A METHOD OF DETERMINING THE QUADRATURES OF FIGURES WITH RIGHT AND CURVED LINES

Author John Craig.

LONDON:

Sold by *Moses Pitt*, at the sign of the Angel in the cemetery of St. Paul's, MDCLXXXV.

THIS TREATISE IS DEDICATED AND

CONSECRATED

Therefore,

TO THE HONORABLE

MASTER

ROBERT DAWES,

ENGLISH BARONET,

for his Benevolence and Observance.

JOHN CRAIG.

Method of Figures, &c.

Recently the best geometers have observed there are certain figures capable of indefinite quadrature: which are quadrable, both with respect to the whole as well as to individual parts of figures [*i.e.* integrable or squarable for any part thereof] ; Truly there are others which do not permit an indefinite quadrature of this kind, yet some have a quadrable part, and indeed sometimes others with the whole figure able to be squared, when any part of it cannot be squared. Yet it is possible that an error [in judgment] may have arisen from another source, for those who had considered the quadrature of the circle, hyperbola, and certain other figures to be impossible to be evaluated [by geometrical means]: because they had not considered this class of figures. [For at this time curves were either algebraic, represented by finite equations in two variables up to some power, such as 2, 3, 4, etc., following *Descarte*s, while other curves, such as the cycloid, were geometric or mechanical in nature and could not be squared, at least according to *Descartes*.] For by using the methods which supposed figures to be indefinitely quadrable, since they might have been assumed largely to be quadrable, but with which methods [employed at the time] of squaring rejected, at once they were believed to be impossible of undergoing that squaring; thence accordingly the squaring could not be completed, as the methods used were imperfect, and did not extend to all figures.

But when my method is put in place, errors of these other kinds shall not discovered, apart from a few that I have considered to have explained; Here I will treat a method (not deduced from arithmetical but from geometrical principles) which will determine the quadratures of all kinds of figures. Geometrical quadratures of the first kind will be shown, truly algebraic quadratures of the latter kind will be shown by infinite series. And because which special methods may determine the quadrature of such figures, not hitherto published by anyone, we may hope that the outstanding German [*i.e.* E. W. Tschirnhaus] (who has promised publicly, and generally it may be asserted to be in his power, in a publication of the *Acta Eruditorum Lipsae*) will shortly be publishing his ideas.

Theorem 1.

VH shall be a certain curve, (the axis of which is VD, the applied line HD perpendicular to VD) likewise VZS shall be such a line, that if from a point of the curve VH freely assumed, for example E, the right line EP may be drawn to the curve, & EAZ perpendicular to the axes, the right line AZ shall be equal to the intercept AP, the area $\mathrm{VDS} = \frac{\mathrm{DHq}}{2} \left[= \frac{\mathrm{DH}^2}{2} \right]$.

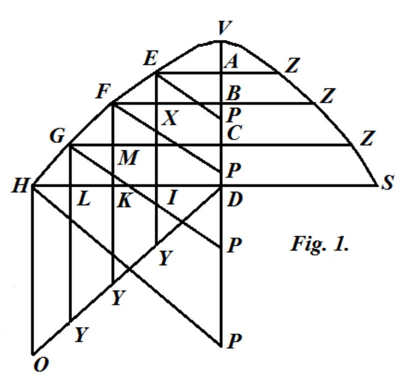


Fig. 1.

[Note: If we let the vertical axis be the x-axis, while

the right-hand curve VZS is $f(x)$ while the left-hand curve GEFGH is $g(x)$, for appropriately defined functions, then treating the horizontal lines as separated by the differentials dx, then *e.g.* from triangle EAP we have
$\frac{dg}{dx} = \frac{AP}{g} = \frac{f}{g} \therefore \int gdg = \int fdx$ and $\frac{1}{2} g^2 = \int fdx = area$ VDS.]

Demonstration :

HDO shall be a semi-right angle, and VD shall be cut equally at the indefinite points A, B, C through which may be drawn EAZ, FBZ, GCZ parallel to HDl, and crossing the curve at E, F, G from which EIY, FKY, GLY, shall be drawn parallel to VD, and indeed the right lines EP, FP, GP, HP shall be perpendicular to the curve VH . The triangle HLG is similar to the triangle PDH (for on account of the indefinite section the curvelet is able to take GH as a right line) whereby there is, following Barrow's notation :

$\mathrm{HL} \cdot \mathrm{LG} :: \mathrm{PD} \cdot \mathrm{DH}$ [Note the manner of writing $\mathrm{HL} : \mathrm{LG} :: \mathrm{PD} : \mathrm{DH}$ or $\frac{\mathrm{HL}}{\mathrm{LG}} = \frac{\mathrm{PD}}{\mathrm{DH}}$],

and thus $\mathrm{HL} \times \mathrm{DH} = \mathrm{LG} \times \mathrm{PD}$, that is, $\mathrm{HL} \times \mathrm{HO} = \mathrm{DC} \times \mathrm{DS}$, and by a similar discussion it will be shown, because triangle GMF shall be similar to triangle PCG, there becomes $\mathrm{LK} \times \mathrm{LY} = \mathrm{CB} \times \mathrm{CZ}$ and similarly $\mathrm{KI} \times \mathrm{KY} = \mathrm{BA} \times \mathrm{BZ}$, and likewise [finally] there will be $\mathrm{ID} \times \mathrm{IY} = \mathrm{AV} \times \mathrm{AZ}$; from which it is agreed the triangle HDO (because it differs minimally from these rectangles $\mathrm{HL} \times \mathrm{HO} + \mathrm{LK} \times \mathrm{LY} + \mathrm{KI} \times \mathrm{KY} + \mathrm{ID} \times \mathrm{IY}$) is equal to the area VDS (because likewise it differs minimally from the rectangles :
$\mathrm{DC} \times \mathrm{DS} + \mathrm{CB} \times \mathrm{CZ} + \mathrm{BA} \times \mathrm{BZ} + \mathrm{AV} \times \mathrm{AZ}$), that is: $\mathrm{VDS} = \frac{\mathrm{DHq}}{2}$. Q. E. D.

[Thus, the sum of all the infinitesimal strips making up the area of the triangle HOD is equal to the area bounded by the curve VZS and the lines VD and DS.]

This noble theorem is due to the most celebrated *Dr. Barrow,* who has innumerable sublime theorems about the properties of curved lines : nor by me to have seen any (of which the writing have been published) who touched on the subject with so much judgment (nor indeed I think to have been touched on by others), and with such a great success, to have treated and promoted this more abstruse and less cultivated part of geometry.

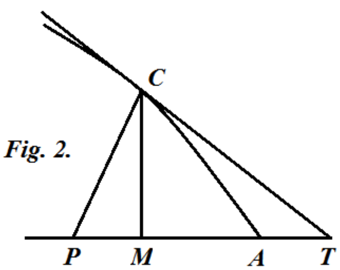
[J. M. Child has some illuminating comments to make on this theorem by Barrow, though he seems to be unaware of Craig's work, as indeed Leibniz appears to be of Barrow at this stage ; see notes on p.24-25 of *The Early Manuscripts of Leibniz*.]

PROBLEM I.

From a given relation between PM (*which designates the distance between the perpendicular of the curve PC and the applied ordinate MC* [i.e. the sub-normal]) *and the abscissa AM* (*which designates the distance between the applied line and the vertex A*) *to find the nature of the equation defining curved line AC.*

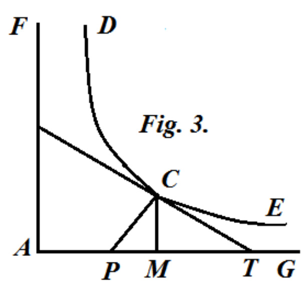
So that I may deal with all curves under one general rule, I designate on any curved line AC to be always $\mathrm{PM} \times \mathrm{MT} = \mathrm{CM}^2$, on account of the right angle PCT. Whereby I multiply the individual terms specifying PM by the term AM (the former multiplied by diverse unknown numbers) and I put the product equal to the square of the applied line CM.
[Note: The sub-normal CM is the geometric mean of PM and MT. This particular way of finding the tangent at a point on a curve was investigated by Van Heuraet and Hudde. See e.g., *A History of Mathematics*, V.J. Katz, Ch. 12]
An account of this rule can be gathered from the method of finding tangents published by the most illustrious *de Sluse* in the *Transactions of the Royal Society of England* (1672.)
I illustrate the rule by examples.
*Ex.*1: *Fig.* 2. Given [the sub-normal] $\mathrm{PM} = \frac{1}{2}r$, and calling AM *y*, CM *x*; a, b, c, i, &c. denote known and determined quantities, likewise *l*, *m*, *n*, *h*, *k* &c. denote unknown numbers.
Now I multiply $\frac{1}{2}$r by *ny* by the nearby rule, & the product $\frac{nry}{2} = x^2$,which is the equation for a parabola.
[Note that here the dependent and independent variables are *x* and *y* respectively; here *n* is a constant yet to be determined.]

Fig. 2.



Ex. 2: Fig.2. There shall be $\mathrm{PM} = y + \frac{1}{2}r$, and the equation determining that curve shall be sought : proceeding I multiply $\frac{1}{2}r + y$ by *ny, my* following the rule, and I put the product $\frac{nry}{2} + my^2$ equal to the square from *x*, truly $\frac{nry}{2} + my^2 = x^2$, which is the equation for the curve sought.
Ex. 3. Let there be $\mathrm{PM} = \frac{y^2}{a} + a$ and the curve AC is sought, in which there shall be $\mathrm{PM} = \frac{y^2}{a} + a$. I multiply $\frac{y^2}{a} + a$ by *ny*, *my*, and the product will be $\frac{ny^3}{a} + may = x^2$.

Fig. 3.



Ex.4. There shall be $\mathrm{PM} = \frac{y^4}{aaa} + \frac{y^3}{aa} + \frac{y^2}{a} + y,$ and the equation shall be sought defining the nature of this curve, following the rule I multiply $\frac{y^4}{aaa}+\frac{y^3}{aa}+\frac{y^2}{a}+y$ by *ny, my, ly, hy*, and there will be $\frac{ny^5}{a^3}+\frac{my^4}{a^2}+\frac{ly^3}{a}+hy^2=x^2$, which is the equation sought.

Finally there shall be the case $\mathrm{PM}=\frac{a^3}{y^2}$, I multiply $\frac{a^3}{y^2}$ per *ny*, and the product will be $\frac{na^3}{y}=x^2$, or $na^3=yx^2$.
but because *n, m, l, h;* &c. hitherto shall be unknown, I show a way of determining those.

PROBLEM II.

To determine the quantities l, m, n, &c. used in the preceding problem.

I find [the sub-normal] PM through the equation found (by proceeding along some common method of finding tangents) and compare its value with the value given, clearly a single one of those which a comparison will determine from the individual terms *l, m, n,* &c.

As in the first example $\frac{nry}{2}=x^2\left[=CM^2\right]$, I find $\mathrm{PM}=\frac{3}{4}nr$, hence I compare the value with the value given, thus $\frac{1}{2}r=\frac{1}{4}nr$ from which after reduction, $n=2$, whereby with this value substituted for (n) in the equation $\frac{nxy}{2}=x^2$ it becomes $ry=x^2$.

[For the gradient is $\frac{nr}{4x}=\frac{PM}{CM}=\frac{PM}{x}\therefore PM=\frac{nr}{4}\equiv\frac{r}{2}\therefore n=2.$]

Thus in the second example, $\frac{nry}{a}+my^2=x^2$ I find PM to be $=\frac{nx}{4}+my$, now I compare these terms each with the corresponding values of the terms given, thus $\frac{nr}{4}=\frac{r}{2}$ from which $n=2,$ secondly $my=y,$ from which $m=1$; if these values may be substituted the equation sought will be fully determined $ry+y^2=x^2$.

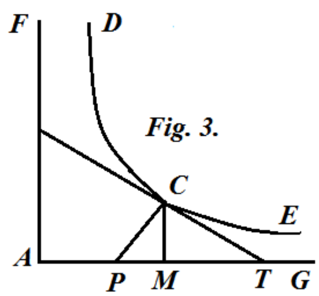


Fig. 3.

