VARIAE SPECVLATIONES SVPER AREA TRIANGVLORVM SPHAERICORVM

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§ 1. Primus, qui aream trianguli sphaerici definire docuit, erat, teste Wallisio, Albertus Girard, qui demonstrauit, aream trianguli sphaerici semper proportionalem esse excessui summae ternorum angulorum super duobus rectis, atque adeo ipsam aream inueniri, si iste excessus, in arcum circuli maximi conuersus, per radium sphaerae multiplicetur. Quemadmodum autem area trianguli sphaerici ex eius lateribus sit determinanda, inuestigationem multo difficiliorem postulat. Inueni autem iam olim egregium theorema, quo ista determinatio facile institui potest, quo ita se habet: *Si latera trianguli sphaerici denotentur litteris a, b, c, area vero eiusdem trianguli ponatur $= \Delta$, tum semper erit*

$$\cos\frac{1}{2}\Delta = \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c},$$

cuius veritas non nisi per longas ambages, siue ex theoremate Girardi, siue immediate per calculum integralem ostendi potest. Vtramque igitur demonstrationem hic in medium attulisse operae erit pretium.

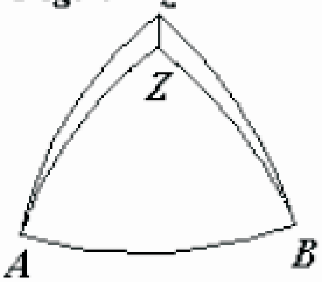
Problema.

Si trianguli sphaerici AZB super basi AB exstructi bina latera AZ et BZ suis differentialibus augeantur, vt inde oriatur triangulum AzB, inuestigare augmentum quod hinc areae trianguli AZB accessit.

Solutio.

§ 2. Posita basi huius trianguli $AB = a$ (vide Tab. I, Fig. 4), vocentur eius latera $AZ = x$ et $BZ = y$, ita vt latera trianguli aucti futura sint $Az = x + \partial x$ et $Bz = y + \partial y$. Porro vero vocentur anguli $BAZ = \phi$ et $ABZ = \psi$, vt prodeant anguli elementares $ZAz = \partial\phi$ et $ZBz = \partial\psi$, quibus positis constat trianguli elementaris Zaz aream esse $= \partial\phi(1 - \cos x)$,

Fig. 4



trianguli vero $= \partial\psi(1-\cos y)$. Quoniam igitur haec duo triangula elementaria exhibibent augmentum areae trianguli Δ, habebimus hanc aequationem:

$$\partial\Delta = \partial\phi(1-\cos x) + \partial\psi(1-\cos y).$$

§ 3. Nunc igitur angulos ϕ et ψ ex calculo eliminemus, eorumque loco ipsa latera x et y introducamus ope praeceptorum Trigonometricorum, quae nobis praebent

$$\cos\phi = \frac{\cos y - \cos a\cos x}{\sin a\sin x} \qquad \text{et} \qquad \cos\psi = \frac{\cos x - \cos a\cos y}{\sin a\sin y},$$

hinc igitur per differentiationem colligimus

$$-\partial\phi\sin\phi = \frac{\partial x\cos a - \partial x\cos x\cos y - \partial y\sin x\sin y}{\sin a\sin^2 x},$$

eodemque modo erit

$$-\partial\psi\sin\psi = \frac{\partial y\cos a - \partial y\cos x\cos y - \partial x\sin x\sin y}{\sin a\sin^2 y}.$$

At vero cum sit $\cos\phi = \frac{\cos y - \cos a\cos x}{\sin a\sin x}$, erit

$$\sin\phi = \frac{\sqrt{1-\cos^2 a - \cos^2 x - \cos^2 y + 2\cos a\cos x\cos y}}{\sin a\sin x},$$

simileque modo erit

$$\sin\psi = \frac{\sqrt{1-\cos^2 a - \cos^2 x - \cos^2 y + 2\cos a\cos x\cos y}}{\sin a\sin y}.$$

Quoniam hae ambae formulae radicales sunt eadem, ponamus breuitatis gratia

$$\sqrt{1-\cos^2 a - \cos^2 x - \cos^2 y + 2\cos a\cos x\cos y} = v,$$

vt habeamus

$$\sin\phi = \frac{v}{\sin a \sin x} \qquad \text{et} \qquad \sin\psi = \frac{v}{\sin a \sin y}.$$

§ 4. His igitur valoribus substitutis nanciscemur hos valores differentiales:

$$\begin{aligned} \partial\phi &= -\frac{\partial x \cos a + \partial x \cos x \cos y + \partial y \sin x \sin y}{v \sin x} \quad \text{et} \\ \partial\psi &= -\frac{\partial y \cos a + \partial y \cos x \cos y + \partial x \sin x \sin y}{v \sin y}. \end{aligned}$$

Hinc igitur incrementum areae quaesitum erit

$$\partial\Delta = \frac{\left\{ \begin{array}{c} -\cos a[\partial x \sin y(1-\cos x) + \partial y \sin x(1-\cos y)] \\ +\partial x \sin y[\cos x \cos y(1-\cos x) + \sin^2 x(1-\cos y)] \\ +\partial y \sin x[\cos x \cos y(1-\cos y) + \sin^2 y(1-\cos x)] \end{array} \right\}}{v \sin x \sin y},$$

quod euolutum induet hanc formam:

$$v\partial\Delta = \left\{ \begin{array}{l} +\partial x \sin x(1-\cos y) + \frac{\partial x \cos x \cos y(1-\cos x)}{\sin x} - \frac{\partial x \cos a(1-\cos x)}{\sin x} \\ +\partial y \sin y(1-\cos x) + \frac{\partial y \cos x \cos y(1-\cos y)}{\sin y} - \frac{\partial y \cos a(1-\cos y)}{\sin y} \end{array} \right\}.$$

Hic iam notetur esse

$$\frac{1-\cos x}{\sin x} = \tan\frac{1}{2}x \qquad \text{et} \qquad \frac{1-\cos y}{\sin y} = \tan\frac{1}{2}y,$$

hinc ergo termini elementum ∂x inuoluentes erunt

$$\partial x \sin x(1-\cos y) + \partial x \cos x \cos y \tan\frac{1}{2}x - \partial x \cos a \tan\frac{1}{2}x.$$

Quoniam autem non solum est $\tan\frac{1}{2}x = \frac{1-\cos x}{\sin x}$, sed etiam $\tan\frac{1}{2}x = \frac{\sin x}{1+\cos x}$, in primo membro loco $\sin x$ scribatur $(1+\cos x)\tan\frac{1}{2}x$, vt ∂x vbique multiplicatum sit per $\tan\frac{1}{2}x$, sicque istud membrum reducetur ad hanc formam:

$$\partial x \tan\frac{1}{2}x(1+\cos x - \cos y - \cos a).$$

Eodem modo alterum membrum erit

$$\partial y \tan\frac{1}{2}y(1+\cos y - \cos x - \cos a),$$

sicque tota nostra aequatio ita erit expressa:

$$v\partial\Delta = \partial x \tan\frac{1}{2}x(1+\cos x - \cos y - \cos a) + \partial y \tan\frac{1}{2}y(1+\cos y - \cos x - \cos a).$$

§ 5. Quodsi iam breutatis gratia ponamus $\cos a + \cos x + \cos y = s$, erit

$$v\partial\Delta = \partial x \tan\frac{1}{2}x(1 - s + 2\cos x) + \partial y \tan\frac{1}{2}y(1 - s + 2\cos y),$$

quae aequatio hoc modo repraesentari potest:

$$v\partial\Delta = (1-s)(\partial x \tan\frac{1}{2}x + \partial y \tan\frac{1}{2}y) + 2\partial x \cos x \tan\frac{1}{2}x + 2\partial y \cos y \tan\frac{1}{2}y.$$

