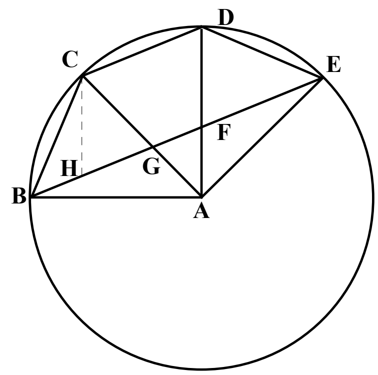
§3.1 Chapter Three

Concerning [a General Method of] Triplication

And these are the particular theorems to be had from the writings of *Ptolemy* and the Ancient Mathematicians. Let us now move on to these which by the favour of God have been found by us a few years ago, from which the whole *Table of Subtended Chords* can be constructed with great ease and accuracy.



[Figure 3 - 1]

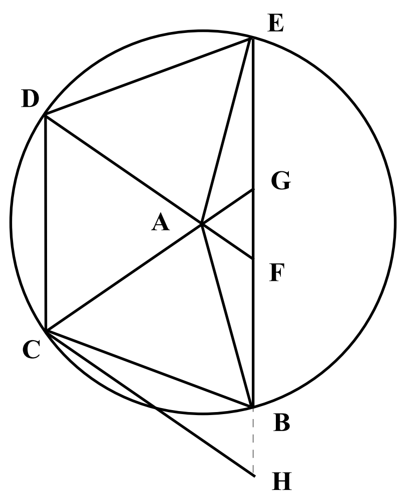
To Trisect a Given Arc.

If the straight line AC drawn through the centre cutting the base of the section EB [the given minor arc] in the point G and the arc in the point C, at equal distances from the vertex B; then this line cuts the arc length in the duplicate ratio; and the arc [or angle EDC] is twice the arc or angle CB[1]. [Figure 3 - 1.]

For if BC, BG are equal then the angles BCG, BGC, CBA are equal, *by Prop. 4, Book 1*; and CBG, CAB also are equal, *by Prop. 32, Book 1*: and therefore CDE, the base of the angle CBE in the Arc, is twice the Arc CB, the base of the Angle CAB in the Centre, *by Prop.20, Book 3.*

FIRST CONCLUSION.

If the Line CH is drawn, [Figure 3-2, for the major arc case], parallel to the line DA, then the triangles ABC, BCG, GCH are similar: and the lines AB, BC, CG, GH are continued proportionals. And if the radius AB is one ; BC is the side; CG is the square [of the side]; and GH is the cube [of the side]. We thus know the second and the third continued proportionals for the square and the cube from unity: indeed from the radius AB, which we put as one.



[Figure 3-2]

SECOND CONCLUSION.

The chord BE (composed of the three equal subtending chords BC, CD, DE) inscribed together with the cube GH is equal to these three Subtended Chords taken together: for ED, EF; BC, BG; and DC, FH are equal from the construction[2], and from *Prop. 33, Book 1.*

THIRD CONCLUSION.

Given any subtended chord BC whatsoever ; one can find the subtended chord BE of the triple arc, and conversely.

The square and the cube of a given subtended chord are sought. The cube taken from the triple of the subtended chord leaves the subtended chord of three times the arc. For let these be[3]:

Radius of the Circle…………………..	10000000000	*Continued Proportions*
Subtended Chord of 16 degrees…….	02783462019	
Square of the same Subtended Chord	0077476608112	
Cube……………………………………	0021565319604	
Subtended Chord of 16:0': tripled……	08350386057	
Cube of Subtended Chord……………	00215653196	
Subtended Chord of 48:0':……………	08134732861	
Radius…………………………………	100000	*Continued Proportions*
Subtended 74:0': ……………………..	1203630046304	
Square…………………………………	14487252883658	
Cube……………………………………	17437292859173	
Subtended Chord of 74:0': tripled……	3610890138912	
Subtended Chord of 222:0':	18671608529947	
Radius…………………………………	10000000000	*Continued Proportions*
Subtended Chord of 2:26':………..	004246648681	
Square…………………………………	0001803402502	
Cube……………………………………	00000765841685	
Subtended Chord of 2:26' tripled……	012739946043	
Subtended Chord of 7:18':	012732287626	