Similarly in example 3 : $\frac{ny^3}{a}+may=x^2$ I have found $\mathrm{PM}=\frac{3ny^3}{2a}+\frac{ma}{2}$, therefore with a comparison made thus:

$\frac{3ny^2}{2}+\frac{y^2}{a}$ there will be $n=\frac{2}{3}$,

& from $\frac{ma}{2}=a,$ there will be $m=2$, and with these values substituted there will be $\frac{2y^3}{3}+2ay=x^2$.

And in *ex*. 4 there will be $\mathrm{PM}=\frac{5ny^4}{2a^3}+\frac{2my^3}{a^2}+\frac{2ly^2}{2}+by.$

and from a comparison made of these terms with the given terms, there will be :

$\frac{5ny^4}{2a^3}=\frac{y^4}{a^3}$, thence $n=\frac{2}{5}$, from $\frac{2my^3}{a^2}=\frac{y^3}{2a}$ there will be $m=\frac{1}{2}$,
and from $\frac{3ly^2}{2a}=\frac{y^2}{a}$ there will be $l=\frac{2}{3}$,
and from $hy=y$ there will be $h=1$. And by substituting these values, the equation will be

$$\frac{2ny^5}{5a^3}+\frac{y^4}{2a^2}+\frac{2y^3}{3}+y^3=x.$$

Finally in *Ex*.5. there will be $\mathrm{PM}=\frac{ny^3}{2y^2}\;=\frac{a^3}{y^2}$ from which $n=2$, [*Fig*. 3]

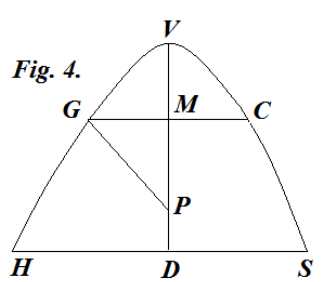
and thus $2a^3=yx^2$, which has the form of a hyperbola DCE.
I have pursued these two problems in more detail (or rather the two parts of the one problem), because hitherto they shall not have been treated there by anyone, at any rate the pages of which have come to hand; then especially, because with the aid of these I may be able to begin determining the quadratures of figures.

PROBLEM III.

To determine the quadrature of the parabola.

Let *VCS* be the parabola whose latus rectum [*Fig*. 4] shall be *r*. VM may be called *y*, MC *z*.
from which the nature of the parabola $\sqrt{ry}=z=\mathrm{MC}$, then by the first problem the curve VH may be found, such that $\mathrm{PM}=\sqrt{ry}$ (by PG here and in the following it is required to understand the perpendicular of the curve sought) but by the method now treated I have found the curve sought to be

Fig. 4.



defined by this equation $nry^3=x^4$ (by *x* I designate the applied lines GM, HD of the curve sought) and on determining *n* by Prob. 2, you find $n=\frac{16}{9}$ from which $\frac{16}{9}ry^3=x^4$, and therefore $\sqrt{\frac{4}{9}ry^3} = \frac{x^2}{2} = \frac{GM^2}{2} = VMC$, so that it may agree with the theorem now presented.

PROBLEM IV.

To determine the quadrature of the Cubic Parabola.

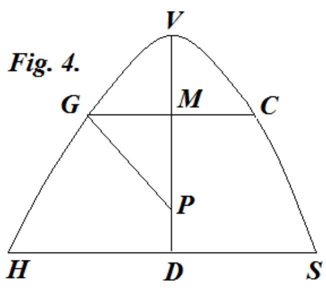
VCS shall be a cubic parabola [*Fig.* 4], VD the axis and the latus rectum *r*, and $\mathrm{VM} = y$,
from which the nature of the curve itself will be $rry = z^3$ and thus $\sqrt[3]{rry} = z$, therefore for the determination of the area of the curve VMC it is required to find the curve (by Prob. 1.) VH such that always there shall be [the sub-normal] $\mathrm{PM} = \sqrt[3]{rry}$, & proceeding following the rule proposed there I find the curve VH to be defined by this equation $nr^2y^4 = x^6$ and by determining *n* (by Prob. 2.) finding $n = \frac{27}{8}$, and therefore the equation sought to be $\frac{27}{8}r^2y^4 = x^6$ from which $\frac{3}{4}\sqrt[3]{r^2y^4} = \frac{x^2}{2} = \frac{\mathrm{GM}^2}{2} = \mathrm{VMC}.$ And in this way the quadratures of an infinitude of parabolas which are defined by $r^3y = z^4;\ r^4y = z^5,\ r^5y = z^6,$ &c.

PROBLEM V.

To find the Quadrature of the Semi-cubic Parabola.

VCS shall be a semi cubic parabola, of which this is a property $ry^2 = z^3$, from which $\sqrt[3]{ry^2} = z^1$, therefore it is required to find the curve VH in which $\mathrm{PM} = \sqrt[3]{ry^2} = \mathrm{MC}$; and by the first problem I find that to be defined by this equation $nry^3 = x^6$; and so that *n* may be determined I proceed in this manner by Prob. 2. I find PM from the equation found $ny^5r = x^6$ and I find $\mathrm{PM} = \dfrac{5nry^4}{\sqrt[3]{216\,n^2r^2y^{10}}}$ and with a comparison made with the given value it will become $\dfrac{5nry^4}{\sqrt[3]{216\,n^2r^2y^{10}}} = \sqrt[3]{ry^2}$, from which there comes about $n = \frac{216}{125}$ after due reduction, and thus the curve VH may be defined by $\frac{216ry^5}{125} = x^6$, from which there will be $\frac{3}{5}\sqrt[3]{ry^5} = \frac{x^2}{2} = \frac{\mathrm{GM}^2}{2} = \mathrm{VMC}.$ And in the same manner the quadratures

Fig. 4.



of an infinitude of parabolic forms can be found, which are defined by
$ry^3 = z^4;\ ry^4 = z^5,\ ry^5 = z^6$, &c.

PROBLEM VI.

For the Hyperbola OCN *the quadrature shall be the indeterminate Area* OCMVL.

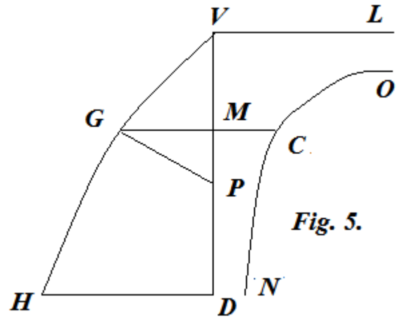


Fig. 5.

Let the power of the hyperbola $= a^2$, so that $\frac{a^2}{y} = z$, therefore a curve VH is required to be found such that always there shall be $\mathrm{PM} = \frac{a^2}{y}$ but we see that at once no treatment of this kind can be used [*Fig.* 5]: for just as the rule requires $\frac{a^2}{y}$ to be multiplied by *ny*, and the product na^2 cannot be put equal to x^2, the determined square cannot be put equal to the indeterminate square, and hence it is required to be concluded the unbounded area is not squarable : for if its quadrature may be given, also a certain curve VH shall be given in which always there shall be $\mathrm{PM} = \mathrm{MC}$.

PROBLEM VII.

Hyperbolas of the form OCN *of which this shall be the property* $yz^2 = a^3$.
To determine the indeterminate area OCMVL.

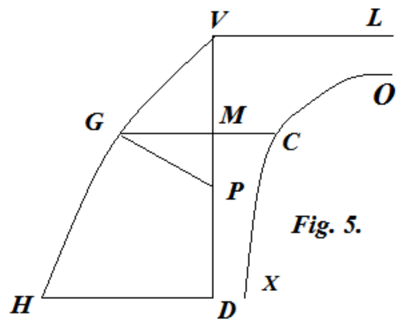


Fig. 5.

Because, from the nature of the curve
$\sqrt{\frac{a^3}{y}} = z = \mathrm{MC}$
[*Fig.*5] the curve VH is required to be found in which there shall be always, $\mathrm{PM} = \sqrt{\frac{a^3}{y}}$, and by *Prob.* 1, I find the curve VH to be defined by $n\sqrt{a^3y} = x^2$ and on determining *n* by *Prob.* 2. You will find $n = 4$, and thus $4a^3y = x^2$ from which the unbounded area $\mathrm{OCMVL} = 2\sqrt{a^3y}$.

[For, $\sqrt{\frac{a^3}{y}} = z; \int zdy = \int \sqrt{\frac{a^3}{y}}dy = \sqrt{a^3}\left[2\sqrt{y}\right] = 2\sqrt{a^3 y}$; while the integrated function is the curve $na^3 y = x^2$ or $x = \sqrt{na^3 y}$ for which the gradient is $\frac{dx}{dy} = \frac{1}{2}\sqrt{na^3}\, y^{-\frac{1}{2}}$, which must be the original function $yz^2 = a^3$ or $z = \sqrt{\frac{a^3}{y}}$ $\therefore \frac{dx}{dy} = \frac{1}{2}\sqrt{na^3}\, y^{-\frac{1}{2}} = \sqrt{\frac{a^3}{y}} \therefore n = 4.$ There are mistakes in the original.]

PROBLEM VIII.

The nature of the hyperbola shall be defined by this equation $yz^3 = a^4$, and the quadrature shall be the unbounded area OCMVL.

From the nature of the curve $\sqrt[3]{\frac{a^4}{y}} = z$, and the curve VH may be found in which PM $= \sqrt[3]{\frac{a^4}{y}}$ to be defined by $27a^4y^2 = x^6,$ from which $\frac{3}{2}\cdot 3\sqrt[3]{a^4y^2} = \frac{x^2}{2} = \frac{GM^2}{2} =$ OCMVL. And thus the quadrature of an infinity of hyperbolic forms defined by $yz^4 = x^5,\ yz^5 = x^6,\ yz^6 = x^7,$&c. can be performed.

$$[\sqrt[3]{\frac{a^4}{y}} = z; \int zdy = \int \sqrt[3]{\frac{a^4}{y}}dy = \sqrt[3]{a^4}\left[3\sqrt[3]{y^2}\right] = \sqrt[3]{27a^4y^2}.\ ]$$

PROBLEM IX.

In the Hyperbolic form OCK of which this shall be the property $y^2z = a^3$, the quadrature shall be of the unbounded area OCMVL.

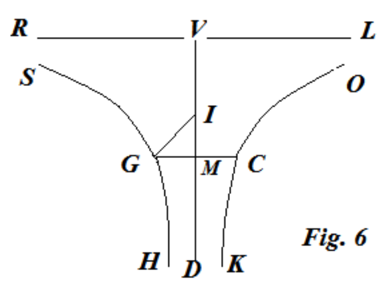


Fig. 6

From the kind of this curve it is evident that $z = \frac{a^3}{y^2} =$ MC whereby some curve is required to be found, so that the perpendicular distance of this and of the applied line shall be equal to $\frac{a^3}{y^2}$, and by the commonly used method I find the curve sought to be defined by $yx^2 = na^3$ and by determining (n) by the second problem there will be $n = 2$ and thus the equation is $2a^3 = yx^2$, which likewise is the equation for the hyperbolic forms (but of other kinds) SGH ; and because of that the perpendicular GP, falls between the vertex and the applied line, and thus rises towards $\frac{a^3}{y} = \frac{x^2}{2} = \text{KCMD}$. And the area OCMVL to be from a number of those which the geometers call more than infinite, now considered by our most distinguished *David Gregory* in his most attractive treatise *de Dimensione Figurarum*.

PROBLEM X.

ACD shall be a curve such that with MC drawn to the normal AD, any power of AD shall be to a similar power of the part AM, so that the power of any part DM shall be to the similar power of the applied line MC, and the quadrature of the area AMC shall be required to be determined.