Cum nunc sit $\tan\frac{1}{2}x = \frac{1-\cos x}{\sin x}$, erit

$$\tan\frac{1}{2}x\cos x = \frac{\cos x - \cos^2 x}{\sin x} = \frac{\cos x - 1 + \sin^2 x}{\sin x} = \sin x - \tan\frac{1}{2}x.$$

Eodem modo erit $\tan\frac{1}{2}y\cos y = \sin y - \tan\frac{1}{2}y$, hisque valoribus substitutis orietur haec aequatio:

$$v\partial\Delta = -(1+s)(\partial x \tan\frac{1}{2}x + \partial y \tan\frac{1}{2}y) + 2\partial x \sin x + 2\partial y \sin y.$$

§ 6. Haec postrema forma ideo notatu maxime est digna, quod membrum dextrum absolute sit integrabile, si diuidatur per $1+s$. Facta enim hac diuisione nostra aequatio erit

$$\frac{v\partial\Delta}{(1+s)} = -\partial x \tan\frac{1}{2}x - \partial y \tan\frac{1}{2}y + \frac{2\partial x \sin x + 2\partial y \sin y}{1+\cos a + \cos x + \cos y},$$

vbi notetur esse $\int \partial x \tan\frac{1}{2}x = -2\log(\cos\frac{1}{2}x)$, similique modo $\int \partial x \tan\frac{1}{2}y = -2\log(\cos\frac{1}{2}y)$, ac denique

$$2\int \frac{\partial x \sin x + \partial y \sin y}{1+\cos a + \cos x + \cos y} = -2\log(1+\cos a + \cos x + \cos y) = -2\log(1+s),$$

sic igitur per integrationem reperimus

$$\int \frac{v\partial\Delta}{(1+s)} = 2\log(\cos\frac{1}{2}x) + 2\log(\cos\frac{1}{2}y) - 2\log(1+s) = 2\log\Big(\frac{\cos\frac{1}{2}x\cos\frac{1}{2}y}{1+s}\Big),$$

at vero hoc modo membrum sinistrum non est integrabile, cui ergo sequenti modo remedium afferetur.

§ 7. Cum enim posuerimus $s = \cos a + \cos x + \cos y$, statuamus insuper $\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y = q$, vt habeamus hanc aequationem:

$$\int \frac{v\partial\Delta}{1+s} = 2\log(\frac{q}{(1+s)\cos\frac{1}{2}a}),$$

vbi porro fiat $\frac{q}{1+s} = p$, ita vt sit

$$\int \frac{v\partial\Delta}{1+s} = 2\log(\frac{p}{\cos\frac{1}{2}a}),$$

quae aequatio denuo differentiata praebet $\frac{v\partial\Delta}{1+s} = \frac{2\partial p}{p}$, vnde consicitur $\partial\Delta = \frac{2\partial p(1+s)}{pv}$, quae ergo formula integrationem admittet, si modo fuerit $\frac{v}{1+s}$ functio quaedam ipsus p, id quod iam certo asseuerare possumus, propterea quod $\partial\Delta$ designat differentiale ipsus areae trianguli.

§ 8. Ad hoc ostendendum obseruasse iuuabit esse

$$vv+(1+s)^2 = 2(1+\cos a+\cos x+\cos y+\cos a\cos x+\cos a\cos y$$
$$+\cos x\cos y+\cos a\cos x\cos y) = 2(1+\cos a)(1+\cos x)(1+\cos y).$$

Constat autem esse $1+\cos a = 2\cos^2\frac{1}{2}a$;

$$1+\cos x = 2\cos^2\frac{1}{2}x \qquad \text{et} \qquad 1+\cos y = 2\cos^2\frac{1}{2}y,$$

quare cum posuerimus $q = \cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y$, erit

$$vv+(1+s)^2 = 16qq \qquad \text{ideoque} \qquad v = \sqrt{16qq-(1+s)^2},$$

hincque porro

$$\frac{v}{1+s} = \sqrt{\frac{16qq}{(1+s)^2}-1}.$$

§ 9. Cum igitur aequatio nostra differentialis fuisset $\frac{v\partial\Delta}{1+s} = \frac{2\partial p}{p}$, ob $p = \frac{q}{1+s}$ ea induet hanc formam:

$$\partial\Delta\sqrt{16pp-1} = \frac{2\partial p}{p}, \quad \text{ideoque} \quad \partial\Delta = \frac{2\partial p}{p\sqrt{16pp-1}}.$$

Fiat iam $p = \frac{1}{r}$, vt habeatur $\partial\Delta = -\frac{2\partial r}{\sqrt{16-rr}}$, vnde integrando colligimus $\Delta = C + 2\arccos\frac{r}{4}$ et loco r valore substituto, qui est

$$r = \frac{1}{p} = \frac{1+s}{q} = \frac{1+\cos a+\cos x+\cos y}{\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y},$$

nostra aequatio integralis erit

$$\Delta = C + 2\arccos\frac{1+\cos a+\cos x+\cos y}{4\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y}.$$

§ 10. Nunc ergo totum negotium eo redit, vt valor constantis per integrationem ingressae C indagetur, quem scilicet ex casu quodam cognito erui oportet; manifestum autem est aream trianguli euanescere debere, quando alterum binorum erurum x vel y euanescit. Ponamus igitur esse $y = 0$, tum vero necesse est, vt fiat $x = 0$, hoc ergo casu constituto nostra aequatio erit

$$0 = C + 2\arccos\frac{2+2\cos a}{4\cos^2\frac{1}{2}a}.$$

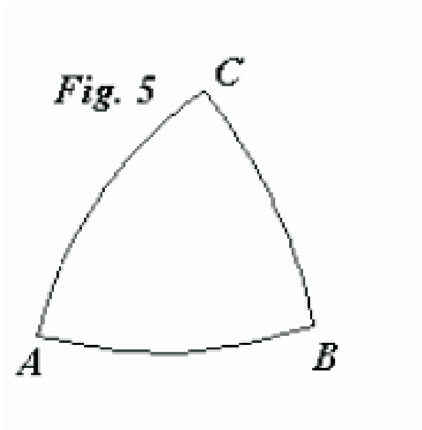
Quoniam vero $4\cos^2\frac{1}{2}a^2 = 2+2\cos\frac{1}{2}a$ et $\arccos 1 = 0$, euidens est statui debere $C = 0$, ita vt habeamus

$$\Delta = 2\arccos\frac{1+\cos a+\cos x+\cos y}{4\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y},$$

vnde concluditur

$$\cos\frac{1}{2}\Delta = \frac{1+\cos a+\cos x+\cos y}{4\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y},$$

quae ipsa expressio cum theoremate supra memorato egregie conuenit, si modo loco x et y scribantur litterae b et c.



Alia demonstratio Geometrica theorematis initio allati.

