AN AID FOR THE CALCULATION OF SINES

E246

Novi commentarii academiae scientiarum Petropolitanae 5 (1754/65), 1760, p.164-204.

LEMMA

1. *The value of this imaginary formula*

$$\left(cos.\varphi+\sqrt{-1}\cdot sin.\varphi\right)^n$$

is

$$cos.n\varphi+\sqrt{-1}\cdot sin.n\varphi\,,$$

moreover the value of this imaginary formula

$$\left(cos.\varphi-\sqrt{-1}\cdot sin.\varphi\right)^n$$

is

$$cos.n\varphi-\sqrt{-1}\cdot sin.n\varphi.$$

DEMONSTRATION

For if the two angles φ and α may be had, the product of these two formulas will be

$$\left(cos.\varphi+\sqrt{-1}\cdot sin.\varphi\right)\left(cos.\varphi+\sqrt{-1}\cdot sin.\varphi\right)$$
$$=cos.\varphi\, cos.\alpha-sin.\varphi\, sin.\varphi+\left(\sin.\varphi\, cos.\alpha+\cos.\varphi\, sin.\alpha\right)\sqrt{-1}$$

But it is agreed to be

$$\cos.\varphi\cos.\alpha-\sin.\varphi\sin.\alpha=\cos.(\varphi+\alpha)$$

and

$$\sin.\varphi\ \cos.\alpha+\cos.\varphi\ \sin.\alpha=\sin.(\varphi+\alpha),$$

from which the former product will become:

$$\cos.\left(\varphi+\alpha\right)+\sqrt{-1}\cdot\sin.\left(\varphi+\alpha\right).$$

Now there may be put $\alpha=\varphi$ and there will become:

$$\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^2=\cos.2\varphi+\sqrt{-1}\cdot\sin.2\varphi.$$

This formula again multiplied by $\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$ will give

$$\cos.(2\varphi+\alpha)+\sqrt{-1}\cdot\sin.\ (2\varphi+\alpha)$$

and on putting $\alpha=\varphi$:

$$\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^3=\cos.3\varphi+\sqrt{-1}\cdot\sin.3\varphi.$$

Which if it may be multiplied by $\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$ once more and there may be put $\alpha=\varphi$, will give:

$$\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^4=\cos.4\varphi+\sqrt{-1}\cdot\sin.4\varphi.$$

and in this manner it is deduced to become generally:

$$\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^n=\cos.n\varphi+\sqrt{-1}\cdot\sin.n\varphi.$$

But since the expression $\sqrt{-1}$ from its nature shall involve an ambiguity of sign, on that account by the same reasoning there will become :

$$\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)^n=\cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi.\quad \text{Q. E. D.}$$

COROLLARY

2. Therefore, if for the sake of brevity there may be put

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u \text{ and } \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon,$$

[Here and forthwith the Greek letter υ is taken for the italic form of the letter v rather than the Greek letter *v*, which is used to express the superscripts : this confusion arises from the Matlab version used in these translations, which uses *v* both for the italic 'vee' as well as in superscript: thus v, *v*, and ^{v}.]
then there shall become

$$u^n=\cos.n\varphi+\sqrt{-1}\cdot\ \sin.n\varphi \text{ and } \upsilon^n=\cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi,$$

and there will be

$$u^n+\upsilon^n=2\cos.n\varphi \text{ and } u^n-\upsilon^n=2\sqrt{-1}\cdot\sin.n\varphi,$$

Moreover there is agreed to be $u\upsilon=1.$

PROBLEM 1

3. *To convert whatever power of the cosine of some angle into simple cosines, thus so that two or more cosines never occur multiplied by each other in turn.*

SOLUTION

The proposed power shall be $(\cos.\varphi)^n$ or $\cos.\varphi^n$ for the proposed power requiring to be converted according to the prescribed manner (for I have these synonymous designations for these powers). [In this translation, the accepted modern usage $\cos^n.\varphi$ is used.] We may put as before

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u \text{ and } \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon,$$

and there will become

$$\cos.\varphi=\tfrac{1}{2}(u+\upsilon)$$

and thus

$$\cos^n.\varphi=\tfrac{(u+\upsilon)^n}{2^n}$$

or

$$2^n\cos^n.\varphi=(u+\upsilon)^n$$

Which power of the binomial may be expanded out in the usual manner, so that it may produce

$$2^n\cos^n.\varphi=u^n+nu^{n-1}\upsilon+\tfrac{n(n-1)}{1\cdot 2}u^{n-2}\upsilon^2+\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}u^{n-3}\upsilon^3+\text{etc.};$$

and a similar expression arises, if the letters u and υ may be interchanged. Therefore with these two expressions added together there is produced

$$2^{n+1}\cos^n.\varphi=u^n+\upsilon^n+\tfrac{n}{1}\left(u^{n-2}+\upsilon^{n-2}\right)u\upsilon+\tfrac{n(n-1)}{1\cdot 2}\left(u^{n-4}+\upsilon^{n-4}\right)u^{n-2}\upsilon^2+\text{etc.}$$

and on account of $uv=1$, there will be had, on dividing by 2 :

$$2^n\cos^n.\varphi=\tfrac{1}{2}\left(u^n+\upsilon^n\right)+\tfrac{n}{1}\cdot\tfrac{1}{2}\left(u^{n-2}+\upsilon^{n-2}\right)+\tfrac{n(n-1)}{1\cdot 2}\cdot\tfrac{1}{2}\left(u^{n-4}+\upsilon^{n-4}\right)u^{n-2}\upsilon^2$$
$$+\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\tfrac{1}{2}\left(u^{n-6}+\upsilon^{n-6}\right)+\text{etc.}$$

Truly since there shall become $u^n+\upsilon^n=2\cos.n\varphi$, there is seen to become

$$2^n \cos^n .\varphi = \cos.n\varphi + \tfrac{n}{1}\cos.(n-2)\varphi + \tfrac{n(n-1)}{1\cdot 2}\cdot\cos.(n-4)\varphi$$
$$+\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\cos.(n-6)\varphi + \text{etc.}$$

In which series since the cosine of negative numbers may arise, it is required to observe these agree with the cosines of the same accepted positive angles, or to become

$$\cos.(n-m)\varphi = \cos.(m-n)\varphi.$$

Q. E. I.

COROLLARY 1

4. If there shall be $n=1$, there will become:

$$2\cos.\varphi = \cos.\varphi + \cos.\varphi = 2\cos.\varphi;$$

but if $n=2$, there is had:

$$2^2\cos^2.\varphi = \cos.2\varphi + 2\cos.0\varphi + \cos.2\varphi = 2\cos.2\varphi + 2.$$

Let $n=3$ then there will become:

$$2^3\cos^3.\varphi = \cos.3\varphi + 3\cos.\varphi + 3\cos.\varphi + \cos.3\varphi = 2\cos.3\varphi + 6\cos.\varphi.$$

Let $n=4$; then there will become:

$$2^4\cos^4.\varphi = \cos.4\varphi + 4\cos.2\varphi + 6\cos.0\varphi + 4\cos.2\varphi + \cos.4\varphi$$

and thus, since the individual terms apart from the middle term occur twice, on account of $\cos.0\varphi = 1$ there will become :

$$2^4\cos^4.\varphi = 2\cos.4\varphi + 8\cos.2\varphi + 6$$

COROLLARY 2

5. This same always arises in use, as often as n is a positive whole number, so that the same series written from the end may be produced and thus besides the individual terms occur twice. But the middle term present, as often as n is an even number, and the cosine expressed by this term will be changed into one.

COROLLARY 3

6. Therefore so that if the equal terms from the beginning and the end may be joined together and the whole series may be divided by 2 , the same coefficients will be had as before, except for the constant terms, which if present, the coefficient shall be required to be changed into half of its value. From which these transformations, as often as *n* is a whole positibe number, themselves will be had thus:

$$1\cos.\varphi=\cos.\varphi,$$
$$2\cos^2.\varphi=\cos.2\varphi+\tfrac{1}{2}\cdot 2,$$
$$4\cos^3.\varphi=\cos.3\varphi+3\cos.\varphi,$$
$$8\cos^4.\varphi=\cos.4\varphi+4\cos.2\varphi+\tfrac{1}{2}\cdot 6,$$
$$16\cos^5.\varphi=\cos.5\varphi+5\cos.3\varphi+10\cos.\varphi,$$
$$32\cos^6.\varphi=\cos.6\varphi+6\cos.4\varphi+15\cos.2\varphi+\tfrac{1}{2}\cdot 20,$$
$$64\cos^7.\varphi=\cos.7\varphi+7\cos.5\varphi+21\cos.3\varphi+35\cos.\varphi,$$
$$128\cos^8.\varphi=\cos.8\varphi+8\cos.6\varphi+28\cos.4\varphi+56\cos.2\varphi+\tfrac{1}{2}\cdot 70\ \text{ etc.}$$

COROLLARIUM 4

7. If the exponent *n* shall be a negative number, the expression found in series will become infinite and thus will become :

$$\frac{1}{2\cos.\varphi}=\cos.\varphi-\cos.3\varphi+\cos.5\varphi-\cos.7\varphi+\cos.9\varphi-\text{etc.},$$
$$\frac{1}{4\cos^2.\varphi}=\cos.2\varphi-2\cos.4\varphi+3\cos.6\varphi-4\cos.8\varphi+5\cos.10\varphi-6\cos.12\varphi+\text{etc.},$$
$$\frac{1}{8\cos^3.\varphi}=\cos.3\varphi-3\cos.5\varphi+6\cos.7\varphi-10\cos.9\varphi+15\cos11\varphi-21\cos.13\varphi+\text{etc.},$$
$$\frac{1}{16\cos^4.\varphi}=\cos.4\varphi-4\cos.6\varphi+10\cos.8\varphi-20\cos10\varphi+35\cos.12\varphi-56\cos14\varphi+\text{etc.}$$

etc.

COROLLARY 5

8. But if also *n* were a fractional number, the noteworthy series will be produced :

$$\sqrt{2\cos.\varphi}=\cos.\tfrac{1}{2}\varphi+\tfrac{1}{2}\cos.\tfrac{3}{2}\varphi-\tfrac{1}{2}\cdot\tfrac{1}{4}\cos.\tfrac{7}{2}\varphi+\tfrac{1\cdot1\cdot3}{2\cdot4\cdot6}\cos.\tfrac{11}{2}\varphi-\tfrac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\cos.\tfrac{15}{2}\varphi+\text{ etc.},$$

$$\frac{1}{\sqrt{2\cos.\varphi}}=\cos.\tfrac{1}{2}\varphi-\tfrac{1}{2}\cos.\tfrac{5}{2}\varphi+\tfrac{1\cdot3}{2\cdot4}\cos.\tfrac{9}{2}\varphi-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos.\tfrac{13}{2}\varphi+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cos.\tfrac{17}{2}\varphi-\text{etc.},$$

where the coefficients planely are the same, as those which are accustomed to be deduced in the customary extraction of the root from the usual binomial.

SCHOLIUM

9. Generally the formula treated in corollary 3 is more convenient, if *n* is a positive whole number, with the order shown inverted. Moreover then it will be agreed these to be distributed into two classes, as the exponent *n* were an even or odd number.
Indeed in the case, where *n* is an even number, these will themselves be had thus:

$$2\cos^2.\varphi = 1 + \cos.2\varphi,$$
$$8\cos^4.\varphi = 3 + 4\cos.2\varphi + \cos.4\varphi,$$
$$32\cos^6.\varphi = 10 + 15\cos.2\varphi + 6\cos.4\varphi + \cos.6\varphi,$$
$$128\cos^8.\varphi = 35 + 56\cos.2\varphi + 28\cos.4\varphi + 8\cos.6\varphi + \cos.8\varphi,$$
$$512\cos^{10}.\varphi = 126 + 210\cos.2\varphi + 120\cos.4\varphi + 45\cos.6\varphi + 10\cos.8\varphi + \cos.10\varphi,$$
$$2048\cos^{12}.\varphi = 462 + 792\cos.2\varphi + 495\cos.4\varphi + 220\cos.6\varphi + 66\cos.8\varphi + 12\cos.10\varphi + \cos.12\varphi,$$
etc.;

moreover in general, if there were $n = 2v$, there will become:

$$\begin{aligned} 2^{2v-1}\cos^{2v}.\varphi &= \frac{2v(2v-1)(2v-2)\cdots(v+1)}{1\cdot 2\cdot 3\cdots v} + \frac{2v(2v-1)(2\cdot v-2)\cdots(v+2)}{1\cdot 2\cdot 3\cdots(v-1)}\cos.2\varphi \\ &+ \frac{2v(2v-1)(2v-2)\cdots(v+3)}{1\cdot 2\cdot 3\cdots(v-2)}\cos.4\varphi + \frac{2v(2v-1)(2\cdot v-2)\cdots(v+4)}{1\cdot 2\cdot 3\cdots(v-3)}\cos.6\varphi \\ &+ \frac{2v(2v-1)(2v-2)\cdots(v+5)}{1\cdot 2\cdot 3\cdots(v-4)}\cos.8\varphi + \frac{2v(2v-1)(2\cdot v-2)\cdots(v+6)}{1\cdot 2\cdot 3\cdots(v-5)}\cos.10\varphi \end{aligned}$$
etc.

Then on the other case, where *n* is an odd number, there will become:

$$1\cos.\varphi = \cos.\varphi,$$
$$4\cos.^{3}\varphi = 3\cos.\varphi + \cos.3\varphi,$$
$$16\cos.^{5}\varphi = 10\cos.\varphi + 5\cos.3\varphi + \cos.5\varphi,$$
$$64\cos.^{7}\varphi = 35\cos.\varphi + 21\cos.3\varphi + 7\cos.5\varphi + \cos.7\varphi,$$
$$256\cos.^{9}\varphi = 126\cos.\varphi + 84\cos.3\varphi + 36\cos.5\varphi + 9\cos.7\varphi + \cos.9\varphi,$$
$$1024\cos.^{11}\varphi = 462\cos.\varphi + 330\cos.3\varphi + 165\cos.5\varphi + 55\cos.7\varphi + 11\cos.9\varphi + \cos.11\varphi$$

etc.;

moreover in general, if there shall be $n = 2v-1$, there will become :

$$\begin{aligned} 2^{2v-2}\cos^{2v-1}.\varphi &= \frac{(2v-1)(2v-2)\cdots(v+1)}{1\cdot 2\cdot 3\cdots(v-1)}\cos.\varphi + \frac{(2v-1)(2\cdot v-2)\cdots(v+2)}{1\cdot 2\cdot 3\cdots(v-2)}\cos.3\varphi \\ &+ \frac{(2v-1)(2v-2)\cdots(v+3)}{1\cdot 2\cdot 3\cdots(v-3)}\cos.5\varphi + \frac{(2v-1)(2\cdot v-2)\cdots(v+4)}{1\cdot 2\cdot 3\cdots(v-4)}\cos.7\varphi \\ &+ \frac{(2v-1)(2v-2)\cdots(v+5)}{1\cdot 2\cdot 3\cdots(v-5)}\cos.9\varphi + \frac{(2v-1)(2\cdot v-2)\cdots(v+6)}{1\cdot 2\cdot 3\cdots(v-6)}\cos.11\varphi \end{aligned}$$

etc.

PROBLEM 2

10. *To conver any power of the sine of the any angle into simple sines or cosines, thus so that never do two sines or cosines occur multiplied by each other in turn.*

SOLUTION

This problem is solved easily from the preceeding. For by putting $\varphi = 90° - \psi$ there becomes $\cos.\varphi = \sin.\psi$ and thus the expression for the power $\cos^{n}.\varphi$ found now will prevail for the power $\sin^{n}.\varphi$. Then moreover there will become :

$$\cos.2\varphi = -\cos.2\psi,\ \cos.3\varphi = -\sin.3\psi,$$
$$\cos.4\varphi = +\cos.4\psi,\ \cos.5\varphi = +\sin.5\psi,$$
$$\cos.6\varphi = -\cos.6\psi,\ \cos.7\varphi = -\sin.7\psi$$

etc.

Therefore as often as *n* is a whole number, indeed this reduction is less convenient to be able to be put in place for fractions, the following reductions will be obtained:

$$
\begin{aligned}
1\sin.\psi &= +\sin.\psi,\\
2\sin^2.\psi &= -\cos.2\psi + \tfrac{1}{2}\cdot 2,\\
4\sin^3\psi &= -\sin.3\psi + 3\sin.\psi,\\
8\sin^4.\psi &= +\cos.4\psi - 4\cos.2\psi + \tfrac{1}{2}\cdot 6,\\
16\sin^5.\psi &= +\sin.5\psi - 5\sin.3\psi + 10\sin.\psi,\\
32\sin^6.\psi &= -\cos.6\psi + 6\cos.4\psi - 15\cos.2\psi + \tfrac{1}{2}\cdot 20,\\
64\sin^7.\psi &= -\sin.7\psi + 7\sin.5\psi - 21\sin.3\psi + 35\sin.\psi,\\
128\sin^8.\psi &= +\cos.8\psi - 8\cos.6\psi + 28\cos.4\psi - 56\cos.2\psi + \tfrac{1}{2}\cdot 70
\end{aligned}
$$

etc.

But for negative values of *n* there will be had :

$$
\begin{aligned}
\frac{1}{2\sin.\psi} &= +\sin.\psi + \sin.3\psi + \sin.5\psi + \sin.7\psi + \sin.9\psi + \text{etc.},\\
\frac{1}{4\sin^2.\psi} &= -\cos.2\psi - 2\cos.4\psi - 3\cos.6\psi - 4\cos.8\psi - 5\cos.10\psi - \text{etc.},\\
\frac{1}{8\sin^3.\psi} &= -\sin.3\psi - 3\sin.5\psi - 6\sin.7\psi - 10\sin.9\psi - 15\sin.11\psi - \text{etc.},\\
\frac{1}{16\sin^4.\psi} &= +\cos.4\psi + 4\cos.6\psi + 10\cos.8\psi + 20\cos.10\psi + 35\cos.12\psi + \text{etc.},\\
\frac{1}{32\sin^5.\psi} &= +\sin.5\psi + 5\sin.7\psi + 15\sin.9\psi + 35\sin.11\psi + 70\sin.13\psi + \text{etc.},
\end{aligned}
$$

etc.

Hence therefore the quadruple general formulas will be elicited, provided *n* were a number of the form , either $4m$, $4m-1$, $4m-2$, or $4m-3$, and these will become :

$$2^{4m-1}\sin^{4m}.\psi = \cos.4m\psi - 4m\cos.(4m-2)\psi$$
$$+\tfrac{4m(4m-1)}{1\cdot 2}\cos.(4m-4)\psi - \tfrac{4m(4m-1)(4m-2)}{1\cdot 2\cdot 3}\cos.(4m-6)\psi + \cdots$$
$$+\tfrac{1}{2}\cdot\tfrac{4m(4m-1)(4m-2)\cdots(2m+1)}{1\cdot 2\cdot 3\cdots 2m},$$

$$2^{4m-2}\sin^{4m-1}.\psi = -\sin.(4m-1)\psi + (4m-1)\sin.(4m-3)\psi$$
$$-\tfrac{4(4m-1)(4m-2)}{1\cdot 2}\sin.(4m-5)\psi + \tfrac{(4m-1)(4m-2)(4m-3)}{1\cdot 2\cdot 3}\sin.(4m-7)\psi + \cdots$$
$$+\tfrac{(4m-1)(4m-2)(4m-3)\cdots(2m+1)}{1\cdot 2\cdot 3\cdots(2m-1)}\sin.\psi,$$

$$2^{4m-3}\sin^{4m-2}.\psi = -\cos.(4m-2)\psi + (4m-2)\cos.(4m-4)\psi$$
$$-\tfrac{4(4m-2)(4m-3)}{1\cdot 2}\cos.(4m-6)\psi + \tfrac{(4m-2)(4m-3)(4m-4)}{1\cdot 2\cdot 3}\cos.(4m-8)\psi + \cdots$$
$$+\tfrac{1}{2}\cdot\tfrac{(4m-2)(4m-3)(4m-4)\cdots 2m}{1\cdot 2\cdot 3\cdots(2m-1)},$$

$$2^{4m-4}\sin^{4m-3}.\psi = \sin.(4m-3)\psi - (4m-3)\sin.(4m-5)\psi$$
$$+\tfrac{(4m-3)(4m-4)}{1\cdot 2}\sin.(4m-7)\psi - \tfrac{(4m-3)(4m-4)(4m-5)}{1\cdot 2\cdot 3}\sin.(4m-9)\psi + \cdots$$
$$+\tfrac{(4m-3)(4m-4)(4m-5)\cdots 2m}{1\cdot 2\cdot 3\cdots(2m-2)}\sin.\psi.$$

In a similar manner, if n shall be a negative integer, we will have the four general formulas :

$$\frac{1}{2^{4m}\sin^{4m}.\psi} = +\cos.4m\psi + 4m\cos.(4m+2)\psi$$
$$+\frac{4m(4m+1)}{1\cdot 2}\cos.(4m+4)\psi + \frac{4m(4m+1)(4m+2)}{1\cdot 2\cdot 3}\cos.(4m+6)\psi + \text{etc.},$$

$$\frac{1}{2^{4m+1}\sin^{4m+1}.\psi} = +\sin.(4m+1)\psi + (4m+1)\sin.(4m+3)\psi$$
$$+\frac{(4m+1)(4m+2)}{1\cdot 2}\sin.(4m+5)\psi + \frac{(4m+1)(4m+2)(4m+3)}{1\cdot 2\cdot 3}\sin.(4m+7)\psi + \text{etc.},$$

$$\frac{1}{2^{4m+2}\sin^{4m+2}.\psi} = -\cos.(4m+2)\psi - (4m+2)\cos.(4m+4)\psi$$
$$-\frac{4(4m+2)(4m+3)}{1\cdot 2}\cos.(4m+6)\psi - \frac{(4m+2)(4m+3)(4m+4)}{1\cdot 2\cdot 3}\cos.(4m+8)\psi - \text{etc.},$$

$$\frac{1}{2^{4m+3}\sin^{4m+3}.\psi} = -\sin.(4m+3)\psi - (4m+3)\sin.(4m+5)\psi$$
$$-\frac{(4m+3)(4m+4)}{1\cdot 2}\sin.(4m+7)\psi - \frac{(4m+3)(4m+4)(4m+5)}{1\cdot 2\cdot 3}\sin.(4m+9)\psi + \text{etc.}$$

And thus, as often as n is a whole number, either positive or negative, the power $\sin^{n}.\psi$ is resolved in the desired manner. Q. E. I.