Let AD = b, Fig. 7, and the exponent of the power 2, AM may be called y from which the exponent of its power also is 2, in addition 2 shall be the exponent of the power of the line DM or $b-y$, (1) and thus the exponent of the applied line MC or z is 1, then from the nature of the line the curve $b^2 \cdot y^2 :: b-y \cdot z$ $\left[i.e.\ \frac{b^2}{y^2} = \frac{b-y}{z} \right]$ from which $z = \frac{by^2 - y^3}{b^2}$, Therefore the curve AH is sought in which there shall be $\text{PM} = \frac{by^2 - y^3}{b^2}$, and it is found by Prob. 1. & 2. that to be

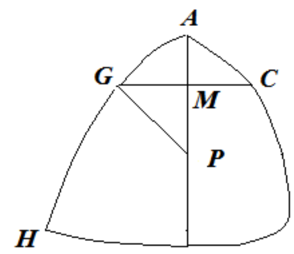


Fig. 7

defined by $\frac{2}{3}by^3 - 1y^4 = b^2x^2$, from which $\frac{y^3}{3b} - \frac{y^4}{4bb} = \frac{x^2}{2} = \text{AMC}.$ And this is the same curve which Descartes is discussing in *Book* 3 of his Letters, page 219 that he preferred to think of the curve, as the French call it, (on account of the ease of construction) *la Galande.*

PROBLEM XI.

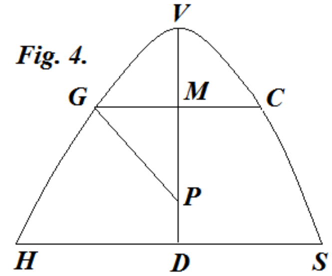
The quadrature of the area AMC shall be required to be determined, and the nature of the curve may be defined by $y^5 + ay^4 + a^2y^3 + a^3y^2 + a = a^4z.$

The curve is required to be found VH, Fig. 4, such that on that there shall be always :

$$\text{PM} = \text{MC} = z = \frac{y^5}{a^4} + \frac{y^4}{a^3} + \frac{y^3}{a^2} + \frac{y^2}{a} + a.$$

From Prob. 1. it may be defined by :

$$ny^6 + may^5 + la^2y^4 + ka^3y^3 + ha^3y = a^4x^2$$

Fig. 4.


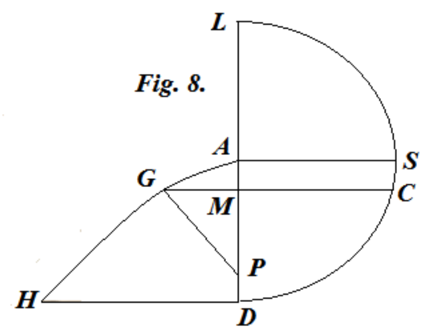
and on determining *n, m, l, k, h* (by *Prob. 2.)* there will be
$n=\frac{1}{3},\ m=\frac{2}{5},\ l=\frac{1}{2},\ k=\frac{2}{3},\ h=2$; and thus the equation sought is
$\frac{1}{3}y^6+\frac{2}{5}ay^5+\frac{1}{2}a^2y^4+\frac{2}{3}a^3y^3+2a^3y=a^4x^2,$ and thus :

$$\frac{y^6}{6a^4}+\frac{y^5}{5a^3}+\frac{y^4}{4a^2}+\frac{y^3}{3a}+ay=\frac{x^2}{2}=\frac{GM^2}{2}=\text{AMC.}$$

And up to the present I have treated only these figures which are indefinite quadrable, and with a little labor, the quadratures of these may be determined by this method, I leave the rest to be judged by others: Now I move on to these which reject quadrature of this kind : and I warn expressly quadratures which are going to be shown by infinite series, not to be found from geometry, but from algebra or arithmetic.

PROBLEM XII.

To determine the Quadrature of the Circle.



I shall make a beginning from a circle, which is the simplest of all the curved lines, if the simplicity of the curve may be judged not from an equation but from the simple description (as indeed must be the case).
And thus the quadrant of the circle shall be ASD in which AM may be called *y*, and the ordinate MC *z*, and the radius $\text{AL}=r$, then from the nature of the circle there will be $z^2=r^2-y^2$, and therefore $z=\sqrt{r^2-y^2}$ hence I resolve the value into a series following the method of the celebrated *Isaac Newton*, and I find
$z=r-\frac{y^2}{2r}-\frac{y^4}{8r^3}-\frac{y^6}{16r^5}-\text{etc.}$, therefore the curve AGH is found in which

$$\text{PM }=r-\frac{y^2}{2r}-\frac{y^4}{8r^3}-\frac{y^6}{16r^5}-\text{.....etc.}$$

and may be found by the first Problem.
The curve sought is defined by this equation:
$nry-\frac{my^3}{2r}-\frac{ly^5}{8r^3}-\frac{ky^7}{16r^5}=x^2$; and by determining the quantities *n, m, l, k,* according to Prob.2 there will be found : $n=2,\ m=3,\ l=\frac{2}{7},$ and on substituting the values in this, the equation will be $2ry-\frac{y^3}{3r}-\frac{y^5}{20r^3}-\frac{y^7}{56r^5}=x^2.$ From which,

$$ry-\frac{y^3}{6r}-\frac{y^5}{40r^3}-\frac{y^7}{112r^5}=\frac{x^2}{2}=\frac{\text{GM}^2}{2}=\text{AMCS.}$$

Or if the quadrature of the whole quadrant may be sought, there will be
$\text{ASD} = r^2 - \frac{1}{6}rr - \frac{1}{40}rr - \frac{1}{112}rr - ...\text{etc.}$ From which
$4rr - \frac{2}{3}rr - \frac{1}{8}r^2 - \frac{1}{28}r^2 - ... =$ whole circle. And if this series may be expressed by numbers, on putting $r = \frac{1}{2}$ the area of the circle will be : $1 - \frac{1}{4} - \frac{1}{32} - \frac{1}{112} -$ etc. ad infinitum
I observe it is worth noting it is possible hence to elicit the dimension of a zone of the circle, which was found by the most celebrated geometer *Isaac Newton* , which the most illustrious *David Gregory* referred to in his memorable discussion. Let ABCD be a zone of which the latitude $\text{VL} = y$, and the radius of the circle $= r$; by the preceding quadrature :

$$\text{VBCL} = ry - \frac{y^3}{6r} - \frac{y^5}{40r^3} - \frac{y^7}{112r^5} - .....\text{and thus}$$

$$2\text{VBCL} = \text{ABCD} = 2ry - \frac{y^3}{3r} - \frac{y^5}{20r^3} - \frac{y^7}{56r^5}.$$

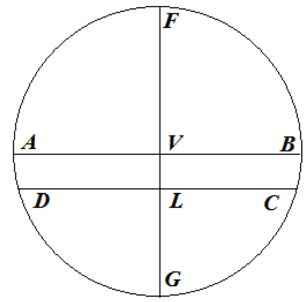


Fig. 9

PROBLEM XIII.

To determine the Quadrature of the Hyperbola.

LSC shall be the Hyperbola the asymptotes of which are VD, VP, and in which $\text{VE} = \text{EL} = a$,and $\text{VA} = c$, the abscissa A*M* *y*, and the applied ordinate *z*, but from the nature of the Hyperbola $\text{VE} \times \text{EL} = \text{VM} \times \text{MC}$, that is $a^2 = yz + cz$; and thus there is $z = \frac{a^2}{c+y}$, and with the division done, following the received method, there will be
$z = \frac{a^2}{c} - \frac{a^2y}{c^2} + \frac{a^2y^2}{c^3} \cdots \text{etc.}$

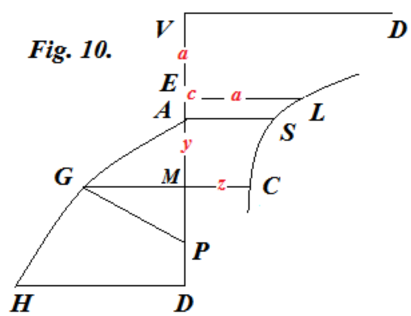


Fig. 10.

Therefore the curve AGH is sought, in which there shall be $\text{PM} = \frac{a^2}{c} - \frac{a^2y}{c^2} + \frac{a^2y^2}{c^3} \cdots \text{etc.}$, and from the first Problem, that will be found to be defined by this equation $\frac{na^2y}{c} - \frac{ma^2y^2}{c^2} + \frac{la^2y^3}{c^3} = x^2$ and on determining *n*, *m*, *l*, by Prob. 2. there will be $n = 2,\ m = 1,\ l = \frac{2}{3}$, and hence the equation sought is
$\frac{2a^2y}{c} - \frac{a^2y^2}{c^2} + \frac{2a^2y^3}{3c^3} = x^2$ from which $\frac{a^2y}{c} - \frac{a^2y^2}{2c^2} + \frac{a^2y^3}{3c^3} = \text{ASCM} = \frac{x^2}{2} = \frac{\text{GM}^2}{2}$. This is the same quadrature of the Hyperbola that the most celebrated *Nicolas Mercator* had shown

in his _Logarithmo-technia_, although by the method I have used clearly to be different from his.

By considering another property of the Hyperbola ; we will discover also another property of the quadrature. Therefore in the opposite figure SCL shall be an equilateral Hyperbola whose centre is A and the transverse axis RS, putting
$\mathrm{AM}=y=\mathrm{KC},\ \mathrm{MC}=z,\ \mathrm{AR}=\mathrm{AS}=r$, so that the nature of this Hyperbola will be $rr+yy=zz$, and thus $z=\sqrt{rr+yy}$, by extracting the square root from $rr+yy$ there will be

$$z=r+\frac{y^2}{2r}-\frac{y^4}{8r^3}+\frac{y^6}{16r^5}\cdots \text{ etc.}$$

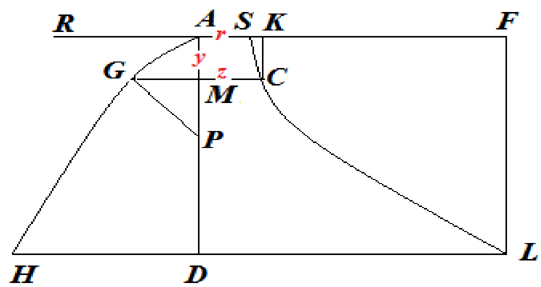


Fig. 11

Therefore the curve AH is sought in which there shall be

$$\mathrm{PM}=z=r+\frac{y^2}{2r}-\frac{y^4}{8r^3}+\frac{y^6}{16r^5}\cdots \text{ etc.}$$

and by proceeding according to the first Problem that will be found to be defined by this equation $nry+\frac{my^3}{2r}-\frac{ly^5}{8r^3}+\frac{ky^7}{16r^5}=x^2$, and on determining n, m, l, k, by the second Problem $n=2,\ m=\frac{2}{3},\ l=\frac{2}{5},\ k=\frac{2}{7},$ there will be this equation fully determined for the curve sought, with these values substituted : $2ry+\frac{y^3}{3r}-\frac{y^5}{20r^3}+\frac{y^7}{56r^5}=x^2$, and thus there will be

$$2ry+\frac{y^3}{6r}-\frac{y^5}{40r^3}+\frac{y^7}{112r^5}=\frac{x^2}{2}=\frac{GM^2}{2}=ASCM.$$

It is easy to determine the quadrature of the zones from the quadrature of this Hyperbola. EDA and GCB shall be opposite sides of the Hyperbola, of which the centre is K and the vertices A and B, with the zone ABCD the latitude of which KL = y, the transverse semi-axis AK, or $\mathrm{KB}=r$, from which by the preceding quadrature :

$\mathrm{KLCB}=ry+\frac{y^3}{6r}-\frac{y^5}{40r^3}+\frac{y^7}{112r^5}$ and hence there will be

$$\mathrm{ABCD}=2ry+\frac{y^3}{3r}-\frac{y^5}{20r^3}+\frac{y^7}{56r^5}\cdots\text{etc.}$$

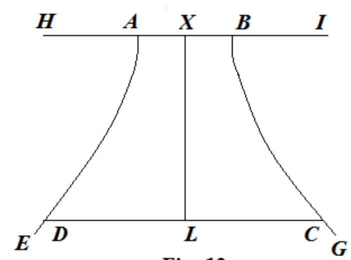


Fig. 12.

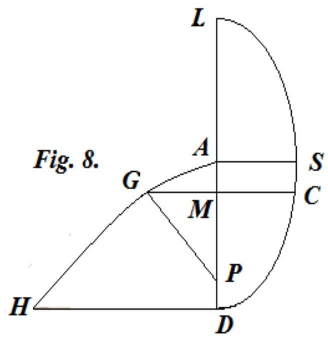
PROB. XIV.

To determine the Quadrature of the Ellipse.