[Table 3-1]

With these chords be exceedingly careful, lest the [triple of the] subtended chord or the cube should be taken from the supplementary arc [i.e. the signs have reversed in the defining equation at the start of the chapter], and not from its own arc; of course it is permitted that the numbers keep the same digits [*nota*], of these however the value [of the resultant chord] for the different arc is changed the most, and the result by the subtraction agrees with the remaining subtended chord, if the position of the cube truly is correct: otherwise all will become frustration[4].

Radius…………………………………	10000	*Continued Proportions*
Subtended Chord of 117:34':………..	171042707205	
Square…………………………………	292556076878	
Cube……………………………………	500395833987	
Subtended Chord of117:34': tripled……	513128121615	
Subtended Chord of 352:42':	012732287626	
7:18':		

[Table 3-2]

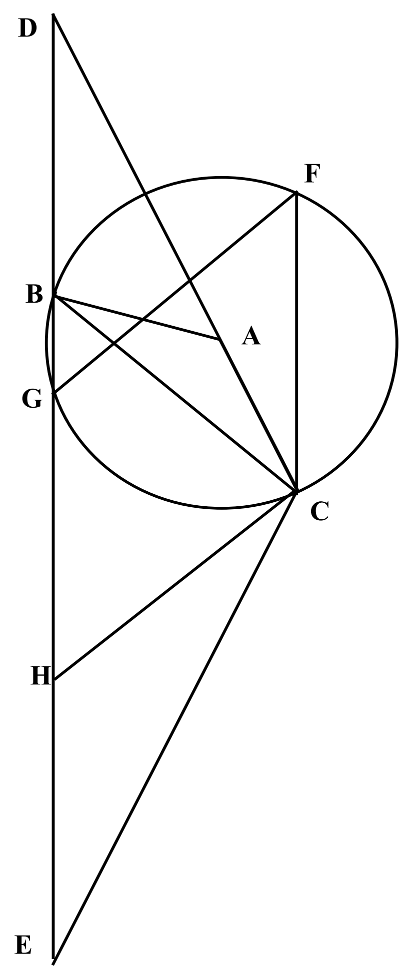
For if the arc of which the subtended chord is given, the triplicate is greater than the whole circle, the triplicate of the given subtended chord is taken from the cube of the same; the remainder is the departure of the subtended chord of the given arc tripled over the whole circle, for:

Subtended Arc 147:0'	1917639469736386	
The triple Arc 441:0' exceeds the	*whole circle by 81:0*	
Radius…………………………………	1000000000000000	*Continued Proportions*
Subtended Chord of 147:0'………	1917639469736386	
Square………………………………	3677341135890848	
Cube…………………………………	7051814505869524	
Subtended Chord of147:0': tripled	5752918409209158	
Subtended Chord of 81:0':	1298896096660366	

[Table 3-3]

This is shown as follows. Let BC, CF, FG be inscribed equal; BG, CF are parallel: and BG continued in either direction makes BC, BD; CF, GH; CH, HE equal: ABC, BCD, CDE are similar triangles; and AB, BC, CD, DE, continued proportionals [in the continued proportion $1 : p : p^2 : p^3$]; and by subtraction, DB,GH, HE, (of which each is equal to the given subtended chord BC) leaves BG, the subtended chord the departure above of the whole circle.

And according to this method which we can call *Triplationem* [a general method of Triplication], the lines subtended for any Arc whatsoever are found most easily, if the Subtended Chord of a third of the Arc is given.



[Figure 3 - 3]

§3.2 Notes on Chapter Three.