§ 11. Sit igitur ABC triangulum sphaericum propositum (vide Tab. I, Fig. 5), cuius latera vocentur a, b, c, et anguli iis oppositi α, β, γ, area vero, quam querimus, designemus charactere Δ. Cum igitur ex theoremate Girardi sit

$$\Delta = \alpha + \beta + \gamma - 180^\circ, \quad \text{erit} \quad \cos\Delta = -\cos(\alpha+\beta+\gamma).$$

Nunc vero ex compositione angulorum constat esse

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \text{et} \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

vnde colligitur

$$\begin{aligned}\cos(\alpha+\beta+\gamma) &= \cos(\alpha+\beta)\cos\gamma - \sin(\alpha+\beta)\sin\gamma \\ &= \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma - \cos\beta\sin\alpha\sin\gamma - \cos\gamma\sin\alpha\sin\beta,\end{aligned}$$

consequenter habebimus

$$\cos\Delta = \cos\alpha\sin\beta\sin\gamma + \cos\beta\sin\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta - \cos\alpha\cos\beta\cos\gamma.$$

§ 12. Ex Trigonometria sphaerica autem nouimus esse

$$\begin{aligned}\cos\alpha &= \frac{\cos a - \cos b\cos c}{\sin b\sin c}, \\ \cos\beta &= \frac{\cos b - \cos a\cos c}{\sin a\sin c} \quad \text{et} \\ \cos\gamma &= \frac{\cos c - \cos a\cos b}{\sin a\sin b},\end{aligned}$$

hincque colligimus porro

$$\begin{aligned}\sin\alpha &= \frac{\sqrt{1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c}}{\sin b\sin c}; \\ \sin\beta &= \frac{\sqrt{1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c}}{\sin a\sin c} \quad \text{et} \\ \sin\gamma &= \frac{\sqrt{1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c}}{\sin a\sin b}.\end{aligned}$$

Ponamus igitur breuitatis gratia

$$\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}=v,$$

ita vt sit

$$\sin\alpha=\frac{v}{\sin b\sin c};\quad \sin\beta=\frac{v}{\sin a\sin c};\quad \text{et}\quad \sin\gamma=\frac{v}{\sin a\sin b};$$

quibus valoribus substitutis fiet

$$\cos\Delta=\frac{vv(\cos a-\cos b\cos c)}{\sin^2 a\sin^2 b\sin^2 c}+\frac{vv(\cos b-\cos a\cos c)}{\sin^2 a\sin^2 b\sin^2 c}$$
$$+\frac{vv(\cos c-\cos a\cos b)}{\sin^2 a\sin^2 b\sin^2 c}-\frac{(\cos a-\cos b\cos c)(\cos b-\cos a\cos c)(\cos c-\cos a\cos b)}{\sin^2 a\sin^2 b\sin^2 c},$$

sicque erit

$$\begin{aligned}\sin^2 a\sin^2 b\sin^2 c\cos\Delta &= vv(\cos a+\cos b+\cos c-\cos a\cos b-\cos a\cos c-\cos b\cos c)\\ &- \cos a\cos b\cos c+\cos^2 a\cos^2 b+\cos^2 a\cos^2 c+\cos^2 b\cos^2 c\\ &- \cos a\cos b\cos c(\cos^2 a+\cos^2 b+\cos^2 c)+\cos^2 a\cos^2 b\cos^2 c.\end{aligned}$$

§ 13. Quo nunc has formulas non parum complicatas commodius tractare liceat, ponamus primo breuitatis gratia $\cos a=A$; $\cos b=B$; $\cos c=C$; vt habeamus

$$\begin{aligned}&(1-A^2)(1-B^2)(1-C^2)\cos\Delta=vv(A+B+C-AB-AC-BC)\\ -\ &ABC+AABB+AACC+BBCC-ABC(AA+BB+CC)+AABBCC,\end{aligned}$$

vbi iam erit

$$vv=1-A^2-B^2-C^2+2ABC.$$

§ 14. Quoniam hic ternae litterae A, B, C aequaliter in calculum ingrediuntur, ita vt tanquam radices cuiuspiam aequationis cubicae spectari queant; ad calculum contrahendum non parum conferet statui

$$\begin{aligned}&A+B+C=P,\\ &AB+AC+BC=Q,\\ &ABC=R,\end{aligned}$$

hincque facile colligitur fore

$$AA+BB+CC=PP-2Q\quad\text{ideoque}\quad vv=1-PP+2Q+2R.$$

Deinde notetur formulam $(1-A^2)(1-B^2)(1-C^2)$ esse productum ex his duabus formulis:

$$(1+A)(1+B)(1+C)=1+P+Q+R,\quad\text{et ex}\quad(1-A)(1-B)(1-C)=1-P+Q-R,$$

sicque nostra aequatio hanc induet formam:

$$\begin{aligned}&(1+P+Q+R)(1-P+Q-R)\cos\Delta\\ =\ &(1-PP+2Q+2R)(P-Q)-R+QQ-2PR-R(PP-2Q)+RR,\end{aligned}$$

cuius membrum dextrum euolutum dat

$$P - Q - R - QQ + 2PQ + RR - P^3 + PPQ - PPR,$$

quod per $1 - P + Q - R$ diuisum praebet quotientem $P - Q - R + PP$, consequenter nostra aequatio hanc induet formam:

$$(1 + P + Q + R) \cos \Delta = P - Q - R + PP.$$

§ 15. Hactenus igitur deducti sumus ad hanc aequationem: $\cos \Delta = \frac{P-Q-R+PP}{2+P+Q+R}$, vnde porro colligimus

$$1 + \cos \Delta = \frac{(1 + P)^2}{1 + P + Q + R} = 2 \cos^2 \frac{1}{2}\Delta,$$

consequenter habebimus

$$\cos \frac{1}{2}\Delta = \frac{1 + P}{\sqrt{2(1 + P + Q + R)}}.$$

Cum igitur sit

$$(1 + P + Q + R) = (1 + A)(1 + B)(1 + C) = (1 + \cos a)(1 + \cos b)(1 + \cos c),$$

angulis dimidiis introductis erit

$$1 + P + Q + R = 8 \cos^2 \frac{1}{2}a \cos^2 \frac{1}{2}b \cos^2 \frac{1}{2}c.$$

Quare cum sit

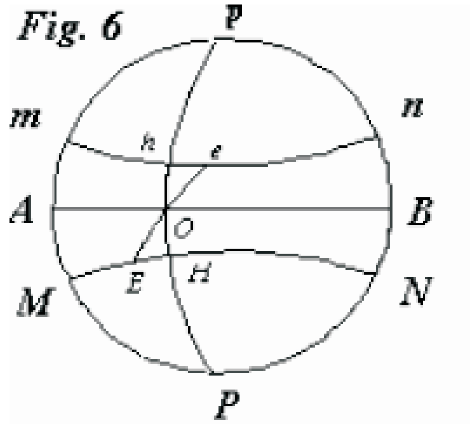
$$1 + P = 1 + \cos a + \cos b + \cos c,$$

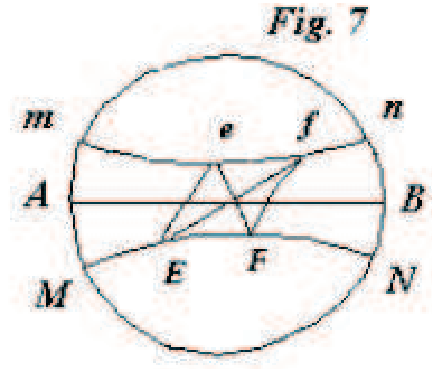
hinc tandem impetramus istum valorem:

$$\cos \frac{1}{2}\Delta = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c},$$

quae est altera demonstratio theorematis initio commemorati.

§ 16. Haec cogitandi occasionem mihi dedit theorema a Celeberrima Professore Lexell in medium allatum, circa omnia triangula sphaerica eiusdem areae super eadem basi exstruenda, quo acutissime demonstrauit, omnes vertices horum triangolum semper in quodam circulo minore sphaere esse sitos, quae elegantissima proprietas non nisi per plures ambages ex nostro theoremate deriuari potest; verum sequens consideratio viam planissimam ad hoc praestandum aperiet.



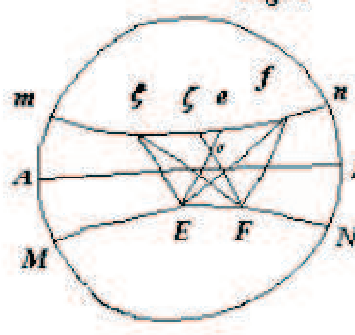


De eximiis proprietaribus binorum circulorum parallelorum inter se aequalium in superficie sphaerica.