In the semi-ellipse LSCD the semi transverse axis shall be $\mathrm{AS} = b$ and the conjugate semi axis $\mathrm{AL} = a$, and the abscissa may be put $\mathrm{AM} = y$, the applied ordinate $\mathrm{MC} = z$, from which the nature of the ellipse is :

$z = \frac{b}{a}\sqrt{a^2 - y^2}$, therefore so that the area AMCS may be determined, in the first place $\frac{b}{a}\sqrt{a^2 - y^2}$ is required to be resolved into a series by extracting the root from $\sqrt{a^2 - y^2}$, from which there will be found :

Fig. 8.



$z = b - \frac{by^2}{2a^2} + \frac{by^4}{8a^4} - \frac{by^6}{16a^6} \cdots$ etc. Therefore some curve AH is required to be found in which there shall be always $\mathrm{PM} = b - \frac{by^2}{2a^2} - \frac{by^4}{8a^4} - \frac{by^6}{16a^6} \cdots$ etc.

and that may be defined by Problem 1 to be defined by this equation

$nby - \frac{mby^3}{2a^2} - \frac{lby^5}{8a^4} - \frac{kby^7}{16a^6} = x$ and by determining the quantities, *n, m, l, k.* by Problem 2, there will be $n = 2,\ m = \frac{2}{3},\ l = \frac{2}{5},\ k = \frac{2}{7}$, and thus $2by - \frac{mby^3}{3a^2} - \frac{lby^5}{20a^4} - \frac{kby^7}{56a^6} = x^2$ from which $by - \frac{mby^3}{6a^2} - \frac{lby^5}{40a^4} - \frac{by^7}{112a^6} = \frac{x^2}{2} = \frac{\mathrm{GM}^2}{2} = \mathrm{AMSC}.$

Thence the dimension of an elliptic zone is elicited easily, as if the latitude of the zone shall be $\mathrm{AM} = y$, with everything put in place as before, the zone will be :

$$2\mathrm{AMCS} = 2by - \frac{mby^3}{3a^2} - \frac{lby^5}{20a^4} - \frac{by^7}{56a^6} \cdots \text{etc.}$$

PROBLEM XV.

$AD(=d)$ shall be given in position and magnitude, and of such a curve SCD that with some right line drawn $MC(=z)$ to the perpendicular AD there shall be $d^3 = z^3 + y^3$, of which the area AMCS shall be required to be determined.

Because the nature of this curve shall be $z = \sqrt[3]{d^3 - y^3}$; the cube root is to be extracted from $d^3 - y^3$, and it will be found to become :

$z = d - \frac{y^3}{3d^2} - \frac{y^6}{9d^5} - \cdots \text{etc.}$

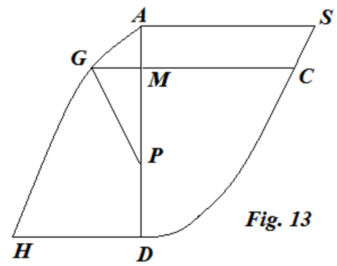


Fig. 13

The curved line is required to be found AGH in which $\text{PM} = d - \frac{y^3}{3d^2} - \frac{y^6}{9d^5} - \cdots \text{etc.}$, and the sought curve AH will be found from this equation $ndy - \frac{my^4}{3d^2} - \frac{ly^7}{9d^5} = x^2$, and on determining *n, m, l,* by the second problem there will be $n = 2,\ m = \frac{1}{2},\ l = \frac{2}{7}$, and thus $2dy - \frac{y^4}{6d^2} - \frac{2y^7}{63d^5} = x^2$, from which

$dy - \frac{y^4}{12d^2} - \frac{y^7}{63d^5} = \frac{x^2}{2} = \frac{\text{GM}^2}{2} = \text{AMCS}.$

And thus the quadrature of an infinitude of cyclic shapes which are defined by $d^4 - y^4 = z^4,\ d^5 - y^5 = z^5 \cdots \text{etc.}$ may be found.

PROBLEM XVI.

Let $AD(=d)$ be a right line with position and magnitude given, and SCD shall be a curved line such that for any right line $MC(=z)$ drawn normal to AD, the cube from AD with the cube from shall be equal to the cube from MC thus: $d^3+y^3=z^3$ and the quadrature of the area AMC shall be required to be determined. [*Fig.13.*]

Because $z=\sqrt[3]{d^3+y^3}$ is resolved by extracting the cube root of $\sqrt[3]{d^3+y^3}$ in a series, and there may be found $z=d+\frac{y^3}{3d^2}-\frac{y^6}{9d^5}+\cdots\text{etc.}$ The curve AH is required to be found in which always there shall be $\text{PM}=d+\frac{y^3}{3d^2}-\frac{y^6}{9d^5}+\cdots\text{etc.}$, and by Problems 1. & 2. the equation for the curve sought will be $\cdots 2dy+\frac{y^4}{6d^2}-\frac{2y^7}{63d^5}=x^2$ from which

$dy+\frac{y^4}{12d^2}-\frac{y^7}{126d^5}=\frac{x^2}{2}=\text{AMCS.}$

And thus the quadratures can be found of an infinitude of hyperbolic forms which are defined by

$$d^4+y^4=z^4,\ d^5+y^5=z^5,\ \text{etc.}$$

PROBLEM XVII.

Let $AD=a$, $AS=b$, and the curve shall be SCD [*Fig.13.*] *such that with some $MC(=z)$ drawn perpendicular to AD there shall be $z^3.a^3-y^3::b^3.a^3$, and the area AMCS shall be required to be determined.*

Because from the nature of the curve $z=\frac{b}{a}\sqrt[3]{a^3-y^3}$, with the root being extracted from a^3-y^3, there will be found $z=b-\frac{by^3}{3a^3}-\frac{by^6}{9b^6}-\cdots\text{etc.}$ The curve AH is sought on which $\text{PM}=\text{MC}=b-\frac{by^3}{3a^3}-\frac{by^6}{9b^6},\text{etc.}$ and by Problems 1 & 2, the curve sought is defined by this equation : $2by-\frac{by^4}{6a^3}-\frac{2by^7}{63a^6}=x^2$. Therefore $by-\frac{by^4}{12a^3}-\frac{2by^7}{126a^6}=\frac{x^2}{2}=\frac{\text{GM}^2}{2}=\text{AMCS.}$
And thus the quadrature may be found for an infinitude of elliptical forms, which are defined by the equations :

$$\frac{b}{a}\sqrt[4]{a^4-y^4}=z,\ \frac{b}{a}\sqrt[5]{a^5-y^5}=z,\text{etc.}$$

PROBLEM XVIII.

$AD(=d)$ is a right line put in place [*Fig.13.*] *and with the magnitude given, and SCD a curve such that with any MC* [$=z$] *drawn normal to AD there shall be $d^2z+y^2z=r^3$, and the quadrature shall be the area AMCS.*

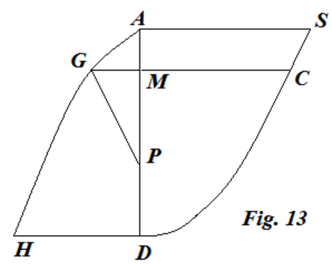


Fig. 13

Because $z=\frac{r^3}{d^2+y^2}$; with the division made, and there may be found : $z=\frac{r^3}{d^2}-\frac{r^3y^2}{d^4}+\frac{r^3y^4}{d^6}$; and there becomes : $\frac{2r^3y}{d^2}-\frac{2r^3y^3}{3d^4}+\frac{2r^3y^5}{5d^6}=x^2$, the equation for the curve AH in which $\mathrm{PM}=\frac{r^3}{d^2}-\frac{r^3y^2}{d^4}+\frac{r^3y^4}{d^6}$, from which

$$\frac{r^3y}{d^2}-\frac{r^3y^3}{3d^4}+\frac{r^3y^5}{5d^6}=\frac{x^2}{2}=\frac{\mathrm{GM}^2}{2}=\mathrm{AMCS}.$$

PROBLEM XIX.

To find an Infinitude of Quadratures of any Figure.

ACE shall be a certain curve (whose axis is AB, base BE and which we will call $y(x)$), YXZ and BRF shall be two curves [$f(x)$ and $g(y)$] related thus, with AB the common x-axis, such that from any point C on the curve ACE the tangent CT may be drawn, and CP may be drawn parallel to EB, & CR parallel to AB, so that there shall become $\mathrm{TP.PC}::\mathrm{DR.PX}$ $\left[i.e.\ \frac{\mathrm{TP}}{\mathrm{PC}}=\frac{dx}{dy}=\frac{g}{f}=\frac{\mathrm{DR}}{\mathrm{PX}}\right]$, then the areas will be $\mathrm{ABZY}=\mathrm{BEF}$, and $\mathrm{APXY}=\mathrm{DBR}$. [For $gdy=fdx$ and hence $\int gdy=\int fdx$, on choosing appropriate limits of integration.]

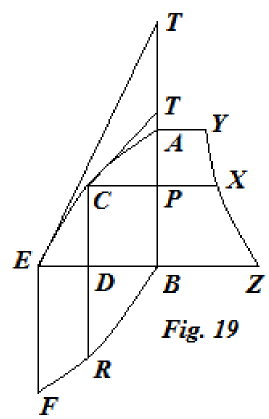


Fig. 19

This excellent Theorem also is due to the most celebrated *Dr. Barrow*. [Fig. 201 in Barrow's *Lectiones Geometricae*.]

There may be sought, for example, the indefinite quadrature of the cubic paraboloid BRF. Any curve may be assumed by choice for ACE, for example the common parabola whose parameter shall be r (which likewise shall be the case for the parameter of the paraboloid) and putting $\mathrm{AP}=y$, $\mathrm{PX}=z$, [adopting the author's choice of co-ordinates] and there will be $\mathrm{TP}=2y$, $\mathrm{PC}=\sqrt{ry}$ [$\mathrm{PC}^2=ry\therefore 2PC\frac{dPC}{dy}=r$] and from the nature of the paraboloid, $\mathrm{DR}=\sqrt[6]{ry}$, and thus the proportionality will be

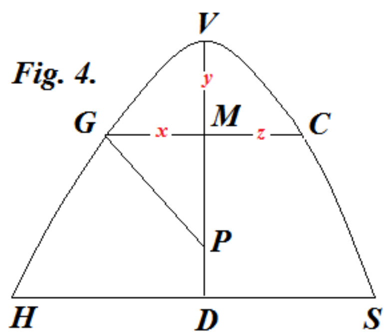
$2y.\sqrt{ry} :: \sqrt[6]{ry}.z$ $\left[i.e.\ \frac{2y}{\sqrt{ry}} = \frac{\sqrt[6]{ry}}{z} \right]$ from which $z = \sqrt[6]{\frac{r^8}{64y^2}}$ which is the equation for the curve YXZ, the quadrature of which may be found by the method discussed above, thus $\frac{3}{4}\sqrt[6]{r^8y^4} = \text{APYZ} = \text{DBR}$. Which is different from the quadrature of the paraboloid that I gave in Prob. 4. And more and more quadratures can be found in the same manner and with a different curve ACE. And thus all the other curves can be treated, whatever the paraboloid treated here.

Hence also it is evident the figures can be suppressed and with the quadrature made simpler and easier, for in figure ABZY with the curve YXZ defined by this equation $z^6 = \frac{r^8}{64y^2}$, it is more composite than the curve BRF. And thus anyone may advance geometry considerably, who may give the method reduced to the simplest figures.

PROBLEM XX.

To find the curve the area of which may be assigned by some given equation.

Fig. 4.



The area may be designated by this equation $\sqrt{r^3y} = \text{VMC}$ (by considering VCS to be the curve sought.) Then from what has been shown above, it is apparent that $\sqrt{r^3y} = \frac{x^2}{2}$ is the equation for some other curve VGH in which $\text{PM} = \text{MC}$ (which is the ordinate of the curve sought) ; therefore the value of the line PM may be sought, and it will be found that $\text{PM} = \sqrt{\frac{r^3}{4y}} = z$ or $4z^2y = r^3$, which is the equation for the curve sought VCS, of which the area $= \sqrt{r^3y}$. Because here as before it is required to note that *y* denotes the abscissa VM, *z* the ordinate MC, and *x* the ordinate GM.

