On the Resolution of Indeterminate Quadratic Formulas by Integers.

E 279 : L. Euler.

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PROBLEM 1

1. *For the proposed irrational formula*

$$\sqrt{(\alpha xx+\beta x+\gamma)}$$

to find the numbers requiring to be substituted for x ,which shall render that rational.

SOLUTION

Before all it is required to be observed that this investigation is susceptible to difficulties, except perhaps in the one case where that formula may be agreed to become rational. Therefore we may put here $x=a$ for this case to occur, and the formula shall become

$$\sqrt{(\alpha aa+\beta a+\gamma)}=b\ ,$$

thus so that *b* shall be a rational number. But from a single known case of this kind to occur innumerable others will be enabled to be derived from that. To this end there may be put

$$x=a+mz \quad \text{and} \quad \sqrt{(\alpha xx+\beta x+\gamma)}=b+nz$$

and from this quadratic equation there shall become :

$$\begin{aligned}&+\alpha aa+2\alpha maz+ammzz=bb+2nbz+nnzz\\&+\ \beta a+\beta mz\quad +\gamma\end{aligned}$$

Now since by the hypothesis there shall be $bb=\alpha aa+\beta a+\gamma$, the remaining equation divided by *z* will give :

$$2\alpha ma+\beta m+\alpha mmz=2nb+nnz,$$

from which there is deduced:

$$z=\tfrac{2\alpha ma-2nb+\beta m}{nn-\alpha mm}.$$

With which value substituted we conclude, if there may be put

$$x=\left[a+mz=a+m\cdot\tfrac{2\alpha ma-2nb+\beta m}{nn-\alpha mm}\right]=\tfrac{(nn+\alpha mm)a-2mnb+\beta mm}{nn-\alpha mm},$$

to become

$$\sqrt{(\alpha xx+\beta x+\gamma)}=\tfrac{2\alpha mna-(nn+\alpha mm)b+\beta mn}{nn-\alpha mm}.$$

Therefore whatever numbers may be taken for *m* and *n*, from the known case

$$\sqrt{(\alpha aa+\beta a+\gamma)}=b$$

the formula $\sqrt{(\alpha xx+\beta x+\gamma)}$ can be made rational in infinitely many other ways, and since the number *b* may be allowed to take both negative as well as positive values, for the found numbers *a* and *b* and for the assumed numbers *m* and *n* taken as it pleases there may be taken:

$$x=\tfrac{(nn+\alpha mm)a\pm 2mnb+\beta mm}{nn-\alpha mm},$$

and there will become

$$\sqrt{(\alpha xx+\beta x+\gamma)}=\tfrac{2\alpha mna\pm(nn+\alpha mm)b+\beta mn}{nn-\alpha mm},$$

SCHOLIUM

2. Therefore it is necessary for solving this problem, that at least one case shall be known, so that the proposed formula may become rational. Nor indeed for any case of this kind requiring to be found can any certain rule be prescribed, since also formulas of this kind may be given, which has shown clearly that no rational case is able to occur. If for the sake of an example this formula $\sqrt{(3xx+2)}$ may be proposed, then no rational number can be found for x , whereby that may become rational. But nevertheless enough cases are known, by which the formula $\alpha xx+\beta x+\gamma$ is capable of such a reduction, certainly it eventuates, whenever it may be retained in this general formula $(px+q)^2+(rx+s)(tx+u)$, yet I will not attend to this formula here, from which that case shall be drawn, which I have assumed known, will have become known either by some reasoning or by guesswork. Truly from one case found an infinitude of other cases can be found without any further difficulty, here mainly I consider the solutions which are going to be resolved by whole numbers. For since the values for *x* found shall be expressed by a fraction, a new question now arises, how the numbers *m* and *n* may be required to be assumed, so that thence whole numbers may be found for x .

PROBLEM 2

3. *If $\alpha,\ \beta,\ \gamma$ shall be given integers, to find the whole numbers requiring to be taken for x , by which the formula $\alpha xx+\beta x+\gamma$ may become a square.*

SOLUTION

Again I assume one whole number *a* to be present, which shall satisfy the question, thus so that there shall be

$$\alpha aa+\beta a+\gamma=b,$$

and in the manner we have seen, if there may be taken

$$x=\frac{(nn+\alpha mm)a\pm 2mnb+\beta mm}{nn-\alpha mm},$$

to become :

$$\sqrt{(\alpha xx+\beta x+\gamma)}=\frac{2\alpha mna\pm(nn+\alpha mm)b+\beta mn}{nn-\alpha mm}.$$

Therefore it remains only, as we have seen, that it shall be required to assume numbers of this kind for m and n , so that these whole number formulas may emerge. Which indeed is observed to happen at once, if the denominator of each $nn-\alpha mm$ may be established equal to one.
Therefore there shall be

$$nn-\alpha mm=1 \quad \text{or} \quad nn=\alpha mm+1$$

and thus

$$n=\sqrt{(\alpha mm+1)};$$

moreover, unless α shall be a square or a negative number, it will always be able for this formula to be satisfied; but if α shall be either a square or negative, indeed it will not be allowed to resolve this problem. Now indeed whenever two or more cases may be able to be assigned, still they do not constitute an infinitude of cases, yet it is agreed to generate such a kind of outcome. Therefore α shall be a non-square positive integer and the numbers m and n are able to be assigned always, in order that there may become $n=\sqrt{(\alpha mm+1)}$; so that even if it can happen in an infinite number of ways, yet it suffices to know the smallest number only. Therefore there will become

$$x=(nn+\alpha mm)a\pm 2mnb+\beta mm$$

and

$$\sqrt{(\alpha xx+\beta x+\gamma)}=2\alpha mma\pm(nn+\alpha mm)b+\beta mn$$

and thus a new case satisfying the question is obtained. Truly in a similar manner from this, as that a new result will be derived arising from a and b, and hence repeated again continually for others indefinitely. For we may put the values for x arising in this way successively

$$a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},\ \text{etc.},$$

truly the corresponding values of the formula shall be

$$b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},\ \text{etc.},$$

and in the following manner any two latter results will be defined from the two preceding ones in the following manner:

$$a^{\mathrm{I}}=(nn+\alpha mm)a\pm 2mnb+\beta mm,\ \ b^{\mathrm{I}}=2\alpha mna\pm(nn+\alpha mm)b+\beta mn,$$
$$a^{\mathrm{II}}=(nn+\alpha mm)a^{\mathrm{I}}\pm 2mnb^{\mathrm{I}}+\beta mm,\ \ b^{\mathrm{II}}=2\alpha mna^{\mathrm{I}}\pm(nn+\alpha mm)b^{\mathrm{I}}+\beta mn,$$
$$a^{\mathrm{II}}=(nn+\alpha mm)a^{\mathrm{I}}\pm 2mnb^{\mathrm{I}}+\beta mm,\ \ b^{\mathrm{II}}=2\alpha mna^{\mathrm{I}}\pm(nn+\alpha mm)b^{\mathrm{I}}+\beta mn,$$
$$a^{\mathrm{III}}=(nn+\alpha mm)a^{\mathrm{II}}\pm 2mnb^{\mathrm{II}}+\beta mm,\ \ b^{\mathrm{III}}=2\alpha mna^{\mathrm{II}}\pm(nn+\alpha mm)b^{\mathrm{II}}+\beta mn,$$
etc.

Therefore it will be permitted for this account to continually progress further and thus from one solution in terms of known whole numbers likewise innumerable others may be elicited in terms of whole numbers.

COROLLARY 1

4. Therefore in order that the formula $\alpha xx+\beta x+\gamma$ may be able to produce an infinitude of square whole numbers, it is necessary, that α shall be neither a square number or a negative number, and besides in order that one case, where that becomes a square shall be known, from whatever point.

COROLLARY 2

5. But if *a* were not a positive square, then at first two numbers *m* and *n* may be sought so that there shall be $n=\sqrt{(\alpha mm+1)}$, which can always be done. With which found, if there may be put

$$\sqrt{(\alpha xx+\beta x+\gamma)}=y,$$

and now there will have been a known case satisfying the question, which shall be $x=a$ and $y=b$, from that by the first operation not only one, but two new numbers will be found on account of the sign ambiguity.
Certainly there will be

$$x=(nn+\alpha mm)a\pm 2mnb+\beta mm$$

and

$$y=2\alpha mna\pm\left(nn+\alpha mm\right)b+\beta mn.$$

COROLLARY 3

6. If only the upper of the two ambiguous signs may be taken, in order that by continuing we may come upon greater satisfying numbers, and the values for *x* successively being produced we may designate by $a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},$ etc., moreover the corresponding values for y by $b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},$ etc., there will become :

$$a^{\mathrm{I}} = (nn+\alpha mm)a + 2mnb + \beta mm, \quad b^{\mathrm{I}} = 2\alpha mna + (nn+\alpha mm)b + \beta mn,$$
$$a^{\mathrm{II}} = (nn+\alpha mm)a^{\mathrm{I}} + 2mnb^{\mathrm{I}} + \beta mm, \quad b^{\mathrm{II}} = 2\alpha mna^{\mathrm{I}} + (nn+\alpha mm)b^{\mathrm{I}} + \beta mn,$$
$$a^{\mathrm{III}} = (nn+\alpha mm)a^{\mathrm{II}} + 2mnb^{\mathrm{II}} + \beta mm, \quad b^{\mathrm{III}} = 2\alpha mna^{\mathrm{II}} + (nn+\alpha mm)b^{\mathrm{II}} + \beta mn,$$
etc.

COROLLARY 4

7. Therefore hence we come upon this twofold progression of numbers $a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},$ etc., et $b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},$ etc., the continuation of which depends on each of which, yet each nevertheless can be separated from the other, so that the terms of each may be able to be increased slowly without depending on the other; but then any term in whatever series will be formed from the two preceding terms in both series.

COROLLARY 5

8. Indeed if the value for b^{I} may be substituted into a^{II}, there will be had

$$a^{\mathrm{II}} = (nn+\alpha mm)a^{\mathrm{I}} + 4mmnna + 2mn(nn+\alpha mm)b + 2\beta mmnn + \beta mn.$$

Truly from the value of a^{I} there becomes $2mnb = a^{\mathrm{I}} - (nn+\alpha mm)a - \beta mm,$ from which the value of $2mnb$ substituted there in a^{II} will produce

$$a^{\mathrm{II}} = (nn+\alpha mm)a^{\mathrm{I}} + 4\alpha mmnna + (nn+\alpha mm)a^{\mathrm{I}}$$
$$-(nn+\alpha mm)^2 a - \beta mm(nn+\alpha mm) + 2\beta mmnn + \beta mm.$$

But on account of $nn = \alpha mm + 1$ there becomes

$$4\alpha mmnn - \left(nn+\alpha mm\right)^2 = -\left(nn-\alpha mm\right)^2 = -1$$

and

$$2\beta mmnn - \beta mm\left(nn+\alpha mm\right) = \beta mm(nn-\alpha mm) = \beta mm,$$

from which there becomes

$$a^{\mathrm{II}} = 2(nn+\alpha mm)a^{\mathrm{I}} - a + 2\beta mm.$$

COROLLARY 6

9. Therefore since in a like manner there shall be

$$a^{\mathrm{III}} = 2(nn+\alpha mm)a^{\mathrm{II}} - a^{\mathrm{I}} + 2\beta mm \ \text{ etc.},$$

and immediately the two first terms will be had in the series $a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},$ etc., the first clearly *a* from wherever and the second from the formula $a^{\mathrm{I}} = (nn+\alpha mm)a + 2mmb + \beta mm,$ and from these all the following will be defined by these formulas :

$$a^{\mathrm{II}} = 2(nn+\alpha mm)a^{\mathrm{I}} - a + 2\beta mm,$$
$$a^{\mathrm{III}} = 2(nn+\alpha mm)a^{\mathrm{II}} - a^{\mathrm{I}} + 2\beta mm,$$
$$a^{\mathrm{IV}} = 2(nn+\alpha mm)a^{\mathrm{III}} - a^{\mathrm{II}} + 2\beta mm,$$
etc.

COROLLARY 7

10. Moreover the progression of the numbers $b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},$ etc. is prepared in a similar manner. For indeed the first term is known from elsewhere, and the second by the formula $b^{\mathrm{I}} = 2\alpha mna + (nn+\alpha mm)b + \beta mn,$ if the value for a^{I} may be substituted into b^{II}, there will become

$$b^{\mathrm{II}} = 2\alpha mn(nn+\alpha mm)a + 4\alpha mmnnb + 2\alpha\beta m^3 n + (nn+\alpha mm)b^{\mathrm{I}} + \beta mn;$$

but from the value of b^{I} itself there becomes $2\alpha mna = b^{\mathrm{I}} - (nn+\alpha mm)b - \beta mn,$ with which substituted, on account of $nn - \alpha mm = 1$, there becomes

$$b^{\mathrm{II}} = 2(nn+\alpha mm)b^{\mathrm{I}} - b$$

and similarly

$$b^{\mathrm{III}} = 2(nn+\alpha mm)b^{\mathrm{II}} - b^{\mathrm{I}},$$
$$b^{\mathrm{IV}} = 2(nn+\alpha mm)b^{\mathrm{III}} - b^{\mathrm{II}},$$
etc.

COROLLARY 8

11. Therefore since each series shall be prepared thus, so that any term may be defined from the two preceding terms according to a certain rule, and each series will be recurring with the scale of the relation present being $2(nn+\alpha mm), -1$. Hence therefore with the equation formed

$$zz = 2(nn+\alpha mm)z - 1$$

the roots of which will be

$$z = 2nn - 1 \pm 2n\sqrt{(nn-1)} = \left(n \pm m\sqrt{a}\right)^2, \ [\text{since } nn - \alpha mm = 1.]$$

COROLLARY 9

12. Hence therefore from the theory for the recurring series of the progression $a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},\ a^{\mathrm{IV}}$ etc. any indefinite term will be expressed by the following formula:

$$\left(\tfrac{a}{2}+\tfrac{\beta}{4\alpha}+\tfrac{b}{2\sqrt{\alpha}}\right)\left(n+m\sqrt{\alpha}\right)^{2v}+\left(\tfrac{a}{2}+\tfrac{\beta}{4\alpha}-\tfrac{b}{2\sqrt{\alpha}}\right)\left(n-m\sqrt{\alpha}\right)^{2v}-\tfrac{\beta}{2\alpha}=x,$$

truly any term of the other series $b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},$ etc., with this taken for some whole number v :

$$\left(\tfrac{b}{2}+\tfrac{a\sqrt{\alpha}}{2}+\tfrac{\beta}{4\sqrt{\alpha}}\right)\left(n+m\sqrt{\alpha}\right)^{2v}+\left(\tfrac{b}{2}-\tfrac{a\sqrt{\alpha}}{2}-\tfrac{\beta}{4\sqrt{\alpha}}\right)\left(n-m\sqrt{\alpha}\right)^{2v}=y,$$

SCHOLIUM

13. If here we may substitute successively all the integer numbers 0, 1, 2, 3, 4, 5 etc. for $2v$, each progression will produce interpolations, the mean terms of which will satisfy equally that sought, provided they were integers. Moreover we will find on putting

$$2v=0\begin{cases}x=a,\\ y=b,\end{cases}$$

$$2v=1\begin{cases}x=na+mb+\tfrac{\beta}{2\alpha}(n-1),\\ y=nb+\alpha ma+\tfrac{1}{2}\beta m,\end{cases}$$

$$2v=2\begin{cases}x=(nn+\alpha mm)a+2mnb+\beta mm,\\ y=(nn+\alpha mm)b+2\alpha mna+\beta mm.\end{cases}$$

Each series has $2n,-1$ as the recurring step of the relation ; and indeed for the first value x, if the three consecutive terms shall be $P, Q, R,$ there will become

$$R=2nQ-P+\tfrac{\beta(n-1)}{\alpha};$$

but if three terms following each other in order in the progression of the values of y shall be P, Q and R, there will become

$$R=2nQ-P.\ \text{[As used in } E29.]$$

Therefore if $\frac{\beta}{2\alpha}(n-1)$ were a whole number, all the terms associated with the problem will be resolved equally and thus we will obtain two more solutions, which method used will have supplied the required needs of the problem. But so that more solutions may be obtained than we have found, then it is easily deduced, because due to necessity we have put the first root of the of the formula $nn-\alpha mm$ equal to unity, since yet often without doubt also the numerator may be able to be divided by the denominator, even if this shall be greater than unity. Therefore just as clearly all the solutions shall be able to be found in terms of whole numbers, we will examine more closely in the following problem.