COROLLARY 1

11. Therefore as long as n is a positive number, either positive of negative, the power $\sin^{n}.\psi$ is resolved into simple cosines of the multiple angles of ψ. But if n were an odd number, the power $\sin^{n}.\psi$ is resolved into simple sines of the multiple angles of ψ.

COROLLARY 2

12. But if n were a positive integer and the expressions found were set out backwards, the four general formulas given above shall occur in pairs. Indeed, for the even exponents there will become :

$$2\sin^2.\psi = 1-\cos.2\psi,$$
$$8\sin^4.\psi = 3-4\cos.2\psi+\cos.4\psi,$$
$$32\sin^6.\psi = 10-15\cos.2\psi+6\cos.4\psi-\cos.6\psi,$$
$$128\sin^8.\psi = 35-56\cos.2\psi+28\cos.4\psi-8\cos.6\psi+10\cos.8\psi-\cos.10\psi,$$
$$512\sin^{10}.\psi = 126-210\cos.2\psi+120\cos.4\psi-45\cos.6\psi+10\cos.8\psi$$
$$-\cos.10\psi,$$
$$2048\sin.^{12}\psi = 462-792\cos.2\psi+495\cos.4\psi-220\cos.6\psi+66\cos.8\psi$$
$$-12\cos.10\psi+\cos.12\psi,$$
etc.

and generally there will become :

$$2^{2v-2}\sin^{2v}.\psi$$
$$=\frac{1}{2}\cdot\frac{2v(2v-1)(2v-2)\cdots(v+1)}{1\cdot2\cdot3\cdots v} + \frac{2v(2v-1)(2v-2)\cdots(v+2)}{1\cdot2\cdot3\cdots(v-1)}\cos.2\psi$$
$$+\frac{2v(2v-1)(2v-2)\cdots(v+3)}{1\cdot2\cdot3\cdots(v-2)}\cos.4\psi - \frac{2v(2v-1)(2v-2)\cdots(v+4)}{1\cdot2\cdot3\cdots(v-3)}\cos.6\psi$$
$$+\frac{2v(2v-1)(2v-2)\cdots(v+5)}{1\cdot2\cdot3\cdots(v-4)}\cos.8\psi + \frac{2v(2v-1)(2v-2)\cdots(v+6)}{1\cdot2\cdot3\cdots(v-5)}\cos.10\psi$$
etc.

COROLLARY 3

13. Moreover, for the odd exponents there will be had :

$$1\sin.\psi = \sin.\psi,$$
$$4\sin^3.\psi = 3\sin.\psi-\sin.3\psi,$$
$$16\sin.^5\psi = 10\sin.\psi-5\sin.3\psi+\sin.5\psi,$$
$$64\sin.^7\psi = 35\sin.\psi-21\sin.3\psi+7\sin.5\psi-\sin.7\psi,$$
$$256\sin^9.\psi = 126\sin.\psi-84\sin.3\psi+36\sin.5\psi-9\sin.7\psi+\sin.9\psi,$$
$$1024\sin^{11}.\psi = 462\sin.\psi-330\sin.3\psi+165\sin.5\psi-55\sin.7\psi$$
$$+11\sin.9\psi-\sin.11\psi$$
etc.,

for which the general formula becomes :

$$2^{2v-2}\sin^{2v-1}.\psi$$

$$=\frac{(2v-1)(2v-2)\cdots(v+1)}{1\cdot 2\cdot 3\cdots(v-1)}\sin.\psi \quad + \quad \frac{(2v-1)(2v-2)\cdots(v+2)}{1\cdot 2\cdot 3\cdots(v-2)}\sin.3\psi$$

$$+\frac{(2v-1)(2v-2)\cdots(v+3)}{1\cdot 2\cdot 3\cdots(v-3)}\sin.5\psi \quad - \quad \frac{(2v-1)(2v-2)\cdots(v+4)}{1\cdot 2\cdot 3\cdots(v-4)}\sin.7\psi$$

$$+\frac{(2v-1)(2v-2)\cdots(v+5)}{1\cdot 2\cdot 3\cdots(v-5)}\sin.9\psi \quad + \quad \frac{(2v-1)(2v-2)\cdots(v+6)}{1\cdot 2\cdot 3\cdots(v-6)}\sin.11\psi$$

etc.

SCHOLIUM

14. Therefore it is apparent, if a power of the sine of any angle such as $\sin^{n}.\psi$ may occur, a convenient resolution cannot be established, unless n shall be a positive integer, either positive or negative, moreover in this case four formulas are produced, provided the exponent n were any number, either $4a$, or $4a+1$, or $4a+2$, or $4a+3$; which distinction is not necessary, if the question is concerned with a power of some cosine. Yet meanwhile, if n is a fractional number, the formulas for the resolution of the cosine here may be treated without difficulty, since the sine may be changed into the cosine. Indeed on putting $\varphi = 90^{\circ}-\psi$ there will become :

$$\sqrt{2\sin.\psi} = \cos.\tfrac{1}{2}\varphi + \tfrac{1}{2}\cos.\tfrac{3}{2}\varphi - \tfrac{1\cdot 1}{2\cdot 4}\cos.\tfrac{7}{2}\varphi + \tfrac{1\cdot 1\cdot 6}{2\cdot 4\cdot 6}\cos.\tfrac{11}{2}\varphi - \text{etc.},$$

$$\frac{1}{\sqrt{2\sin.\psi}} = \cos.\tfrac{1}{2}\varphi - \tfrac{1}{2}\cos.\tfrac{5}{2}\varphi + \tfrac{1\cdot 3}{2\cdot 4}\cos.\tfrac{9}{2}\varphi - \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos.\tfrac{13}{2}\varphi + \text{etc.}$$

Truly if a product may be proposed of this kind $\sin^{m}.\varphi\cdot\cos^{n}.\varphi$, which may be required to be resolved into simple sines or cosines, this cannot be done conveniently, unless the exponent m shall be a whole number, eithe positive or negative, and then four casess are required to be put in place, provided that m were a number of the form either $4a$, or $4a+1$, or $4a+2$, or $4a+3$. Therefore following these four equation I shall elicit the resolution of the formulas $\sin^{m}.\varphi\cdot\cos^{n}.\varphi$, where indeed it is required to note the exponent n to be subjected to no restriction, thus so that not only whole numbers, but also fractions and thus irrational numbers may be able to be specified.

PROBLEM 3

15. *To resolve a product of this kind* $\sin^m.\varphi\cdot\cos^n.\varphi$ *into simple sines and cosines, in which the exponent m is a whole number of the form* 4α.

SOLUTION

There may be put

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u \text{ and } \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon;$$

there will become

$$\cos.\varphi=\tfrac{u+\upsilon}{2} \text{ and } \sin.\varphi=\tfrac{u-\upsilon}{2\sqrt{-1}}$$

and

$$\cos.\nu\varphi=\tfrac{u^\nu+\upsilon^\nu}{2} \text{ and } \sin.\nu\varphi=\tfrac{u^\nu-\upsilon^\nu}{2\sqrt{-1}},$$

therefore by the lemma, we will have

$$\cos.\nu\varphi+\sqrt{-1}\cdot\sin.\nu\varphi=u^\nu \text{ and } \cos.\nu\varphi-\sqrt{-1}\cdot\sin.\nu\varphi=\upsilon^\nu.$$

Therefore the proposed formula $\sin^m.\varphi\cos^n.\varphi$ will be changed into

$$\tfrac{(u-\upsilon)^m}{2^m\left(\sqrt{-1}\right)^m}\cdot\tfrac{(u+\upsilon)^n}{2^n},$$

and since *m* is a whole number of the form 4α, there will become $\left(\sqrt{-1}\right)^m=+1$ and thus there will be had :

$$\sin^m.\varphi\cos^n.\varphi=\tfrac{(u-\upsilon)^m(u+\upsilon)^n}{2^{m+n}}$$

or

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi=\left(u-\upsilon\right)^m\left(u+\upsilon\right)^n=u^{m+n}\left(1-\tfrac{\upsilon}{u}\right)^m\left(1+\tfrac{\upsilon}{u}\right)^n.$$

For the sake of brevity there shall be $\tfrac{\upsilon}{u}=z$, and it will be required to change this expression into a series $(1-z)^m(1+z)^n$, which may be called $=S$, and there will become

$$lS=ml(1-z)+nl(1+z)$$

and on differentiating,

$$\tfrac{dS}{S}=-\tfrac{mdz}{(1-z)}+\tfrac{ndz}{(1+z)}=\tfrac{(n-m)dz-(m+n)zdz}{1-zz}.$$

There may be put

$$n-m=f \quad \text{and} \quad m+n=g,$$

so that there shall become

$$(1-zz)\tfrac{dS}{dz}-fS+gSz=0.$$

Now there may be put

$$S=1+Az+Bzz+Cz^3+Dz^4+Ez^5+Fz^6+Gz^7+\text{etc.}$$

and with the factors substituted there will be produced :

$$\begin{array}{lllllllll}
A+ & 2Bz & +3Cz^2 & +4Dz^3 & +5Ez^4 & +6Fz^5 & +7Gz^6 & +\text{ etc.}=0. \\
 & & -A & -2B & -3C & -4D & -5E & \\
-f & -fA & -fB & -fC & -fD & -fE & -fF & \\
 & +g & +gA & +gB & +gC & +gD & +gE &
\end{array}$$

Therefore the assumed coefficients *A, B, C* etc. will be determined thus, so that there shall become:

$$\begin{aligned}
A &= f, \\
2B &= fA-g, \\
3C &= fB-(g-1)A, \\
4D &= fC-(g-2)B, \\
5E &= fD-(g-3)C, \\
6F &= fE-(g-4)D \\
&\text{etc.}
\end{aligned}$$

and from these values found there will become

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi=u^g+Au^{g-1}\upsilon+Bu^{g-2}\upsilon^2+Cu^{g-3}\upsilon^3+Du^{g-4}\upsilon^4+\text{ etc.}$$

But since on account of the equal number *m* there shall become :

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi=(v-\upsilon)^m(v+\upsilon)^n,$$

and in a like manner

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi=\upsilon^g+A\upsilon^{g-1}u+B\upsilon^{g-2}u^2+C\upsilon^{g-3}u^3+D\upsilon^{g-4}u^4+\text{ etc.}$$

Therefore with these formulas added there will become on account of $\upsilon u=1$

$$2\cdot 2^{m+n}\sin^m.\varphi\cos^n.\varphi = \upsilon^g + A\upsilon^g u + A\left(u^{g-2}+\upsilon^{g-2}\right) + B\upsilon^{g-3}u^3 + D\upsilon^{g-4}u^4 + \text{etc.}$$

and since in general there shall be $u^\nu + \upsilon^\nu = 2\cos.\nu\varphi$, there will become

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi = \cos.g\varphi + A\cos.\left(g-2\right)\varphi + B\cos.(g-4)\varphi + C\cos.(g-6)\varphi + \text{ etc.}$$

on putting for the sake of brevity $m+n=g$ and $n-m=f$, and with the values indicated before substituted in place of the coefficients *A, B, C, D* etc.. Q. E. I.

PROBLEM 4

16. *If the exponent m were a number of this form* $4\alpha+2$ *or unequally even, to resolve* $\sin^m.\varphi\cos^n.\varphi$ *into simple sines or cosines.*

SOLUTION

On putting as before:

$$\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi = u \quad\text{and}\quad \cos.\varphi - \sqrt{-1}\cdot\sin.\varphi = \upsilon,$$

there will be produced:

$$\sin^m.\varphi\cos^n.\varphi = \frac{(u-\upsilon)^m}{2^m\left(\sqrt{-1}\right)^m}\cdot\frac{(u+\upsilon)^n}{2^n}.$$

Since *m* is twice an odd number, there will become $\left(\sqrt{-1}\right)^m = -1$ and thus there will become:

$$-2^{m+n}\sin^m.\varphi\cos^n.\varphi = \left(u-\upsilon\right)^m\left(u+\upsilon\right)^n$$

and on account of *m* being an even number there will be also

$$-2^{m+n}\sin^m.\varphi\cos^n.\varphi = \left(\upsilon-u\right)^m\left(\upsilon+u\right)^n$$

each of which formulas is resolved as before ; evidently by putting $n-m=f$ and $m+n=g$ and with the coefficients *A, B, C* etc. assumed thus, so that there shall become :

$$\begin{aligned} A &= f,\\ 2B &= fA-g,\\ 3C &= fB-\left(g-1\right)A,\\ 4D &= fC-(g-2)B,\\ 5E &= fD-\left(g-3\right)C,\\ 6F &= fE-\left(g-4\right)D \end{aligned}$$

etc.,

the sum of these formulas will produce

$$-2\cdot 2^{m+n}\sin^{m}.\varphi\cos^{n}.\varphi = u^{g}+\upsilon^{g}+A\left(u^{g-2}+\upsilon^{g-2}\right)+B\left(u^{g-4}+\upsilon^{g-4}\right)+\text{etc.},$$

which progression as before is reduced to the simple cosines of mutiples of the angle φ, thus so that there shall become :

$$2^{m+n}\sin^{m}.\varphi\cos^{n}.\varphi = -\cos.g\varphi - A\cos.(g-2)\varphi - B\cos.(g-4)\varphi - C\cos.(g-6)\varphi - \text{etc.}$$

Q. E. I.

PROBLEM 5

17. *If the exponent m were an odd number of the form* $4\alpha+1$**,** *to resolve the product* $\sin^{m}.\varphi\cos^{n}.\varphi$ *into simple sines or cosines.*

SOLUTION

Again on putting

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi = u \quad\text{and}\quad \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon,$$

so that there may become

$$\sin^{m}.\varphi\cos^{n}.\varphi = \frac{(u-\upsilon)^{m}}{2^{m}\left(\sqrt{-1}\right)^{m}}\cdot\frac{(u+\upsilon)^{n}}{2^{n}},$$

on account of *m* of the form $4\alpha+1$ there will become $(-\sqrt{1})^{m}=\sqrt{-1}$ and thus there will be had

$$2^{m+n}\sqrt{-1}\cdot\sin^{m}.\varphi\cos^{n}.\varphi = \left(u-\upsilon\right)^{m}\left(u+\upsilon\right)^{n}.$$

But on account of the odd number *m* there will become $(u-\upsilon)^{m}=-(\upsilon-u)^{m}$ and hence

$$2^{m+n}\sqrt{-1}\cdot\sin^{m}.\varphi\cos^{n}.\varphi = -\left(\upsilon-u\right)^{m}\left(\upsilon+u\right)^{n}.$$

On this account with these formulas requiring to be added and divided by $2\sqrt{-1}$ there shall become

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi = \frac{(u-\upsilon)^{m}(u+\upsilon)^{n}-(\upsilon-u)^{m}(\upsilon+u)^{n}}{2\sqrt{-1}}.$$

and on putting $m+n=g$ and $n-m=f$ and by taking

$$\begin{aligned}
&A = f,\\
&2B = fA - g,\\
&3C = fB - (g-1)A,\\
&4D = fC - (g-2)B,\\
&5E = fD - (g-3)C,\\
&6F = fE - (g-4)D\\
&\text{etc.}
\end{aligned}$$

there will be obtained:

$$2^{m+n}\cdot\sin^m.\varphi\cos^n.\varphi = \frac{u^g-\upsilon^g}{2\sqrt{-1}} + \frac{A\left(u^{g-2}-\upsilon^{g-2}\right)}{2\sqrt{-1}} + \frac{B\left(u^{g-4}-\upsilon^{g-4}\right)}{2\sqrt{-1}} + \text{etc.}$$

Truly, from the lemma there is :

$$\frac{u^v-\upsilon^v}{2\sqrt{-1}} = \sin.v\varphi,$$

from which there arises :

$$2^{m+n}\cdot\sin^m.\varphi\cos^n.\varphi = \sin.g\varphi + A\sin.(g-2)\varphi + B\sin.(g-4)\varphi + C\sin.(g-6)\varphi + \text{etc.}$$

Q. E. I.

PROBLEM 6

18. *If the exponent m shall be an odd number of the form* $4\alpha+1$, *to resolve this product* $\sin^m.\varphi\cos^n.\varphi$ *into simple sines and cosines.*

SOLUTION

Putting anew

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi = u \quad\text{and}\quad \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi = \upsilon,$$

so that there shall become

$$\sin^m.\varphi\cos^n.\varphi = \frac{(u-\upsilon)^m}{2^m\left(\sqrt{-1}\right)^m}\cdot\frac{(u+\upsilon)^n}{2^n},$$

on account of the number *m* of the form $4\alpha+1$, there will become $(-\sqrt{1})^m = -\sqrt{-1}$ and thus

$$2^{m+n}\cdot\sin^m.\varphi\cos^n.\varphi = -\frac{(u-\upsilon)^m(u+\upsilon)^n}{\sqrt{-1}}.$$

But on account of the odd number *m* there will become $(u-\upsilon)^m = -(\upsilon-u)^m$; therefore

$$2^{m+n}\cdot\sin^m.\varphi\cos^n.\varphi = +\frac{(u-\upsilon)^m(u+\upsilon)^n}{\sqrt{-1}},$$

the half sum of which expression is

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi=\frac{-(u-\upsilon)^{m}(u+\upsilon)^{n}+(\upsilon-u)^{m}(\upsilon+u)^{n}}{2\sqrt{-1}}.$$

If now there maybe put $n-m=f$, $m+n=g$ and the coefficients *A, B, C* etc. will be determined by the following formulas:

$$\begin{aligned}A&=f,\\2B&=fA-g,\\3C&=fB-(g-1)A,\\4D&=fC-(g-2)B,\\5E&=fD-(g-3)C,\\6F&=fE-(g-4)D\\&\text{etc.,}\end{aligned}$$

there will be found:

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi=-\frac{(u^{g}-\upsilon^{g})}{2\sqrt{-1}}-\frac{A(u^{g-2}-\upsilon^{g-2})}{2\sqrt{-1}}-\frac{B(u^{g-4}-\upsilon^{g-4})}{2\sqrt{-1}}-\text{etc.}$$

and on account of

$$\frac{u^{\nu}-\upsilon^{\nu}}{2\sqrt{-1}}=\sin.\nu\varphi$$

there will be obtained finally:

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi=-\sin.g\varphi-A\sin.(g-2)\varphi-B\sin.(g-4)\varphi-C\sin.(g-6)\varphi-\text{etc.}$$

Q. E. I.

COROLLARY 1

19. Therefore the product $\sin^{m}.\varphi\cos^{n}.\varphi$ is resolved into simple cosines, if the exponent *m* were an even number, but into simple sines, if the exponent *m* were an odd number. And if the exponent *m* shall be either 4α or $4\alpha+1$, the individual terms will be positive, but if m shall be either $4\alpha+2$, or $4\alpha-1$, or $4\alpha+3$, the terms with the negative sign have been affected.

COROLLARY 2

20. From these rules observed, both on account of the signs as well as whether a sine or cosine must be taken, the resolution of these four cases requires the same determination of the coefficients *A, B, C* etc., which is itself had thus, so that on putting $n-m=f$ et $m+n=g$ there must become

$$\begin{aligned}
A &= f, \\
2B &= fA - g, \\
3C &= fB - (g-1)A, \\
4D &= fC - (g-2)B, \\
5E &= fD - (g-3)C, \\
6F &= fE - (g-4)D
\end{aligned}$$

etc.

COROLLARY 3

21. Or it will be required to define these coefficients thus, so that there shall be

$$(1-z)^m (1+z)^n = 1 + Az + Bzz + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \text{etc.},$$

from which resolution these coefficients often will be elicited more easily.

COROLLARY 4

22. Since these coefficients in general will become fractions, if we wish to avoid this inconvenience, there may be put at once

$$(1-z)^m (1+z)^n = 1 + \tfrac{\alpha}{1} z + \tfrac{\beta}{1\cdot 2} z^2 + \tfrac{\gamma}{1\cdot 2\cdot 3} z^3 + \tfrac{\delta}{1\cdot 2\cdot 3\cdot 4} z^4 + \text{etc.},$$

so that there shall become

$$A = \tfrac{\alpha}{1},\ \ B = \tfrac{\beta}{1\cdot 2},\ \ C = \tfrac{\gamma}{1\cdot 2\cdot 3},\ \ D = \tfrac{\delta}{1\cdot 2\cdot 3\cdot 4} z^4,$$
$$E = \tfrac{\varepsilon}{1\cdot 2\cdot 3\cdot 4\cdot 5} z^5,\ \ F = \tfrac{\varepsilon}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} z^6 \ \text{ etc.},$$

then moreover, there shall become:

$$\begin{aligned}
\alpha &= f, \\
\beta &= fa - g, \\
\gamma &= f\beta - 2(g-1)\alpha, \\
\delta &= f\gamma - 3(g-2)\beta, \\
\varepsilon &= f\gamma - 4(g-3)\gamma, \\
\zeta &= fe - 5(g-4)\delta
\end{aligned}$$

etc.

COROLLARY 5

23. If these values may be established, there will be found :

$$\alpha = f,$$
$$\beta = ff - g,$$
$$\gamma = f^3 - (3g-2)f,$$
$$\delta = f^4 - (6g-8)ff + 3g(g-2),$$
$$\varepsilon = f^5 - (10g-20)f^3 + (15gg-50g+24)f,$$
$$\zeta = f^6 - (15g-40)f^4 + (45gg-210g+184)ff - 15g(g-2)(g-4)$$

etc.

Truly it is difficult to see the progression of these formulas and that to be continued, unless the determinations specified before may be called in to help.

