac{5^4+1}{5^4} \cdot \frac{7^4+1}{7^4} \cdot \frac{11^4+1}{11^4} \cdot \text{etc.}$$

and

$$\frac{7}{6} = \frac{2^4+1}{2^4-1} \cdot \frac{3^4+1}{3^4-1} \cdot \frac{5^4+1}{5^4-1} \cdot \frac{7^4+1}{7^4-1} \cdot \frac{11^4+1}{11^4-1} \cdot \text{etc.}$$

or

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \text{etc.}$$

In these factors the numerators exceed the denominators by one, truly likewise they provide the biquadratics of the numerators of the odd primes 3, 5, 7, 11 etc.

278. Because here we have reduced the sum of the series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

to factors, it will be advantageous to progress to logarithms. For since there shall be

$$M = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}},$$

there will be

$$lM = -l\left(1-\frac{1}{2^n}\right) - l\left(1-\frac{1}{3^n}\right) - l\left(1-\frac{1}{5^n}\right) - l\left(1-\frac{1}{7^n}\right) - l\left(1-\frac{1}{11^n}\right) - \text{etc.}$$

Hence with the hyperbolic logarithms taken there will be

$$\begin{aligned} lM = &+ 1\left(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \text{etc.}\right) \\ &+ \frac{1}{2}\left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \text{etc.}\right) \\ &+ \frac{1}{3}\left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \text{etc.}\right) \\ &+ \frac{1}{4}\left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \text{etc.}\right) \\ &\qquad \text{etc.} \end{aligned}$$

But if above we may put

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \text{etc.},$$

so that there shall be

$$N = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\left(1-\frac{1}{11^{2n}}\right)\text{ etc.}},$$

it becomes with the logarithms taken

$$\begin{aligned} lN = &+1\left(\tfrac{1}{2^{2n}} + \tfrac{1}{3^{2n}} + \tfrac{1}{5^{2n}} + \tfrac{1}{7^{2n}} + \tfrac{1}{11^{2n}} + \text{etc.}\right) \\ &+\tfrac{1}{2}\left(\tfrac{1}{2^{4n}} + \tfrac{1}{3^{4n}} + \tfrac{1}{5^{4n}} + \tfrac{1}{7^{4n}} + \tfrac{1}{11^{4n}} + \text{etc.}\right) \\ &+\tfrac{1}{3}\left(\tfrac{1}{2^{6n}} + \tfrac{1}{3^{6n}} + \tfrac{1}{5^{6n}} + \tfrac{1}{7^{6n}} + \tfrac{1}{11^{6n}} + \text{etc.}\right) \\ &+\tfrac{1}{4}\left(\tfrac{1}{2^{8n}} + \tfrac{1}{3^{8n}} + \tfrac{1}{5^{8n}} + \tfrac{1}{7^{8n}} + \tfrac{1}{11^{8n}} + \text{etc.}\right) \\ &\qquad \text{etc.} \end{aligned}$$

From these taken together there becomes

$$\begin{aligned} lM - \tfrac{1}{2}lN = &+1\left(\tfrac{1}{2^{n}} + \tfrac{1}{3^{n}} + \tfrac{1}{5^{n}} + \tfrac{1}{7^{n}} + \tfrac{1}{11^{n}} + \text{etc.}\right) \\ &+\tfrac{1}{3}\left(\tfrac{1}{2^{3n}} + \tfrac{1}{3^{3n}} + \tfrac{1}{5^{3n}} + \tfrac{1}{7^{3n}} + \tfrac{1}{11^{3n}} + \text{etc.}\right) \\ &+\tfrac{1}{5}\left(\tfrac{1}{2^{5n}} + \tfrac{1}{3^{5n}} + \tfrac{1}{5^{5n}} + \tfrac{1}{7^{5n}} + \tfrac{1}{11^{5n}} + \text{etc.}\right) \\ &+\tfrac{1}{7}\left(\tfrac{1}{2^{7n}} + \tfrac{1}{3^{7n}} + \tfrac{1}{5^{7n}} + \tfrac{1}{7^{7n}} + \tfrac{1}{11^{7n}} + \text{etc.}\right) \\ &\qquad \text{etc.} \end{aligned}$$

279. If $n = 1$, there will be

$$M = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \text{etc.} = l\infty$$

and

$$N = \tfrac{\pi\pi}{6};$$

and hence there will be

$$\begin{aligned} l.l\infty - \tfrac{1}{2}l\tfrac{\pi\pi}{6} = &+1\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \text{etc.}\right) \\ &+\tfrac{1}{3}\left(\tfrac{1}{2^{3}} + \tfrac{1}{3^{3}} + \tfrac{1}{5^{3}} + \tfrac{1}{7^{3}} + \tfrac{1}{11^{3}} + \text{etc.}\right) \\ &+\tfrac{1}{5}\left(\tfrac{1}{2^{5}} + \tfrac{1}{3^{5}} + \tfrac{1}{5^{5}} + \tfrac{1}{7^{5}} + \tfrac{1}{11^{5}} + \text{etc.}\right) \\ &+\tfrac{1}{7}\left(\tfrac{1}{2^{7}} + \tfrac{1}{3^{7}} + \tfrac{1}{5^{7}} + \tfrac{1}{7^{7}} + \tfrac{1}{11^{7}} + \text{etc.}\right) \\ &\qquad \text{etc.} \end{aligned}$$

Truly these series besides the first not only have finite sums, but also all likewise taken make a finite sum and that small enough ; from which it is necessary, that the sum of the first series shall be infinitely large

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \text{etc.}$$

Clearly with a small enough quantity taken from the hyperbolic logarithm of the series :

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.}$$

280. Let $n=2$; there will be

$$M=\frac{\pi\pi}{6} \text{ and } N=\frac{\pi^4}{90},$$

from which there becomes

$$\begin{aligned}2l\pi - l6 = &+1\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\text{etc.}\right)\\ &+\frac{1}{2}\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{11^4}+\text{etc.}\right)\\ &+\frac{1}{3}\left(\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\text{etc.}\right)\\ &\text{etc.},\end{aligned}$$

$$\begin{aligned}4l\pi - l90 = &+1\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{11^4}+\text{etc.}\right)\\ &+\frac{1}{2}\left(\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{11^8}+\text{etc.}\right)\\ &+\frac{1}{3}\left(\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{5^{12}}+\frac{1}{7^{12}}+\frac{1}{11^{12}}+\text{etc.}\right)\\ &\text{etc.},\end{aligned}$$

$$\begin{aligned}\frac{1}{2}l\frac{5}{2} = &+1\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\text{etc.}\right)\\ &+\frac{1}{3}\left(\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\text{etc.}\right)\\ &+\frac{1}{5}\left(\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\frac{1}{11^{10}}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

281. Although the law, by which the prime numbers are progressing, is not in existence, yet the sums of the series of the higher powers can be assigned approximately without difficulty. Indeed let this series be

$$M=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\text{etc.}$$

and

$$S=\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{11^n}+\frac{1}{13^n}+\text{etc.};$$

there will be

$$S=M-1-\frac{1}{4^n}-\frac{1}{6^n}-\frac{1}{8^n}-\frac{1}{9^n}-\frac{1}{10^n}-\text{etc.}$$

and on account of

$$\frac{M}{2^n}=\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\frac{1}{8^n}+\frac{1}{10^n}+\frac{1}{12^n}+\text{etc.}$$

there will be

$$S=M-\frac{M}{2^n}-1+\frac{1}{2^n}-\frac{1}{9^n}-\frac{1}{15^n}-\frac{1}{21^n}-\text{etc.}$$

or

$$S=(M-1)\left(1-\frac{1}{2^n}\right)-\frac{1}{9^n}-\frac{1}{15^n}-\frac{1}{21^n}-\frac{1}{25^n}-\frac{1}{27^n}-\text{etc.}$$

and on account of

$$M\left(1-\frac{1}{2^n}\right)\frac{1}{3^n}=\frac{1}{3^n}+\frac{1}{9^n}+\frac{1}{15^n}+\frac{1}{21^n}+\text{etc.}$$

there will be

$$S=(M-1)\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)+\frac{1}{6^n}-\frac{1}{25^n}-\frac{1}{35^n}-\frac{1}{49^n}\ \text{etc.}$$

Hence on account of the given sum M [see § 168] the value of S is found conveniently, if indeed n were moderately large.

282. Moreover from the sums of the higher powers also the sums of the lower powers can be shown from the formulas found. And by this method the following sums of the series have appeared

$$\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{11^n}+\frac{1}{13^n}+\frac{1}{17^n}+\text{etc.}:$$

If	the sum of the series will be
$n=2$	0,452247420041065
$n=4$	0,076993139764247
$n=6$	0,017070086850637
$n=8$	0,004061405366518
$n=10$	0,000993603574437
$n=12$	0,000246026470035
$n=14$	0,000061244396725
$n=16$	0,000015282026219
$n=18$	0,000003817278703
$n=20$	0,000000953961124
$n=22$	0,000000238450446
$n=24$	0,000000059608185
$n=26$	0,000000014901555
$n=28$	0,000000003725334
$n=30$	0,000000000931327
$n=32$	0,000000000232831
$n=34$	0,000000000058208
$n=36$	0,000000000014552

The remaining sums of the even powers decrease in the quadruple ratio.