PROBLEM 3

14. *If a shall be a none-square whole number, with one given integer value a, which put in place for x may return a square value of the formula* $\alpha xx+\beta x+\gamma$ *, to find infinitely many other whole numbers, which taken for x shall establish the same outcome.*

SOLUTION

In general there may be put $\sqrt{(\alpha xx+\beta x+\gamma)}=y$, and in the known case where $x=a$ moreover, there becomes $\sqrt{(\alpha aa+\beta a+\gamma)}=b$, and hence we have seen in general to become for non-exclusive fractions

$$x=\frac{(nn+\alpha mm)a+2mnb+\beta mm}{nn-\alpha mm},$$
$$y=\frac{(nn+\alpha mm)b+2\alpha mna+\beta mm}{nn-\alpha mm}.$$

Now indeed, so that these numbers may become integers, it is not absolutely necessary, that the denominator $nn-\alpha mm$ may be reduced to unity, in order that the fractions $\frac{nn+\alpha mm}{nn-\alpha mm}$ and $\frac{2mn}{nn-\alpha mm}$ may be changed into whole numbers. Therefore we may put

$$\frac{nn+\alpha mm}{nn-\alpha mm}=p \text{ and } \frac{2mn}{nn-\alpha mm}=q,$$

from which there becomes

$$p-1=\frac{2\alpha mm}{nn-\alpha mm} \text{ and thus}$$

$$\frac{\beta mm}{nn-\alpha mm}=\frac{\beta}{2\alpha}(p-1) \text{ and } \frac{\beta mn}{nn-\alpha mm}=\frac{1}{2}\beta q.$$

But then there becomes from the assumed formulas,

$$pp-\alpha qq=\frac{(nn+\alpha mm)^2-4\alpha m^2n^2}{(nn-\alpha mm)^2}=1,$$

thus so that there shall be

$$pp=\alpha qq+1 \quad \text{and} \quad p=\sqrt{(\alpha qq+1)}.$$

Therefore again as before it will be required to assign the two numbers *p* and *q* from the number α, so that there shall be $p=\sqrt{(\alpha qq+1)}$; with which found there will be obtained :

$$x = pa+qb+\tfrac{\beta}{2\alpha}\left(p-1\right) \text{ and } y = pb+\alpha qa+\tfrac{1}{2}\beta q.$$

Therefore provided $\frac{\beta}{2\alpha}\left(p-1\right)$ were a whole number, these values will be satisfactory. But since it is permitted to take the numbers *p* and *q* both negative as well as positive, these formulas in addition satisfy these three other formulas :

$$x = pa-qb+\tfrac{\beta}{2\alpha}\left(p-1\right) \text{ and } y = pb-\alpha qa-\tfrac{1}{2}\beta q,$$
$$x = -pa+qb-\tfrac{\beta}{2\alpha}\left(p+1\right) \text{ and } y = -pb+\alpha qa+\tfrac{1}{2}\beta q,$$
$$x = -pa-qb-\tfrac{\beta}{2\alpha}\left(p+1\right) \text{ and } y = -pb-\alpha qa-\tfrac{1}{2}\beta q.$$

So that if again whichever of these two may be assumed for *a* and *b*, from which for four new solutions will arise as it pleases. Hence nevertheless not 16, but only six different solutions arise, among which thus in the first place $x=a$ and $y=b$ are known, and $x=-a-\frac{\beta}{\alpha}$ and $y=-b$ are present next to this ; truly the four remaining are:

$$x=\left(pp+\alpha qq\right)a\pm 2pqb+\beta qq, \qquad y=\left(pp+\alpha qq\right)b\pm 2\alpha pqa\pm\beta pq,$$
$$x=-\left(pp+\alpha qq\right)a\pm 2pqb-\tfrac{\beta}{\alpha}pp, \quad y=\left(pp+\alpha qq\right)b\mp 2\alpha pqa\mp\beta pq,$$

from which henceforth other new integers can be found indefinitely.

COROLLARY 1

15. Therefore so that if there were either $\beta=0$ or there shall be a number of such a kind present that either $\beta(p-1)$ or $\beta(p+1)$ may be divisible by 2α, then in this manner more solutions in integers will be obtained than established in the previous manner.

COROLLARY 2

16. But in general it is required to be observed, if some case $x=v$ were satisfied , then also the case $x=-v-\frac{\beta}{\alpha}$ will be going to be satisfied ; for the same value *y* arises from each. Whereby since these cases may be elicited so easily from those, with these omitted the investigation of the solutions agreed on is reduced by half.

COROLLARY 3

17. Therefore with the cases $x=-v-\frac{\beta}{\alpha}$ rejected, certainly which are produced at once from the cases found $x=v$, from the case $x=a$ et $y=b$ at once the two numbers will be found

$$x= pa\pm qb+\tfrac{\beta}{2\alpha}(p-1),\ y=\alpha qa\pm pb+\tfrac{1}{2}\beta q$$

and hence again, by the following operation of the two

$$x=(pp+\alpha qq)\pm 2pqb+\beta qq,\ y=2\alpha pqa\pm(pp+\alpha qq)b+\beta pq,$$

which becomes twofold from the ambiguity of the signs of the number b .

COROLLARY 4

18. If these may be compared with §12 and §13, it will become apparent all these formulas to be contained in the following general expressions, if indeed all the whole numbers may be substituted in turn for μ :

$$\text{I.}\ \begin{cases} x=\frac{1}{4\alpha}\left(2\alpha a+\beta+2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}+\frac{1}{4\alpha}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}-\frac{\beta}{2\alpha}, \\ y=\frac{1}{4\sqrt{\alpha}}\left(2\alpha a+\beta+2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}-\frac{1}{4\alpha\sqrt{\alpha}}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p-q\sqrt{\alpha}\right)^{\mu} \end{cases}$$

and

$$\text{II.}\ \begin{cases} x=\frac{1}{4\alpha}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}+\frac{1}{4\alpha}\left(2\alpha a+\beta+2b\sqrt{\alpha}\right)\left(p-q\sqrt{\alpha}\right)^{\mu}-\frac{\beta}{2\alpha}, \\ y=\frac{1}{4\sqrt{\alpha}}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}-\frac{1}{4\alpha\sqrt{\alpha}}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p-q\sqrt{\alpha}\right)^{\mu}. \end{cases}$$

COROLLARY 5

19. Hence therefore duplicate series are found for the values of the numbers *x* and *y*, which will maintain the same law of the progression. Indeed if we may put

$$\begin{aligned} x&=a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},\ a^{\mathrm{IV}},a^{\mathrm{V}},\ \text{etc.},\ P,\ Q,\ R, \\ y&=b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},\ b^{\mathrm{IV}},b^{\mathrm{V}},\ \text{etc.},\ S,\ T,\ V, \end{aligned}$$

there will be for the one:

$$a^{\mathrm{I}}=pa+qb+\tfrac{\beta}{2\alpha}(p-1)\ \text{ et }\ b^{\mathrm{I}}=\alpha qa+pb+\tfrac{1}{2}\beta q$$

and for the other:

$$a^{\mathrm{I}}=pa-qb+\tfrac{\beta}{2\alpha}(p-1)\ \text{ et }\ b^{\mathrm{I}}=\alpha qa-pb+\tfrac{1}{2}\beta q,$$

and truly this common law of the progression will prevail for each, so that there shall become

$$R = 2pQ - P + \tfrac{\beta}{\alpha}(p-1) \text{ and } V = 2pT - S.$$

COROLLARY 6

20. Since there shall be $pp - \alpha qq = 1$, there will become

$$\left(p + q\sqrt{\alpha}\right)^{\mu} = \left(p - q\sqrt{\alpha}\right)^{-\mu} \text{ and } \left(p - q\sqrt{\alpha}\right)^{\mu} = \left(p + q\sqrt{\alpha}\right)^{-\mu}$$

and hence, if the second series may be expanded backwards, they will produce the first series. Therefore it suffices for each case these series to have these series constructed, which continued both forwards as well as backwards will contain within themselves all the solutions arising from the ambiguity of the signs of the numbers *b*.

SCHOLIUM

21. Therefore if there were $\beta = 0$, so that this formula $\sqrt{(\alpha xx + \gamma)} = y$ may be obtained by being rendered rational, and the case may be constructed, where there shall be $\sqrt{(\alpha xx + \gamma)} = b$, with the numbers p and q taken thus, so that there shall become $p = \sqrt{(\alpha qq + 1)}$ and innumerable other satisfying values will be contained in these series :

$$x = a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},\ a^{\mathrm{IV}}, a^{\mathrm{V}},\ \text{etc.},\ P,\ Q,\ R,$$
$$y = b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},\ b^{\mathrm{IV}}, b^{\mathrm{V}},\ \text{etc.},\ S,\ T,\ V,$$

where the second terms must be taken thus, so that there shall be :

$$a^{\mathrm{I}} = pa + qb,\ b^{\mathrm{I}} = \alpha qa + pb;$$

then each series is recurring with the scale of the relation being $2p, -1$. Clearly there will become:

$$a^{\mathrm{II}} = 2pa^{\mathrm{I}} - a \text{ and in general } R = 2pQ - P,$$
$$b^{\mathrm{II}} = 2pb^{\mathrm{I}} - b \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot \quad V = 2pT - S;$$

truly also both these series should be continued backwards and thus two more [new] solutions will be produced: unless there shall be either $a = 0$ or $b = 0$. Moreover here neither do negative solutions arise in agreement, since if $x = v$ may be satisfied, also $x = -v$ is satisfied . Again all these solutions will be contained in these general formulas

$$\begin{cases} x=\frac{1}{2\sqrt{\alpha}}\left(a\sqrt{\alpha}+b\right)\left(p+q\sqrt{\alpha}\right)^{\mu}+\frac{1}{2\sqrt{\alpha}}\left(a\sqrt{\alpha}-b\right)\left(p-q\sqrt{\alpha}\right)^{\mu}, \\ y=\frac{1}{2}\left(a\sqrt{\alpha}+b\right)\left(p+q\sqrt{\alpha}\right)^{\mu} \qquad -\frac{1}{2}\left(a\sqrt{\alpha}-b\right)\left(p-q\sqrt{\alpha}\right)^{\mu}. \end{cases}$$

Therefore we will establish the following examples for the various numbers which constitute the coefficient α, and indeed more generally, so that also the ratio of the coefficient β may be found, evidently for the cases , for which perhaps $\frac{\beta}{2\alpha}(p-1)$ were a whole number.

EXAMPLE 1

22. *To find infinitely many integer values of* x *which emerge from the proposed formula*

$$\sqrt{(2xx+\beta x+\gamma)}=y,$$

if indeed a single solution may be agreed on.

The known solution shall be $x=a$ and $y=b$ and on account of $\alpha=2$ we will have $p=\sqrt{(2qq+1)}$ and thus $q=2$ and $p=3$. Hence the following values will be :

$$a^{\mathrm{I}}=3a\pm 2b+\frac{\beta}{2},\ b^{\mathrm{I}}=4a\pm 3b+\beta.$$

Therefore since in §19 there shall become

$$R=6Q-P+\beta \text{ and } V=6T-S,$$

we will have the following series of satisfying values, and indeed of whole numbers, if β were an even number :

Values of x	Values of y
$a,$	$\pm b,$
$3a\pm 2b+\frac{\beta}{2}$	$4a\pm 3b+\beta,$
$17a\pm 12b+4\beta,$	$24a\pm 17b+6\beta,$
$99a\pm 70b+\frac{49}{2}\beta,$	$140a\pm 99b+35\beta,$
$577a\pm 408b+144\beta,$	$816a\pm 577b+204\beta,$
$3363a\pm 2378b+\frac{1681}{2}\beta$	$4756a\pm 3363b+1189\beta$
etc.	etc.

Then truly, since y may retain these same values, if there may be written $-x-\frac{1}{2}\beta$ for x, these solutions will have a place :

Values of x	Values of y
$-a-\frac{1}{2}\beta$,	$\pm b$,
$-3a\mp 2b-\beta$	$4a\pm 3b+\beta$,
$-17a\mp 12b-\frac{9}{2}\beta$,	$24a\pm 17b+6\beta$,
$-99a\mp 70b-25\beta$,	$140a\pm 99b+35\beta$,
$-577a\mp 408b-\frac{289}{2}\beta$,	$816a\pm 577b+204\beta$,
$-3363a\mp 2378b-841\beta$	$4756a\pm 3363b+1189\beta$
etc.	etc.

Therefore even if β were not an even number, yet in each order the value of half of x would be a whole number.

EXAMPLE 2

23. *To find infinitely many integer values of* x *which emerge from the proposed formula*

$$\sqrt{(3xx+\beta x+\gamma)}=y\,,$$

if indeed a single solution may be agreed on.

A known case $x=a$ and $y=b$ may be provided , then truly on account of $\alpha=3$ there may be taken $p=\sqrt{(3qq+1)}$ and there will be $q=1$ and $p=2$. Hence we will have for the second case:

$$a^{\mathrm{I}}=2a\pm b+\tfrac{\beta}{6};\qquad b^{\mathrm{I}}=3a\pm 2b+\tfrac{\beta}{2},$$

from which the two recurring series will be formed following these scales of the relation from which there will be obtained :

$$R=4Q-P+\tfrac{\beta}{3},\quad V=4T-S,$$

Values of x	Values of y
a,	$\pm b$,
$2a\pm b+\frac{\beta}{6}$,	$3a\pm 2b+\frac{1}{2}\beta$,
$7a\pm 4b+\beta$,	$12a\pm 7b+2\beta$,
$26a\pm 15b+\frac{25}{6}\beta$,	$45a\pm 26b+\frac{15}{2}\beta$,
$97a\pm 56b+16\beta$,	$168a\pm 97b+28\beta$,
$362a\pm 209b+\frac{361}{6}\beta$,	$627a\pm 362b+\frac{209}{2}\beta$,
$1351a\pm 780b+225\beta$	$2340a\pm 1351b+390\beta$

etc.	etc.

Truly in addition by writing $-x-\frac{\beta}{3}$ for x there will be produce :

Values of x	Values of y
$-a-\frac{1}{3}\beta$,	$\pm b$,
$-2a\mp b-\frac{1}{2}\beta$	$3a\pm 2b+\frac{1}{2}\beta$,
$-7a\mp 4b-\frac{4}{3}\beta$	$12a\pm 7b+2\beta$,
$-26a\mp 15b-\frac{9}{2}\beta$,	$45a\pm 26b+\frac{15}{2}\beta$,
$-97a\mp 56b-\frac{49}{3}\beta$,	$168a\pm 97b+28\beta$,
$-362a\mp 209b-\frac{121}{2}\beta$,	$627a\pm 362b+\frac{209}{2}\beta$,
$-1351a\mp 780b-\frac{676}{3}\beta$	$2340a\pm 1351b+390\beta$
etc.	etc.

Therefore just as the number β was divisible by 2 or 3 or by each, hence there more solutions in integers will have been elicited.

EXAMPLE 3

24. *To find infinitely many integer values of x which emerge from the proposed formula*

$$(5xx+\beta x+\gamma)=y,$$

if indeed a single case were known.

For the single known case $x=a$ and $y=b$, and on account of $\alpha=5$ the numbers p and q may be sought, so that there shall be $p=\sqrt{(5qq+1)}$. Therefore there will become $q=4$ and $p=9$ and hence the following solution will be produced:

$$a^{\mathrm{I}}=9a\pm 4b+\tfrac{4}{5}\beta, \qquad b^{\mathrm{I}}=20a\pm 9b+2\beta.$$

Therefore since there shall be

$$a^{\mathrm{II}}=18a^{\mathrm{I}}-a+\tfrac{8}{5}\beta, \qquad b^{\mathrm{II}}=18b^{\mathrm{I}}-b,$$

the following solutions will be had :

Values of x	Values of y
a,	$\pm b$,
$9a\pm 4b+\frac{4}{5}\beta$,	$20a\pm 9b+2\beta$,
$161a\pm 72b+16\beta$,	$360a\pm 161b+36\beta$,

$2889a \pm 1292b + \frac{1444}{5}\beta,$	$6460a \pm 2889b + 646\beta,$
etc.	etc.

where for any value of x also $-x-\frac{\beta}{5}$ can be put in place.

SCHOLIUM 1

25. Since in this manner from one solution in known whole numbers infinitely many other solutions also in whole numbers may be elicited, the question arises, whether or not in this manner plainly all integral solutions may be obtained. And indeed in the first and second examples there will be no doubt, why by this method all the integral solutions may not be obtained. Truly in the third example certainly cases are given, in which many more solutions in integers can be shown, as indeed may be found by this method. Just as if the formula $\sqrt{(5xx+4)} = y$ were proposed, which for the known case produces $a = 0$ and $b = 2$, our solution gives:

Values of x	Values or y
0,	2,
8,	18,
144,	322,
2584	5778
etc.	etc.

Truly it will be apparent on being scrutinized with more care not only for these cases $\sqrt{(5xx+4)}$ to become rational, but also with these numbers substituted for x

$$x = 0,\ 1,\ 3,\ 8,\ 21,\ 55,\ 144,\ 377,\ 987 \text{ etc.},$$

from which the number of solutions is tripled. The reason for this being, because for the formula $p = \sqrt{(5qq+4)}$ required to be resolved, we may put $q = 4$, from which there becomes $p = 9$; which indeed is the simplest solution in whole numbers. But since $2p$ is present in the scale of the relations, that will consist of whole numbers, even if p shall be a fraction having the denominator 2. On account of this we will arrive at the simpler solutions, if we may put $q = \frac{1}{2}$, from which there becomes $p = \frac{3}{2}$ and thus on account of $\alpha = 5$ the following values will be

$$a^{\mathrm{I}} = \tfrac{3}{2}a \pm \tfrac{1}{2}b + \tfrac{1}{20}\beta, \quad b^{\mathrm{I}} = \tfrac{5}{2}a \pm \tfrac{3}{2}b + \tfrac{1}{4}\beta$$

and the third with the following will be supplied by this law:

$$a^{\mathrm{II}} = 3a^{\mathrm{I}} - a + \tfrac{1}{10}\beta, \quad b^{\mathrm{II}} = 3b^{\mathrm{I}} - b,$$

from which we obtain these values:

Values of x	Values of y
$a,$	$\pm b\,,$
$\frac{3}{2}a\pm\frac{1}{2}b+\frac{1}{20}\beta,$	$\frac{5}{2}a\pm\frac{3}{2}b+\frac{1}{4}\beta,$
$\frac{7}{2}a\pm\frac{3}{2}b+\frac{1}{4}\beta,$	$\frac{15}{2}a\pm\frac{7}{2}b+\frac{3}{4}\beta,$
$9a\pm 4b+\frac{4}{5}\beta,$	$20a\pm 9b+2\beta,$
$\frac{47}{2}a\pm\frac{21}{2}b+\frac{9}{4}\beta,$	$\frac{105}{2}a\pm\frac{47}{2}b+\frac{21}{4}\beta,$
$\frac{123}{2}a\pm\frac{55}{2}b+\frac{121}{20}\beta,$	$\frac{275}{2}a\pm\frac{123}{2}b+\frac{55}{4}\beta,$
$161a\pm 72b+16\beta$	$360a\pm 161b+36\beta$
etc.	etc.

