The Solution of the General Isoperimetric Problem, Taken in the Widest Sense.

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§.1. The problems, which require curves with the stated property of maxima or minima, and hitherto have been treated by the Geometers, conveniently are referred to two classes. The first of which contains all these, which among all the curves postulated, shall have the common property of the maximum or minimum that they require. Truly for the other class, all these are concerned with problems, which enjoy a certain property not common to all curves, but only to that class of problems, having a certain maximum or minimum property, which they are appointed to determine. So that they are understood to deal with all these later isoperimetric problems, taken in the widest sense, the solution of which now have been given some time ago, by the most celebrated Geometers *Jacob* and *Johan Bernoulli, Taylor* and *Hermann*. Although indeed these men were able to investigate only that curve among all the curves of the same length, which would have the property of being a maximum or minimum; yet the method also can be extended to the questions, which require all the other common properties mentioned. So that, if from all the curves, which turned about the axis, equally those shall be found which shall produce a volume of the smallest surface area.

§.2. Two problems mainly of the first kind are to be investigated, for defining the curves of the quickest descent and of the minimum resistance, which among all the curves for that class certainly will possess their maxima or minima. Moreover it is evident to have that same prerogative amongst all the curves of the same length, to which curve amongst all will experience the minimum resistance, or which shall produce the maximum speed of descent, or have a maximum or minimum according to some other property defined. Truly a succession of such outcomes shall not prevail, so that, though the brachristochrome is one amongst all these curves of the same length, the same shall not be true among all such curves generally, even if innumerable curves of the shortest length may be given, since besides the cycloid, no other curve shall be satisfactory here. From which it is understood, the first class to be as if an example of the second class, and this in turn is allowed to be much wider than the former.

§. 3. Considering this according to that way of thinking, whether or not a certain third class may exist, of which the second class may be only a certain kind? and by progressing in that manner, also a fourth, or fifth or even more following classes of this kind may be given ? And indeed these classes themselves may be taken to become known themselves, so that these higher orders may be reached, and of these curves, from which a maximum or minimum property may be had, several may be determined, which shall have more than one purpose: so that if amongst all the curves of the same length, and indeed considering the same area that shall be required, that which turned about the axis will generate the maximum volume. Moreover the equation, which I have adapted for this curve, was more general, than if the curves either were only of the same length, or for which I may have put of the same volume. And without doubt a more general equation would be going to be produced, if I were going to add one or more properties to these two properties. From which perhaps it may be seen as a paradox, it is understood that a greater number of curves may be restricted, and therefore the question will be found more satisfying.

§.4. Therefore I shall consider these most general classes in the following questions.

I. *Therefore, from all the curves arising, to determine that one which shall hold the property A with a maximum or minimum value.*

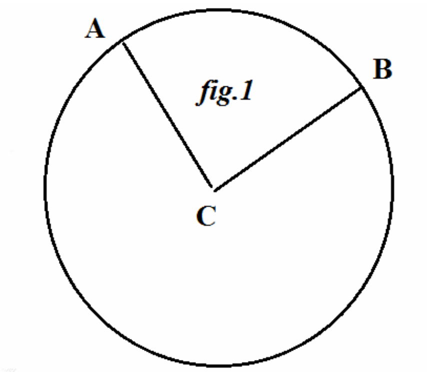
II. *From all the curves with the aforementioned property A to determine that, which shall contain the property B, to be the maximum or minimum of the order.*

III. *From all the curves provided equally with the aforementioned properties A and B, to determine that curve, which shall contain the property B to the maximum or minimum order.*

IV. *From all the curves with the aforementioned individual properties A and B and C contained equally, to determine that which shall contain the property D to the maximum or minimum order.*

In a similar manner curves of the fifth class will consider curves with the four aforementioned properties present, and thus again in the following.

§. 5. Of these questions, this property is required to be observed properly, so that the properties of the given curves shall be able to commute with those, which the sought curve must have. Thus the following question agrees with this, since this commutative property cannot find a place in the first property :*To determine that commutative property from all the aforestated properties B, which the property may have from the maximum or minimum order A.* And the third question can be commuted in three ways, just as the curve requiring to be found shall be able to contain either the property A, or B, or C for a certain sum in the order, provided meanwhile the two remaining properties of the curve sought may hold equally. But this will be clear from the manner of solving, since that curve shall have the property of being the maximum or minimum, which it retains approximately; that which also is found in curves enjoying the same property.

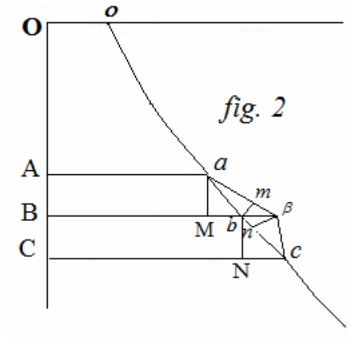


§. 6. Truly the property of being a maximum or minimum, which the curve sought in these problems must have, thus is required to be understood, so that nothing shall be given between these two ends of the curve, except for that sought, which prescribed may have a useful purpose, and the proposition order may contain either with a very large or very small distinctive property. Thus the cycloid has this nature, so that no other curve shall be given between the same ends, upon which a body descending from a higher to lower shall be able to arrive in a shorter time. And besides the catenary passing through two points, no other curve of the same length is given within the same two points, the centre of gravity of which shall be placed at a lower location. Truly any two points can be assumed for these ends, through which the curve sought passes. And thus in the circle, for which it shall be agreed, to be the largest of all figures, for with any two points assumed AB, it is not possible to find another curve of the same length between these two points, which will hold a greater sector than ABC.

§. 7. For the first class of problems requiring to be solved, it suffices to consider two contiguous elements of the curve, such as that is evident from the solutions which are found generally for the brachistochrone, and the solid of minimum resistance. Truly problems of the second class cannot be resolved unless three elements of the curve shall be drawn into the calculation. And for these combined for the solution of the problem pertaining to the third class, there is a need for four elements of the curve. And thus again for the fourth class five elements, moreover for the fifth class, six elements are required and so forth thence. From which it is understood the solution of the problem continually to emerge more difficult, where you proceed to nearby classes. Indeed the difficulty in the extent of the calculation becomes so apparent, since that shall be more painstaking, where more elements of the curve must be considered. But that will be shortened considerably, if the elements can be added together.

§. 8. But so that it may be understood more easily, with which method it shall be convenient to use in the individual classes, also with these I shall not touch on here, for the most part it will help also to run through the first two classes. Though these truly have now been treated satisfactorily, so that scarcely anything new may be seen detected in these; yet I am going to pursue these for a little while by this method which will be different and clearly much more general, which is adapted better for the following classes. Moreover also thence this usefulness will arise , so that this property arises, which the curve sought must have, in the first and second class produced by the calculation, and in the remaining classes also it may be of service, if it may be changed a little. But there shall be need of a much greater labour for this calculation to be performed at last in the following classes.

§. 9. Therefore it will be required to determine that first curve arising *oa*, which shall contain the given property for the greatest or least order. For this outstanding curve the axis OA may be taken as it pleases, to which the curve sought shall be referred. The elements AB, BC of this may be taken equal, and to this the two elements *ab* and *bc* on the curve shall correspond, which shall provide the elements of the applied lines *b*M and *c*N. The arc *oa* shall be called *s*; the abscissa OA, *x*; and the applied line A*a*, *y*. There will become $AB = BC = x$, $bM = dy$, and $ab = ds$. And again $cN = dy + ddy$ and $bc = ds + dds$. Then it is evident, as the whole curve *oa* shall have the maximum or minimum



property, the same must be had for any part you please ; therefore also for the two elements *ab* and *bc*. On account of which two other elements such as $a\beta$ and βc will not be able to be drawn, which shall contain the prescribed property between the ends *a* and *c* to a greater or lesser order. Truly so that the distinctive maxima and minima property shall be present in this, so that with everything in place nearby translated, the prescribed property shall retain its prescribed state only; two nearby elements $a\beta$ and βc will have to be considered between the ends *a* and *c*. Therefore here the prescribed property must agree equally, and in the beginning to become *ab* and *bc* . From which the position of the elements *ab* and *bc* and hence that curve itself sought *oa* will become known.

§.10. Moreover in this approximate position *ab* passes into $a\beta$, bc into βc; and $\mathrm{M}\beta$, *ac* *c*N into $cN - b\beta$.
Therefore the element *ab* will increase into the particle βm , truly the element *bc* decreases into the particle *bn*. Similarly *b*M will be increased by the increments $b\beta$ and *bn* Truly the first particles βm and *bn* also are able to be reduced to $b\beta$ by the similar triangles βbm, *ba*M and βbn, *cb*N, from which there becomes
$\beta m = \frac{b\mathrm{M}\cdot b\beta}{ab}$, and $bn = \frac{c\mathrm{N}\cdot b\beta}{ac}$. Therefore in the approximate situation, *ab* changes into $ab + \frac{b\mathrm{M}\cdot b\beta}{bc}$; *bc* into $bc - \frac{c\mathrm{N}\cdot b\beta}{bc}$; *b*M into $b\mathrm{M} + b\beta$; and *c*N in $c\mathrm{N} - b\beta$, truly the abscissas of elements AB et BC meanwhile remain invariant. Then also the applied line B*b* increases by the element $b\beta$; and the arc *oab* by the particle βm , *i.e.* $\frac{b\mathrm{M}\cdot b\beta}{ab}$.
Which, since most often they can be ignored, yet in general they must be retained.

§.11. But since the prescribed properties, which the curve *oa* must contain in the maximum or minimum order, as great as may be able to be found in the elements *ad*, *bc*, with the amounts in the approximate $a\beta$, βc ; in each case it will be agreed to recall that property to the calculation, and from that in turn to be subtracted; indeed that, which remains will be required to be put equal to zero. Truly the individual terms of this to be affected will be either the particles $b\beta$, or βm and *bn*; but which, since they can be reduced to $b\beta$, the whole remainder will be divisible by $b\beta$ divisible, with which done an equation will be produced in which clearly no quantity will be found depending on the point β , but the whole will be constructed from *x, y* and δ with constants. From this therefore the nature of the curve sought will be determined.

§.12. We shall consider a curve *va* that must have that property, so that $\int x^n ds$ may have a smaller value, than in any other line passing through the points *o* et *a*. Therefore the elements *ab, bc* will have this same property. Whereby $\mathrm{OA}^n.ab + \mathrm{OB}^n.\beta c$ also will have to be a minimum or equal to that quantity $\mathrm{OA}^n.a\beta + \mathrm{OB}^n.\beta c.$ With this taken from that this equation will remain : $\mathrm{OA}^n.\beta m = \mathrm{OB}^n.bn$, or in place of βm and *bn* with the values found substituted, this equation:

$$\frac{\mathrm{OA}^n.b\mathrm{M}.b\beta}{ab}=\frac{\mathrm{OB}^n.c\mathrm{N}.b\beta}{bc}, \text{ or } \frac{\mathrm{OA}^n.b\mathrm{M}}{ab}=\frac{\mathrm{OB}^n.c\mathrm{N}}{bc}.$$

Which equation has been prepared thus, so that the latter member shall be of the first differential itself increased: Therefore the differential of this $\frac{\mathrm{OA}^n.b\mathrm{M}}{ab}$ will be $=0,$ and thus, this quantity itself will be equal to a constant quantity, which shall be a^n. Therefore the following equation will be had in symbols $x^n dy=a^n ds,$ by which the curve sought will be known. From these it is observed likewise, if a curve such as $\int Pds$ is required, where P shall designates some function of x, in that there shall be a minimum, this equation $\mathrm{P}dy=\mathrm{A}ds$ is going to be produced, with A put for a homogeneous constant of P. But if P were a function of y, with the coordinates x and y interchanged, this equation will be produced $\mathrm{P}dx=\mathrm{A}ds$.

§.13. If it shall be required so that in the curve sought $\int x^m y^n ds$ shall be always a maximum or minimum, there will become,

$$\mathrm{OA}^m.Aa^n.ab+\mathrm{O}B^m b^n.bc=\mathrm{OA}^m.Aa^n.a\beta+\mathrm{O}B^m.\ \mathrm{B}\beta^n.\beta c=\mathrm{OA}^m.Aa^n.a\beta$$
$$+\mathrm{O}B^m.\ \mathrm{B}\beta^{n-1}b\beta.\beta c+\mathrm{O}B^m.\ \mathrm{B}\beta^{n-1}.\mathrm{B}\beta.\beta c,$$

on putting $\mathrm{B}b+b\beta$ in place of $b\beta$. From which this equation arises:

$$\frac{\mathrm{OB}^m.\mathrm{B}b^n.c\mathrm{N}.b\beta}{bc}-\frac{\mathrm{OA}^m.\mathrm{A}a^n.b\mathrm{M}.b\beta}{ab}=n.\mathrm{OB}^m.Bb^{n-1}bc.b\beta-n.\mathrm{OA}^m.Aa^{n-1}.ab.b\beta.$$

Moreover the first part divided by $b\beta$ everywhere, is expressed by the differential of this quantity $\frac{\mathrm{OA}^m.\mathrm{A}a^n.b\mathrm{M}}{ab}$. Whereby with the symbols substituted, this equation will be produced $d.\frac{x^m y^n dy}{ds}=nx^m y^{n-1}ds,$ which on being taken there for the differential quantity itself, and for dds with the value being put equal to $\frac{dyddy}{ds}$, will be changed into this $\frac{xyddy}{ds^2}+mydy-nxdx=0.$ Indeed this can be changed into a differential equation of the first order ; but that thus shall be complicated, so that it may not be evident, whether or not it shall be able to be separated. If $\int x^m s^n ds$ must be a maximum or minimum, this equation is found:

$$d.\frac{x^m s^n dy}{ds}=nx^m s^{n-1}dy,$$

which again may be changed into this : $xdxddy + mds^2dy = 0.$ From which the exponent *n* appears to vanish from the calculation, thus so that the same curve shall be produced, and if it were required for $\int x^m ds$ to become a maximum; truly that on being integrated is reduced to this: $x^m dy = a^m d\mathrm{s}$, as now has been found above.

§.14. It shall be required to find the curve, in which $\int \frac{ds^m dy^n}{dx^{m+n-1}}$ shall become a maximum or minimum. This shall be apparent at once, since *dx* may be put constant, only that must be done so that $\int ds^m dy^n$ shall become either a minimum or a maximum.

On this account there will become $ab^m.b\mathrm{M}^n + bc^m.c\mathrm{N}^n = a\beta^m.\beta\mathrm{M}^n + \beta c^m (c\mathrm{N} - b\beta)^n.$ Hence this equation will result :

$$nab^m \cdot b\mathrm{M}^{m-1} + mab^{m-2} \cdot b\mathrm{M}^{m+1} = n \cdot\ bc^m \cdot\ c\mathrm{N}^{n-1} + m \cdot bc^{m-2} \cdot c\mathrm{N}^{n+1},$$

from which again it is concluded $nab^m \cdot b\mathrm{M}^{m-1} + mab^{m-2} \cdot b\mathrm{M}^{m+1}$ must be constant. On account of which for the curve sought, this equation shall be found : $nds^m dy^{n-1} + mds^{m-2}dy^{n+1} = adx^{m+n-1},$ which is for a right line always. Truly if this equation were given for a maximum or minimum, $\frac{\int x^h ds^m dy^n}{dx^{m+n-1}},$ then this equation would be produced $nx^k ds^m dy^{n-1} + mx^k ds^{m-2}dy^{n+1} = x^k dx^{m+n-1}.$ And generally if this quantity were proposed $\frac{\int \mathrm{P}ds^m dy^n}{dx^{m+n-1}}$, the curve sought will be determined by the following equation: $n\mathrm{P}ds^m dy^{n-1} + m\mathrm{P}ds^{m-2}dy^{n+1} = Adx^{m+n-1};$ where P shall denote some function of *x*. And then that, which must become a maximum or minimum, will have this form $s\int xds,$ will be found in the same way, clearly with the elements *ab* and *bc* thus being put in place so that $\int \left(ds\int xds + sxds\right)$ shall be a maximum or minimum, this equation : $sds^2dy + dxddy\left(sx + \int xds\right) = 0.$

§.15. Truly with these primary cases of the first class established, I go on to the second class, in which not only from all the curves arising, but only from these which shall have a certain common property, the curve must be determined, which shall have some maximum or minimum property. Therefore it shall be required to find the curve *oa*, which amongst all the curves having equally a certain property A, it will be required to find another having some property B in the maximum or minimum order. For this problem requiring to be solved it is necessary to consider three elements of the curve sought. On this account on the axis OA as it pleases , three elements shall be taken AB, BC, CD, which shall be equal to each other, and from these three elements *ab*, *bc*, and *cd* shall correspond on the curve. Again, as before, three applied lines A*a*, B*b*, C*c*, D*d* shall be drawn parallel to the axis *a*M, *b*N, *c*P. Therefore on calling OA, *x*; A*a*, *y*; and *oa*, *s*: ; there will become $AB = BC = CD = dx$; $bM = dy$; and $ab = \pm ds$; and again $cN = dy + ddy$; $bc = ds + dds$; and $dP = dy + 2ddy + d^3y$ and $cd = ds + 2dds + d^3s$.

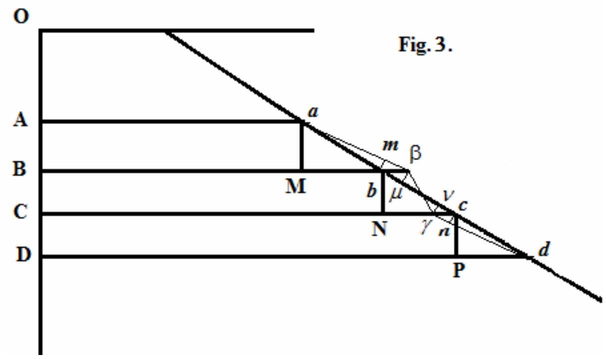


Fig. 3.

$.16. Then other curves of a certain kind shall be drawn passing through the points *a* and *d*, the elements $\alpha\beta$, $\beta\gamma$, and γd, related to the same axis. Moreover, these must be prepared thus, so that *ab*, *bc*, and *cd* equally shall contain the property A, as before ; indeed here other curves shall not be considered, unless the property A shall be present. But nevertheless these elements are able to be modified in an infinite number of ways, since they will depend on the position of the two points β an γ . On account of which the other property B at this stage, so that, if only two elements as in the preceding case were drawn into the calculation, it would not be able to happen. But from the nature of the maxima and minima the elements $\alpha\beta$, $\beta\gamma$, γd , from which it is understood the curve sought to produce *oa*, thus these two triads of elements may themselves be assumed, and each property A and B taken with the same order; and in these likewise the same account of the commutation property of A and B is in place, of which now mention has been made.

$.17. But when the elements *ab*, *bc*, and *cd* placed nearby are transformed into $\alpha\beta$, $\beta\gamma$, γd : *ab* will be increased to the particle βm, *bc* is diminished to the sum of the elements $b\gamma + cv$, and *cd* again is increased to the element γn. Similarly *bM* increases to the element $b\beta$, and *c*N decreases to the sum $b\beta + c\gamma$, and *d*P increases to the element $c\gamma$. Truly also these accruements and decrements can be reduced to $b\beta$ and $c\gamma$ by similar triangles: indeed there shall become

$$\beta m = \frac{bm.b\beta}{ab},\ b\mu = \frac{cN.b\beta}{bc},\ cv = \frac{cN.c\gamma}{bc} \text{ and } c\gamma = \frac{dP.c\gamma}{cd}.$$

Moreover, on account of these two properties A and B, which must be common to each of the elements of the tirade, two equations will be produced of which the individual terms affected will be either the particle $b\beta$ or the particle $c\gamma$. Therefore with these eliminated an equation will be elicited, in which no further quantities are present depending on the points β and γ , or with the symbols introduced, the whole will be constructed from *x*, *y*, *s* and constants. From which therefore the nature of the curve sought is known.