283. But this conversion of the series

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}$$

into an infinite product can also be put in place directly in this manner. Let

$$A=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{8^n}+\text{etc.};$$

by subtracting

$$\frac{1}{2^n}A=\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{6^n}+\frac{1}{8^n}+\text{etc.};$$

there will be

$$\left(1-\frac{1}{2^n}\right)A=1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{9^n}+\frac{1}{11^n}+\text{etc.}=B.$$

Thus all the terms divisible by two have been subtracted. By subtracting

$$\frac{1}{3^n}B=\frac{1}{3^n}+\frac{1}{9^n}+\frac{1}{15^n}+\frac{1}{21^n}+\text{etc.};$$

there will be

$$\left(1-\frac{1}{3^n}\right)B=1+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{11^n}+\frac{1}{13^n}+\text{etc.}=C.$$

Thus all the above terms divisible by 3 above have been taken away. By subtracting

$$\frac{1}{5^n}B=\frac{1}{5^n}+\frac{1}{25^n}+\frac{1}{35^n}+\frac{1}{55^n}+\text{etc.};$$

there will be

$$\left(1-\frac{1}{5^n}\right)C=1+\frac{1}{7^n}+\frac{1}{11^n}+\frac{1}{13^n}+\frac{1}{17^n}+\text{etc.}$$

Thus also all the terms divisible by 5 have been removed. In an equal manner the terms divisible by 7, 11 and the remaining prime numbers have been removed ; moreover it is evident with all the terms subtracted, which shall be divisible by prime numbers, only unity remains. Whereby for *B, C, D, E* etc. with the values constituted, finally there becomes

$$A\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}=1,$$

from which the sum of the proposed series will be

$$A = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right) \text{ etc.}}$$

or

$$A = \frac{2^n}{2^n-1} \cdot \frac{3^n}{3^n-1} \cdot \frac{5^n}{5^n-1} \cdot \frac{7^n}{7^n-1} \cdot \frac{11^n}{11^n-1} \cdot \text{ etc.}$$

284. This method now will be able to be used conveniently for converting other series into infinite products, the sums of which we have found above. But above (§ 175) we have found the sums of these series

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \text{etc.},$$

if n were an odd number. Indeed the sum is $= N\pi^n$ and we have given the values of N in the place mentioned. But it is to be observed, since here only odd numbers occur, these, which shall be of the form $4m+1$, have a + sign, the rest of the form $4m-1$ have a – sign. There since there shall be

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} + \text{ etc.}$$

The expression may be added:

$$\frac{1}{3^n} A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \text{ etc.};$$

and there will be

$$\left(1 + \frac{1}{3^n}\right) A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{etc.} = B.$$

On subtracting

$$\frac{1}{5^n} B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n} + \text{ etc.};$$

there will be

$$\left(1 - \frac{1}{5^n}\right) B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{etc.} = C,$$

where now the numbers divisible by 3 and 5 are absent. On adding

$$\frac{1}{7^n} C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \text{ etc.};$$

there will be

$$\left(1+\tfrac{1}{7^n}\right)C = 1-\tfrac{1}{11^n}+\tfrac{1}{13^n}+\tfrac{1}{17^n}-\text{etc.}=D,$$

Thus now the numbers divisible by 7 have been removed. On adding

$$\tfrac{1}{11^n}D=\tfrac{1}{11^n}-\tfrac{1}{121^n}+\text{etc.};$$

there will be

$$\left(1+\tfrac{1}{11^n}\right)D = 1+\tfrac{1}{13^n}+\tfrac{1}{17^n}-\text{etc.}=E.$$

Thus now the numbers divisible by 11 also have been removed. But with all the remaining numbers divisible by the remaining prime numbers taken away finally there will be produced

$$A\left(1+\tfrac{1}{3^n}\right)\left(1-\tfrac{1}{5^n}\right)\left(1+\tfrac{1}{7^n}\right)\left(1+\tfrac{1}{11^n}\right)\left(1-\tfrac{1}{13^n}\right)\ \text{etc.}=1$$

or

$$A=\tfrac{3^n}{3^n+1}\cdot\tfrac{5^n}{5^n-1}\cdot\tfrac{7^n}{7^n+1}\cdot\tfrac{11^n}{11^n+1}\cdot\tfrac{13^n}{13^n-1}\cdot\tfrac{17^n}{17^n-1}\cdot\text{etc.},$$

where the powers of all the prime numbers occur in the numerators, which are present increased or decreased by one in the denominators, provided the prime numbers should be of the form $4m-1$ or $4m+1$.

285. Therefore on putting $n=1$ on account of $A=\tfrac{\pi}{4}$ [see § 176] there will be

$$\tfrac{\pi}{4}=\tfrac{3}{4}\cdot\tfrac{5}{4}\cdot\tfrac{7}{8}\cdot\tfrac{11}{12}\cdot\tfrac{13}{12}\cdot\tfrac{17}{16}\cdot\tfrac{19}{20}\cdot\tfrac{23}{24}\cdot\text{etc.}$$

But above we have found

$$\tfrac{\pi\pi}{6}=\tfrac{4}{3}\cdot\tfrac{3^2}{2\cdot 4}\cdot\tfrac{5^2}{4\cdot 6}\cdot\tfrac{7^2}{6\cdot 8}\cdot\tfrac{11^2}{10\cdot 12}\cdot\tfrac{13^2}{12\cdot 14}\cdot\tfrac{17^2}{16\cdot 18}\cdot\tfrac{19^2}{18\cdot 20}\cdot\text{etc.}$$

The second may be divided by the first and there becomes

$$\tfrac{2\pi}{3}=\tfrac{4}{3}\cdot\tfrac{3}{2}\cdot\tfrac{5}{6}\cdot\tfrac{7}{6}\cdot\tfrac{11}{10}\cdot\tfrac{13}{14}\cdot\tfrac{17}{18}\cdot\tfrac{19}{18}\cdot\tfrac{23}{22}\cdot\text{etc.}$$

or

$$\tfrac{\pi}{2}=\tfrac{3}{2}\cdot\tfrac{5}{6}\cdot\tfrac{7}{6}\cdot\tfrac{11}{10}\cdot\tfrac{13}{14}\cdot\tfrac{17}{18}\cdot\tfrac{19}{18}\cdot\tfrac{23}{22}\cdot\text{etc.},$$

where prime numbers constitute the numerators and the denominators truly are even numbers unequally different from the numerators by one. But if these may be divided by $\frac{\pi}{4}$ anew, there will be

$$2=\frac{4}{2}\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{16}{18}\cdot\frac{20}{18}\cdot\frac{24}{22}\cdot\text{etc.}$$

or

$$2=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdot\text{etc.},$$

which fractions arise from the odd prime numbers 3, 5, 7, 11, 13, 17 etc. each requiring to be separated into two different parts by unity and on taking the even parts for the numerators, with odd parts for the denominators.

286. If these expressions may be compared with those of Wallis :

$$\frac{\pi}{2}=\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot11}\text{ etc.}$$

or

$$\frac{4}{\pi}=\frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{9\cdot9}{8\cdot10}\cdot\frac{11\cdot11}{10\cdot12}\cdot\text{etc.},$$

since there shall be [see § 277]

$$\frac{\pi\pi}{8}=\frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{10\cdot12}\cdot\frac{13\cdot13}{12\cdot14}\cdot\text{etc.},$$

these will give by division

$$\frac{32}{\pi^3}=\frac{9\cdot9}{8\cdot10}\cdot\frac{15\cdot15}{14\cdot16}\cdot\frac{21\cdot21}{20\cdot22}\cdot\frac{25\cdot25}{24\cdot26}\cdot\text{etc.},$$

where all the non-prime odd numbers occur in the numerators.

287. Now let $n=3$; there will be $A=\frac{\pi^3}{32}$, from which there becomes

$$\frac{\pi^3}{32}=\frac{3^3}{3^3+1}\cdot\frac{5^3}{5^3-1}\cdot\frac{7^3}{7^3+1}\cdot\frac{11^3}{11^3+1}\cdot\frac{13^3}{13^3-1}\cdot\frac{17^3}{17^3-1}\cdot\text{etc.}$$

But from the series

$$\frac{\pi^6}{945}=1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}$$

there becomes

$$\frac{\pi^6}{945}=\frac{2^6}{2^6-1}\cdot\frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot\frac{13^6}{13^6-1}\cdot\text{etc.}$$

or

$$\frac{\pi^6}{960}=\frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot\frac{13^6}{13^6-1}\cdot\text{etc.},$$

which divided by the first above will give

$$\frac{\pi^3}{30}=\frac{3^3}{3^3-1}\cdot\frac{5^3}{5^3+1}\cdot\frac{7^3}{7^3-1}\cdot\frac{11^3}{11^3-1}\cdot\frac{13^3}{13^3+1}\cdot\frac{17^3}{17^3+1}\cdot\text{etc.}$$

This truly divided anew by the first will give

$$\frac{16}{15}=\frac{3^3+1}{3^3-1}\cdot\frac{5^3-1}{5^3+1}\cdot\frac{7^3+1}{7^3-1}\cdot\frac{11^3+1}{11^3-1}\cdot\frac{13^3-1}{13^3+1}\cdot\frac{17^3-1}{17^3+1}\cdot\text{etc.}$$

or

$$\frac{16}{15}=\frac{14}{13}\cdot\frac{62}{63}\cdot\frac{172}{171}\cdot\frac{666}{665}\cdot\frac{1098}{1099}\cdot\text{etc.},$$

which fractions are formed from the cubes of the uneven prime numbers each separated into two parts different by one and on taking the even parts for the numerators and the odd parts for the denominators.

288. From these expressions new series can be formed anew, in which all the natural numbers constitute the denominators. For since there shall be from § 285 :

$$\frac{\pi}{4}=\frac{3}{3+1}\cdot\frac{5}{5-1}\cdot\frac{7}{7+1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdot\text{etc.},$$

there will be

$$\frac{\pi}{6}=\frac{1}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

from which by expansion this series will be formed :

$$\frac{\pi}{6}=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\text{etc.},$$

where the reasoning of the signs thus is prepared, so that of the second may be had as $-$, prime numbers of the form $4m-1$ have a $-$ sign, and prime numbers of the form $4m+1$ a $+$ sign; but composite numbers have that sign, which agrees with the account of the multiplication from the primes. Thus it will be apparent, that the sign of the fraction $\frac{1}{60}$ on account of

$$60=\overset{-}{2}\cdot\overset{-}{2}\cdot\overset{-}{2}\cdot\overset{+}{5},$$

will be $-$.