[As before, we have area $VMC = \int zdy$; hence $\frac{dx}{dy} = \frac{MP}{x}$ and $\frac{1}{2}x^2 = \int zdy$; in this case $\int zdy = \frac{1}{2}x^2 = \sqrt{r^3y} \therefore z = \frac{1}{2}\sqrt{\frac{r^3}{y}} = \sqrt{\frac{r^3}{4y}}$.]

PROBLEM XXI.

To find an Infinitude of Curves the Area of which may be designated by a given Equation.

The solution of this problem depends on the two preceding problems ; on curve may be found of which the area may be expressed by the given equation (by Problem 20) and thus an infinitude may be found by Problem 19.

PROBLEM XXII.

For any given curve AHD to find another curve AFB whose area AGF is equal to the rectangle contained under the ordinate GH and the abscissa AG of the given curve.

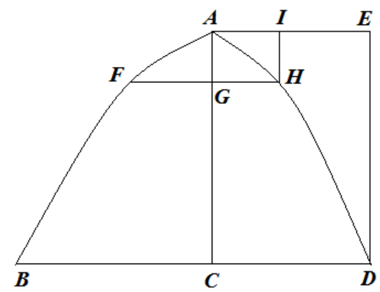


Fig. 20.

For the curve AHD there shall be $\mathrm{AG} = y$, $\mathrm{GH} = x$, and the nature of this may be expressed by this equation $2ay - yy = x^2$, from which $\sqrt{2ay - yy} = x$ and thus $\sqrt{2ay^3 - y^4} = xy = \mathrm{AH}$. Therefore the area of the figure AGF may be found, from which the curve AFB is defined easily by Problem 20. Thus : $z^2 = \frac{9a^2y - 12ay^2 + 4y^3}{2a - y}$

[*i.e.* $z = D \cdot \sqrt{2ay^3 - y^4} = \frac{1}{2} \cdot \frac{1}{\sqrt{2ay^3 - y^4}} \cdot \left(6ay^2 - 4y^3\right) = \frac{1}{\sqrt{2ay^3 - y^4}} \cdot \left(3ay^2 - 2y^3\right)$

$z^2 = \frac{\left(9a^2y^4 - 6ay^5 + 4y^6\right)}{2ay^3 - y^4} = \frac{\left(9a^2y - 6ay^2 + 4y^3\right)}{2a - y}$.]

PROBLEM XXIII.

With some curve AHD given, to find another curve AFB whose area AGF shall be equal to the rectangle contained under the ordinate GH of the curve AHD, [Fig. 20.] *and to some given right line* (a)

The curve AHD may be defined as before as $\sqrt{2ay-yy}=x$ from which
$\sqrt{2a^3y-a^2y^2}=ax=\text{AGF};$,
therefore the nature of the curve AFB may be had according to Problem 20. Thus :

$$z^2=\frac{a^4-2a^2y^2+a^2y^4}{2ay-y^2}$$

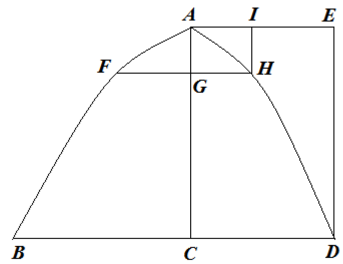


Fig. 20.

here and in the preceding, z indicates the ordinate of the curve sought AFB.

And indeed it is possible to find a curve of this area by infinitely many other ways (besides the two now treated), with the aid of another given curve which shall be able to be squared, by Problem 20. Which the praise-worthy German asserts can be done, but he has not shown in what manner it shall be done.

Another solution of the preceding problem.

ACB shall be the given curve, CT the tangent at some point C, with the ordinate CF; and there becomes : $\text{TF.FC}::a.\text{FZ}\ \left[i.e.\ \frac{\text{TF}}{\text{FC}}=\frac{a}{\text{FZ}}\right]$, hence the curve AZZ arises such that $a+\text{FC}=\text{AFZ}$; as has been shown by the most illustrious *Dr.Barrow*.

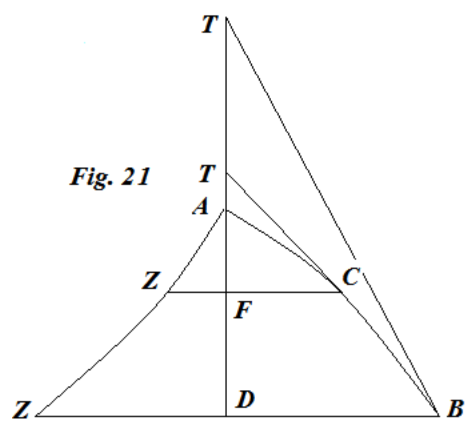
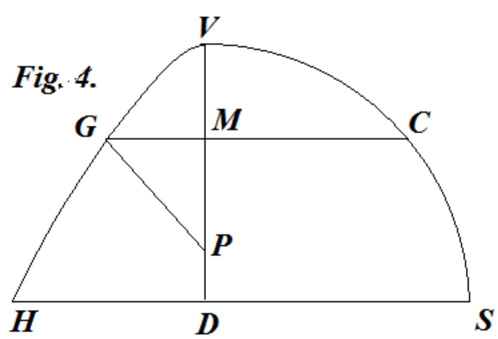


Fig. 21

Nor now does the method that I used for determining the squares of figures lack anything and can be extended to all figures (with those excepted which are defined by transcending curves, which no one hitherto has dealt with by the common method) except that I may remove two difficulties ; which can appertain to certain cases ; the first of which occurs when a certain figure cannot be squared, and the root cannot be extracted from the equation produced (and with the squares rising above), in which case a single remedy has occurred to me: the root of this equation shall be resolved in an infinite series (the method of the most enlightened Isaac Newton, not only the most outstanding of the Geometers but also of the Analysists), that we hear has been put together for the printing press by the most distinguished Wallis, and that the worthy Newton has communicated to me in manuscripts by his great kindness : For the *General Method of determining the Roots of Equations analytically* (in *Actis Eruditorum Lipzig* publish. in the year 1683, in the month of May produced by that outstanding German) [The only article to be found in this issue, by D.T., *i.e.* E. W. von Tschirnhaus, with the title translated : *A method for removing all intermediate terms from a given equation*, from the *Acta Eruditorum*, May 1683, pp 204 – 207] serves little or no purpose in this matter; so that I say nothing about the insurmountable trouble in that calculation. But nevertheless among the issues particular examples of the analytical art are worthy of merit and require to be enumerated.

The second difficulty is with the value of the applied ordinate agreed on with asymmetric terms, for the result shall be an equation of immense labor to free the equation from asymmetry, if more than four terms shall be affected with root signs, as skilled analysts have got to know well. But an optimum remedy for this difficulty has been supplied by the outstanding Geometer W. G. Leibniz in his New Method of finding Tangents published in the *Actis Eruditorum* of the earlier year, where indeed the bright fellow has shown a way set out of finding the tangents, whenever the equation expressing the nature of a curve shall be especially involved with irrational terms, without removing the irrationalities. I will show by an example how this method shall be applied to the present circumstances.



Let VCS be the quadrant of a circle whose diameter shall be (r) and VM may be called y, likewise the ordinate MC is z [Fig. 4 : not to scale], then from the nature of the circle $z=\sqrt{ry-y^2}$ and by resolving $ry-y^2$ in series by extracting the root, there will be found

$$z=\sqrt{ry}\left[1-\tfrac{1}{2}\tfrac{y}{r}-\tfrac{1}{4}\tfrac{y^2}{r^2}-\right]=\sqrt{ry}-\sqrt{\frac{y^3}{4r}}-\sqrt{\frac{y^5}{16r^3}}\cdots\text{etc.}.$$

So that the quadrature of the area VMC may be determined, the curve VH is required to be found, in which $\text{PM}=\sqrt{ry}-\sqrt{\frac{y^3}{4r}}-\sqrt{\frac{y^5}{16r^3}}$, and by Problem 1 the equation for the curve VH sought will be $\sqrt{nry^3}-\sqrt{\frac{my^5}{4r}}-\sqrt{\frac{ly^7}{16r^3}}=x^2;$ and by removing the fractional quantities (which still is not absolutely necessary, but here it shall be on account of greater facility) on multiplying by $\sqrt{16r^3}$: there will be

$$\sqrt{16nr^4y^3}-\sqrt{4mr^2y^5}-\sqrt{ly^7}=x^2\sqrt{16r^3};$$

and by determining $n, m, l,$ (by Problem 2) which alone is difficult and I proceed thus : in order to shorten the calculations, I put $p=16nr^4y^3,\ q=4mr^2y^5,\ s=ly^7$; and there will be

$$\sqrt{p}-\sqrt{q}-\sqrt{s}=x^{2}\sqrt{16r^{3}};$$

but by the calculation [to be] explained here, there will be found :

$$\sqrt{p}=\frac{dp}{\sqrt{4p}},\ \sqrt{q}=\frac{dq}{\sqrt{4q}},\ \sqrt{s}=\frac{ds}{\sqrt{4s}},\text{ and } x^{2}\sqrt{16r^{3}}=2x\sqrt{16r^{3}}dx$$

$$[\,i.e.\ d\sqrt{p}=\frac{dp}{\sqrt{4p}},\ d\sqrt{q}=\frac{dq}{\sqrt{4q}},\ d\sqrt{s}=\frac{ds}{\sqrt{4s}},\ d.x^{2}\sqrt{16r^{3}}=2x\sqrt{16r^{3}}.dx;]$$

and with these values substituted there will be : $\frac{dp}{\sqrt{4p}}-\frac{dq}{\sqrt{4q}}-\frac{ds}{\sqrt{4s}}=2x\sqrt{16r^{3}}dx,$

but in the same manner, the calculation may become

$$dp=48nr^{4}y^{2}dy,\ dq=20mr^{2}y^{4}dy\text{ and finally ; } ds=7ly^{6}dy,$$

and with these values substituted, with the values of the quantities $\sqrt{4p},\ \sqrt{4q},\ \sqrt{4s},$ the equation becomes :

$$\frac{48nr^{4}y^{2}dy}{\sqrt{64nr^{4}y^{3}}}-\frac{20mr^{2}y^{4}dy}{\sqrt{16mr^{2}y^{5}}}-\frac{7ly^{6}dy}{\sqrt{4ly^{7}}}=2x\sqrt{16r^{3}}dx,$$

Which the most illustrious Author calls a differential equation : and this equation resolved into proportions gives :

$$dy.dx::x\sqrt{64r^{3}}.\left[\frac{48nr^{4}y^{2}}{\sqrt{64nr^{4}y^{3}}}-\frac{20mr^{2}y^{4}}{\sqrt{16mr^{2}y^{5}}}-\frac{7ly^{6}}{\sqrt{4ly^{7}}}\right]::x.\mathrm{PM},$$

$$[i.e.\ \frac{dy}{dx}=\frac{x\sqrt{64r^{3}}}{\frac{48nr^{4}y^{2}}{\sqrt{64nr^{4}y^{3}}}-\frac{20mr^{2}y^{4}}{\sqrt{16mr^{2}y^{5}}}-\frac{7ly^{6}}{\sqrt{4ly^{7}}}}=\frac{x}{\mathrm{PM}}]$$

as is evident from the same calculation, and thus there will be

$$\mathrm{PM}=\frac{48nr^{4}y^{2}}{\sqrt{4096nr^{7}y^{3}}}-\frac{20mr^{2}y^{4}}{\sqrt{1024mr^{5}y^{5}}}-\frac{7ly^{6}}{\sqrt{256lr^{3}y^{7}}}=\text{etc.}$$

From a comparison made of these terms with the terms denoted by the previous PM, [i.e. $z=\sqrt{ry}-\sqrt{\frac{y^{3}}{4r}}-\sqrt{\frac{y^{5}}{16r^{3}}}\cdots\text{etc.}$] you read of the known comparison nearby thus : $\frac{48nr^{4}y^{2}}{\sqrt{4096nr^{7}y^{3}}}=\sqrt{ry}$, thence $n=\frac{16}{9}$; similarly $m=\frac{16}{25}$;& $l=\frac{16}{49}$; with which substituted there will be

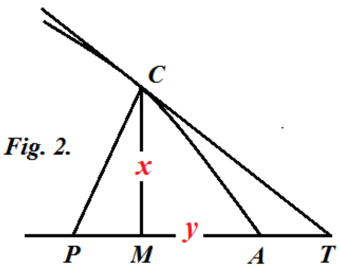
$$\frac{\sqrt{16ry}}{9}-\frac{\sqrt{16y^5}}{100r}-\frac{\sqrt{y^7}}{496r^3}=x^2=GM^2,$$

and thus

$$\frac{\sqrt{4ry}}{9}-\frac{\sqrt{16y^5}}{200r}-\frac{\sqrt{y^7}}{392r^3}=\frac{x^2}{2}=\frac{CM^2}{2}=\text{VMC}.$$

And thus in this manner the square of the circle also may be had. [This series converges very slowly.] And it will not be difficult to use a similar argument in other problems, for any in this versatile kind of singular calculation, thus so that I have considered it superfluous to illustrate the use of this outstanding method by more examples. Yet there is one which I consider to be worthy of note here, to be able to show briefly the truth of the rule I gave for the solution of the first problem by this method of tangents. In as much as : $dy.dx::\text{TM.MC}\left[i.e.\ \frac{dy}{dx}=\frac{\text{TM}}{\text{MC}}\right]$ (as is evident from that method) but

Fig. 2.