§.18. Truly these two equations, which arise from the consideration of the properties A and B of these two , will have a form of this kind $P.b\beta - Q.c\gamma = 0$ and $R.b\beta - S.c\gamma = 0$, in which the quantities Q and S and several more thus are prepared, so that there shall be

$\mathrm{Q}=\mathrm{P}+d\mathrm{P}$ and $\mathrm{S}=\mathrm{R}+d\mathrm{R}$. Truly if a form of this kind will not be had, the equations will be able to be reduced to such a form of this kind on being multiplied or divided. Truly if this were done, I say always to be of the form $\mathrm{P}+a\mathrm{R}=0,$ where for the quantity a some constant can be taken. For with $b\beta$ and $c\gamma$ excluded this equation $\mathrm{QR}=\mathrm{PS}$ arises, which with $\mathrm{P}+d\mathrm{P}$ and $\mathrm{R}+d\mathrm{R}$ substituted in place of S, it will be changed into this form $RdP=PdR$ from which integrated again it will become $\mathrm{P}+a\mathrm{R}=0.$ This equation produced in this manner will be that for the curve sought; whereby if these equations deduced from the properties proposed may be put in place in the manner prescribed, in any case the equation of the curve sought will be brought into view.

§.19. Therefore it will suffice for the individual properties, which shall be able to be proposed, to set out equations in the said manner so that they shall have the form $\mathrm{P}.b\beta-(\mathrm{P}+d\mathrm{P}).c\gamma=0.$ Indeed, from two of this kind taken together, the equation for the curve sought will be obtained; these equations are provided with the property, which of all the curves from which it is required to be determined, must be in common, and from that, which is sought requiring to be found, must contain the maximum order. Therefore the first quantity proposed, $\int y^n dx,$ which either must be of the common curves given, or the max. or min. sought in question : and indeed each returns the same. Hence on this account, on account of dx being constant, there must become $\mathrm{A}a^n+\mathrm{B}b^n+\mathrm{C}c^n=Aa^n+Bb^n+\mathrm{C}\gamma^n$, from which this equation is produced $\mathrm{B}\beta^{n-1}.b\beta-\mathrm{C}\gamma^{n-1}.c\gamma=0,$ which now has the prescribed property. And in symbols the corresponding magnitude of P is y^{n-1}. If there were put $\int x^n dy$ there would be produced x^{n-1} corresponding to the magnitude of P. And from the assumed quantity $\int \mathrm{T}dx$, with T some function of y, this fraction $\frac{d\mathrm{T}}{dx}$ will be found for the letter P. In a similar manner $\frac{d\mathrm{T}}{dy}$ will be produced from $\int \mathrm{T}dy$, if T were some function of x.

§. 20. $\int x^n ds$ shall express the property, where the elements ab, bc, cd, and $\alpha\beta,\ \beta\gamma,\ \gamma d$ shall be equal to each other ; there will become

$$\mathrm{OA}^n.ab+\mathrm{OB}^n.bc+\mathrm{OC}^n.cd=\mathrm{OA}^n.\alpha\beta+\mathrm{OB}^n.\beta\gamma+\mathrm{OC}^n.\gamma d\ .$$

But there becomes

$$a\beta-ab=\frac{b\mathrm{M}.b\beta}{ab},\quad bc-\beta\gamma=\frac{c\mathrm{N}.b\beta}{bc}+\frac{c\mathrm{N}.c\gamma}{bc}\ \text{ and }\ \gamma d-cd=\frac{d\mathrm{P}.c\gamma}{cd},$$

whereby this equation is produced:

$$\mathrm{OA}^n\cdot\frac{b\mathrm{M}\cdot b\beta}{ab}-\frac{\mathrm{OB}^n\cdot c\mathrm{N}}{bc}-\frac{\mathrm{OB}^n\cdot c\mathrm{N}\cdot c\gamma}{bc}+\frac{\mathrm{OC}^n\cdot d\mathrm{P}\cdot c\gamma}{cd}=0.$$

Now this expression is required to have the property, since indeed the factor multiplied by $b\beta$ shall be equal to its own differential must equal the factor, by which $-c\gamma$ is multiplied. Therefore in this case there becomes

$$\mathrm{P} = \frac{\mathrm{OA}^n \cdot b\mathrm{M}}{ab} - \frac{\mathrm{OB}^n \cdot c\mathrm{N}}{bc},$$

or by the negative of this, which in symbols is $d \cdot \frac{x^n dy}{ds}$. In a similar manner, if the proposition were $\int \mathrm{X}ds$ and X shall denote some function of *x*, for P there would be found , $d \cdot \frac{\mathrm{X}dy}{ds}$. But if $\int y^n ds$ were proposed, then for P there would be found,

$$d \cdot \frac{y^n dy}{ds} - ny^{n-1}ds.$$

And from this again it is seen from the quantity $\int \mathrm{Y}ds$, with Y designating some function of y, this quantity will be going to be found for P

$$d \cdot \frac{\mathrm{Y}dy}{ds} - \frac{d\mathrm{Y}ds}{dy}.$$

§. 21. With these requiring to be examined with care, we will be able to define which quantity shall be going to be produced corresponding to P, if more general formulas shall be taken. So that the formula $\int \mathrm{T}dx$, where T will denote some function of *x* and *y*, and there shall become $d\mathrm{T} = +\mathrm{M}dy + \mathrm{N}dx,$ there is found $\mathrm{P} = +\mathrm{M}dx.$ And the proposed formula $\int \mathrm{T}dy$, with T remaining as before, there will become $\mathrm{P} = +\mathrm{N}dx.$ And then, if the formula were proposed $\int \mathrm{T}ds$, there will be found

$$P = +\mathrm{M}ds - d \cdot \frac{\mathrm{T}dy}{ds} = \frac{\mathrm{M}dx^2}{ds} - \frac{\mathrm{N}dxdy}{ds} - \mathrm{T}d \cdot \frac{dy}{ds}.$$

These values given here, just as have been found at once from the calculation, if the prescribed properties for the elements *ab*, *bc*, *cd* may be adapted; and neither the sign to be changed nor to be multiplied or divided by constant quantities. From the same considerable usefulness arises, because the value of P also shall be able to be found, if the prescribed formula shall have a composite value, such as $\int \mathrm{T}ds + \int tdy$. Indeed if there were $dt = +mdy + ndx,$ with $d\mathrm{T}$ remaining as before, P will be the sum of these, which for any member is able to be found separately. evidently

$$P=\frac{\mathrm{M}dx^2}{ds}-\frac{\mathrm{N}dxdy}{ds}-\mathrm{T}d\cdot\frac{dy}{ds}-ndx.$$

§.22. These are the cases, when the magnitude T in the proposed formula, which is multiplied either by *dx, dy* or *ds*, is some function of *x* and *y* themselves. And in these, it is agreed in order to be used, to come upon some equation, which has the form $\mathrm{P}.b\beta-(\mathrm{P}+d\mathrm{P})c\gamma.$ But if *s* also shall be contained in T, it will not result in an equation of this kind, but to one that can be reduced to that at last. In order that this formula will be required to be proposed $\int s^n dx$ on account of the *dx*,

$oa^n+ob^n+oc^n=oa^n+o\beta^n+o\gamma^n$, or $ob^n+oc^n=o\beta^n+o\gamma^n$. Truly there becomes $o\beta-ob+\beta m=ob+\frac{b\mathrm{M}.b\beta}{ab}$, and $o\gamma=oc+\beta m-\beta\mu-cv=oc+\frac{b\mathrm{M}.b\beta}{ab}-\frac{c\mathrm{N}.b\beta}{bc}-\frac{c\mathrm{N}.c\gamma}{bc}$.

Hence, $o\beta^n-ob^n=\frac{n.ob^{n-1}b\mathrm{M}.b\beta}{ab}$ and

$$o\gamma^n-oc^n=\frac{n.oc^{n-1}b\mathrm{M}.b\beta}{ab}-\frac{n.oc^{n-1}cN.b\beta}{bc}-\frac{n.oc^{n-1}cN.c\gamma}{bc}.$$

Of which the sum of the residues, since they must vanish, there will become :

$$\left(\left(ob^{n-1}+oc^{n-1}\right)\frac{bM}{ab}-\frac{oc^{n-1}cN}{bc}\right)b\beta=\frac{oc^{n-1}cN.c\gamma}{bc}.$$

There shall be put $\frac{b\mathrm{M}}{ab}=q,$ there becomes $\frac{c\mathrm{N}}{bc}=q+dq,$ and with *s* put for *ob*, there will become $oc=s+ds$, and there will be had,

$$\left((2s^{n-1}q+\ n-1)s^{n-2}qds-s^{n-1}\left(q+dq\right)-\left(n-1\right)s^{n-2}qds\right)b\beta=\left(s^{n-1}q+s^{n-1}dq+\left(n-1\right)s^{n-2}qds\right)c\gamma,$$

From which this equation is formed, $\mathrm{P}.b\beta=\mathrm{P}.c\gamma.c^{\frac{sq+sdq+(n-1)qds}{sq-sdq}}$. Therefore so that $\mathrm{P}\frac{sq+sd\pm\left(n-1\right)qds}{sq-sdq}=\mathrm{P}+dP,$ must become the required value of P. Moreover from this equation there shall become $2\mathrm{P}sdq+\left(n-1\right)Pqds=sqdP.$ And with the integral of this $s^{n-1}q^2=\mathrm{P}$, or $\mathrm{P}=\frac{s^{n-1}dy^2}{ds^2}$.

If this formula were proposed, $\int Sdx$, where S shall denote some function of *s*, there will be produced $\mathrm{P}=\frac{s^{n-1}dy^2}{ds^2}$. And to this formulae $\int SXdx$ the value will correspond $\mathrm{P}=\frac{\mathrm{E}dsdy^2}{ds^2}$.

And generally if T were some function of *s, y* and *x;* there will become on putting $d\mathrm{T}=\mathrm{P}ds+\mathrm{M}dy+\mathrm{N}dx$, and $\mathrm{P}=c^{\int\frac{\mathrm{L}dq}{\mathrm{L}q+\mathrm{M}}}(\mathrm{L}q+\mathrm{M})$, on writing *q* in place of $\frac{dy}{ds}$.

§.23. Now this case shall be proposed, where $\int \mathrm{T}ds$ (where T as before is some function of x, y and *s*, and $d\mathrm{T}=\mathrm{L}ds+\mathrm{M}dy+\mathrm{N}dx$, in which two nearby curves ought to be the same. Therefore there will become

$$\mathrm{T}.ab+(\mathrm{T}+d\mathrm{T})bc+(\mathrm{T}+2d\mathrm{T}+dd\mathrm{T}).cd=\mathrm{T}.a\beta+(\mathrm{T}+d\mathrm{T})\beta\gamma+(\mathrm{T}+2d\mathrm{T}+dd\mathrm{T})\gamma d.$$

But the differentials *d*T and *dd*T in each member are not equal, but are different for the points β and γ. But the differences may be put equal to the first if that member shall be shall be taken away from this $-b\beta.\,d.\mathrm{T}q+c\gamma d(\mathrm{T}+d\mathrm{T})(q+dq)$; with *q* put in place of $\frac{dy}{ds}$. Now $a\beta$, $\beta\gamma$ et γd may be put equal to *ab, bc,* and *cd* themselves, and the difference may be sought, which arises from the significant variations *d*T and *dd*T. Truly for L*ds*, in the transition made from the elements *ab*, *bc, cd* to the elements $a\beta$, $\beta\gamma$, γd the increment $\mathrm{L}.\beta m=\mathrm{L}q.b\beta$, of *Mdy* itself truly $\mathrm{M}b\beta.\mathrm{N}dx$ shall not be changed. In a similar manner of $2Lds+d.Lds$ itself the increment is $(\mathrm{L}+d\mathrm{L})(\beta m-b\mu-c\gamma)=(\mathrm{L}+d\mathrm{L})(-dq.b\beta-(q+dq)c\gamma)$, and of $2\mathrm{M}dy+d.\mathrm{M}dy$ the increment is $-(\mathrm{M}+d\mathrm{M})c\gamma$. With these individual increments multiplied respectively by *bc* et *cd* , since before with the remainder found in one sum taken together, and put $=0$, this equation

$$b\beta(-\mathrm{d}.\mathrm{T}q+\mathrm{L}qds+\mathrm{M}ds-\mathrm{L}dsdq)+c\gamma\begin{pmatrix}\mathrm{d}.(\mathrm{T}+d\mathrm{T})(q+dq)-\mathrm{L}qds-\mathrm{L}qdds\\-qd\mathrm{L}ds-\mathrm{M}ds-\mathrm{M}dds-d\mathrm{M}ds\end{pmatrix}=0.$$ But here

with *bc* assumed for *ds*, and *cd* for $ds+dds$, since *ab* does not occur. If this equation may be compared with $\mathrm{P}b\beta-(\mathrm{P}+d\mathrm{P})c\gamma=0$, there will be found

$$\mathrm{P}=c^{\int\frac{\mathrm{L}ds^2dq}{\mathrm{M}dx^2-\mathrm{N}dxdy-\mathrm{T}dsdq}}\left(\frac{Mdx^2-\mathrm{N}dxdy-Tdsdq}{ds}\right),$$ where *c* shall signify the number, of which the logarithm is 1. Similarly, if the formula proposed were $\int\mathrm{T}dy$, there shall be found

$$\mathrm{P}=c^{-\int\frac{\mathrm{L}dsdydq}{\mathrm{L}dx^2+\mathrm{N}dxds}}\left(\frac{\mathrm{L}dx^2+\mathrm{N}dxds}{ds}\right).$$ Here if there were $\mathrm{N}=0$, there will become $\mathrm{P}=\frac{\mathrm{L}dx^2}{ds^2}$.

§.24. We will now consider the single formula at this point $\int Xds^m dy^n dx^{1-m-n}$, in which X is specified as a function of *x* only. Therefore with *dx* ignored as a constant, there will become:

$$\mathrm{X}.ab^m.b\mathrm{M}^n + \left(\mathrm{X}+d\mathrm{X}\right)bc^m.c\mathrm{N}^n + \left(\mathrm{X}+2d\mathrm{X}+dd\mathrm{X}\right)cd^m.d\mathrm{P}^m =$$
$$\mathrm{X}.a\beta^m.\beta\mathrm{M}^n + \left(\mathrm{X}+d\mathrm{X}\right)\beta\gamma^m.\left(c\mathrm{N}-b\beta-c\gamma\right)^n + \left(\mathrm{X}+2d\mathrm{X}+dd\mathrm{X}\right)\gamma d^m.\left(d\mathrm{P}+c\gamma\right)^m.$$

Of which that part of the equation subtracted from this there will remain :

$$\mathrm{d.X}\left(mab^{m-2}.bM^{n+1} + n.\,ab^m.\,bM^{n-1}\right) + c\gamma d.(\mathrm{X}+dX)\left(m.bc^{m-2}.c\mathrm{N}^{n+1} + n.bc^m\mathrm{N}^{n-1}\right) = 0.$$

Which equation since now it shall be had of this form: $\mathrm{P}.b\beta - (\mathrm{P}+dP)c\gamma = 0$,
there will become: $\mathrm{P} = -\mathrm{d.X}\left(mds^{m-2}dy^{n+1} + nds^m dy^{n-1}\right) = -\mathrm{d.}Xds^{m-2}dy^{n-1}(mdy^2+nds^2)$.
In a similar manner, if this formula were proposed $\int Tds^m dy^n dx^{1-m-n}$, in which T were some function of x and y, thus so that there shall become $d\mathrm{T} = \mathrm{M}dy + \mathrm{N}dx$, this equation would be produced $\mathrm{P} = \mathrm{M}ds^m dy^n - \mathrm{d.T}ds^{m-2}dy^{n-1}(mdy^2+nds^2)$. And generally, if in $\int Tds^m dy^n dx^{1-m-n}$ T were some function of *x*, *y* and *s*, and therefore
$d\mathrm{T} = \mathrm{L}ds + \mathrm{M}dy + \mathrm{N}dx$ there will become

$$P = c^{\int \frac{\mathrm{L}ds^m dy^n dq}{\mathrm{M}ds^m dy^n + Lqds^m dy^n - \mathrm{d.T}ds^{m-2}dy^{n-1}\left(mdy^2+nds^2\right)}} \cdot \mathrm{M}ds^m dy^n + Lqds^m dy^n - \mathrm{d.T}ds^{m-2}dy^{n-1}\left(mdy^2+nds^2\right).$$

And this is the most general formula including all as well as the earlier formulas.

§.25. These formulas found, as well as the corresponding values of P, which can be proposed for all the properties, by involving only one sign of the summation, so that they may be viewed more clearly, and which shall be able to be adapted easily to some cases, I have gathered these together, and set out in the following table.