SCHOLIUM 1

26. But if successive positive integers may be taken for the exponents *m* and *n*, the following resolutions will be produced:

$$1\cos.\varphi = \cos.\varphi,$$
$$1\sin.\varphi = \sin\varphi;$$
$$2\cos^2.\varphi = +\cos.2\varphi + 1,$$
$$2\sin.\varphi\cos.\varphi = +\sin.2\varphi,$$
$$2\sin^2.\varphi = -\cos.2\varphi + 1;$$
$$4\cos^3.\varphi = +\cos.3\varphi + 3\cos.\varphi,$$
$$4\sin.\varphi\cos^2.\varphi = +\sin.3\varphi + sin.\varphi,$$

$4\sin.\varphi\cos^2.\varphi = -\cos.3\varphi + \cos.\varphi,$

$4\sin^3.\varphi = -\sin.3\varphi + 3\sin.\varphi;$

$8\cos^4.\varphi = +\cos.4\varphi + 4\cos.2\varphi + 3,$

$8\sin.\varphi\cos^3.\varphi = +\sin.4\varphi + 2\sin.2\varphi,$

$8\sin^2.\varphi\cos^2.\varphi = \cos.4\varphi \quad * \quad +1,$

$8\sin^3.\varphi\cos.\varphi = -\sin.4\varphi + 2\sin.2\varphi,$

$8\sin^4.\varphi = +\cos.4\varphi - 4\cos.2\varphi + 3;$

$16\cos^5.\varphi = +\cos.5\varphi + 5\cos.3\varphi + 10\cos.\varphi,$

$16\sin.\varphi\cos^4.\varphi = +\sin.5\varphi + 3\sin.3\varphi + 2\sin.\varphi,$

$16\sin^2.\varphi\cos^3.\varphi = -\cos.5\varphi + \cos.3\varphi + 2\cos.\varphi,$

$16\sin^3.\varphi\cos^2.\varphi = -\sin.5\varphi + \sin.3\varphi + 2\sin.\varphi,$

$16\sin^4.\varphi\cos.\varphi = +\cos.5\varphi - 3\cos.3\varphi + 2\cos.\varphi,$

$16\sin^5.\varphi = +\sin.5\varphi + 5\sin.3\varphi + 10\sin.\varphi;$

$32\cos^6.\varphi = +\cos.6\varphi + 6\cos.4\varphi + 15\cos.2\varphi + 10,$

$32\sin.\varphi\cos^5.\varphi = +\sin.6\varphi + 4\sin.4\varphi + 5\sin.2\varphi,$

$32\sin^2.\varphi\cos^4.\varphi = -\cos.6\varphi - 2\cos.4\varphi + \cos.2\varphi + 2,$

$32\sin^3.\varphi\cos^3.\varphi = -\sin.6\varphi \quad * \quad + 3\sin.2\varphi,$

$32\sin^4.\varphi\cos^2.\varphi = +\cos.6\varphi - 2\cos.4\varphi - \cos.2\varphi + 2,$

$32\sin^5.\varphi\cos.\varphi = +\sin.6\varphi - 4\sin.4\varphi + 5\sin.2\varphi,$

$32\sin^6.\varphi = -\cos.6\varphi + 6\cos.4\varphi - 15\cos.2\varphi + 10,$

$64\cos^7.\varphi = +\cos.7\varphi + 7\cos.5\varphi + 21\cos.3\varphi + 35\cos.\varphi,$

$64\sin.\varphi\cos^6.\varphi = +\sin.7\varphi + 5\sin.5\varphi + 9\sin.3\varphi + 5\sin.\varphi,$

$64\sin^2.\varphi\cos^5.\varphi = -\cos.7\varphi - 3\cos.5\varphi - \cos.3\varphi + 5\cos.\varphi,$

$64\sin^3.\varphi\cos^4.\varphi = -\sin.7\varphi - \sin.5\varphi + 3\sin.3\varphi + 3\sin.\varphi,$

$64\sin^4.\varphi\cos^3.\varphi = +\cos.7\varphi - \cos.5\varphi - 3\cos.3\varphi + 3\cos.\varphi,$

$64\sin^5.\varphi\cos^2.\varphi = +\sin.7\varphi - 3\sin.5\varphi + \sin.3\varphi + 5\sin.\varphi,$

$64\sin^6.\varphi\cos.\varphi = -\cos.7\varphi + 5\cos.5\varphi - 9\cos.3\varphi + 5\cos\varphi,$

$64\sin^7.\varphi \quad = -\sin.7\varphi + 7\sin.5\varphi - 21\sin.3\varphi + 35\sin.\varphi.$

But just as these formulas may be allowed to be elicited more conveniently, thence I will demonstrate successively, so that from the first, the series of each order is known from those preceding.

SCHOLION 2

27. But moreover if either of the exponents m and n shall be a negative number, the expression found will produce an infinite series, the form of which will be worth the effort to be investigated in some cases. In this end the following examples has been seen to be added.

EXAMPLE 1

28. *To convert the tangent of each angle φ or this expression* $\frac{\sin.\varphi}{\cos.\varphi}$ *into a series, which proceeds following the simple sines.*

For this form prepared from the general form $\sin^m.\varphi\cos^n.\varphi$ there will be $m=1$ and $n=-1$, from which there becomes $f=-2$ and $g=0$, and hence the following values will be elicited :

$$\begin{array}{ll} A=-2, & A=-2, \\ 2B=+4, & B=+\ 2, \\ 3C=-4-2=-6, & C=-2, \\ 4D=+\ 4+4=+8, & D=+2, \\ 5E=-4-6=-10, & E=-2, \\ 6F=+4+8=+12, & F=+2 \end{array}$$

etc.

Now since there shall be $m=1$, the case will pertain to problem 5 and there will become

$$\frac{2^0\sin.\varphi}{\cos.\varphi}=\sin.0\varphi-2\sin.(-2\varphi)+2\sin.(-4\varphi)-2\sin.(-6\varphi)+\text{etc.},$$

from which there is concluded to become:

$$\frac{\sin.\varphi}{\cos.\varphi}=2\sin.2\varphi-2\sin.4\varphi+2\sin.6\varphi-2\sin.8\varphi+2\sin.10\varphi-\text{etc.}$$

EXAMPLE 2

29. *To convert the cotangent of any angle φ or this expression* $\frac{\sin.\varphi}{\cos.\varphi}$ *into a series, which shall produce the following simple sines.*

For this case there will be $m = -1$ and $n = 1$, from which $f = 2$ and $g = 0$, and thus there will be obtained

$$\begin{array}{ll} A = 2, & A = 2, \\ 2B = 4 - 0, & B = 2, \\ 3C = 4 + 2 = 6, & C = 2, \\ 4D = 4 + 4 = 8, & D = 2, \\ 5E = 4+6 = 10, & E = 2, \\ \text{etc.} & \end{array}$$

But on account of $m = -1$ this case pertains to problem 6 and there will become

$$\frac{2^0 \cos.\varphi}{\sin.\varphi} = -2\sin.0\varphi - 2\sin.(-2\varphi) - 2\sin.(-4\varphi) + 2\sin.(-6\varphi) - \text{etc.},$$

which is reduced to

$$\frac{\cos.\varphi}{\sin.\varphi} = 2\sin.2\varphi + 2\sin.4\varphi + 2\sin.6\varphi + 2\sin.8\varphi + 2\sin.10\varphi + \text{etc.}$$

EXAMPLE 3

30. *To convert this expression* $\frac{\sin.\varphi}{\cos^2.\varphi}$ *into a series.*

On account of $m = 1$ and $n = -2$ there will become $f = -3$ and $g = -1$, from which

$$\begin{array}{ll} A = -3, & A = -3, \\ 2\text{B} = +\ 9+1 = \quad 10, & B = +5, \\ 3C = -15 - 6 = \ -21, & C = -7, \\ 4D = +21+15 = \quad 36, & D = +9, \\ 5E = -27 - 28 = -55, & E = -11 \\ \text{etc.} & \end{array}$$

Therefore on account of $m = 1$ there will be had from problem 5

$$\frac{\sin.\varphi}{2\cos^2.\varphi} = \sin.(-\varphi) - 3\sin.(-3\varphi) + 5\sin.(-5\varphi) - 7\sin.(-7\varphi) + \text{etc.},$$

which is reduced to this form

$$\frac{\sin.\varphi}{\cos^2.\varphi} = -2\sin.\varphi + 6\sin.3\varphi - 10\sin.5\varphi + 14\sin.7\varphi - 18\sin.9\varphi + \text{etc.},$$

the law of which progression is at once apparent.

SCHOLIUM 3

31. Since in these series the coefficients *A, B, C, D* etc. have been found to constitute a progression either of equal terms or of arithmetical terms, in generel I observe these coefficients to progress initially according to equal terms, as often as there were $2f+g=0$ or $g-f=2$, that is, if there shall be $m=-1$, and in this case all the terms to have the same sign attached; but if there shall be $n=-1$, indeed the terms to become equal, but to be endowed with opposite signs. Then I observe, if there shall be either $m=-2,$ or $n=-2$, the series of the coefficients *A, B, C* etc. to become arithmetical and in the first case with all the terms equal, in the latter truly with the adjoining signs alternating; but if there shall be either $m=-3,$ or $n=-3$, a series of the second order to be produced, to be progressing with with the same signs or with the signs alternating, and thus so on. Truly this is to be kept in mind, so that series of this kind may arise, such as I have said, if a negative integer may be taken for *m* or *n*, the other number required to be a positive integer not greater than that.

PROBLEM 7

32. *If there were*

$$S = A + B\cos.2\varphi + C\cos.4\varphi + D\cos.6\varphi + \text{etc.},$$

to find the series equal to $\frac{S\sin.\varphi}{\cos.\varphi}$.

SOLUTION

There may be put

$$\frac{S\sin.\varphi}{\cos.\varphi} = \beta\sin.2\varphi + \gamma\sin.4\varphi + \delta\sin.6\varphi + \varepsilon\sin.8\varphi + \text{etc.}$$

and with that being multiplied by $2\cos.\varphi$

$$\begin{aligned} 2S\sin.\varphi = \beta\sin.\varphi + \beta&\sin.3\varphi + \gamma\sin.5\varphi + \delta\sin.7\varphi + \text{etc.} \\ +\gamma& \qquad\quad +\delta \qquad\quad +\varepsilon \end{aligned}$$

But if moreover the proposed series itself may be multiplied by $2\sin.\varphi$, there will be produced :

$$\begin{aligned} 2S\sin.\varphi = 2A&\sin.\varphi + B\sin.3\varphi + C\sin.5\varphi + D\sin.7\varphi + \text{etc.} \\ -B& \qquad\quad -C \qquad\quad -D \qquad\quad -E \end{aligned}$$

Therefore in a similar manner with the terms equated amongst themselves there will be obtained

$$\beta = 2A - B,$$
$$\gamma = B - C - \beta = -C + 2B - 2A,$$
$$\delta = C - D - \gamma = -D + 2C - 2B + 2A,$$
$$e = D - E - \delta = -E + 2D - 2C + 2B - 2A$$
etc.,

from which the values of the coefficients $\beta,\ \gamma,\ \delta$ etc. will be defined readily. Q. E. I.

COROLLARY 1

33. If the series S may agreed with a finite number of terms, the other series $S\text{tang.}\varphi$ eiher may pass through an infinite number of terms or will be terminated somewhere, so that the latter will occur, if there were $A - B + C - D + \text{etc.} = 0$.

COROLLARY 2

34. But if $A - B + C - D + \text{etc.}$ is the value of the proposed series S in the case, where the angle φ is right; the series $S\text{tang.}\varphi$ therefore cannot be finite, unless the series S may be prepared thus, so that the case $\varphi =$ right angle may be changed into zero.

PROBLEM 8

35. *If this series were proposed*

$$S = B\sin.2\varphi + C\sin.4\varphi + D\sin.6\varphi + \text{E}\sin.8\varphi + \text{etc.},$$

to find the series, which may express the value of the formula $\frac{S\sin.\varphi}{\cos.\varphi}$.

SOLUTION

The series sought may be put

$$\frac{S\sin.\varphi}{\cos.\varphi} = \alpha + \beta\cos.2\varphi + \gamma\cos.4\varphi + \delta\cos.6\varphi + \varepsilon\cos.8\varphi + \text{etc.},$$

which multiplied by $2\cos.\varphi$ gives

$$\begin{aligned} 2S\sin.\varphi = 2\alpha\cos.\varphi &+ \beta\cos.3\varphi + \gamma\cos.5\varphi + \delta\cos.7\varphi + \varepsilon\cos.9\varphi + \text{etc.}, \\ &+ \beta \qquad\quad + \gamma \qquad\quad + \delta \qquad\quad + \varepsilon \qquad\quad + \zeta \end{aligned}$$

from which with the similar terms equated there is elicited

$$\beta = B - 2\alpha,$$
$$\gamma = C - B - \beta = C - 2B + 2\alpha,$$
$$\delta = D - C - \gamma = D - 2C + 2B - 2\alpha,$$
$$\varepsilon = E - D - \delta = E - 2D + 2C - 2B + 2\alpha$$
etc.

Therefore the coefficient α there remains indeterminate, according to the value that expression can assume as it pleases . Q. E. I.

COROLLARY 1

36. Therefore in the proposed series there may be put $B = 0$, $C = 0$, $D = 0$ etc., thus so that also there shall be $S = 0$, there becomes

$\beta = -2\alpha$, $\gamma = +2\alpha$, $\delta = -2\alpha$, $\varepsilon = +2\alpha$ etc. ad infinitum; from which there is produced

$$0 = \alpha - 2\alpha\cos.2\varphi + 2\alpha\cos.4\varphi - 2\alpha\cos.6\varphi + 2\alpha\cos.8\varphi - \text{etc.}$$

Therefore a series of this kind added to any series do not change its sum, from which the reason is apparent, why the value of α may not be determined.

COROLLARY 2

37. If the series S may not extend indefinitely, then a value of this kind will be accepted always for α, so that also the series for $\frac{S\sin.\varphi}{\cos.\varphi}$ may not be extended indefinitely. Evidently if all the terms of the series S may vanish, so that there shall become $S = 0$, then there may be taken $\alpha = 0$ and there will become also $\frac{S\sin.\varphi}{\cos.\varphi} = 0$.

COROLLARY 3

38. If the series S corresponds to a single term there shall be

$$S = B\sin.2\varphi,$$

there becomes $\alpha = B$, so that there shall be $\beta = -B$, and there will be found $\gamma = 0$, $\delta = 0$, $\varepsilon = 0$ etc., and thus there will be produced

$$S\tang.\varphi = B - B\cos.2\varphi.$$

COROLLARY 4

39. If the series S may have only two terms, so that there shall become

$$S = B\sin.2\varphi + C\sin.4\varphi\,,$$

there may be taken $\alpha = B - C$ and the coefficients $\delta,\ \varepsilon,\ \zeta$ etc. will become equal to zero, so that there shall become:

$$S\text{tang}.\varphi = \alpha + \beta\cos.2\varphi + \gamma\cos.4\varphi.$$

COROLLARY 5

40. Hence therefore it is clear, if the series S consists of a finite number of terms, so that the series $S\text{tang}.\varphi = \frac{S\sin.\varphi}{\cos.\varphi}$ may be finite also, then the value of α must be taken thus, so that there shall become:

$$\alpha = B - C + D - E + F - G + \text{etc.},$$

with which assumed, the remaining coefficients will be found easily.

PROBLEM 9

41. *If this series shall be proposed*

$$S = A\cos.\varphi + B\cos.3\varphi + C\cos.5\varphi + D\cos.7\varphi + E\cos.9\varphi + \text{ etc.},$$

to find the series, which may express the value of the formula $\frac{S\sin.\varphi}{\cos.\varphi}$.

SOLUTIO

The series sought may be put

$$\frac{S\sin.\varphi}{\cos.\varphi} = \alpha\cos.\varphi + \beta\cos.3\varphi + \gamma\cos.5\varphi + \delta\cos.7\varphi + \varepsilon\cos.9\varphi + \text{ etc.},$$

which multiplied by $2\cos.\varphi$ gives

$$\begin{aligned}2S\sin.\varphi = \alpha\sin.2\varphi &+ \beta\sin.4\varphi + \gamma\sin.6\varphi + \delta\sin.8\varphi + \varepsilon\sin.10\varphi \text{ etc.}\\ &+\beta \qquad\quad +\gamma \qquad\quad +\delta \qquad\quad +\varepsilon \qquad\quad +\zeta\end{aligned}$$

But the proposited series itself multiplied by $2\sin.\varphi$ gives

$$\begin{aligned}2S\sin.\varphi = {} & \alpha\sin.2\varphi + \beta\sin.4\varphi + \gamma\sin.6\varphi + \delta\sin.8\varphi + \varepsilon\sin.10\varphi \text{ etc.,}\\ & \quad -B \qquad\quad -C \qquad\quad -D \qquad\quad -E \qquad\quad -F\end{aligned}$$

from which the following determinations will be produced

$$\begin{aligned}&\beta = A - B - \alpha,\\ &\gamma = B - C - \beta = -C + 2B - A + \alpha,\\ &\delta = C - D - \gamma = -D + 2C - 2B + A - \alpha,\\ &\varepsilon = D - E - \delta = -E + 2D - 2C + 2B - A + \alpha,\\ &\zeta = E - F - \varepsilon = -F + 2E - 2D + 2C - 2B + A - \alpha\end{aligned}$$

etc.,

where again the coefficient α is not determined, but is left to our choice.

Q. E. I.

COROLLARY 1

42. If all the coefficients *A, B, C* etc. may vanish, so that there shall be $S = 0$, there will become

$\beta = -\alpha,\ \ \gamma = +\alpha,\ \ \delta = -\alpha,\ \ \varepsilon = +\alpha$ etc. and thus there will become:

$$0 = \alpha\sin.\varphi - \alpha\sin.3\varphi + \alpha\sin.5\varphi - \alpha\sin.7\varphi + \alpha\sin.9\varphi - \text{etc.}$$

or

$$\sin.\varphi - \sin.3\varphi + \sin.5\varphi - \sin.7\varphi + \sin.9\varphi - \text{etc.} = 0$$

Moreover above (§36) we have found to be

$$\cos.2\varphi - \cos.4\varphi + \cos.6\varphi - \cos.8\varphi + \cos.10\varphi - \text{etc.} = \tfrac{1}{2}.$$

COROLLARY 2

43. Therefore if the proposed series *S* may consist of a finite number of terms, a value of this kind can be taken for α, so that also the series $S\text{tang.}\varphi$ may consist of a finite number of terms. Evidently there must be taken

$$\alpha = A - 2B + 2C - 2D + 2E - \text{etc.}$$

PROBLEM 10

44. *If this series shall be proposed*

$$S = A\sin.\varphi + B\sin.3\varphi + C\sin.5\varphi + D\sin.7\varphi + E\sin.9\varphi + \text{ etc.,}$$

to find the series, which shall express the value of the formula $\frac{S\sin.\varphi}{\cos.\varphi}$.

SOLUTION

The series sought may be put:

$$\frac{S\sin.\varphi}{\cos.\varphi}=\alpha\cos.\varphi+\beta\cos.3\varphi+\gamma\cos.5\varphi+\delta\cos.7\varphi+\varepsilon\cos.9\varphi+\text{etc.},$$

which multiplied by $2\cos.\varphi$ gives

$$\begin{aligned}2S\sin.\varphi=\alpha+\alpha&\cos.2\varphi+\beta\cos.4\varphi+\gamma\cos.6\varphi+\delta\cos.8\varphi+\varepsilon\cos.10\varphi+\text{etc.}\\ +\beta&\qquad\quad +\gamma\qquad\quad +\delta\qquad\quad +\varepsilon\qquad\quad +\zeta\end{aligned}$$

But if that proposed series itself may be multiplied by $2\sin.\varphi$, there will be had

$$\begin{aligned}2S\sin.\varphi=A-A&\cos.2\varphi-B\cos.4\varphi+C\cos.6\varphi-D\cos.8\varphi-E\cos.10\varphi-\text{etc.},\\ +B&\qquad\quad +C\qquad\quad +D\qquad\quad +E\qquad\quad +F\end{aligned}$$

from which the following determinations of the coefficients sought may be elicited:

$$\begin{aligned}
\alpha&=A,\\
\beta&=B-A-\alpha=B-2A,\\
\gamma&=C-B-\beta=C-2B+2A,\\
\delta&=D-C-\gamma=D-2C+2B-2A,\\
\varepsilon&=E-D-\delta=E-2D+2C-2B+2A,\\
\zeta&=F-E-\varepsilon=F-2E+2D-2C+2B-2A\\
&\text{etc.}
\end{aligned}$$

Therefore in this case all the coefficients sought are determined and none of these is left to our choice. Q. E. I.

COROLLARY 1

45. If the proposed series S may be prepared from a finite number of terms, it can happen, so that the series $\frac{S\sin.\varphi}{\cos.\varphi}$ shall be finite too, or may be extended indefinitely. The first will eventuate, if the coefficientes *A, B, C* etc. were prepared thus, so that there shall become

$$A-B+C-D+E-\text{etc.}=0.$$

COROLLARIUM 2

46. Moreover the proposed series *S* will be changed into $A-B+C-D+\text{etc.}$, if the angle φ were placed right; whereby if the value of the series *S* may vanish on putting $\varphi=90°$, then the series $\frac{S\sin.\varphi}{\cos.\varphi}$ will be agreed to consist of a finite number of terms, if indeed the series *S* were of such a kind.