And hence these solutions arise three times greater, as often as $a\pm b$ will have been even and β either $=0$ or to be divisible by 20 .

SCHOLIUM 2

26. Therefore when several solutions are found in whole numbers, if fractions with the denominator 2 are assumed for p and q; so that when in general it may eventuate, it will be worth the effort to have investigated. But furthermore these cases are not worth investigating, unless there shall be $\beta=0$ or the formula may be able to be reduced to such a form. The proposed formula shall be $\sqrt{(\alpha xx+\gamma)}=y$, for which $x=a$ and $y=b$ may satisfy the case ; then there may be put $p=\frac{m}{2}$ and $q=\frac{n}{2}$, or the numbers m and n may be sought , so that there shall be $mm=\alpha nn+4$ and $m=\sqrt{(\alpha nn+4)}.$ Then truly the first solution at once gives the second :

$$a^{\mathrm{I}}=\frac{ma+nb}{2}\quad\text{and}\quad b^{\mathrm{I}}=\frac{\alpha na+mb}{2},$$

where indeed the numbers m and n can be taken both negative and positive.
Finally, from these two found initially, the following will be found by the following rule

$$a^{\mathrm{II}}=ma^{\mathrm{I}}-a\quad\text{and}\quad b^{\mathrm{II}}=mb^{\mathrm{I}}-b.$$

Moreover in general whatever satisfying number for x is contained by this formula :

$$x=\frac{1}{2\sqrt{\alpha}}\left(a\sqrt{\alpha}+b\right)\left(\frac{m+n\sqrt{\alpha}}{2}\right)^{\mu}+\frac{1}{2\sqrt{\alpha}}\left(a\sqrt{\alpha}-b\right)\left(\frac{m-n\sqrt{\alpha}}{2}\right)^{\mu},$$

from which there becomes

$$y=\tfrac{1}{2}\left(a\sqrt{\alpha}+b\right)\left(\tfrac{m+n\sqrt{\alpha}}{2}\right)^{\mu}-\tfrac{1}{2}\left(a\sqrt{\alpha}-b\right)\left(\tfrac{m-n\sqrt{\alpha}}{2}\right)^{\mu}.$$

Therefore however often an even number $ma+nb$ has been produced yet with neither m or n even, three times as many integral solutions will be produced as by the preceding method. Truly these solutions thus will be had:

$$a=a,\qquad b=b,$$

$$a^{\mathrm{I}}=\tfrac{ma+nb}{2},\qquad b^{\mathrm{I}}=\tfrac{mb+\alpha na}{2},$$

$$a^{\mathrm{II}}=\frac{(mm-2)a+mnb}{2},\qquad b^{\mathrm{II}}=\frac{(mm-2)b+\alpha mna}{2},$$

$$a^{\mathrm{III}}=\frac{\left(m^3-3m\right)a+(mm-1)nb}{2},\qquad b^{\mathrm{III}}=\frac{\left(m^3-3m\right)b+\alpha(mm-1)na}{2},$$

$$a^{\mathrm{IV}}=\frac{\left(m^4-4mm+2\right)a+\left(m^3-2m\right)nb}{2},\qquad b^{\mathrm{IV}}=\frac{\left(m^4-4mm+2\right)b+\alpha\left(m^3-2m\right)na}{2},$$

$$a^{\mathrm{V}}=\frac{\left(m^5-5m^3+5m\right)a+\left(m^4-3mm\right)nb}{2},\qquad b^{\mathrm{V}}=\frac{\left(m^5-5m^3+5m\right)b+\alpha\left(m^4-3mm+1\right)na}{2},$$

etc.

OBSERVATION 1

27. Then by this other method at last more solutions in integral numbers is supplied than before, since *m* and *n* will have become odd numbers and likewise *a* and *b* both either even or odd. Indeed if *m* and *n* shall be even numbers, *p* and *q* will be integers and the formula $m=\sqrt{(\alpha nn+4)}$ will present the same solutions as the formula $p=\sqrt{(aqq+1)}$. Then if $ma\pm nb$ will not have been even, the values $a^{\mathrm{I}},a^{\mathrm{II}}$ may not emerge whole nor therefore will more solutions be found than by the earlier method, while the formula $p=\sqrt{(aqq+1)}$ is used. Therefore it will be required to distinguish these cases, for which the formula can be satisfied with odd numbers being taken for *m* and *n* in the formula $m=\sqrt{(\alpha nn+4)}$, which at once is apparent cannot be done, if α were a number of the form $4z-1$ or also of this form $8z+1$. Whereby other odd numbers for α need not be abandoned, unless they shall be of the form $8z+5$. Therefore for these cases the following table shows the smallest values satisfying the formula $m=\sqrt{(\alpha nn+4)}$:

If there were	there may be taken	and there will become
$\alpha=5,$	$n=1$	$m=3,$
$\alpha=13,$	$n=3$	$m=11,$
$\alpha=21,$	$n=1$	$m=5,$
$\alpha=29,$	$n=5$	$m=27,$
$\alpha=37,$	$n=-$	$n=-$
$\alpha=45,$	$n=1$	$m=7,$

$$\begin{array}{lll}
\alpha = 53, & n = 7 & m = 51, \\
\alpha = 61, & n = 195 & m = 1523, \\
\alpha = 69, & n = 75 & m = 623, \\
\alpha = 77, & n = 1 & m = 9, \\
\alpha = 85, & n = 9 & m = 83, \\
\alpha = 93, & n = 87 & m = 839.
\end{array}$$

Here the reason is sought , why the case $\alpha = 37$ may not take odd values for *m* and *n.* Therefore it is evident here, if there shall be $\alpha = 37$, odd numbers not to be given for *n* and *m;* but for the remaining cases the resolution is successful.

Thus if this formula may be proposed :

$$\sqrt{(53xx + 28)} = y,$$

at once there will be had $a = 1$ and $b = 9$. Then on account of $n = 7$ and $m = 51$, there will become not only

$$a^{\mathrm{I}} = \tfrac{51+63}{2} = 57 \text{ and } b^{\mathrm{I}} = \tfrac{371+459}{2} = 415,$$

but also

$$a^{\mathrm{I}} = -6 \text{ and } b^{\mathrm{I}} = -44;$$

and the recurring series for *x* and *y,* of which the scale of the relation is 51, -1, will be :

$$\begin{array}{llllll}
x = \text{etc.} & -307, & -6, & 1, & 57, & 2906 \quad \text{etc.,} \\
y = \text{etc.} & 2235, & 44, & 9, & 415, & 21156 \quad \text{etc.}
\end{array}$$

OBSERVATION 2

28. But it suffices to establish the case, for which the second term is missing in the general formula $\alpha xx + \beta x + \gamma$, because this can be reduced to such a safe form from the integrity of the numbers. Certainly the common manner, where the second term is accustomed to be taken from equations by putting $x = y - \frac{\beta}{2\alpha}$, here this cannot happen, unless β shall be a number divisible by 2α . Truly if $\alpha xx + \beta x + \gamma$ must be a square, there may be put

$$\alpha xx + \beta x + \gamma = yy$$

and by multiplying by 4α there will be produced

$$4\alpha\alpha xx + 4\alpha\beta x + 4\alpha\gamma = 4\alpha yy$$

and thence

$$4\alpha yy + \beta\beta - 4\alpha\gamma = \left(2\alpha x + \beta\right)^2 .$$

Therefore the cases may be sought, for which the formula $4\alpha yy+\beta\beta-4\alpha\gamma$ becomes a square, and thence the values requiring to be substituted for x, which shall return the square formula $\alpha xx+\beta x+\gamma$; evidently if there were

$$\sqrt{(4\alpha yy+\beta\beta-4\alpha\gamma)}=z,$$

there will be $2\alpha x+\beta=z$ and hence

$$x=\tfrac{z-\beta}{2\alpha}.$$

But if β were an even number, such as 2δ, on putting $\alpha xx+2\delta x+\gamma=yy$ there will become

$$(\alpha x+\delta)^2=\alpha yy+\delta\delta-\alpha\gamma$$

and thus the formula $\alpha yy+\delta\delta-\alpha\gamma$ is required to be recalled square; and if we find

$$\sqrt{(\alpha yy+\delta\delta-\alpha\gamma)}=z,$$

there will become $\alpha x+\delta=z$ and

$$x=\tfrac{z-\delta}{\alpha},$$

from which generally the whole numbers x are found ; for perhaps even if $\frac{z-\delta}{\alpha}$ were not an integer, yet from one known value z, if others may be elicited in the manner treated indefinitely, perhaps some will be whole numbers. From which it is evident the resolution of the root of the quadratic formulas $\sqrt{(\alpha xx+\beta x+\gamma)}$ may not to be affected by some limitation, evidently even if the term βx may be omitted, and thus the whole work may be reduced to this, so that rational formulas of this kind $\sqrt{(\alpha xx+\gamma)}$ indeed may be rendered as whole numbers.

OBSERVATION 3

29. At least now I have observed the formula $\alpha xx+\gamma$ to be unable to be put into the form of a square with whole numbers and in an indefinite number of ways, unless α shall be a non-square positive number. Moreover without such a number α the problem thus cannot be resolved, so that for some number assumed for γ the solution shall succeed ; for certainly numbers of this kind can be given for γ, so that the problem evidently will allow no solution, and perhaps on this account I have demanded a single solution known must be known, on which account the insolvable case to be excluded.

Truly for a given α the characters can be shown, from which it may be permitted to discern whether or not a number γ may be of this kind, which may allow a solution. And certainly it is evident in the first place no solution can have a place, unless γ shall be contained in such a formula $bb-\alpha aa$. Hence for a given number α a series of all the numbers both positive and negative may be formed, which indeed shall be contained in the formula $bb-\alpha aa$; and unless γ may be found in this series, certainly to indicate that in no manner can the formula $\sqrt{(\alpha xx+\gamma)}$ be rendered rational; moreover in turn as many times as γ is understood to be in this series, since then there is $\gamma = bb-\alpha aa$, the formula $\alpha aa+\gamma$ shall become a square by putting $x=a$ and there will become $\sqrt{(\alpha xx+\gamma)}=b.$

Therefore this series, as if the general term of which is $bb-\alpha aa$, initially with $a=0$ taken, will contain all the squares

$$1, 4, 9, 16, 25 \text{ etc.},$$

then truly all the squares multiplied by $-\alpha$, surely

$$-\alpha,\ -4\alpha,\ -9\alpha,\ -16\alpha \text{ etc.}$$

Besides if p and q were numbers contained in this series, in that also will be found the product of these pq; for if there shall be

$$p=bb-\alpha aa \text{ and } q=dd-\alpha cc,$$

there will become

$$pq=(bd\pm\alpha ac)^2-\alpha(bc\pm ad)^2$$

and on account of the ambiguity of the signs this product is a number of the form $bb-\alpha aa$ in a two-fold way and thus at once these two solutions will be had :

$$x=bc+ad \text{ and } x=bc-ad.$$

OBSERVATION 4

30. Hence therefore we are to follow with this excellent theorem, which includes the above fundamental solution :

If there were

$$\alpha xx+p=yy$$

in the case $x=a$ *and* $y=b$ *, then truly also*

$$\alpha xx+q=yy$$

in the case $x=c$ *and* $y=d$ *, this formula*

$$\alpha xx+pq=yy$$

will be satisfied by taking

$$x = bc \pm ad \quad and \quad y = bd \pm \alpha ac.$$

If indeed there shall become $q = 1$ and $dd = \alpha cc + 1$, truly besides for the formula $\alpha xx + p = yy$ may be satisfied by the case $x = a$ and $y = b$, which is the case assumed above as being known, then the values

$$x = bc \pm ad \quad \text{and} \quad y = bd \pm \alpha ac,$$

will satisfy the same formula, from which generally the same solution is composed, as we have shown above and which we have elicited above at length from different principles ; whereupon this latter method of investigation is noteworthy on account of its neatness and clarity. Therefore here it is agreed, that this method may appear to be much wider than the preceding, certainly than which would have been restricted to the case $q = 1$. Moreover the demonstration of this most elegant theorem thus will be had most briefly. Since there shall be

$$\alpha aa + p = bb,$$

there will become

$$p = bb - \alpha aa$$

and on account of

$$\alpha cc + q = dd$$

there will become

$$q = dd - \alpha cc;$$

hence there will become $pq = (bb - \alpha aa)(dd - \alpha cc)$, which expression is reduced to this:

$$pq = (bd \pm \alpha ac)^2 - \alpha (bc \pm ad)^2 .$$

Therefore so that if there were

$$x = bc \pm ad \quad \text{and} \quad y = bd \pm \alpha ac,$$

there will become $pq = yy - \alpha xx$ and thus

$$\alpha xx + pq = yy.$$

Q. E. D.

OBSERVATION 5

31. Therefore since for any number α of the formula $\alpha xx + \gamma = yy$ the number γ must be of the form $bb - \alpha aa$, the numbers contained in this form deserve to be examined with care; and because, if the numbers *p* and *q* are present among these, likewise the product of these *pq* occurs also, as well as the square numbers 1, 4, 9, 16, 25 etc. and the negative multiples of these $-\alpha,\ -4\alpha,\ -9\alpha,\ -16\alpha,\ -25\alpha$ etc., especially the prime numbers

contained in this form are required to be observed, clearly from which henceforth the composite numbers arise by multiplication.

I. Let there be $\alpha = 2$ and the prime numbers of the form $bb - 2aa$ are :

positive
$+1, \ +2, \ +7, \ +17, \ +23, \ +31, \ +41, \ +47, \ +71, \ +73, \ +79, \ +89, \ +97$ etc.,

negative
$-1, \ -2, \ -7, \ -17, \ -23, \ -31, \ -41, \ -47, \ -71, \ -73, \ -79, \ -89, \ -97$ etc.,
which besides $+2$ and -2 all are contained in the forms $\pm(8n \pm 1)$.

II. Let there be $\alpha = 3$ and the prime numbers of the form $bb - 3aa$ are :

positive $+1, \ +13, \ +37, \ +61, \ +73, \ +97, \ +109$ etc.,
negative $-2, \ -3, \ -11, \ -23, \ -47, \ -59, \ -71, \ -83, \ -107$ etc.,

which besides -2 and -3 are all contained in the form $12n+1$, if indeed both positive as well as negative numbers may be taken for n .

III. Let there be $\alpha = 5$ and the prime numbers $bb - 5aa$ are:

positive: $+1, \ +5, \ +11, \ +19, \ +29, \ +31, \ +41, \ +59,$
$+61, \ +71, \ +79, \ +89, +101$ etc.,

negative: $-1, \ -5, \ -11, \ -19, \ -29, \ -31, \ -41, \ -59,$
$-61, \ -71, \ -79, \ -89, \ -101$ etc.,

which besides $+5$ and -5 are all contained in the form $10n+1$.

IV. Let there be $\alpha = 6$ and the prime numbers of the form $bb - 6aa$ are:

positive: $+1, \ +3, \ +19, \ +43, \ +67, \ +73, \ +97$ etc.,

negative: $-2, \ -23, \ -29, \ -47, \ -53, \ -71, \ -101$ etc.,

which besides -2 and $+3$, all are contained in either of the forms $24n+1$ and $24n-5$ by taking both negative as well as positive numbers *n*.

V. Let there be $\alpha = 7$ and the number of primes of the form $bb - 7aa$ are :

positive: $+1,\ +2,\ +29,\ +37,\ +53,\ +109$ etc.,

negative: $-7,\ -3,\ -19,\ -31,\ -47,\ -59,\ -83$ etc.,

which besides $+2$ and -7, all are contained in one of these forms
$28n+1,\ 28n+9,\ 28n+25.$

OBSERVATION 6

32. Hence we deduce all the prime numbers contained in the formula $bb-\alpha aa$ likewise to be contained in certain formulas of this kind $4\alpha n+A$, while certain numbers are substituted for *A*. Just as the same can be shown also in this manner.
There may be put

$$b=2\alpha p+r \text{ and } a=2q+s$$

and the formula $bb-\alpha aa$ changes into this form:

$$4\alpha\alpha pp+4\alpha pr+rr-4\alpha qq-4\alpha qs-\alpha ss;$$

there may be put $\alpha pp+pr-qq-qs=n$ and we will have

$$bb-\alpha aa=4\alpha n+rr-\alpha ss.$$

Therefore all the prime numbers of the form $bb-\alpha aa$ also are contained in this form $4\alpha n+rr-\alpha ss$; and as these numbers shall be prime, it will be required to take *r* and *s*, so that the number $rr-\alpha ss$ shall be either prime or perhaps prime to 4α . Therefore initially on taking $s=0$ successively odd numbers prime relative to α can be taken for *r*, and if rr were greater than 4α , then 4α may be subtracted just as many times as it can be done, so that the remainder shall be less than the remainder shall be less than 4α , and so that in this manner different numbers are produced, these may be placed in the formula $4\alpha n+A$ in place of A . Then also in a similar manner the numbers may be deduced from the formulas $rr-\alpha$, which, since they are different, may be added to those above. But there is no need to assume other values for *s* besides unity ; for if *s* shall be an even number, the number $-\alpha ss$ will now be contained in the form $4\alpha n$, and if *s* were odd, the number $-\alpha ss$ will have the form $-4aN-\alpha$, of which the part $-4aN$ now is contained in $4\alpha n$, and thus it suffices for the formula $4\alpha n+A$ to set out both the cases $4\alpha n+rr$ and $4\alpha n+rr-\alpha$, and now all the prime numbers which indeed are present, are to be included in the formula $bb-\alpha aa$. But now in turn all the prime numbers to be contained in these formulas $4\alpha n+rr$ and $4\alpha n+rr-\alpha$ shall be numbers of the form $bb-\alpha aa$, the question requiring a deeper investigation, which yet is seen to be required to be confirmed.