Proposed properties $\left(q=\frac{dy}{ds},\ \text{et}\ ddx=0.\right)$		*Corresponding values of the letter* P.
I.	$\int \mathrm{T}dx,\quad d\mathrm{T}=\mathrm{M}dy$ – –	$\mathrm{P}=\mathrm{M}dx.$
II.	$\int \mathrm{T}dy,\quad d\mathrm{T}=\mathrm{N}dx$ – –	$\mathrm{P}=\mathrm{N}dx.$
III.	$\int \mathrm{T}ds,\quad d\mathrm{T}=\mathrm{N}dx$ – –	$\mathrm{P}=\mathrm{d.T}q.$
IV.	$\int \mathrm{T}ds,\quad d\mathrm{T}=\mathrm{M}dy$ – –	$\mathrm{P}=\mathrm{d.T}q-\mathrm{M}ds.$
V.	$\int \mathrm{T}ds,\quad d\mathrm{T}=\mathrm{M}dy+\mathrm{N}dx$ –	$\mathrm{P}=\mathrm{M}dx.$
VI.	$\int \mathrm{T}ds,\quad d\mathrm{T}=\mathrm{M}dy+\mathrm{N}dx$ –	$\mathrm{P}=\mathrm{M}dx.$
VII.	$\int \mathrm{T}ds,\quad d\mathrm{T}=\mathrm{M}dy+\mathrm{N}dx$ –	$\mathrm{P}=\mathrm{d.T}q-\mathrm{M}ds.$
VIII.	$\int \mathrm{T}dx,\quad d\mathrm{T}=Lds+\mathrm{N}dx$ –	$\mathrm{P}=\mathrm{L}q^2.$
IX.	$\int \mathrm{T}dy,\quad d\mathrm{T}=Lds+\mathrm{M}dy$ –	$\mathrm{P}=\mathrm{L}dx^2:ds^2.$
X.	$\int \mathrm{T}dx,\quad d\mathrm{T}=Lds+\mathrm{M}dy+\mathrm{N}dx.$	$\mathrm{P}=c^{\frac{\int \mathrm{L}dq}{\mathrm{L}q+\mathrm{M}}}\left(\mathrm{L}q+\mathrm{M}\right).$
XI.	$\int \mathrm{T}dy,\quad d\mathrm{T}=Lds+\mathrm{M}dy+\mathrm{N}dx.$	$\mathrm{P}=c^{-\int\frac{\mathrm{L}dsdydq}{\mathrm{L}dx^2+\mathrm{N}dxds}}\left(\frac{\mathrm{L}dx+\mathrm{N}ds}{ds}\right).$
XII.	$\int \mathrm{T}ds,\quad d\mathrm{T}=Lds+\mathrm{M}dy+\mathrm{N}dx,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}ds^2dq}{\mathrm{M}dx^2-\mathrm{N}dxdy-Tdsdq}}\left(\frac{Mdx^2-\mathrm{N}dxdy-\mathrm{T}dsdq}{ds}\right).$
XIII.	$\int \frac{\mathrm{T}ds^m dy^n}{dx^{m+n-1}} d\mathrm{T}=\mathrm{N}dx,$	$\mathrm{P}=\mathrm{d.T}ds^{m-2}dy^{m-1}\left(mdy^2+nds^2\right).$
XIV.	$\int \frac{\mathrm{T}ds^m dy^n}{dx^{m+n-1}},\ d\mathrm{T}=\mathrm{M}dy+\mathrm{N}dx,$	$\mathrm{P}=\mathrm{d.T}ds^{m-2}dy^{m-1}\left(mdy^2+nds^2\right)-Mds^m dy^n.$
XV.	$\int \frac{\mathrm{T}ds^m dy^n}{dx^{m+n-1}},\quad d\mathrm{T}=\mathrm{L}ds+\mathrm{M}dy+\mathrm{N}dx,$	

$$\mathrm{P}=c^{\int\frac{\mathrm{L}ds^m dy^n dq}{(\mathrm{LQ+M})ds^m dy^n-\mathrm{d.T}ds^{m-2}dy^{n-1}\left(mdy^2+nds^2\right)}}\left(\left(\mathrm{L}q+\mathrm{M}\right)ds^m dy^n-\mathrm{d.T}ds^{m-2}dy^{m-1}\left(dy^2+ds^2\right)\right).$$

§.26. Now with the aid of this table it will be made easy to resolve this problem both for the first as well as the second order. So that indeed as far as it concerns the first order, in which the curve is sought, which shall have the maximum or minimum of all the values A proposed; for this the following rule is required to be had; The property A is sought in the table, and with the function T adapted to that, the corresponding value of P accepted, and that may be put $=0$, which will be the equation for the curve sought. So that if the brachistochrone curve shall be sought, the curve must become the time of the descent, which is expressed by $\int\frac{ds}{\sqrt{x}}$ to be a minimum. But this formula is held in row X, and there shall be $\mathrm{T}=\frac{ds}{\sqrt{x}}$ to which there will correspond $\mathrm{P}=\mathrm{d}.\frac{q}{\sqrt{x}}$, which value since it must be $=0$, there will become $\frac{q}{\sqrt{x}}=\text{const.}$ or $dy\sqrt{a}=ds\sqrt{x}$, and $ads^2-adx^2=xds^2$. Therefore there will become $ds=\frac{dx\sqrt{a}}{\sqrt{a-x}}$ and $s=C-2\sqrt{a(a-x)}$, from which it is understood, the curve sought to be a cycloid. For finding the curve *oa* which rotated around the axis O*o* normal to *o*A, shall generate a solid figure, so that it may experience the minimum resistance in a fluid in a motion following the direction of its axis, $\int\frac{xdx^3}{ds^2}$, this is in formula XIII, where there must become $\mathrm{T}=x$, $m=-2$, $n=0$, Hence there shall be $\mathrm{P}=\mathrm{d}.-\frac{2xdy}{ds^4}=0.$ Therefore $xdx^3dy=ads^4$, from which the curve generating the solid of the minimum resistance may be determined.

§.27. The following rule will help for problems of the second kind requiring to be solved. If from all the curves with the property A set out equally that must be found, which shall contain the property B with the maximum or minimum position; the properties A and B shall be sought in the table and the corresponding values of P shall be taken, and the sum of these multiplied by constants shall be put equal to zero ; with which done the equation arising will express the nature of the curve sought: It will help to illustrate this rule by several examples. The curve *oa* shall be sought, which amongst all the curves of the same length shall enclose the maximum area; there will become $\mathrm{A}=s-\int ds,$ and $B=\int ydx$. But for these there will correspond from formula III, there will correspond $\mathrm{P}=dq$, and for this from the first $P=dx$. On account of which *adq=dx* will be the equation for the curve sought. Truly from that, this equation will be produced: $aq=\frac{ady}{ds}=x,$ or $dy=\frac{xdx}{\sqrt{a^2-x^2}}$, *i.e.* $y^2+x^2=a^2$. Which is the equation for a circle.

Now it shall be required to find the curve *oa*, which amongst all the other curves of the same length, if it may be rotated about the axis O*o*, it shall produce the solid of the maximum volume. Therefore there will become $A=\int ds$ and $B=\int x^2ds$:

: whereby for A there will become $\mathrm{P}=2xdx.$ From which, just as the rule shall be $a^2dq=2xdx.$ Which integrated gives $a^2dy=x^2ds\mp b^2ds,$ from which the nature of the stretched curve is expressed. Again the curve *oa* shall be required to be found, which amongst all the others producing a solid of the same volume on being rotated about the axis 0*o*, which shall generate the minimum surface. Therefore there will $\mathrm{A}=\int xxdy$ et $B=\int xds.$ Therefore to that from the table there will correspond $\mathrm{P}=2xdx,$

to which truly $\mathrm{P} = \mathrm{d}.xq$. Hence the equation arises $2xdx = ad.xq$ or $x^2 \pm b^2 = \frac{axdy}{ds}$.
Which is reduced to this:

$$dy = \frac{\left(x^2 + b^2\right)dx}{\sqrt{\left(a^2x^2 - \left(x^2 \pm b^2\right)^2\right)}}.$$

Which is for the circle of $b = 0$, and for the catenary if a shall become infinite, and $bb = ae.$ Also the curve oa shall be sought, which among all the curves of the same length, its centre of gravity shall be the maximum distance away from the axes Oo.
Therefore there will become $\mathrm{A} = \int ds$ and $\mathrm{B} = \frac{\int xds}{s}$. Moreover since s in all the curves is put of the same magnitude, for B there will be taken $\int xds$. Therefore for A there shall be $\mathrm{P} = dq$ and for B, $\mathrm{P} = d : xq.$ From which this equation arises $adq = d.xq$, or $aq = xq - b.$ There may be written x in place of $x - a$, there will be had $xq = b$ or $xdy = bds,$ which is the equation for the catenary.

§.28. Here I may note why I am unable to resolve a problem from the formulas because a form of this kind $\frac{\int xds}{t}$ is not found in these, unless s were in all the curves of the same length and therefore might have been able to be found in B. Indeed it can be more appropriate for the differentials to be taken again and to be reduced and with the summation sign to be attached again, so that the total quantity may have the $\int$ sign prefixed, and in this manner, $\mathrm{B} = \int \frac{ds\int sdx}{s^2}.$

Truly, since this quantity $\frac{\int sdx}{s^2}$ in T, certainly which letter always denotes an integral quantity, may not be present in this table, nothing may be useful in this table. For since in T the differentials are not present, it is easily understood nor can the integrals be present.
On this account for cases of this kind the formulas also are required to be made available. Moreover I have found, if this $\int (s^n \int sdx)ds$ were proposed, to become

$$\mathrm{P} = c^{\int \frac{sqdsdq - ndsdq\int sdx}{ds^2qdx + sdq\int dx}} \left(s^{n+1}qdx + s^n dq\int sdx\right).$$

Which in the case proposed, where there is $n = -2$, gives

$$\mathrm{P} = c^{\int \frac{sqdsdq + 2dsdq\int sdx}{s^2qdx + sdq\int sdx}} . \left(\frac{sqdx + dq\int sdx}{s^2}\right).$$

This problem after being solved must be put equal to $adq.$

Therefore with the logarithms taken and then with the differential, there will be produced:

$$\frac{ddq}{dq}=\frac{sqdsdq+2dsdq\int sdx}{ds^2qdx+sdq\int dx}+\frac{2sdqdx+qdsdx+ddq\int sdx}{sqdx+dq\int sdx}-\frac{2ds}{s}.$$

it will be changed into this: $\frac{ssqdxddy}{dq}=2ssdqdx,$

and this divided by $ssdx$ becomes $qddq=2dq^2.$ On being integrated from this there becomes $qqdx=-adq$, and again, $x=\frac{a}{q}=\frac{ads}{dy}$ or $xdy=ads,$ which is for the catenary as before.

§.29. Moreover so that we may follow with more general formulas, this shall be proposed $\int Tdx\int Vdy$. In which T and V denote some functions of *x, y·* and *s*, thus so that there shall become
$d\mathrm{T}=\mathrm{L}ds+\mathrm{M}dy+\mathrm{N}dx$ et $\mathrm{dV}=\mathrm{G}ds+\mathrm{H}dy+\mathrm{K}dx.$ From this formula there is found

$$\mathrm{P}=c^{\int\frac{Ldq\int \mathrm{V}dx-\mathrm{TG}qdy-\mathrm{TH}dx}{(\mathrm{L}q+\mathrm{M})\int \mathrm{V}dx}}\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}dx.$$

Now this proposed formula shall become $\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}dx.$ in which T and V have the preceding values, there will become

$$\mathrm{P}=c^{\int\frac{Ldq\int \mathrm{V}dy+\mathrm{T}d\mathrm{V}-\mathrm{TG}qdy-\mathrm{TH}dy}{(\mathrm{L}q+\mathrm{M})\int \mathrm{V}dy+\mathrm{TV}}}\left(\mathrm{TV}+\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}dy\right).$$

And for this formula $\int Tdx\int Vdy$ there is found

$$\mathrm{P}=c^{\int\frac{Ldq\int \mathrm{V}ds+\mathrm{Tq}d\mathrm{V}-\mathrm{TG}qds-\mathrm{TH}ds}{(\mathrm{L}q+\mathrm{M})\int \mathrm{V}ds+\mathrm{TVq}}}\left(\mathrm{TV}q+\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}ds\right).$$

Now also three of this kind are found, if there may be taken $\mathrm{T}ds$ in place of $\mathrm{T}dx.$ Moreover all these I have found, I have added as a continuation of the table.

Proposed properties *Corresponding values of the letter* P.

XVI. $\int \mathrm{T}dx\int \mathrm{V}dx,$ $\mathrm{P}=c^{\int\frac{\mathrm{L}dq\int \mathrm{V}dx-\mathrm{TG}qdx-\mathrm{TH}dx}{(\mathrm{LQ}+\mathrm{M})\int Vdx}}\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}dx.$

XVII. $\int \mathrm{T}dx\int \mathrm{V}dy,$ $\mathrm{P}=c^{\int\frac{\mathrm{L}dq\int \mathrm{V}dy+\mathrm{TdV}-\mathrm{TG}qdy-\mathrm{TH}dy}{(\mathrm{L}q+\mathrm{M})\int Vdy+\mathrm{TV}}}\left(\mathrm{TV}+\left(\mathrm{L}q+\mathrm{M}\right)\right)\int \mathrm{V}dy.$

XVIII. $\int \mathrm{T}dx\int \mathrm{V}ds,$ $\mathrm{P}=c^{\int\frac{\mathrm{L}dq\int \mathrm{V}ds+\mathrm{T}q\mathrm{dV}-\mathrm{TG}qds-\mathrm{TH}ds}{(\mathrm{L}q+\mathrm{M})\int Vds+\mathrm{TV}q}}\left(\mathrm{TV}q+\left(\mathrm{L}q+\mathrm{M}\right)\right)\int \mathrm{V}ds.$

XIX. $\int \mathrm{T}dy\int \mathrm{V}dx,$ $\mathrm{P}=c^{\int\frac{\mathrm{L}dqdy\int \mathrm{V}dx-\mathrm{TG}qdxdy-\mathrm{TH}dxdy}{(\mathrm{L}qdy+\mathrm{M}dy)\mathrm{V}dx-d\cdot\mathrm{T}\int \mathrm{V}dx-\mathrm{TV}dx}}\left(\left(\mathrm{L}qdy+\mathrm{M}dy\right)\int \mathrm{V}dx-\mathrm{d}\cdot\mathrm{T}\int Vdx\right)$

XX. $\int \mathrm{T}dy\int \mathrm{V}dy,$ $\mathrm{P}=c^{\int\frac{\mathrm{L}dydq\int \mathrm{V}dy+\mathrm{T}d\mathrm{V}qdy-\mathrm{TG}qdy^2-\mathrm{TH}dy^2}{(\mathrm{L}qdy+\mathrm{M}dy)\int \mathrm{V}dy+\mathrm{TV}dy-d\cdot\mathrm{T}\int Vdy-\mathrm{VT}dy}}\left(\left(\mathrm{L}qdy+\mathrm{M}dy\right)\int \mathrm{V}dy+\mathrm{TV}dy-\mathrm{d}\cdot\int Vdy\right)$

XXI. $\int \mathrm{T}dy \int \mathrm{V}ds,$ $\quad \mathrm{P} = c^{\int \frac{\mathrm{L}dydq\int \mathrm{V}ds + \mathrm{T}qd\mathrm{V}dy - \mathrm{TG}qdsdy - \mathrm{TH}dsdy}{(\mathrm{L}qdy + \mathrm{M}dy)\int \mathrm{V}ds + \mathrm{TV}qdy - d\cdot \mathrm{T}\int Vds}} \left((\mathrm{L}qdy + \mathrm{M}dy)\int \mathrm{V}ds + \mathrm{TV}qdy - \mathrm{d}\cdot \mathrm{T}\int Vds \right)$

XXII. $\int \mathrm{T}ds \int \mathrm{V}dx,$ $\quad \mathrm{P} = c^{\int \frac{\mathrm{L}dsdq\int \mathrm{V}dx - \mathrm{TG}dsdx - \mathrm{TH}dsdx}{(\mathrm{L}qds + \mathrm{M}ds)\int \mathrm{V}dx - d\cdot \mathrm{T}q\int Vdx}} \left((\mathrm{L}qds + \mathrm{M}ds)\int \mathrm{V}dx - \mathrm{d}\cdot \mathrm{T}q\int Vdx \right)$

XXIII. $\int \mathrm{T}ds \int \mathrm{V}dy,$ $\quad \mathrm{P} = c^{\int \frac{\mathrm{L}dsdq\int \mathrm{V}dy + \mathrm{T}d\mathrm{V}ds - \mathrm{TG}qdsdy - \mathrm{TH}dsdy}{(\mathrm{L}qds + \mathrm{M}ds)\int \mathrm{V}dy + \mathrm{TV}ds - d\cdot \mathrm{T}q\int Vdy}} \left((\mathrm{L}qds + \mathrm{M}ds)\int \mathrm{V}dy + \mathrm{TV}ds - \mathrm{d}\cdot \mathrm{T}q\int Vdy \right)$

XXIV. $\int \mathrm{T}ds \int \mathrm{V}ds,$

$$\mathrm{P} = c^{\int \frac{\mathrm{L}dsdq\int \mathrm{V}ds + \mathrm{T}qd\mathrm{V}ds - \mathrm{TG}qds^2 - \mathrm{TH}ds^2}{(\mathrm{L}qds + \mathrm{M}ds)\int \mathrm{V}ds + \mathrm{TV}qds - d\,.\mathrm{T}q\int Vds}} \left((\mathrm{L}qds + \mathrm{M}ds)\int \mathrm{V}ds + \mathrm{TV}qds - \mathrm{d}\cdot \mathrm{T}q\int \mathrm{V}dy \right)$$

And throughout there becomes $d\mathrm{T} = \mathrm{L}ds + \mathrm{M}dy + \mathrm{N}dx,$ and $dV = \mathrm{G}ds + \mathrm{H}dy + \mathrm{K}dx.$

§. 30. So that the use of these formulas may be understood better, it shall be required to enquire amongst all the curves, where catenaries of whatever thickness can be formed, the one which shall have its centre of gravity furthest away from the right line *Oo*. Here it is evident that the other condition that all the curves are to be of the same length, from which there arises $\mathrm{P} = dq$, the other condition being to consider the distance of the centre of gravity from the right line *Oo*, which is expressed by this formula, $\frac{\int xd\mathrm{S}}{\mathrm{S}}$, where S shall represent the weight of the chain *oa*, is to be differentiated, and will produce $\frac{\mathrm{S}xd\mathrm{S} - d\mathrm{S}\int xdS}{\mathrm{SS}}$, or by putting $\int Sdx + \int xdS$ in place of S*x*, this becomes $\frac{d\mathrm{S}\int xdS}{\mathrm{SS}}$. Whereby the integral of this $\int \frac{d\mathrm{S}\int xdS}{\mathrm{SS}}$ will become $= \frac{\int xdS}{\mathrm{S}}$. Let $dS = tds$, for S is a function of s, with this expression compared with the 22nd, $\mathrm{T} = \frac{t}{ss}$ and $\mathrm{V} = \mathrm{S}$. And $\mathrm{L} = \frac{\mathrm{S}\sigma - 2tt}{\mathrm{S}^2}$ on putting $dt = \sigma ds$, $\mathrm{G} = t,$ and $\mathrm{M} = \mathrm{N} = \mathrm{H} = \mathrm{K} = 0.$ From which there is produced :

$$\mathrm{P} = c^{-\int \frac{(2tt - \mathrm{S}\sigma)dsdq\int \mathrm{S}dx + \mathrm{S}t^2qdsdx}{\mathrm{S}^2tqdx + \mathrm{S}tdq\int \mathrm{S}dx}} .\left(\frac{\mathrm{S}tqdx + tdq\int \mathrm{S}dx}{SS} \right),$$

which therefore must be put equal to *adq*. Logarithms are taken and then the differentials, there will be produced

$$2\mathrm{SS}tdqdx + \mathrm{SS}\sigma qdsdx = \frac{\mathrm{SS}tqdxddq}{dq},$$

which divided by SS*tqdx* and integrated gives $qqtddx = adq$; which with $\frac{dy}{ds}$ substituted for *q* and $\frac{d\mathrm{S}}{ds}$ for *t*, can be integrated again, and produces $\mathrm{S}dy = adx$. And this expresses the nature of the catenary, of which the weight is had to the length as S to *s*. Indeed a much easier equation can be found from the same, if in $\frac{\int xdS}{S}$ we may neglect the denominator, certainly which must be the same for all the curves by the first condition. Truly since happens accidentally, I have preferred to use the direct method, especially since I have been able to show the use of these formulas.

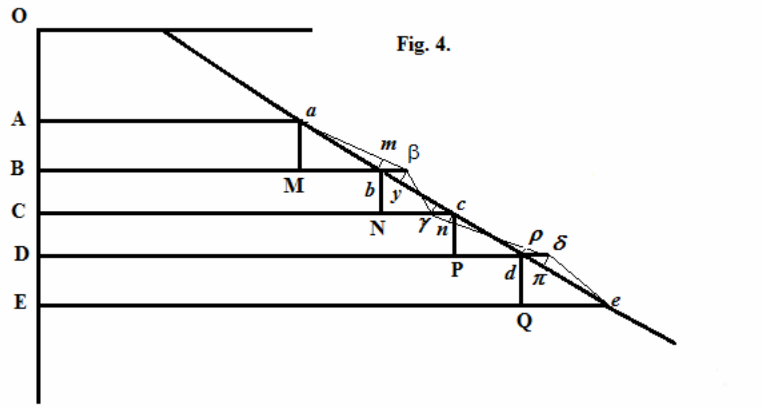


Fig. 4.