$\text{TM.MC}::\text{MC.PM}\left[i.e.\frac{\text{TM}}{\text{MC}}=\frac{\text{MC}}{\text{PM}}\right]$, on account of the right angle TMC. Therefore $dy.dx::\text{MC.PM}\left[i.e.\ \frac{dy}{dx}=\frac{\text{MC}}{\text{PM}}=\frac{\text{ordinate}}{\text{sub-normal}}\right]$ (or with x substituted for MC) there will be $dy.dx::x.\text{PM}\left[i.e.\ \frac{dy}{dx}=\frac{x}{\text{PM}}\right]$. From which $\text{PM}\times dy=xdx$, and by substituting y and x for the differences of these dy, dx there will be $\text{PM}\times y=x^2$. Q.E.D.

I conclude now by saying that, if there shall not be curve, in which the distance between the perpendicular of that and the ordinate [*i.e.* the subnormal PM] shall be equal to the corresponding ordinate in the figure of the curve taken (with a right line or with right lines), that figure is not to be squarable indefinitely [i.e. the indefinite integral will not exist, as for a finite algebraic form]; for if the indefinite quadrature of that may be given, a curve of this kind may also be given, as it is apparent from Problem 20. And it is easy to show that there is no such curve for a circle or hyperbola [*i.e*. not an algebraic curve with a finite number of terms, but only transcendental curves, involving an infinite series expansion ; recall that the function concept did not yet exist as such], but I omit the demonstration here on account of the excessive extent of the calculation.

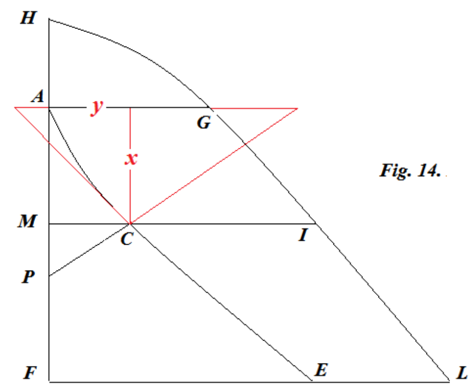
Concerning the Rectification of Curves.

Pray who was the first amongst many some time ago to find a right line equal to a curve ? There was a dispute between the English and the Dutch ; and anyone who would wish to satisfy themselves further about this matter, can see the whole dispute in the little book on cycloids published by the most enlightened Wallis, pages 91, 92, 93, &c. ; and likewise, in the *Horologio Oscillatorio* of the most illustrious Huygens, page 72, 73, and finally, in a letter of Wallis published in the Transactions of the Royal Soc., Number 98 ; for the matter may be seen not of so great as to be worth further dispute, especially by me who is neither English nor Dutch. Yet I will note briefly that which can well be considered about the matter on both sides:

1. Because William Neil, the son of an English knight, was the first of all who discovered the equation of a right line equal to the length of a curve [the semi-cubical parabola] ;
2. Because not only had he shown that a curve could be rectified but also had shown the ability to rectify a curve.
3. Because Christopher Wren, the most worthy and most skilled Geometer, was the first to determined the right line equal to an oblate curve (*i.e*. equal to the Cycloid).
4. Because Hendrik van Heuraet first showed how to rectify any given curve, from the supposition of the quadrature of an associated figure. And in van Heuraet's method it is quite evident how at once what that figure shall be whose curvature may give the rectification of the curve ; And thus since now I have set out the general method of the quadratures of figures to be determined ; it will be easy to change some curve into a right line; And from the right line, that either by a finite equation (when surely the figure is indefinitely squarable) or may be expressed by an infinite series. For Heuraet lacked such a method, was not able to extend his method of rectification of curves to all these curves, of which the rectification depends on the quadratures from definite figures; and furthermore, and less since the ability to be quadratures depend so much on special figures.

THEOREM 2.

There shall be two curves ACE and GIL, [Fig.14.], and the right line AF of that kind so that (drawn from the point M selected freely with the perpendicular MI cutting the curves at C and I, and so that CP shall be perpendicular to the curve ACE) there shall be $\mathrm{MC.CP :: R.MI}\ \left[\frac{\mathrm{MC}}{\mathrm{CP}} = \frac{\mathrm{R}}{\mathrm{MI}}\right]$(here R is any given or assumed right line) there will be $\mathrm{AGILEF = R \times ACE}$. The demonstration of this theorem may be found in the letter of van Heuraet to Schooten. [The modern reader can find this theorem explained in modern terms in the excellent work by Victor Katz, *The History of Mathematics*, Harper Collins, 1993; p.454 onwards. Basically the length of the first curve, can be found from the integral, between appropriate limits :



$\int ds = \int \sqrt{dx^2 + dy^2} = \int dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \int dx / \cos\psi$, where x is the independent variable y is the ordinate of the curve, and $\tan\psi = \frac{dy}{dx}$; van Heuraet's ratio $\frac{\mathrm{MC}}{\mathrm{CP}} = \frac{\mathrm{R}}{\mathrm{MI}}$ or $\frac{dy}{ds} = \frac{\mathrm{R}}{\mathrm{MI}(=z)} = \cos\psi$, on taking y as the independent variable, and applied to the characteristic elemental triangle with sides *dx*, *dy*, and *ds*, (which is not shown in the diagram), gives $\int ds = R\int zdy$; the latter integral represents the area under the curve with the ordinate z with the same limits as the first curve, *R* being a line of constant length that gives the curves in the same fixed ratio.]

PROBLEM I.

To Determine the Length of the Parabola ACE.

The vertex of the parabola shall be A, its axis AG and the parameter (*a*); *AM* may be called *x*, and *MC* may be called *y* [Fig. 14.]; from which $x^2 = ay$ from the nature of the parabola ; the tangents are required to be found by some common method, there will be agreed to become $\mathrm{PM} = \frac{2x^3}{a^2}$ [for $\mathrm{PM} = y\frac{dy}{dx} = \frac{x^2}{a} \times \frac{2x}{a} = \frac{2x^3}{a^2}$], and thus $\mathrm{PM}^2 = \frac{4x^6}{a^4}$, from which $\mathrm{PC} = \sqrt{\frac{4x^6}{a^4} + \frac{x^4}{a^2}}$ because now $\mathrm{CM.CP} :: a.\mathrm{MI}$ $\left[i.e.\ \frac{\mathrm{CM}}{\mathrm{CP}} = \frac{a}{\mathrm{MI}} \text{ by Th. 2.}\right]$; or in analytical terms, $\frac{x^2}{a}.\sqrt{\frac{4x^6}{a^4}+\frac{x^4}{a^2}}::a.z$ $\left[i.e.\ \frac{\frac{x^2}{a}}{\sqrt{\frac{4x^6}{a^4}+\frac{x^4}{a^2}}}=\frac{a}{z}. \right]$ (clearly on putting $MI=z$) from which $z=\sqrt{a^2+4x^2}$, which is the equation for a Hyperbola; and thus, for the determination of the length of the parabolic line ACE, the quadrature is the hyperbolic area AGILEF (as in Prob.13 .) and there will be,

$$\text{AGILEF}=ax+\frac{2x^3}{3a}-\frac{2x^5}{3a^3}+\frac{4x^7}{3a^5}-\text{etc.}$$

From which $ax\text{ACE}=ax+\frac{2x^3}{3a}-\frac{2x^5}{3a^3}+\frac{4x^7}{3a^5}-\text{etc.}$ by Th. 2.

And thus $\text{ACE}=x+\frac{2x^3}{3a^2}-\frac{2x^5}{5a^4}+\frac{4x^7}{7a^6}-\text{etc.}$

PROBLEM II.

To show the right line equal to the periphery of a circle.

ACF shall be the quadrant of a circle of which the radius shall be d and PM may be called y, MC x; and MI z, [Fig. 15.] and GIL shall be such a curve so that with some normal CMI drawn to the right line PF there shall be

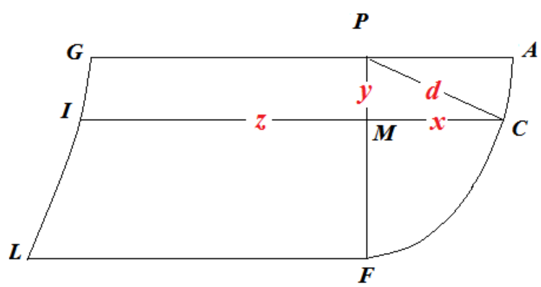


Fig. 15.

$\text{MC.PC}::d.(\text{with the right line freely assumed}).\text{MI.}$ that is $\sqrt{d^2-y^2}.\,d::d.z.$ $\left[\frac{\sqrt{d^2-y^2}}{d}=\frac{d}{z}\right]$, from which $z=\frac{dd}{\sqrt{dd-yy}}$ which is the equation for the curve GIL and thus $\text{PM}=dy+\frac{y^3}{6d}+\frac{3y^5}{40d^3}\cdots\text{etc.}$ But by Theorem 2 there becomes

$d x\text{AC}=dy+\frac{y^3}{6d}+\frac{3y^5}{40d^3}+\frac{5y^7}{112d^5}\cdots\text{etc.}$ Therefore $\text{AC}=y+\frac{y^3}{6d^2}+\frac{3y^5}{40d^4}+\frac{5y^7}{112d^4}\cdots\text{etc.}.$

PROBLEM III.

To show the Right Line equal to a Hyperbola.

ACE shall be [the branch of an] equilateral hyperbola of which the semi axis $BA = a$ and the centre B ; and BM may be called y, AC x, so that from the nature of the hyperbolae $a^2 + y^2 = x^2$; PC is put perpendicular to the Hyperbola at C [Fig. 16.] ; there will be found: $PM = y$ [For $\frac{PM}{x} = \frac{dx}{dy} = \frac{y}{x}$] and thus

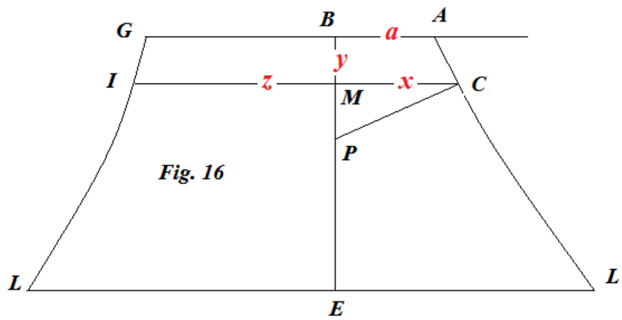


Fig. 16

$PC = \sqrt{a^2 + 2y^2}$; if there may be made $MC.CP :: a.MI$ that is, $\frac{\sqrt{a^2+y^2}}{\sqrt{a^2+2y^2}} = \frac{a}{z}$; there will be

$z = \frac{\sqrt{a^4+2a^2y^2}}{\sqrt{a^2+y^2}}$, which is the equation for the curve GIL.

But $\sqrt{a^4 + 2a^2y^2} = a^2 + y^2 - \frac{y^4}{2a^2} - \cdots$etc.