OBSERVATION 7

33. So that we may illustrate these examples, there shall become $\alpha = 13$ and from these formulas for prime numbers $4\alpha n + rr$ and $4\alpha n + rr - \alpha$ there will arise :

from $4\alpha n + rr$	from $4\alpha n + rr - \alpha$
$52n+1,$	$52n-9,$
$52n+9,$	$52n+3,$
$52n+25,$	$52n+23,$
$52n+49 = 52n-3,$	$52n+51 = 52n-1,$
$52n+81 = 52n-23,$	$52n+87 = 52n-17,$
$52n+121 = 52n+17,$	$52n+131 = 52n-25,$

which formulas are reduced to these :

$$52n \pm 1,\ 52n \pm 3,\ 52n \pm 9,\ 52n \pm 17,\ 52n \pm 23,\ 52n \pm 25;$$

and the prime numbers contained in these are :

$$1,\ \pm 3,\ \pm 17,\ \pm 23,\ \pm 29,\ \pm 43,\ \pm 53,\ \pm 61,\ \pm 79,\ \pm 101,\ \pm 103,$$

to which ± 13 must be added, then truly all the numbers are squares ; and if in addition the products from two or more of these numbers may be added, certainly in this case all the numbers will have been obtained, which substituted for γ produce the formula $13xx + \gamma = yy$, resolvable in whole numbers ; or some one of these numbers may be taken for γ, the first prime, then infinitely many whole numbers can be found for *x*, for which the formula $13xx + \gamma$ may be returned square. Indeed all these numbers likewise are contained in the form $bb - 13aa$; for here the more difficult are seen to be reduced, which are:

$-1 = 18^2 - 13\cdot 5^2,$	$+13 = 65^2 - 13\cdot 18^2,$	$-3 = 7 - 13\cdot 2^2,$
$+17 = 15^2 - 13\cdot 4^2,$	$17 = 10^2 - 13\cdot 3^2,$	$-23 = 43^2 - 13\cdot 12^2,$
$+29 = 9^2 - 13\cdot 2^2,$	$-29 = 32^2 - 13.\ 9^2,$	$+43 = 76^2 - 13\cdot 21^2,$
$-43 = 3^2 - 13\cdot 2^2,$	$+53 = 51^2 - 13\cdot 14^2,$	$-53 = 8^2 - 13\cdot 3^2,$
$+61 = 23^2 - 13\cdot 6^2,$	$-61 = 24^2 - 13\cdot 7^2,$	$+79 = 14^2 - 13\cdot 3^2,$
	$-79 = 57^2 - 13\cdot 16^2$	etc.

Therefore since there shall be $-1 = 18^2 - 13\cdot 5^2$, if there were $+\gamma = bb - 13aa,$ there will become

$$-\gamma = \left(18b \pm 65a\right)^2 - 13\left(18a \pm 5b\right)^2,$$

from which the more difficult cases are resolved.
Therefore this equation is required to be resolved

$$13xx+43\cdot 79=yy$$

since there shall be $\gamma=43\cdot 79\ =-\ 43\cdot -79,$ hence there will be had by the composition

therefore :
$$\text{I. } \gamma=\left(14\cdot 76\pm 13\cdot 63\right)^2-13\cdot\left(14\cdot 21\pm 3\cdot 76\right)^2,$$
$$x=294\pm 228 \text{ and } y=1064\pm 819,$$

therefore:
$$\text{II. } \gamma=\left(3\cdot 57\pm 13\cdot 32\right)^2-13\ \left(2\cdot 57\pm 3\cdot 16\right)^2,$$
$$x=114\pm 48 \quad\text{and}\quad y=416\pm 171,$$

from which at once three solutions are obtained.

[These are : $x=522,\ \ y=1883;\ x=66,\ \ y=245;\ x=162,\ \ y=487.$]

OBSERVATION 8

34. Truly not always from these prime numbers, which we have shown how to be investigated only, since clearly the numbers of all the squares are to be found, which can be assumed for γ, of which an example is the case $\alpha=10$, for which the values of γ are held in this form $bb-10aa$; and these are to be taken both negative and positive

1, 4, 6, 9, 10, 15, 16, 24, 25, 26, 31, 36, 39, 40, 41, 49, 54, 60, 64,65, 71, 74, 79, 81, 86, 89, 90, 96, 100, 104, 106, 111, 121, 124, 129, 134, 135, 144, 150, 151, 156, 159, 160, 164, 166, 169, 185, 186, 191, 196, 199, 201 etc.,

among which all the square numbers occur

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196 etc.,

then also a number of primes

31, 41, 71, 79, 89, 151, 191, 199 etc.,

which may be contained in these formulas $40n\pm 1$ and $40n\pm 9,$ and in addition the products from two or more of these numbers to be present. Truly in the third place besides these the numbers composed from two prime numbers are present, which are:

$$2\cdot 3,\ 2\cdot 5,\ 2\cdot 13,\ 2\cdot 37,\ 2\cdot 43,\ 2\cdot 53,\ 2\cdot 67,\ 2\cdot 83\ \text{ etc.},$$
$$3\cdot 5,\ 3\cdot 13,\ \ 3\cdot 37,\ \ 3\cdot 43,\ 3\cdot 53,\ \ 3\cdot 67\ \text{ etc.},$$
$$5\cdot 13,\ \ 5\cdot 37\ \text{ etc.}$$

But these prime numbers, of which two always are required to be multiplied by each other, are in the first place 2 et 5, truly the rest are held in these formulas $40n\pm3$ and $40n\pm13.$ Finally also the following general rule must be applied to the products from two or more numbers, which they themselves satisfy.

Thus this equation can be resolved :

$$10xx+13\cdot53\cdot151=yy,$$

for $13\cdot53=bb-10aa$ is given with $b=27,$ $a=2$ and $151=dd-10cc$ with $d=31$ and $c=9$, and hence

$$13\cdot53\cdot151=\left(bd\pm10ac\right)^2-10\left(ad\pm bc\right)^2$$

with

$$x=ad\pm\ bc \text{ and } y=bd\pm10ac.$$

Since then also there shall be $-13\cdot53=\mathrm{BB}-10AA$ and $-151=DD-10CC,$ hence two other solutions are found. But since there shall be $-1=3^2-10\cdot1^2$, if there were $\gamma=bb-10aa,$ there will become $\gamma=\left(3b\pm10a\right)^2-10(3a+b)^2.$ Moreover the solutions hence arising are :

$$x=181,\qquad y=657,$$
$$x=305,\qquad y=1017,$$
$$x=307,\qquad y=1023;$$

for the two may agree between themselves, thus so that hence only three can be found.

[Actually 6, due to the sign ambiguity of *A*, *B*, *C*, *&* *D*, the other three being :

$$x=503,\qquad y=1623,$$
$$x=7381,\qquad y=23343,$$
$$x=11897,\qquad y=37623.]$$

OBSERVATION 9

35. Therefore in the case $\alpha=10$, we find primitive numbers of three kinds for γ : in the first place evidently all the square numbers, then certain prime numbers present in the formulas $40n\pm1$ and $40n\pm9$, moreover in the third place products from the two certain prime numbers, which are 2, 5 and the rest being desired from these formulas $40n\pm3$ and $40n\pm13$, and at last from this threefold order all the suitable numbers for γ are formed, so that it shall be possible to satisfy this equation $10xx\pm\gamma=yy$. But appropriate prime numbers themselves may not be found in the formulas

$40n \pm 3$ and $40n \pm 13$, since they are not of the form $bb - 10aa$, but yet these numbers are all of the form $2bb - 5aa$, also to be used from these two taken together, 2 and 5. Moreover it is evident, if two numbers of this kind may be had $2bb - 5aa$ and $2dd - 5cc$, the product of these to become $= (2bd \pm 5ac)^2 - 10(bc \pm ad)^2$ and thus can be used for γ. Therefore products of this kind of two prime numbers, which themselves may not be satisfactory, cannot occur, if α were a prime number, but only to be used here, if α were a composite number ; which still also may not be appropriate, as we have seen in the case $\alpha = 6 = 2 \cdot 3$, from which numbers of the form $3bb - 2aa$ agree with numbers of the form $bb - 6aa$. So that therefore if there were in general $\alpha = pq$ and the equation $pqxx + \gamma = yy$ must be resolved, the number γ either must be a square number or a prime number of the form $bb - pqaa$, or the product from two prime numbers of the form $pbb - qaa$, therefore so that the product of this kind is :

$$(pbb - qaa)(pdd - qcc) = (pbd \pm qac)^2 - pq(bc \pm ad)^2.$$

Therefore unless such prime numbers $pbb - qaa$ may themselves be contained in the form $bb - pqaa$, that third order of the numbers is to be added by merging the two first orders. Just as then solitary prime numbers are contained in the formulas $4pqn + rr$ and $4pqn + rr - pq$, thus other prime numbers requiring to be used must be derived from this formula

$$4pqn + prr - qss.$$

EXAMPLE 1

36. *All the suitable values of γ shall be investigated, so that this equation*

$$30xx + \gamma = yy$$

may admit a resolution.

Indeed in the first place all the square numbers can be assumed for γ, then all the prime numbers contained in these forms $120n + rr$ and $120n + rr - 30$, which are reduced to these

$$120n+1,\ 120n+49,\ 120n+19,\ 120n-29 \text{ with } -5,$$

from which these prime number less than 200 arise :

$$\text{positive } +19,\ +139 \text{ and negative } -5,\ -29,\ -71,\ -101,\ -149,\ -191.$$

Thirdly on account of $\alpha = 2 \cdot 3 \cdot 5$ the product of the first two prime numbers may be taken, which may contain either both or one of these formulas

$$\text{I. } 120n+2rr-15ss,\ \text{II. } 120n+3rr-10ss,\ \text{III. } 120n+5rr-6ss;$$

of which the first two give the same prime numbers, which are $+2,\ +3,$ and the rest are contained in these formulas:

$$120n-7,\ 120n-13,\ 120n+17,\ 120n-37,$$

from which there prime numbers less than 200 arise:

$$\begin{array}{ll} \text{positive} & +2,\ +3,\ +17,\ +83,\ +107,\ +113,\ +137, \\ \text{negative} & -7,\ -13,\ -37,\ -103,\ -127, \end{array}$$

of which the product of two are required to be taken for γ

$$\begin{array}{c} +6,\ +34,\ +51,\ +91,\ +166, \\ -14,\ -21,\ -26,\ -39,\ -74,\ -111,\ -119. \end{array}$$

But the third formula contains the prime number +5 with these forms

$$120n-1,\ 120n-19,\ 120n+29,\ 120n-49,$$

from which these prime numbers arise less than 200

$$\begin{array}{ll} \text{positive} & +5,\ +29,\ +71,\ +101,\ +149,\ +191, \\ \text{negative} & -1,\ -19,\ -139. \end{array}$$

But, from the combination of these, the same numbers arise, which now arise from the first prime numbers.
On account of which all the numbers, all the numbers, which can be substituted for γ, below 200 will become

$$+1,\ +4,\ +9,\ +16,\ +25,\ +36,\ +49,\ +64,\ +81,\ +100,\ +121,\ +144,\ +169,\ +196;$$

$$\begin{array}{l} -5,\ +19,\ -29,\ -71,\ -101,\ +139,\ -149,\ -191; \\ +6,\ -14,\ -21,\ -26,\ +34,\ -39,\ +51,\ -74,\ +91,\ -111,\ -119,\ +166; \end{array}$$

$$\begin{array}{l} -20,\ +24,\ -30,\ -45,\ +54,\ -56,\ +70,\ +76,\ -80,\ -84,\ -95, \\ +96,\ -104,\ +105,\ +114,\ -116,\ -125,\ -126,\ +130,\ +136,\ 145, \\ +150,\ -156,\ -170,\ +171,\ -189,\ +195. \end{array}$$

Moreover all the remaining numbers assumed for γ will return an impossible problem.

EXAMPLE 2

37. *To resolve the equation into whole numbers*

$$5xx+11\cdot 19\cdot 29=yy.$$

Since there is $\alpha=5$ and $\gamma=11\cdot 19\cdot 29$, factors with this form $bb-5aa$ are agreed and the individual factors contained in that are taken; for

for 11 there is $b=4,\quad a=1,$
for 19 there is $b=8,\quad a=3,$
for 29 there is $b=7,\quad a=2,$

from which also the products from the second in the same form are contained; for $11\cdot 19$ there is

$$b=17,\quad a=4,$$
$$b=47,\quad a=20;$$

therefore for the third requiring to be added on for $11\cdot 19\cdot 29$ there is

$$b=79,\quad a=6,\quad b=129,\quad a=46,$$
$$b=159,\quad a=62,\quad b=529,\quad a=234.$$

Now since there shall be $1=9^2-5\cdot 4^2$ or $b=9$ and $a=4$ for 1, these formulas above multiplied by 1 will be duplicated and there will become for $11\cdot 19\cdot 29$

$$b=591,\quad a=262,\quad b=241,\quad a=102,$$
$$b=831,\quad a=370,\quad b=2081,\quad a=930,$$
$$b=191,\quad a=78,\quad b=81,\quad a=10,$$
$$b=2671,\quad a=1194,\quad b=9441,\quad a=4222.$$

Hence therefore now we have obtained twelve solutions of the problem, which are:

I. $x=6$,	$y=79$,	VII. $x=234$,	$y=529$,
II. $x=10$,	$y=81$,	VIII. $x=262$,	$y=591$,
III. $x=46$,	$y=129$,	IX. $x=370$,	$y=831$,
IV. $x=62$,	$y=159$,	X. $x=930$,	$y=2081$,
V. $x=78$,	$y=191$,	XI. $x=1194$,	$y=2671$,
VI. $x=102$,	$y=241$,	XII. $x=4222$,	$y=9441$,

from which again since for the formula $1=9^2-5\cdot 4^2$ infinitely many are required to be adjoined and all these will be elicited; evidently from the second there will be produced $x=414,\ \ y=929$ and from the sixth
$x=1882,\ y=4209,$ from the fifth $x=1466,\ y=3279,$ from the eighth $x=4722,\ \ y=10559;$ and thus now we have gained sixteen solutions.

CONCLUSION

38. From these established we are unable to collect further propositions from an equation of this kind $\alpha xx+\gamma=yy$ as if at first to seek by requiring to divine one satisfactory case, but by examining the number γ we are able to announce at once from the following formulas whether or not an equation may admit to a resolution ; and if it allowed, by the same principles perhaps one solution will be permitted to be elicited, which certainly can be done at once, if the number γ were resolvable into not too great factors. Truly if the number γ shall be prime and exceedingly large, indeed the judgment of solubility is equally easy, but will discovery will require the effort of finding one larger solution. Just as if there were proposed

$$30xx+1459=yy,$$

since 1459 is a prime number of the form $120n+19$, the equation is resolvable; truly that will be satisfied by taking $x=39$ and $y=217$ which is not so very easy to find. Yet the investigation is raised, if we may put $y=30z\pm 7$, from which there becomes $xx=30zz\pm 14z-47,$ and now we may find more quickly $z=7$ and $x=39$, from which $y=217$ is produced. But if we may put $y=30z\pm 13$, there becomes $xx=30zz\pm 26z-43$ and there is found more quickly $x=5$ and $y=47$.
Truly insurmountable labor emerges with much larger numbers and a method certainly is desired to be put in place at this point for handling the problem; then also, because all the prime numbers in the above formula $4\alpha n+A$ likewise shall contain numbers of this form $bb-\alpha aa$, pertaining to these propositions, which we believe true, even if we may not prevail to demonstrate these.

DE RESOLUTIONE FORMULARUM QUADRATICARUM INDETERMINATARUM PER NUMEROS INTEGROS

Commentatio 279 indicis ENESTROEMIANI

Novi commentarii academiae scientiarum Petropolitanae 9 (i 762/3), 1764, p. 3-39

PROBLEMA 1

1. *Proposita formula irrationali*

$$\sqrt{(\alpha xx+\beta x+\gamma)}$$

invenire numeros pro x substituendos, qui eam rationalem reddant.