SCHOLIUM 1

47. These four problems supply a method available for finding the above formulas shown (§ 26) more easily ; and thus I have set out these problems, so that these formulas may be provided written in the reverse order. Indeed since the value of the above expression $2^{n-1}\cos^{n}.\varphi$ may now be elicited from a progression of simple cosines, thence the solution of these problems will be able to be changed into similar progressions with the aid of these formulas

$$2^{n-1}\sin.\varphi\cos^{n-1}.\varphi,\quad 2^{n-1}\sin^{2}.\varphi\cos^{n-2}.\varphi,\quad 2^{n-1}\sin^{3}.\varphi\cos^{n-3}.\varphi\ \text{ etc.};$$

and indeed if the exponent *n* were an even number, the matter may be resolved by the first two problems, but if *n* shall be an odd number, by the latter two. Since we have shown these formulas now to the seventh powe , we may take the eighth power from § 9:

$$128\cos^{8}.\varphi=35+56\cos.2\varphi+28\cos.4\varphi+8\cos.6\varphi+\cos.8\varphi$$

and in problem 7 there shall be:

$$S=128\cos^{8}.\varphi;$$

there will become

$$A=35,\quad B=56,\quad C=28,\quad D=8,\quad E=1,$$

from which there is elicited

$$\begin{aligned}\beta&=70-56=14,\\ \gamma&=56-28-14=14,\\ \delta&=28-8-14=6,\\ \varepsilon&=8-1-6=1,\end{aligned}$$

and thus there will become:

$$128\sin.\varphi\cos^{7}.\varphi=14\sin.2\varphi+14\sin.4\varphi+6\sin.6\varphi+\sin.8\varphi.$$

If now in problem 8:

$$S=128\sin.\varphi\cos^{7}.\varphi,$$

and thus

$$B=14,\quad C=14,\quad D=6,\quad E=1,$$

and there may be taken (§ 40)

$$\alpha = B - C + D - E;$$

there will become

$$\begin{aligned}\alpha &= B - C + D - E = 5;\\ \beta &= 14 - 10 = 4,\\ \gamma &= 14 - 14 - 4 = -4,\\ \delta &= 6 - 14 + 4 = -4,\\ \varepsilon &= 1 - 6 + 4 = -1\end{aligned}$$

and on this account:

$$128 \sin^2.\varphi\cos^6.\varphi = 5 + 4\cos.2\varphi - 4\cos.4\varphi - 4\cos.6\varphi - \cos.8\varphi.$$

Anew in problem 7 there shall be

$$S = 128\sin^2.\varphi\cos^6.\varphi$$

and

$$\mathrm{A} = 5,\quad B = 4,\quad C = -4,\quad D = -4,\quad E = -1;$$

and there will be found:

$$\begin{aligned}\beta &= 10 - 4 = 6,\\ \gamma &= 4 + 4 - 6 = 2,\\ \delta &= -4 + 4 - 2 = -2,\\ \varepsilon &= -4 + 1 + 2 = -1;\end{aligned}$$

therefore

$$128 \sin^3.\varphi\cos^5.\varphi = 6\sin.2\varphi + 2\sin.4\varphi - 2\sin.6\varphi - \sin.8\varphi.$$

Now in problem 8 there shall become

$$S = 128\sin^3.\varphi\cos^5.\varphi$$

and

$$B = 6,\quad C = 2,\quad D = -2,\quad E = -1$$

and there may be taken

$$\begin{aligned}\alpha &= B - C + D - E = 3,\\ \beta &= 6 - 6 = 0,\\ \gamma &= 2 - 6 - 0 = -4,\\ \delta &= -2 - 2 + 4 = 0,\\ \varepsilon &= -1 + 2 - 0 = 1;\end{aligned}$$

therefore

$$128\sin^4.\varphi\cos^4.\varphi = 3 \;\; * \;\; -4\cos.4\varphi \;\; * \;\; +\cos.8\varphi.$$

Now in problem 7 there shall become :

$$S = 128\sin^4.\varphi\cos^4.\varphi\,,$$

and

$$A = 3,\quad B = 0,\quad C = -4,\quad D = 0,\quad E = 1\,,$$

and there may be taken

$$\begin{aligned}\beta &= 6 - 0 \qquad = 6,\\ \gamma &= 0 + 4 - 6 = -2,\\ \delta &= 4 - 0 + 2 = -2,\\ \varepsilon &= \ 0 - 1 + 2 = +1;\end{aligned}$$

therefore

$$128\sin^5.\varphi\cos.^3\varphi = 6\sin.2\varphi - 2\sin.4\varphi - 2\sin.6\varphi + \sin.8\varphi.$$

And thus on progressing further we will obtain these formulas in supplement § 26 on being inverted :

$$\begin{aligned}
128\cos^8.\varphi &= +\cos.8\varphi + 8\cos.6\varphi + 28\cos.4\varphi + 56\cos.2\varphi + 36,\\
128\sin.\varphi\cos^7.\varphi &= +\sin.8\varphi + 6\sin.6\varphi + 14\sin.4\varphi + 14\sin.2\varphi,\\
128\sin^2.\varphi\cos^6.\varphi &= -\cos.8\varphi - 4\cos.6\varphi - 4\cos.4\varphi + 4\cos.2\varphi + 5,\\
128\sin^3.\varphi\cos^5.\varphi &= -\sin.8\varphi - 2\sin.6\varphi + 2\sin.4\varphi + 6\sin.2\varphi,\\
128\sin^4.\varphi\cos^4.\varphi &= +\cos.8\varphi \quad * \quad 4\cos.4p \quad * \quad + 3,\\
128\sin^5.\varphi\cos^3.\varphi &= +\sin.8\varphi - 2\sin.6\varphi - 2\sin.4\varphi + 6\sin.2\varphi,\\
128\sin^6.\varphi\cos^2.\varphi &= -\cos.8\varphi + 4\cos.6\varphi - 4\cos.4\varphi - 4\cos.2\varphi + 5,\\
128\sin^7.\varphi\cos.\varphi &= -\sin.8\varphi + 6\sin.6\varphi - 14\sin.4\varphi + 14\sin.2\varphi,\\
128\sin^8.\varphi &= +\cos.8\varphi - 8\cos.6\varphi + 28\cos.4\varphi - 56\cos.2\varphi + 35.
\end{aligned}$$

SCHOLIUM 2

48. In a similar manner for the use of problems 9 and10 by requiring to show there shall be

$$256\cos.\varphi^9 = 126\cos.\varphi + 84\cos.3\varphi + 36\cos.5\varphi + 9\cos.7\varphi + \cos.9\varphi.$$

And there shall be in problemat 9

$$S = 256\cos^9.\varphi$$

and

$$A = 126,\quad B = 84,\quad C = 36,\quad D = 9,\quad E = \ 1;$$

there may be taken

$$\alpha = A - 2B + 2C - 2D + 2E = 14;$$

$$\beta = 126 - 84 - 14 = 28,$$
$$\gamma = 84 - 36 - 28 = 20,$$
$$\delta = 36 - 9 - 20 = 7,$$
$$\varepsilon = 9 - 1 - 7 = 1;$$

therefore

$$256\sin.\varphi\cos^8.\varphi = 14\sin.\varphi + 28\sin.3\varphi + 20\sin.5\varphi + 7\sin.7\varphi + \sin.9\varphi.$$

In problem 10 there shall be

$$S = 256\sin.\varphi\cos^8.\varphi$$

and

$$A = 14, \quad B = 28, \quad C = 20, \quad D = 7, \quad E = 1;$$

there may be taken

$$\alpha = 14;$$

therefore

$$\beta = 28 - 14 - 14 = 0,$$
$$\gamma = 20 - 28 - 0 = -8,$$
$$\delta = 7 - 20 + 8 = -5,$$
$$\varepsilon = 1 - 7 + 5 = -1;$$

$$256\sin^2.\varphi\cos^7.\varphi = 14\cos.\varphi \quad * \quad -8\cos.5\varphi - 5\cos.7\varphi - \cos.9\varphi.$$

In problem 9 there shall be

$$S = 256\sin^2.\varphi\cos^7.\varphi$$

and

$$A = 14, \quad B = 0, \quad C = -8, \quad D = -5, \quad E = -1;$$

there may be taken

$$\alpha = 14 - 0 - 16 + 10 - 2 = +6;$$

therefore

$$\beta = 14 - 0 - 6 = +8,$$
$$\gamma = 0 + 8 - 8 = 0,$$
$$\delta = -8 + 5 - 0 = -3,$$
$$\varepsilon = -5 + 1 + 3 = -1;$$

$$256\sin^3.\varphi\cos^6.\varphi = 6\sin.\varphi + 8\sin.3\varphi \quad * \quad -3\sin.7\varphi - \sin.9\varphi.$$

In problem 10 there shall be

$$S = 256\sin^3.\varphi^6\cos.\varphi$$

and

$$A=6,\quad B=8,\quad C=0,\quad D=-3,\quad E=-1\ ;$$

there will become

$$\alpha=6;$$
$$\beta=\ \ 8-6-6=-4,$$
$$\gamma=\ \ 0-8+4=-4,$$
$$\delta=-3-0+4=+1,$$
$$\varepsilon=-1+3-1=+1;$$

therefore

$$256\sin^4.\varphi\cos^5.\varphi=6\cos.\varphi-4\cos.3\varphi-4\cos.5\varphi+\cos.7\varphi+\cos.9\varphi,$$

from which there will be had in in supplementum § 26 :

$$256\cos^9.\varphi=+\cos.9\varphi+9\cos.7\varphi+36\cos.5\varphi+84\cos.3\varphi+126\cos.\varphi,$$
$$256\sin.\varphi\cos^8.\varphi=+\sin.9\varphi+7\sin.7\varphi+20\sin.5\varphi+28\sin.3\varphi\quad+14\sin.\varphi,$$
$$256\sin^2.\varphi\cos^7.\varphi=-\cos.9\varphi-5\cos.7\varphi-8\cos.5\varphi\qquad *\qquad+14\cos.\varphi,$$
$$256\sin^3.\varphi\cos^6.\varphi=-\sin.9\varphi-3\sin.7\varphi\qquad *\qquad+8\sin.3\varphi\quad+6\sin.\varphi,$$
$$256\sin^4.\varphi\cos^5.\varphi=+\cos.9\varphi+\cos.7\varphi-4\cos.5\varphi-4\cos.3\varphi\quad+6\cos.\varphi,$$
$$256\sin^5.\varphi\cos^4.\varphi=+\sin.9\varphi-\sin.7\varphi-4\sin.5\varphi+4\sin.3\varphi\quad+6\sin.\varphi,$$
$$256\sin^6.\varphi\cos^3.\varphi=-\cos.9\varphi+3\cos.7\varphi\qquad *\qquad-8\cos.3\varphi\quad+6\cos.\varphi,$$
$$256\sin^7.\varphi\cos^2.\varphi=-\sin.9\varphi+5\sin.7\varphi-8\sin.5\varphi\qquad *\qquad+14\sin.\varphi,$$
$$256\sin^8.\varphi\cos.\varphi=+\cos.9\varphi-7\cos.7\varphi+20\cos.5\varphi-28\cos.3\varphi+14\cos.\varphi,$$
$$256\sin^9.\varphi\qquad=+\sin.9\varphi-9\sin.7\varphi+36\sin.5\varphi-84\sin.3\varphi+126\sin.\varphi.$$

Therefore these formulas will be allowed to be continued in this manner for as long as it may be wished.

THEOREM

49. *If it may be able to assign the sum of this series*

$$Az^m+Bz^{m+n}+Cz^{m+2n}+Dz^{m+3n}+Ez^{m+4n}+\text{etc.}=Z,$$

always will be be able also to show the sum of these series

$$A\cos.m\varphi+B\cos.(m+n)\varphi+a\cos.(m+2n)\varphi+D\cos.(m+3n)\varphi+\text{etc.},$$
$$A\sin.m\varphi+B\sin.(m+n)\varphi+a\sin.(m+2n)\varphi+D\sin.(m+3n)\varphi+\text{etc.}$$

DEMONSTRATION

The sums of these series may be put :

$$A\cos.m\varphi+B\cos.(m+n)\varphi+a\cos.(m+2n)\varphi+D\cos.(m+3n)\varphi+\text{etc.}=S,$$
$$A\sin.m\varphi+B\sin.(m+n)\varphi+a\sin.(m+2n)\varphi+D\sin.(m+3n)\varphi+\text{etc.}=T$$

and as above there shall become:

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u \text{ and } \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon;$$

there will become

$$\cos.v\varphi+\sqrt{-1}\cdot\sin.v\varphi=u^{v} \text{ and } \cos.v\varphi-\sqrt{-1}\cdot\sin.v\varphi=\upsilon^{v}.$$

Hence therefore there will become

$$S+T\sqrt{-1}=Au^{m}+Bu^{m+n}+Cu^{m+2n}+Du^{m+3n}+\text{etc.}=U,$$
$$S-T\sqrt{-1}=A\upsilon^{m}+B\upsilon^{m+n}+C\upsilon^{m+2n}+D\upsilon^{m+3n}+\text{etc.}=V.$$

Evidently the sums of these series U and V are given by hypothesis, since U and V shall be such functions of u et υ, just as the function Z is of z. And thus hence there is elicited

$$S=\tfrac{U+V}{2} \text{ and } S=\tfrac{U-V}{2\sqrt{-1}}$$

and thus the sums of the proposed series S and T become known. Q. E. D.

COROLLARY 1

50. Since there shall be

$$z^{m}+az^{m+n}+a^{2}z^{m+2n}+a^{3}z^{m+3n}+\text{ etc.}=\tfrac{z^{m}}{1-az^{n}},$$

there will become

$$U=\tfrac{u^{m}}{1-au^{n}} \quad\text{and}\quad V=\tfrac{\upsilon^{m}}{1-a\upsilon^{n}},$$

hence

$$U+V=\frac{u^{m}+\upsilon^{m}-a\left(u^{m-n}+\upsilon^{m-n}\right)u^{n}\upsilon^{n}}{1-a\left(u^{n}+\upsilon^{m}\right)+aau^{n}\upsilon^{n}},$$

$$U-V=\frac{u^{m}-\upsilon^{m}-a\left(u^{m-n}-\upsilon^{m-n}\right)u^{n}\upsilon^{n}}{1-a\left(u^{n}+\upsilon^{m}\right)+aau^{n}\upsilon^{n}}.$$

But there is

$$uv=1,\ \ u^{v}+\upsilon^{v}=2\cos.v\varphi,\ \ u^{v}-\upsilon^{v}=2\sqrt{-1}\sin.v\varphi,$$

from which there becomes

$$\frac{U+V}{2}=\frac{\cos.m\varphi-a\cos.(m-n)\varphi}{1+aa-2a\cos.n\varphi}=S$$

and

$$\frac{U-V}{2\sqrt{-1}}=\frac{\sin.m\varphi-a\sin.(m-n)\varphi}{1+aa-2a\cos.n\varphi}=T.$$

From which the following summations will be had:

$$\cos.m\varphi+a\cos.(m+n)\varphi+a^2\cos.(m+2n)\varphi+a^3\cos.(m+3n)\varphi+\text{etc.}$$
$$=\frac{\cos.m\varphi-a\cos.(m-n)\varphi}{1+aa-2a\cos.n\varphi},$$

$$\sin.m\varphi+a\sin.(m+n)\varphi+a^2\sin.(m+2n)\varphi+a^3\sin.(m+3n)\varphi+\text{etc.}$$
$$=\frac{\sin.m\varphi-a\sin.(m-n)\varphi}{1+aa-2a\cos.n\varphi},$$

COROLLARY 2

51. Let there be $m=1$ and $n=1$; there will become

$$\cos.\varphi+a\cos.2\varphi+a^2\cos.3\varphi+a^3\cos.4\varphi+\text{etc.}$$
$$=\frac{\cos.\varphi-a}{1+aa-2a\cos.\varphi},$$

$$\sin.\varphi+a\sin.2\varphi+a^2\sin.3\varphi+a^3\sin.4\varphi+\text{etc.}$$
$$=\frac{\sin.m\varphi-a\sin.(m-n)\varphi}{1+aa-2a\cos.\varphi}.$$

If in addition there shall be $a=1$, there will become

$$\cos.\varphi+\cos.2\varphi+\cos.3\varphi+\cos.4\varphi+\text{etc.}$$
$$=\frac{\cos.\varphi-1}{2-2a\cos.\varphi}=-\frac{1}{2},$$

$$\sin.\varphi+\sin.2\varphi+\sin.3\varphi+\sin.4\varphi+\text{etc.}$$
$$=\frac{\sin.m\varphi}{2-2\cos\varphi}=\frac{1}{2\operatorname{tang}\frac{1}{2}\varphi}.$$

But if there shall be $a=-1$, there will become

$$\cos.\varphi-\cos.2\varphi+\cos.3\varphi-\cos.4\varphi+\text{etc.}$$
$$=\frac{\cos.\varphi+1}{2+2\cos.\varphi}=\frac{1}{2},$$

$$\sin.\varphi - \sin.2\varphi + \sin.3\varphi - \sin.4\varphi + \text{etc.}$$
$$= \tfrac{\sin.\varphi}{2+2\cos\varphi} = \tfrac{1}{2}\text{tang}\tfrac{1}{2}\varphi.$$

COROLLARY 3

52. Let there be $m=1$ and $n=2$; there shall become

$$\cos.\varphi + a\cos.3\varphi + a^2\cos.5\varphi + a^3\cos.7\varphi + \text{etc.} = \tfrac{\cos.\varphi - a\cos.\varphi}{1+aa-2a\cos.2\varphi},$$

$$\sin.\varphi + a\sin.3\varphi + a^2\sin.5\varphi + a^3\sin.7\varphi + \text{etc.} = \tfrac{\sin.\varphi + a\sin.\varphi}{1+aa-2a\cos.2\varphi}.$$

But if therefore there shall become $a=1$, then there will become

$$\cos.\varphi + \cos.3\varphi + \cos.5\varphi + \cos.7\varphi + \text{etc.} = 0,$$

$$\sin.\varphi + \sin.3\varphi + \sin.5\varphi + \sin.7\varphi + \text{etc.} = \tfrac{\sin.\varphi}{1-\cos.2\varphi} = \tfrac{1}{2\sin.\varphi}.$$

But if there shall become $a=-1$, there will become

$$\cos.\varphi - \cos.3\varphi + \cos.5\varphi - \cos.7\varphi + \text{etc.} = \tfrac{2\cos.\varphi}{2+2\cos.2\varphi} = \tfrac{1}{2\cos.2\varphi},$$

$$\sin.\varphi - \sin.3\varphi + \sin.5\varphi - \sin.7\varphi + \text{etc.} = 0.$$

SCHOLIUM

53. Therefore with the help of this theorem, the use of which appears the widest, innumerable series are able to be shown requiring to be progressing following either the sine or cosine of multiples of any angle, of which the sum may be agreed. Here indeed only the case to be established , where the coefficients *A, B, C, D* etc. are proceeding in a geometrical progression, truly in a like manner the calculation may be adapted to other series. But besides here it may suffice to be noted now from the series found innumerable others to be able to be elicited both by differentiation as well as integration. Just as, since there shall be

$$\cos.\varphi - \cos.2\varphi + \cos.3\varphi - \cos.4\varphi + \cos.5\varphi - \text{etc.} = \tfrac{1}{2},$$

by differentiation there will become

$$\sin.\varphi-2\sin.2\varphi+3\sin.3\varphi-4\ \sin.4\varphi+5\sin.5\varphi-\ \text{etc.}=0$$

and by differentiaton again,

$$\cos.\varphi-4\cos.2\varphi+9\cos.3\varphi-16\cos.4\varphi+25\cos.5\varphi-\text{etc.}=0,$$

and thus so on.
But that series multied by $d\varphi$ and integrated gives

$$\sin.\varphi-\tfrac{1}{2}\sin.2\varphi+\tfrac{1}{3}\sin.3\varphi-\tfrac{1}{4}\ \sin.4\varphi+\tfrac{1}{5}\sin.5\varphi-\text{etc.}=\tfrac{\varphi}{2}$$

where there is no need for the addition of a constant, since on putting $\varphi=0$ the sum vanishes at once.
If this series multiplied by $-d\varphi$ may be integrated again, there will be produced

$$\cos.\varphi-\tfrac{1}{4}\cos.2\varphi+\tfrac{1}{9}\cos.3\varphi-\tfrac{1}{16}\cos.4\varphi+\text{etc.}=\alpha-\tfrac{\varphi\varphi}{4}$$

and thus on putting $\varphi=0$

$$1-\tfrac{1}{4}+\tfrac{1}{9}-\tfrac{1}{16}+\text{etc.}=\alpha=\tfrac{\pi\pi}{12},$$

as agreed from elsewhere.
Whereby if there shall be

$$\varphi=\tfrac{\pi}{\sqrt{3}}=103^{0}55^{\text{I}}22^{\text{II}}58^{\text{III}}28^{\text{IV}},$$

the sum of this series vanishes. But many other conspicuous dispositions of series of this kind I will pass over her, lest I may become exceedingly tedious.

SUBSIDIUM CALCULI SINUUM

Commentatio E246
Novi commentarii academiae scientiarum Petropolitanae 5 (1754/65), 1760, p.164-204.