§ 17. Sint MN et mn duo huiusmodi circuli paralleli (vide Tab. I, Fig. 6), et quia sumuntur inter se aequales, ab aequatore AB vtrinque aequaliter erunt remoti, perinde atque ab vtroque polo P et p. Hic prima proprietas, quae se offert, in hoc consistit, quod quilibet arcus circuli maximi Ee, inter hos duos parallelos interceptus, ad vtrumque aequaliter inclinetur, atque ab aequatore in O in duas partes aequales secetur. Si enim per punctum O ducatur meridianus OPp, binos parallelos secans in H et h, ob angulos HOE et hOe inter se aequales ambo trilinea HOE et hOe manifesto inter se erunt aequalia et similia, ideoque tam erit $OE = Oe$, quam angulus $OEH = Oeh$. Praeterea hic observasse iuuabit, si iste arcus Ee vsque ad semicirculum continuetur, eum iterum in circulum minorem mn incidere, scilicet in eius puncto, quod puncto E diametraliter opponitur.

§ 18. Ducatur nunc inter eosdem parallelos insuper alius arcus circuli maximi Ff (vide Tab. I, Fig. 7), ad vtrumque perinde inclinatus atque arcus Ee, et manifestum est non solum hos duos arcus Ee et Ff inter se esse aequales, sed etiam circulorum minorum arcus EF et ef. Quare cum in hoc quadrilineo $EFef$ non solum latera opposita, sed etiam anguli oppositi sint inter se aequales, istud quadrilineum rite vocari posset parallelogrammum sphaericum, propterea quod omnibus proprietaribus parallelogrammorum est praeditum. Euidens enim

Fig. 8



est, istud quadrilineum etiam ab vtraque diagonali Ef et Fe in duo trilinea aequalia secari, scilicet tam area trilinei efF quam EeF erit semissis areae parallelogrammi $EefF$.

§ 19. Exstrui nunc concipiatur super eodem arcu EF (vide Tab. I, Fig. 8), tanquam basi, aliud huiusmodi parallelogrammum sphaericum $EF\zeta\epsilon$, atque facile intelligitur, areas horum duorum parallelogrammorum $EFfe$ et $EF\zeta\epsilon$ esse inter se aequales. Hic enim prorsus eodem modo, vti in plano, ambo trilinea $Ee\epsilon$ et $Ff\zeta$ inter se perfecte sunt aequalia, a quibus si trilineum commune $oe\zeta$ auferatur, quadrilinea residua $Eo\zeta\epsilon$ et $Foef$ erunt inter se aequalia; quibus si addatur trilineum EoF, ambo parallelogramma integra erunt etiam aequalia; sicque etiam euictum est, omnia parallelogramma sphaerica, inter binos circulos parallelos et aequales, super eadem basi EF exstructa, esse inter se aequalia.

§ 20. Cum igitur talia paralelogramma sphaerica a diagonalibus in duas partes aequales diuidantur, etiam omnia trilinea super eadem basi EF exstructa, et in altero parallelo mn terminata, areas habebunt inter se aequales; in hac scilicet figura quattuor habebuntur trilinea inter se aequales, scilicet: 1. EfF; 2. EeF; 3. $E\zeta F$; 4. $E\epsilon F$.

§ 21. Haec autem trilinea ideo non vocamus triangula, quia eorum basis EF non est arcus circuli maximi, quemadmodum in triangulis sphaericis statui solet. Facile autem haec trilinea in triangula sphaerica conuertuntur, si ab E ad F ducatur arcus circuli maximi $E\alpha F$, quo predictis trilineis, idem augmentum $EF\alpha E$ accedit, ita vt nunc etiam omnia triangula sphaerica super eadem basi $E\alpha F$ exstructa, quorum vertices in alterum parallelum mn incidunt areas habeant aequales, si modo termini baseos E et F in altero parallelo illi opposito MN fuerint assumti; sique iam clare euictum est, si super basi quacunque innumera constituantur triangula sphaerica, quorum areae sint inter se aequales, eorum vertices semper sitos esse in circulo quodam sphaere minore. Hoc obseruato problema clarissimi Professoris Lexell sequenti modo facillime resoluti poterit.

Problema.

In superficie sphaerica super data basi EF omnia triangula sphaerica exstruere, quorum area sit data $= \Delta$, vbi quidem Δ designat arcum circuli maximi, qui per radium sphaerae multiplicatus producat aream praescriptam.

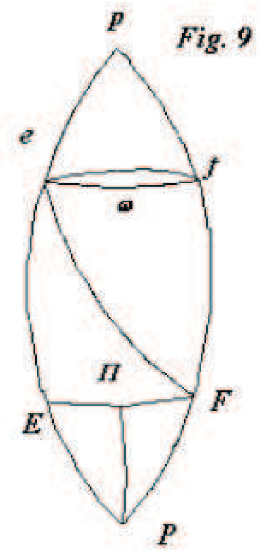


Fig. 9

Solutio.

§ 22. Sit igitur EF ipsa basis proposita $= \alpha$ (vide Tab. I, Fig. 9), et totum negotium huc redit, vt inueniantur poli P et p, qui quaesito satisfaciant; his enim inuentis fi ex polo p interuallo $pe = PE$ describatur circulus minor ef, omnium triangulorum super basi EF exstructorum, et in circulo minori ef terminatorum areae erunt inter se aequales; tantumque superest, vt ex area proposita Δ positio polorum P et p determinetur.

§ 23. Cum igitur EF sit arcus circuli maximi $= a$, ponatur $EP = FP = x$ et angulus $EPF = \omega$, eritque ex Trigonometria sphaerica $\cos\omega = \frac{\cos a - \cos^2 x}{\sin^2 x}$, ideoque

$$1 - \cos\omega = \frac{1 - \cos\alpha}{\sin^2 x} = 2\sin^2\frac{1}{2}\omega.$$

Quare cum sit $1 - \cos a = 2\sin^2\frac{1}{2}a$, erit $\sin\frac{1}{2}\omega = \frac{\sin\frac{1}{2}a}{\sin x}$, hincque vicissim $\sin x = \frac{\sin\frac{1}{2}a}{\sin\frac{1}{2}\omega}$. Ex cognito autem angulo ω innotescit toto area segmenti sphaerici inter binos semicirculos PEp et PFp, quippe quae erit $= 2\omega$. Scilicet arcus circuli maximi $= 2\omega$ per radium sphaere $= 1$ multiplicatus dabit aream huius segmenti.

§ 24. Quaeramus nunc etiam aream trianguli EPF, quem in finem vocetur angulus $PEF = PFE = \phi$, ita vt summa trium angulorum huius trianguli sit $= \omega + 2\phi$, vnde area huius trianguli erit $= \omega + 2\phi - \pi$. Quare si etiam ab e ad f arcus circuli maximi $e\omega f$ ducatur, erit quoque area trianguli sphaerici $pe\omega f = \omega + 2\phi - \pi$. Hinc ergo area quadrilateri sphaerici $EFfe$ inter arcus circulorum maximorum Ee; Ff; EF et $e\omega f$ comprehensi erit $2\omega - 2(\omega + 2\phi - \pi) = 2\pi - 4\phi$, cuius semissis manifesto praebet aream trianguli sphaerici EFe.

§ 25. Cum igitur punctum e sit etiam in circulo minori ef, erit triangulum EeF vnum ex illis triagulis infinitis, quae super basi EF exstruere oportet, cuius area debet esse $= \Delta$, sique adepti sumus hanc aequationem $\Delta = \pi - 2\phi$, vnde colligimus angulum $\phi = \frac{1}{2}\pi - \frac{1}{2}\Delta$. Cum igitur angulus Δ detur, super basi data EF exstruantur vtrinque anguli aequales $FEP = EFP = 90^\circ - \frac{1}{2}\Delta$. Sicque innotescet polus P, ideoque et ei oppositus p, ex quo si interuallo $pe = PE$ descibatur circulus minor ef, omnia triangula super basi EF exstructa et in peripheria circuli minoris ef terminata habebunt ipsam aream propositam $= \Delta$.