SOLUTIO

Ante omnia notandum est hanc investigationem frustra suscipi, nisi unus saltem casus constet, quo ea fiat rationalis. Ponamus ergo hoc evenire casu $x=a$ eoque esse

$$\sqrt{(\alpha aa+\beta a+\gamma)}=b\,,$$

ita ut *b* sit numerus rationalis. Huiusmodi autem casus unico cognito innumerabiles alios ex eo derivare licet. Ponatur in hunc finem

$$x=a+mz \quad \text{et} \quad \sqrt{(\alpha xx+\beta x+\gamma)}=b+nz$$

et hac aequatione quadrata fit

$$\begin{aligned}&+\alpha aa+2\alpha maz+ammzz=bb+2nbz+nnzz\\&+\beta a+\beta mz\\&+\ \gamma\end{aligned}$$

Cum iam per hypothesin sit $bb=\alpha aa+\beta a+\gamma$, reliqua aequatio per *z* divisa dabit

$$2\alpha ma+\beta m+\ \alpha mmz=2nb+nnz,$$

ex qua elicitur

$$z=\frac{2\alpha ma-2nb+\beta m}{nn-\alpha mm}.$$

Quo valore substituto concludimus, si ponatur

$$x=\frac{(nn+\alpha mm)a-2mnb+\beta mm}{nn-\alpha mm},$$

fore

$$\sqrt{(\alpha xx+\beta x+\gamma)}=\frac{2\alpha mna-(nn+\alpha mm)b+\beta mn}{nn-\alpha mm},$$

Quicunque ergo numeri pro *m* et *n* accipiantur, ex casu cognito

$$\sqrt{(\alpha aa+\beta a+\gamma)}=b$$

infinitis aliis modis formula $\sqrt{(\alpha xx+\beta x+\gamma)}$ rationalis effici potest, et quia numerum *b* tam negative quam affirmative assumere licet, exploratis numeris *a* et *b* ac pro lubitu assumtis numeris *m* et *n* capiatur

$$x=\frac{(nn+\alpha mm)a\pm 2mnb+\beta mm}{nn-\alpha mm},$$

eritque

$$\sqrt{(\alpha xx+\beta x+\gamma)}=\frac{2\alpha mna\pm(nn+\alpha mm)b+\beta mn}{nn-\alpha mm},$$

SCHOLION

2. Ad hoc ergo problema solvendum necesse est, ut aliunde unus saltem casus sit cognitus, quo formula proposita fiat rationalis. Neque vero pro huiusmodi casu explorando ulla certa regula praescribi potest, cum etiam dentur eiusmodi formulae, quas nullo plane casu rationales fieri posse demonstratum est. Si enim verbi gratia haec formula $\sqrt{(3xx+2)}$ proponeretur, certum est nullum numerum rationalem pro *x* inveniri posse, quo ea fieret rationalis. Quanquam autem satis noti sunt casus, quibus formula $\alpha xx+\beta x+\gamma$ talis reductionis est capax, quippe quod evenit, quoties in hac formula generali $(px+q)^2+(rx+s)(tx+u)$ continetur, tamen hic non curo, unde casus ille, quem cognitum assumo, sit haustus, sive certa quadam ratione sive divinatione innotuerit. Verum cum cognito uno casu inventio infinitorum aliorum nulla laboret difficultate, hic potissimum ad solutiones, quae numeris integris absolvuntur, respicio. Cum enim valores pro *x* inventi per fractionem exprimantur, nova iam oritur quaestio, quomodo numeros *m* et *n* assumi oporteat, ut inde numeri integri pro *x* obtineantur.

PROBLEMA 2

3. *Si* α, β, γ *sint numeri integri dati, invenire numeros integros pro x sumendos, qui formulam* $\alpha xx+\beta x+\gamma$ *quadratam reddant.*

SOLUTIO

Iterum assumo unum numerum integrum *a* constare, qui quaesito satisfaciat, ita ut sit

$$\alpha aa+\beta a+\gamma=b,$$

ac modo vidimus, si sumatur

$$x=\frac{(nn+\alpha mm)a\pm 2mnb+\beta mm}{nn-\alpha mm},$$

fore

$$\sqrt{(\alpha xx+\beta x+\gamma)}=\frac{2\alpha mna\pm(nn+\alpha mm)b+\beta mn}{nn-\alpha mm}.$$

Superest ergo tantum, ut videamus, cuiusmodi numeros pro *m* et *n* assumi oporteat, ut hae formulae integrae evadant. Quod quidem statim fieri perspicuum est, si utriusque denominator $nn-\alpha mm$ statuatur unitati aequalis.
Sit igitur

$$nn-\alpha mm=1 \quad \text{seu} \quad nn=\alpha mm+1$$

ideoque

$$n=\sqrt{(\alpha mm+1)};$$

nisi autem sit α vel numerus quadratus vel negativus, huic formulae semper satisfieri potest; sin autem sit vel quadratus vel negativus, ne problema quidem propositum resolvere licet. Etsi enim quandoque duo pluresve casus assignari queant, tamen infiniti non dantur, cuiusmodi tamen hic evolvi convenit. Sit ergo α numerus integer positivus non quadratus ac semper numeri *m* et *n* assignari possunt, ut fiat $n=\sqrt{(\alpha mm+1)}$; quod etsi infinitis modis fieri potest, tamen sufficit minimos solos nosse. Erit ergo

$$x=(nn+\alpha mm)a\pm 2mnb+\beta mm$$

et

$$\sqrt{(\alpha xx+\beta x+\gamma)}=2\alpha mma\pm(nn+\alpha mm)b+\beta mn$$

sicque habetur novus casus quaestioni satisfaciens. Ex hoc vero simili modo, quo is ex *a* et *b* prodiit, novus derivabitur hincque porro continuo alii in infinitum. Ponantur enim valores hoc modo pro *x* oriundi successive

$$a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},\ \text{etc.},$$

respondentes vero valores formulae $\sqrt{(\alpha xx+\beta x+\gamma)}$ sint

$$b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},\ \text{etc.},$$

ac sequenti modo bini quique posteriores ex binis antecedentibus definientur:

$$a^{\mathrm{I}}=(nn+\alpha mm)a\pm 2mnb+\beta mm,\ \ b^{\mathrm{I}}=2\alpha mna\pm(nn+\alpha mm)b+\beta mn,$$
$$a^{\mathrm{II}}=(nn+\alpha mm)a^{\mathrm{I}}\pm 2mnb^{\mathrm{I}}+\beta mm,\ \ b^{\mathrm{II}}=2\alpha mna^{\mathrm{I}}\pm(nn+\alpha mm)b^{\mathrm{I}}+\beta mn,$$
$$a^{\mathrm{II}}=(nn+\alpha mm)a^{\mathrm{I}}\pm 2mnb^{\mathrm{I}}+\beta mm,\ \ b^{\mathrm{II}}=2\alpha mna^{\mathrm{I}}\pm(nn+\alpha mm)b^{\mathrm{I}}+\beta mn,$$
$$a^{\mathrm{III}}=(nn+\alpha mm)a^{\mathrm{II}}\pm 2mnb^{\mathrm{II}}+\beta mm,\ \ b^{\mathrm{III}}=2\alpha mna^{\mathrm{II}}\pm(nn+\alpha mm)b^{\mathrm{II}}+\beta mn,$$
$$\text{etc.}$$

Hac igitur ratione continuo ulterius progredi licet sicque ex una solutione in numeris integris cognita innumerabiles aliae in numeris integris quoque elicientur.

COROLLARIUM 1

4. Ut igitur formula $\alpha xx+\beta x+\gamma$ infinitis modis in numeris integris quadratum effici possit, necesse est, ut α neque sit numerus quadratus neque negativus, ac praeterea, ut unus casus, quo ea fit quadratum, undecunque sit cognitus.

COROLLARIUM 2

5. At si *a* fuerit numerus positivus non quadratus, tum primum quaerantur duo numeri *m* et *n*, ut sit $n=\sqrt{(\alpha mm+1)}$, id quod semper fieri potest. Quibus inventis si ponatur

$$\sqrt{(\alpha xx+\beta x+\gamma)}=y$$

atque iam cognitus fuerit casus quaestioni satisfaciens, qui sit $x=a$ et $y=b$, ex eo per primam operationem non solum unus, sed duo novi invenientur ob signi ambiguitatem. Erit quippe

$$x=(nn+\alpha mm)a\pm 2mnb+\beta mm$$

et

$$y=2\alpha mna\pm\left(nn+\alpha mm\right)b+\beta mn.$$

COROLLARIUM 3

6. Si sumantur tantum signorum ambiguorum superiora, ut continuo ad maiores numeros satisfacientes perveniamus, atque valores pro *x* hoc modo successive prodeuntes designentur per $a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},$ etc., valores autem pro *y* respondentes per $b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},$ etc., erit

$$\begin{aligned}
a^{\mathrm{I}} &= (nn+\alpha mm)a+2mnb+\beta mm, & b^{\mathrm{I}} &= 2\alpha mna+(nn+\alpha mm)b+\beta mn,\\
a^{\mathrm{II}} &= (nn+\alpha mm)a^{\mathrm{I}}+2mnb^{\mathrm{I}}+\beta mm, & b^{\mathrm{II}} &= 2\alpha mna^{\mathrm{I}}+(nn+\alpha mm)b^{\mathrm{I}}+\beta mn,\\
a^{\mathrm{III}} &= (nn+\alpha mm)a^{\mathrm{II}}+2mnb^{\mathrm{II}}+\beta mm, & b^{\mathrm{III}} &= 2\alpha mna^{\mathrm{II}}+(nn+\alpha mm)b^{\mathrm{II}}+\beta mn,
\end{aligned}$$

etc.

COROLLARIUM 4

7. Duplicem ergo hinc progressionem numerorum $a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},$ etc., et $b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},$ etc., adipiscimur, quarum utriusque continuatio ab utraque

pendet, utraque tamen ab altera ita seiungi potest, ut termini utriusque sensim sine adminiculo alterius continuari queant; formabitur autem tum in utraque serie quilibet terminus ex binis praecedentibus.

COROLLARIUM 5

8. Si enim in valore a^{II} au pro b^{I} eius valor substituatur, habebitur

$$a^{\mathrm{II}} = (nn+\alpha mm)a^{\mathrm{I}} + 4mmnna + 2mn(nn+\alpha mm)b + 2\beta mmnn + \beta mn.$$

Verum ex valore ipsius a^{I} est $2mnb = \mathrm{a}^{\mathrm{I}} - (nn+\alpha mm)a - \beta mm,$ quo valore ipsius *2mnb* ibi substituto prodibit

$$\begin{aligned} a^{\mathrm{II}} &= (nn+\alpha mm)a^{\mathrm{I}} + 4\alpha mmnna + (nn+\alpha mm)a^{\mathrm{I}} \\ &\quad -(nn+\alpha mm)^2 a - \beta mm(nn+\alpha mm) + 2\beta mmnn + \beta mm. \end{aligned}$$

At ob $nn = \alpha mm + 1$ est

$$4\alpha mmnn - \left(nn + \alpha mm\right)^2 = -\left(nn - \alpha mm\right)^2 = -1$$

et

$$2\beta mmnn - \beta mm\left(nn + \alpha mm\right) = \beta mm(nn - \alpha mm) = \beta mm,$$

unde fit

$$a^{\mathrm{II}} = 2(nn+\alpha mm)a^{\mathrm{I}} - a + 2\beta mm.$$

COROLLARIUM 6

9. Cum igitur simili modo sit

$$a^{\mathrm{III}} = 2(nn+\alpha mm)a^{\mathrm{II}} - a^{\mathrm{I}} + 2\beta mm \quad \text{etc.},$$

statim atque in serie $a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},$ etc. duo primi termini habentur, primus scilicet *a* undecunque et secundus ex formula $a^{\mathrm{I}} = (nn+\alpha mm)a + 2mmb + \beta mm,$ ex his sequentes omnes per has formulas definientur

$$\begin{aligned} a^{\mathrm{II}} &= 2(nn+\alpha mm)a^{\mathrm{I}} - a + 2\beta mm, \\ a^{\mathrm{III}} &= 2(nn+\alpha mm)a^{\mathrm{II}} - a^{\mathrm{I}} + 2\beta mm, \\ a^{\mathrm{IV}} &= 2(nn+\alpha mm)a^{\mathrm{III}} - a^{\mathrm{II}} + 2\beta mm, \\ &\text{etc.} \end{aligned}$$

COROLLARIUM 7

10. Pari autem modo progressio numerorum $b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},$ etc. est comparata. Primo enim eius termino aliunde cognito et secundo per formulam

$b^{\mathrm{I}} = 2\alpha mna + (nn+\alpha mm)b + \beta mn,$ si in b^{II} pro a^{I} valor substituatur, erit

$$b^{\mathrm{II}} = 2\alpha mn(nn+\alpha mm)a + 4\alpha mmnnb + 2a\beta m^3 n + (nn+\alpha mm)b^{\mathrm{I}} + \beta mn;$$

at ex valore ipsius b^{I} est $2\alpha mna = b^{\mathrm{I}} - (nn+\alpha mm)b - \beta mn,$ quo substituto fit ob $nn - \alpha mm = 1$

$$b^{\mathrm{II}} = 2(nn+\alpha mm)b^{\mathrm{I}} - b$$

similiterque

$$b^{\mathrm{III}} = 2(nn+\alpha mm)b^{\mathrm{II}} - b^{\mathrm{I}},$$
$$b^{\mathrm{IV}} = 2(nn+\alpha mm)b^{\mathrm{III}} - b^{\mathrm{II}},$$
etc.

COROLLARIUM 8

11. Cum igitur utraque series ita sit comparata, ut quilibet terminus ex binis praecedentibus secundum certam legem definiatur, utraque series erit recurrens scala relationis existente $2(nn+\alpha mm), -1$. Hinc ergo formata aequatione

$$zz = 2(nn+\alpha mm)z - 1$$

eius radices erunt

$$z = 2nn - 1 \pm 2n\sqrt{(nn-1)} = \left(n \pm m\sqrt{a}\right)^2.$$

COROLLARIUM 9

12. Hinc ergo ex doctrina serierum recurrentium progressionis $a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},\ a^{\mathrm{IV}}$ etc. terminus quicunque indefinite per sequentem formulam exprimetur

$$\left(\tfrac{a}{2} + \tfrac{\beta}{4\alpha} + \tfrac{b}{2\sqrt{\alpha}}\right)\left(n + m\sqrt{\alpha}\right)^{2v} + \left(\tfrac{a}{2} + \tfrac{\beta}{4\alpha} - \tfrac{b}{2\sqrt{\alpha}}\right)\left(n - m\sqrt{\alpha}\right)^{2v} - \tfrac{\beta}{2\alpha} = x,$$

alterius vero seriei $b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},$ etc. terminus quicunque per hanc sumto pro v numero quocunque integro.

$$\left(\tfrac{b}{2} + \tfrac{a\sqrt{\alpha}}{2} + \tfrac{\beta}{4\sqrt{\alpha}}\right)\left(n + m\sqrt{\alpha}\right)^{2v} + \left(\tfrac{b}{2} - \tfrac{a\sqrt{\alpha}}{2} - \tfrac{\beta}{4\sqrt{\alpha}}\right)\left(n - m\sqrt{\alpha}\right)^{2v} = y,$$

SCHOLION

13. Si hic pro $2v$ substituamus successive omnes numeros integros 0, 1, 2, 3, 4, 5 etc., utraque progressio prodibit interpolata, cuius termini medii quaesito aeque satisfacient, dummodo fuerint integri. At reperiemus posito

$$2v=0\begin{cases}x=a,\\ y=b,\end{cases}$$
$$2v=1\begin{cases}x=na+mb+\frac{\beta}{2\alpha}(n-1),\\ y=nb+\alpha ma+\frac{1}{2}\beta m,\end{cases}$$
$$2v=2\begin{cases}x=(nn+\alpha mm)a+2mnb+\beta mm,\\ y=(nn+\alpha mm)b+2\alpha mna+\beta mm.\end{cases}$$

Quae utraque series est recurrens scalam relationis habens $2n$, - 1; ac pro priori quidem valorum ipsius x, si terni termini consecutivi sint P, Q, R, erit

$$R=2nQ-P+\frac{\beta(n-1)}{\alpha};$$

at si in progressione valorum ipsius y terni termini se ordine sequentes sint P, Q et R, erit

$$R=2nQ-P.$$

Quodsi ergo fuerit $\frac{\beta}{2\alpha}(n-1)$ numerus integer, omnes hi termini problema aeque resolvent sicque duplo plures obtinebimus solutiones, quam methodus adhibita suppeditaverat. Quod autem plures locum habere possint solutiones, quam invenimus, inde facile colligitur, quod praeter necessitatem primam erutarum formularum $nn-\alpha mm$ unitati aequalem posuimus, cum tamen sine dubio saepe etiam numerator per denominatorem dividi possit, etiamsi hic unitate sit maior. Quemadmodum igitur omnes plane solutiones in numeris integris inveniri queant, sequenti problemate accuratius examinemus.

PROBLEMA 3

14. *Si a sit numerus integer positivus non quadratus, dato uno numero integro a, qui pro x positus reddat formulam* $\alpha xx+\beta x+\gamma$ *quadratam, invenire infinitos alios numeros integros, qui pro x sumti idem sint praestituri.*

SOLUTIO

Ponatur in genere $\sqrt{(\alpha xx+\beta x+\gamma)}=y$, casu autem cognito, quo $x=a$, esse $\sqrt{(\alpha aa+\beta a+\gamma)}=b$ atque hinc in genere fractionibus non exclusis fore vidimus

$$x=\frac{(nn+\alpha mm)a+2mnb+\beta mm}{nn-\alpha mm},$$
$$y=\frac{(nn+\alpha mm)b+2\alpha mna+\beta mm}{nn-\alpha mm}.$$

Iam quidem, ut hi numeri fiant integri, non absolute necesse est, ut denominator $nn-\alpha mm$ ad unitatem revocetur, verum sufficit, ut fractiones

$\frac{nn+\alpha mm}{nn-\alpha mm}$ et $\frac{2mn}{nn-\alpha mm}$ in numeros integras abeant. Ponamus ergo esse

$$\frac{nn+\alpha mm}{nn-\alpha mm} = p \text{ et } \frac{2mn}{nn-\alpha mm} = q,$$

unde fit

$$p-1 = \frac{2\alpha mm}{nn-\alpha mm} \text{ ideoque}$$

$$\frac{\beta mm}{nn-\alpha mm} = \frac{\beta}{2\alpha}(p-1) \text{ et } \frac{\beta mn}{nn-\alpha mm} = \frac{1}{2}\beta q.$$

Deinde autem ex formulis assumtis fiet

$$pp-\alpha qq = \frac{(nn+\alpha mm)^2-4\alpha m^2n^2}{(nn-\alpha mm)^2} = 1,$$

ita ut sit

$$pp = \alpha qq+1 \text{ et } p = \sqrt{(\alpha qq+1)}.$$

Iterum igitur ut ante ex numero α binos numeros p et q assignari oportet, ut sit $p = \sqrt{(\alpha qq+1)}$; quibus inventis habebitur

$$x = pa+qb+\frac{\beta}{2\alpha}(p-1) \text{ et } y = pb+\alpha qa+\frac{1}{2}\beta q.$$

Dummodo ergo fuerit $\frac{\beta}{2\alpha}(p-1)$ numerus integer, hi valores satisfaciunt. Quia autem numeros p et q tam negative quam positive sumere licet, hae formulae insuper tres alias solutiones suppeditant

$$x = pa-qb+\frac{\beta}{2\alpha}(p-1) \text{ et } y = pb-\alpha qa-\frac{1}{2}\beta q,$$

$$x = -pa+qb-\frac{\beta}{2\alpha}(p+1) \text{ et } y = -pb+\alpha qa+\frac{1}{2}\beta q,$$

$$x = -pa-qb-\frac{\beta}{2\alpha}(p+1) \text{ et } y = -pb-\alpha qa-\frac{1}{2}\beta q.$$

Quodsi porro horum bini quicunque pro a et b assumantur, ex quolibet quatuor novae solutiones orientur. Hinc tamen non 16, sed tantum sex diversae oriuntur, inter quas adeo prima cognita $x = a$ et $y = b$ et, quae huic est affinis, $x = -a-\frac{\beta}{\alpha}$ et $y = -b$ continentur; reliquae vero quatuor sunt

$$x = (pp+\alpha qq)a \pm 2pqb+\beta qq, \quad y = (pp+\alpha qq)b \pm 2\alpha pqa \pm \beta pq,$$

$$x = -(pp+\alpha qq)a \pm 2pqb-\frac{\beta}{\alpha}pp, \quad y = (pp+\alpha qq)b \mp 2\alpha pqa \mp \beta pq,$$

ex quibus deinceps novae aliae in infinitum inveniri possunt.