In a similar manner again there will be

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

from which this series arises :

$$\frac{\pi}{2}=1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\text{etc.},$$

were the sign of the second term has a + sign, prime numbers of the form $4m-1$ have a – sign, prime numbers of the form $4m+1$ a + sign ; and some composite number will have that sign, which will agree with the account of the composition itself from primes, following the rules of multiplication.

289. Since then there shall be

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

by expansion there becomes:

$$\frac{\pi}{2}=1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}-\text{etc.},$$

where just as many odd numbers occur, but thus the signs have been prepared, so that the prime numbers of the form $4m-1$ may have a + sign, prime numbers of the form $4m+1$ a – sign, from which likewise the signs of composite numbers are defined.

Again two series hence can be formed, where all the numbers occur. Clearly there will be

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

from which by expansion there becomes

$$\pi=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\text{etc.},$$

where the sign of the second term has +, prime numbers of the form $4m-1$ a + sign, truly numbers of the form $4m+1$ a – sign.

Then truly also there will be

$$\frac{\pi}{3}=\frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

from which by expansion there becomes

$$\frac{\pi}{3}=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\text{etc.},$$

where the second term has a – sign, prime numbers of the form $4m-1$ a + positive sign, and prime numbers of the form $4m+1$ a – sign.

290. Hence innumerable other conditions of the sign are able to be shown, thus so that the sum of the series

$$1, \quad \tfrac{1}{2}, \quad \tfrac{1}{3}, \quad \tfrac{1}{4}, \quad \tfrac{1}{5}, \quad \tfrac{1}{6}, \quad \tfrac{1}{7}, \quad \tfrac{1}{8} \text{ etc.}$$

may be assigned. Since clearly there shall be

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\text{ etc.}},$$

this expression may be multiplied by $\frac{1+\frac{1}{3}}{1-\frac{1}{3}}=2$; there will become

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\text{ etc.}}$$

and

$$\pi=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}\text{ etc.},$$

where the second term has a $+$ sign, the third $+$, all the remaining prime numbers of the form $4m-1$ a – sign, but prime numbers of the form $4m+1$ a + sign; from which the account of the signs of the composite numbers is understood.

In a similar manner since there shall be

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{ etc.}},$$

it may be multiplied by $\frac{1+\frac{1}{5}}{1-\frac{1}{5}}=\frac{3}{2}$; there will be

$$\frac{3\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\text{ etc.}},$$

from which by expansion there becomes

$$\frac{3\pi}{2}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}-\frac{1}{13}+\text{ etc.},$$

where the second term has a + sign, prime numbers of the form $4m-1$ have a + sign, and prime numbers of the form $4m+1$ beyond the fifth term have a – sign.

291. Also innumerable series of this kind can be shown, the sum of which shall be $=0$. For since there shall be, by § 277,

$$0=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\text{etc.},$$

there will be

$$0=\frac{1}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}}$$

from which, as we have seen above, there emerges

$$0=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\text{etc.},$$

where all the prime numbers have a $-$ sign, and the signs of the composite numbers follow the rule of multiplication.

Moreover we may multiply that expression by $\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=3$; equally there will be

$$0=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}}$$

from which by expansion there arises

$$0=1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\text{etc.},$$

where the second term has a + sign, and all the remaining prime numbers have a – sign.

Also in a similar manner there will be

$$0=\frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

from which this series arises

$$0=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\text{etc.},$$

where all the prime numbers except for 3 and 5 have a – sign .

Moreover in general it is to be observed, as often as all the prime numbers with only a few excepted which may have a $-$ sign, the sum of the series becomes $=0$, but on the contrary, as often as all the prime numbers with only some excepted that may have a + sign, then the sum of the series becomes infinitely great.

292. Also above (§ 176) we have given the sum of the series

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n} - \text{etc.},$$

if n were an odd number. Therefore there will be

$$\frac{1}{2^n}A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \text{etc.},$$

which added gives

$$B = \left(1 + \frac{1}{2^n}\right)A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \text{etc.},$$

There is added

$$\frac{1}{5^n}B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n} + \text{etc.};$$

and there will be

$$C = \left(1 + \frac{1}{5^n}\right)B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} - \text{etc.}$$

On subtracting

$$\frac{1}{7^n}C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} + \text{etc.};$$

there will be

$$D = \left(1 - \frac{1}{7^n}\right)C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \text{etc.}$$

From these at last arises

$$A\left(1 + \frac{1}{2^n}\right)\left(1 + \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 + \frac{1}{11^n}\right)\left(1 - \frac{1}{13^n}\right) \text{ etc.} = 1,$$

where the prime numbers exceeding a multiple of six by one have a – sign, but those deficient by one a + sign. And there will be

$$A = \frac{2^n}{2^n+1} \cdot \frac{5^n}{5^n+1} \cdot \frac{7^n}{7^n-1} \cdot \frac{11^n}{11^n+1} \cdot \frac{13^n}{13^n-1} \cdot \text{etc.}$$

293. We may consider the case $n = 1$, so that $A = \frac{\pi}{3\sqrt{3}}$, and there will be

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \text{etc.},$$

where in the numerators all the prime numbers occurring after 3, truly with the denominators disagreeing with the numerators by one, and they are all divisible by 6. Since now there shall be

$$\frac{\pi\pi}{6} = \frac{4}{3}\cdot\frac{9}{8}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{12\cdot10}\cdot\frac{13\cdot13}{12\cdot14}\cdot\text{etc.},$$

with this expression divided by that there will be

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\text{etc.},$$

where the denominators are not divisible by 6. Or there will be

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{24}\cdot\text{etc.},$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot\text{etc.},$$

of which the latter divided by the former gives

$$\frac{4}{3} = \frac{6}{4}\cdot\frac{6}{8}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{18}{16}\cdot\frac{18}{20}\cdot\frac{24}{22}\cdot\text{etc.}$$

or

$$\frac{4}{3} = \frac{3}{2}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{9}{10}\cdot\frac{12}{11}\cdot\text{etc.},$$

where the individual fractions are formed from the individual prime numbers 5, 7, 11 etc. by separating each prime number in two parts, differing by unity and by taking the parts constantly divisible by 3 for the numerators.

294. Since truly above we have seen that there is

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\text{etc.}$$

or

$$\frac{\pi}{3} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\text{etc.},$$

if the above $\frac{\pi}{2\sqrt{3}}$ and $\frac{2\pi}{3\sqrt{3}}$ are divided by this, there becomes

$$\frac{\sqrt{3}}{2} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{14}{15} \cdot \frac{16}{15} \cdot \text{etc.},$$

$$\frac{2}{\sqrt{3}} = \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{18}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \text{etc.}$$

In the first expression the fractions are formed from prime numbers of the form $12m+6\pm1$, in the second from prime numbers of the form $12m\pm1$, by taking the individual fractions separating into two parts differing by one, and even parts for the numerators, and truly by taking odd parts for the denominators.

295. At this point we may consider the series found above (§ 179), which will progress thus

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.} = A;$$

there will be

$$\frac{1}{3}A = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} - \frac{1}{21} + \frac{1}{27} + \frac{1}{33} - \text{etc.}$$

The series may be subtracted :

$$\left(1-\frac{1}{3}\right)A = 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} - \text{etc.} = B.$$

There may be added

$$\frac{1}{5}B = \frac{1}{5} - \frac{1}{25} - \frac{1}{35} + \frac{1}{55} - \text{etc.};$$

and there will be

$$\left(1+\frac{1}{5}\right)B = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} - \text{etc.} = C.$$

And thus by progressing, the product at last comes to

$$\frac{\pi}{2\sqrt{2}}\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1-\frac{1}{17}\right)\left(1-\frac{1}{19}\right)\text{etc.} = 1,$$

where the signs thus are to be considered, so that the forms of the prime numbers $8m+1$ or $8m+3$ shall have a $-$ sign, truly the forms of the prime numbers $8m+5$ or $8m+7$ shall have a + sign. And hence thus there will be

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \text{etc.},$$

where all the denominators either are divisible by 8 or are only in pairs of unequal numbers. Since there shall be

$$\frac{\pi}{4}=\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\text{etc.},$$

$$\frac{\pi}{2}=\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot\text{etc.},$$

therefore

$$\frac{\pi\pi}{8}=\frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{10\cdot12}\cdot\frac{13\cdot13}{12\cdot14}\cdot\text{etc.},$$

there will be

$$\frac{\pi}{2\sqrt{2}}=\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot\text{etc.}$$

where no denominators divisible by 8 occur, equally pairs truly are present, as often as they differ by one from the numerators. Truly the first divided by the last gives

$$1=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{10}{9}\cdot\frac{11}{12}\cdot\text{etc.};$$

which fractions are formed from the individual prime numbers on separating each into two parts differing by unity, and by taking even parts for the numerators (unless they shall be equal even numbers).

296. In a similar manner the remaining series, which we have found above for the expression of circular arcs (§ 179 et seq.) can be transformed into factors, which can be constituted from prime numbers. And thus many other conspicuous properties both of the factors of this kind as well as of the infinite series they are able to elicit. Because truly now that I have mentioned this particular situation, I will not linger here giving more explanations. But I shall proceed to another related argument. Just as clearly in this chapter the numbers, in as much as they arise from multiplication, have been considered, thus in the following the generation of numbers by addition will be investigated carefully.

CAPUT XV

DE SERIEBUS EX EVOLUTIONE FACTORUM ORTIS

264. Sit propositum productum ex factoribus numero sive finitis sive infinitis constans huiusmodi

$$(1+\alpha z)(1+\beta z)(1+\gamma z)(1+\delta z)(1+\varepsilon z)(1+\zeta z) \text{ etc.},$$

quod, si per multiplicationem actualem evolvatur, det

$$1+Az+Bz^2+Cz^3+Dz^4+Ez^5+Fz^6+\text{etc.},$$

atque manifestum est coefficientes *A*, *B*, *C*, *D*, *E* etc. ita formari ex numeris

$\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ etc., ut sit

$A=\alpha+\beta+\gamma+\delta+\varepsilon+\zeta+\text{etc.}=$ summae singulorum,
B = summae factorum ex binis diversis,
C = summae factorum ex ternis diversis,
D = summae faetorum ex quaternis diversis,
E = summae factorum ex quinis diversis
etc.,

donec perveniatur ad productum ex omnibus.