LEMMA

1. *Valor huius formulae imaginariae*

$$\left(cos.\varphi+\sqrt{-1}\cdot sin.\varphi\right)^{n}$$

est

$$cos.n\varphi+\sqrt{-1}\cdot sin.n\varphi,$$

huius autem formulae imaginariae

$$\left(cos.\varphi-\sqrt{-1}\cdot sin.\varphi\right)^n$$

valor est

$$cos.n\varphi-\sqrt{-1}\cdot sin.n\varphi.$$

DEMONSTRATIO

Si enim habeantur duo anguli φ et α, erit harum duarum formularum productum

$$\left(cos.\varphi+\sqrt{-1}\cdot sin.\varphi\right)\left(cos.\varphi+\sqrt{-1}\cdot sin.\varphi\right)$$
$$=cos.\varphi cos.\alpha-sin.\varphi sin.\varphi+\left(\sin.\varphi cos.\alpha+\cos.\varphi sin.\alpha\right)\sqrt{-1}$$

Constat autem esse

$$\cos.\varphi\cos.\alpha-\sin.\varphi\sin.\alpha=\cos.(\varphi+\alpha)$$

et

$$\sin.\varphi\ \cos.\alpha+\cos.\varphi\ \sin.\alpha=\sin.(\varphi+\alpha),$$

unde illud productum erit

$$\cos.\left(\varphi+\alpha\right)+\sqrt{-1}\cdot\sin.\left(\varphi+\alpha\right).$$

Sit iam $\alpha=\varphi$ eritque

$$\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^2=\cos.2\varphi+\sqrt{-1}\cdot\sin.2\varphi.$$

Haec formula denuo per $\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$ multiplicata dabit

$$\cos.(2\varphi+\alpha)+\sqrt{-1}\cdot\sin.\ (2\varphi+\alpha)$$

ac posito $\alpha=\varphi$

$$\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^3=\cos.3\varphi+\sqrt{-1}\cdot\sin.3\varphi.$$

Quae si denuo per $\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi$ multiplicetur ac ponatur $\alpha=\varphi$, dabit

$$\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^4=\cos.4\varphi+\sqrt{-1}\cdot\sin.4\varphi.$$

hocque modo generatim colligitur fore

$$\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^n=\cos.n\varphi+\sqrt{-1}\cdot\sin.n\varphi.$$

Cum autem expressio $\sqrt{-1}$ natura sua signi ambiguitatem involvat, erit ob eandem rationem

$$\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)^n=\cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi.\ \text{Q. E. D.}$$

COROLLARIUM

2. Si ergo brevitatis gratia ponatur

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u\ \text{ et }\ \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon,$$

cum sit

$$u^n=\cos.n\varphi+\sqrt{-1}\cdot\ \sin.n\varphi\ \text{et}\ \upsilon^n=\cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi,,$$

erit

$$u^n+v^n=2\cos.n\varphi\ \text{et}\ u^n-v^n=2\sqrt{-1}\cdot\sin.n\varphi,$$

Constat autem esse $u\upsilon=1$.

PROBLEMA 1

3. *Potestatem quamcunque cosinus cuiuspiam anguli in cosinus simplices convertere, ita ut nusquam duo pluresve occurrant cosinus in se invicem multiplicati.*

SOLUTIO

Sit $(\cos.\varphi)^n$ seu $\cos^n.\varphi$ (has enim designationes pro synonimis habeo) potestas proposita ad modum praescriptum convertenda. Ponatur ut ante

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u\ \text{ et }\ \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon,$$

eritque

$$\cos.\varphi=\tfrac{1}{2}(u+\upsilon)$$

ideoque

$$\cos^n.\varphi=\tfrac{(u+\upsilon)^n}{2^n}$$

seu

$$2^n\cos^n.\varphi=(u+\upsilon)^n$$

Quae potestas binomialis solito modo evolvatur, ut prodeat

$$2^n\cos^n.\varphi=u^n+nu^{n-1}v+\tfrac{n(n-1)}{1\cdot2}u^{n-2}v^2+\tfrac{n(n-1)(n-2)}{1\cdot2\cdot3}u^{n-3}v^3+\text{etc.};$$

similisque expressio prodit, si litterae u et υ permutentur. Additis ergo his duabus expressionibus prodit

$$2^{n+1}\cos^n.\varphi = u^n+\upsilon^n+\tfrac{n}{1}\left(u^{n-2}+\upsilon^{n-2}\right)u\upsilon+\tfrac{n(n-1)}{1\cdot 2}\left(u^{n-4}+\upsilon^{n-4}\right)u^{n-2}\upsilon^2+\text{etc.}$$

et ob $uv=1$ habebitur dividendo per 2

$$2^n\cos^n.\varphi=\tfrac{1}{2}\left(u^n+\upsilon^n\right)+\tfrac{n}{1}\cdot\tfrac{1}{2}\left(u^{n-2}+\upsilon^{n-2}\right)+\tfrac{n(n-1)}{1\cdot 2}\cdot\tfrac{1}{2}\left(u^{n-4}+\upsilon^{n-4}\right)u^{n-2}\upsilon^2$$
$$+\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\tfrac{1}{2}\left(u^{n-6}+\upsilon^{n-6}\right)+\text{etc.}$$

Verum cum sit $u^n+\upsilon^n=2\cos.n\varphi$, perspicuum est fore

$$2^n\cos^n.\varphi=\cos.n\varphi+\tfrac{n}{1}\cos.(n-2)\varphi+\tfrac{n(n-1)}{1\cdot 2}\cdot\cos.(n-4)\varphi$$
$$+\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\cos.(n-6)\varphi+\text{etc.}$$

In qua serie cum ad cosinus angulorum negativorum pervenitur, notandum est eos convenire cum cosinibus eorundem angulorum affirmative sumtorum seu esse

$$\cos.(n-m)\varphi=\cos.(m-n)\varphi.$$

Q. E. I.

COROLLARIUM 1

4. Si sit $n=1$, erit

$$2\cos.\varphi=\cos.\varphi+\cos.\varphi=2\cos.\varphi;$$

at si $n=2$, habetur

$$2^2\cos^2.\varphi=\cos.2\varphi+2\cos.0\varphi+\cos.2\varphi=2\cos.2\varphi+2.$$

Sit $n=3$ eritque

$$2^3\cos^3.\varphi=\cos.3\varphi+3\cos.\varphi+3\cos.\varphi+\cos.3\varphi=2\cos.3\varphi+6\cos.\varphi.$$

Sit $n=4$; erit

$$2^4\cos^4.\varphi=\cos.4\varphi+4\cos.2\varphi+6\cos.0\varphi+4\cos.2\varphi+\cos.4\varphi$$

ideoque, cum singuli termini praeter medium bis occurrant, ob $\cos.0\varphi=1$ erit

$$2^4\cos^4.\varphi=2\cos.4\varphi+8\cos.2\varphi+6$$

COROLLARIUM 2

5. Idem hoc semper usu venit, quoties *n* est numerus integer affirmativus, ut series a fine scripta eadem prodeat ideoque singuli termini praeter medium bis occurrant. Medius autem terminus adest, quoties *n* est numerus par, cosinusque hoc termino contentus abit in unitatem.

COROLLARlUM 3

6. Quodsi ergo termini aequales ab initio et fine coniungantur et tota series per 2 dividatur, iidem habebuntur coefficientes qui ante, nisi quod termini constantis, si quis adest, coefficiens in sui semissem sit transmutandus. Unde hae transformationes, quoties *n* fuerit numerus integer positivus, ita se habebunt:

$$1\cos.\varphi=\cos.\varphi,$$
$$2\cos^2.\varphi=\cos.2\varphi+\tfrac{1}{2}\cdot 2,$$
$$4\cos^3.\varphi=\cos.3\varphi+3\cos.\varphi,$$
$$8\cos^4.\varphi=\cos.4\varphi+4\cos.2\varphi+\tfrac{1}{2}\cdot 6,$$
$$16\cos^5.\varphi=\cos.5\varphi+5\cos.3\varphi+10\cos.\varphi,$$
$$32\cos^6.\varphi=\cos.6\varphi+6\cos.4\varphi+15\cos.2\varphi+\tfrac{1}{2}\cdot 20,$$
$$64\cos^7.\varphi=\cos.7\varphi+7\cos.5\varphi+21\cos.3\varphi+35\cos.\varphi,$$
$$128\cos^8.\varphi=\cos.8\varphi+8\cos.6\varphi+28\cos.4\varphi+56\cos.2\varphi+\tfrac{1}{2}\cdot 70\ \text{etc.}$$

COROLLARIUM 4

7. Si exponens *n* sit numerus negativus, expressio inventa in seriem abit infinitam sicque fiet:

$$\frac{1}{2\cos.\varphi}=\cos.\varphi-\cos.3\varphi+\cos.5\varphi-\cos.7\varphi+\cos.9\varphi-\text{etc.},$$
$$\frac{1}{4\cos^2.\varphi}=\cos.2\varphi-2\cos.4\varphi+3\cos.6\varphi-4\cos.8\varphi+5\cos.10\varphi-6\cos.12\varphi+\text{etc.},$$
$$\frac{1}{8\cos^3.\varphi}=\cos.3\varphi-3\cos.5\varphi+6\cos.7\varphi-10\cos.9\varphi+15\cos11\varphi-21\cos.13\varphi+\text{etc.},$$
$$\frac{1}{16\cos^4.\varphi}=\cos.4\varphi-4\cos.6\varphi+10\cos.8\varphi-20\cos10\varphi+35\cos.12\varphi-56\cos14\varphi+\text{etc.}$$

etc.

COROLLARIUM 5

8. Quin etiam si *n* fuerit numerus fractus, series notatu dignae prodeunt

$$\sqrt{2\cos.\varphi} = \cos.\ \tfrac{1}{2}\varphi + \tfrac{1}{2}\cos.\tfrac{3}{2}\varphi - \tfrac{1}{2}\cdot\tfrac{1}{4}\cos.\tfrac{7}{2}\varphi + \tfrac{1\cdot1\cdot3}{2\cdot4\cdot6}\cos.\tfrac{11}{2}\varphi - \tfrac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\cos.\tfrac{15}{2}\varphi + \text{ etc.},$$

$$\frac{1}{\sqrt{2\cos.\varphi}} = \cos.\tfrac{1}{2}\varphi - \tfrac{1}{2}\cos.\tfrac{5}{2}\varphi + \tfrac{1\cdot3}{2\cdot4}\cos.\tfrac{9}{2}\varphi - \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos.\tfrac{13}{2}\varphi + \tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cos.\tfrac{17}{2}\varphi - \text{etc.},$$

ubi coefficientes plane sunt iidem, qui in extractione radicis ex binomio more consueto erui solent.

SCHOLION

9. Plerumque commodius est formulas in corollario 3 traditas, si n est numerus integer positivus, ordine inverso exhibera. Tum autem conveniet eas in duas classes distribui, prout exponens n fuerit numerus par vel impar.
Casu quidem, quo n est numerus par, eae ita se habebunt:

$$\begin{aligned}
2\cos^2.\varphi &= 1+\cos.2\varphi,\\
8\cos^4.\varphi &= 3+4\cos.2\varphi+\cos.4\varphi,\\
32\cos^6.\varphi &= 10+15\cos.2\varphi+6\cos.4\varphi+\cos.6\varphi,\\
128\cos^8.\varphi &= 35+56\cos.2\varphi+28\cos.4\varphi+8\cos.6\varphi+\cos.8\varphi,\\
512\cos^{10}.\varphi &= 126+210\cos.2\varphi+120\cos.4\varphi+45\cos.6\varphi\\
&\quad +10\cos.8\varphi+\cos.10\varphi,\\
2048\cos^{12}.\varphi &= 462+792\cos.2\varphi+495\cos.4\varphi+220\cos.6\varphi\\
&\quad +66\cos.8\varphi+12\cos.10\varphi+\cos.12\varphi,\\
&\text{etc.};
\end{aligned}$$

in genere autem, si fuerit $n=2\nu$, erit

$$\begin{aligned}
&2^{2\nu-1}\cos^{2\nu}.\varphi\\
&= \frac{2\nu(2\nu-1)(2\nu-2)\cdots(\nu+1)}{1\cdot2\cdot3\cdots\nu} + \frac{2\nu(2\nu-1)(2\cdot\nu-2)\cdots(\nu+2)}{1\cdot2\cdot3\cdots(\nu-1)}\cos.2\varphi\\
&+\frac{2\nu(2\nu-1)(2\nu-2)\cdots(\nu+3)}{1\cdot2\cdot3\cdots(\nu-2)}\cos.4\varphi + \frac{2\nu(2\nu-1)(2\cdot\nu-2)\cdots(\nu+4)}{1\cdot2\cdot3\cdots(\nu-3)}\cos.6\varphi\\
&+\frac{2\nu(2\nu-1)(2\nu-2)\cdots(\nu+5)}{1\cdot2\cdot3\cdots(\nu-4)}\cos.8\varphi + \frac{2\nu(2\nu-1)(2\cdot\nu-2)\cdots(\nu+6)}{1\cdot2\cdot3\cdots(\nu-5)}\cos.10\varphi\\
&\text{etc.}
\end{aligned}$$

Altero deinde casu, quo est n numerus impar, erit

$$1\cos.\varphi = \cos.\varphi,$$

$$4\cos.^3\varphi = 3\cos.\varphi + \cos.3\varphi,$$

$$16\cos.^5\varphi = 10\cos.\varphi + 5\cos.3\varphi + \cos.5\varphi,$$

$$64\cos.^7\varphi = 35\cos.\varphi + 21\cos.3\varphi + 7\cos.5\varphi + \cos.7\varphi,$$

$$256\cos.^9\varphi = 126\cos.\varphi + 84\cos.3\varphi + 36\cos.5\varphi + 9\cos.7\varphi + \cos.9\varphi,$$

$$1024\cos.^{11}\varphi = 462\cos.\varphi + 330\cos.3\varphi + 165\cos.5\varphi + 55\cos.7\varphi + 11\cos.9\varphi + \cos.11\varphi$$

etc.;

in genere autem, si sit $n = 2\nu - 1$, erit

$$2^{2\nu-2}\cos^{2\nu-1}.\varphi = \frac{(2\nu-1)(2\nu-2)\cdots(\nu+1)}{1\cdot2\cdot3\cdots(\nu-1)}\cos.\varphi + \frac{(2\nu-1)(2\cdot\nu-2)\cdots(\nu+2)}{1\cdot2\cdot3\cdots(\nu-2)}\cos.3\varphi$$
$$+\frac{(2\nu-1)(2\nu-2)\cdots(\nu+3)}{1\cdot2\cdot3\cdots(\nu-3)}\cos.5\varphi + \frac{(2\nu-1)(2\cdot\nu-2)\cdots(\nu+4)}{1\cdot2\cdot3\cdots(\nu-4)}\cos.7\varphi$$
$$+\frac{(2\nu-1)(2\nu-2)\cdots(\nu+5)}{1\cdot2\cdot3\cdots(\nu-5)}\cos.9\varphi + \frac{(2\nu-1)(2\cdot\nu-2)\cdots(\nu+6)}{1\cdot2\cdot3\cdots(\nu-6)}\cos.11\varphi$$

etc.

PROBLEMA 2

10. *Potestatem quamcunque sinus cuiuspiam anguli in sinus cosinusve* sim*plices convertere, ita ut nusquam duo sinus vel cosinus occurrant in se invicem multiplicati.*

SOLUTIO

Hoc problema ex praecedenti facile solvitur. Posito enim $\varphi = 90° - \psi$ fit $\cos.\varphi = \sin.\psi$ ideoque expressio pro potestate $\cos^n.\varphi$ inventa iam pro potestate $\sin^n.\varphi$ valebit. Tum autem erit

$$\cos.2\varphi = -\cos.2\psi,\ \cos.3\varphi = -\sin.3\psi,$$
$$\cos.4\varphi = +\cos.4\psi,\ \cos.5\varphi = +\sin.5\psi,$$
$$\cos.6\varphi = -\cos.6\psi,\ \cos.7\varphi = -\sin.7\psi$$

etc.

Quoties ergo *n* est numerus integer, pro fractis enim haec reductio minus commode institui potest, sequentes obtinebuntur reductiones:

$$1\sin.\psi = +\sin.\psi,$$
$$2\sin^2.\psi = -\cos.2\psi + \tfrac{1}{2}\cdot 2,$$
$$4\sin^3\psi = -\sin.3\psi + 3\sin.\psi,$$
$$8\sin^4.\psi = +\cos.4\psi - 4\cos.2\psi + \tfrac{1}{2}\cdot 6,$$
$$16\sin^5.\psi = +\sin.5\psi - 5\sin.3\psi + 10\sin.\psi,$$
$$32\sin^6.\psi = -\cos.6\psi + 6\cos.4\psi - 15\cos.2\psi + \tfrac{1}{2}\cdot 20,$$
$$64\sin^7.\psi = -\sin.7\psi + 7\sin.5\psi - 21\sin.3\psi + 35\sin.\psi,$$
$$128\sin^8.\psi = +\cos.8\psi - 8\cos.6\psi + 28\cos.4\psi - 56\cos.2\psi + \tfrac{1}{2}\cdot 70$$

etc.

Pro valoribus autem negativis ipsius *n* habebitur:

$$\frac{1}{2\sin.\psi} = +\sin.\psi + \sin.3\psi + \sin.5\psi + \sin.7\psi + \sin.9\psi + \text{etc.},$$
$$\frac{1}{4\sin^2.\psi} = -\cos.2\psi - 2\cos.4\psi - 3\cos.6\psi - 4\cos.8\psi - 5\cos.10\psi - \text{etc.},$$
$$\frac{1}{8\sin^3.\psi} = -\sin.3\psi - 3\sin.5\psi - 6\sin.7\psi - 10\sin.9\psi - 15\sin.11\psi - \text{etc.},$$
$$\frac{1}{16\sin^4.\psi} = +\cos.4\psi + 4\cos.6\psi + 10\cos.8\psi + 20\cos.10\psi + 35\cos.12\psi + \text{etc.},$$
$$\frac{1}{32\sin^5.\psi} = +\sin.5\psi + 5\sin.7\psi + 15\sin.9\psi + 35\sin.11\psi + 70\sin.13\psi + \text{etc.},$$

etc.

Hinc ergo quadruplices formulae generales eliciuntur, prout *n* fuerit numerus formae vel *4m* vel $4m-1$ vel $4m-2$ vel $4m-3$, eaeque erunt:

$$2^{4m-1}\sin^{4m}.\psi = \cos.4m\psi - 4m\cos.(4m-2)\psi$$
$$+\tfrac{4m(4m-1)}{1\cdot 2}\cos.(4m-4)\psi - \tfrac{4m(4m-1)(4m-2)}{1\cdot 2\cdot 3}\cos.(4m-6)\psi + \cdots$$
$$+\tfrac{1}{2}\cdot\tfrac{4m(4m-1)(4m-2)\cdots(2m+1)}{1\cdot 2\cdot 3\cdots 2m},$$

$$2^{4m-2}\sin^{4m-1}.\psi = -\sin.(4m-1)\psi + (4m-1)\sin.(4m-3)\psi$$
$$-\tfrac{4(4m-1)(4m-2)}{1\cdot 2}\sin.(4m-5)\psi + \tfrac{(4m-1)(4m-2)(4m-3)}{1\cdot 2\cdot 3}\sin.(4m-7)\psi + \cdots$$
$$+\tfrac{(4m-1)(4m-2)(4m-3)\cdots(2m+1)}{1\cdot 2\cdot 3\cdots(2m-1)}\sin.\psi,$$

$$2^{4m-3}\sin^{4m-2}.\psi = -\cos.(4m-2)\psi + (4m-2)\cos.(4m-4)\psi$$
$$-\tfrac{4(4m-2)(4m-3)}{1\cdot 2}\cos.(4m-6)\psi + \tfrac{(4m-2)(4m-3)(4m-4)}{1\cdot 2\cdot 3}\cos.(4m-8)\psi + \cdots$$
$$+\tfrac{1}{2}\cdot\tfrac{(4m-2)(4m-3)(4m-4)\cdots 2m}{1\cdot 2\cdot 3\cdots(2m-1)},$$

$$2^{4m-4}\sin^{4m-3}.\psi = \sin.(4m-3)\psi - (4m-3)\sin.(4m-5)\psi$$
$$+\tfrac{(4m-3)(4m-4)}{1\cdot 2}\sin.(4m-7)\psi - \tfrac{(4m-3)(4m-4)(4m-5)}{1\cdot 2\cdot 3}\sin.(4m-9)\psi + \cdots$$
$$+\tfrac{(4m-3)(4m-4)(4m-5)\cdots 2m}{1\cdot 2\cdot 3\cdots(2m-2)}\sin.\psi.$$

Simili modo, si n sit numerus negativus integer, quaternas habebimus formulas generales:

$$\frac{1}{2^{4m}\sin^{4m}.\psi} = +\cos.4m\psi + 4m\cos.(4m+2)\psi$$
$$+\tfrac{4m(4m+1)}{1\cdot 2}\cos.(4m+4)\psi + \tfrac{4m(4m+1)(4m+2)}{1\cdot 2\cdot 3}\cos.(4m+6)\psi + \text{etc.},$$

$$\frac{1}{2^{4m+1}\sin^{4m+1}.\psi} = +\sin.(4m+1)\psi + (4m+1)\sin.(4m+3)\psi$$
$$+\tfrac{(4m+1)(4m+2)}{1\cdot 2}\sin.(4m+5)\psi + \tfrac{(4m+1)(4m+2)(4m+3)}{1\cdot 2\cdot 3}\sin.(4m+7)\psi + \text{etc.},$$

$$\frac{1}{2^{4m+2}\sin^{4m+2}.\psi} = -\cos.(4m+2)\psi - (4m+2)\cos.(4m+4)\psi$$
$$-\tfrac{4(4m+2)(4m+3)}{1\cdot 2}\cos.(4m+6)\psi - \tfrac{(4m+2)(4m+3)(4m+4)}{1\cdot 2\cdot 3}\cos.(4m+8)\psi - \text{etc.},$$

$$\frac{1}{2^{4m+3}\sin^{4m+3}.\psi} = -\sin.(4m+3)\psi - (4m+3)\sin.(4m+5)\psi$$
$$-\tfrac{(4m+3)(4m+4)}{1\cdot 2}\sin.(4m+7)\psi - \tfrac{(4m+3)(4m+4)(4m+5)}{1\cdot 2\cdot 3}\sin.(4m+9)\psi + \text{etc.}$$

Sicque, quoties n est numerus integer, sive positivus sive negativus, potestas $\sin^{n}.\psi$ desiderato modo resolvitur. Q. E. I.

COROLLARIUM 1

11. Quoties ergo n est numerus par, sive positivus sive negativus, potestas $\sin^{n}.\psi$ resolvitur in cosinus simplices angulorum multiplorum ipsius ψ. Sin autem n fuerit numerus impar, potestas $\sin^{n}.\psi$ resolvitur in sinus simplices angulorum multiplorum ipsius ψ.