§.31. With these established for the first and second order, it will be much easier to approach the third and the following. And by beginning from the third, as we see now, all the problems pertaining to that are completely understood, so that the curve may be sought, which among all the curves which maintain the property A and likewise the aforementioned property B, shall maintain property C either to a maximum or minimum value. For problems of this kind requiring to be solved, it is necessary to consider four elements of the curve requiring to be solved (Fig.4). Hence on this account the figure is established, so that the four elements, *ab, bc, cd,* and *de* shall be shown, which as in the preceding figures, the equal elements AB , BC, CD and DE are referred to the axis OA. Therefore there will be just as many applied lines for the elements of the applied lines *b*M, *c*N, *d*P and *e*Q required to be considered. There shall remain, as before $oa = s,$ $\mathrm{OA} = x$ and $\mathrm{A}a = y,$ there will be
$\mathrm{AB} = \mathrm{BC} = \mathrm{CD} = \mathrm{DE} = dx,$ $b\mathrm{M} = dy,$ $cN = dy + ddy,$ $dP = dy + 2ddy + d^3y$ and
$eQ = dy + 3ddy + 3d^3y + d^4y$; likewise $ab = ds,$ $bc = ds + dds$,
$cd = ds + 2dds + d^3s,$ and $de = ds + 3dds + 3d^3s + d^4s.$

§.32. Then also the related elements $\alpha\beta,$ $\beta\gamma,$ $\gamma\delta,$ δe are drawn passing through the ends a and *e* to the same axis OA of the approximate curve. Therefore they will have also on account of the previous ratio brought forth of the three properties present in these two sets of four elements to be agreed equally. On account of which a formula may be expressed of any proposed property both for the elements *ab, bc, cd, de,* and for the elements $\alpha\beta,$ $\beta\gamma,$ $\gamma\delta,$ δe; and then the latter may be taken from the former, and the remainder put $= 0$. Therefore in any case three expressions will be produced in this manner, which all will have such a form $\mathrm{P}.b\beta - \mathrm{Q}.c\gamma + \mathrm{R}.d\delta = 0.$ And indeed the increments both of the individual applied lines as well as the increments of the arcs are able to be increased or decreased, as has been done before, and to be reduced to these

$b\beta,\ c\gamma,$ and $d\delta$. And P, Q and R clearly will be given by *s*, *y* and *x*, nor will they be assumed to depend on being situated nearby. Whereby since three equations of this kind $\mathrm{P}.b\beta - \mathrm{Q}.c\gamma + \mathrm{R}.d\delta = 0$ shall be obtained, the increments $b\beta,\ c\gamma,$ and $d\delta$ will be able to be eliminated ; with which done with these found individually, and in this way the nature of the curve sought *oa* will be determined.

§.33. Often and mainly in the simpler cases it happens, that there shall become $\mathrm{Q} = \mathrm{P} + d\mathrm{P}$, et $\mathrm{R} = \mathrm{P} + 2d\mathrm{P} + dd\mathrm{P}$. And it will be agreed for an equation of this form as often as the other shall be present, to be reduced, if it can be done, either by being multiplied of divided by that. But if it were possible to arrive at such equations from all three proportions proposed, it will be easy from these to form the equation for the curve sought: indeed there is only a need for this, so that the quantity in the individual terms being produced for P may be taken together multiplied in some manner, and the sum of these shall be put $= 0$. For if three equations of this kind may be had

$$\mathrm{P}.b\beta - (\mathrm{P} + d\mathrm{P})c\gamma + \left(\mathrm{P} + 2d\mathrm{P} + dd\mathrm{P}\right)d\delta = 0,$$
$$p.b\beta - (p + dp)c\gamma + \left(p + 2dp + ddp\right)d\delta = 0, \text{et} \ ,$$
$$\pi.b\beta - (\pi + d\pi)c\gamma + \left(p + 2d\pi + dd\pi\right)d\delta = 0,$$

with $b\beta, c\gamma$ and $d\delta$ eliminated, this equation will be produced:

$$pd\pi dd\mathrm{P} - \pi dpdd\mathrm{P} + \pi d\mathrm{P}ddp - \mathrm{P}d\pi ddp + \mathrm{P}dpdd\pi - pdPdd\pi = 0.$$

From which integrated there is found: $\mathrm{P} + mp + n\pi = 0,$ in which *m* and *n* designate some constant quantities. Therefore the truth of the given rules is evident.

§.34. This quantity $\int y^n dx$ shall be proposed for a certain property. On account of dx being constant,

$$\mathrm{A}a^n + \mathrm{B}b^n + \mathrm{C}c^n + \mathrm{D}d^n = Aa^n + Bb^n + \mathrm{C}\gamma^n + \mathrm{D}\delta^n,$$

of which equation, with one member taken from the other, there will remain,

$$n.\mathrm{B}b^{n-1}.b\beta - n.\mathrm{C}c^{n-1}.c\gamma + n.\mathrm{D}d^{n-1}.d\delta = 0.$$

Since which shall have now the prescribed form, there will become $\mathrm{P} = n.\mathrm{B}b^{n-1} = ny^{n-1}$. Again it is understood, if that were assumed to become $\int \mathrm{Y}dx$, where Y shall denote some function of *y*, there were going to be produced $\mathrm{P} = d\mathrm{Y} : dy$. On account of the first formula in the above table, also for the order here the third will prevail. If this formula were proposed $\int x^n dy,$ there will become :

$$\mathrm{OA}^n.b\mathrm{M} + \mathrm{O}B^n.c\mathrm{N} + \mathrm{OC}^n.d\mathrm{P} + \mathrm{OD}^n.\ e\mathrm{Q}. = \mathrm{OA}^n(b\mathrm{M} + b\beta) + \mathrm{OB}^n(c\mathrm{N} - b\beta - c\gamma)$$
$$+ \mathrm{OC}^n\left(d\mathrm{P} + c\gamma + d\delta\right) + \mathrm{OD}^n\left(eQ - d\delta\right).$$

Hence there shall become :

$$b\beta(\mathrm{OA}^n-\mathrm{OB}^n)-c\gamma\left(\mathrm{OB}^n-\mathrm{OC}^n\right)+d\delta.\left(\mathrm{OC}^n-\mathrm{OD}^n\right)=0,$$

which since they have the prescribed form, it is evident to become $\mathrm{P}=-nx^n dx$, and likewise it is understood the second formula of the table also has a place in this third class. Generally truly it is permitted to be seen, if in the prescribed formula $\int \mathrm{T}dx$, $\int \mathrm{T}dy$, $\int \mathrm{T}ds$, T may not depend on *s*, at once to arrive at an equation of the required form, and P to retain the same value, which it has in the preceding table. Therefore in the third order, the formulas I, II, III , IV, V, VI, VII prevail, and indeed also XIII and XIV. And not only in the third class, but also within all the following classes present.

§.35. And thus since the said formulas shall be able to be used in all the grades, the problems of any grade will be brought out to be resolved, but only if the properties which occur in that, shall be contained in these formulas of the table. Then truly the following rule, which is similar to the first, must be used ; clearly for the individual properties, which are brought forth in the problem, the values of the letter P are taken from the table, and of these then some multiple shall be taken and the sum of these shall be made equal to zero; and with that done the equation arising will set out the nature of the curve sought. So that if the curve shall be required to be found, amongst all the curves which are of the same length, and which enclose the same area, and which rotated about the axis O*o* generate equal volumes of the minimum surface. Here four properties occur, which are present in all the formulas designated. From the first, which gives the formula $\int ds$, there shall be $\mathrm{P}=dq$; from the second, which give $\int ydx$ there shall be $\mathrm{P}=dx$, from the third which contains the formula $\int xxdy$, there shall be $\mathrm{P}=xdx$; and from the fourth with the formula contained $\int xds$ there shall be $\mathrm{P}=d.xq$. On account of which the equation for the curve sought will be $adq+bdx+2cxdx+\mathrm{d}\cdot xq=0$, or this: $aq+bx+cx^2+xq=f$, that is $ady+bxds+cx^2+xdy=fds$, which deals with innumerable curves amongst themselves.

§.36. But before I consider these formulas for the third class, in which T also depends on *s*, I shall present certain examples requiring to be solved for which the aforementioned formulas suffice. Therefore the proposed curve shall be found, among all the curves of the same length and enclosing the same area, which will produce the maximum volume on rotating about the axis *Oo*. The three properties which occur here, are $\int ds$, $\int ydx$, and $\int x^2dy$, to which the values of P will correspond *dq*, *dx*, and *xdx*. Therefore the curve sought hence shall have the equation $adq+bdx+2xdx=0$ or

$$aq+bx+xx=c,\ i.e.\ ady+bxds+xxds=cds.$$

Now amongst all the curves again all curves of the same length and of the same area is sought, upon which a weight shall descend the fastest or in the shortest time. Here for these first two properties P will have these values dq and dx, but for the third, of which this is the formula $\int \frac{ds}{\sqrt{x}}$, hence $d\cdot\frac{q}{\sqrt{x}}$. Therefore this equation will be had for the curve sought, $adq + dx + bd\cdot\frac{q}{\sqrt{x}} = 0$ or $aq + x + \frac{bq}{\sqrt{x}} = c$, that is, $ady + xds + \frac{bdy}{\sqrt{x}} = cds$. Also the curve requiring to be found shall be also the curve, which amongst all the curves of the same area, and rotating around the axis *Oo* generating equal volumes, which shall experience the minimum resistance in a fluid moving along the direction of its axis. The first two conditions give the values dx and xdx for P, the latter truly, the formula of which is $\int \frac{xdx^3}{dx^4}$ give this: $\mathrm{d}.\frac{xdy}{ds^4}$. Therefore the curve sought will become:

$$adx + 2bxdx + \mathrm{d}.\frac{xdx^3dy}{ds^4} = 0, \text{ or } ax + bxx + \frac{xdx^3dy}{ds^4} = c.$$

Which if there shall be put $c = 0$, and x may be increased or decreased by some constant amount, will go into this: $xds^4 = adx^3dy$, which is the equation for an algebraic curve, which indeed gives on integration this equation of the fourth order:

$$y^4 - 2by^3 + 2x^2y^2 - 18bx^2y + x^4 + 27b^2x^2 = 0,$$

or

$$y^4 + 6by^3 + 2x^2y^2 + 12b^2y^2 - 10bx^2y + 8b^3y - b^2x^2 + x^4 = 0.$$

This also shall be produced if a and c were put $= 0$. From which it follows this equation to give the curve generating the volume of the minimum resistance, amongst all the curves producing the solid of the same capacity.

§.37. However problems of this kind, just as often as the formula arises, which is required to be referred to in the table by which T as well as s shall be determined, shall not be able to be solved, because the value of P shall not be had for this third order : yet often it can happen, that nevertheless the solution shall be perfected easily. Indeed often from the collection of the formulas, the given properties shown can be changed into others, which shall be contained in the formulas defined. Just as shall be required if it shall be required to determine all the curves of the same length and of the same area, in which $\int sdx$ shall be a maximum or minimum. This formula $\int sdx$ pertains as far as the eighth, because it is not allowed in the third and following classes. Truly since $\int sdx = sx - \int xds$, and the first property s prescribed must be the same in all the curves of the same length, sx will be required to have a constant value, and thus $\int xds$ also must become a minimum or a maximum. On this account for this problem, for these three formulas there can be taken $\int ds, \int ydx$ and $\int xds$, and from these the solution can be found. Indeed P will have these three values dq, dx and d.xq, from which the following equation is obtained for the power sought: $aq + bx + xq = c$ or $ady + bxds + xdy = cds$. Which, unless we shall make use of this shortcut, it would be solved with difficulty. Indeed, when reductions of this kind shall have a place, some case may be seen to be offered, which can be defined by the rule.

§.38 Yet again, we will consider formulas of this kind, in which s also is present in T ; and it shall be proposed so that $\int s^n dx$ shall be the same in each part of the quaternion. Therefore there will become on account of dx being constant

$$oa^n + ob^n + oc^n + od^n = o\alpha^n + o\beta^n + o\gamma^n + o\delta^n.$$

Truly there becomes:

$$o\beta^n - ob^n = n \cdot ob^{n-1} q \cdot b\beta,\ o\gamma^n - oc^n = n \cdot oc^{n-1}\left(-dq \cdot b\beta - \left(q + dq\right)c\gamma\right) \text{ and}$$
$$o\delta^n - od^n = n \cdot od^{n-1}\left(-dq \cdot b\beta + \left(dq + ddq\right)c\gamma + \left(q + 2dq + ddq\right)d\delta\right).$$

Therefore there shall become:

$$b\beta(ob^{n-1} \cdot q - oc^{n-1} \cdot dq - od^{n-1} \cdot dq) - c\gamma(oc^{n-1}(q + dq) - od^{n-1}\left(dq + ddq\right)$$
$$+ d\delta \cdot od^{n-1}\left(q + 2dq + ddq\right) = 0.$$

And generally if this formula were assumed $\int Tdx$ and T shall be proposed to be some function s, thus so that there shall be $d\mathrm{T} = \mathrm{L}ds$, this equation shall be produced:

$$b\beta(Lq - (2\mathrm{L} + 2d\mathrm{L} + dd\mathrm{L})dq) - c\gamma(\mathrm{L}q + qd\mathrm{L} - d\mathrm{L}dq - \mathrm{L}ddq - 2d\mathrm{L}ddq - dqdd\mathrm{L} -$$
$$dd\mathrm{L}ddq) + d\delta(\mathrm{L} + 2d\mathrm{L} + dd\mathrm{L})\left(q + 2dq + ddq\right) = 0.$$

These equations truly in no way can be reduced to such a form:

$$b\beta.\mathrm{P} - c\gamma\ (\mathrm{P} + d\mathrm{P}) + d\delta\left(\mathrm{P} + 2d\mathrm{P} + dd\mathrm{P}\right) = 0.$$

On account of which these other equations cannot be used, except that the two remaining equations, which supply the other conditions, may be taken jointly, from which these elements $b\beta$, $c\gamma$, and $d\delta$ may be eliminated. But the remaining two equations shall have such a form, and they shall become

$$b\beta p - c\gamma\left(p + dp\right) + d\delta\left(p + 2\ dp + ddp\right) = 0$$
$$\text{et } b\beta.\ r - c\gamma\left(r + dr\right) + d\delta\left(r + 2dr + ddr\right) = 0.$$

Truly this equation shall be shortened thanks to $b\beta \cdot \mathrm{A} - \gamma \cdot \mathrm{B} + d\delta \cdot \mathrm{C} = 0.$ From these if $b\beta$, $c\gamma$ et $d\delta$ shall be eliminated, this equation will be produced:

$$\mathrm{A}(pdr - rdp + pddr - rddp + dpddr - drddp) -$$
$$\mathrm{B}\left(2pdr - 2rdp + pddr - rddp\right) + \mathrm{C}\left(pdr - rdp\right) = 0,$$

or if there were put $r = pt$, this will become:

$$\mathrm{A}(ppdt + ppddt + 2pdpdt + pdpddt - pdtddp + 2dp^2dt)$$
$$-\mathrm{B}\left(2ppdt + ppddt + 2pdpdt\right) + \mathrm{C}ppdt = 0.$$

Which will determine the nature of the curve sought. Moreover in place of the equation $b\beta.\mathrm{A} - c\gamma.\mathrm{B} + d\delta.\mathrm{C} = 0$, all the equations can be substituted, which arise from whatever the formulas. And in this manner all the problems of the three classes shall be solved, in which perhaps two conditions can be deduced for the formulas having a place in this class.

§. 39. Indeed in our case, if a problem were proposed, so that between all the curves having the properties A and B may be found there, in which $\int Tdx$ (where $d\mathrm{T} = \mathrm{L}ds$) shall be a maximum or minimum, and the properties A and B may be put in place for these equations:

$b\beta \cdot p - c\gamma(p + dp) + d\delta(p + 2dp + ddp) = 0$ and $b\beta \cdot r - c\gamma(r + dr) + d\delta(r + 2drddr) = 0$;

the equation for the curve sought will be found:

$$\begin{aligned}&3\mathrm{L}pdrddq - 3\mathrm{L}rdpddq + pqdrdd\mathrm{L} - rqpdpdd\mathrm{L}\\&+2\mathrm{L}rdqddp - \mathrm{L}qdrddp + rqd\mathrm{L}ddp - 2\mathrm{L}pdqddr\\&+\mathrm{L}qdpddr - pqd\mathrm{L}ddr + 4pd\mathrm{L}drdq - 4rd\mathrm{L}dpdq = 0,\end{aligned}$$

which with the substitution made $r = pt$ will be turned into this:

$$\begin{aligned}&\mathrm{L}pqdpddt - 2\mathrm{L}ppdqddt - ppqd\mathrm{L}ddt + 3\mathrm{L}ppdtddq + ppqdtdd\mathrm{L}\\&-\mathrm{L}pqdtddp + 2\mathrm{L}qdp^2dt - 4\mathrm{L}pdpdqdt + 4ppd\mathrm{L}dqdt - 2pqd\mathrm{L}dpdt = 0.\end{aligned}$$

If the other equation shall put the areas equal, thus so that there shall be $p = ax$ and dp as well as $ddp = 0$, this equation will be produced:

$$\frac{ddr}{dr} = \frac{3\mathrm{L}ddq + qdd\mathrm{L} + 4d\mathrm{L}dq}{2\mathrm{L}dq + qd\mathrm{L}}.$$

If in addition there were $\mathrm{T} = s$, there will become $\mathrm{L} = 1$, $d\mathrm{L}$ and $dd\mathrm{L} = 0$. On account of which this equation will be found for the curve $\frac{2ddr}{dr} = \frac{3ddq}{dq}$, and on integrating, $dxdr^2 = adq^3$. If the third condition shall require all the curves to be of the same length, there will become $r = dq$, therefore this equation will arise $addq^2 = dq^3 dx$ or $\frac{addq}{\surd dq} = dq \surd dx$, which integrated gives $a \surd dq = q \surd dx + b \surd dx$, or $\frac{adq}{(b+q)^2} = dx$ and $x = \frac{a}{b+q} + c = \frac{a+cq}{b+q}$. Which is the same, as we have found for the same case in paragraph 37 .

Problematis Isoperimetrici in Latissimo Sensu Accepti Solutio Generalis.

Author
Leonhard Euler.

§. 1.

Problemata, quae curvas maximi minimive proprietate praeditas requirunt, et adhuc a Geometris tractata sunt, ad duas classes commode referuntur. Quarum prima omnia ea complectitur, quae inter omnes prorsus curvas eam postulant, quae maximi, vel minimi cuiusdam habeat proprietatem. Ad altera, vero classem omnia illa pertinent problemata, quae non ex omnibus, sed iis tantum curvis, quae communi quadam gaudent affectione, maximi minimive proprietatem habentem determinare iubent. Comprehendunt ut hae posteriora omnia in famoso isoperimetrico problemate latiori sensu accepto, cuius solutionem Celeberrimi Geometrae *Iacobus* et *Ioannes Bernoullii, Taylorus* et *Hermannus* iam pridem dederunt. Quanquam enim hi Viri inter omnes tantum curvas eiusdem longitudinis, eam, quae maximi minimiue proprietatem habeat, quaestiones possunt; tamen eorum methodi facile ad eas quoque quaestiones extendi possunt, quae quaesitam curvam ex omnibus alia communi proprietate praeditis requirunt. Ut si ex omnibus curvis, quae circa axem conversae solida generant aequalia ea invenienda sit, quae solidum minimae supersiciei producat.