265. Quodsi ergo ponatur $z=1$, productum hoc

$$(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon) \text{ etc.}$$

aequabitur unitati cum serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. vel sumendis singulis vel duobus pluribusve diversis in se multiplicandis nascuntur. Atque si idem numerus duobus pluribusve modis resultare queat, etiam idem bis pluriesve in hac numerorum serie occurret.

266. Si ponatur $z=-1$, productum hoc

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon) \text{ etc.}$$

aequabitur unitati cum serie numerorum omnium, qui ex his $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. vel sumendis singulis vel duobus pluribusve diversis in se multiplicandis nascuntur; ut ante quidem, verum hoc discrimine, ut ii numeri, qui vel ex singulis vel ternis vel quinis vel numero imparibus nascuntur, sint negativi, illi vero, qui vel ex binis vel quaternis vel senis vel numero paribus resultant, sint affirmativi.

267. Scribantur pro $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. numeri primi omnes

$$2, \quad 3, \quad 5, \quad 7, \quad 11, \quad 13 \quad \text{etc}.$$

atque hoc productum

$$(1+2)(1+3)(1+5)(1+7)(1+11)(1+13) \text{ etc.} = P$$

aequabitur unitati cum serie omnium numerorum vel primorum ipsorum vel ex primis diversis per multiplicationem ortorum. Erit ergo

$$P = 1+2+3+5+6+7+10+11+13+14+15+17+ \text{ etc.},$$

in qua serie omnes occurrunt numeri naturales exceptis potestatibus iisque, qui per quamvis potestatem sunt divisibiles. Desunt scilicet numeri 4, 8, 9,12, 16, 18 etc., quoniam sunt vel potestates ut 4, 8, 9, 16 etc., vel per potestates divisibiles ut 12, 18 etc.

268. Simili modo res se habebit, si pro $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. potestates quaecunque numerorum primorum substituantur, scilicet si ponamus

$$P = \left(1+\tfrac{1}{2^n}\right)\left(1+\tfrac{1}{3^n}\right)\left(1+\tfrac{1}{5^n}\right)\left(1+\tfrac{1}{7^n}\right)\left(1+\tfrac{1}{11^n}\right) \text{ etc.}$$

Erit enim multiplicatione instituta

$$P = 1+\tfrac{1}{2^n}+\tfrac{1}{3^n}+\tfrac{1}{5^n}+\tfrac{1}{6^n}+\tfrac{1}{7^n}+\tfrac{1}{10^n}+\tfrac{1}{11^n}+ \text{ etc.},$$

in quibus fractionibus omnes occurrunt numeri praeter illos, qui vel ipsi sunt potestates vel per potestatem quampiam divisibiles. Cum enim omnes numeri integri sint vel primi vel ex primis per multiplicationem compositi, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur.

269. Si numeri $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. negative capiantur, ut ante (§ 266) fecimus, atque ponatur

$$P=\left(1-\tfrac{1}{2^n}\right)\left(1-\tfrac{1}{3^n}\right)\left(1-\tfrac{1}{5^n}\right)\left(1-\tfrac{1}{7^n}\right)\left(1-\tfrac{1}{11^n}\right)\text{ etc.},$$

erit

$$P=1-\tfrac{1}{2^n}-\tfrac{1}{3^n}-\tfrac{1}{5^n}+\tfrac{1}{6^n}-\tfrac{1}{7^n}+\tfrac{1}{10^n}-\tfrac{1}{11^n}+\tfrac{1}{14^n}+\tfrac{1}{15^n}-\text{ etc.},$$

ubi iterum ut ante omnes occurrunt numeri praeter potestates ac divisibiles per potestates. Verum ipsi numeri primi et qui ex ternis, quinis numerove imparibus constant, signum habent praefixum –, qui autem ex binis vel quaternis vel senis vel numero paribus formantur, signum habent +. Sic in hac serie occurret terminus $\tfrac{1}{30^n}$ quia est $30=2\cdot3\cdot5$ neque adeo potestatem complectitur ; habebit vero hic terminus $\tfrac{1}{30^n}$ signum –, quia 30 est productum ex tribus numeris primis.

270. Consideremus iam hanc expressionem

$$\frac{1}{(1-\alpha z)(1-\beta z)(1-\gamma z)(1-\delta z)(1-\varepsilon z)\text{ etc.}},$$

quae per divisionem actualem evoluta praebeat hanc seriem

$$1+Az+Bz^2+Cz^3+Dz^4+Ez^5+Fz^6+\text{etc.}$$

atque manifestum est coefficientes *A*, *B*, *C*, *D*, *E* etc. sequenti modo ex numeris $\alpha,\beta,\gamma,\delta,\varepsilon$ etc. componi, ut sit

$$\begin{aligned}A&=\text{summae singulorum},\\ B&=\text{summae factorum ex binis},\\ C&=\text{summae factorum ex ternis},\\ D&=\text{summae factorum ex quaternis}\\ &\text{etc.}\end{aligned}$$

non exclusis factoribus iisdem.

271. Posito ergo $z=1$ ista expressio

$$\frac{1}{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)(1-\varepsilon)\text{ etc.}}$$

aequabitur unitati cum serie numerorum omnium, qui ex his $\alpha,\beta,\gamma,\delta,\varepsilon$ etc. vel sumendis singulis vel duobus pluribusve in se multiplicandis oriuntur non exclusis aequalibus. Hoc ergo differt ista numerorum series ab illa, quae § 265 prodiit, quod ibi factores tantum diversi sumi debebant, hic autem idem factor bis pluriesve occurrere possit. Hic scilicet omnes numeri occurrunt, qui per multiplicationem ex his $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. provenire possunt.

272. Hanc ob rem series semper ex terminorum numero infinito constat, sive factorum numerus fuerit infinitus sive finitus. Sic erit

$$\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{etc.},$$

ubi omnes numeri adsunt, qui ex binario solo per multiplicationem oriuntur, seu omnes binarii potestates. Deinde erit

$$\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \text{etc.},$$

ubi alii numeri non occurrunt, nisi qui ex his duobus 2 et 3 per multiplicationem originem trahunt, seu qui alios divisores praeter 2 et 3 non habent.

273. Si igitur pro $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. unitas per singulos omnes numeros primos scribatur ac ponatur

$$P = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \text{etc.},$$

ubi omnes numeri, tam primi quam qui ex primis per multiplicationem nascuntur, occurrunt. Cum autem omnes numeri vel sint ipsi primi vel ex primis per multiplicationem oriundi, manifestum est hic omnes omnino numeros integros in denominatoribus adesse debere.

274. Idem evenit, si numerorum primorum potestates quaecunque accipiantur.
Si enim ponatur

$$P = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}},$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{etc.},$$

ubi omnes numeri naturales nullo excepto occurrunt. Quodsi autem in factoribus ubique signum + statuatur, ut sit

$$P=\frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.}},$$

erit

$$P=1-\frac{1}{2^n}-\frac{1}{3^n}+\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}-\frac{1}{8^n}+\frac{1}{9^n}+\frac{1}{10^n}-\text{etc.},$$

ubi numeri primi habent signum – ; qui sunt producti ex duobus primis, sive iisdem sive diversis, signum habent + ; et generatim, quorum numerorum numerus factorum primorum est par, signum habent +, qui autem ex factoribus primis numero imparibus constant, habent signum –. Sic terminus $\frac{1}{240^n}$ ob $240=2\cdot2\cdot2\cdot2\cdot3\cdot5$ habebit signum +. Cuius legis ratio percipitur ex § 270, si ponatur $z=-1$.

275. Si haec cum superioribus conferantur, nascentur binae series, quarum productum unitati aequatur. Sit enim

$$P=\frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}}$$

et

$$Q=\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.};$$

erit

$$P=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{8^n}+\text{etc.},$$
$$Q=1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\frac{1}{11^n}-\text{etc.},$$

(§ 269) atque manifestum est fore $PQ=1$.

276. Sin autem ponatur

$$P=\frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.}}$$

et

$$Q=\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\left(1+\frac{1}{11^n}\right)\text{ etc.},$$

erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \text{ etc.},$$
$$Q = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{ etc.}$$

similique modo habebitur $PQ = 1$. Cognita ergo alterius seriei summa simul alterius innotescet.

277. Vicissim porro ex cognitis summis harum serierum assignari poterunt valores factorum infinitorum. Sit nimirum

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \text{ etc.},$$
$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \text{ etc.}$$

eritque

$$M = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{ etc.}}$$
$$N = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\left(1-\frac{1}{11^{2n}}\right)\text{ etc.}}$$

Hinc per divisionem nascitur

$$\frac{M}{N} = \left(1 + \frac{1}{2^n}\right)\left(1 + \frac{1}{3^n}\right)\left(1 + \frac{1}{5^n}\right)\left(1 + \frac{1}{7^n}\right)\left(1 + \frac{1}{11^n}\right)\text{etc.},$$

denique vero erit

$$\frac{MM}{N} = \left(\frac{2^n+1}{2^n-1}\right)\left(\frac{3^n+1}{3^n-1}\right)\left(\frac{5^n+1}{5^n-1}\right)\left(\frac{7^n+1}{7^n-1}\right)\left(\frac{11^n+1}{11^n-1}\right)\cdot\text{ etc.}$$

Ex cognitis ergo M et N praeter valores horum productorum summae harum serierum habebuntur:

$$\frac{1}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \text{ etc.},$$
$$\frac{1}{N} = 1 - \frac{1}{2^{2n}} - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{6^{2n}} - \frac{1}{7^{2n}} + \frac{1}{10^{2n}} - \frac{1}{11^{2n}} - \text{ etc.},$$
$$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{ etc.},$$
$$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \text{ etc.},$$

ex quarum combinatione multae aliae deduci possunt.