COROLLARIUM 2

12. Quodsi n fuerit numerus integer positivus atque expressiones inventae retro disponantur, quaternae formulae supra datae in binas incidunt. Pro paribus enim exponentibus erit

$$2\sin^{2}.\psi = 1-\cos.2\psi,$$
$$8\sin^{4}.\psi = 3-4\cos.2\psi+\cos.4\psi,$$
$$32\sin^{6}.\psi = 10-15\cos.2\psi+6\cos.4\psi-\cos.6\psi,$$
$$128\sin^{8}.\psi = 35-56\cos.2\psi+28\cos.4\psi-8\cos.6\psi+10\cos.8\psi-\cos.10\psi,$$
$$512\sin^{10}.\psi = 126-210\cos.2\psi+120\cos.4\psi-45\cos.6\psi+10\cos.8\psi -\cos.10\psi,$$
$$2048\sin.^{12}\psi = 462-792\cos.2\psi+495\cos.4\psi-220\cos.6\psi+66\cos.8\psi -12\cos.10\psi+\cos.12\psi,$$
etc.

atque generatim erit

$$\begin{aligned}2^{2v-2}\sin^{2v}.\psi &= \tfrac{1}{2}\cdot\frac{2v(2v-1)(2v-2)\cdots(v+1)}{1\cdot2\cdot3\cdots v} + \frac{2v(2v-1)(2v-2)\cdots(v+2)}{1\cdot2\cdot3\cdots(v-1)}\cos.2\psi\\ &+\frac{2v(2v-1)(2v-2)\cdots(v+3)}{1\cdot2\cdot3\cdots(v-2)}\cos.4\psi - \frac{2v(2v-1)(2v-2)\cdots(v+4)}{1\cdot2\cdot3\cdots(v-3)}\cos.6\psi\\ &+\frac{2v(2v-1)(2v-2)\cdots(v+5)}{1\cdot2\cdot3\cdots(v-4)}\cos.8\psi + \frac{2v(2v-1)(2v-2)\cdots(v+6)}{1\cdot2\cdot3\cdots(v-5)}\cos.10\psi\end{aligned}$$
etc.

COROLLARIUM 3

13. Pro imparibus autem exponentibus habebitur

$$1\sin.\psi = \sin.\psi,$$

$$4\sin^3.\psi = 3\sin.\psi - \sin.3\psi,$$

$$16\sin.^5\psi = 10\sin.\psi - 5\sin.3\psi + \sin.5\psi,$$

$$64\sin.^7\psi = 35\sin.\psi - 21\sin.3\psi + 7\sin.5\psi - \sin.7\psi,$$

$$256\sin^9.\psi = 126\sin.\psi - 84\sin.3\psi + 36\sin.5\psi - 9\sin.7\psi + \sin.9\psi,$$

$$1024\sin^{11}.\psi = 462\sin.\psi - 330\sin.3\psi + 165\sin.5\psi - 55\sin.7\psi$$
$$+11\sin.9\psi - \sin.11\psi$$

etc.,

pro quibus formula generalis est

$$2^{2\nu-2}\sin^{2\nu-1}.\psi$$
$$= \frac{(2\nu-1)(2\nu-2)\cdots(\nu+1)}{1\cdot2\cdot3\cdots(\nu-1)}\sin.\psi \quad + \quad \frac{(2\nu-1)(2\nu-2)\cdots(\nu+2)}{1\cdot2\cdot3\cdots(\nu-2)}\sin.3\psi$$
$$+\frac{(2\nu-1)(2\nu-2)\cdots(\nu+3)}{1\cdot2\cdot3\cdots(\nu-3)}\sin.5\psi \quad - \quad \frac{(2\nu-1)(2\nu-2)\cdots(\nu+4)}{1\cdot2\cdot3\cdots(\nu-4)}\sin.7\psi$$
$$+\frac{(2\nu-1)(2\nu-2)\cdots(\nu+5)}{1\cdot2\cdot3\cdots(\nu-5)}\sin.9\psi \quad + \quad \frac{(2\nu-1)(2\nu-2)\cdots(\nu+6)}{1\cdot2\cdot3\cdots(\nu-6)}\sin.11\psi$$

etc.

SCHOLION

14. Patet ergo, si potestas sinus cuiuspiam anguli velut $\sin^n.\psi$ occurrat, resolutionem commode institui non posse, nisi n sit numerus integer, sive sit positivus sive negativus, hoc autem casu quadruplices prodire formulas, prout exponens n fuerit numerus formae vel $4a$ vel $4a+1$ vel $4a+2$ vel $4a+3$; quae distinctio non est necessaria, si quaestio est de potestate cosinus cuiuspiam. Interim tamen, si n est numerus fractus, formulae pro resolutione potestatum cosinus huc non difficulter traducuntur, cum sinus in cosinum transmutari possit. Posito enim $\varphi = 90° - \psi$ erit

$$\sqrt{2}\,\sin.\psi = \cos.\tfrac{1}{2}\varphi + \tfrac{1}{2}\cos.\tfrac{3}{2}\varphi - \tfrac{1\cdot1}{2\cdot4}\cos.\tfrac{7}{2}\varphi + \tfrac{1\cdot1\cdot6}{2\cdot4\cdot6}\cos.\tfrac{11}{2}\varphi - \text{etc.},$$

$$\frac{1}{\sqrt{2}\,\sin.\psi} = \cos.\tfrac{1}{2}\varphi - \tfrac{1}{2}\cos.\tfrac{5}{2}\varphi + \tfrac{1\cdot3}{2\cdot4}\cos.\tfrac{9}{2}\varphi - \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos.\tfrac{13}{2}\varphi + \text{etc.}$$

Verum si productum proponatur huiusmodi $\sin^m.\varphi\cdot\cos^n.\varphi$, quod in simplices sinus cosinusve sit resolvendum, hoc commode fieri nequit, nisi exponens m sit numerus integer, sive positivus sive negativus, tumque quatuor constituendi sunt casus, prout m fuerit numerus formae vel $4a$ vel $4a+1$ vel $4a+2$ vel $4a+3$. Secundum hos ergo quaternos casus resolutionem formulae $\sin^m.\varphi\cdot\cos^n.\varphi$ eruam, ubi quidem notandum est exponentem *n* nulli restrictioni esse subiectum, ita ut non solum numeros integros, sed etiam fractos atque adeo irrationales denotare possit.

PROBLEMA 3

15. *Huiusmodi productum* $\sin^m .\varphi \cdot \cos^n .\varphi$ *, in quo exponens m est numerus integer formae* 4α*, in sinus cosinusve simplices resolvere.*

SOLUTIO

Ponatur

$$\cos.\varphi + \sqrt{-1} \cdot \sin.\varphi = u \text{ et } \cos.\varphi - \sqrt{-1} \cdot \sin.\varphi = \upsilon;$$

erit

$$\cos.\varphi = \tfrac{u+\upsilon}{2} \text{ et } \sin.\varphi = \tfrac{u-\upsilon}{2\sqrt{-1}}$$

et

$$\cos.\nu\varphi = \tfrac{u^\nu+\upsilon^\nu}{2} \text{ et } \sin.\nu\varphi = \tfrac{u^\nu-\upsilon^\nu}{2\sqrt{-1}}$$

propterea quod per lemma habemus

$$\cos.\nu\varphi + \sqrt{-1}\cdot \sin.\nu\varphi = u^\nu \text{ et } \cos.\nu\varphi - \sqrt{-1}\cdot \sin.\nu\varphi = \upsilon^\nu .$$

Formula ergo proposita $\sin^m .\varphi \cos^n .\varphi$ abit in

$$\frac{(u-\upsilon)^m}{2^m\left(\sqrt{-1}\right)^m} \cdot \frac{(u+\upsilon)^n}{2^n},$$

et quia *m* est numerus integer formae 4α*,* erit $\left(\sqrt{-1}\right)^m = +1$ ideoque habebitur

$$\sin^m .\varphi \cos^n .\varphi = \frac{(u-\upsilon)^m (u+\upsilon)^n}{2^{m+n}}$$

sive

$$2^{m+n} \sin^m .\varphi \cos^n .\varphi = (u-\upsilon)^m \,(u+\upsilon)^n = u^{m+n} \left(1-\tfrac{\upsilon}{u}\right)^m \left(1+\tfrac{\upsilon}{u}\right)^n .$$

Sit brevitatis gratia $\frac{\upsilon}{u} = z$, atque in seriem converti oportet hanc expressionem $(1-z)^m (1+z)^n$, quae vocetur $= S$, eritque

$$lS = ml(1-z) + nl(1+z)$$

et differentiando

$$\frac{dS}{S} = -\frac{mdz}{(1-z)} + \frac{ndz}{(1+z)} = \frac{(n-m)dz-(m+n)zdz}{1-zz}.$$

Ponatur

$$n-m = f \text{ et } m+n = g,$$

ut sit

$$(1-zz)\tfrac{dS}{dz}-fS+gSz=0.$$

Iam statuatur

$$S=1+Az+Bzz+Cz^3+Dz^4+Ez^5+Fz^6+Gz^7+\text{etc.}$$

ac facta substitutione prodibit

$$\begin{array}{llllllll}
A+ & 2Bz & +3Cz^2 & +4Dz^3 & +5Ez^4 & +6Fz^5 & +7Gz^6 & +\text{ etc.}=0.\\
 & -A & -2B & -3C & -4D & -5E & & \\
-f & -fA & -fB & -fC & -fD & -fE & -fF & \\
 & +g & +gA & +gB & +gC & +gD & +gE &
\end{array}$$

Coefficientes ergo assumti *A, B, C* etc. ita determinabuntur, ut sit

$$\begin{aligned}
A&=f,\\
2B&=fA-g,\\
3C&=fB-(g-1)A,\\
4D&=fC-(g-2)B,\\
5E&=fD-(g-3)C,\\
6F&=fE-(g-4)D\\
&\text{etc.}
\end{aligned}$$

hisque valoribus inventis erit

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi=u^g+Au^{g-1}\upsilon+Bu^{g-2}\upsilon^2+Cu^{g-3}\upsilon^3+Du^{g-4}\upsilon^4+\text{ etc.}$$

Cum autem ob m numerum parem sit

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi=(v-\upsilon)^m(v+\upsilon)^n,$$

erit simili modo

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi=\upsilon^g+A\upsilon^{g-1}u+B\upsilon^{g-2}u^2+C\upsilon^{g-3}u^3+D\upsilon^{g-4}u^4+\text{ etc.}$$

His igitur formulis addendis erit ob $\upsilon u=1$

$$2\cdot2^{m+n}\sin^m.\varphi\cos^n.\varphi=\upsilon^g+A\upsilon^g u+A\left(u^{g-2}+\upsilon^{g-2}\right)+B\upsilon^{g-3}u^3+D\upsilon^{g-4}u^4+\text{ etc.}$$

et cum in genere sit $u^\nu+\upsilon^\nu=2\cos.\nu\varphi,$ erit

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi=\cos.g\varphi+A\cos.(g-2)\varphi+B\cos.(g-4)\varphi+C\cos.(g-6)\varphi+\text{ etc.}$$

posito brevitatis gratia $m+n=g$ et $n-m=f$ substitutisque in locum coefficientum *A, B, C, D* etc. valoribus ante indicatis. Q. E. I.

PROBLEMA 4

16. *Si exponens m fuerit numerus huius formae* $4\alpha+2$ *seu impariter par, productum* $\sin^m.\varphi\cos^n.\varphi$ *in sinus cosinusve simplices resolvere.*

SOLUTIO

Posito ut ante

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u \quad\text{et}\quad \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon,$$

prodibit

$$\sin^m.\varphi\cos^n.\varphi=\frac{(u-\upsilon)^m}{2^m\left(\sqrt{-1}\right)^m}\cdot\frac{(u+\upsilon)^n}{2^n}.$$

Quia autem *m* est numerus impariter par, erit $\left(\sqrt{-1}\right)^m=-1$ ideoque erit

$$-2^{m+n}\sin^m.\varphi\cos^n.\varphi=\left(u-\upsilon\right)^m\left(u+\upsilon\right)^n$$

et ob *m* numerum parem erit quoque

$$-2^{m+n}\sin^m.\varphi\cos^n.\varphi=\left(\upsilon-u\right)^m\left(\upsilon+u\right)^n$$

quarum formularum utraque ut ante resolvatur; scilicet posito $n-m=f$ et $m+n=g$ et coefficientibus *A, B, C* etc. ita assumtis, ut sit

$$\begin{aligned}A&=f,\\ 2B&=fA-g,\\ 3C&=fB-\left(g-1\right)A,\\ 4D&=fC-(\mathrm{g}-2)B,\\ 5E&=fD-\left(g-3\right)C,\\ 6F&=fE-\left(g-4\right)D\end{aligned}$$

etc.,

summa illarum formularum praebebit

$$-2\cdot2^{m+n}\sin^m.\varphi\cos^n.\varphi=u^g+\upsilon^g+A\left(u^{g-2}+\upsilon^{g-2}\right)+B\left(u^{g-4}+\upsilon^{g-4}\right)+\text{etc.},$$

quae progressio ut ante ad cosinus simplices angulorum multiplorum ipsius φ reducitur, ita ut sit

$$2^{m+n}\sin^m.\varphi\cos^n.\varphi = -\cos.g\varphi - A\cos.(g-2)\varphi - B\cos.(g-4)\varphi - C\cos.(g-6)\varphi - \text{etc.}$$

Q. E. I.

PROBLEMA 5

17. *Si exponens m fuerit numerus impar formae* $4\alpha+1$, *productum* $\sin^m.\varphi\cos^n.\varphi$ *in sinus cosinusve simplices resolvere.*

SOLUTIO

Posito iterum

$$\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi = u \quad \text{et} \quad \cos.\varphi - \sqrt{-1}\cdot\sin.\varphi = \upsilon,$$

ut fiat

$$\sin^m.\varphi\cos^n.\varphi = \frac{(u-\upsilon)^m}{2^m\left(\sqrt{-1}\right)^m}\cdot\frac{(u+\upsilon)^n}{2^n},$$

ob m numerum formae $4\alpha+1$ erit $(-\sqrt{1})^m = \sqrt{-1}$ ideoque habebitur

$$2^{m+n}\sqrt{-1}\cdot\sin^m.\varphi\cos^n.\varphi = \left(u-\upsilon\right)^m\left(u+\upsilon\right)^n.$$

At ob m numerum imparem erit $(u-\upsilon)^m = -(\upsilon-u)^m$ hineque

$$2^{m+n}\sqrt{-1}\cdot\sin^m.\varphi\cos^n.\varphi = -\left(\upsilon-u\right)^m\left(\upsilon+u\right)^n.$$

Hanc ob rem his formulis addendis et per $2\sqrt{-1}$ dividendis fiat

$$2^{m+n}\cdot\sin^m.\varphi\cos^n.\varphi = \frac{(u-\upsilon)^m(u+\upsilon)^n-(\upsilon-u)^m(\upsilon+u)^n}{2\sqrt{-1}}.$$

positoque $m+n=g$ et $n-m=f$ sumtisque

$$\begin{aligned}
A &= f,\\
2B &= fA-g,\\
3C &= fB-\left(g-1\right)A,\\
4D &= fC-\left(g-2\right)B,\\
5E &= fD-(g-3)C,\\
6F &= fE-\left(g-4\right)D\\
&\text{etc.}
\end{aligned}$$

obtinebitur

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi=\frac{u^{g}-\upsilon^{g}}{2\sqrt{-1}}+\frac{A\left(u^{g-2}-\upsilon^{g-2}\right)}{2\sqrt{-1}}+\frac{B\left(u^{g-4}-\upsilon^{g-4}\right)}{2\sqrt{-1}}+\text{etc.}$$

Verum ex lemmate est

$$\frac{u^{\nu}-\upsilon^{\nu}}{2\sqrt{-1}}=\sin.\nu\varphi,$$

unde oritur

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi=\sin.g\varphi+A\sin.(g-2)\varphi+B\sin.(g-4)\varphi+C\sin.(g-6)\varphi+\text{etc.}$$

Q. E. I.

PROBLEMA 6

18. *Si exponens m sit numerus impar formae* $4\alpha+1$, *resolvere hoc productum* $\sin^{m}.\varphi\cos^{n}.\varphi$ *in sinus cosinusve simplices.*

SOLUTIO

Posito denuo

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u \ \text{ et } \ \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon,$$

ut sit

$$\sin^{m}.\varphi\cos^{n}.\varphi=\frac{(u-\upsilon)^{m}}{2^{m}\left(\sqrt{-1}\right)^{m}}\cdot\frac{(u+\upsilon)^{n}}{2^{n}},$$

ob m numerum formae $4\alpha+1$ erit $(-\sqrt{1})^{m}=-\sqrt{-1}$ ideoque

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi=-\frac{(u-\upsilon)^{m}(u+\upsilon)^{n}}{\sqrt{-1}}.$$

At ob m numerum imparem erit $(u-\upsilon)^{m}=-(\upsilon-u)^{m}$*;* ergo

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi=+\frac{(u-\upsilon)^{m}(u+\upsilon)^{n}}{\sqrt{-1}},$$

quarum expressionum semisumma est

$$2^{m+n}\cdot\sin^{m}.\varphi\cos^{n}.\varphi=\frac{-(u-\upsilon)^{m}(u+\upsilon)^{n}+(\upsilon-u)^{m}(\upsilon+u)^{n}}{2\sqrt{-1}}.$$

Si iam ponatur $n-m=f$, $m+n=g$ coefficientesque *A, B, C* etc. per sequentes formulas determinentur

$$A = f,$$
$$2B = fA - g,$$
$$3C = fB - (g-1)A,$$
$$4D = fC - (g-2)B,$$
$$5E = fD - (g-3)C,$$
$$6F = fE - (g-4)D$$
etc.,

reperietur

$$2^{m+n}\cdot\sin^m.\varphi\cos^n.\varphi = -\frac{(u^g - \upsilon^g)}{2\sqrt{-1}} - \frac{A(u^{g-2} - \upsilon^{g-2})}{2\sqrt{-1}} - \frac{B(u^{g-4} - \upsilon^{g-4})}{2\sqrt{-1}} - \text{etc.}$$

atque ob

$$\frac{u^v - \upsilon^v}{2\sqrt{-1}} = \sin.v\varphi$$

obtinebitur tandem

$$2^{m+n}\cdot\sin^m.\varphi\cos^n.\varphi = -\sin.g\varphi - A\sin.(g-2)\varphi - B\sin.(g-4)\varphi - C\sin.(g-6)\varphi - \text{etc.}$$

Q. E. I.

COROLLARIUM 1

19. Productum ergo $\sin^m.\varphi\cos^n.\varphi$ in cosinus simplices resolvitur, si exponens *m* fuerit numerus par, in sinus autem simplices, si exponens *m* fuerit numerus impar. Atque si exponens *m* sit vel 4α vel $4\alpha+1$, singuli termini erunt positivi, sin autem *m* sit vel $4\alpha+2$ vel $4\alpha-1$ seu $4\alpha+3$, termini signo negativo sunt affecti.

COROLLARIUM 2

20. His regulis tam ratione signorum quam utrum sinus an cosinus accipi debeant, observatis resolutio horum quaternorum casuum requirit eandem coefficientium *A, B, C* etc. determinationem, quae ita se habet, ut posito $n-m=f$ et $m+n=g$ esse debeat

$$A = f,$$
$$2B = fA - g,$$
$$3C = fB - (g-1)A,$$
$$4D = fC - (g-2)B,$$
$$5E = fD - (g-3)C,$$
$$6F = fE - (g-4)D$$
etc.

COROLLARIUM 3

21. Vel hos coefficientes ita definiri oportet, ut sit

$$(1-z)^m(1+z)^n = 1+Az+Bzz+Cz^3+Dz^4+Ez^5+Fz^6+\text{etc.},$$

ex qua resolutione hi coefficientes saepe facilius eruentur.

COROLLARIUM 4

22. Quoniam hi coefficientes in genere in fractiones abeunt, si hoc incommodum vitare velimus, statim ponatur

$$\left(1-z\right)^m\left(1+z\right)^n = 1+\tfrac{\alpha}{1}z+\tfrac{\beta}{1\cdot 2}z^2+\tfrac{\gamma}{1\cdot 2\cdot 3}z^3+\tfrac{\delta}{1\cdot 2\cdot 3\cdot 4}z^4+\text{etc.},$$

ut sit

$$A=\tfrac{\alpha}{1},\ \ B=\tfrac{\beta}{1\cdot 2},\ \ C=\tfrac{\gamma}{1\cdot 2\cdot 3},\ \ D=\tfrac{\delta}{1\cdot 2\cdot 3\cdot 4}z^4,$$
$$E=\tfrac{\varepsilon}{1\cdot 2\cdot 3\cdot 4\cdot 5}z^5,\ \ F=\tfrac{\varepsilon}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}z^6\ \ \text{etc.},$$

tum autem erit

$$\begin{aligned}
&\alpha = f,\\
&\beta = fa-g,\\
&\gamma = f\beta-2(g-1)\alpha,\\
&\delta = f\gamma-3(g-2)\beta,\\
&\varepsilon = f\gamma-4(g-3)\gamma,\\
&\zeta = fe-5\left(g-4\right)\delta\\
&\qquad\text{etc.}
\end{aligned}$$

COROLLARIUM 5

23. Si hi valores evolvuntur, reperietur

$$\begin{aligned}
&\alpha = f,\\
&\beta = ff-g,\\
&\gamma = f^3-\left(3g-2\right)f,\\
&\delta = f^4-\left(6g-8\right)ff+3g\left(g-2\right),\\
&\varepsilon = f^5-\left(10\text{g}-20\right)f^3+(15gg-50g+24)f,\\
&\zeta = f^6-\left(15g-40\right)f^4+\left(45gg-210g+184\right)ff-15g(g-2)(g-4)\\
&\qquad\qquad\text{etc.}
\end{aligned}$$

Verum difficile est harum formularum progressionem perspicere eamque continuare, nisi determinationes ante indicatae in subsidium vocentur.