§ 26. Quo haec constructio facilior reddatur, ex polo P in medium basis Π ducatur arcus normalis $P\Pi$, et quia in triangulo $EP\Pi$ habetur latus $E\Pi = \frac{1}{2}a$, cum angulo $PE\Pi = 90^\circ - \frac{1}{2}\Delta$, hinc colligitur latus $EP = x$, cuius tangens est $\tan x = \frac{\tan\frac{1}{2}a}{\tan\frac{1}{2}\Delta}$. Nunc igitur inuenta quantitate arcuum EP et FP eorum intersectio dabit polum P, ex cuius opposito p circulus minor interualle $pe = x$ descriptus praebet loca verticum omnium triangulorum super basi EF describendorum, quae constructio egregie conuenit cum ea, quam clarissimus Lexell inuenit.

Translated and edited by J. C.-E. Sten

Different Speculations on the Area of Spherical Triangles
By Leonhard Euler
presented at the meeting of the 29th January 1778

§ 1. The first to learn how to define the area of a spherical triangle was, according to Wallis, Albert Girard, who demonstrated that the area of spherical triangles always be proportional to the excess of the sum of the three angles over two right angles, and also that the area is found, if this excess of the angles converted into arcs of a great circle, be multiplied by the radius of the sphere. But how to determine the area of spherical triangles from their sides requires a more intricate investigation. However, a long time ago I discovered an outstandig theorem, by which this determination can be made easily. It goes as follows: *If the sides of a spherical triangle be denoted by the letters a, b, c, the area of the triangle is set $= \Delta$, then always*

$$\cos\frac{1}{2}\Delta = \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c},$$

the truth of which may be shown not without great detours, either by Girard's theorem or immediately by means of integral calculus. And so, both proofs are worthwhile of bringing into light.

Problem

If, in a spherical triangle AZB having the base AB, the two sides AZ and BZ are augmented by their differentials, so that there arises a triangle AzB, investigate the growth of the area gained by the triangle AZB.

Solution

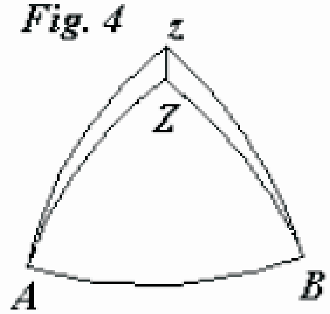
§ 2. Let the base of this triangle $AB = a$ (see Fig. 4), call its sides $AZ = x$ and $BZ = y$, such that the sides of the future augmented triangle will be $Az = x + \partial x$ and $Bz = y + \partial y$. Moreover, the angles are called $BAZ = \phi$ and $ABZ = \psi$, so that there arises the elementary angles $ZAz = \partial\phi$ and $ZBz = \partial\psi$, which being set, the areas of the elementary triangles ZAz are $= \partial\phi(1-\cos x)$ and $ZBz = \partial\psi(1-\cos y)$. Hence, as these two elementary triangles exhibit an augmentation of the area Δ of the triangle, we have the following equation

$$\partial\Delta = \partial\phi(1-\cos x) + \partial\psi(1-\cos y).$$

§ 3. Next we eliminate the angles ϕ and ψ from the calculation and introduce in their place the sides x and y by the aid of the Trigonometric Rules, which give us

$$\cos\phi = \frac{\cos y - \cos a\cos x}{\sin a\sin x} \qquad \text{and} \qquad \cos\psi = \frac{\cos x - \cos a\cos y}{\sin a\sin y}$$

Fig. 4



whence, by differentiation, we obtain

$$-\partial\phi \sin\phi = \frac{\partial x \cos a - \partial x \cos x \cos y - \partial y \sin x \sin y}{\sin a \sin^2 x},$$

and in a similar way

$$-\partial\psi \sin\psi = \frac{\partial y \cos a - \partial y \cos x \cos y - \partial x \sin x \sin y}{\sin a \sin^2 y}.$$

But because of $\cos\phi = \frac{\cos y - \cos a \cos x}{\sin a \sin x}$,

$$\sin\phi = \frac{\sqrt{1 - \cos^2 a - \cos^2 x - \cos^2 y + 2\cos a \cos x \cos y}}{\sin a \sin x},$$

and in a similar way

$$\sin\psi = \frac{\sqrt{1 - \cos^2 a - \cos^2 x - \cos^2 y + 2\cos a \cos x \cos y}}{\sin a \sin y}.$$

As the roots of the two formulas are equal, we may put owing to brevity

$$\sqrt{1 - \cos^2 a - \cos^2 x - \cos^2 y + 2\cos a \cos x \cos y} = v,$$

to have

$$\sin\phi = \frac{v}{\sin a \sin x} \qquad \text{and} \qquad \sin\psi = \frac{v}{\sin a \sin y}.$$

§ 4. By substituting these values, the differentials:

$$\begin{aligned} \partial\phi &= -\frac{\partial x \cos a + \partial x \cos x \cos y + \partial y \sin x \sin y}{v \sin x} \quad \text{and} \\ \partial\psi &= -\frac{\partial y \cos a + \partial y \cos x \cos y + \partial x \sin x \sin y}{v \sin y}, \end{aligned}$$

are obtained. Hence, the incremental area searched for is

$$\partial\Delta = \frac{\left\{\begin{array}{c} -\cos a[\partial x \sin y(1-\cos x) + \partial y \sin x(1-\cos y)] \\ +\partial x \sin y[\cos x \cos y(1-\cos x) + \sin^2 x(1-\cos y)] \\ +\partial y \sin x[\cos x \cos y(1-\cos y) + \sin^2 y(1-\cos x)] \end{array}\right\}}{v \sin x \sin y},$$

which, when expanded, takes this form:

$$v\partial\Delta = \left\{\begin{array}{l} +\partial x \sin x(1-\cos y) + \frac{\partial x \cos x \cos y(1-\cos x)}{\sin x} - \frac{\partial x \cos a(1-\cos x)}{\sin x} \\ +\partial y \sin y(1-\cos x) + \frac{\partial y \cos x \cos y(1-\cos y)}{\sin y} - \frac{\partial y \cos a(1-\cos y)}{\sin y} \end{array}\right\}.$$

Here, it is noted that

$$\frac{1-\cos x}{\sin x} = \tan\frac{1}{2}x \qquad \text{and} \qquad \frac{1-\cos y}{\sin y} = \tan\frac{1}{2}y,$$

the terms involving the elements ∂x hence become

$$\partial x \sin x(1-\cos y) + \partial x \cos x \cos y \tan\frac{1}{2}x - \partial x \cos a \tan\frac{1}{2}x.$$

But because not only is $\tan\frac{1}{2}x = \frac{1-\cos x}{\sin x}$, but also $\tan\frac{1}{2}x = \frac{\sin x}{1+\cos x}$, in the first member is $\sin x$ written as $(1+\cos x)\tan\frac{1}{2}x$, so that ∂x is everywhere multiplied by $\tan\frac{1}{2}x$, and so this member is reduced in the form:

$$\partial x \tan\frac{1}{2}x(1+\cos x - \cos y - \cos a).$$

In the same way the other member is

$$\partial y \tan\frac{1}{2}y(1+\cos y - \cos x - \cos a),$$

and so our whole equation is expressed as:

$$v\partial\Delta = \partial x \tan\frac{1}{2}x(1+\cos x - \cos y - \cos a) + \partial y \tan\frac{1}{2}y(1+\cos y - \cos x - \cos a)$$