EXEMPLUM 1

Sit $n=1$, et quoniam supra demonstravimus esse

$$l\frac{1}{1-x}=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\text{etc.},$$

erit posito $x=1$

$$l\frac{1}{1-1}=l\infty=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.},$$

At logarithmus numeri infinite magni ∞ ipse est infinite magnus, ex quo erit

$$M=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\text{etc}=\infty.$$

Hinc ob $\frac{1}{M}=\frac{1}{\infty}=0$ fiet

$$0=1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{10}-\frac{1}{11}-\frac{1}{13}+\frac{1}{14}+\frac{1}{15}-\text{etc.},$$

Tum vero in productis habebitur

$$M=\infty=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{ etc.}}$$

unde fit

$$\infty=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\text{etc.}$$

et

$$0=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{10}{11}\cdot\frac{12}{13}\cdot\frac{16}{17}\cdot\frac{18}{19}\cdot\text{etc.}$$

Deinde per summationem serierum supra [§ 167] traditam erit

$$N=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\text{etc.}=\frac{\pi\pi}{6}.$$

Hinc obtinentur istae summae serierum:

$$\frac{6}{\pi\pi}=1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}+\frac{1}{10^2}-\frac{1}{11^2}-\text{etc.},$$

$$\infty=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\frac{1}{11}+\text{etc.},$$

$$0=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\text{etc.}$$

Denique pro factoribus orietur

$$\frac{\pi\pi}{6}=\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdot\text{etc.}$$

seu

$$\frac{\pi\pi}{6}=\frac{4}{3}\cdot\frac{9}{8}\cdot\frac{25}{24}\cdot\frac{49}{48}\cdot\frac{121}{120}\cdot\frac{169}{168}\cdot\text{etc.}$$

et ob $\frac{M}{N}=\infty$ seu $\frac{N}{M}=0$ habebitur

$$\infty=\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{12}{11}\cdot\frac{14}{13}\cdot\frac{18}{17}\cdot\frac{20}{19}\cdot\text{etc.}$$

seu

$$0=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{20}\cdot\text{etc.},$$

atque

$$\infty=\frac{3}{1}\cdot\frac{4}{2}\cdot\frac{6}{4}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{14}{12}\cdot\frac{18}{16}\cdot\frac{20}{18}\cdot\text{etc.},$$

seu

$$0=\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{9}{10}\cdot\text{etc.},$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus, summae autem ex numeratoribus et denominatoribus cuiusque fractionis constanter praebent numeros primos 3, 5, 7, 11, 13, 17, 19 etc.

EXAMPLUM 2

Sit $n=2$ eritque ex superioribus

$$M=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\text{etc.}=\frac{\pi\pi}{6},$$

$$N=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4}+\text{etc.}=\frac{\pi^4}{90}.$$

Hinc primo istae series summantur:

$$\frac{6}{\pi\pi}=1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{10^2}-\frac{1}{11^2}-\text{etc.},$$

$$\frac{90}{\pi^4}=1-\frac{1}{2^4}-\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{6^4}-\frac{1}{7^4}+\frac{1}{10^4}-\frac{1}{11^4}\text{etc.},$$

$$\frac{15}{\pi^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{10^2}+\frac{1}{11^2}+\text{etc.},$$

$$\frac{\pi\pi}{15}=1-\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}-\text{etc.}$$

Deinde valores sequentium productorum innotescunt:

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot\frac{7^2}{7^2-1}\cdot\frac{11^2}{11^2-1}\cdot\text{etc.},$$
$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1}\cdot\frac{3^4}{3^4-1}\cdot\frac{5^4}{5^4-1}\cdot\frac{7^4}{7^4-1}\cdot\frac{11^4}{11^4-1}\cdot\text{etc.},$$
$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2}\cdot\frac{3^2+1}{3^2}\cdot\frac{5^2+1}{5^2}\cdot\frac{7^2+1}{7^2}\cdot\frac{11^2+1}{11^2}\cdot\text{etc.},$$

seu

$$\frac{\pi\pi}{15} = \frac{4}{5}\cdot\frac{9}{10}\cdot\frac{25}{26}\cdot\frac{49}{50}\cdot\frac{121}{122}\cdot\frac{169}{170}\cdot\text{etc.}$$

et

$$\frac{5}{2} = \frac{2^2+1}{2^2-1}\cdot\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\cdot\frac{7^2+1}{7^2-1}\cdot\frac{11^2+1}{11^2-1}\cdot\text{etc.}$$

sive

$$\frac{5}{2} = \frac{5}{3}\cdot\frac{5}{4}\cdot\frac{13}{12}\cdot\frac{25}{24}\cdot\frac{61}{60}\cdot\frac{85}{84}\cdot\text{etc.}$$

vel

$$\frac{3}{2} = \frac{5}{4}\cdot\frac{13}{12}\cdot\frac{25}{24}\cdot\frac{61}{60}\cdot\frac{85}{84}\cdot\text{etc.}$$

In his fractionibus numeratores unitate superant denominatores, simul vero sumpti praebent quadrata numerorum primorum $3^2, 5^2, 7^2, 11^2$ etc.

EXEMPLUM 3

Quia ex superioribus valores ipsius M tantum, si n sit numerus par, assignare licet, ponamus $n = 4$ eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} = \frac{\pi^4}{90},$$
$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \text{etc.} = \frac{\pi^8}{9450}.$$

Hinc primo sequentes series summantur:

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{etc.},$$
$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8} - \text{etc.},$$
$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4} + \text{etc.},$$
$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} + \frac{1}{9^4} + \text{etc.}$$

Deinde etiam valores sequentium productorum obtinentur:

$$\frac{\pi^4}{90} = \frac{2^4}{2^4-1}\cdot\frac{3^4}{3^4-1}\cdot\frac{5^4}{5^4-1}\cdot\frac{7^4}{7^4-1}\cdot\frac{11^4}{11^4-1}\cdot\text{etc.},$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8-1}\cdot\frac{3^8}{3^8-1}\cdot\frac{5^8}{5^8-1}\cdot\frac{7^8}{7^8-1}\cdot\frac{11^8}{11^8-1}\cdot\text{etc.},$$

$$\frac{105}{\pi^4} = \frac{2^4+1}{2^4}\cdot\frac{3^4+1}{3^4}\cdot\frac{5^4+1}{5^4}\cdot\frac{7^4+1}{7^4}\cdot\frac{11^4+1}{11^4}\cdot\text{etc.}$$

et

$$\frac{7}{6} = \frac{2^4+1}{2^4-1}\cdot\frac{3^4+1}{3^4-1}\cdot\frac{5^4+1}{5^4-1}\cdot\frac{7^4+1}{7^4-1}\cdot\frac{11^4+1}{11^4-1}\cdot\text{etc.}$$

seu

$$\frac{35}{34} = \frac{41}{40}\cdot\frac{313}{312}\cdot\frac{1201}{1200}\cdot\frac{7321}{7320}\cdot\text{etc.}$$

In his factoribus numeratores unitate superant denominatores, simul vero sumpti praebent biquadrata numerorum primorum imparium 3, 5, 7, 11 etc.

278. Quoniam hic summam seriei

$$M = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.}$$

ad factores reduximus, ad logarithmos commode progredi licebit. Nam cum sit

$$M = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right)\text{etc.}},$$

erit

$$lM = -l\left(1-\frac{1}{2^n}\right)-l\left(1-\frac{1}{3^n}\right)-l\left(1-\frac{1}{5^n}\right)-l\left(1-\frac{1}{7^n}\right)-l\left(1-\frac{1}{11^n}\right)-\text{etc.}$$

Hinc sumendis logarithmis hyperbolicis erit

$$\begin{aligned} lM = &+1\left(\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{11^n}+\text{etc.}\right)\\ &+\frac{1}{2}\left(\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{11^{2n}}+\text{etc.}\right)\\ &+\frac{1}{3}\left(\frac{1}{2^{3n}}+\frac{1}{3^{3n}}+\frac{1}{5^{3n}}+\frac{1}{7^{3n}}+\frac{1}{11^{3n}}+\text{etc.}\right)\\ &+\frac{1}{4}\left(\frac{1}{2^{4n}}+\frac{1}{3^{4n}}+\frac{1}{5^{4n}}+\frac{1}{7^{4n}}+\frac{1}{11^{4n}}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

Quodsi insuper ponamus

$$N = 1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{6^{2n}}+\text{etc.},$$

ut sit

$$N = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\left(1-\frac{1}{11^{2n}}\right)\text{ etc.}},$$

fiet logarithmis hyperbolicis sumendis

$$\begin{aligned} lN = &+1\left(\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{11^{2n}}+\text{etc.}\right)\\ &+\frac{1}{2}\left(\frac{1}{2^{4n}}+\frac{1}{3^{4n}}+\frac{1}{5^{4n}}+\frac{1}{7^{4n}}+\frac{1}{11^{4n}}+\text{etc.}\right)\\ &+\frac{1}{3}\left(\frac{1}{2^{6n}}+\frac{1}{3^{6n}}+\frac{1}{5^{6n}}+\frac{1}{7^{6n}}+\frac{1}{11^{6n}}+\text{etc.}\right)\\ &+\frac{1}{4}\left(\frac{1}{2^{8n}}+\frac{1}{3^{8n}}+\frac{1}{5^{8n}}+\frac{1}{7^{8n}}+\frac{1}{11^{8n}}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

Ex his coniunctis fiet

$$\begin{aligned} lM-\frac{1}{2}lN = &+1\left(\frac{1}{2^{n}}+\frac{1}{3^{n}}+\frac{1}{5^{n}}+\frac{1}{7^{n}}+\frac{1}{11^{n}}+\text{etc.}\right)\\ &+\frac{1}{3}\left(\frac{1}{2^{3n}}+\frac{1}{3^{3n}}+\frac{1}{5^{3n}}+\frac{1}{7^{3n}}+\frac{1}{11^{3n}}+\text{etc.}\right)\\ &+\frac{1}{5}\left(\frac{1}{2^{5n}}+\frac{1}{3^{5n}}+\frac{1}{5^{5n}}+\frac{1}{7^{5n}}+\frac{1}{11^{5n}}+\text{etc.}\right)\\ &+\frac{1}{7}\left(\frac{1}{2^{7n}}+\frac{1}{3^{7n}}+\frac{1}{5^{7n}}+\frac{1}{7^{7n}}+\frac{1}{11^{7n}}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