SCHOLION 1

26. Quodsi pro exponentibus *m* et *n* successive numeri integri affirmativi capiantur, sequentes prodibunt resolutiones:

$$1\cos.\varphi=\cos.\varphi,$$
$$1\sin.\varphi=\sin\varphi;$$
$$2\cos^2.\varphi=+\cos.2\varphi+1,$$
$$2\sin.\varphi\cos.\varphi=+\sin.2\varphi,$$
$$2\sin^2.\varphi=-\cos.2\varphi+1;$$
$$4\cos^3.\varphi=+\cos.3\varphi+3\cos.\varphi,$$
$$4\sin.\varphi\cos^2.\varphi=+\sin.3\varphi+\mathit{sin}.\varphi,$$
$$4\sin.\varphi\cos^2.\varphi=-\cos.3\varphi+\cos.\varphi,$$
$$4\sin^3.\varphi=-\sin.3\varphi+3\sin.\varphi;$$
$$8\cos^4.\varphi=+\cos.4\varphi+4\cos.2\varphi+3,$$
$$8\sin.\varphi\cos^3.\varphi=+\sin.4\varphi+2\sin.2\varphi,$$
$$8\sin^2.\varphi\cos^2.\varphi=\cos.4\varphi\ \ *\ \ +1,$$
$$8\sin^3.\varphi\cos.\varphi=-\sin.4\varphi+2\sin.2\varphi,$$
$$8\sin^4.\varphi=+\cos.4\varphi-4\cos.2\varphi+3;$$
$$16\cos^5.\varphi=+\cos.5\varphi+5\cos.3\varphi+10\cos.\varphi,$$
$$16\sin.\varphi\cos^4.\varphi=+\sin.5\varphi+3\sin.3\varphi+2\sin.\varphi,$$
$$16\sin^2.\varphi\cos^3.\varphi=-\cos.5\varphi+\cos.3\varphi+2\cos.\varphi,$$
$$16\sin^3.\varphi\cos^2.\varphi=-\sin.5\varphi+\sin.3\varphi+2\sin.\varphi,$$
$$16\sin^4.\varphi\cos.\varphi=+\cos.5\varphi-3\cos.3\varphi+2\cos.\varphi,$$
$$16\sin^5.\varphi=+\sin.5\varphi+5\sin.3\varphi+10\sin.\varphi;$$

$$32\cos^6.\varphi = +\cos.6\varphi + 6\cos.4\varphi + 15\cos.2\varphi + 10,$$
$$32\sin.\varphi\cos^5.\varphi = +\sin.6\varphi + 4\sin.4\varphi + 5\sin.2\varphi,$$
$$32\sin^2.\varphi\cos^4.\varphi = -\cos.6\varphi - 2\cos.4\varphi + \cos.2\varphi + 2,$$
$$32\sin^3.\varphi\cos^3.\varphi = -\sin.6\varphi \quad * \quad + 3\sin.2\varphi,$$
$$32\sin^4.\varphi\cos^2.\varphi = +\cos.6\varphi - 2\cos.4\varphi - \cos.2\varphi + 2,$$
$$32\sin^5.\varphi\cos.\varphi = +\sin.6\varphi - 4\sin.4\varphi + 5\sin.2\varphi,$$
$$32\sin^6.\varphi = -\cos.6\varphi + 6\cos.4\varphi - 15\cos.2\varphi + 10,$$
$$64\cos^7.\varphi = +\cos.7\varphi + 7\cos.5\varphi + 21\cos.3\varphi + 35\cos.\varphi,$$
$$64\sin.\varphi\cos^6.\varphi = +\sin.7\varphi + 5\sin.5\varphi + 9\sin.3\varphi + 5\sin.\varphi,$$
$$64\sin^2.\varphi\cos^5.\varphi = -\cos.7\varphi - 3\cos.5\varphi - \cos.3\varphi + 5\cos.\varphi,$$
$$64\sin^3.\varphi\cos^4.\varphi = -\sin.7\varphi - \sin.5\varphi + 3\sin.3\varphi + 3\sin.\varphi,$$
$$64\sin^4.\varphi\cos^3.\varphi = +\cos.7\varphi - \cos.5\varphi - 3\cos.3\varphi + 3\cos.\varphi,$$
$$64\sin^5.\varphi\cos^2.\varphi = +\sin.7\varphi - 3\sin.5\varphi + \sin.3\varphi + 5\sin.\varphi,$$
$$64\sin^6.\varphi\cos.\varphi = -\cos.7\varphi + 5\cos.5\varphi - 9\cos.3\varphi + 5\cos\varphi,$$
$$64\sin^7.\varphi \qquad = -\sin.7\varphi + 7\sin.5\varphi - 21\sin.3\varphi + 35\sin.\varphi.$$

Quemadmodum autem formulas has commodius eruere liceat, deinceps docebo, inde quod cuiusque ordinis prima series ex praecedentibus est cognita.

SCHOLION 2

27. Sin autem alter exponentium *m* et *n* sit numerus negativus, expressio inventa seriem exhibebit infinitam, cuius formam in aliquot casibus investigare operae erit pretium. In hunc finem sequentia exempla adiungere visum est.

EXEMPLUM 1

28. *Tangentem cuiusque anguli φ seu hanc expressionem* $\frac{\sin.\varphi}{\cos.\varphi}$ *in seriem convertere, quae secundum sinus simplices procedat.*

Forma hac comparata cum generali $\sin^m.\varphi\cos^n.\varphi$ erit $m = 1$ et $n = -1$, unde fit $f = -2$ et $g = 0$, hincque elicientur valores sequentes:

$$\begin{aligned} A &= -2, & A &= -2, \\ 2B &= +4, & B &= +\ 2, \\ 3C &= -4-2=-6, & C &= -2, \\ 4D &= +\ 4+4=+8, & D &= +2, \\ 5E &= -4-6=-10, & E &= -2, \\ 6F &= +4+8=+12, & F &= +2 \end{aligned}$$

etc.

Cum nunc sit $m=1$, casus ad problema 5 pertinet eritque

$$\frac{2^0 \sin.\varphi}{\cos.\varphi} = \sin.0\varphi - 2\sin.(-2\varphi) + 2\ \sin.(-4\varphi) - 2\ \sin.(-6\varphi) + \text{etc.},$$

unde concluditur fore

$$\frac{\sin.\varphi}{\cos.\varphi} = 2\sin.2\varphi - 2\ \sin.4\varphi + 2\ \sin.6\varphi - 2\ \sin.8\varphi + 2\ \sin.10\varphi - \text{etc.}$$

EXEMPLUM 2

29. *Cotangentem cuiusvis anguli φ seu hanc expressionem* $\frac{\sin.\varphi}{\cos.\varphi}$ *in seriem convertere, quae secundum sinus simplices procedat.*

Pro hoc casu erit $m=-1$ et $n=1$, unde $f=2$ et $g=0$, ideoque obtinebitur

$$\begin{aligned} A &= 2, & A &= 2, \\ 2B &= 4-0, & B &= 2, \\ 3C &= 4+2=6, & C &= 2, \\ 4D &= 4+4=8, & D &= 2, \\ 5E &= 4+6=\ 10, & E &= 2, \end{aligned}$$

etc.

At ob $m=-1$ hic casus ad problema 6 pertinet eritque

$$\frac{2^0 \cos.\varphi}{\sin.\varphi} = -2\sin.0\varphi - 2\sin.(-2\varphi) - 2\sin.(-4\varphi) + 2\sin.(-6\varphi) - \text{etc.},$$

quae reducitur ad

$$\frac{\cos.\varphi}{\sin.\varphi} = 2\sin.2\varphi + 2\sin.4\varphi + 2\sin.6\varphi + 2\sin.8\varphi + 2\sin.10\varphi + \text{etc.}$$

EXEMPLUM 3

30. *Hanc expressionem* $\frac{\sin.\varphi}{\cos^2.\varphi}$ *in seriem convertere.*

Ob $m=1$ et $n=-2$ erit $f=-3$ et $g=-1$, unde

$$\begin{array}{ll}
A=-3, & A=-3,\\
2B=+9+1=\quad 10, & B=+5,\\
3C=-15-6=\ -21, & C=-7,\\
4D=+21+15=\quad 36, & D=+9,\\
5E=-27-28=-55, & E=-11
\end{array}$$

etc.

Ergo ob $m=1$ ex problemate 5 habebitur

$$\frac{\sin.\varphi}{2\cos^2.\varphi}=\sin.(-\varphi)-3\sin.\left(-3\varphi\right)+5\sin.\left(-5\varphi\right)-7\sin.(-7\varphi)+\text{etc.},$$

quae reducitur ad hanc formam

$$\frac{\sin.\varphi}{\cos^2.\varphi}=-2\sin.\varphi+6\sin.3\varphi-10\sin.5\varphi+14\sin.7\varphi-18\sin.9\varphi+\text{etc.},$$

cuius progressionis lex sponte patet.

SCHOLION 3

31. Quoniam in his seriebus coefficientes *A, B, C, D* etc. progressionem vel terminorum aequalium vel arithmeticam constituere sunt inventi, in genere observo primo hos coefficientes secundum terminos aequales progredi, quoties fuerit $2f+g=0$ seu $g-f=2$, hoc est, si sit $m=-1$, hocque casu omnes terminos eodem signo fore affectos; sin autem sit $n=-1$, terminos quidem fore aequales sed signis alternis praeditos. Deinde noto, si sit vel $m=-2$ vel $n=-2$, seriem coefficientium *A, B, C* etc. fore arithmeticam priorique casu omnes terminos paribus, posteriori vero alternantibus signis affectos; sin autem sit vel $m=-3$ vel $n=-3$, seriem prodire secundi ordinis, vel eodem vel alternantibus signis progredientem, et ita porro. Verum hic est animadvertendum, ut huiusmodi series, quales dixi, proveniant, si pro *m* vel *n* numerus negativus integer accipiatur, alterum numerum oportere esse affirmativum integrum illo non maiorem.

PROBLEMA 7

32. *Si fuerit*

$$S=A+B\cos.2\varphi+C\cos.4\varphi+D\cos.6\varphi+\text{etc.},$$

invenire seriem ipsi $\frac{S\sin.\varphi}{\cos.\varphi}$ *aequalem.*

SOLUTIO

Ponatur

$$\frac{S\sin.\varphi}{\cos.\varphi} = \beta\sin.2\varphi + \gamma\sin.4\varphi + \delta\sin.6\varphi + \varepsilon\sin.8\varphi + \text{etc.}$$

eritque per $2\cos.\varphi$ multiplicando

$$\begin{aligned}2S\sin.\varphi = \beta\sin.\varphi + \beta&\sin.3\varphi + \gamma\sin.5\varphi + \delta\sin.7\varphi + \text{etc.}\\ +\gamma& \qquad\quad +\delta \qquad\quad +\varepsilon\end{aligned}$$

Quodsi autem ipsa series proposita per $2\sin.\varphi$ multiplicetur, prodibit

$$\begin{aligned}2S\sin.\varphi = 2A\sin.\varphi &+ B\sin.3\varphi + C\sin.5\varphi + D\sin.7\varphi + \text{etc.}\\ -B \qquad &-C \qquad\quad -D \qquad\quad -E\end{aligned}$$

Similibus ergo terminis inter se aequandis obtinebitur

$$\begin{aligned}&\beta = 2A - B,\\ &\gamma = B - C - \beta = -C + 2B - 2A,\\ &\delta = C - D - \gamma = -D + 2C - 2B + 2A,\\ &e = D - E - \delta = -E + 2D - 2C + 2B - 2A\\ &\qquad\qquad \text{etc.},\end{aligned}$$

unde valores coefficientium $\beta,\ \gamma,\ \delta$ etc. facile definiuntur. Q. E. I.

COROLLARIUM 1

33. Si series S finito terminorum numero constet, altera series $S\text{tang.}\varphi$ vel in infinitum excurret vel alicubi terminabitur, quod posterius eveniet, si fuerit $A - B + C - D + \text{etc.} = 0$.

COROLLARIUM 2

34. At $A - B + C - D + \text{etc.}$ est valor seriei propositae S casu, quo angulus φ sit rectus; series ergo $S\text{tang.}\varphi$ non abrumpitur, nisi series S ita sit comparata, ut casu $\varphi =$ angulo recto in nihilum abeat.

PROBLEMA 8

35. *Si proposita fuerit haec series*

$$S = B\sin.2\varphi + C\sin.4\varphi + D\sin.6\varphi + \text{E}\sin.8\varphi + \text{etc.},$$

invenire seriem, quae exprimat valorem formulae $\frac{S\sin.\varphi}{\cos.\varphi}$.

SOLUTIO

Ponatur series quaesita

$$\frac{S\sin.\varphi}{\cos.\varphi} = \alpha + \beta\cos.2\varphi + \gamma\cos.4\varphi + \delta\cos.6\varphi + \varepsilon\cos.8\varphi + \text{etc.},$$

quae per $2\cos.\varphi$ multiplicata dat

$$\begin{aligned} 2S\sin.\varphi = 2\alpha\cos.\varphi &+ \beta\cos.3\varphi + \gamma\cos.5\varphi + \delta\cos.7\varphi + \varepsilon\cos.9\varphi + \text{etc.},\\ &+\beta \qquad\quad +\gamma \qquad\quad +\delta \qquad\quad +\varepsilon \qquad\quad +\zeta \end{aligned}$$

unde terminis similibus aequandis elicitur

$$\begin{aligned} &\beta = B - 2\alpha,\\ &\gamma = C - B - \beta = C - 2B + 2\alpha,\\ &\delta = D - C - \gamma = D - 2C + 2B - 2\alpha,\\ &\varepsilon = E - D - \delta = E - 2D + 2C - 2B + 2\alpha\\ &\qquad\qquad \text{etc.} \end{aligned}$$

Coefficiens ergo α manet indeterminatus pro eoque pro lubitu valor assumi potest. Q. E. I.

COROLLARIUM 1

36. Si ergo in serie proposita ponatur $B = 0,\ \ C = 0,\ \ D = 0$ etc., ita ut quoque sit $S = 0$, fiet
$\beta = -2\alpha,\ \ \gamma = +2\alpha,\ \ \delta = -2\alpha,\ \ \varepsilon = +2\alpha$ etc. in infinitum; unde prodibit

$$0 = \alpha - 2\alpha\cos.2\varphi + 2\alpha\cos.4\varphi - 2\alpha\cos.6\varphi + 2\alpha\cos.8\varphi - \text{etc.}$$

Huiusmodi ergo series seriei cuicunque addita eius summam non mutat, unde ratio patet, cur valor ipsius α non determinetur.

COROLLARIUM 2

37. Si series S non in infinitum excurrat, tum semper pro α eiusmodi valor accipi poterit, ut etiam series pro $\frac{S\sin.\varphi}{\cos.\varphi}$ non in infinitum excurrat. Scilicet si seriei S omnes termini evanescant, ut sit $S = 0$, tum capiatur $\alpha = 0$ fietque etiam $\frac{S\sin.\varphi}{\cos.\varphi} = 0$.

COROLLARIUM 3

38. Si series S unico termino constet seu sit

$$S = B\sin.2\varphi,$$

fiat $\alpha = B$, ut sit $\beta = -B$, reperienturque $\gamma = 0,\ \delta = 0,\ \varepsilon = 0$ etc. sicque prodibit

$$\text{Stang.}\varphi = B - B\cos.2\varphi\,.$$

COROLLARIUM 4

39. Si series S duos tantum habeat terminos, ut sit

$$S = B\sin.2\varphi + C\sin.4\varphi\,,$$

capiatur $\alpha = B - C$ fientque coefficientes $\delta,\ \varepsilon,\ \zeta$ etc. nihilo aequales, ita ut sit

$$\text{Stang.}\varphi = \alpha + \beta\cos.2\varphi + \gamma\cos.4\varphi.$$

COROLLARIUM 5

40. Hinc igitur patet, si series S finito terminorum numero constat, ut etiam series $S\text{tang.}\varphi = \frac{\text{Ssin.}\varphi}{\cos.\varphi}$ fiat finita, tum valorem ipsius α ita capi oportere, ut sit

$$\alpha = B - C + D - E + F - G + \text{etc.},$$

quo assumto reliqui coefficientes facile reperientur.

PROBLEMA 9

41. *Si proposita sit haec series*

$$S = A\cos.\varphi + B\cos.3\varphi + C\cos.5\varphi + D\cos.7\varphi + \text{E}\cos.9\varphi + \text{ etc.},$$

invenire seriem, quae exprimat valorem formulae $\frac{\text{Ssin.}\varphi}{\cos.\varphi}$.

SOLUTIO

Ponatur series quaesita

$$\frac{\text{Ssin.}\varphi}{\cos.\varphi} = \alpha\cos.\varphi + \beta\cos.3\varphi + \gamma\cos.5\varphi + \delta\cos.7\varphi + \varepsilon\cos.9\varphi + \text{ etc.},$$

quae per $2\cos.\varphi$ multiplicata dat

$$\begin{array}{l} 2S\sin.\varphi = \alpha\sin.2\varphi + \beta\sin.4\varphi + \gamma\sin.6\varphi + \delta\sin.8\varphi + \varepsilon\sin.10\varphi \text{ etc.} \\ \qquad\qquad\quad +\beta \qquad\quad +\gamma \qquad\quad +\delta \qquad\quad +\varepsilon \qquad\quad +\zeta \end{array}$$

At ipsa series proposita per $2\sin.\varphi$ multiplicata dat

$$\begin{array}{l} 2S\sin.\varphi = \alpha\sin.2\varphi + \beta\sin.4\varphi + \gamma\sin.6\varphi + \delta\sin.8\varphi + \varepsilon\sin.10\varphi \text{ etc.,} \\ \qquad\qquad\quad -B \qquad\quad -C \qquad\quad -D \qquad\quad -E \qquad\quad -F \end{array}$$

unde sequentes prodeunt determinationes

$$\begin{aligned} &\beta = A - B - \alpha, \\ &\gamma = B - C - \beta = -C + 2B - A + \alpha, \\ &\delta = C - D - \gamma = -D + 2C - 2B + A - \alpha, \\ &\varepsilon = D - E - \delta = -E + 2D - 2C + 2B - A + \alpha, \\ &\zeta = E - F - \varepsilon = -F + 2E - 2D + 2C - 2B + A - \alpha \\ &\qquad\qquad\qquad \text{etc.,} \end{aligned}$$

ubi iterum coefficiens α non determinatur, sed arbitrio nostro relinquitur
Q. E. I.

COROLLARIUM 1

42. Si omnes coefficientes *A, B, C* etc. evanescant, ut sit $S = 0$, fiet
$\beta = -\alpha,\ \ \gamma = +\alpha,\ \ \delta = -\alpha,\ \ \varepsilon = +\alpha$ etc. ideoque erit

$$0 = \alpha\sin.\varphi - \alpha\sin.3\varphi + \alpha\sin.5\varphi - \alpha\sin.7\varphi + \alpha\sin.9\varphi - \text{etc.}$$

seu

$$\sin.\varphi - \sin.3\varphi + \sin.5\varphi - \sin.7\varphi + \sin.9\varphi - \text{etc.} = 0$$

Supra autem invenimus (§36) esse

$$\cos.2\varphi - \cos.4\varphi + \cos.6\varphi - \cos.8\varphi + \cos.10\varphi - \text{etc.} = \tfrac{1}{2}.$$

COROLLARIUM 2

43. Si ergo series proposita *S* finito terminorum numero constet, pro α eiusmodi valor accipi potest, ut etiam series $S\text{tang.}\varphi$ finito terminorum numero constet. Capi scilicet debet

$$\alpha = A - 2B + 2C - 2D + 2E - \text{etc.}$$

PROBLEMA 10

44. *Si proposita sit haec series*

$$S = A\sin.\varphi + B\sin.3\varphi + C\sin.5\varphi + D\sin.7\varphi + E\sin.9\varphi + \text{ etc.},$$

invenire seriem, quae exprimat valorem formulae $\frac{S\sin.\varphi}{\cos.\varphi}$.

SOLUTIO

Ponatur series quaesita

$$\frac{S\sin.\varphi}{\cos.\varphi} = \alpha\cos.\varphi + \beta\cos.3\varphi + \gamma\cos.5\varphi + \delta\cos.7\varphi + \varepsilon\cos.9\varphi + \text{etc.},$$

quae per $2\cos.\varphi$ multiplicata dat

$$\begin{aligned} 2S\sin.\varphi = \alpha + \alpha\cos.2\varphi &+ \beta\cos.4\varphi + \gamma\cos.6\varphi + \delta\cos.8\varphi + \varepsilon\cos.10\varphi + \text{etc.} \\ +\beta \quad &+\gamma \qquad\quad +\delta \qquad\quad +\varepsilon \qquad\quad +\zeta \end{aligned}$$

Si autem ipsa series proposita per $2\sin.\varphi$ multiplicetur, habebitur

$$\begin{aligned} 2S\sin.\varphi = A - A\cos.2\varphi &- B\cos.4\varphi + C\cos.6\varphi - D\cos.8\varphi - E\cos.10\varphi - \text{etc.}, \\ +B \quad &+C \qquad\quad +D \qquad\quad +E \qquad\quad +F \end{aligned}$$

unde sequentes eliciuntur coefficientium quaesitorum determinationes:

$$\begin{aligned} &\alpha = A, \\ &\beta = B - A - \alpha = B - 2A, \\ &\gamma = C - B - \beta = C - 2B + 2A, \\ &\delta = D - C - \gamma = D - 2C + 2B - 2A, \\ &\varepsilon = E - D - \delta = E - 2D + 2C - 2B + 2A, \\ &\zeta = F - E - \varepsilon = F - 2E + 2D - 2C + 2B - 2A \\ &\qquad\qquad\qquad \text{etc.} \end{aligned}$$

Hoc igitur casu omnes coefficientes quaesiti determinantur nullusque eorum arbitrio nostro relinquitur. Q. E. I.

COROLLARIUM 1

45. Si series proposita S finito terminorum numero constet, fieri potest, ut series $\frac{S\sin.\varphi}{\cos.\varphi}$ vel quoque sit finita vel in infinitum excurrat. Prius eveniet, si coefficientes A, B, C etc. ita fuerint comparati, ut sit

$$A - B + C - D + E - \text{etc.} = 0.$$

COROLLARIUM 2

46. Series autem proposita S abit in $A-B+C-D+\text{etc.}$, si angulus φ statuatur rectus; quare si valor seriei S evanescat posito $\varphi=90°$, tum series $\frac{S\sin.\varphi}{\cos.\varphi}$ finito constabit terminorum numero, siquidem series S fuerit talis.