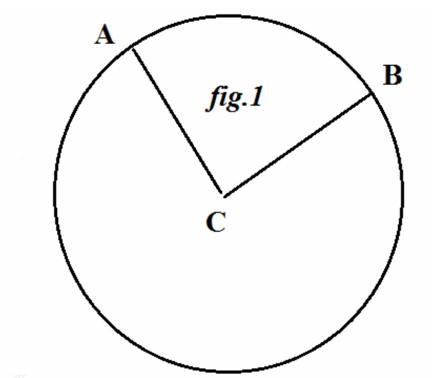
§.2 Prioris generis problematae duo potissimum agitata sunt, ad definiendas curvas celerrimi descensus et minimae resistentiae, quae utique inter omnes prorsus curvas suam proprietatem maximo sive minima possident gradu. Perspicuum autem est, quae curva inter omnes minimam patiatur resistentiam, vel celerrimum producat descensum, eandem hanc praerogativam habere inter omnes curvas eiusdem longitudinis, vel alia quacunque proprietate praeditas. Vicissim vero non valet consequentia, ut, quae inter omnes curvas eiusdem longitudinis est brachystochrona, eadem inter omnes omnino curvas talis sit ; Illius enim generis dantur innumerabiles, cum tamen in hoc praeter cycloidem nulla alia satisfaciat. Ex quo colligitur, priorem classem esse quasi speciem posterioris, hancque multo latius patere quam illam.

§.3. Haec considerans in eam incidi cogitationem, an forte tertia quaedam classis existat, cuius secunda tantum esset aliqua species ? et hoc modo progrediendo, an dentur etiam quarta, quinta , pluresque hoc ordine sequentes huiusmodi classes ? Atque reipsa ita se rem habere deprehendi cognovi enim ad has ulteriores classes perveniri, si curvae eae , ex quibus, quae maximi minimive proprietatem habeat, determinari debet, plures una habuerint affectiones: ut si inter omnes curvas eiusdem longitudinis et eandem comprehendentes aream ea requiratur, quae circa axem conversa maximum generet solidum. Aequatio autem, quam pro hac curva adeptus sum, magis erat generalis, quam si curvas tantum vel eiusdem longitudinis, vel eiusdem capacitatis posuissem. Atque sine dubio aequatio magis generalis proditura fuisset, si ad duas has proprietates adhuc unam pluresue superaddidissem. Ex quibus, quod forte admodum paradoxum videbitur, intelligitur, quo magis curvarum propositarum numerus restringatur, eo plures quaesito satisfacientes reperiri.

§.4. Has igitur classes in sequentibus quaestionibus complectar maxime universalibus. I. *Ex omnibus prorsus curvis eam determinare, quae proprietatem A maximo vel minimo gradu contineat.* II. *Ex omnibus curvis proprietate A aequaliter praeditis, eam determinare, quae proprietatem B maximo vel minimo gradu contineat.* III. *Ex omnibus curvis et proprietate A et proprietate B aequaliter praeditis, eam determinare, quae proprietatem B maximo vel minimo gradu contineat.* IV. *Ex omnibus curvisproprietatibus A et B et C singulis aequaliter praeditis, eam determinare, quae proprietatem D maximi minimove gradu contineat.* Simuli modo quinta classis curvas proprietatibus praeditas contemplabitur et ita porro sequentes.

§. 5. Harum quarum quaestionum probe est notanda proprietas ista, quod proprietates curvarum datarum cum ea, quam quaesita habere debet, possint commutari. Ita secunda quaestio, nam in prima haec commutatio locum habere nequit, congruit cum hac: *Ex omnibus curvis proprietate B praeditis, eam determinare, quae proprietatem A maximo minimove gradu habeat.* Et tertia quaestio tribus modis potest commutari, prout curva invenienda vel proprietatem A vel B vel C in summo quodam gradu continere debeat, dum interim curvae propositae duas reliquas proprietates aequaliter possideant. Hoc autem ex modo solvenai apparet, cum ea curva maximi vel minimi habeat proprietatem, quae eandem in situ proximo retinet; id quod etiam in curvas eadem proprietate gaudentes comperit.

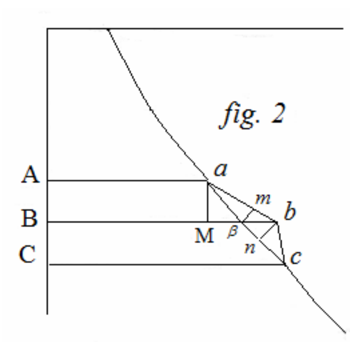
§. 6. Proprietas vero maximi vel minimi, quam curva in his problematibus quaesita habere debet, ita intelligenda est, ut nulla intra eosdem terminos detur curva, nisi ipsa quasita, quae praescriptas habeat affectiones, et tam magno vel tam parvo gradu propositam proprietatem contineat. Ita cyclois hanc habet naturam, ut nulla alia curva intra eosdem terminos dari possit, super qua corpus descendens ab altero ad alterum minori tempore perveniat. Et praeter catenariam per duo puncta transeuntem, nulla datur alia curva eiusdem longitudinis et intra eadem duo puncta contenta, cuius centrum gravitatis in inferiore loco sit positum. Assumi vero possunt pro terminis his duo quaecunquae puncta , per quae curva quaesita transit. Sicque in circulo, qui ut constat, est omnium figuratum capacissima, assumtis duobus quibuscunque punctis AB, non potest inveniri inter ea puncta alia curva eiusdem longitudinis, quae maiorem sectorem quam ABC, comprehendat.



§. 7. Ad problemata primae classis soluenda sufficit duo curvae elementa contigua considerare, quemadmodum ex solutionibus, quae passim inveniuntur, lineae brachystochromae, et solidi minimae resistentiae apparet. Secundae vero classis problemata resolvi non possunt nisi tria elementa curvae in computum ducantur. Ex hisque collegi ad solutionem problematum ad tertiam classem pertinentium quatuor opus esse curvae elementis. Atque ita porro pro quarta classe quinque elementa, pro quinta autem sex requiruntur et sic deinceps. Ex quo intelligitur solutionem problematum continuo evadere difficiliorem, quo magis iuxta has classes progrediaris. Difficultas quidem in prolixitate calculi tantum consistit, quia eo sit operosior, quo plura elementa curvae debent considerari. Vehementer is autem poterit abbreviari si debita compendia adhibeantur.

§. 8. Quo autem facilius intelligatur, qua methodo in singulis classibus uti conveniat, iis etiam quas hic non attingam, plurimum iuvabit duas priores etiam classes precurrere. Quanquam hae vero iam satis sunt tractatae, ut vix quicquam novi in iis detegi posse videatur; tamen eas methodo paulisper diversa et multo latius patente sum persecuturus, quae ad sequentes etiam classes magis est accommodata. Praeterea quoque haec inde nascetur utilitas, quod quae libet proprietas, quam curva quaesita habere debet, in prima et secunda classe ad calculum perducta, in reliquis etiam si parum immutetur, possit inservire. Multo autem maiore labore opus esset hunc calculum in sequentibus demum classibus de novo perficere.

§. 9. Oporteat igitur inter omnes priorsus curvas determinare eam *oa*, quae datam proprietatem maximo vel minimo gradu contineat. Ad hoc praestandum sumatur pro lubitu axis OA, ad quem curva quaesita referatur. Accipiantur huius elementa AB, BC aequalia, hisque respondeant in curva ipsa duo elementa *ab* et *bc*, quae in applicatis β praebeant elementa *b*M, et *c*N. Vocetur arcus *oa*, *s*; abscissa OA, *x*; applicata A*a*, *y*. Erunt $\mathrm{AB} = \mathrm{BC} = x$, $b\mathrm{M} = dy$, et $ab = ds$. Atque porro $c\mathrm{N} = dy + ddy$ et $bc = ds + dds$. Deinde manifestum est, quam maximi minimive proprietatem habeat tota curva oa, eandem habere debere quamvis eius partem; ergo etiam duo elementa *ab* et *bc*. Quamobrem non duci poterunt alia duo elementa ut $a\beta$ et βc, inter terminos *a* et *c*, quae contineant praescriptam proprietatem maiore vel minore grada. Cum vero maximi et minimi proprietas in hoc consistat, ut omnibus in situm proximum translatis, proprietas praescripta statum suum tamen retineat; considerari debebunt duo elementa proxima $a\beta$ et βc intra eosdem terminos *a* et *c*, contenta. In haec igitur praescripia proprietas aeque competere elementorum *ab* et *bc* hincque ipsa quaesita *oa* innotescet.



§. 10. In hoc autem situ proximo ab transit in $a\beta$, bc in βc; et Mβ, *ac* *c*N in $cN - b\beta$. Crescit igitur elementum *ab* particula βm, elementum vero *bc* decrescit particula *bn*. Similiter *b*M augetur particula $b\beta$ et *bn* possunt etiam ad $b\beta$ reduci per similitudinem triangulorum βbm, *ba*M et βbn, *cb*N, ex qua reperitur $\beta m = \frac{b\mathrm{M} \cdot b\beta}{ab}$, et $bn = \frac{c\mathrm{N} \cdot b\beta}{ac}$. Situ ergo proximo migrat *ab* in $ab + \frac{b\mathrm{M} \cdot b\beta}{bc}$; *bc* in $bc - \frac{c\mathrm{N} \cdot b\beta}{bc}$; *b*M in $b\mathrm{M} + b\beta$; and *c*N in $c\mathrm{N} - b\beta$, abscissae vero elementa AB et BC interim manent invariata. Deinde etiam ipsa applicata B*b* crescit elemento $b\beta$; et arcus *oab* particula βm, i.e. $\frac{bM \cdot b\beta}{ab}$. Quae, quanquam saepissime negligi possunt, tamen in genere retineri debent.

§. 11. Cum autem praescripta proprietas, quam curva *oa* maximo vel minimo gradu continere, tanta debeat reperiri in elementis *ad*, *bc*, quanta in proximis $a\beta$, βc; in utroque casu eam proprietatem ad calculum revocare conveniet, et expressiones resultantes a se invicem subtrahere; id enim, quod restat aequale erit ponendum nihilo. Singuli vero huius residui termini vel affecti erunt particula $b\beta$, vel βm et *bn*; quae autem, quia ad $b\beta$ reduci possunt, totum residuum erit per $b\beta$ divisibile, quo facto prodibit aequatio, in qua nulla prorsus quantitas a puncto β pendens reperietur, sed tota ex *x*, *y* et δ cum constantibus constabit. Ex hac igitur natura curvae quaesitae determinabitur.

§. 12. Ponamus curvam *va* eam habere debere proprietatem , ut in ea *frPd s* minorem habeatrem , quam in alia quaecunque linea per pun era *o* etr. '
a transeunte. Hanc eandem igitur proprietatem habebunt elementa *ab, bc.* Quare $\mathrm{OA}^n.ab + \mathrm{OB}^n.\beta c.$ debebit etiam esse minimum vel aequale huic quantitati $\mathrm{OA}^n.a\beta + \mathrm{OB}^n.\beta c.$ His a se invicem subtractis restabit haec aequatio $\mathrm{OA}^n.\beta m = \mathrm{OB}^n.bn$, vel loco βm et bn valoribus inventis substitutis, haec

$$\frac{\mathrm{OA}^n.b\mathrm{M}.b\beta}{ab} = \frac{\mathrm{OB}^n.c\mathrm{N}.b\beta}{bc}, \text{ sive } \frac{\mathrm{OA}^n.b\mathrm{M}}{ab} = \frac{\mathrm{OB}^n.c\mathrm{N}}{bc}.$$

Quae aequatio ita est comparata, ut posterius membrum sit ipsum prius differentiali suo auctum: Propterea differentiale huius $\frac{\mathrm{OA}^n.b\mathrm{M}}{ab} = 0$, ideoque ipsa haec quantitatis aequalis erit quantitati constanti, quae sit a^n. In symbolis igitur sequens habebitur aequatio $x^n dy = a^n ds,$ ex qua curva quaesita cognoscitur. Perspicitur ex his simul, si talis requiratur curva, ut $\int Pds$, ubi P functionem quamcunque ipsius designae, in ea sit minimum, prodituram esse aequationem hanc $\mathrm{P}dy = \mathrm{A}ds$, posito A pro constanti homogenea ipsi P. At si P fuerit functio ipsius *y*, permutatis coordinatis *x* et *y*, prodibit aequatio $\mathrm{P}dx = \mathrm{A}ds$.

§.13. Si requiratur ut in curva quaesita sit semper $\int x^m y^n ds$ maximum vel minimum, erit, posito *Bb*

$$\mathrm{OA}^m.Aa^n.ab + \mathrm{O}B^m b^n.bc = \mathrm{OA}^m.Aa^n.a\beta + \mathrm{O}B^m.\,\mathrm{B}\beta^n.\beta c = \mathrm{OA}^m.Aa^n.a\beta$$
$$+\mathrm{O}B^m.\,\mathrm{B}\beta^{n-1}b\beta.\beta c + \mathrm{O}B^m.\,\mathrm{B}\beta^{n-1}.\mathrm{B}\beta.\beta c,$$

posito $\mathrm{B}b + b\beta$ loco $b\beta$. Ex qua aequatione oritur ista

$$\frac{\mathrm{OB}^m.\mathrm{B}b^n.c\mathrm{N}.b\beta}{bc} - \frac{\mathrm{OA}^m.\mathrm{A}a^n.b\mathrm{M}.b\beta}{ab} = n.\mathrm{OB}^m.Bb^{n-1}bc.b\beta - n.\mathrm{OA}^m.Aa^{n-1}.ab.b\beta.$$

Prior autem pars ubique per $b\beta$ divisa, exprimit differentiale huius quantitatis $\frac{\mathrm{OA}^m.\mathrm{A}a^n.b\mathrm{M}}{ab}$. Quare symbolis substitutis. prodibit ista equatio $d.\frac{x^m y^n dy}{ds} = nx^m y^{n-1}ds,$ quae sumendo ibi re ipsa differentiale, et pro *dds* ponendo valorem $\frac{dyddy}{ds}$, abit in hanc $\frac{xyddy}{ds^2} + mydy - nxdx = 0.$

Reduci haec quidem .ad differentialem aequationem primi gradus potest; sed ea sit ita complicata, ut, an separari possit, non appareat. Si $\int x^m s^n ds$ debeat esse maximum vel minimum, reperitur haec aequatio

$$d.\frac{x^m s^n dy}{ds} = nx^m s^{n-1}dy,$$

quae porro mutatur in istam $xdxddy + mds^2 dy = o.$ Ex quo apparet exponentem *n* ex calculo evanescere, ita ut eadem prodeat curva , ac si requireretur $\int x^m ds$ pro maximo; ea vero integrando reducitur ad hanc $x^m dy = a^m d$s ut iam supra est inventum.

§. 14. Oporteat invenire curvam, in qua sit $\int \frac{ds^m dy^n}{dx^{m+n-1}}$ maximum vel minimum.

Hic statim apparet, quia dx ponitur constans, effici debere ut $\int \frac{ds^m dy^n}{dx^{m+n-1}}$ sit minimum vel maximum. Propterea erit $ab^m.b\mathrm{M}^n + bc^m.c\mathrm{N}^n = a\beta^m.\beta\mathrm{M}^n + \beta c^m(c\mathrm{N} - b\beta)^n.$

Hinc resultat ista aequatio

$$nab^m \cdot b\mathrm{M}^{m-1} + mab^{m-2} \cdot b\mathrm{M}^{m+1} = n \cdot bc^m \cdot c\mathrm{N}^{n-1} + m \cdot bc^{m-2} \cdot c\mathrm{N}^{n+1},$$

ex qua iterum concluditur $nab^m \cdot b\mathrm{M}^{m-1} + mab^{m-2} \cdot b\mathrm{M}^{m+1}$ deberc esse constans. Quamobrem pro curva quaesita haec invenietur aequatio $nds^m dy^{n-1} + mds^{m-2}dy^{n+1} = adx^{m+n-1},$ quae semper est pro linea recta. Si vero pro maximo minimove haec quantitas data fuisset, $\frac{\int x^h ds^m dy^n}{dx^{m+n-1}}$, tum prodiisset ista aequatio $nx^k ds^m dy^{n-1} + mx^k ds^{m-2}dy^{n+1} = x^k dx^{m+n-1}.$ Atque generatim si proponeretur ista quantitas

$\frac{\int Pds^m dy^n}{dx^{m+n-1}}$, curva quaesita sequentei determinabitur aequatione
$n\text{P}ds^m dy^{n-1} + m\text{P}ds^{m-2}dy^{n+1} = Adx^{m+n-1}$; ubi P denotat functionem ipsiuis x quamcunque. Si denique id, quod maximum vel minimum esse debet, habuerit hanc formam $s\int xds$, reperietur eodem modo, scilicet elementis ab et bc ita constituendis ut $\int\left(ds\int xds + sxds\right)$ sit maximum vel minimum, haec aequatio : $sds^2dy + dxddy\left(sx + \int xds\right) = 0.$

§. 15. His autem primariis casibus primae classis expositis, pergo ad secundam , in qua non ex omnibus prorsus curvis, sed iis solum quae communem quandan habeat proprietatem, determinari debet curva , quae maximi vel minimi quandam habeat proprietatem. Quaeri igitur oporteat curvam *oa,* quae inter omnes curvas affectionem quandam A aequalaliter continentes, habeat aliam quandam proprietatem B in maximo vel minimo gradu. Ad hoc problema soluendum tria necesse est considerare elementa curvae quaesitae. Hancobrem in axe pro lubito OA accipiantur tria elementae AB, BC, CE, quae sint inter se aequalia, hisque respondeant in curva tria elementa *ab*, *bc*, et *cd*. Ducantur porro, ut ante,tria elementae A*a*, B*b*, C*c*, D*d*, axique parallelae $\text{A}a, \text{B}b, \text{C}c, \text{D}d$, axique parallae *a*M, *b*N, *c*P. Dictis ergo OA,*x*; A*a*,*y*; et $\pm\, ds$; erit $AB = BC = CD = dx$; $bM = dy$; et $ab = ds$; porroque $c\text{N} = dy + ddy$; $bc = ds + dds$; atque $dP = dy + 2ddy + d^3y$ et $cd = ds + 2dds + d^3s.$

\$. 16. Deinde ducantur alius cuisdam curvae per puncti *a* et *d* transeuntis,elementa $\alpha\beta,\ \beta\gamma,$ et γd, ad eadem axis elementa relata. Haec autem ita debent esse comparata, ut proprietatem A aeque contineant ac priora *ab, bc*, et *cd*; aliae enim hic curvae non considerantur, nisi in quas proprietas A aequaliter competat. At nihilominus haec elementa infinitis modis inflecti possunt, quia a positione duorum punctorum β et γ pendent. Quocirca altera proprietas B adhuc iti computum duci potest, quod, si duo tantum elementa ut in antecedenta casu assumta suissent, fieri not potuisset. At ex natura maximorum et minimorum elementa $\alpha\beta,\ \beta\gamma,\ \gamma d$, ex quo intelligitur curvam quaesitam *oa* prodire, se hae duae elementorum triades ita assumantur, utramque proprietatem A et B aequali gradu comprehendant; in hocque simul ratio commutationis proprietatum A et B , cuius mentio iam est facta, consistit.