279. Si $n=1$, erit

$$M = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{etc.} = l\infty$$

et

$$N = \frac{\pi\pi}{6};$$

hincque erit

$$\begin{aligned} l.l\infty-\frac{1}{2}l\frac{\pi\pi}{6} = &+1\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\text{etc.}\right)\\ &+\frac{1}{3}\left(\frac{1}{2^{3}}+\frac{1}{3^{3}}+\frac{1}{5^{3}}+\frac{1}{7^{3}}+\frac{1}{11^{3}}+\text{etc.}\right)\\ &+\frac{1}{5}\left(\frac{1}{2^{5}}+\frac{1}{3^{5}}+\frac{1}{5^{5}}+\frac{1}{7^{5}}+\frac{1}{11^{5}}+\text{etc.}\right)\\ &+\frac{1}{7}\left(\frac{1}{2^{7}}+\frac{1}{3^{7}}+\frac{1}{5^{7}}+\frac{1}{7^{7}}+\frac{1}{11^{7}}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

Verum hae series praeter primam non solum summas habent finitas, sed etiam cunctae simul sumptae summam efficiunt finitam eamque satis parvam; unde necesse est, ut seriei primae

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\text{etc.}$$

summa sit infinite magna. Quantitate scilicet satis parva deficiet a logarithmo hyperbolico seriei

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.}$$

280. Sit $n=2$; erit

$$M=\frac{\pi\pi}{6} \quad \text{et} \quad N=\frac{\pi^4}{90},$$

unde fit

$$\begin{aligned}
2l\pi-l6=&+1\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\text{etc.}\right)\\
&+\frac{1}{2}\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{11^4}+\text{etc.}\right)\\
&+\frac{1}{3}\left(\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\text{etc.}\right)\\
&\text{etc.},
\end{aligned}$$

$$\begin{aligned}
4l\pi-l90=&+1\left(\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{11^4}+\text{etc.}\right)\\
&+\frac{1}{2}\left(\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{11^8}+\text{etc.}\right)\\
&+\frac{1}{3}\left(\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{5^{12}}+\frac{1}{7^{12}}+\frac{1}{11^{12}}+\text{etc.}\right)\\
&\text{etc.},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}l\frac{5}{2}=&+1\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\text{etc.}\right)\\
&+\frac{1}{3}\left(\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{11^6}+\text{etc.}\right)\\
&+\frac{1}{5}\left(\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\frac{1}{11^{10}}+\text{etc.}\right)\\
&\text{etc.}
\end{aligned}$$

281. Quanquam lex, qua numeri primi progrediuntur, non constat, tamen harum serierum altiorum potestatum summae non difficulter proxime assignari poterunt. Sit enim haec series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \text{etc.}$$

et

$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \text{etc.};$$

erit

$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \text{etc.}$$

et ob

$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \text{etc.}$$

erit

$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \text{etc.}$$

seu

$$S = (M-1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} - \frac{1}{27^n} - \text{etc.}$$

et ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \text{etc.}$$

erit

$$S = (M-1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{49^n} \ \text{etc.}$$

Hinc ob datam summam M [§ 168] valor ipsius S commode invenitur, siquidem n fuerit numerus mediocriter magnus.

282. Inventis autem summis altiorum potestatum etiam summae potestatum minorum ex formulis inventis exhiberi possunt. Atque hac methodo sequentes prodierunt summae seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \text{etc.}:$$

Si sit	erit summa seriei
$n = 2$	0,452247420041065
$n = 4$	0,076993139764247
$n = 6$	0,017070086850637
$n = 8$	0,004061405366518
$n = 10$	0,000993603574437
$n = 12$	0,000246026470035
$n = 14$	0,000061244396725
$n = 16$	0,000015282026219
$n = 18$	0,000003817278703
$n = 20$	0,000000953961124
$n = 22$	0,000000238450446

$n=24$	0,000000059608185
$n=26$	0,000000014901555
$n=28$	0,000000003725334
$n=30$	0,000000000931327
$n=32$	0,000000000232831
$n=34$	0,000000000058208
$n=36$	0,000000000014552

Reliquae summae parium potestatum in ratione quadrupla decrescunt.

283. Haec autem seriei

$$1+\tfrac{1}{2^n}+\tfrac{1}{3^n}+\tfrac{1}{4^n}+\text{etc.}$$

in productum infinitum conversio etiam directe institui potest hoc modo. Sit

$$A=1+\tfrac{1}{2^n}+\tfrac{1}{3^n}+\tfrac{1}{4^n}+\tfrac{1}{5^n}+\tfrac{1}{6^n}+\tfrac{1}{7^n}+\tfrac{1}{8^n}+\text{etc.};$$

subtrahe

$$\tfrac{1}{2^n}A=\tfrac{1}{2^n}+\tfrac{1}{4^n}+\tfrac{1}{6^n}+\tfrac{1}{8^n}+\text{etc.};$$

erit

$$\left(1-\tfrac{1}{2^n}\right)A=1+\tfrac{1}{3^n}+\tfrac{1}{5^n}+\tfrac{1}{7^n}+\tfrac{1}{9^n}+\tfrac{1}{11^n}+\text{etc.}=B.$$

Sic sublati sunt omnes termini per 2 divisibiles. Subtrahe

$$\tfrac{1}{3^n}B=\tfrac{1}{3^n}+\tfrac{1}{9^n}+\tfrac{1}{15^n}+\tfrac{1}{21^n}+\text{etc.};$$

erit

$$\left(1-\tfrac{1}{3^n}\right)B=1+\tfrac{1}{5^n}+\tfrac{1}{7^n}+\tfrac{1}{11^n}+\tfrac{1}{13^n}+\text{etc.}=C.$$

Sic insuper sublati sunt omnes termini per 3 divisibiles. Subtrahe

$$\tfrac{1}{5^n}B=\tfrac{1}{5^n}+\tfrac{1}{25^n}+\tfrac{1}{35^n}+\tfrac{1}{55^n}+\text{etc.};$$

erit

$$\left(1-\tfrac{1}{5^n}\right)C=1+\tfrac{1}{7^n}+\tfrac{1}{11^n}+\tfrac{1}{13^n}+\tfrac{1}{17^n}+\text{etc.}$$

Sic sublati etiam sunt omnes termini per 5 divisibiles. Pari modo tolluntur termini divisibiles per 7, 11 reliquosque numeros primos; manifestum autem est sublatis omnibus terminis, qui per numeros primos divisibiles sint, solam unitatem relinqui. Quare pro *B*, *C, D, E* etc. valoribus restitutis tandem orietur

$$A\left(1-\tfrac{1}{2^n}\right)\left(1-\tfrac{1}{3^n}\right)\left(1-\tfrac{1}{5^n}\right)\left(1-\tfrac{1}{7^n}\right)\left(1-\tfrac{1}{11^n}\right) \text{ etc.} = 1,$$

unde seriei propositae summa erit

$$A = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\left(1-\frac{1}{11^n}\right) \text{ etc.}}$$

seu

$$A = \frac{2^n}{2^n-1}\cdot\frac{3^n}{3^n-1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n-1}\cdot\frac{11^n}{11^n-1}\cdot \text{ etc.}$$

284. Haec methodus iam commode adhiberi poterit ad alias series, quarum summas supra invenimus, in producta infinita convertendas. Invenimus autem supra (§ 175) summas harum serierum

$$1-\tfrac{1}{3^n}+\tfrac{1}{5^n}-\tfrac{1}{7^n}+\tfrac{1}{9^n}-\tfrac{1}{11^n}+\tfrac{1}{13^n}-\text{etc.},$$

si *n* fuerit numerus impar. Summa enim est $= N\pi^n$ et valores ipsius *N* loco citato dedimus. Notandum autem est, cum hic tantum numeri impares occurrunt, eos, qui sint formae $4m+1$, habere signum +, reliquos formae $4m-1$ signum –. Sit igitur

$$A = 1-\tfrac{1}{3^n}+\tfrac{1}{5^n}-\tfrac{1}{7^n}+\tfrac{1}{9^n}-\tfrac{1}{11^n}+\tfrac{1}{13^n}-\tfrac{1}{15^n}+\text{ etc.}$$

Addatur

$$\tfrac{1}{3^n}A = \tfrac{1}{3^n}-\tfrac{1}{9^n}+\tfrac{1}{15^n}-\tfrac{1}{21^n}+\tfrac{1}{27^n}-\text{ etc.};$$

erit

$$\left(1+\tfrac{1}{3^n}\right)A = 1+\tfrac{1}{5^n}-\tfrac{1}{7^n}-\tfrac{1}{11^n}+\tfrac{1}{13^n}+\tfrac{1}{17^n}-\text{etc.} = B.$$

Subtrahatur

$$\tfrac{1}{5^n}B = \tfrac{1}{5^n}+\tfrac{1}{25^n}-\tfrac{1}{35^n}-\tfrac{1}{55^n}+\text{ etc.};$$

erit

$$\left(1-\tfrac{1}{5^n}\right)B = 1-\tfrac{1}{7^n}-\tfrac{1}{11^n}+\tfrac{1}{13^n}+\tfrac{1}{17^n}-\text{etc.} = C,$$

ubi iam numeri per 3 et 5 divisibiles desunt. Addatur

$$\tfrac{1}{7^n}C = \tfrac{1}{7^n}-\tfrac{1}{49^n}-\tfrac{1}{77^n}+\text{ etc.};$$

erit

$$\left(1+\tfrac{1}{7^n}\right)C = 1-\tfrac{1}{11^n}+\tfrac{1}{13^n}+\tfrac{1}{17^n}-\text{etc.} = D,$$