SCHOLION 1

47. Quatuor haec problemata methodum suppeditant formulas supra (§ 26) exhibitas facilius inveniendi; atque haec problemata ita adornavi, ut has formulas ordine retrograda scriptas praeberent. Cum enim valor expressionis $2^{n-1}\cos^{n}.\varphi$ iam supra per progressionem cosinuum simplicium sit erutus, inde horum problematum ope istae formulae

$$2^{n-1}\sin.\varphi\cos^{n-1}.\varphi,\ \ 2^{n-1}\sin^{2}.\varphi\cos^{n-2}.\varphi,\ \ 2^{n-1}\sin^{3}.\varphi\cos^{n-3}.\varphi\ \ \text{etc.}$$

in similes progressiones converti poterunt; ac si quidem exponens n fuerit numerus par, negotium per bina problemata priora conficietur, si autem n sit numerus impar, per bina posteriora. Quoniam has formulas iam ad potestatem septimam exhibuimus, sumamus potestatem octavam ex § 9

$$128\cos^{8}.\varphi=35+56\cos.2\varphi+28\cos.4\varphi+8\cos.6\varphi+\cos.8\varphi$$

et in problemate 7 sit

$$S=128\cos^{8}.\varphi;$$

erit

$$\mathrm{A}=35,\quad \mathrm{B}=56,\quad C=28,\quad D=8,\quad \mathrm{E}=1,$$

unde eruitur

$$\begin{aligned}\beta&=70-56&&=14,\\ \gamma&=56-28-14&&=14,\\ \delta&=28-\ 8-14&&=\ \ 6,\\ \varepsilon&=8-1-6&&=\ \ 1,\end{aligned}$$

sicque erit

$$128\sin.\varphi\cos^{7}.\varphi=14\sin.2\varphi+14\sin.4\varphi+6\sin.6\varphi+\sin.8\varphi.$$

Sit nunc in problemate 8

$$S=128\sin.\varphi\cos^{7}.\varphi$$

ideoque

$$B=14,\ \ C=14,\ \ D=6,\ \ E=1$$

et capiatur (§ 40)

$$\alpha=B-C+D-E;$$

erit

$$\begin{aligned}\alpha &= B-C+D-E=5;\\ \beta &= 14-10 = 4,\\ \gamma &= 14-14-4 = -4,\\ \delta &= 6-14+4 = -4,\\ \varepsilon &= 1-6+4 = -1\end{aligned}$$

et hanc ob rem

$$128\sin^2.\varphi\cos^6.\varphi = 5+4\cos.2\varphi-4\cos.4\varphi-4\cos.6\varphi-\cos.8\varphi.$$

Sit denuo in problemate 7

$$S=128\sin^2.\varphi\cos^6.\varphi$$

et

$$\mathrm{A}=5,\quad B=4,\quad C=-4,\quad D=-4,\quad E=-1$$

reperieturque

$$\begin{aligned}\beta &= 10-4 = 6,\\ \gamma &= 4+4-6 = 2,\\ \delta &= -4+4-2 = -2,\\ \varepsilon &= -4+1+2 = -1;\end{aligned}$$

ergo

$$128\sin^3.\varphi\cos^5.\varphi = 6\sin.2\varphi+2\sin.4\varphi-2\sin.6\varphi-\sin.8\varphi.$$

Nunc in problemate 8 sit

$$S=128\sin^3.\varphi\cos^5.\varphi$$

atque

$$B=6,\quad C=2,\quad D=-2,\quad E=-1$$

et capiatur

$$\begin{aligned}\alpha &= B-C+D-E=3\\ \beta &= 6-6 = 0,\\ \gamma &= 2-6-0 = -4,\\ \delta &= -2-2+4 = 0,\\ \varepsilon &= -1+2-0 = 1;\end{aligned}$$

ergo

$$128\sin^4.\varphi\cos^4.\varphi = 3 \quad * \quad -4\cos.4\varphi \quad * \quad +\cos.8\varphi.$$

Sit nunc in problemate 7

$$S=128\sin^4.\varphi\cos^4.\varphi$$

et

$$A=3,\quad B=0,\quad C=-4,\quad D=0,\quad E=1$$

sumaturque

$$\beta = 6-0 \qquad = 6,$$
$$\gamma = 0+4-6 = -2,$$
$$\delta = 4-0+2 = -2,$$
$$\varepsilon = \ 0-1+2 = +1;$$

ergo

$$128\sin^5.\varphi\cos.^3\varphi = 6\ \sin.2\varphi - 2\sin.4\varphi - 2\sin.6\varphi + \sin.8\varphi.$$

Sicque uulterius progrediendo obtinebimus in supplementum § 26 has formulas invertendo :

$$128\cos^8.\varphi \quad = +\cos.8\varphi + 8\cos.6\varphi + 28\cos.4\varphi + 56\cos.2\varphi + 36,$$
$$128\sin.\varphi\cos^7.\varphi \quad = +\ \sin.8\varphi + 6\ \sin.\ 6\varphi + 14\ \sin.4\varphi + 14\ \sin.2\varphi,$$
$$128\sin^2.\varphi\cos^6.\varphi = -\cos.8\varphi - 4\cos.6\varphi - 4\cos.4\varphi + 4\cos.2\varphi + 5,$$
$$128\ \sin^3.\varphi\cos^5.\varphi = -\sin.8\varphi - 2\sin.6\varphi + 2\ \sin.4\varphi + 6\sin.2\varphi,$$
$$128\sin^4.\varphi\cos^4.\varphi = +\cos.8\varphi \qquad * \qquad 4\cos.4p \qquad * \quad + 3,$$
$$128\sin^5.\varphi\cos^3.\varphi = +\ \sin.8\varphi - 2\sin.6\varphi - 2\sin.4\varphi + 6\ \sin.2\varphi,$$
$$128\sin^6.\varphi\cos^2.\varphi = -\cos.8\varphi + 4\cos.6\varphi - 4\cos.4\varphi - 4\cos.2\varphi + 5,$$
$$128\ \sin^7.\varphi\cos.\varphi = -\sin.8\varphi + 6\sin.6\varphi - 14\sin.4\varphi + 14\sin.2\varphi,$$
$$128\ \sin^8.\varphi = +\cos.8\varphi - 8\cos.6\varphi + 28\ \cos.4\varphi - 56\ \cos.2\varphi\ + 35.$$

SCHOLION 2

48. Simili modo pro usu problematum 9 et 10 ostendendo sit

$$256\cos.\varphi^9 = 126\cos.\varphi + 84\cos.3\varphi + 36\cos.5\varphi + 9\cos.7\varphi + \cos.9\varphi.$$

Sitque in problemate 9

$$S = 256\ \cos^9.\varphi$$

atque

$$A = 126, \quad B = 84, \quad C = 36, \quad D = 9, \quad E = \ 1;$$

capiatur

$$\alpha = A - 2B + 2C - 2D + 2E = 14;$$

$$\beta = 126-84-14 = 28,$$
$$\gamma = 84-36-28 \quad = 20,$$
$$\delta = 36-9\ -20 \quad = \quad 7,$$
$$\varepsilon = \quad 9-1\ -\ 7 \quad = \quad 1;$$

ergo

$$256\sin.\varphi\cos^8.\varphi = 14\ \sin.\varphi + 28\ \sin.3\varphi + 20\ \sin.5\varphi + 7\sin.7\varphi + \sin.9\varphi.$$

In problemate 10 sit

$$S = 256\sin.\varphi\cos^{8}.\varphi$$

et

$$A = 14,\quad B = 28,\quad C = 20,\quad D = 7,\quad E = 1;$$

capiatur

$$\alpha = 14;$$

ergo

$$\begin{aligned}\beta &= 28 - 14 - 14 = \quad 0,\\ \gamma &= 20 - 28 - \quad 0 = -8,\\ \delta &= \ 7 - 20 + \quad 8 = -5,\\ \varepsilon &= \quad 1 - \ 7 + \quad 5 = -1;\end{aligned}$$

$$256\sin^{2}.\varphi\cos^{7}.\varphi = 14\cos.\varphi \quad * \quad - 8\cos.5\varphi - 5\cos.7\varphi - \cos.9\varphi.$$

In problemate 9 sit

$$S = 256\sin^{2}.\varphi\cos^{7}.\varphi$$

et

$$A = 14,\quad B = 0,\quad C = -8,\quad D = -5,\quad E = -1;$$

capiatur

$$\alpha = 14 - 0 - 16 + 10 - 2 = +\ 6;$$

ergo

$$\begin{aligned}\beta &= 14 - 0 - 6 = +\ 8,\\ \gamma &= \quad 0 + 8 - 8 = \quad 0,\\ \delta &= -8 + 5 - 0 = \ -3,\\ \varepsilon &= -5 + 1 + 3 = \ -1;\end{aligned}$$

$$256\sin^{3}.\varphi\cos^{6}.\varphi = 6\sin.\varphi + 8\sin.3\varphi \quad * \quad - 3\sin.7\varphi - \sin.9\varphi.$$

In problemate 10 sit

$$S = 256\sin^{3}.\varphi^{6}\cos.\varphi$$

et

$$A = 6,\quad B = 8,\quad C = 0,\quad D = -\ 3,\quad E = -1\ ;$$

fiet

$$\alpha = 6;$$

$$\beta = \quad 8-6-6=-4,$$
$$\gamma = \quad 0-8+4=-4,$$
$$\delta = -3-0+4=+1,$$
$$\varepsilon = -1+3-1=+1;$$

ergo

$$256\sin^4.\varphi\cos^5.\varphi = 6\cos.\varphi - 4\cos.3\varphi - 4\cos.5\varphi + \cos.7\varphi + \cos.9\varphi,$$

unde in supplementum § 26 habebitur

$$\begin{aligned}
256\cos^9.\varphi &= +\cos.9\varphi + 9\cos.7\varphi + 36\cos.5\varphi + 84\cos.3\varphi + 126\cos.\varphi,\\
256\sin.\varphi\cos^8.\varphi &= +\sin.9\varphi + 7\sin.7\varphi + 20\sin.5\varphi + 28\sin.3\varphi + 14\sin.\varphi,\\
256\sin^2.\varphi\cos^7.\varphi &= -\cos.9\varphi - 5\cos.7\varphi - 8\cos.5\varphi \quad * \quad + 14\cos.\varphi,\\
256\sin^3.\varphi\cos^6.\varphi &= -\sin.9\varphi - 3\sin.7\varphi \quad * \quad + 8\sin.3\varphi + 6\sin.\varphi,\\
256\sin^4.\varphi\cos^5.\varphi &= +\cos.9\varphi + \cos.7\varphi - 4\cos.5\varphi - 4\cos.3\varphi + 6\cos.\varphi,\\
256\sin^5.\varphi\cos^4.\varphi &= +\sin.9\varphi - \sin.7\varphi - 4\sin.5\varphi + 4\sin.3\varphi + 6\sin.\varphi,\\
256\sin^6.\varphi\cos^3.\varphi &= -\cos.9\varphi + 3\cos.7\varphi \quad * \quad - 8\cos.3\varphi + 6\cos.\varphi,\\
256\sin^7.\varphi\cos^2.\varphi &= -\sin.9\varphi + 5\sin.7\varphi - 8\sin.5\varphi \quad * \quad + 14\sin.\varphi,\\
256\sin^8.\varphi\cos.\varphi &= +\cos.9\varphi - 7\cos.7\varphi + 20\cos.5\varphi - 28\cos.3\varphi + 14\cos.\varphi,\\
256\sin^9.\varphi &= +\sin.9\varphi - 9\sin.7\varphi + 36\sin.5\varphi - 84\sin.3\varphi + 126\sin.\varphi.
\end{aligned}$$

Hoc igitur modo istas formulas, quousque libuerit, continuare licet.

THEOREMA

49. *Si assignari queat summa huius seriei*

$$Az^m + Bz^{m+n} + Cz^{m+2n} + Dz^{m+3n} + Ez^{m+4n} + \text{etc.} = Z,$$

semper quoque exhiberi poterunt summae harum serierum

$$A\cos.m\varphi + B\cos.(m+n)\varphi + a\cos.(m+2n)\varphi + D\cos.(m+3n)\varphi + \text{etc.},$$
$$A\sin.m\varphi + B\sin.(m+n)\varphi + a\sin.(m+2n)\varphi + D\sin.(m+3n)\varphi + \text{etc.}$$

DEMONSTRATIO

Ponantur summae harum serierum

$$A\cos.m\varphi + B\cos.(m+n)\varphi + a\cos.(m+2n)\varphi + D\cos.(m+3n)\varphi + \text{etc.} = S,$$
$$A\sin.m\varphi + B\sin.(m+n)\varphi + a\sin.(m+2n)\varphi + D\sin.(m+3n)\varphi + \text{etc.} = T$$

sitque ut supra

$$\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi=u \text{ et } \cos.\varphi-\sqrt{-1}\cdot\sin.\varphi=\upsilon;$$

erit

$$\cos.\nu\varphi+\sqrt{-1}\cdot\sin.\nu\varphi=u^{\nu} \text{ et } \cos.\nu\varphi-\sqrt{-1}\cdot\sin.\nu\varphi=\upsilon^{\nu}.$$

Hinc ergo erit

$$S+T\sqrt{-1}=Au^{m}+Bu^{m+n}+Cu^{m+2n}+Du^{m+3n}+\text{etc.}=U,$$
$$S-T\sqrt{-1}=A\upsilon^{m}+B\upsilon^{m+n}+C\upsilon^{m+2n}+D\upsilon^{m+3n}+\text{etc.}=V.$$

Summae scilicet harum serierum U et V per hypothesin dantur, cum U et V tales sint functiones ipsarum u et υ, qualis functio Z est ipsius z. Hinc itaque elicitur

$$S=\frac{U+V}{2} \quad \text{et} \quad S=\frac{U-V}{2\sqrt{-1}}$$

ideoque summae proposita serierum S et T innotescunt. Q. E. D.

COROLLARIUM 1

50. Cum sit

$$z^{m}+az^{m+n}+a^{2}z^{m+2n}+a^{3}z^{m+3n}+\text{ etc.}=\frac{z^{m}}{1-az^{n}},$$

erit

$$U=\frac{u^{m}}{1-au^{n}} \quad \text{et} \quad V=\frac{\upsilon^{m}}{1-a\upsilon^{n}},$$

hinc

$$U+V=\frac{u^{m}+\upsilon^{m}-a\left(u^{m-n}+\upsilon^{m-n}\right)u^{n}\upsilon^{n}}{1-a\left(u^{n}+\upsilon^{m}\right)+aau^{n}\upsilon^{n}},$$

$$U-V=\frac{u^{m}-\upsilon^{m}-a\left(u^{m-n}-\upsilon^{m-n}\right)u^{n}\upsilon^{n}}{1-a\left(u^{n}+\upsilon^{m}\right)+aau^{n}\upsilon^{n}}.$$

At est

$$uv=1,\quad u^{\nu}+\upsilon^{\nu}=2\cos.\nu\varphi,\quad u^{\nu}-\upsilon^{\nu}=2\sqrt{-1}\sin.\nu\varphi,$$

unde fit

$$\frac{U+V}{2}=\frac{\cos.m\varphi-a\cos.(m-n)\varphi}{1+aa-2a\cos.n\varphi}=S$$

et

$$\frac{U-V}{2\sqrt{-1}}=\frac{\sin.m\varphi-a\sin.(m-n)\varphi}{1+aa-2a\cos.n\varphi}=T.$$

Ex quo sequentes habentur summationes:

$$\cos.m\varphi + a\cos.(m+n)\varphi + a^2\cos.(m+2n)\varphi + a^3\cos.(m+3n)\varphi + \text{etc.}$$
$$= \tfrac{\cos.m\varphi - a\cos.(m-n)\varphi}{1+aa-2a\cos.n\varphi},$$
$$\sin.m\varphi + a\sin.(m+n)\varphi + a^2\sin.(m+2n)\varphi + a^3\sin.(m+3n)\varphi + \text{etc.}$$
$$= \tfrac{\sin.m\varphi - a\sin.(m-n)\varphi}{1+aa-2a\cos.n\varphi},$$

COROLLARIUM 2

51. Sit $m=1$ et $n=1$; erit

$$\cos.\varphi + a\cos.2\varphi + a^2\cos.3\varphi + a^3\cos.4\varphi + \text{etc.}$$
$$= \tfrac{\cos.\varphi - a}{1+aa-2a\cos.\varphi},$$
$$\sin.\varphi + a\sin.2\varphi + a^2\sin.3\varphi + a^3\sin.4\varphi + \text{etc.}$$
$$= \tfrac{\sin.m\varphi - a\sin.(m-n)\varphi}{1+aa-2a\cos.\varphi}.$$

Si insuper sit $a=1$, erit

$$\cos.\varphi + \cos.2\varphi + \cos.3\varphi + \cos.4\varphi + \text{etc.}$$
$$= \tfrac{\cos.\varphi - 1}{2-2a\cos.\varphi} = -\tfrac{1}{2},$$
$$\sin.\varphi + \sin.2\varphi + \sin.3\varphi + \sin.4\varphi + \text{etc.}$$
$$= \tfrac{\sin.m\varphi}{2-2\cos\varphi} = \tfrac{1}{2\,\text{tang}\frac{1}{2}\varphi}.$$

Sin autem sit $a=-1$, erit

$$\cos.\varphi - \cos.2\varphi + \cos.3\varphi - \cos.4\varphi + \text{etc.}$$
$$= \tfrac{\cos.\varphi + 1}{2+2\cos.\varphi} = \tfrac{1}{2},$$
$$\sin.\varphi - \sin.2\varphi + \sin.3\varphi - \sin.4\varphi + \text{etc.}$$
$$= \tfrac{\sin.\varphi}{2+2\cos\varphi} = \tfrac{1}{2}\,\text{tang}\,\tfrac{1}{2}\varphi.$$

COROLLARIUM 3

52. Sit $m=1$ et $n=2$; erit

$$\cos.\varphi + a\cos.3\varphi + a^2\cos.5\varphi + a^3\cos.7\varphi + \text{etc.} = \tfrac{\cos.\varphi - a\cos.\varphi}{1+aa-2a\cos.2\varphi},$$

$$\sin.\varphi + a\sin.3\varphi + a^2\sin.5\varphi + a^3\sin.7\varphi + \text{etc.} = \tfrac{\sin.\varphi + a\sin.\varphi}{1+aa-2a\cos.2\varphi}.$$

Quodsi ergo sit $a=1$, erit

$$\cos.\varphi + \cos.3\varphi + \cos.5\varphi + \cos.7\varphi + \text{etc.} = 0,$$

$$\sin.\varphi+\sin.3\varphi+\sin.5\varphi+\sin.7\varphi+\text{etc.}=\frac{\sin.\varphi}{1-\cos.2\varphi}=\frac{1}{2\sin.\varphi}.$$

Sin autem sit $a=-1$, erit

$$\cos.\varphi-\cos.3\varphi+\cos.5\varphi-\cos.7\varphi+\text{etc.}=\frac{2\cos.\varphi}{2+2\cos.2\varphi}=\frac{1}{2\cos.2\varphi},$$
$$\sin.\varphi-\sin.3\varphi+\sin.5\varphi-\sin.7\varphi+\text{etc.}=0.$$

SCHOLION

53. Ope huius theorematis ergo, cuius usus latissime patet, innumerabiles exhiberi possunt series secundum vel sinus vel cosinus multiplorum cuiuspiam anguli progredientes, quarum summa constat. Casum hic quidem tantum evolvi, quo coefficientes *A, B, C, D* etc. in geometrica progressione progrediuntur, verum pari modo calculus ad alias series accommodatur. Praeterea autem hic notasse sufficiat ex seriebus iam inventis innumerabiles alias tam per differentiationem quam integrationem elici posse. Veluti, cum sit

$$\cos.\varphi-\cos.2\varphi+\cos.3\varphi-\cos.4\varphi+\cos.5\varphi-\text{etc.}=\tfrac{1}{2},$$

erit differentiando

$$\sin.\varphi-2\sin.2\varphi+3\sin.3\varphi-4\sin.4\varphi+5\sin.5\varphi-\text{etc.}=0$$

denuoque differentiando

$$\cos.\varphi-4\cos.2\varphi+9\cos.3\varphi-16\cos.4\varphi+25\cos.5\varphi-\text{etc.}=0,$$

et ita porro.

Illa autem series per $d\varphi$ multiplicata et integrata dat

$$\sin.\varphi-\tfrac{1}{2}\sin.2\varphi+\tfrac{1}{3}\sin.3\varphi-\tfrac{1}{4}\sin.4\varphi+\tfrac{1}{5}\sin.5\varphi-\text{etc.}=\tfrac{\varphi}{2}$$

ubi additione constantis non est opus, cum posito $\varphi=0$ summa sponte evanescat. Si haec per $-d\varphi$ multiplicata denuo integretur, prodibit

$$\cos.\varphi-\tfrac{1}{4}\cos.2\varphi+\tfrac{1}{9}\cos.3\varphi-\tfrac{1}{16}\cos.4\varphi+\text{etc.}=\alpha-\tfrac{\varphi\varphi}{4}$$

ideoque posito $\varphi=0$

$$1-\tfrac{1}{4}+\tfrac{1}{9}-\tfrac{1}{16}+\text{etc.}=\alpha=\tfrac{\pi\pi}{12},$$

ut aliunde constat.

Quare si sit

$$\varphi=\frac{\pi}{\sqrt{3}}=103^{0}55^{\mathrm{I}}22^{\mathrm{II}}58^{\mathrm{III}}28^{\mathrm{IV}},$$

summa istius serei evanescit. Plurimas alias autem insignes huiusmodi serierum affectiones, ne nimis sim longus, hic praetermitto.