\$. 17. Quando autem elementa situm proximum *ab*, *bc*, et *cd* in situm $\alpha\beta,\ \beta\gamma,\ \gamma d$ transferuntur, augetur *ab* particula βm, *bc* minuitur summa particularum $b\gamma + cv$, et *cd* iterum augetur particula $\gamma n.$ Similiter *bM* crescit particula $b\beta$, et *c*N decrescit summa $b\beta + c\gamma$, atque *d*P crescit particula $c\gamma.$ Possunt vero illa etiam accrementa et decrementa reduci ad $b\beta$ et $c\gamma$ per similia triangula, sit enim $\beta m = \frac{bm.b\beta}{ab},\ b\mu = \frac{c\text{N}.b\beta}{bc},\ cv = \frac{c\text{N}.c\gamma}{bc}$ et $c\gamma = \frac{d\text{P}.c\gamma}{cd}.$ Propter duas autem proprietates A et B propositas, quae communes esse debent utrique elementorum tiradi, prodibunt duae aequationes quarum singuli termini affecti erunt vel particula $b\beta$ vel $c\gamma$. His igitur eliminatis elicitur aequatio, in qua nullae amplius insunt quantitates a punctis β et γ pendentes, seu symbolis introductis, tota constabit ex *x, y, s* et constantibus. Ex qua propterea curva quaesita cognoscitur.

§. 18. Duae vero illae aequationes, quae ex consideratione duarum propositarum proprietatum A et B oriuntur, huiusmodi habebunt formam $\mathrm{P}.b\beta - \mathrm{Q}.c\gamma = o$ et $\mathrm{R}.b\beta - \mathrm{S}.c\gamma = 0,$ in quibus quantitates Q et S plerumque ita sunt comparatae, ut sit $\mathrm{Q} = \mathrm{P} + d\mathrm{P}$ et $\mathrm{S} = \mathrm{R} + d\mathrm{R}$. Si vero huiusmodi formam non habuerint, poterint semper multiplicando vel dividendo aequationes ad talem reduci. Hoc si factum erit, dico fore semper $\mathrm{P} + a\mathrm{R} = 0,$ ubi pro a quantitas constans quaecunque accipi potest. Nam expulsis $b\beta$ et $c\gamma$ oritur ista aequatio $\mathrm{QR} = \mathrm{PS}$, quae substitutis $\mathrm{P} + d\mathrm{P}$ et $\mathrm{R} + d\mathrm{R}$ loco et *S,* abit in hanc $RdP = PdR$ ex qua integrata provenit $\mathrm{P} + a\mathrm{R} = 0.$ Haec aequatio hoc modo producta erit pro ipsa curva quaesita ; quare si illae aequationes ex proprietatibus propositis deductae praescripto modo instituantur, in promtu erit in quivis casu aequationem curvae quaesitae exhibere.

§ 19. Sufficiet igitur pro singulis proprietatibus, quae proponi possint, aequationes dicto modo adornasse ut habeant formam $\mathrm{P}.b\beta - (\mathrm{P} + d\mathrm{P}).c\gamma = 0.$ Huiusmodi enim duabus coniunctis obtinetur aequatio pro curva quaesita; dummodo eae aequationes ex proprietate, quae omnium curvarum ex quibus quaesita determinanda est, communis esse debet, et ex ea, quam quaesita maximo gradu continere debet, et ex ea, quam quaesita maximo gradu continere debet, eliciantur. Proposita ergo sit primo quantitas $\int y^n dx,$ quae vel communis esse debeat curvarum datarum, vel maxima minimave in quaesita: utrumque enim eodem redit. Hanc ob rem ob *dx* constans, debebit esse $\mathrm{A}a^n + \mathrm{B}b^n + \mathrm{C}c^n = Aa^n + Bb^n + \mathrm{C}\gamma^n$, unde prodit ista aequatio $\mathrm{B}\beta^{n-1}.b\beta - \mathrm{C}\gamma^{n-1}.c\gamma = 0,$ quae praescriptam iam habet proprietatem. Atque in symbolis quantitas ipsi P respondens est y^{n-1}. Si positum fuisset $\int x^n dy$ prodiisset x^{n-1} respondens quantitati P. Et ex assumta quantitate $\int \mathrm{T}dx$ designante T functionem quamcunque ipsius y, invenietur pro littera P haec fractio $\frac{d\mathrm{T}}{dx}$. Simili modo prodiisset $\frac{d\mathrm{T}}{dy}$ ex $\int \mathrm{T}dy$, si T fuerit functio ipsius *x.*

§ 20. Exprimat $\int x^n ds$ proprietatem, quae in elementis *ab, bc, cd,* et $\alpha\beta,\ \beta\gamma,\ \gamma d$ equaliter inesse debeat; erit $\mathrm{OA}^n.ab + \mathrm{OB}^n.bc + \mathrm{OC}^n.cd = \mathrm{OA}^n$ prout calculo immediate invenuuntur, si proprietas praescripta ad elementa $\alpha\beta,\ \beta\gamma,\ \gamma d$ accommodata ; neque figua mutani, neque per quantitates constantes vel multiplicavi vel divisi. Ex hoc non parva nascitur utilitas ista, quod valor ipsius P etiam invenire queat, si formula praescripta habeat valorm compositum ut ab eadem ad elementa, ut $\int Tds + \int ttdy$. Si enim fuerit $dt = mdy + ndx,$ manente dT ut ante, erit P summa eorum, quae pro quolibet membro seorsim inveniuntur, scilicet

$$\mathrm{P} = \frac{\mathrm{M}dx^2}{ds} - \frac{Ndxdy}{ds} - \mathrm{Td}.\frac{dy}{ds} - ndx.$$

§. 22. Hi sunt casus, quando in formula proposita quantitas T, quae vel in dx vel dy vel ds est ducta, est functio quaecunque ipsarum x et y. In hisque, uti constat, statim ad aequationem pervenitur, quae formam habet $\mathrm{P}.b\beta - (\mathrm{P} + d\mathrm{P})c\gamma.$ At si etiam s in T contineatur, non pervenitur ad huiusmodi aequationem, sed ea demum ad talem debet reduci. Ut sit proposita haec formula $\int s^n dx$ "dx .oportebit esse ob dx constans,

$oa^n + ob^n + oc^n = oa^n + o\beta^n + o\gamma^n$, seu $ob^n + oc^n = o\beta^n + o\gamma^n$. Est vero

$o\beta - ob + \beta m = ob + \frac{b\mathrm{M}.b\beta}{ab}$, et $o\gamma = oc + \beta m - \beta\mu - cv = oc + \frac{b\mathrm{M}.b\beta}{ab} - \frac{c\mathrm{N}.b\beta}{bc} - \frac{c\mathrm{N}.c\gamma}{bc}.$

Ergo $o\beta^n - ob^n = \frac{n.ob^{n-1}b\mathrm{M}.b\beta}{ab}$ et

$o\gamma^n - oc^n = \frac{n.oc^{n-1}b\mathrm{M}.b\beta}{ab} - \frac{noc^{n-1}cN.b\beta}{bc} - \frac{noc^{n-1}cN.c\gamma}{bc}.$

Quorum residuorum summa , cum debent evanescere erit

$$\left(\left(ob^{n-1} + oc^{n-1}\right)\frac{bM}{ab} - \frac{oc^{n-1}cN}{bc}\right)b\beta = \frac{oc^{n-1}cN.c\gamma}{bc}.$$

Ponatur $\frac{b\mathrm{M}}{ab} = q,$ erit $\frac{c\mathrm{N}}{bc} = q$ et pro ob posito s, erit $oc = s + ds$, habebiturque

$\left((2s^{n-1}q + \ n-1)s^{n-2}qds - s^{n-1}\left(q + dq\right) - \left(n-1\right)s^{n-2}qds\right)b\beta = \left(s^{n-1}q + s^{n-1}dq + \left(n-1\right)s^{n-2}qds\right)c\gamma,$

Ex qua formatur ista aequatio $\mathrm{P}.b\beta = \mathrm{P}.c\gamma c\frac{sq + sdq + \left(n-1\right)qds}{sq - sdq}.$ Debet igitur esse

$\mathrm{P}\frac{sq + sd \pm \left(n-1\right)qds}{sq - sdq} = \mathrm{P} + dP,$ ut prodeat requisitus valor ipsius P. Fiet autem ex ista

aequatione $2\mathrm{P}sdq + \left(n-1\right)Pqds = sqdP.$ Huiusque integrale $s^{n-1}q^2 = \mathrm{P}$, seu $\mathrm{P} = \frac{s^{n-1}dy^2}{ds^2}.$

Si proposita fuiiset haec formula $\int Sdx$, ubi S denotat functionem quamcumque ipsius s,

prodisset $\mathrm{P} = \frac{s^{n-1}dy^2}{ds^2}.$ Et huic formulae $\int SXdx$ respondet valor $\mathrm{P} = \frac{\mathrm{E}dsdy^2}{ds^2}.$

Atque generatim si fuerit T functio quaecunque ipsarum *s, y* et *x;* erit posito

$$dT = Pds + Mdy + Ndx,\ P = c^{\int \frac{Ldq}{Lq+M}}(Lq+M),\ \text{scripto } q \text{ loco } \frac{dy}{ds}.$$

§. 23. Propositus nunc sit hic casus, quo $\int Tds$ (ubi T ut ante est functio quaecunque ipsarum *x* ,*y* et *s*, et $dT = Lds + Mdy + Ndx$, in duabus curvis proximis debeat esse idem. Erit ergo

$$T.ab + (T + dT)bc + (T + 2dT + ddT).cd = T.a\beta + (T + dT)\beta\gamma + (T + 2dT + ddT)\gamma d.$$

At differentialia *d*T et *ddT* in utroque membra non sunt aequalia, sed differunt pro punctis β et γ. Ponantur autem primo equalia erit residuum si. illud membrum ab hoc subtrahatur $-b\beta.\,d.Tq + c\gamma d(T + dT)(q + dq);$ posito *q* loco $\frac{d\gamma}{ds}$. Ponantur iam $a\beta,\ \beta\gamma$ et γd in aequalia ipsis *ab, bc,* et *cd,* et quaeratur differentia , quae ex varia significatione *d*T et *dd*T oritur. Est vero ipsius L*ds*, transitu facto ab elementis *ab*, *bc*, *cd* ad elementa $a\beta,\ \beta\gamma,\ \gamma d$ incrementun $L.\beta m = Lq.b\beta$, ipsius *Mdy* vero $Mb\beta.Ndx$ non mutatur. Simili modo ipsius $2Lds + d.Lds$ incrementum est (L+ dL)($\beta m - b\mu - c\gamma = (L + dL)(-dq.b\beta - (q + dq)c\gamma)$, et ipsius $2Mdy + d.Mdy$ incrementum est $(M + dM)c\gamma$. His singulis incremcntis per *bc et cd* respective multiplicatis, cum ante invento residuo in unam summam coniectis, et $= 0$ positis, prodibit ista aequatio

$$b\beta(-d.Tq + Lqds + Mds - Ldsdq) + c\gamma\begin{pmatrix} d.(T + dT)(q + dq) - Lqds - Lqdds \\ -qdLds - Mds - Mdds - dMds \end{pmatrix} = 0.$$ Assumo

hic·autem *bc pro ds*, et *cd* pro $ds + dds$, quia *ab* non occurrit. Si haec aequatio .cum $Pb\beta - (P + dP)c\gamma = 0$ conferatur, reperietur

$$P = c^{\int \frac{Lds^2dq}{Mdx^2 - Ndxdy - Tdsdq}}\left(\frac{Mdx^2 - Ndxdy - Tdsdq}{ds}\right),$$ ubi *c* significat numerum, cuius logarithmus est 1.

Similiter, si

formula proposita fuerit $\int Tdy$, reperietur

$$P = c^{-\int \frac{Ldsdydq}{Ldx^2 + Ndxds}}\left(\frac{Ldx^2 + Ndxds}{ds}\right).$$

Hic si fuerit $N = 0$, erit $P = \frac{Ldx^2}{ds^2}$.

§. 24. Consideremus adhuc unicam formulam $\int Xds^m dy^n dx^{1-m-n}$, in qua X functionem tantum ipsius *x* denotat. Neglecto igitur *dx* ut constante erit

$$\mathrm{X}.ab^m.b\mathrm{M}^n + (\mathrm{X}+d\mathrm{X})bc^m.c\mathrm{N}^n + (\mathrm{X}+2d\mathrm{X}+dd\mathrm{X})cd^m.d\mathrm{P}^m =$$
$$\mathrm{X}.a\beta^m.\beta\mathrm{M}^n + (\mathrm{X}+d\mathrm{X})\beta\gamma^m.(c\mathrm{N}-b\beta-c\gamma)^n + (\mathrm{X}+2d\mathrm{X}+dd\mathrm{X})\gamma d^m.(d\mathrm{P}+c\gamma)^m.$$

Cuius aequationis illa parte ab hac subtracta restabit :

$$d.\mathrm{X}\left(mab^{m-2}.bM^{n+1} + n.\,ab^m.\,bM^{n-1}\right) + c\gamma d.(\mathrm{X}+dX)\left(m.bc^{m-2}.c\mathrm{N}^{n+1} + n.bc^m\mathrm{N}^{n-1}\right) = 0.$$

Quae aequatio cum iam habeat formam huius $\mathrm{P}.b\beta - (\mathrm{P}+dP)c\gamma = 0$,
erit $\mathrm{P} = -d.\mathrm{X}\left(mds^{m-2}dy^{n+1} + n.ds^m dy^{n-1}\right) = -d.Xds^{m-2}dy^{n-1}(mdy^2 + nds^2)$.
Simili modo si proposita fuisset haec formula $\int Tds^m dy^n dx^{1-m-n}$, in qua T fuerit functio quaecunque *x* et *y*, ita ut sit $d\mathrm{T} = \mathrm{M}dy + \mathrm{N}dx$, proditura fuisset haec aequatio $P = Mds^m dy^n - d.\mathrm{T}ds^{m-2}dy^{n-1}(mdy^2 + nds^2)$. Atque generalissime, si in $\int Tds^m dy^n dx^{1-m-n}$ fuerit T functio quaecunque ipsarum *xy* et *s*, atque propterea $d\mathrm{T} = \mathrm{L}ds + \mathrm{M}dy + \mathrm{N}dx$ erit

$$P = c^{\int \frac{\mathrm{L}ds^m dy^n dq}{\mathrm{M}ds^m dy^n + Lqds^m dy^n - d.\mathrm{T}ds^{m-2}dy^{n-1}\left(mdy^2 + nds^2\right)}} \mathrm{M}ds^m dy^n + Lqds^m dy^n - d.\mathrm{T}ds^{m-2}dy^{n-1}\left(mdy^2 + nds^2\right).$$

Haecque est formula generalissima omnes & priores in se complectens.

§. 25. Hae formulae inventae seu valores ipsius P respondentes omnibus, quae proponi possunt proprietatibus, unum tantum signum summatorium involuentibus, quo clarius in conspectum cadant, atque facilius ad casus quosvis possint accommodari, collegi eas, et in sequentem tabulam disposui.

Pr*oprietates proposiae* $\left(q = \dfrac{dy}{ds},\ \text{et } ddx = 0.\right)$ *Valores litterae* P *respondentes.*

I. $\int Tdx,\quad d\mathrm{T} = \mathrm{M}dy \quad - \quad - \quad \mathrm{P} = \mathrm{M}dx.$

II. $\int Tdy,\quad d\mathrm{T} = \mathrm{N}dx \quad - \quad - \quad \mathrm{P} = \mathrm{N}dx.$

III. $\int Tds,\quad d\mathrm{T} = \mathrm{N}dx \quad - \quad - \quad \mathrm{P} = \mathrm{d.T}q.$

IV. $\int Tds,\quad d\mathrm{T} = \mathrm{M}dy \quad - \quad - \quad \mathrm{P} = \mathrm{d.T}q - \mathrm{M}ds.$

V. $\int Tds,\quad d\mathrm{T} = \mathrm{M}dy + \mathrm{N}dx \quad - \quad \mathrm{P} = \mathrm{M}dx.$

VI. $\int Tds,\quad d\mathrm{T} = \mathrm{M}dy + \mathrm{N}dx \quad - \quad \mathrm{P} = \mathrm{M}dx.$

VII. $\int Tds,\quad d\mathrm{T} = \mathrm{M}dy + \mathrm{N}dx \quad - \quad \mathrm{P} = \mathrm{d.T}q - \mathrm{M}ds.$

VIII. $\int Tdx,\quad d\mathrm{T} = Lds + \mathrm{N}dx \quad - \quad \mathrm{P} = \mathrm{L}q^2.$

IX. $\int Tdy,\quad d\mathrm{T} = Lds + \mathrm{M}dy \quad - \quad \mathrm{P} = \mathrm{L}dx^2 : ds^2.$

X. $\int Tdx,\quad d\mathrm{T} = Lds + \mathrm{M}dy + \mathrm{N}dx.\mathrm{P} = c^{\frac{\int \mathrm{L}dq}{\mathrm{L}q+\mathrm{M}}}\left(\mathrm{L}q + \mathrm{M}\right).$

XI. $\int Tdy,\quad d\mathrm{T} = Lds + \mathrm{M}dy + \mathrm{N}dx.\mathrm{P} = c^{-\int \frac{\mathrm{L}dsdydq}{\mathrm{L}dx^2+\mathrm{N}dxds}}\left(\frac{\mathrm{L}dx+\mathrm{N}ds}{ds}\right).$

XII. $\int Tds\quad d\mathrm{T} = Lds + \mathrm{M}dy + \mathrm{N}dx.\mathrm{P} = c^{\int \frac{\mathrm{L}ds^2dq}{\mathrm{M}dx^2-\mathrm{N}dxdy-Tdsdq}}\left(\frac{Mdx^2-\mathrm{N}dxdy-\mathrm{T}dsdq}{ds}\right).$

XIII. $\int \frac{Tds^m dy^n}{dx^{m+n-1}} d\mathrm{T} = \mathrm{N}dx \quad - \quad \mathrm{P} = \mathrm{d.T}ds^{m-2}dy^{m-1}\left(mdy^2 + nds^2\right).$

XIV. $\int \frac{Tds^m dy^n}{dx^{m+n-1}} d\mathrm{T} = \mathrm{M}dy + \mathrm{N}dx,\quad \mathrm{P} = \mathrm{d.T}ds^{m-2}dy^{m-1}\left(mdy^2 + nds^2\right) - Mds^m dy^n.$

XV. $\int \frac{Tds^m dy^n}{dx^{m+n-1}},\ d\mathrm{T} = \mathrm{L}ds + \mathrm{M}dy + \mathrm{N}dx,$

$$\mathrm{P} = c^{\int \frac{\mathrm{L}ds^m dy^n dq}{(\mathrm{LQ}+\mathrm{M})ds^m dy^n - \mathrm{d.T}ds^{m-2}dy^{n-1}\left(mdy^2+nds^2\right)}}\left(\left(\mathrm{L}q + \mathrm{M}\right)ds^m dy^n - \mathrm{d.T}ds^{m-2}dy^{n-1}\left(dy^2 + ds^2\right)\right).$$

§. 26. Ope huius tabulae nunc perfacile erit problemata tum primae tum secundae classis resolvere. Quod quidem ad primam attinet, in qua quaeritur curva, quae omnium maximum vel minimum habeat valorem propositae A; ad hanc inveniendam sequens habetur regula; Quaeratur proprietates A in tabula, et functione T ad eam accomodata , accipiatur valor ipsius P respondens, isque ponatur $= 0$, quae aequatio erit pro curva quaesita. Ut si quaerenda sit curva brachystrochrona debebit tempus descensus, quod per $\int \frac{ds}{\sqrt{x}}$ exprimitur, esse minimum. Continetur autem haec formula in tertia, sitque $\mathrm{T} = \frac{ds}{\sqrt{x}}$ cui respondet $\mathrm{P} = \mathrm{d}.\frac{q}{\sqrt{x}}$, qui valor cum debeat esse $= 0$ erit $\frac{q}{\sqrt{x}} = \text{const.}$ seu

$dy\sqrt{a} = ds\sqrt{x}$, et $ads^2 - adx^2 = xds^2$. Sit igitur $ds = \dfrac{dx\sqrt{a}}{\sqrt{a-x}}$ et $s = C - 2\sqrt{a(a-x)}$, ex qua intclligitur, curvam quaestam, esse cycloidem. Ad inveniendam curvam *oa* quae circa axem O*o* ipsi *o*A normalem rotata, generat solidum, quod in fluido secundum huius axis directionem motum patiatur minimam resistentiam debebit $\int \dfrac{xdx^3}{ds^2}$ esse minimum, continetur hoc in formula XIII, ubi esse debet $T = x,\ m = -2,\ n = 0,$ Hinc sit $\mathrm{P} = \mathrm{d}.-\dfrac{2xdy}{ds^4} = 0.$ Ergo $xdx^3dy = ads^4$, ex qua curva generans folidum minimae resistentiae determinatur.