Sic etiam numeri per 7 divisibiles sunt sublati. Addatur

$$\tfrac{1}{11^n}D = \tfrac{1}{11^n}-\tfrac{1}{121^n}+\text{etc.};$$

erit

$$\left(1+\tfrac{1}{11^n}\right)D = 1+\tfrac{1}{13^n}+\tfrac{1}{17^n}-\text{etc.} = E.$$

Sic numeri per 11 divisibiles quoque sunt sublati. Auferendis autem hoc modo reliquis numeris omnibus per reliquos numeros primos divisibilibus tandem prodibit

$$A\left(1+\tfrac{1}{3^n}\right)\left(1-\tfrac{1}{5^n}\right)\left(1+\tfrac{1}{7^n}\right)\left(1+\tfrac{1}{11^n}\right)\left(1-\tfrac{1}{13^n}\right)\ \text{etc.} = 1$$

seu

$$A = \tfrac{3^n}{3^n+1}\cdot\tfrac{5^n}{5^n-1}\cdot\tfrac{7^n}{7^n+1}\cdot\tfrac{11^n}{11^n+1}\cdot\tfrac{13^n}{13^n-1}\cdot\tfrac{17^n}{17^n-1}\cdot\text{etc.},$$

ubi in numeratoribus occurrunt potestates omnium numerorum primorum, quae in denominatoribus insunt unitate sive auctae sive minutae, prout numeri primi fuerint formae $4m-1$ vel $4m+1$.

285. Posito ergo $n=1$ ob $A=\frac{\pi}{4}$ erit

$$\tfrac{\pi}{4} = \tfrac{3}{4}\cdot\tfrac{5}{4}\cdot\tfrac{7}{8}\cdot\tfrac{11}{12}\cdot\tfrac{13}{12}\cdot\tfrac{17}{16}\cdot\tfrac{19}{20}\cdot\tfrac{23}{24}\cdot\text{etc.}$$

Supra autem invenimus esse

$$\tfrac{\pi\pi}{6} = \tfrac{4}{3}\cdot\tfrac{3^2}{2\cdot4}\cdot\tfrac{5^2}{4\cdot6}\cdot\tfrac{7^2}{6\cdot8}\cdot\tfrac{11^2}{10\cdot12}\cdot\tfrac{13^2}{12\cdot14}\cdot\tfrac{17^2}{16\cdot18}\cdot\tfrac{19^2}{18\cdot20}\cdot\text{etc.}$$

Dividatur secunda per primam et orietur

$$\tfrac{2\pi}{3} = \tfrac{4}{3}\cdot\tfrac{3}{2}\cdot\tfrac{5}{6}\cdot\tfrac{7}{6}\cdot\tfrac{11}{10}\cdot\tfrac{13}{14}\cdot\tfrac{17}{18}\cdot\tfrac{19}{18}\cdot\tfrac{23}{22}\cdot\text{etc.}$$

seu

$$\tfrac{\pi}{2} = \tfrac{3}{2}\cdot\tfrac{5}{6}\cdot\tfrac{7}{6}\cdot\tfrac{11}{10}\cdot\tfrac{13}{14}\cdot\tfrac{17}{18}\cdot\tfrac{19}{18}\cdot\tfrac{23}{22}\cdot\text{etc.},$$

ubi numeri primi constituunt numeratores, denominatores vero sunt numeri impariter pares unitate differentes a numeratoribus. Quodsi haec denuo per primam $\frac{\pi}{4}$ dividatur, erit

$$2 = \frac{4}{2}\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{16}{18}\cdot\frac{20}{18}\cdot\frac{24}{22}\cdot\text{etc.}$$

seu

$$2 = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdot\text{etc.},$$

quae fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17 etc. quemque in duas partes unitate differentes dispescendo et partes pares pro numeratoribus, impares pro denominatoribus sumendo.

286. Si hae expressiones cum WALLISIANA comparentur

$$\frac{\pi}{2} = \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot11}\ \text{etc.}$$

seu

$$\frac{4}{\pi} = \frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{9\cdot9}{8\cdot10}\cdot\frac{11\cdot11}{10\cdot12}\cdot\text{etc.},$$

cum sit

$$\frac{\pi\pi}{8} = \frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{10\cdot12}\cdot\frac{13\cdot13}{12\cdot14}\cdot\text{etc.},$$

illa per hanc divisa dabit

$$\frac{32}{\pi^3} = \frac{9\cdot9}{8\cdot10}\cdot\frac{15\cdot15}{14\cdot16}\cdot\frac{21\cdot21}{20\cdot22}\cdot\frac{25\cdot25}{24\cdot26}\cdot\text{etc.},$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit iam $n=3$; erit $A=\frac{\pi^3}{32}$, unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3+1}\cdot\frac{5^3}{5^3-1}\cdot\frac{7^3}{7^3+1}\cdot\frac{11^3}{11^3+1}\cdot\frac{13^3}{13^3-1}\cdot\frac{17^3}{17^3-1}\cdot\text{etc.}$$

At ex serie

$$\frac{\pi^6}{945} = 1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}$$

fit

$$\frac{\pi^6}{945} = \frac{2^6}{2^6-1}\cdot\frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot\frac{13^6}{13^6-1}\cdot\text{etc.}$$

seu

$$\frac{\pi^6}{960} = \frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot\frac{13^6}{13^6-1}\cdot\text{etc.},$$

quae per primam divisa dabit

$$\frac{\pi^3}{30}=\frac{3^3}{3^3-1}\cdot\frac{5^3}{5^3+1}\cdot\frac{7^3}{7^3-1}\cdot\frac{11^3}{11^3-1}\cdot\frac{13^3}{13^3+1}\cdot\frac{17^3}{17^3+1}\cdot\text{etc.}$$

Haec vero denuo per primam divisa dabit

$$\frac{16}{15}=\frac{3^3+1}{3^3-1}\cdot\frac{5^3-1}{5^3+1}\cdot\frac{7^3+1}{7^3-1}\cdot\frac{11^3+1}{11^3-1}\cdot\frac{13^3-1}{13^3+1}\cdot\frac{17^3-1}{17^3+1}\cdot\text{etc.}$$

seu

$$\frac{16}{15}=\frac{14}{13}\cdot\frac{62}{63}\cdot\frac{172}{171}\cdot\frac{666}{665}\cdot\frac{1098}{1099}\cdot\text{etc.},$$

quae fractiones formantur ex cubis numerorum primorum imparium quemque in duas partes unitate differentes dispescendo ac partes pares pro numeratoribus, impares pro denominatoribus sumendo.

288. Ex his expressionibus denuo novae series formari possunt, in quibus omnes numeri naturales denominatores constituunt. Cum enim sit

$$\frac{\pi}{4}=\frac{3}{3+1}\cdot\frac{5}{5-1}\cdot\frac{7}{7+1}\cdot\frac{11}{11+1}\cdot\frac{13}{13-1}\cdot\text{etc.}$$

erit

$$\frac{\pi}{6}=\frac{1}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem haec series nascetur

$$\frac{\pi}{6}=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\text{etc.},$$

ubi ratio signorum ita est comparata, ut binarius habeat –, numeri primi formae $4m-1$ signum – et numeri primi formae $4m+1$ signum + ; numeri autem compositi ea habent signa, quae ipsis ratione multiplicationis ex primis conveniunt. Sic patebit signum fractionis $\frac{1}{60}$ ob

$$60=\overset{-}{2}\cdot\overset{-}{2}\cdot\overset{-}{2}\cdot\overset{+}{5},$$

quod erit – .
Simili modo porro erit

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\text{ etc.}},$$

unde orietur haec series

$$\frac{\pi}{2}=1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\text{etc.},$$

ubi binarius habet signum + , numeri primi formae $4m-1$ signum – , numeri primi formae $4m+1$ signum + ; et numerus quisque compositus id habet signum, quod ipsi ratione compositionis ex primis convenit secundum regulas multiplicationis.

289. Cum deinde sit

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

erit per evolutionem

$$\frac{\pi}{2}=1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}-\text{etc.},$$

ubi tantum numeri impares occurrunt, signa autem ita sunt comparata, ut numeri primi formae $4m-1$ signum habeant +, numeri primi formae $4m+1$ signum –, unde simul numerorum compositorum signa definiuntur.

Binae porro series hinc formari possunt, ubi omnes numeri occurrunt.
Erit scilicet

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\pi=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\text{etc.},$$

ubi binarius signum habet + , numeri primi formae $4m-1$ signum +, numeri vero primi formae $4m+1$ signum –.

Tum vero etiam erit

$$\frac{\pi}{3}=\frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\frac{\pi}{3}=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\text{etc.},$$

ubi binarius habet signum –, numeri primi formae $4m-1$ signum + et numeri primi formae $4m+1$ signum –.

290. Possunt hinc etiam innumerabiles aliae signorum conditiones exhiberi, ita ut seriei

$$1,\quad \tfrac{1}{2},\quad \tfrac{1}{3},\quad \tfrac{1}{4},\quad \tfrac{1}{5},\quad \tfrac{1}{6},\quad \tfrac{1}{7},\quad \tfrac{1}{8}\text{ etc.}$$

summa assignari queat. Cum scilicet sit

$$\frac{\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\text{ etc.}},$$

multliplicetur haec expressio per $\frac{1+\frac{1}{3}}{1-\frac{1}{3}}=2$; erit

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\text{ etc.}}$$

et

$$\pi=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}\text{ etc.},$$

ubi binarius signum habet + , ternarius + , reliqui numeri primi omnes formae $4m-1$ signum –, at numeri primi formae $4m+1$ signum + ; unde pro numeris compositis ratio signorum intelligitur.
Simili modo cum sit

$$\pi=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\text{ etc.}},$$

multiplicetur per $\frac{1+\frac{1}{5}}{1-\frac{1}{5}}=\frac{3}{2}$; erit

$$\frac{3\pi}{2}=\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\left(1+\frac{1}{17}\right)\text{ etc.}},$$

unde per evolutionem oritur

$$\frac{3\pi}{2}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}-\frac{1}{13}+\text{ etc.},$$

ubi binarius habet signum + , numeri primi formae $4m-1$ signum + et
numeri – primi formae $4m+1$ praeter quinarium signum

291. Possunt etiam innumerabiles huiusmodi series exhiberi, quarum summa sit $=0$.
Cum enim sit

$$0=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\text{etc.},$$

erit

$$0=\frac{1}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}}$$

unde, ut supra vidimus, oritur

$$0=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\text{etc.},$$

ubi omnes numeri primi signum habent – compositorumque numerorum signa regulam multiplicationis sequuntur.