§ 5. But if we now, owing to brevity, put $\cos a + \cos x + \cos y = s$, then

$$v\partial\Delta = \partial x \tan\frac{1}{2}x(1 - s + 2\cos x) + \partial y \tan\frac{1}{2}y(1 - s + 2\cos y),$$

which equation may be represented as:

$$v\partial\Delta = (1-s)(\partial x \tan\frac{1}{2}x + \partial y \tan\frac{1}{2}y) + 2\partial x \cos x \tan\frac{1}{2}x + 2\partial y \cos y \tan\frac{1}{2}y.$$

Now, since $\tan\frac{1}{2}x = \frac{1-\cos x}{\sin x}$, we have

$$\tan\frac{1}{2}x \cos x = \frac{\cos x - \cos^2 x}{\sin x} = \frac{\cos x - 1 + \sin^2 x}{\sin x} = \sin x - \tan\frac{1}{2}x.$$

In the same way $\tan\frac{1}{2}y\cos y = \sin y - \tan\frac{1}{2}y$, and substituting these values gives rise to the following equation:

$$v\partial\Delta = -(1+s)(\partial x\tan\frac{1}{2}x + \partial y\tan\frac{1}{2}y) + 2\partial x\sin x + 2\partial y\sin y.$$

§ 6. This last form thus appears in the most suitable way, as its right member is absolutely integrable after a division with $1+s$. Indeed, having performed the division, our equation becomes

$$\frac{v\partial\Delta}{(1+s)} = -\partial x\tan\frac{1}{2}x - \partial y\tan\frac{1}{2}y + \frac{2\partial x\sin x + 2\partial y\sin y}{1+\cos a+\cos x+\cos y},$$

where it is noted that $\int\partial x\tan\frac{1}{2}x = -2\log(\cos\frac{1}{2}x)$, and in the same way $\int\partial y\tan\frac{1}{2}y = -2\log(\cos\frac{1}{2}y)$, and finally

$$2\int\frac{\partial x\sin x + \partial y\sin y}{1+\cos a+\cos x+\cos y} = -2\log(1+\cos a+\cos x+\cos y) = -2\log(1+s)$$

so that by integration we find

$$\int\frac{v\partial\Delta}{(1+s)} = 2\log(\cos\frac{1}{2}x) + 2\log(\cos\frac{1}{2}y) - 2\log(1+s) = 2\log\Big(\frac{\cos\frac{1}{2}x\cos\frac{1}{2}y}{1+s}\Big).$$

In this form, however, the left member is not integrable, to which therefore a remedy will be brougt in the following way.

§ 7. Namely, while we put $s = \cos a + \cos x + \cos y$ and moreover set $\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y = q$, we have this equation:

$$\int\frac{v\partial\Delta}{1+s} = 2\log(\frac{q}{(1+s)\cos\frac{1}{2}a}),$$

where it further be $\frac{q}{1+s} = p$, so that

$$\int\frac{v\partial\Delta}{1+s} = 2\log(\frac{p}{\cos\frac{1}{2}a}),$$

which equation differentiated once more renders $\frac{v\partial\Delta}{1+s} = \frac{2\partial p}{p}$, from which it is obtained that $\partial\Delta = \frac{2\partial p(1+s)}{pv}$, which formula admits integration, if only $\frac{v}{1+s}$ be some function of p itself, which we now can surely assert, because $\partial\Delta$ denotes the differential area of the triangle.

§ 8. To make this clear, it is helpful to observe that

$$vv + (1+s)^2 = 2(1+\cos a+\cos x+\cos y+\cos a\cos x+\cos a\cos y$$

$$+\cos x\cos y + \cos a\cos x\cos y) = 2(1+\cos a)(1+\cos x)(1+\cos y).$$

But as $1+\cos a = 2\cos^2\frac{1}{2}a$;

$$1+\cos x = 2\cos^2\frac{1}{2}x \qquad \text{and} \qquad 1+\cos y = 2\cos^2\frac{1}{2}y,$$

whereby we put $q = \cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y$, to have

$$vv + (1+s)^2 = 16qq \qquad \text{and therefore} \qquad v = \sqrt{16qq - (1+s)^2}$$

and hence further

$$\frac{v}{1+s} = \sqrt{\frac{16qq}{(1+s)^2} - 1}.$$

§ 9. Thus, as our differential equation was $\frac{v\partial\Delta}{1+s} = \frac{2\partial p}{p}$, on account of $p = \frac{q}{1+s}$ it takes the following form:

$$\partial\Delta\sqrt{16pp-1} = \frac{2\partial p}{p}, \quad \text{and therefore} \quad \partial\Delta = \frac{2\partial p}{p\sqrt{16pp-1}}.$$

Now, let $p = \frac{1}{r}$, to have $\partial\Delta = -\frac{2\partial r}{\sqrt{16-rr}}$, from which we get through integration $\Delta = C + 2\arccos\frac{r}{4}$ and by substituting for r its value, which is

$$r = \frac{1}{p} = \frac{1+s}{q} = \frac{1+\cos a+\cos x+\cos y}{\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y},$$

our integral equation becomes

$$\Delta = C + 2\arccos\frac{1+\cos a+\cos x+\cos y}{4\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y}.$$

§ 10. Thus, the whole work now goes back to finding the value of the integration constant C, which of course can be found by some known case; but it is evident that the area of the triangle should vanish when either of the two sides, x or y, vanishes. Let us therefore put $y = 0$, wherupon it is necessary to let $x = 0$, in which case the state of our equation is

$$0 = C + 2\arccos\frac{2+2\cos a}{4\cos^2\frac{1}{2}a}.$$

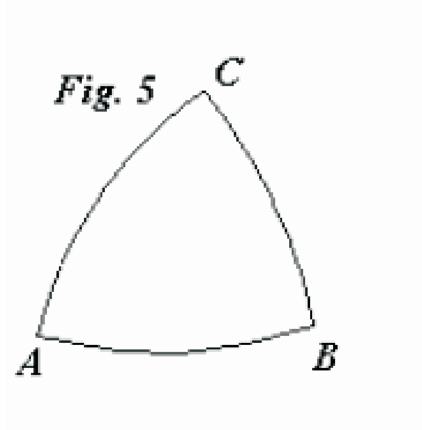
But because $4\cos^2\frac{1}{2}a^2 = 2 + 2\cos\frac{1}{2}a$ and $\arccos 1 = 0$, it is evident that $C = 0$, so that we have

$$\Delta = 2\arccos\frac{1+\cos a+\cos x+\cos y}{4\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y},$$

whence it is concluded that

$$\cos\frac{1}{2}\Delta = \frac{1+\cos a+\cos x+\cos y}{4\cos\frac{1}{2}a\cos\frac{1}{2}x\cos\frac{1}{2}y},$$

which actually agrees excellently with the expression of the theorem mentioned above in the case that x and y be written with the letters b and c.