COROLLARIUM 1

15. Quodsi ergo fuerit vel $\beta=0$ vel eiusmodi numerus, ut $\beta(p-1)$ vel etiam $\beta(p+1)$ per 2α divisibile existat, tum hoc modo plures solutiones in integris obtinentur quam modo ante exposito.

COROLLARIUM 2

16. In genere autem observandum est, si satisfecerit casus quicunque $x=v$, tum etiam satisfacturum esse casum $x=-v-\frac{\beta}{\alpha}$; ex utroque enim y eundem valorem nanciscitur. Quare cum hi casus ex illis tam facile eliciantur, his omissis investigatio solutionum convenientium ad dimidium reducitur.

COROLLARIUM 3

17. Reiectis ergo casibus $x=-v-\frac{\beta}{\alpha}$, quippe qui sponte se produnt inventis casibus $x=v$, ex casu $x=a$ et $y=b$ statim bini reperiuntur

$$x=pa\pm qb+\tfrac{\beta}{2\alpha}(p-1),\ y=\alpha qa\pm pb+\tfrac{1}{2}\beta q$$

hincque porro per operationem secundam bini

$$x=(pp+\alpha qq)\pm 2pqb+\beta qq,\ y=2\alpha pqa\pm(pp+\alpha qq)b+\beta pq,$$

quae duplicitas ex signo ambiguo numeri b nascitur.

COROLLARIUM 4

18. Si haec cum §12 et §13 conferantur, patebit omnes has formulas in sequentibus expressionibus generalibus contineri, siquidem pro μ successive omnes numeri integri substituantur:

$$\text{I.}\ \begin{cases} x=\frac{1}{4\alpha}\left(2\alpha a+\beta+2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}+\frac{1}{4\alpha}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}-\frac{\beta}{2\alpha}, \\ y=\frac{1}{4\sqrt{\alpha}}\left(2\alpha a+\beta+2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}-\frac{1}{4\alpha\sqrt{\alpha}}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p-q\sqrt{\alpha}\right)^{\mu} \end{cases}$$

et

$$\text{II.}\ \begin{cases} x=\frac{1}{4\alpha}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}+\frac{1}{4\alpha}\left(2\alpha a+\beta+2b\sqrt{\alpha}\right)\left(p-q\sqrt{\alpha}\right)^{\mu}-\frac{\beta}{2\alpha}, \\ y=\frac{1}{4\sqrt{\alpha}}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p+q\sqrt{\alpha}\right)^{\mu}-\frac{1}{4\alpha\sqrt{\alpha}}\left(2\alpha a+\beta-2b\sqrt{\alpha}\right)\left(p-q\sqrt{\alpha}\right)^{\mu}. \end{cases}$$

COROLLARIUM 5

19. Hinc igitur duplices series pro valoribus numerorum x et y reperiuntur, quae eandem progressionis legem tenebunt. Si enim ponamus

$$x = a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},\ a^{\mathrm{IV}}, a^{\mathrm{V}},\ \text{etc.},\ P,\ Q,\ R,$$
$$y = b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},\ b^{\mathrm{IV}}, b^{\mathrm{V}},\ \text{etc.},\ S,\ T,\ V,$$

erit pro altera

$$a^{\mathrm{I}} = pa + qb + \tfrac{\beta}{2\alpha}(p-1) \ \text{ et } \ b^{\mathrm{I}} = \alpha qa + pb + \tfrac{1}{2}\beta q$$

et pro altera

$$a^{\mathrm{I}} = pa - qb + \tfrac{\beta}{2\alpha}(p-1) \ \text{ et } \ b^{\mathrm{I}} = \alpha qa - pb + \tfrac{1}{2}\beta q,$$

pro utraque vero haec communis progressionis lex valebit, ut sit

$$R = 2pQ - P + \tfrac{\beta}{\alpha}(p-1) \ \text{ et } \ V = 2pT - S.$$

COROLLARIUM 6

20. Cum sit $pp - \alpha qq = 1$, erit

$$\left(p + q\sqrt{\alpha}\right)^{\mu} = \left(p - q\sqrt{\alpha}\right)^{-\mu} \ \text{ et } \ \left(p - q\sqrt{\alpha}\right)^{\mu} = \left(p + q\sqrt{\alpha}\right)^{-\mu}$$

hincque, si alterae series retrorsum continuentur, prodibunt alterae. Sufficit ergo pro altero casu has series instruxisse, quae tam antrorsum quam retrorsum continuatae omnes solutiones ex ambiguitate numeri b oriundas in se continebunt.

SCHOLION

21. Si ergo fuerit $\beta = 0$, ut habeatur haec formula $\sqrt{(\alpha xx + \gamma)} = y$ rationalis reddenda, casusque constet, quo sit $\sqrt{(\alpha xx + \gamma)} = b$, sumtis numeris p et q ita, ut sit $p = \sqrt{(\alpha qq + 1)}$ innumerabiles alii valores satisfacientes continebuntur in his seriebus

$$x = a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},\ a^{\mathrm{III}},\ a^{\mathrm{IV}}, a^{\mathrm{V}},\ \text{etc.},\ P,\ Q,\ R,$$
$$y = b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},\ b^{\mathrm{IV}}, b^{\mathrm{V}},\ \text{etc.},\ S,\ T,\ V,$$

ubi secundi termini ita debent accipi, ut sit

$$a^{\mathrm{I}} = pa + qb,\ b^{\mathrm{I}} = \alpha qa + pb;$$

deinde utraque series est recurrens scala relationis existente $2p, -1$. Erit scilicet

$$a^{\text{II}} = 2pa^{\text{I}} - a \text{ et in genere } R = 2pQ - P,$$
$$b^{\text{II}} = 2pb^{\text{I}} - b \cdot \cdot \cdot \cdot \cdot V = 2pT - S;$$

ambae vero series etiam retrorsum continuari debent sicque duplo plures prodibunt solutiones: nisi sit vel $a = 0$ vel $b = 0$. Neque autem hic in censum veniunt solutiones negativae, quia, si satisfecerit $x = v$, etiam satisfacit $x = -v$. Omnes porro istae solutiones continentur in his formulis generalibus

$$\begin{cases} x = \frac{1}{2\sqrt{\alpha}}\left(a\sqrt{\alpha}+b\right)\left(p+q\sqrt{\alpha}\right)^{\mu} + \frac{1}{2\sqrt{\alpha}}\left(a\sqrt{\alpha}-b\right)\left(p-q\sqrt{\alpha}\right)^{\mu}, \\ y = \frac{1}{2}\left(a\sqrt{\alpha}+b\right)\left(p+q\sqrt{\alpha}\right)^{\mu} \qquad - \frac{1}{2}\left(a\sqrt{\alpha}-b\right)\left(p-q\sqrt{\alpha}\right)^{\mu}. \end{cases}$$

Pro variis igitur numeris, qui coefficientem α constituunt, sequentia exempla evolvamus, et quidem generalius, ut etiam coefficientis β ratio habeatur, pro casibus scilicet, quibus forte $\frac{\beta}{2\alpha}(p-1)$ fuerit numerus integer.

EXEMPLUM 1

22. *Proposita formula*

$$\sqrt{(2xx+\beta x+\gamma)} = y$$

invenire infinitos valores integros ipsius x, quibus haec formula rationalis evadit, siquidem una solutio constet.

Sit solutio cognita $x = a$ et $y = b$ et ob $\alpha = 2$ habebimus $p = \sqrt{(2qq+1)}$ ideoque $q = 2$ et $p = 3$. Hinc secundi valores erunt

$$a^{\text{I}} = 3a \pm 2b + \tfrac{\beta}{2},\ b^{\text{I}} = 4a \pm 3b + \beta.$$

Cum igitur in §19 sit

$$R = 6Q - P + \beta \text{ et } V = 6T - S,$$

habebimus sequentes series valorum satisfacientium, et quidem integrorum, si β fuerit numerus par:

Valores ipsius x	Valores ipsius y
$a,$	$\pm b,$
$3a \pm 2b + \frac{\beta}{2}$	$4a \pm 3b + \beta,$
$17a \pm 12b + 4\beta,$	$24a \pm 17b + 6\beta,$
$99a \pm 70b + \frac{49}{2}\beta,$	$140a \pm 99b + 35\beta,$

$577a \pm 408b + 144\beta,$	$816a \pm 577b + 204\beta,$
$3363a \pm 2378b + \frac{1681}{2}\beta$	$4756a \pm 3363b + 1189\beta$
etc.	etc.

Tum vero, cum y eosdem retineat valores, si pro x scribatur $-x-\frac{1}{2}\beta$, hae solutiones locum habebunt:

Valores ipsius x	Valores ipsius y
$-a-\frac{1}{2}\beta,$	$\pm b,$
$-3a \mp 2b - \beta$	$4a \pm 3b + \beta,$
$-17a \mp 12b - \frac{9}{2}\beta,$	$24a \pm 17b + 6\beta,$
$-99a \mp 70b - 25\beta,$	$140a \pm 99b + 35\beta,$
$-577a \mp 408b - \frac{289}{2}\beta,$	$816a \pm 577b + 204\beta,$
$-3363a \mp 2378b - 841\beta$	$4756a \pm 3363b + 1189\beta$
etc.	etc.

Etiamsi ergo β non fuerit numerus par, tamen in utroque ordine semissis valorum ipsius x fuerit numeri integri.

EXEMPLUM 2

23. *Proposita formula*

$$\sqrt{(3xx+\beta x+\gamma)} = y$$

invenire infinitos valores integros ipsius x, quibus haec formula rationalis evadit, siquidem unus casus constet.

Praebeat casus cognitus $x=a$ et $y=b$, tum vero ob $\alpha=3$ capiatur $p=\sqrt{(3qq+1)}$ eritque $q=1$ et $p=2$. Hinc pro secundo casu habebimus

$$a^{\mathrm{I}} = 2a \pm b + \frac{\beta}{6}; \qquad b^{\mathrm{I}} = 3a \pm 2b + \frac{\beta}{2},$$

ex quibus formentur binae series recurrentes secundum has scalas relationis
unde obtinentur:

$$R = 4Q - P + \frac{\beta}{3}, \quad V = 4T - S,$$

Valores ipsius x	Valores ipsius y
$a,$	$\pm b,$
$2a \pm b + \frac{\beta}{6},$	$3a \pm 2b + \frac{1}{2}\beta,$
$7a \pm 4b + \beta,$	$12a \pm 7b + 2\beta,$

$26a\pm15b+\frac{25}{6}\beta,$	$45a\pm26b+\frac{15}{2}\beta,$
$97a\pm56b+16\beta,$	$168a\pm97b+28\beta,$
$362a\pm209b+\frac{361}{6}\beta,$	$627a\pm362b+\frac{209}{2}\beta,$
$1351a\pm780b+225\beta$	$2340a\pm1351b+390\beta$
etc.	etc.

Praeterea vero scribendo $-x-\frac{\beta}{3}$ pro x prodibunt:

Valores ipsius x	Valores ipsius y
$-a-\frac{1}{3}\beta,$	$\pm b,$
$-2a\mp b-\frac{1}{2}\beta$	$3a\pm2b+\frac{1}{2}\beta,$
$-7a\mp4b-\frac{4}{3}\beta$	$12a\pm7b+2\beta,$
$-26a\mp15b-\frac{9}{2}\beta,$	$45a\pm26b+\frac{15}{2}\beta,$
$-97a\mp56b-\frac{49}{3}\beta,$	$168a\pm97b+28\beta,$
$-362a\mp209b-\frac{121}{2}\beta,$	$627a\pm362b+\frac{209}{2}\beta,$
$-1351a\mp780b-\frac{676}{3}\beta$	$2340a\pm1351b+390\beta$
etc.	etc.

Prout ergo numerus β divisibilis fuerit per 2 vel 3 vel utrumque, hinc eo plures solutiones in integris eliciuntur.

EXEMPLUM 3

24. *Proposita formula*

$$(5xx+\beta x+\gamma)=y$$

invenire infinitos valores integros ipsius x, quibus haec formula rationalis evadat, siquidem unus casus fuerit cognitus.

Pro casu cognito sit $x=a$ et $y=b$ et ob $\alpha=5$ quaerantur numeri p et q, ut sit $p=\sqrt{(5qq+1)}$. Fiet ergo q = 4 et p = 9 et hinc secunda solutio prodibit

$$a^{\mathrm{I}}=9a\pm4b+\frac{4}{5}\beta,\qquad b^{\mathrm{I}}=20a\pm9b+2\beta.$$

Cum ergo sit

$$a^{\mathrm{II}}=18a^{\mathrm{I}}-a+\frac{8}{5}\beta,\qquad b^{\mathrm{II}}=18b^{\mathrm{I}}-b,$$

sequentes solutiones habebuntur:

Valores ipsius x	Valores ipsius y
$a,$	$\pm b,$
$9a\pm4b+\frac{4}{5}\beta,$	$20a\pm9b+2\beta,$
$161a\pm72b+16\beta,$	$360a\pm161b+36\beta,$

$$2889a \pm 1292b + \tfrac{1444}{5}\beta, \qquad 6460a \pm 2889b + 646\beta,$$
etc. etc.

ubi pro quolibet valore ipsius x etiam poni potest $-x-\frac{\beta}{5}$.

SCHOLION 1

25. Cum hoc modo ex una solutione in integris cognita infinitae aliae solutiones etiam in integris eliciantur, quaestio nascitur, an hoc modo omnes plane solutiones integrae obtineantur necne. Ac in exemplis quidem primo et secundo nullum erit dubium, quin hac methodo omnes solutiones integrae obtineantur. Verum in exemplo tertio utique dantur casus, quibus multo plures solutiones in integris exhiberi possunt, quam quidem hac methodo reperiuntur. Veluti si proposita fuerit formula $\sqrt{(5xx+4)}=y$, quae pro casu cognito praebet $a=0$ et $b=2$, nostra solutio dat:

Valores ipsius x	Valores ipsius y
0,	2,
8,	18,
144,	322,
2584	5778
etc.	etc.

Verum hanc formulam diligentius scrutanti patebit non solum his casibus $\sqrt{(5xx+4)}$ fieri rationalem, sed etiam istis numeris pro x substituendis

$$x = 0,\ 1,\ 3,\ 8,\ 21,\ 55,\ 144,\ 377,\ 987 \text{ etc.},$$

unde solutionum numerus triplicatur. Cuius rei ratio est, quod ad formulam $p=\sqrt{(5qq+4)}$ resolvendam posuimus $q=4$, unde fit $p=9$; quae quidem est simplicissima solutio in numeris integris. At quoniam in scala relationis inest 2*p*, ea numeris integris constabit, etiamsi *p* sit fractio denominatorem habens 2. Hanc ob rem istas simpliciores solutiones nanciscemur, si ponamus $q=\frac{1}{2}$, unde fit $p=\frac{3}{2}$ sicque ob $\alpha=5$ secundi valores erunt

$$a^{\mathrm{I}} = \tfrac{3}{2}a \pm \tfrac{1}{2}b + \tfrac{1}{20}\beta, \quad b^{\mathrm{I}} = \tfrac{5}{2}a \pm \tfrac{3}{2}b + \tfrac{1}{4}\beta$$

ac tertii cum sequentibus per hanc legem suppeditabuntur

$$a^{\mathrm{II}} = 3a^{\mathrm{I}} - a + \tfrac{1}{10}\beta, \quad b^{\mathrm{II}} = 3b^{\mathrm{I}} - b,$$

unde nanciscimur hos valores:

Valores ipsius x	Valores ipsius y
$a,$	$\pm b,$
$\frac{3}{2}a\pm\frac{1}{2}b+\frac{1}{20}\beta,$	$\frac{5}{2}a\pm\frac{3}{2}b+\frac{1}{4}\beta,$
$\frac{7}{2}a\pm\frac{3}{2}b+\frac{1}{4}\beta,$	$\frac{15}{2}a\pm\frac{7}{2}b+\frac{3}{4}\beta,$
$9a\pm 4b+\frac{4}{5}\beta,$	$20a\pm 9b+2\beta,$
$\frac{47}{2}a\pm\frac{21}{2}b+\frac{9}{4}\beta,$	$\frac{105}{2}a\pm\frac{47}{2}b+\frac{21}{4}\beta,$
$\frac{123}{2}a\pm\frac{55}{2}b+\frac{121}{20}\beta,$	$\frac{275}{2}a\pm\frac{123}{2}b+\frac{55}{4}\beta,$
$161a\pm 72b+16\beta$	$360a\pm 161b+36\beta$
etc.	etc.