And $\sqrt{a^2 + y^2} = a + \frac{y^2}{2a} - \frac{5y^4}{8a^3} \cdots$ etc.

From which $BELG = ay + \frac{y^3}{6a} + \frac{5y^5}{40a^3} \cdots$ etc.

And $ACE = y + \frac{y^3}{6a^2} - \frac{5y^5}{40a^4} \cdots$ etc.

On the Measurement of Curved Surfaces.

Just as the lengths of curved lines, thus also of the surfaces, which may be generated by the rotation of these, the measurement will depend on the quadratures of certain figures, as may be agreed on from the following theorem.

THEOREM 3.

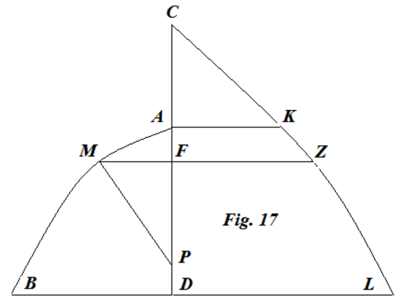


Fig. 17

MP shall be the perpendicular to a certain curve AMB and the line KZL such that (with MFZ drawn normal to the axis AD) MP shall be equal to the corresponding FZ ; the surface produced by the rotating of the curve AMB about the axis AD will be, to the area ADLK, as the circumference of the circle to its radius.

[*i.e.* , in modern terms : $\frac{\text{curved surface formed on rotating about axis}}{\text{area ADLK}} = 2\pi$; we note that π had not been so defined at this time.]

This theorem too is from the innumerable and outstanding theorems if the most celebrated of men, Isaac Barrow.

PROBLEM I.

To determine the Surface of a Sphere.

AMB shall be a semicircle, from the rotation of which a given sphere is produced : and r shall designate the radius and c the circumference of any circle ; and thus AB (the diameter of the semicircle AMB) $= 2d$ now because all the lines MP perpendicular to the curve of the circle arrive at the centre of the circle P; therefore AKZLB shall be a rectangular parallelogram of which the length shall be the diameter AB and the height $\text{AK} = d$, the radius of the semicircle AMB [*i.e.* the function defining the area is a constant equal to the radius of the circle]; from which $\text{AL} = 2d^2$ (I designate the surface of the curve everywhere by the letter *s*) ; therefore by the third theorem : $s.2d^2 :: c.r.$ from which $s = \frac{2d^2c}{r}$ or by putting $r = d$ there will be $s = 2dc \left[= 4\pi r^2 \right]$; and therefore the surface of the sphere is equal to the rectangle whose length is the circumference and the width the diameter of the great circle in the sphere.

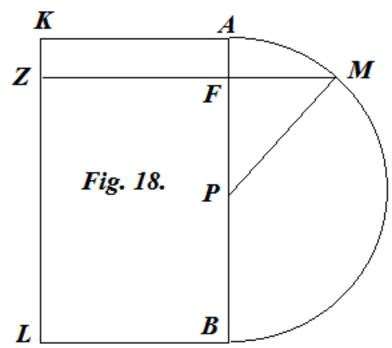


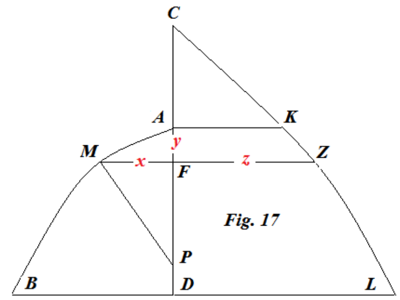
Fig. 18.

It is worth noting, I consider, hence this theorem following all the theorems, and to be by far the most noble, and by which the Prince of Geometers *Archimedes* himself had acquired eternal fame ; Because it is evident the surface of a sphere shall be equal to four of the great circles in that. For there shall be $\text{Q} =$ the greatest area in a circle in the sphere ; but which $\text{Q} = \frac{dc}{2}$ as had been shown by Archimedes ; Therefore $2\text{Q} = dc,$ and $4\text{Q} = 2dc$; but now it has been found that : $s = 2dc$; Therefore $4\text{Q} = s$.

Q.E.D.

PROBLEM II.

To determine the Surface of a Parabolic Conoid.



The latus rectum of the parabola AMB shall be r , by the rotation of which the conoid is produced, the axis shall be AD, the vertex A and there may be called AF, y ; FM x; [i.e. the equation of the parabola is $x^2 = ry$] by some method of tangents there may be found : $\mathrm{PM}^2 = \frac{1}{4}r^2 + ry$;

$\frac{dx}{dy} = \frac{FP}{x} = \frac{r}{2x} \therefore MP^2 = \frac{r^2}{4} + 4ry$,or by putting $\mathrm{FZ} = z$, because there is supposed $\mathrm{PM} = \mathrm{FZ}$, or $\frac{1}{4}r^2 + ry\left[= r\left(\frac{1}{4}r + y\right)\right] = z^2$ which is the equation for a parabola the axis of which is the same as the axis of the given parabola AMB ; the vertex of which is C, with there being $\mathrm{AC} = \frac{1}{4}r$; and latus rectum of this also is r, there will be found

$\mathrm{AKLD} = \sqrt{\frac{4}{9}rv^3} - \frac{1}{12}r^2$, with there being $\mathrm{CD} = v$;

[In modern terms, the area AKLD is given by :

$$\int \sqrt{\tfrac{1}{4}r^2 + ry}.dy = \left[\tfrac{2}{3r}\left(\tfrac{1}{4}r^2 + ry\right)^{\frac{3}{2}}\right]_0^{v-\frac{r}{4}} = \tfrac{2}{3r}(rv)^{\frac{3}{2}} - \tfrac{2}{3r}\left(\tfrac{1}{4}r^2\right)^{\frac{3}{2}} = \sqrt{\tfrac{4}{9}rv^3} - \tfrac{1}{12}r^2. \text{]}$$

but $s.\sqrt{\frac{4}{9}rv^3} - \frac{1}{12}r^2 :: c.r$ $\left[i.e.\ \frac{s}{\sqrt{\frac{4}{9}rv^3 - \frac{1}{12}r^2}} = \frac{c}{r} \right]$ by Theorem 3.

Therefore $s = \sqrt{\frac{4c^2v^3}{9r}} - \frac{1}{12}rc.$

In this manner not only the surfaces of hyperbolic conoids and spheroids can be determined, but any other curved surface which is generated by the rotation of the curve, and these two examples show well enough how in some manner the same method shall be applied to all other curved surfaces.

An observation in the method of measuring figures, produced by a certain most enlightened German gentleman, and published in the Actis Eruditorum of Leipzig.

[The final of 3 papers of interest by D.T. is reproduced below, in an early English translation.]

The most learned author of this paper [D.T. : E.W.Tschirnhaus, a friend of Leibniz in Paris and equally devoted to the advancement of the calculus] had proposed the method in the month of October for the year 1683, as thus he believed it to be complete; so that either the quadrature of a figure, or of its impossibility, could be determined ; and from that he concluded the geometrical quadrature of the circle and of the hyperbola to be impossible. Truly later the most dist. man saw that the proof provided was not completed to perfection, so that thence it would be possible to approve the impossibility of the quadrature of the circle, hyperbola or other figures, as he admits himself freely in the Acta of the following year, where he has said he has been forced by his love of the truth to give this single warning. From which he considers there are certain figures which are not capable of indefinite quadratures and he gives an example of a figure in which he says a particular quadrature can succeed without the general one doing so : yet here the most dist. man is nevertheless talking nonsense, because by his method he concludes the indefinite quadrature to be rejected ; before he had pointed out that his method extended to all indefinitely quadrable figures ; which is impossible from the demonstration, since one from thousands does not make sense ; as will be apparent later. For an infinitude of figures may be given to be indefinitely squarable, which in no manner are squarable by that same method ; and I will put in place an example of which later ; And so that not only will I find the error but the source of the error, also it is considered to add a brief summary of that method.

It makes use of the general equations of curves, of which each and every one of all the curves of the same order may be considered to be expressed: And just as the general quadrature may be sought of such general curves considered; it compares an equation of the specific quadrature with some expressing the nature of the general quadrature ; from which there may be deduced the specific quadrature to be in agreement with the specific general quadrature ; the matter will become evident from an example.

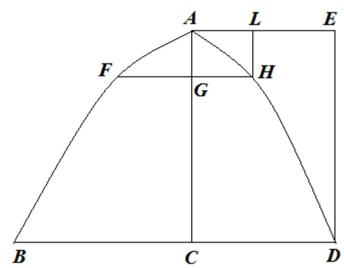


Fig. 20.

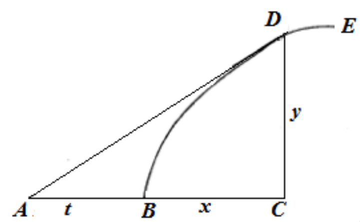
ABC shall be the figure contained by the right lines AC, CB and by the curve AB, and there shall be ACDE = ABC , AGF = AGHL , and the same may be considered everywhere, hence some curve AHD will emerge, that may be called the quadrature, because with its help the area ABC is squared, now an equation is assumed for the general quadrature, AHD, and from that the general quadrature ABC may be deduced: so that if the abscissas AG, AC may be denoted by *x*, and the ordinates of the quadrature CD, GH by *y*, and finally the ordinates for the general quadratic curve by *z*, and the equation may be put for the general squaring curve for which *x* is the ordinate : of two dimensions to be of this kind,

$$\left.\begin{array}{r} by^2+cay+ea^2 \\ +dxy+\ fax \\ +gx^2 \end{array}\right\}=0\,,$$

from which the equation is deduced for the general quadrature, in which the ordinate *z* is also of two dimensions,

$$\left.\begin{array}{l} bz^2+caz+ea^2+\frac{d^2e+c^2g+f^2b-cdf-4beg+a^2x^2}{4bea^2+4bfax+4bgx^2} \\ \quad +2dxz+2fax \\ \qquad +4gx^2-ca^2-2cdax-d^2x^2 \end{array}\right\}=0$$

[Tschirnhaus' s rule, resembling that of *de Sluse* to some extent, can be expressed in more modern terms : for some function of the two variables *x* and *y*, and depending on a constant a , written in the form $F(x,y,a)$, for which the length of the sub tangent can be expressed by $t=x+\frac{aF_a}{F_x}$; see e.g. *Naissance du Calculi…..* p. 84, note 7]



And similarly it is required to investigate the general quadratures for the remaining. Now some particular square figure may be put in place ABC, and the nature of the curve AFB is expressed by this equation $z^2=\frac{9a^2x-12axx+4x^3}{2a-x}$; this equation may be compared with the equation of general squaring now in place, (because the ordinate *z* in the particular squaring curve rises only to the two dimensions) clearly the individual terms of this with the individual terms of that (where *x* obtains the same composition everywhere) and from this comparison there will be $c,\ d,\ e=0$, and $b=\frac{1}{2},\ f=-1$ and $g=\frac{1}{2};$ and he has substituted these values into the equation placed above for the square in which the ordinate *x* is of two dimensions, (because here the ordinate *z* also rises to two dimensions) there will be $\frac{y^2}{2}-ax+\frac{x^2}{2}=0$ or $y^2=2ax-x^2$, the property of the special quadrature AHD in which $\mathrm{AGF}=\mathrm{AGHL}$ and thus the quadrature of the proposed figure will be had.

Yet in that the defect of the reasoning and of the method lies hidden in the reasoning, because he compares all the curves in which z rises to two dimensions (nor beyond) with one and the same general squaring [formula], in which z does not ascent beyond two dimensions; and because he concludes the figure is not indefinitely squarable if this comparable quadratic may not be determined. For the general squarable curves are infinite in number (also deducible from the same method) in which z does not ascend beyond two dimensions, and at some time the equation of the proposed curve may be compared with first, second, third, etc. A quadratic will not be had, and yet a comparison with a thousand quadratics will be able to be determined. For if from the third equation (as he has put for the general quadratic in which x shall be of three dimensions the first of the terms may be removed $by^3 + dxy^2$, it can be deduced from the squarable remaining, in which z does not rise beyond two dimensions, and by which the quadratic may be determined, as that general one put in place does not succeed : And thus from equation four, five, etc. (which one may put for quadratics of higher order) with these terms taken away in which in which y rises beyond two dimensions, the equation of the general squarable may be had from the remainder, which will determine the quadratic, as neither of this being squarable, nor that which I have said it can be determined to be deduced from the third equation, thus so that we do not come upon the required art in the required general quadrature. But because I have said these equations to be deduced for general squaring are deducible from this method ; I wish here to show with a few examples, how here the most distinguished gentleman has found the general equations for quadrature, or perhaps he was able to find these easily.