§. 27. Ad secundae classis problemata solvenda sequens inserviet regula. Si ex omnibus curvis proprietate A aequaliter praeditis ea debetit inveniri, quae proprietatem B maximo minimove gradu contineat; quaerantur proprietates A et B in tabula et sumantur valores ipsius P respondentes, eorumque per quasvis quantitates constantes multiplicatorum summa ponatur aequalis nihilo; quo facto aequatio proveniens exprimet naturam curvae quaesitae; Hanc regulam nonnullis exemplis illustrare iuuabit. Quaeratur curva *oa*, quae inter omnes eiusdem longitudinis maximam comprehendat aream; erit $\mathrm{A} = s - \int ds$, et $B = \int ydx$. Illi autem ex formula III. respondet $\mathrm{P} = dq$; huicque ex prima $P = dx$:. Quamobrem haec aequatio *adq=dx* erit pro curva quaesita. Ex illa vero prodit haec $aq = \dfrac{ady}{ds} = x$, seu $dy = \dfrac{xdx}{\sqrt{a^2 - x^2}}$, *i.e.* $y^2 + x^2 = a^2$. Quae est aequatio ad circulum. Requiratur nunc curva *oa*, quae inter omnes alias eiusdem longitudinis , si circa axem O*o* convertatur, producat maximum solidum. Erit ergo $A = \int ds$ et $B = \int x^2 ds$:

: quare pro A erit $\mathrm{P} = 2xdx.$ Ex quibus iuxta regulam, sit $a^2dq = 2xdx.$ Quae integrata dat $a^2dy = x^2ds + b^2ds,$ qua natura curvae elasticae exprimitur. Invenienda sit porro curva *oa*·, quae circum axem 0*o* rotata inter omnes alias aequalia solida producentes generet minimam superficiem. Erit ergo $\mathrm{A} = \int xxdy$ et $B = \int xds.$ Illi igitur ex Tabula respondet $\mathrm{P} = 2xdx,$ huic vero $\mathrm{P} = \mathrm{d}.xq$. Hinc nascitur aequatio $2xdx = ad.xq$ seu $x^2 \pm b^2 = \dfrac{axdy}{ds}$. Quae reducitur ad hanc

$$dy = \frac{(x^2 \pm b^2)dx}{\surd\left(a^2x^2 - (x^2 \pm b^2)^2\right)}.$$

Haec est ad circulum si $b = 0$,et ad caterariam si fiat *a* infinitum, et $bb = ae.$ Quaeratur etiam curva *oa*, quae inter omnes eiusdem longitudinis habeat centrum suum gravitatis ab axe *Oo* maxime dremotum. Erit ergo $\mathrm{A} = \int ds$ et $\mathrm{B} = \dfrac{\int xds}{s}.$ Quia autem *s* in omnibus curvis ponitur

eiusdem quantitatis, poterit pro B accipi $\int xds$. Sit igitur pro
$\mathrm{A}, \mathrm{P} = dq$ et pro $\mathrm{B}, \mathrm{P} = d : xq.$ Unde haec oritur aequatio $adq = d.xq$,
seu $aq = xq - b.$ Scribatur x loco $x - a$, habebitur $xq = bI$ seu $xdy = bds,$ quae est aequatio pro catenaria.

§. 28. Hic non possum, quin annotem, nisi s fuisset in omnibus curvis eiusdem longitudinis et propterea in B reiici potuisset, problema ex formulis resolvi non potuisse, quia huismodi forma $\frac{\int xds}{t}$ in iis non reperitur. Potest quidem ad propiorem reduci sumendo differentiali iterumque praeponendo signo summatorio, ut tota quantitas signum $\int$ habeat praefixum, sitque hoc modo $\mathrm{B} = \int \frac{ds\int sdx}{s^2}.$
Verum quia haec quantitas $\frac{\int sdx}{s^2}$ in T , quippe quae littera semper quantitatem integratam denotat, non comprehenditur, nihil ivuat ad hoc tabula. Nam quoniam in T non inesse possunt differentialia, facile intelligitur neque integralia inesse posse. Hanc ob rem pro huiusmodi casibus formulae erunt etiam eruendae. Inveni autem , si haec $\int (s^n \int sdx)ds$ fuerit proposita, fore

$$P = c^{\int \frac{sqdsdq - ndsdq\int sdx}{ds^2 qdx + sdq\int dx}} \left(s^{n+1}qdx + s^n dq\int sdx\right).$$

Quae in casu proposito, quo est $n = -a$, dat

$$P = c^{\int \frac{sqdsdq - ndsdq\int sdx}{ds^2 qdx + sdq\int dx}} \left(s^{n+1}qdx + s^n dq\int sdx\right).$$

Haec ad problema postremum soluendum debet aequalis poni $adq.$
Sumtis igitur logarithmis tumque differentialibus, prodibit

$$\frac{ddq}{dq} = \frac{sqdsdq + 2dsdq\int sdx}{ds^2 qdx + sdq\int dx} + \frac{2sqdq + qdsdx + ddq\int sdx}{sqdx + dq\int sdx} - \frac{2ds}{s}.$$

abit in hanc $\frac{ssqdxddy}{dq} = 2ssdqdx,$ haecque per $ssdx$ diuisa in $qddq = 2dq^2.$ Integrando ex hac oritur $qqdx = -adq$, atque iterum $x = \frac{a}{q} = \frac{ads}{dy}$, quae est pro catenaria ut ante.

§. 29. Quo autem generaliores huiusmodi formulas consequamur, sit haec proposita $\int Tdx\int Vdy$. In qua T et V denotant functiones quascunque ipsarum x, y· et s, ita ut sit $d\mathrm{T} = \mathrm{L}ds + \mathrm{M}dy + \mathrm{N}dx$ et $d\mathrm{V} = \mathrm{G}ds + \mathrm{H}dy + \mathrm{K}dx.$ Ex hac formula invenitur

$$\mathrm{P} = c^{\int \frac{Ldq\int \mathrm{V}dy + \mathrm{T}d\mathrm{V} - \mathrm{TG}qdy - \mathrm{TH}dy}{(\mathrm{L}q + \mathrm{M})\int \mathrm{V}dy}} \left(\mathrm{L}q + \mathrm{M}\right)\int \mathrm{V}dy.$$

Sit nunc haec formula proposita $\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}dx.$ in qua T et V praecedentes habent valores, erit

$$\mathrm{P}=c^{\int\frac{Ldq\int \mathrm{V}dy+\mathrm{T}d\mathrm{V}-\mathrm{TG}qdy-\mathrm{TH}dy}{(\mathrm{L}q+\mathrm{M})\int \mathrm{V}dy+\mathrm{TV}}}\left(\mathrm{TV}+\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}dy\right).$$

Atque pro hac formula $\int Tdx\int Vdy$ reperitur

$$\mathrm{P}=c^{\int\frac{Ldq\int \mathrm{V}ds+\mathrm{T}qd\mathrm{V}-\mathrm{TG}qds-\mathrm{TH}ds}{(\mathrm{L}q+\mathrm{M})\int \mathrm{V}ds+\mathrm{TVq}}}\left(\mathrm{TV}q+\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}ds\right).$$

Huiusmodi tres inveniuntur etiam, si sumatur $\mathrm{T}ds$ loco $\mathrm{T}dx.$ Has autem omnes prout eas inveni, tanquam tabulae continuationem adiicio.

Pr*oprietates proposiae*		*Valores litterae* P *respondentes.*
XVI.	$\int \mathrm{T}dx\int \mathrm{V}dx,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dq\int \mathrm{V}dx-\mathrm{TG}qdx-\mathrm{TH}dx}{(\mathrm{LQ+M})\int Vdx}}\left(\mathrm{L}q+\mathrm{M}\right)\int \mathrm{V}dx.$
XVII.	$\int \mathrm{T}dx\int \mathrm{V}dy,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dq\int \mathrm{V}dy+\mathrm{TdV}-\mathrm{TG}qdy-\mathrm{TH}dy}{(\mathrm{L}q+\mathrm{M})\int Vdy+\mathrm{TV}}}\left(\mathrm{TV}+\left(\mathrm{L}q+\mathrm{M}\right)\right)\int \mathrm{V}dy.$
XVIII.	$\int \mathrm{T}dx\int \mathrm{V}ds,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dq\int \mathrm{V}ds+\mathrm{T}q\mathrm{dV}-\mathrm{TG}qds-\mathrm{TH}ds}{(\mathrm{L}q+\mathrm{H})\int Vds+\mathrm{TV}q}}\left(\mathrm{TV}q+\left(\mathrm{L}q+\mathrm{M}\right)\right)\int \mathrm{V}ds.$
XIX.	$\int \mathrm{T}dy\int \mathrm{V}dx,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dqdy\int \mathrm{V}dx-\mathrm{TG}qdxdy-\mathrm{TH}dxdy}{(\mathrm{L}qdy+\mathrm{H}dy)\mathrm{V}dx-d\mathrm{T}\int \mathrm{V}dx-\mathrm{TV}dx}}\left(\left(\mathrm{L}qdy+\mathrm{M}dy\right)\int \mathrm{V}dx-\mathrm{d.T}\int Vdx\right)$
XX.	$\int \mathrm{T}dy\int \mathrm{V}dy,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dydq\int \mathrm{V}dy+\mathrm{T}d\mathrm{V}qdy-\mathrm{TG}qdy^2-\mathrm{TH}dy^2}{(\mathrm{L}qdy+\mathrm{M}dy)\int \mathrm{V}dy+\mathrm{TV}dy-d\mathrm{T}\int Vdy-\mathrm{VT}dy}}\left(\left(\mathrm{L}qdy+\mathrm{M}dy\right)\int \mathrm{V}dy+\mathrm{TV}dy-\mathrm{d.}\int Vdy\right)$
XXI.	$\int \mathrm{T}dy\int \mathrm{V}ds,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dydq\int \mathrm{V}ds+\mathrm{T}d\mathrm{V}qdy-\mathrm{TG}qdsdy-\mathrm{TH}dsdy}{(\mathrm{L}qdy+\mathrm{M}dy)\int \mathrm{V}ds+\mathrm{TV}qdy-d\,.\mathrm{T}\int Vds}}\left(\left(\mathrm{L}qdy+\mathrm{M}dy\right)\int \mathrm{V}ds+\mathrm{TV}qdy-\mathrm{d.T}\int Vds\right)$
XXII.	$\int \mathrm{T}ds\int \mathrm{V}dx,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dsdq\int \mathrm{V}dx-\mathrm{TG}dsdx-\mathrm{TH}dsdx}{(\mathrm{L}qds+\mathrm{M}ds)\int \mathrm{V}dx-d\,.\mathrm{T}q\int Vdx}}\left(\left(\mathrm{L}qds+\mathrm{M}ds\right)\int \mathrm{V}dx-\mathrm{d.T}q\int Vdx\right)$
XXIII.	$\int \mathrm{T}ds\int \mathrm{V}dy,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dsdq\int \mathrm{V}dy+\mathrm{T}d\mathrm{V}ds-\mathrm{TG}qdsdy-\mathrm{TH}dsdy}{(\mathrm{L}qds+\mathrm{M}ds)\int \mathrm{V}dy+\mathrm{TV}ds-d\,.\mathrm{T}q\int Vdy}}\left(\left(\mathrm{L}qds+\mathrm{M}ds\right)\int \mathrm{V}dy+\mathrm{TV}ds-\mathrm{d.T}q\int Vdy\right)$
XXIV.	$\int \mathrm{T}ds\int \mathrm{V}ds,$	$\mathrm{P}=c^{\int\frac{\mathrm{L}dsdq\int \mathrm{V}ds+\mathrm{T}qd\mathrm{V}ds-\mathrm{TG}qds^2-\mathrm{TH}ds^2}{(\mathrm{L}qds+\mathrm{M}ds)\int \mathrm{V}ds+\mathrm{TV}qds-d\,.\mathrm{T}q\int Vds}}\left(\left(\mathrm{L}qds+\mathrm{M}ds\right)\int \mathrm{V}ds+\mathrm{TV}qds-\mathrm{d.T}q\int Vds\right)$

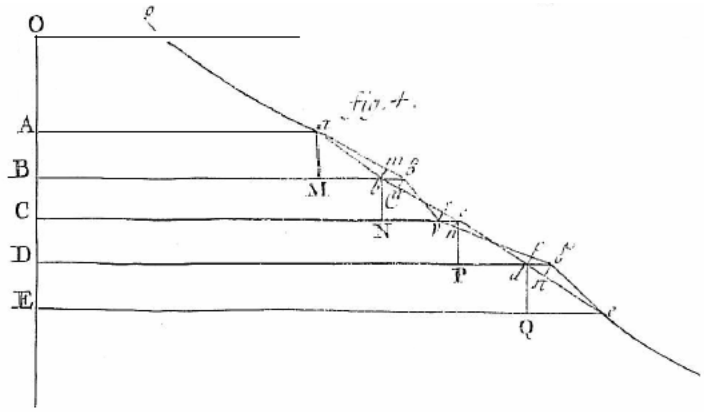
Estque ubique $dT=\mathrm{L}ds+\mathrm{M}dy+\mathrm{N}dx,$ et $dV=\mathrm{G}ds+\mathrm{H}dy+\mathrm{K}dx.$

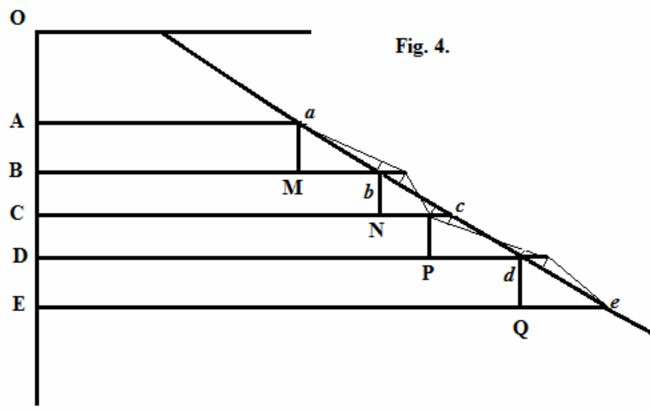
§ 30. Quo usus, harum formularum melius intelligatur, quaeri oporteat inter omnes curvas, quas catena cuiuscunque crassitiei formare potest, eam quae habeat centrum gravitatis suum a recta *Oo* remotissimum. Patet hic alteram conditionem omnes curvas eiusdem ponere longitudinis, ex qua oritur $\mathrm{P}=dq;$ alteram respicere distantiam centri gravitatis a recta *Oo*, quae hac formula exprimitur, $\frac{\int xd\mathrm{S}}{\mathrm{S}}$, ubi S pondus catenae oa representat, differentietur, et prodibit $\frac{\mathrm{S}xd\mathrm{S}-d\mathrm{S}\int xdS}{\mathrm{SS}}$ vel ponendo $\int Sdx+\int xdS$ loco Sx, hoc $\frac{d\mathrm{S}\int xdS}{\mathrm{SS}}$. Quare huius integral $\int\frac{d\mathrm{S}\int xdS}{\mathrm{SS}}$ erit $=\frac{\int xdS}{\mathrm{S}}$. Sit $dS=tds$, est enim S functio ipsius s, erit hac expressione cum vigesima secunda comparata $\mathrm{T}=\frac{t}{ss}$ et $\mathrm{V=S}$. Atque

$\mathrm{L} = \frac{\mathrm{S}c - 2tt}{\mathrm{S}^2}$ posito $dt = \sigma ds$, $\mathrm{G} = t,$ et ut£esima fecunda coli\pamta T == 5t 5 et V= S. Atque $\mathrm{M} = \mathrm{N} = \mathrm{H} = \mathrm{K} = 0.$ Ex quibus prodit

$$\mathrm{P} = c^{-\int \frac{(2tt - \mathrm{S}\sigma)dsdq\int \mathrm{S}dx + \mathrm{ST}^2 qdsdx}{\mathrm{S}^2 tqdx + \mathrm{S}tdq\int \mathrm{S}dx}} \cdot \left(\frac{\mathrm{S}tqdx + tdq\int \mathrm{S}dx}{SS}\right),$$

quod ergo aequale debet poni *adq*. Sumantur logarithmi et deinde differentialia, prodibit $2\mathrm{SS}tdqdx + \mathrm{SS}\sigma qdsdx = \frac{\mathrm{SS}tqdxddq}{dq}$, quae per SS*tqdx* divisa et integrata dat $qqddx = adq;$ quae pro *q* substituto $\frac{dy}{ds}$ et $\frac{d\mathrm{S}}{ds}$ pro *t*, iterum potest integrari, proditque $\mathrm{S}dy = adx$. Haecque .exprimit naturum catenariae, cuius pondus se habet ad longitudinem ut S ad *s*. Potuisset quidem eadem aequatio multo facilius inveniri, si in $\frac{\int xdS}{\mathrm{S}}$ neglexissem denomenatorem, quippe qui per priorem conditionem debet in omnibus curvis esse idem. Verum quia hoc fortuito accidit, malui uti methodo directa, praesertim cum constituissem usum harum formularum ostendere.