Atque hinc illae triplo plures solutiones oriuntur, quoties fuerit $a\pm b$ numerus par ac β vel $=0$ vel per 20 divisibile.

SCHOLION 2

26. Quandoque ergo plures solutiones in numeris integris reperiuntur, si pro p et q fractiones cum denominatore 2 assumuntur; quod quando in genere eveniat, operae pretium erit investigasse. Plerumque autem hi casus locum non habent, nisi sit vel $\beta=0$ vel formula ad talem formam reduci possit. Sit ergo proposita formula $\sqrt{(\alpha xx+\gamma)}=y$, cui satisfaciat casus $x=a$ et $y=b$; tum statuatur $p=\frac{m}{2}$ et $q=\frac{n}{2}$ seu quaerantur numeri m et n, ut sit $mm=\alpha nn+4$ et $m=\sqrt{(\alpha nn+4)}$. Tum vero solutio prima statim dat secundam

$$a^{\mathrm{I}}=\frac{ma+nb}{2}\quad\text{et}\quad b^{\mathrm{I}}=\frac{\alpha na+mb}{2},$$

ubi quidem numeri m et n tam negative quam affirmative accipi possunt. Denique his binis primis inventis sequentes per hanc regulam reperientur

$$a^{\mathrm{II}}=ma^{\mathrm{I}}-a\quad\text{et}\quad b^{\mathrm{II}}=mb^{\mathrm{I}}-b.$$

In genere autem quilibet numerus pro x satisfaciens continetur hac formula

$$x=\frac{1}{2\sqrt{\alpha}}\left(a\sqrt{\alpha}+b\right)\left(\frac{m+n\sqrt{\alpha}}{2}\right)^{\mu}+\frac{1}{2\sqrt{\alpha}}\left(a\sqrt{\alpha}-b\right)\left(\frac{m-n\sqrt{\alpha}}{2}\right)^{\mu},$$

ex qua fit

$$y=\frac{1}{2}\left(a\sqrt{\alpha}+b\right)\left(\frac{m+n\sqrt{\alpha}}{2}\right)^{\mu}-\frac{1}{2}\left(a\sqrt{\alpha}-b\right)\left(\frac{m-n\sqrt{\alpha}}{2}\right)^{\mu}.$$

Quoties igitur $ma+nb$ prodierit numerus par neque tamen m et n sint pares, toties triplo plures solutiones in integris prodeunt quam methodo praecedente. Hae vero solutiones ita se habebunt:

$$a = a, \qquad b = b,$$
$$a^{\mathrm{I}} = \frac{ma+nb}{2}, \qquad b^{\mathrm{I}} = \frac{mb+\alpha na}{2},$$
$$a^{\mathrm{II}} = \frac{(mm-2)a+mnb}{2}, \qquad b^{\mathrm{II}} = \frac{(mm-2)b+\alpha mna}{2},$$
$$a^{\mathrm{III}} = \frac{(m^3-3m)a+(mm-1)nb}{2}, \qquad b^{\mathrm{III}} = \frac{(m^3-3m)b+\alpha(mm-1)na}{2},$$
$$a^{\mathrm{IV}} = \frac{(m^4-4mm+2)a+(m^3-2m)nb}{2}, \qquad b^{\mathrm{IV}} = \frac{(m^4-4mm+2)b+\alpha(m^3-2m)na}{2},$$
$$a^{\mathrm{V}} = \frac{(m^5-5m^3+5m)a+(m^4-3mm)nb}{2}, \qquad b^{\mathrm{V}} = \frac{(m^5-5m^3+5m)b+\alpha(m^4-3mm+1)na}{2},$$

etc.

OBSERVATIO 1

27. Haec altera methodus tum demum plures solutiones in numeris integris suppeditat quam prior, cum m et n fuerint numeri impares simulque a et b ambo vel pares vel impares. Si enim m et n sint numeri pares, p et q erunt integri et formula $m = \sqrt{(\alpha nn+4)}$ easdem solutiones praebebit ac formula $p = \sqrt{(aqq+1)}$. Deinde si $ma \pm nb$ non fuerit numerus par, valores $a^{\mathrm{I}}, a^{\mathrm{II}}$ non evadent integri neque propterea plures solutiones reperiuntur quam priore methodo, dum adhibetur formula $p = \sqrt{(aqq+1)}$. Distingui ergo oportet eos casus, quibus formulae $m = \sqrt{(\alpha nn+4)}$ numeris imparibus pro m et n accipiendis satisfieri potest, id quod statim patet fieri non posse, si α fuerit numerus formae $4z-1$ vel etiam huius $8z+1$. Quare pro α alii numeri impares non relinquuntur, nisi qui sint formae $8z+5$. Pro his ergo casibus minimos valores formulae $m = \sqrt{(\alpha nn+4)}$ satisfacientes sequens tabella exhibet:

Si fuerit	capiatur	eritque
$\alpha = 5,$	$n = 1$	$m = 3,$
$\alpha = 13,$	$n = 3$	$m = 11,$
$\alpha = 21,$	$n = 1$	$m = 5,$
$\alpha = 29,$	$n = 5$	$m = 27,$
$\alpha = 37,$	$n = -$	$n = -$
$\alpha = 45,$	$n = 1$	$m = 7,$
$\alpha = 53,$	$n = 7$	$m = 51,$
$\alpha = 61,$	$n = 195$	$m = 1523,$
$\alpha = 69,$	$n = 75$	$m = 623,$
$\alpha = 77,$	$n = 1$	$m = 9,$
$\alpha = 85,$	$n = 9$	$m = 83,$
$\alpha = 93,$	$n = 87$	$m = 839.$

Quaeritur hic ratio, cur casus $\alpha = 37$ non recipiat valores impares pro *m* et *n*. Hic igitur patet, si sit $\alpha = 37$, non dari numeros impares pro *m* et *n;* pro reliquis autem casibus resolutio succedit.

Ita si proponatur haec formula

$$\sqrt{(53xx+28)} = y,$$

habetur statim $a = 1$ et $b = 9$. Deinde ob $n = 7$ et $m = 51$ erit

$$a^{\mathrm{I}} = \tfrac{51+63}{2} = 57 \text{ et } b^{\mathrm{I}} = \tfrac{371+459}{2} = 415,$$

seu etiam

$$a^{\mathrm{I}} = -6 \text{ et } b^{\mathrm{I}} = -44;$$

et series recurrentes pro *x* et *y*, quarum scala relationis est 51, -1, erunt :

$$\begin{array}{llllll} x = \text{etc.} & -307, & -6, & 1, & 57, & 2906 \ \text{etc.}, \\ y = \text{etc.} & 2235, & 44, & 9, & 415, & 21156 \ \text{etc.} \end{array}$$

OBSERVATIO 2

28. Sufficit autem casus evolvisse, quibus in formula generali $\alpha xx + \beta x + \gamma$ secundus terminus deest, quoniam haec ad talem formam salva numerorum integritate revocari potest. Vulgaris quidem modus, quo ex aequationibus secundus terminus tolli solet ponendo $x = y - \frac{\beta}{2\alpha}$, hic locum habere nequit, nisi β sit numerus per 2α divisibilis. Verum si $\alpha xx + \beta x + \gamma$ debeat esse quadratum, ponatur

$$\alpha xx + \beta x + \gamma = yy$$

ac multiplicando per 4α prodibit

$$4\alpha\alpha xx + 4\alpha\beta x + 4\alpha\gamma = 4\alpha yy$$

ideoque

$$4\alpha yy + \beta\beta - 4\alpha\gamma = \left(2\alpha x + \beta\right)^2.$$

Quaerantur ergo casus, quibus formula $4\alpha yy + \beta\beta - 4\alpha\gamma$ fit quadratum, indeque habebuntur valores pro *x* substituendi, qui formulam $\alpha xx + \beta x + \gamma$ reddant quadratam; scilicet si fuerit

$$\sqrt{\left(4\alpha yy + \beta\beta - 4\alpha\gamma\right)} = z,$$

erit $2\alpha x + \beta = z$ hincque

$$x = \tfrac{z-\beta}{2\alpha}.$$

Quodsi β fuerit numerus par, puta 2δ, posito $\alpha xx+2\delta x+\gamma=yy$ erit

$$\left(\alpha x+\delta\right)^2=\alpha yy+\delta\delta-\alpha\gamma$$

sicque formula $\alpha yy+\delta\delta-\alpha\gamma$ ad quadratum est revocanda; ac si invenimus

$$\sqrt{\left(\alpha yy+\delta\delta-\alpha\gamma\right)}=z,$$

erit $\alpha x+\delta=z$ et

$$x=\tfrac{z-\delta}{\alpha},$$

unde plerumque pro x numeri integri reperiuntur; etsi enim forte $\tfrac{z-\delta}{\alpha}$ non fuerit integer, tamen ex uno valore z cognito, si modo supra tradito alii eliciantur in infinitum, alterni saltem erunt numeri integri. Ex quo perspicuum est resolutionem formularum quadraticarum radicalium $\sqrt{\left(\alpha xx+\beta x+\gamma\right)}$ nulla limitatione affici, etiamsi terminus βx plane omittatur, sicque totum negotium huc redit, ut formulae huiusmodi $\sqrt{\left(\alpha xx+\gamma\right)}$ rationales et quidem in numeris integris reddantur.

OBSERVATIO 3

29. Iam annotavi formulam $\alpha xx+\gamma$ in numeris integris saltem pluribus ac infinitis modis quadratum effici non posse, nisi α sit numerus positivus non quadratus. Existente autem α tali numero problema non ita resolvi potest, ut pro quocunque numero pro γ assumto solutio succedat; possent enim utique eiusmodi numeri pro γ dari, ut problema nullam plane solutionem admitteret, atque hanc ob rem postulavi unam saltem solutionem cognitam esse debere, quo ipso casus insolubiles exclusi.

Verum dato α characteres exhiberi possunt, ex quibus dignosci liceat, utrum numerus γ sit eiusmodi, qui solutionem admittat necne. Ac primo quidem perspicuum est nullam solutionem locum habere posse, nisi γ sit numerus in tali formula $bb-\alpha aa$ contentus. Dato ergo numero α formetur series omnium numerorum tam positivorum quam negativorum, qui quidem in formula $bb-\alpha aa$ sint contenti; ac nisi γ in hac serie reperiatur, certo pronunciare licet formulam $\sqrt{\left(\alpha xx+\gamma\right)}$ nullo modo rationalem reddi posse; vicissim autem quoties γ in hac serie comprehenditur, quia tum est $\gamma=bb-\alpha aa$, formula $\alpha aa+\gamma$ fit quadratum ponendo $x=a$ eritque $\sqrt{\left(\alpha xx+\gamma\right)}=b.$

Haec igitur series, cuius quasi terminus generalis est $bb-\alpha aa$, primo continebit sumto $a=0$ omnes numeros quadratos

$$1,\ 4,\ 9,\ 16,\ 25 \text{ etc.},$$

tum vero omnes quadratos per $-\alpha$ multiplicatos, nempe

$$-\alpha,\ -4\alpha,\ -9\alpha,\ \ -16\alpha \text{ etc.}$$

Praeterea si *p* et *q* fuerint numeri in hac serie contenti, in ea quoque reperietur eorum productum *pq;* nam cum sit

$$p = bb - \alpha aa \text{ et } q = dd - \alpha cc,$$

erit

$$pq = \left(bd \pm \alpha ac\right)^2 - \alpha\left(bc \pm ad\right)^2$$

et ob ambiguitatem signi hoc productum duplici modo est numerus formae $bb - \alpha aa$ ideoque statim habentur duae solutiones

$$x = bc + ad \text{ et } x = bc - ad.$$

OBSERVATIO 4

30. Hinc ergo consecuti sumus hoc Theorema eximium, quod fundamentum superiorum solutionum in se complectitur:

Si fuerit

$$\alpha xx + p = yy$$

casu $x = a$ et $y = b$ *, tum vero etiam*

$$\alpha xx + q = yy$$

casu $x = c$ et $y = d$ *, haec formula*

$$\alpha xx + pq = yy$$

adimplebitur capiendo

$$x = bc \pm ad \text{ et } y = bd \pm \alpha ac.$$

Si enim sit $q = 1$ et $dd = \alpha cc + 1$, praeterea vero formulae $\alpha xx + p = yy$ satisfiat casu $x = a$ et $y = b$, qui est casus supra pro cognito assumtus, tum eidem formulae satisfacient valores

$$x = bc \pm ad \text{ et } y = bd \pm \alpha ac,$$

unde eadem omnino solutio conficitur, quam supra exhibuimus atque ex longe diversis principiis elicuimus; quocirca haec postrema investigationis ratio ob concinnitatem et perspicuitatem eo magis est notatu digna. Hic vero accedit, quod haec ratio multo latius pateat quam praecedens, quippe quae ad casum $q = 1$ fuerat adstricta. Demonstratio autem istius Theorematis elegantissimi ita brevissime se habebit.
Cum sit

$$\alpha aa + p = bb,$$

erit

$$p = bb - \alpha aa$$

et ob

$$\alpha cc + q = dd$$

erit

$$q = dd - \alpha cc;$$

hinc erit $pq = (bb - \alpha aa)(dd - \alpha cc),$ quae expressio reducitur ad hanc

$$pq = (bd + \alpha ac)^2 - \alpha(bc \pm ad)^2.$$

Quodsi ergo fuerit

$$x = bc \pm ad \quad \text{et} \quad y = bd \pm \alpha ac,$$

erit $pq = yy - \alpha xx$ ideoque

$$\alpha xx + pq = yy.$$

Q. E. D.

OBSERVATIO 5

31. Cum igitur pro quolibet numero α formulae $\alpha xx + \gamma = yy$ numerus γ debeat esse formae $bb - \alpha aa$, numeri in hac forma contenti diligentius examinari merentur; et quoniam, si inter eos occurrunt numeri *p* et *q*, simul quoque eorum productum *pq* occurrit, praeter numeros quadratos 1, 4, 9, 16, 25 etc. eorumque multipla negativa $-\alpha,\ -4\alpha,\ -9\alpha,\ -16\alpha,\ -25\alpha$ etc. imprimis numeri primi in hac forma contenti sunt spectandi, quippe ex quibus deinceps per multiplicationem compositi nascuntur.

I. Sit $\alpha = 2$ et numeri primi formae $bb - 2aa$ sunt

positivi
$+1,\ +2,\ +7,\ +17,\ +23,\ +31,\ +41,\ +47,\ +71,\ +73,\ +79,\ +89,\ +97$ etc.,

negativi
$-1,\ -2,\ -7,\ -17,\ -23,\ -31,\ -41,\ -47,\ -71,\ -73,\ -79,\ -89,\ -97$ etc.,
qui praeter $+2$ et -2 omnes in forma $\pm(8n \pm 1)$ continentur.

II. Sit $\alpha = 3$ et numeri primi formae $bb - 3aa$ sunt

positivi $+1,\ +13,\ +37,\ +61,\ +73,\ +97,\ +109$ etc.,
negativi $-2,\ -3,\ -11,\ -23,\ -47,\ -59,\ -71,\ -83,\ -107$ etc.,

qui praeter -2 et -3 omnes continentur in forma $12n+1$, siquidem pro *n* tam numeri positivi quam negativi capiantur.

III. Sit $\alpha = 5$ et numeri primi formae $bb - 5aa$ sunt
positivi $+1,\ +5,\ +11,\ +19,\ +29,\ +31,\ +41,\ +59,$
$+61,\ +71,\ +79,\ +89, +101$ etc.,
negativi $-1,\ -5,\ -11,\ -19,\ -29,\ -31,\ -41,\ -59,$
$-61,\ -71,\ -79,\ -89,\ -101$ etc.,
qui praeter $+5$ et -5 omnes in forma $10n+1$ continentur.

IV. Sit $\alpha = 6$ et numeri primi formae $bb - 6aa$ sunt

positivi $+1,\ +3,\ +19,\ +43,\ +67,\ +73,\ +97$ etc.,
negativi $-2,\ -23,\ -29,\ -47,\ -53,\ -71,\ -101$ etc.,

qui praeter -2 et $+3$ omnes in alterutra harum formarum $24n+1$ et $24n-5$ continentur sumendo pro *n* numeros tam negativos quam positivos.

V. Sit $\alpha = 7$ et numeri primi formae $bb - 7aa$ sunt

positivi $+1,\ +2,\ +29,\ +37,\ +53,\ +109$ etc.,
negativi $-7,\ -3,\ -19,\ -31,\ -47,\ -59,\ -83$ etc.,

qui praeter $+2$ et -7 omnes in una harum formarum continentur
$28n+1,\ 28n+9,\ 28n+25.$

OBSERVATIO 6

32. Hinc colligimus omnes numeros primos in formula $bb - \alpha aa$ contentos simul in quibusdam huiusmodi formulis $4\alpha n + A$ contineri, dum pro *A* certi quidam numeri substituuntur. Quod idem etiam hoc modo ostendi potest.
Ponatur
$$b = 2\alpha p + r \quad \text{et} \quad a = 2q + s$$
ac formula $bb - \alpha aa$ transit in hanc

$$4\alpha app + 4\alpha pr + rr - 4\alpha qq - 4\alpha qs - \alpha ss;\,;$$

statuatur $\alpha pp + pr - qq - qs = n$ et habebimus

$$bb - \alpha aa = 4\alpha n + rr - \alpha ss.$$

Omnes ergo numeri primi formae $bb - \alpha aa$ quoque in hac forma $4\alpha n + rr - \alpha ss$ continentur; atque ut hi numeri sint primi, *r* et *s* ita accipi oportet, ut numerus $rr - \alpha ss$ sit vel ipse primus vel saltem ad 4α primus. Primo ergo sumto $s = 0$ pro *r* successive accipi possunt numeri impares ad α primi, ac si *rr* fuerit maius quam 4α, inde 4α toties subtrahatur, quoties fieri potest, ut residuum sit minus quam 4α, et quot hoc modo diversi numeri resultant, ii in formula $4\alpha n + A$ loco *A* collocentur. Deinde etiam simili modo colligantur numeri ex formulis $rr - \alpha$, qui, quatenus sunt diversi, ad illos insuper adiiciantur. Non autem opus est pro *s* alios numeros praeter unitatem assumere; si enim *s* esset numerus par, numerus $-\alpha ss$ iam in forma $4\alpha n$ contineretur, et si *s* esset impar, numerus $-\alpha ss$ haberet formam $-4\alpha N - \alpha$, cuius pars $-4aN$ iam in $4\alpha n$ continetur, sicque sufficit pro formulis $4\alpha n + A$ quovis casu has $4\alpha n + rr$ et $4\alpha n + rr - \alpha$ evolvere eaeque iam omnes numeros primos, qui quidem in formula $bb - \alpha aa$ comprehenduntur, in se complectentur. Num autem vicissim omnes numeri primi in his formulis $4\alpha n+rr$ et $4\alpha n+rr-\alpha$ contenti simul sint numeri formae $bb-\alpha aa$, quaestio est altioris indaginis, quae tamen affirmanda videtur.