Another Geometric demonstration of the theorem given in the beginning

§ 11. Now let ABC be the proposed spherical triangle (Fig. 5), whose sides are a, b, c, and their opposite angles α, β, γ, the true area searched for being denoted by the letter Δ. Thus, from Girard's theorem

$$\Delta = \alpha + \beta + \gamma - 180^\circ, \quad \text{we get} \quad \cos\Delta = -\cos(\alpha+\beta+\gamma).$$

But from the decomposition of angles it is known that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \text{and} \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

whence it may be inferred that

$$\begin{aligned} \cos(\alpha+\beta+\gamma) &= \cos(\alpha+\beta)\cos\gamma - \sin(\alpha+\beta)\sin\gamma \\ &= \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma - \cos\beta\sin\alpha\sin\gamma - \cos\gamma\sin\alpha\sin\beta, \end{aligned}$$

Consequently we have

$$\cos\Delta = \cos\alpha\sin\beta\sin\gamma + \cos\beta\sin\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta - \cos\alpha\cos\beta\cos\gamma.$$

§ 12. From Spherical Trigonometry we know, however, that

$$\begin{aligned} \cos\alpha &= \frac{\cos a - \cos b\cos c}{\sin b\sin c}, \\ \cos\beta &= \frac{\cos b - \cos a\cos c}{\sin a\sin c} \quad \text{and} \\ \cos\gamma &= \frac{\cos c - \cos a\cos b}{\sin a\sin b}, \end{aligned}$$

and hence we furthermore obtain

$$\begin{aligned} \sin\alpha &= \frac{\sqrt{1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c}}{\sin b\sin c}; \\ \sin\beta &= \frac{\sqrt{1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c}}{\sin a\sin c} \quad \text{and} \\ \sin\gamma &= \frac{\sqrt{1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a\cos b\cos c}}{\sin a\sin b}. \end{aligned}$$

Putting then for brevity

$$\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}=v,$$

so that

$$\sin\alpha=\frac{v}{\sin b\sin c};\qquad \sin\beta=\frac{v}{\sin a\sin c}\quad\text{and}\quad \sin\gamma=\frac{v}{\sin a\sin b};$$

which values substituted gives

$$\cos\Delta=\frac{vv(\cos a-\cos b\cos c)}{\sin^2 a\sin^2 b\sin^2 c}+\frac{vv(\cos b-\cos a\cos c)}{\sin^2 a\sin^2 b\sin^2 c}$$
$$+\frac{vv(\cos c-\cos a\cos b)}{\sin^2 a\sin^2 b\sin^2 c}-\frac{(\cos a-\cos b\cos c)(\cos b-\cos a\cos c)(\cos c-\cos a\cos b)}{\sin^2 a\sin^2 b\sin^2 c},$$

so that

$$\begin{aligned}\sin^2 a\sin^2 b\sin^2 c\cos\Delta &= vv(\cos a+\cos b+\cos c-\cos a\cos b-\cos a\cos c-\cos b\cos c)\\ &- \cos a\cos b\cos c+\cos^2 a\cos^2 b+\cos^2 a\cos^2 c+\cos^2 b\cos^2 c\\ &- \cos a\cos b\cos c(\cos^2 a+\cos^2 b+\cos^2 c)+\cos^2 a\cos^2 b\cos^2 c.\end{aligned}$$

§ 13. Now since these formulas are not little complicated to be treated conveniently, let us first abbreviate $\cos a=A$; $\cos b=B$; $\cos c=C$; so that we have

$$\begin{aligned}&(1-A^2)(1-B^2)(1-C^2)\cos\Delta=vv(A+B+C-AB-AC-BC)\\ -\ &ABC+AABB+AACC+BBCC-ABC(AA+BB+CC)+AABBCC,\end{aligned}$$

wherein already

$$vv=1-A^2-B^2-C^2+2ABC.$$

§ 14. Seeing that each of the three letters A, B, C enter similarly into the calculation, so that they may be regarded as the roots of some cubic equation; to shorten the calculation, not little is achieved by setting

$$\begin{aligned}&A+B+C=P,\\ &AB+AC+BC=Q,\\ &ABC=R,\end{aligned}$$

and hence it is easily obtained that

$$AA+BB+CC=PP-2Q\quad\text{and therefore}\quad vv=1-PP+2Q+2R.$$

Next it is observed that the formula $(1-A^2)(1-B^2)(1-C^2)$ is a product of these two formulas:

$$(1+A)(1+B)(1+C)=1+P+Q+R,\quad\text{and}\quad(1-A)(1-B)(1-C)=1-P+Q-R,$$

and so our equation takes the form:

$$\begin{aligned}&(1+P+Q+R)(1-P+Q-R)\cos\Delta\\ =\ &(1-PP+2Q+2R)(P-Q)-R+QQ-2PR-R(PP-2Q)+RR\end{aligned}$$

whose right member expanded gives

$$P - Q - R - QQ + 2PQ + RR - P^3 + 2PPQ - PPR,$$

which divided by $1 - P + Q - R$ gives the quotient $P - Q - R + PP$. Consequently, our equation takes this form:

$$(1 + P + Q + R)\cos\Delta = P - Q - R + PP.$$

§ 15. Thus far, then, we have been lead to the equation: $\cos\Delta = \frac{P-Q-R+PP}{2+P+Q+R}$, whence we infer further

$$1 + \cos\Delta = \frac{(1+P)^2}{1+P+Q+R} = 2\cos^2\frac{1}{2}\Delta,$$

consequently to have

$$\cos\frac{1}{2}\Delta = \frac{1+P}{\sqrt{2(1+P+Q+R)}}.$$

Also, since

$$(1 + P + Q + R) = (1 + A)(1 + B)(1 + C) = (1 + \cos a)(1 + \cos b)(1 + \cos c),$$

we get by introducing the half angles[1]

$$1 + P + Q + R = 8\cos^2\frac{1}{2}a\cos^2\frac{1}{2}b\cos^2\frac{1}{2}c.$$

whereby, since

$$1 + P = 1 + \cos a + \cos b + \cos c,$$

we finally achieve the value

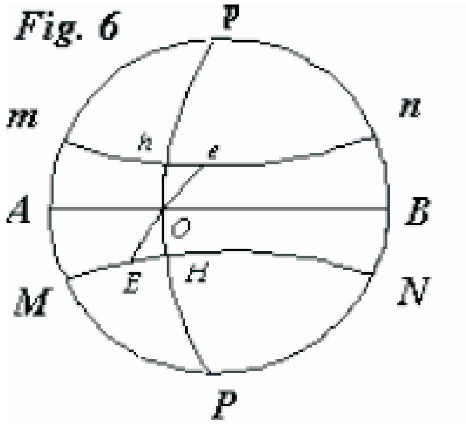
$$\cos\frac{1}{2}\Delta = \frac{1 + \cos a + \cos b + \cos c}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c},$$

which is another demonstration of the theorem stated in the beginning.

§ 16. The occasion for me to start thinking this came from a theorem concerning all the spherical triangles having the same area raised upon the same base, brought into light by the famous Professor Lexell[2], who acutely demonstrated that all the vertices of these triangles always lie on some small circle of the sphere, which most elegant property may be derived not without many detours from our theorem; actually, the following consideration discloses the flattest road for this.

[1] This equation contained a typographical error in the original.

[2] Anders Johan Lexell, mathematician and astronomer, born in Åbo/Turku, Finland, in 1740, deceased in S:t Petersburg, in 1784. The original Theorem appears in *Solutio Problematis Geometrici ex Doctrina Sphaericorum*, Acta Academiae Scientiarum Imperialis Petropolitanae Tomus V Pars I. pp. 112-126, 1784.



On the exceptional properties of two similar parallel circles on a spherical surface.

§ 17. Let MN and mn be two such parallel circles, and as they are taken to be similar to each other (see Fig. 6), they are at all sides equally remote from the equator AB, and similarly from both poles P and p. Here the first property offered consist in this, that whatever be the arc of the great circle Ee taken between these two parallels, it is equally inclined at both, and moreover it is cut at the equator in O in two equal parts. In fact, if through the point O a meridian OPp is drawn, cutting the two parallels at H and h, on account of the equality of the angles HOE and hOe both the three line objects [*trilinea*], HOE and hOe, are clearly equal and similar to each other, and therefore $OE = Oe$ as is also the angle $OEH = Oeh$. In addition to this it is helpful to notice that if this arc Ee be continued all the way up to the semicircle, it falls a second time on the small circle mn, that is, at the point that is diametrically opposite to the point E.