§. 31. His de prima et secunda classe expositis multo erit facilius tertiam sequentesque aggredi. Atque a tertia incipiendo, ut iam vidimus, omnia ad eam pertinentia problemata in hoc universali comprehenduntur, ut quaeratur curva, quae inter omnes et proprietate A et proprietate B simul aequaliter praeditas contineat proprietatem C maximo, minimove gradu. Ad huiusmodi problemata solvenda, necesse est quatuor curvae inveniendae elementa considerare. Hanc ob rem figuram quartam ita institui, ut quatuor elementa, *ab*, *bc*, *cd*, et *de*. exhibeantur, quae ut in praecedentibus figuris ad aequalia axis OA elementa AB , BC, CD et DE referuntur. Totidem igitur erunt etiam applicatarum elementa *bM*, *cN*, *dP* et *e*Q consideranda. Maneant, ut ante $oa = s,$ $\mathrm{OA} = x$ et $\mathrm{Aa} = y,$ erunt

$\mathrm{AB} = \mathrm{BC} = \mathrm{CD} = \mathrm{DE} = dx,$ $b\mathrm{M} = dy,$ $cN = dy + ddy,$ $dP = dy + 2ddy + d^3y$ et
$eQ = dy + 3ddy + 3d^3y + d^4y$; itemque $ab = ds,$ $bc = ds + dds$,
$cd = ds + 2dds + d^3s,$ et $de = ds + 3dds + 3d^3s + d^4s.$

§. 32. Ducantur deinde etiam curvae proximae per terminos a et e transeuntis elementa ad eadem axis OAC elementa relati $\alpha\beta,\ \beta\gamma,\ \gamma\delta$ et $\delta e.$ Debebunt ergo quoque ob rationem ante allatam singulae tres proprietates in has duas elementum quaterniones aequaliter competere. Quamobrem proprietatum propositarum quaelibet et pro elementis *ab, bc, cd, de* formula exprimatur, et pro elementis $\alpha\beta,\ \beta\gamma,\ \gamma\delta,\ \delta e$; tumque illa ab hac subtrahatus, et residuum ponatur $=0$. Huiusmodi ergo tres in quovis casu prodibunt aequationes, quae omnes talem habebunt formulam $\mathrm{P}.b\beta-\mathrm{Q}.c\gamma+\mathrm{R}.d\delta=0.$ Etenim singula tam applicatarum, quam arcuum incrementa vel decrementa possunt, ut ante est factum, ad haec $b\beta,\ c\gamma,$ et $d\delta$ reduci. Atque P, Q et R prorsus in s, y et x dabuntur, neque ab hoc assumto situ proximo pendebunt. Quare cum huiusmodii aequationes $\mathrm{P}.b\beta-\mathrm{Q}.c\gamma+\mathrm{R}.d\delta=0$ tres obtineantur, poterunt particulae $b\beta,\ c\gamma,$ et $d\delta$ eliminari ; quo facto resultabit aequatio ab illis liberata, haecque determinabit curvae quaesitae *oa*.

§.33. Saepe et potissimum in casibus simplicioribus accidit, ut sit $\mathrm{Q}=\mathrm{P}+d\mathrm{P}$, et $\mathrm{R}=\mathrm{P}+2d\mathrm{P}+dd\mathrm{P}.$ Atque ad huiusmodiformam convenit aequationem, quoties aliam habuerit, reducere, si fieri potest, vel multiplicanda vel dividenda ea. Si autcm ex omnibus. tribus proportionibus propositis ad tales aequationes perventum fuerit, facil erit ex iis aequationem pro curva quaesita formae: hoc enim tantum opus est, ut quantitatum in singulis pro P prodeuntium sumantur quaecunque multipla, eorumque summa ponatur $=0$. Nam si tres habeantur huiusmodi aequationes

$$\mathrm{P}.b\beta-(\mathrm{P}+d\mathrm{P})c\gamma+\left(\mathrm{P}+2d\mathrm{P}+dd\mathrm{P}\right)d\delta=0,$$
$$p.b\beta-(p+dp)c\gamma+\left(p+2dp+ddp\right)d\delta=0,\text{et },$$
$$\pi.b\beta-(\pi+d\pi)c\gamma+\left(p+2d\pi+dd\pi\right)d\delta=0,$$

prodibit eliminatis $b\beta, c\gamma$ et $d\delta$, haec aequatio
$pd\pi dd\mathrm{P}-\pi dpdd\mathrm{P}+\pi d\mathrm{P}ddp-\mathrm{P}d\pi ddp+\mathrm{P}dpdd\pi-pdPdd\pi=0.$
Ex qua integrata reperitur $\mathrm{P}+mp+n\pi=0,$ in qua m et n quantities quascunque constantes designant. Patet ergo veritas regulae datae.

§.34. Proposita sit. pro quapiam proprietate haec quantitas $\int y^n dx.$ Erit ob dx constans $\mathrm{A}a^n+\mathrm{B}b^n+\mathrm{C}c^n+\mathrm{D}d^n=Aa^n+Bb^n+\mathrm{C}\gamma^n+\mathrm{D}\delta^n$, cuius aequationis, si illud membrum ab hoc subtrahatur, remanebit $n.\mathrm{B}b^{n-1}.b\beta-n.\mathrm{C}c^{n-1}.c\gamma+n.\mathrm{D}d^{n-1}.d\delta=0.$ Quae cum iam habeat formam praescriptam erit $\mathrm{P}=n.\mathrm{B}b^{n-1}=ny^{n-1}.$ Perspicitur porro si assumta fuisset $\int \mathrm{Y}dx.$ ubi Y denotat functionem quamcunque ipsius y, prodituram fuisse $\mathrm{P}=d\mathrm{Y}:dy$. Quamobrem formula prima in tabula superiore etiam pro classe hac tertia valebit. Si sit proposita haec formula $\int x^n dy,$ erit

$$\mathrm{OA}^n.b\mathrm{M}+\mathrm{O}B^n.c\mathrm{N}+\mathrm{OC}^n.d\mathrm{P}+\mathrm{OD}^n.\ e\mathrm{Q}.=\mathrm{OA}^n(b\mathrm{M}+b\beta)+\mathrm{OB}^n(c\mathrm{N}-b\beta-c\gamma)$$
$$+\mathrm{OC}^n\left(d\mathrm{P}+c\gamma+d\delta\right)+\mathrm{OD}^n\left(eQ-d\delta\right).$$

Hinc sit $b\beta(\mathrm{OA}^n - \mathrm{OB}^n) - c\gamma\left(\mathrm{OB}^n - \mathrm{OC}^n\right) + d\delta.\left(\mathrm{OC}^n - \mathrm{OD}^n\right) = 0,$ quae cum habent formam praescriptam, apparet esse $\mathrm{P} = -nx^n dx,$ atque simul intelligitur formulam secundam tabulae in hac tertia classe etiam locum habere. Generatim vero videre licet, si in praescripta formula $\int \mathrm{T}dx$, $\int \mathrm{T}dy, \int \mathrm{T}ds,$ T ab *s* non pendeat, statim ad aequationem T requisitam formam habentem perveniri, atque P eundem retinere valorem, quam habet in tabula praecedente. Valent ergo in tertia classe etiam formulae I, II, III , IV, V, VI, VII imo quoque XIII, et XIV. Atque non solum in tertia sed etiam omnibus sequentibus classibus subsistunt.

§. 35. Cum itaque dictae formulae in omnibus classibus usurpari possint, in promtu erit problema cuiuscunque classis propositum resolvere, si modo proprietates, quae in illo occurrunt, in istis tabulae formulis contineantur. Tum vero sequens regula, quae priori similis est, debet adhiberi ; pro singulis scilicet proprietatibus, quae in problemate afferuntur quaerendi, sunt valores litterae P ex tabla, eorumque tum sumantur multipla quaecunque et horum summa fiat aequalis nihilo; quo facto aequatio proveniens exponet naturum curvae quastitae. Vt si invenienda esset curva, quae inter omnes, quae sunt eiusdem longitudinis, et eandem comprehendunt aream, et circum axem O*o* conversae generant solida aequalia, producat circa hunc eundem axem rotata solidum minimae superficiei. Occurrunt hic quatuor proprietates, quae omnes in designatis formulis continentur. Ex prima, quae dat formulam $\int ds,$ sit $\mathrm{P} = dq$ ex secunda, quae dat $\int ydx$ sit $\mathrm{P} = dx$, ex tertia contenta formula $\int xxdy,$ sit $\mathrm{P} = xdx$; et ex quarta contenta formula $\int xds$ sit $\mathrm{P} = d.xq.$ Quocirca aequatio pro curva quaesita erit
$adq + bdx + 2cxdx + d.xq = 0,$ seu haec $adq + bdx + 2cxdx + \mathrm{d}.xq = 0,$ seu haec
$aq + bx + cx^2 + xq = f$, hoc est $ady + bxds + cx^2 + xdy = fds,$ quae innumerabiles curvas in se comprehendit.

§. 36. Antequam autem eas formulas pro tertia classe contemplor, in quibus T etiam ab *s* pendet, afferam quaedam exempla ad quae solvenda memoratae formulae sufficiunt. Sit igitur propositum curvam invenire, quae inter omnes eiusdem longitudinis et eandem aream comprehendentes generet circa axem *Oo* conversa maximum solidum. Tres proprietates quae hic occurrunt, funt $\int ds,$ $\int xdx,$ et $\int x^2dy$, quibus respondent hi ipsius P valores *dq*, *dx*, et *xdx*. Ergo curva quaesita hanc habebit aequatationem
$adq + bdx + 2xdx = 0$ seu $aq + bx + xx = c,$ *i.e.* $ady + bxds + xxds = cds.$ Quaeratur nunc inter omnes iterum curvas eiusdem longitudinis et eiusdem areae curva, super qua grave descendat celerrime seu tempore descendat celerrime seu tempore brevissimo. Hic pro prioribus duabus proprietatibus habet P hos valores *dq* et *dx*, pro tertia autem, cuius haec est formula $\int \frac{ds}{\sqrt{x}},$ hunc $\mathrm{d}.\frac{q}{\sqrt{x}}$. Habebitur ergo pro curva quaesita haec aequatio,

$adq + dx + b\mathrm{d}.\frac{q}{\sqrt{x}} = 0$ seu $aq + x + \frac{bq}{\sqrt{x}} = c,$ i.e. $ady + xds + \frac{bdy}{\sqrt{x}} = cds.$ Invenienda sit etiam curva, quae inter omnes eiusdem areae, et circa axem *Oo* rotatas aequalia solida generantes, producat circa eundem axem conversa solidum, quod in fluido secundum huius axis directionem motum miniman patiatur resistentiam. Priores duae conditiones dant pro P hos valores *dx* et *xdx*, posterior, vero, cuius formula est $\int \frac{xdx^3}{dx^4}$ hunc $\mathrm{d}.\frac{xdy}{ds^4}$.

Erit ergo in curva quaesita $adx + 2bxdx + \mathrm{d}.\frac{xdx^3dy}{ds^4} = 0,$ seu $ax + bxx + \frac{xdx^3dy}{ds^4} = c.$ Quae si ponatur $c = 0$, et *x* augeatur constante quadam vel minuatur, abit in hanc $xds^4 = adx^3dy,$ quae aequatio est pro curva algebraica, dat enim integrata hanc aequationem quarti ordinis $y^4 - 2by^3 + 2x^2y^2 - 18bx^2y + x^4 + 27b^2x^2 = 0,$

vel

$$y^4 + 6by^3 + 2x^2y^2 + 12b^2y^2 - 10bx^2y + 8b^3y - b^2x^2 + x^4 = 0.$$

Haec etiam prodiisset si *a* et *c* fuissent positae $= 0.$ Ex quo sequitur hanc aequationem dare curvam minimae resistentiae solidum generautem, inter omnes curvas eiusdem capacitatis solida producentes.

§. 37 Quanquam autem huiusmodi problemata, quoties formula occurrit, quae in tabula ad talem referenda est, in qua T etiam in *s* determinatur, iuxta datam regulam resolvi nequeunt; quia non habetur valor ipsius P pro hac tertia classe : tamen saepe fieri potest, ut nihilominus facile sit solutionem perficere. Ex collatione enim formularum datas proprietates exhibentium saepe eae in alias possunt transmutari, quae in definitis formulis contineantur. Ut si oporteat inter omnes curvas eiusdem longitudinis et eiusdem areae eam determinare, in qua $\int sdx$ sit maximum vel minimum. Pertinet haec formula $\int sdx$ ad octavam, qua uti in tertia et sequentibus classibus non licet. Verum quia $\int sdx = sx - \int xds,$ atque per primam proprietatem praescriptam *s* in omnibus curvis debet esse eiusdem longitudinis, habebit *sx* valorem constantem, adeoque $\int xds$ debebit quoque esse minimum vel maximum. Hanc ob rem pro hoc problemate hae tres formulae poterunt recipi $\int ds, \int ydx$ et $\int xds,$ ex hisque solutio inveniri. Habebit enim P tres hos valores *dq*, *dx* et d.*xq*, ex quibus pro curva quaesita sequens obtinetur aequatio $aq + bx + xq = c$ seu $ady + bxds + xdy = cds$. Quae, nisi hoc compendio usi essemus, difficillime eruta fuiisset. Quando vero huiusmodi reductiones locum habeant facilius est quovis casu oblato perspicere, quam per regulam definire.

§.38. Consideremus tamen huiusmodi formulas, in quibus etiam *s* in T ingreditur ; sitque propositum ut $\int s^n dx$ in utroque elementorum quaternione sit idem. Erit ergo ob *dx* constans $oa^n + ob^n + oc^n + od^n = o\alpha^n + o\beta^n + o\gamma^n + o\delta^n.$ Est vero

$$o\beta^n - ob^n = n\cdot ob^{n-1}q\cdot b\beta,\ o\gamma^n - oc^n = no\cdot c^{n-1}\left(-dq\cdot b\beta-\left(q+dq\right)c\gamma\right) \text{ et}$$
$$o\delta^n - od^n = n\cdot od^{n-1}\left(-dq\cdot b\beta+\left(dq+ddq\right)c\gamma+\left(q+2dq+ddq\right)d\delta\right).$$

Sit igitur

$$b\beta(ob^{n-1}\cdot q - oc^{n-1}\cdot dq - od^{n-1}\cdot dq) - c\gamma(oc^{n-1}(q+dq) - od^{n-1}\left(dq+ddq\right)$$
$$+d\delta\cdot od^{n-1}\left(q+2dq+ddq\right)=0.$$

Et generatim si assumta fuisset haec formula $\int Tdx$ significetque T functionem quamcunque ipsius *s*, ita ut sit $d\mathrm{T}=\mathrm{L}ds$, proditura fuisset aequatio ista

$$b\beta(Lq-(2\mathrm{L}+2d\mathrm{L}\ +dd\mathrm{L})dq)-c\gamma(\mathrm{L}q+qd\mathrm{L}-\ d\mathrm{L}dq-\mathrm{L}ddq-2d\mathrm{L}ddq-dqdd\mathrm{L}-$$
$$dd\mathrm{L}ddq)+d\delta(\mathrm{L}+2d\mathrm{L}+dd\mathrm{L})\left(q+2dq+ddq\right)=0.$$

Hae vero aequationes nullo modo ad talem formam $b\beta.\mathrm{P}-c\gamma\ (\mathrm{P}+d\mathrm{P})+d\delta\left(\mathrm{P}+2d\mathrm{P}+\ dd\mathrm{P}\right)=0$ reduci possunt. Quamobrem eae aliter adhiberi non potenmt, nisi ut cum duabus reliquis aequationibus, quas alterae conditiones. suppeditant, coniungatur, et re ipsa elementa $b\beta,\ c\gamma,$ et $d\delta$ *i* eliminentur. Habeant autem reliquae duae aequationes talem formam, et sint

$$b\beta p-c\gamma\left(p+dp\right)+d\delta\left(p+2\ dp+ddp\right)=0$$
$$\text{et } b\beta.\ r-c\gamma\left(r+dr\right)+d\delta\left(r+2dr+ddr\right)=0.$$

Illa vero aequatio sit brevitatis gratia $b\beta.\mathrm{A}-\gamma.\mathrm{B}+d\delta.\mathrm{C}=0.$ Ex iis si eliminentur $b\beta,\ c\gamma$ et $d\delta$ prodibit ista aequatio:

$$\mathrm{A}(pdr-rdp+pddr-rddp+dpddr-drddp)-$$
$$\mathrm{B}\left(2pdr-2rdp+pddr-rddp\right)+\mathrm{C}\left(pdr-rdp\right)=0,$$

vel si ponatur $r=pt$, haec:

$$\mathrm{A}(ppdt+ppddt+2pdpdt+\ pdpddt-\ pdtddp+2dp^2dt)$$
$$-\mathrm{B}(\ 2ppdt+ppddt+2pdpdt)+\mathrm{C}ppdt=0.$$

Quae determinabit naturam curvae quaesitae. Poterunt autem loco aequationis $b\beta.\mathrm{A}-c\gamma.\mathrm{B}+d\delta.\mathrm{C}=0,$ omnes aequationes, quae ex quibuscunque.; 'Ctmque formnlis oriuntur, si.Jbftitui. Atque hoc modo omnia tertiae classis problemata solventur, in quibus duae saltem conditiones ad formulas in hac classe locum habentes deducunt.

§. 39. In nostro quidem casu, si problema fuerit propositum, ut inter omnes curvas proprietates *A* et B habentes ea inveniatur, in qua $\int Tdx$ (ubi $d\mathrm{T}=\mathrm{L}ds$) sit maximum minimumve, atque proprietates A et B ad has aequationes $b\beta.p-c\gamma(p+dp)+d\delta(p+2dp+ddp)=0$ et $b\beta.r-c\gamma(r+dr)+d\delta(r+2drddr)=0$;

reducantur ; reperietur pro curva quaesita sequens aequatio,

$$3Lpdrddq - 3Lrdpddq + pqdrddL - rqpdpddL + 2Lrdqddp - Lqdrddp$$
$$+rqdLddp - 2Lpdqddr + Lqdpddr - pqdLddr + 4pdLdrdq - 4rdLdpdq = 0,$$

quae facta substitutione $r = pt$ in hanc abit

$$Lpqdpddt - 2Lppdqddt - ppqdLddt + 3Lppdtddq + ppqdtddL$$
$$-Lpqdtddp + 2Lqdp^2dt - 4Lpdpdqdt + 4ppdLdqdt - 2pqdLdpdt = 0.$$

Si altera conditio ponat areas aequales, ita ut sit $p = ax$ et dp et $ddp = 0$ prodibit ista aequatio $\frac{ddr}{dr} = \frac{3Lddq + qddL + 4dLdq}{2Ldq + qdL}$. Si praeterea fuerit $T = s$, erit $L = 1$ et dL et $ddL = 0$. Quocirca habebitur pro curva quaesita ista aequatio $\frac{2ddr}{dr} = \frac{3ddq}{dq}$, et integrando $dxdr^2 = adq^3$. Si tertia conditio requirat omnes curvas eiusdem longitudinis erit $r = dq$, proveniet igitur haec aequatio $addq^2 = dq^3dx$ vel $\frac{addq}{\sqrt{dq}} = dq\sqrt{dx}$, quae integrata dat $a\sqrt{dq} = q\sqrt{dx} + b\sqrt{dx}$, seu $\frac{adq}{(b+q)^2} = dx$ atque $x = \frac{a}{b+q} + c = \frac{a+cq}{b+q}$. Quae est eadem, quam pro eodem casu in §.37 invenimus