OBSERVATIO 7

33. Quo haec exemplo illustremus, sit $\alpha=13$ et ex $4\alpha n+rr$ et $4\alpha n+rr-\alpha$ orientur hae formulae pro numeris primis:

ex $4\alpha n+rr$	ex $4\alpha n+rr-\alpha$
$52n+1,$	$52n-9,$
$52n+9,$	$52n+3,$
$52n+25,$	$52n+23,$
$52n+49=52n-3,$	$52n+51=52n-1,$
$52n+81=52n-23,$	$52n+87=52n-17,$
$52n+121=52n+17,$	$52n+131=52n-25,$

quae formulae reducuntur ad has

$$52n\pm1,\ 52n\pm3,\ 52n\pm9,\ 52n\pm17,\ 52n\pm23,\ 52n\pm25;$$

ac numeri primi in his contenti sunt

$$1,\ \pm3,\ \pm17,\ \pm23,\ \pm29,\ \pm43,\ \pm53,\ \pm61,\ \pm79,\ \pm101,\ \pm103,$$

quibus addi debet ±13, tum vero omnes numeri quadrati; atque si insuper adiiciantur producta ex binis pluribusque horum numerorum, obtinebuntur hoc quidem casu omnes numeri, qui pro γ substituti producunt formulam $13xx+\gamma=yy$ in numeris integris resolubilem; seu quicunque illorum numerorum pro γ accipiatur, unus primo, deinde infiniti numeri integri pro x inveniri possunt, quibus formula $13xx+\gamma$ quadratum reddatur. Omnes enim isti numeri simul in forma $bb-13aa$ continentur; qui enim huc difficiliores reductu videntur, sunt:

$-1=18^2-13\cdot5^2,$	$+13=65^2-13\cdot18^2,$	$-3=7-13\cdot2^2,$
$+17=15^2-13\cdot4^2,$	$17=10^2-13\cdot3^2,$	$-23=43^2-13\cdot12^2,$
$+29=9^2-13\cdot2^2,$	$-29=32^2-13.\,9^2,$	$+43=76^2-13\cdot21^2,$
$-43=3^2-13\cdot2^2,$	$+53=51^2-13\cdot14^2,$	$-53=8^2-13\cdot3^2,$
$+61=23^2-13\cdot6^2,$	$-61=24^2-13\cdot7^2,$	$+79=14^2-13\cdot3^2,$

$$-79 = 57^2 - 13\cdot 16^2 \qquad \text{etc.}$$

Cum ergo sit $-1 = 18^2 - 13\cdot 5^2$, si fuerit $+\gamma = bb - 13aa$, erit

$$-\gamma = (18b \pm 65a)^2 - 13(18a \pm 5b)^2,$$

unde casus difficiliores resolvuntur.

Proposita ergo resolvenda hac aequatione

$$13xx + 43\cdot 79 = yy$$

cum sit $\gamma = 43\cdot 79 \;=-\; 43\cdot -79$, habebitur per compositionem

ergo I. $\gamma = (14\cdot 76 \pm 13\cdot 63)^2 - 13\cdot(14\cdot 21 \pm 3\cdot 76)^2$,
$x = 294 \pm 228$ et $y = 1064 \pm 819$,

ergo II. $\gamma = (3\cdot 57 \pm 13\cdot 32)^2 - 13\ (2\cdot 57 \pm 3\cdot 16)^2$,
$x = 114 \pm 48$ et $y = 416 \pm 171$,

unde statim tres solutiones obtinentur.

OBSERVATIO 8

34. Verum non semper ex his numeris primis, quos modo investigare docuimus, cum quadratis omnes plane numeri, qui pro γ assumi possunt, reperiuntur, cuius rei exemplum est casus $\alpha = 10$, pro quo valores ipsius γ in hac forma $bb - 10aa$ continentur; iique sunt tam negative quam positive sumti

1, 4, 6, 9, 10, 15, 16, 24, 25, 26, 31, 36, 39, 40, 41, 49, 54, 60, 64,65, 71, 74, 79, 81, 86, 89, 90, 96, 100, 104, 106, 111, 121, 124, 129, 134, 135, 144, 150, 151, 156, 159, 160, 164, 166, 169, 185, 186, 191, 196, 199, 201 etc.,

inter quos numeros occurrunt primo omnes quadrati

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196 etc.,

deinde numeri primi

31, 41, 71, 79, 89, 151, 191, 199 etc.,

qui in his formulis continentur $40n \pm 1$ et $40n \pm 9$, insuperque accedunt producta ex binis pluribusve horum numerorum. Tertio vero praeter hos adsunt numeri ex binis numeris primis compositi, qui sunt

$$2\cdot 3,\ 2\cdot 5,\ 2\cdot 13,\ 2\cdot 37,\ 2\cdot 43,\ 2\cdot 53,\ 2\cdot 67,\ 2\cdot 83\ \text{etc.},$$
$$3\cdot 5,\ 3\cdot 13,\ 3\cdot 37,\ 3\cdot 43,\ 3\cdot 53,\ 3\cdot 67\ \text{etc.},$$
$$5\cdot 13,\ 5\cdot 37\ \text{etc.}$$

At hi numeri primi, quorum semper bini sunt in se multiplicandi, sunt primo 2 et 5, reliqui vero in his formulis continentur $40n\pm 3$ et $40n\pm 13.$ Denique etiam secundum regulam generalem adiici debent producta ex binis pluribusve numeris, qui per se satisfaciunt.

Ita resolvi poterit haec aequatio

$$10xx+13\cdot 53\cdot 151=yy,$$

nam est $13\cdot 53=bb-10aa$ existente $b=27$ et $a=2$ et $151=dd-10cc$ existente $d=31$ et $c=9$ hincque

$$13\cdot 53\cdot 151=\left(bd\pm 10ac\right)^{2}-10\left(ad\pm bc\right)^{2}$$

et

$$x=ad\pm\ bc\ \text{ et }\ y=bd\pm 10ac.$$

Deinde cum etiam sit $-13\cdot 53=\mathrm{BB}-10AA$ et $-151=DD-10CC,$ hinc duae aliae solutiones reperiuntur. Cum autem sit $-1=3^{2}-10\cdot 1^{2},$ si fuerit $\gamma=bb-10aa,$ erit $\gamma=\left(3b\pm 10a\right)^{2}-10(3a+b)^{2}.$ Solutiones autem hinc oriundae sunt

$$x=181,\qquad y=657,$$
$$x=305,\qquad y=1017,$$
$$x=307,\qquad y=1023;$$

duae enim inter se conveniunt, ita ut hinc tres tantum reperiantur

OBSERVATIO 9

35. Hoc ergo casu $\alpha=10$ pro γ triplicis generis numeros primitivos invenimus, primo scilicet numeros quadratos omnes, deinde certos numeros primos in formulis $40n\pm 1$ et $40n\pm 9$ contentos, tertio autem producta ex binis certis numeris primis, qui sunt 2, 5 et reliqui ex his formulis $40n\pm 3$ et $40n\pm 13$ petendi, atque ex hoc demum triplici ordine omnes numeri pro γ idonei formantur, ut huic aequationi $10xx\pm\gamma=yy$ satisfieri possit. Ipsi autem numeri primi in formulis $40n\pm 3$ et $40n\pm 13$ contenti non conveniunt, quia non sunt formae $bb-10aa$, sed tamen hi numeri omnes sunt formae $2bb-5aa$, uti etiam duo iis iungendi 2 et 5. Manifestum autem est, si habeantur duo numeri huiusmodi $2bb-5aa$ et $2dd-5cc$, eorum productum fore $=\left(2bd\pm 5ac\right)^{2}-10(bc\pm ad)^{2}$ ideoque pro γ adhiberi posse. Huiusmodi igitur producta binorum numerorum primorum, qui ipsi non satisfaciunt, occurrere nequeunt, si α fuerit numerus primus, sed tantum, uti hic usu venit, si α fuerit numerus compositus; quod

tamen etiam non semper locum habet, uti vidimus casu $\alpha = 6 = 2\cdot 3$, quo numeri formae $3bb - 2aa$ conveniunt cum numeris formae $bb - 6aa.$ Quodsi ergo in genere fuerit $\alpha = pq$ et aequatio $pqxx + \gamma = yy$ resolvi debeat, numerus γ vel esse debet numerus quadratus vel primus formae $bb - pqaa$ vel productum ex duobus numeris primis formae $pbb - qaa$, propterea quod huiusmodi productum est

$$(pbb - qaa)(pdd - qcc) = (pbd \pm qac)^2 - pq(bc \pm ad)^2 .$$

Nisi ergo tales numeri primi iam ipsi $pbb - qaa$ in forma $bb - pqaa$ contineantur, tertius ille ordo numerorum ex binis numeris primis conflatorum accedit. Quemadmodum deinde numeri primi solitarii continentur in formulis $4pqn + rr$ et $4pqn + rr - pq$, ita numeri primi alteri combinandi ex formula hac

$$4pqn + prr - qss$$

derivari debent.

$$x = 503, \quad y = 1623,$$
$$x = 7381, \quad y = 23343,$$
$$x = 11897, \quad y = 37623;$$

utraque autem solutio $x = 181,\ y = 657$ et $x = 305,\ y = 1017$ adeo ter invenitur.

EXEMPLUM 1

36. *Investigentur omnes valores idonei ipsius γ, ut haec aequatio*

$$30xx + \gamma = yy$$

resolutionem admittat.

Primo quidem pro γ assumi possunt omnes numeri quadrati, deinde omnes numeri primi in his formis $120n + rr$ et $120n + rr - 30$ contenti, quae reducuntur ad has

$$120n + 1,\ 120n + 49,\ 120n + 19,\ 120n - 29 \text{ cum } -5,$$

unde oriuntur hi numeri primi infra 200

$$\text{positivi } +19,\ +139 \text{ et negativi } -5,\ -29,\ -71,\ -101,\ -149,\ -191.$$

Tertio ob $\alpha = 2\cdot 3\cdot 5$ sumi possunt producta binorum primorum, qui contineantur vel ambo in una harum formularum

$$\text{I. } 120n + 2rr - 15ss,\ \text{II. } 120n + 3rr - 10ss,\ \text{III. } 120n + 5rr - 6ss;$$

harum autem binae priores eosdem numeros primos dant, qui sunt $+2$, $+3$, et reliqui in his formulis continentur

$$120n-7,\ 120n-13,\ 120n+17,\ 120n-37,$$

unde nascuntur hi numeri primi infra 200

$$\begin{aligned}&\text{positivi}\quad +2,\ +3,\ \ +17,\ +83,\ \ +107,\ +113,\ +137,\\&\text{negativi}\quad -7,\ -13,\ -37,\ -103,\ -127,\end{aligned}$$

quorum binorum producta pro γ capienda sunt

$$\begin{gathered}+6,\ +34,\ +51,\ +91,\ +166,\\-14,\ -21,\ -26,\ -39,\ -74,\ -111,\ -119.\end{gathered}$$

Tertia autem formula continet numerum primum +5 cum his formis

$$120n-1,\ \ 120n-19,\ \ 120n+29,\ \ 120n-49,$$

unde nascuntur hi numeri primi infra 200

$$\begin{aligned}&\text{positivi}\quad +5,\ +29,\ +71,\ +101,\ +149,\ +191,\\&\text{negativi}\quad -1,\ -19,\ -139.\end{aligned}$$

At, ex horum combinatione iidem nascuntur numeri, qui iam ex numeris primis primitivis oriuntur.
Quocirca omnes numeri, qui pro γ substitui possunt, erunt infra 200

$$+1,\ +4,\ +9,\ +16,\ +25,\ +36,\ +49,\ +64,\ +81,\ +100,\ +121,\ +144,\ +169,\ +196;$$

$$\begin{gathered}-5,\ +19,\ -29,\ -71,\ -101,\ +139,\ -149,\ -191,\ +6,\ -14,\\-21,\ -26,\ +34,\ -39,\ +51,\ -74,\ +91,\ -111,\ -119,\ +166;\end{gathered}$$

$$\begin{gathered}-20,\ \ +24,\ -30,\ -45,\ +54,\ -56,\ +70,\ +76,\ -80,\ -84,\ -95,\\+96,\ -104,\ +105,\ +114,\ -116,\ -125,\ -126,\ +130,\ +136,\ 145,\\+150,\ -156,\ -170,\ +171,\ -189,\ +195.\end{gathered}$$

Reliqui autem numeri omnes pro γ assumti reddent problema impossibile.

EXEMPLUM 2

37. *Resolvere in numeris integris aequationem*

$$5xx+11\cdot 19\cdot 29=yy.$$

Quia est $\alpha=5$ et $\gamma=11\cdot 19\cdot 29,$ factores hi cum forma $bb-5aa$ conveniunt et singuli in ea contineri deprehenduntur; nam

$$\begin{aligned}&\text{pro 11 est } b=4,\ \ a=1,\\&\text{pro 19 est } b=8,\ \ a=3,\\&\text{pro 29 est } b=7,\ \ a=2,\end{aligned}$$

unde etiam producta ex binis In eadem forma continentur; pro $11\cdot 19$ est

$$\begin{aligned}&b=17,\ \ a=4,\\&b=47,\ \ a=20;\end{aligned}$$

ergo tertium adiungendo pro $11\cdot 19\cdot 29$ est

$$\begin{aligned}&b=79,\ \ a=6,\ \ b=129,\ \ a=46,\\&b=159,\ \ a=62,\ \ b=529,\ \ a=234.\end{aligned}$$

Cum iam sit $1=9^2-5\cdot 4^2$ seu $b=9$ et $a=4$ pro 1, hae formulae insuper per 1 multiplicatae duplicabuntur fietque pro $11\cdot 19\cdot 29$

$$\begin{aligned}&b=591,\ \ a=262,\ \ b=241,\ \ a=102,\\&b=831,\ \ a=370,\ \ b=2081,\ \ a=930,\\&b=191,\ \ a=78,\ \ b=81,\ \ a=10,\\&b=2671,\ \ a=1194,\ \ b=9441,\ \ a=4222.\end{aligned}$$

Hinc ergo iam duodecim solutiones problematis sumus nacti, quae sunt

I. $x=6$,	$y=79$,	VII. $x=234$,	$y=529$,
II. $x=10$,	$y=81$,	VIII. $x=262$,	$y=591$,
III. $x=46$,	$y=129$,	IX. $x=370$,	$y=831$,
IV. $x=62$,	$y=159$,	X. $x=930$,	$y=2081$,
V. $x=78$,	$y=191$,	XI. $x=1194$,	$y=2671$,
VI. $x=102$,	$y=241$,	XII. $x=4222$,	$y=9441$,

ex quibus porro cum formula $1=9^2-5\cdot 4^2$ coniungendis infinite novae eaeque omnes elicientur; ex secunda scilicet prodit $x=414,\ y=929$ et ex sexta $x=1882,\ y=4209,$ ex quinta $x=1466,\ y=3279,$ ex octava $x=4722,\ y=10559;$ sicque iam sedecim solutiones sumus adepti.

CONCLUSIO

38. His expositis non amplius coacti sumus proposita huiusmodi aequatione $\alpha xx+\gamma=yy$ primum quasi divinando unum casum satisfacientem anquirere, sed numerum γ examinando secundum formulas modo traditas statim pronunciare possumus, utrum aequatio resolutionem admittat necne; ac si admittit, per eadem principia unam saltem solutionem elicere licebit, quod quidem promte fieri poterit, si numerus γ fuerit resolubilis in factores non nimis magnos. Verum si numerus γ sit primus ac praegrandis, iudicium quidem solubilitatis aeque est facile, at inventio unius solutionis maiorem laborem requirit. Veluti si proponatur

$$30xx+1459=yy,$$

quia 1459 est numerus primus formae $120n+19$, aequatio est resolubilis; verum ei satisfieri sumendo $x=39$ et $y=217$ non tam facile investigatur. Investigatio tamen sublevatur, si statuamus $y=30z\pm 7$, unde fit $xx=30zz\pm 14z-47$, et iam citius reperiemus $z=7$ et $x=39$, unde prodit $y=217$. At si ponamus $y=30z\pm 13$, fit $xx=30zz\pm 26z-43$ promtiusque invenitur $x=5$ et $y=47$. Verum in numeris multo maioribus labor evadit insuperabilis methodusque certa adhuc desideratur negotium conficiendi; deinde etiam, quod omnes numeri primi in supra allatis formulis $4\alpha n+A$ contenti simul sint numeri huius formae $bb-\alpha aa$, ad eas propositiones pertinet, quas veras credimus, etiamsi demonstrare non valeamus.