§ 18. Now let yet another arc of a great circle Ff be drawn between the same parallels (see Fig. 7), inclined at both similarly as the arc Ee, whence it is clear that not only are these two arcs Ee and Ff equal to each other, but so are also the arcs of the small circle EF and ef. Hence, as in this four line object $EFef$, not only are the opposite sides equal to each other, but also the opposite angles, so that this four line object can be duly said to be a spherical parallelogram, because every property of a parallelogram is provided. In fact, it is evident that this four line object is also cut into two equal three line objects by either of the diagonals Ef and Fe, of course in such a way that the area of both the three line objects efF and EeF be half the area of the parallelogram $EefF$.

§ 19. Now let us consider raised upon the same arc EF (see Fig. 8), as well as on the same base, another such spherical parallelogram $EF\zeta\epsilon$, and it is easily understood, that the areas of these two parallelograms $EFfe$ and $EF\zeta\epsilon$ will be equal. Here, then, it is quite in the same way as in the plane, that both the three line objects $Ee\epsilon$ and $Ff\zeta$ are perfectly equal to each other, from which, if the common three line object $oe\zeta$ is removed, the residual four line objects $Eo\zeta\epsilon$ and $Foef$ be equal to each other; if a three line object EoF be added, both parallelograms integrated also become equal; and so it has also been established that every

Fig. 7

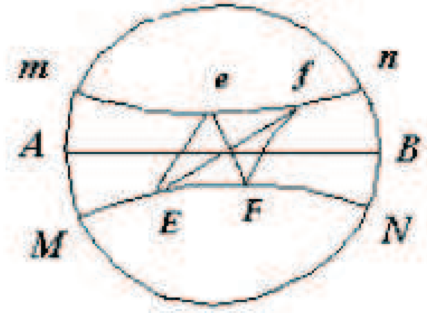
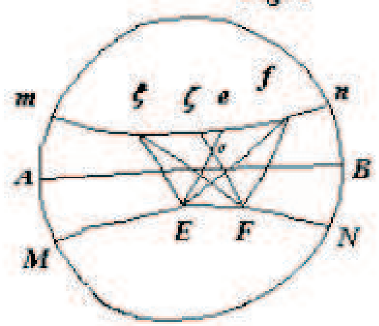


Fig. 8

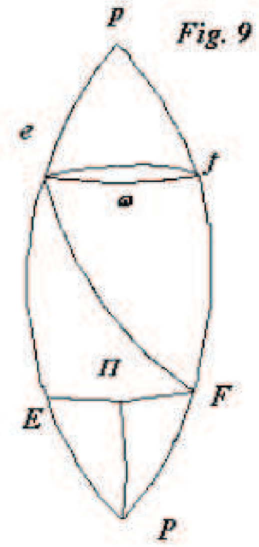


spherical parallelogram between two parallel and equal circles raised upon the same base EF will be equal to each other.

§ 20. While then such spherical parallelograms break up into two equal parts from their diagonals, also every three line object raised over the same base EF and terminated in another parallel mn have equal areas; in the figure, the areas of four of the three lines objects are equal, namely, of: 1. EfF; 2. EeF; 3. $E\zeta F$; 4. $E\epsilon F$. § 21. However, we will not call these three line objects [spherical] triangles, because their base EF is not an arc of a great circle, as it usually is in spherical triangles. But it is easy to convert these three line objects into spherical triangles, if from E to F a great circle $E\alpha F$ is drawn, for which the mentioned three line object receives the same augmentation $EF\alpha E$, so that now also every spherical triangle erected over the same base $E\alpha F$, whose vertices fall upon another parallel mn have equal areas, if the terminal points of the base E and F be assumed on another parallel MN opposite to it; and so it has been clearly established, that if over any base be erected innumerable spherical triangles, whose areas are equal to each other, their vertices will always be located on some small circle. By observing this, the problem due to the famous Professor Lexell may be easily resolved as follows.

Problem

Raise upon a spherical surface over a given base EF all those spherical triangles, of which the



area be given $= \Delta$, *where indeed* Δ *denotes the arc of the great circle, which multiplied by the radius of the sphere produces the prescribed area.*

Solution

§ 22. Let therefore the proposed base itself EF be $= \alpha$ (see Fig. 9), and now the whole work goes back to finding the poles P and p, satisfying the requested; namely having found these, if an interval $pe = PE$ from the pole p be drawn along a small circle ef, then all the triangles raised upon the base EF and terminating in the small circle ef have an equal area; and moreover to determine the position of the poles P and p from the proposed area Δ.

§ 23. Therefore, when EF is an arc of the great circle $= a$, putting $EP = FP = x$ and the angle $EPF = \omega$. From Spherical Trigonometry we have $\cos\omega = \frac{\cos a - \cos^2 x}{\sin^2 x}$, and so

$$1 - \cos\omega = \frac{1 - \cos\alpha}{\sin^2 x} = 2\sin^2\frac{1}{2}\omega.$$

Therefore, when $1 - \cos a = 2\sin^2\frac{1}{2}a$, then $\sin\frac{1}{2}\omega = \frac{\sin\frac{1}{2}a}{\sin x}$, and hence in turn $\sin x = \frac{\sin\frac{1}{2}a}{\sin\frac{1}{2}\omega}$. But by knowing the angle ω, the whole area of the spherical segment between the two semicircles PEp and PFp becomes known, which obviously is $= 2\omega$. Namely, multiplying the arc of the great circle $= 2\omega$ by the radius of the sphere $= 1$ gives the area of this segment.

§ 24. We presently search also for the area of the triangle EPF, which in the end is called by the angle $PEF = PFE = \phi$, so that the sum of the three angles of this triangle would be $= \omega + 2\phi$, and hence the area of this triangle $= \omega + 2\phi - \pi$.[3] Thus, if also from e to f an arc $e\omega f$ of a great circle is drawn, also the area of any spherical triangle $pe\omega f = \omega + 2\phi - \pi$.

[3]The Greek letter π means 3.14159 ..., *i.e.* the semiperimeter of a unit circle. The symbol is due to Euler.

Thus, the area of the four line object $EFfe$ included between the arcs of great circles Ee; Ff; EF and $e\omega f$ becomes $2\omega - 2(\omega + 2\phi - \pi) = 2\pi - 4\phi$, the half of which evidently provides the area of the spherical triangle EFe.

§ 25. Now that the point e is also located on a small circle ef, the triangle EeF will be one of those innumerable triangles, which must be erected over a base EF to have an area $= \Delta$, and so we arrive at the equation $\Delta = \pi - 2\phi$, whence we infer the angle $\phi = \frac{1}{2}\pi - \frac{1}{2}\Delta$. Thus, when the angle Δ be given, over a given base EF erects equal angles $FEP = EFP = 90^\circ - \frac{1}{2}\Delta$ on either side. And so the pole P becomes known and thereby also its opposite p, from which if in the interval $pe = PE$ be drawn a small circle ef, every triangle erected upon the base EF and terminated on the periphery of the small circle ef have the same proposed area $= \Delta$.

§ 26. To render this construction easier, let an arc $P\Pi$ be drawn from the pole P normal to the middle of the base Π, and since in the triangle $EP\Pi$ the side $E\Pi = \frac{1}{2}a$ with the angle $PE\Pi = 90^\circ - \frac{1}{2}\Delta$, it is inferred that the side $EP = x$, the tangent of which is $\tan x = \frac{\tan\frac{1}{2}a}{\tan\frac{1}{2}\Delta}$. Now, the discovered magnitude of the arcs EP and FP render at their intersection the pole P, from the opposite of which p the interval $pe = x$ is drawn on a small circle, giving the locus of the vertex of all those triangles described upon the base EF, which construction agrees wonderfully with the one that was discovered by the famous Lexell.