Multiplicemus autem illam expressionem per $\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=3$; erit pariter

$$0=\frac{1}{\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}}$$

unde per evolutionem nascitur

$$0=1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\text{etc.},$$

ubi binarius habet signum + , reliqui numeri primi omnes signum –.

Simili modo quoque erit

$$0=\frac{1}{\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{13}\right)\text{ etc.}},$$

unde oritur ista series

$$0=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\text{etc.},$$

ubi omnes numeri primi praeter 3 et 5 habent signum – .

In genere autem notandum est, quoties omnes numeri primi exceptis tantum aliquibus habeant signum –, summa seriei fore $=0$, contra autem, quoties omnes numeri primi exceptis tantum aliquibus habeant signum +, tum summam seriei fore infinite magnam.

292. Supra etiam (§ 176) summam dedimus seriei

$$A=1-\frac{1}{2^n}+\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{7^n}-\frac{1}{8^n}+\frac{1}{10^n}-\frac{1}{11^n}+\frac{1}{13^n}-\text{etc.},$$

si fuerit n numerus impar. Erit ergo

$$\frac{1}{2^n}A=\frac{1}{2^n}-\frac{1}{4^n}+\frac{1}{8^n}-\frac{1}{10^n}+\frac{1}{14^n}-\text{etc.},$$

quae addita dat

$$B=\left(1+\tfrac{1}{2^n}\right)A=1-\tfrac{1}{5^n}+\tfrac{1}{7^n}-\tfrac{1}{11^n}+\tfrac{1}{13^n}-\tfrac{1}{17^n}+\tfrac{1}{19^n}-\tfrac{1}{23^n}+\tfrac{1}{25^n}-\text{etc.},$$

Addatur

$$\tfrac{1}{5^n}B=\tfrac{1}{5^n}-\tfrac{1}{25^n}+\tfrac{1}{35^n}-\tfrac{1}{55^n}+\text{etc.};$$

erit

$$C=\left(1+\tfrac{1}{5^n}\right)B=1+\tfrac{1}{7^n}-\tfrac{1}{11^n}+\tfrac{1}{13^n}-\tfrac{1}{17^n}+\tfrac{1}{19^n}-\tfrac{1}{23^n}-\text{etc.}$$

Subtrahatur

$$\tfrac{1}{7^n}C=\tfrac{1}{7^n}+\tfrac{1}{49^n}-\tfrac{1}{77^n}+\text{etc.};$$

erit

$$D=\left(1-\tfrac{1}{7^n}\right)C=1-\tfrac{1}{11^n}+\tfrac{1}{13^n}-\tfrac{1}{17^n}+\tfrac{1}{19^n}-\text{etc.}$$

Ex his tandem fiet

$$A\left(1+\tfrac{1}{2^n}\right)\left(1+\tfrac{1}{5^n}\right)\left(1-\tfrac{1}{7^n}\right)\left(1+\tfrac{1}{11^n}\right)\left(1-\tfrac{1}{13^n}\right)\text{ etc.}=1,$$

ubi numeri primi unitate excedentes multipla senarii habent signum –, deficientes autem signum +. Eritque

$$A=\tfrac{2^n}{2^n+1}\cdot\tfrac{5^n}{5^n+1}\cdot\tfrac{7^n}{7^n-1}\cdot\tfrac{11^n}{11^n+1}\cdot\tfrac{13^n}{13^n-1}\cdot\text{etc.}$$

293. Consideremus casum $n=1$, quo $A=\frac{\pi}{3\sqrt{3}}$, eritque

$$\tfrac{\pi}{3\sqrt{3}}=\tfrac{2}{3}\cdot\tfrac{5}{6}\cdot\tfrac{7}{6}\cdot\tfrac{11}{12}\cdot\tfrac{13}{12}\cdot\tfrac{17}{18}\cdot\tfrac{19}{18}\cdot\text{etc.},$$

ubi in numeratoribus post 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate discrepant suntque omnes per 6 divisibiles.
Cum iam sit

$$\tfrac{\pi\pi}{6}=\tfrac{4}{3}\cdot\tfrac{9}{8}\cdot\tfrac{5\cdot5}{4\cdot6}\cdot\tfrac{7\cdot7}{6\cdot8}\cdot\tfrac{11\cdot11}{12\cdot10}\cdot\tfrac{13\cdot13}{12\cdot14}\cdot\text{etc.},$$

erit hac expressione per illam divisa

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\text{etc.},$$

ubi denominatores non sunt per 6 divisibiles. Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{24}\cdot\text{etc.},$$
$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot\text{etc.},$$

quarum haec per illam divisa dat

$$\frac{4}{3} = \frac{6}{4}\cdot\frac{6}{8}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\frac{18}{16}\cdot\frac{18}{20}\cdot\frac{24}{22}\cdot\text{etc.}$$

seu

$$\frac{4}{3} = \frac{3}{2}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{9}{10}\cdot\frac{12}{11}\cdot\text{etc.},$$

ubi singulae fractiones ex numeris primis 5, 7, 11 etc. formantur singulos numeros primos in duas partes unitate differentes dispescendo et partes per 3 divisibiles constanter pro numeratoribus sumendo.

294. Quoniam vero supra vidimus esse

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\text{etc.}$$

seu

$$\frac{\pi}{3} = \frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\text{etc.},$$

si superiores $\frac{\pi}{2\sqrt{3}}$ et $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2} = \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{8}{9}\cdot\frac{10}{9}\cdot\frac{14}{15}\cdot\frac{16}{15}\cdot\text{etc.},$$
$$\frac{2}{\sqrt{3}} = \frac{6}{5}\cdot\frac{6}{7}\cdot\frac{12}{11}\cdot\frac{18}{19}\cdot\frac{24}{23}\cdot\frac{30}{29}\cdot\text{etc.}$$

In priori expressione fractiones formantur ex numeris primis formae $12m+6\pm1$, in posteriore ex numeris primis formae $12m\pm1$, singulos in duas partes unitate discrepantes dispescendo et partes pares pro numeratoribus, impares vero pro denominatoribus sumendo.

295. Contemplemur adhuc seriem supra (§ 179) inventam, quae ita progrediebatur

$$\frac{\pi}{2\sqrt{2}} = 1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\text{etc.} = A;$$

erit

$$\tfrac{1}{3}A = \tfrac{1}{3} + \tfrac{1}{9} - \tfrac{1}{15} - \tfrac{1}{21} + \tfrac{1}{27} + \tfrac{1}{33} - \text{etc.}$$

Subtrahatur

$$\left(1 - \tfrac{1}{3}\right)A = 1 - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} + \tfrac{1}{17} + \tfrac{1}{19} - \text{etc.} = B.$$

Addatur

$$\tfrac{1}{5}B = \tfrac{1}{5} - \tfrac{1}{25} - \tfrac{1}{35} + \tfrac{1}{55} - \text{etc.};$$

erit

$$\left(1 + \tfrac{1}{5}\right)B = 1 - \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} + \tfrac{1}{17} + \tfrac{1}{19} - \text{etc.} = C.$$

Sicque progrediendo tandem pervenietur ad

$$\tfrac{\pi}{2\sqrt{2}}\left(1 - \tfrac{1}{3}\right)\left(1 + \tfrac{1}{5}\right)\left(1 + \tfrac{1}{7}\right)\left(1 - \tfrac{1}{11}\right)\left(1 + \tfrac{1}{13}\right)\left(1 - \tfrac{1}{17}\right)\left(1 - \tfrac{1}{19}\right)\text{etc.} = 1,$$

ubi signa ita se habent, ut numerorum primorum formae $8m+1$ vel $8m+3$ signa sint –, numerorum primorum vero formae $8m+5$ vel $8m+7$ signa sint +. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{16}\cdot\frac{19}{18}\cdot\frac{23}{24}\cdot\text{etc.},$$

ubi omnes denominatores vel divisibiles sunt per 8 vel tantum sunt numeri impariter pares. Cum igitur sit

$$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\text{etc.},$$
$$\frac{\pi}{2} = \frac{3}{2}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{10}\cdot\frac{13}{14}\cdot\frac{17}{18}\cdot\frac{19}{18}\cdot\frac{23}{22}\cdot\text{etc.},$$

ergo

$$\frac{\pi\pi}{8} = \frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{10\cdot12}\cdot\frac{13\cdot13}{12\cdot14}\cdot\text{etc.},$$

erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot\text{etc.}$$

ubi nulli denominatores per 8 divisibiles occurrunt, pariter pares vero adsunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam divisa dat

$$1 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{10}{9} \cdot \frac{11}{12} \cdot \text{etc.},$$

quae fractiones formantur ex numeris primis singulos in duas partes unitate discrepantes dispescendo et partes pares (nisi sint pariter pares) pro numeratoribus sumendo.

296. Simili modo reliquae series, quas supra pro expressione arcuum circularium invenimus (§ 179 et sq.) in factores transformari possunt, qui ex numeris primis constituantur. Sicque multae aliae insignes proprietates tam huiusmodi factorum quam serierum infinitarum erui poterunt. Quoniam vero praecipuas hic iam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum scilicet in hoc capite numeri, quatenus per multiplicationem oriuntur, sunt considerati, ita in sequenti generatio numerorum per additionem perpendetur.