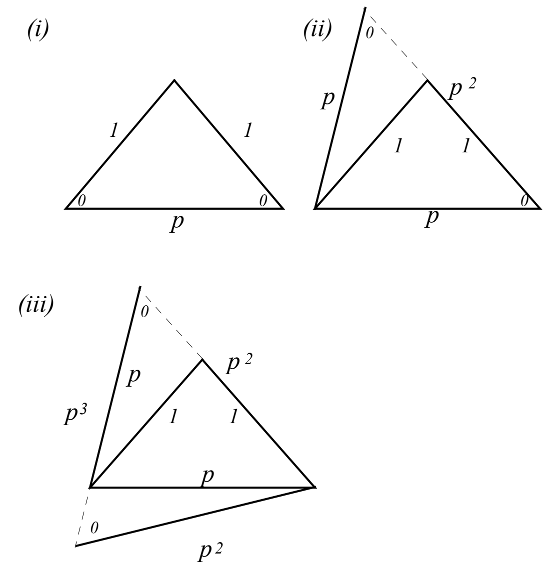


Figure 3-4

[1] For AB = 1, and BC = BG = p; then from the similar triangles ABC, BCG, and CGH in Figures 3-1:
$p/1 = CG/p = GH/CG$; hence, $CG = p^2 = CH$, and $GH = p. CG = p^3$. Thus, the three triangles have sides: $(1, 1, p)$; (p, p, p^2); (p^2, p^2, p^3). It follows that the chord for the triple arc BE has length BG + EF + GF = $2p + p - p^3$; hence, $BE = 3p - p^3$.

We may isolate the method used by Briggs' of constructing nested similar isosceles triangles, as in Figure 3-4 (i), (ii), and (iii); the construction can obviously be extended indefinitely.

Note: Although the sub-heading is for trisection, or cutting into three, the theorem is concerned with the triplication of a chord of length *p*, to find the length of the chord that corresponds to triple the original arc. Obviously, if the angle exceeds 360^0, the defining equation has to be modified, the cause of Brigg's concern in this chapter.

Briggs has distinguished three cases: the first where the triple arc is less than 180^0, or the initial arc $0 < \theta < 60^0$, as in Figure 3-1; the second where the triple arc lies between 180^0 and 360^0, Figure 3-2, corresponding to $60^0 < \theta < 120^0$; and the third where the triple arc is greater than 360^0, as in Figure 3-3, and $\theta > 120^0$.

[2] The analysis of Figure 3-2 is similar to [1]: BE + GH = (EF + FB) + GF + FH = ED + (FB + GF) + CD = ED + BG + CD, as required. Or, $BE + p^3 = 3p$, where BE is the subtended chord for the triple arc.

We recognise this equation to be a form of the sine triplication identity:
$2\sin(3\theta/2) + (2\sin(\theta/2))^3 = 3.(2\sin(\theta/2))$; where we identify $BE = 2\sin(3\theta/2)$, and $p = 2\sin(\theta/2)$. Examples of this are shown in Table 3-1. Briggs has chosen an angle of 117:34' in Table 3-2 to show that the initial scheme still works, as $\theta < 120^0$. However, if $\theta > 120$ In the following Table 3-4, the angle chosen is 147^0, which gives a triple arc of 441^0, or 81^0 beyond the whole circle.

[3] Subtended chord = $2R\sin(\theta/2) = 2\times10^{10}\times\sin8^0$ =2783462019, etc

[4] For $2\sin(3\theta/2) = 3.(2\sin(\theta/2)) - (2\sin(\theta/2))^3 = 3p - p^3 = p(3 - p^2) < 0$ if $p^2 > 3$, or $\theta/2 > 60^0$. In which case the signs are reversed on the right hand side.

From Figure 3-3, according to Note 1: AB, BC, CD, DE, are the continued proportionals $1 : p : p^2 : p^3$. From the construction, $DE = p^3 = (BD + GH + HE) + GB = 3p + GB$: hence $GB = p^3 - 3p$, as required.