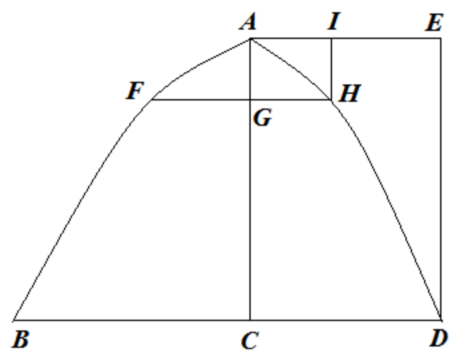


Fig. 20.

It is agreed from *Problem* 22, how from a given equation for some curve [Fig. 20] AHD, another curve AFB shall be required to be found of which the area AGF is equal to the rectangle taken from the below the ordinate GA and with the abscissa AG; that is how from a given quadratic, the quadrature shall be found ; and thus with the assumed equation for the general quadratic (such as I have here from the beginning)
the equation will arise according to the general quadrature. And now here I will consider an example, or another of a figure, in which the quadrature following this method is impossible, and still may be determined by another method. The natural equation of the curve expressing AFB : $z^2 = \dfrac{m^2x^2 + x^4}{p^2}$

[This equation is very indistinct in the two original copies I have seen, and I have assumed the formula shown to agree with the integration set out below as a note.]
in which x may denote the abscissas AC, AG, and z the ordinates BC, GF, m and p are given determinate quantities ; now if the area AGF shall be required to be squarable, this equation is required to be compared with the equation for the general quadrature now discussed, because in this proposed equation z rises to two dimensions; but it is evident

the comparison cannot succeed (as that itself may be reasoned elsewhere) if only the numerator of the fraction present may be compared everywhere, for m^2+x^2 must coincide with $\overline{-d^2e+ag+bf^2-cdf-4beg}+a^2$: an indeterminate with a determinate, which cannot happen, and thus the figure cannot be squared in this manner, and yet this figure itself is indefinitely squarable, evidently

$$\text{AGF}=\sqrt{\frac{m^6+3m^4x^2+3m^2x^4+x^6}{9pp}}=\sqrt{\frac{\left(m^2+x^2\right)^3}{9pp}}=\frac{\left(m^2+x^2\right)^{\frac{3}{2}}}{3p}$$

$$D.AGF=\tfrac{3}{2}\times\tfrac{1}{3p}\times\left(m^2+x^2\right)^{\frac{1}{2}}\times 2x=\frac{\left(m^2+x^2\right)^{\frac{1}{2}}x}{p}\rightarrow z^2=\frac{\left(m^2+x^2\right)x^2}{p^2}.$$

And not only one but an infinity of such squarable figures are able to be found, the quadratures of which are impossible to be found in this manner by Prob. 23. [Fig. 20]. AHD may be defined by this equation $x^9=a^7y^2$, and by *Prob*. 23 the curve AFB may be found of which the area AGF is equal to the rectangle contained under the ordinate GH and with some given right line considered (*a*), and AFB will be defined by this equation $z^2=\frac{81x^7}{4a^3}$;

[$x^9=a^7y^2$ $\therefore$ length of side of square $y=\frac{x^{\frac{9}{2}}}{a^{\frac{7}{2}}}$ & $\frac{dy}{dx}=z=\frac{9}{2}\frac{x^{\frac{7}{2}}}{a^{\frac{7}{2}}}$ & $z^2=\frac{81x^7}{4a^7}$; Note here that *x* and *y* are the abscissa and ordinate as in modern usage.]

and now, if the area AGF shall be squarable following this method, this equation is to be compared with the equation according to the general squaring now treated, because in the proposed equation *z* cannot rise beyond two dimensions ; but a comparison is impossible, because in the proposed curve *x* rises to the seventh power; and in the equation of that for general squaring cannot rise beyond the fourth power ; but the term in which *x* is of the seventh, is unable to be compared with a term in which *x* is of the fourth power ; for following the rule itself, a comparison is thus required to be put in place so that *x* may have the same composition on both sides ; and thus the square cannot be had in this manner, and yet AHD has a square defined by this equation $x^9=a^7y^2$, in which $\text{GH}\cdot a=\text{AGF}$; that is $\sqrt{\frac{x^9}{a^5}}=\text{AGF}$. From which it is agreed abundantly that method does not include all the indefinitely squarable figures ; and infinitely many can be found of which the area cannot be squared according to that method ; for any equation may be assumed in which *z* is not beyond two, and *x* may not be found below four dimensions ; and an equation may be had expressing the nature of the curve by that method which are not squarable: as in these examples $z^2=\frac{x^9}{a^7}$, $z^2=\frac{x^{11}}{a^9}$, $z^2=\frac{x^{13}}{a^{11}}$, etc., which are the equations defining the natures of curves of which the areas may be determined easily and yet in no manner can they be found by that method. But I do not want to digress further into this matter here, hoping I may have given the most distinguished of men some good advice; because the particular reason which impelled me so that I might write this, might not be written by another, which in order that (by showing its errors) I might stimulate him to publish that, by which geometry would be able to assert itself and to move forwards an immense distance beyond the limits put in place by Viet and Descartes.

FINIS.

ADDENDA

A new Method of determining expeditiously the Tangents of Curve Lines.

By D. T. *From the Acta Eruditorum Lips. An.* 1682. *p.*391.

Translated from the LATIN.

How great the use of tangents is, is plain to such, as are appraised, that while the tangents of curves are determined, by that very thing the quadratures of curvilinear spaces may also be exhibited ; Wherefore, general methods of determining the tangents have been invented by the most excellent mathematicians. But as for the most part they require a very prolix calculus, such promote geometry in the best manner, who supply us with some easy method in a thing of so great use; and this *Slusius* in particular has done by a general and very simple rule.

Whence having hit on a rule, which determines the tangents much more generally and with greater simplicity than any hitherto given, I thought I should not begrudge to communicate it to the skillful in these matters.

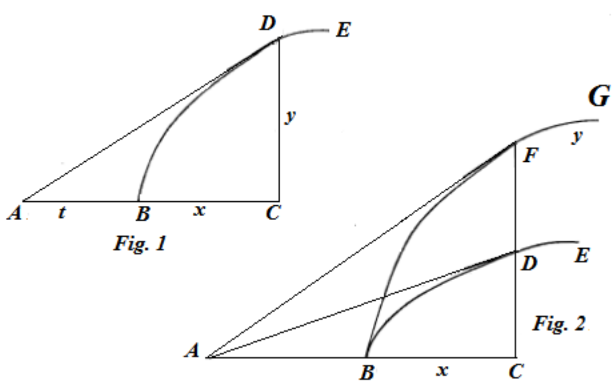


Fig. 1
Fig. 2

Let BDE be any geometrical curve [Fig. 1], as *Descartes* calls them, let the *abscissa* BC be $=x$, the ordinate CD $=y$*;* and let the line determining the tangent AB be $=t$, let moreover the nature of this curve be given, which for instance, let be such

$$y^3+x^3+xxy=xyy-a^3+aay-aax+axx-ayy:$$

These things thus supposed, let the terms of this equation be ordered in such manner, that the greatest power of the ordinate CD or *y* be on one side of the equation: Hence now y^3 will be

$$y^3 = -x^3 - xxy + xyy - a^3 + aay - aax + axx - ayy;$$

but if the greatest power of the quantity *y* be entirely wanting, let all the terms be put $=0$. Let there be made a fraction and its numerator in this manner; namely let all the terms, wherein the known quantity is *a*, be taken with all their signs ; and if the known quantity *a* shall be of one dimension, let unity be prefixed to that term, if of two dimensions, 2, if of three, *&c*. 3, etc. And in the present example the numerator will be

$$-3a^3 + 2aay - 2aax + axx - ayy.$$

Let the denominator be made in this manner, assuming the terms wherein the *abscissa* *x* occurs and retaining the signs, if the quantity *x* be of one dimension, prefix unity as above, but if of two dimensions, 2, and it will be $-3x^3 - 2xxy + xyy - aax + 2axx$. But diminishing each of these by *x*, the denominator will be $-3xx - 2xy + yy - aa + 2ax$.

I say that this fraction is equal to the quantity AB; and therefore in the present case *t* is

$$= \frac{-3a^3 + 2aay - 2aax + axx - ayy}{-3xx - 2xy + yy - aa + 2ax}.$$

And thus in an easy and general way the tangents of all geometrical curves are exhibited. But I shall further show, that the tangents of infinite mechanical curves [Fig. 3 Plate IV.] may be determined by the same method; for, let there be any curve BDE, and let any portion thereof BD be put $=x$; but let DF be *y*; and let any curve BF be formed, for instance, according to the former equation

$$y^3 + x^3 + xxy = xyy - a^3 + aay - aax + axx - ayy.$$

To determine, therefore, the tangent AF of the curve BFG, I proceed thus, let a fraction be assumed as was done above, and to it add the quantity *x*, and this will be equal to AD, a portion of the tangent of the curve BDE : And therefore, in the present case if AD be drawn touching the curve BDE in D, and it be assumed equal to

$$= \frac{-3a^3 + 2aay - 2aax + axx - ayy}{-3xx - 2xy + yy - aa + 2ax} + x,$$

the right line AF being drawn, will touch the curve BFG in the point F. By the same method the tangents of the cycloid and of infinite mechanical curves are very expeditiously determined, and in so general a way, that where hitherto, by any analytical method, the tangent of one curve only is exhibited, here always at one and the same time the tangents of infinite curves may be determined; for, instead of the curve BDE we may suppose infinite curves that the tangent *t* may be always determined by

$$\mathrm{AD} = \frac{-3a^3 + 2aay - 2aax + axx - ayy}{-3xx - 2xy + yy - aa + 2ax} + x,$$

supposing the nature of the curve BFG to be such, that

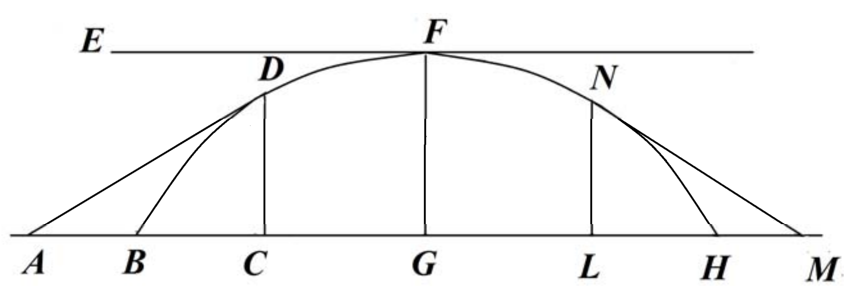
$$y^3 + x^3 + xxy = xyy - a^3 + aay - aax + axx - ayy,$$

and the curve $\mathrm{BD} = x$, the right line $\mathrm{DF} = y$.
I shall give the demonstration of all these things in their proper place, which yet anyone but little conversant in analytics may easily draw from the methods hitherto exhibited by *Descartes*, *Fermat*, *de Sluse*, &c.
[Tschirnhaus' s rule, resembling that of *de Sluse* to some extent, can be expressed in more modern terms : for some function of the two variables *x* and *y*, and depending on a constant *a* , written in the form $F(x, y, a)$, for which the length of the sub tangent can be expressed by $t = x + \frac{aF_a}{F_x}$; see e.g. *Naissance du Calculi…..* p. 84, note 7]

A New Method of determining the Maxima and Minima.

By D. T. From the Acta Eruditorum Lips. An. 1683. p. 122.

As some of my friends, upon seeing my method of tangents*, published in the *Acta Erudit.* for the month of *December* p. 391. *seq. anno* 1682. have greatly desired to know, by what method I might hence determine the *maxima* and *minima*, and whether this might not probably be done by the same method; so I have resolved to communicate some things here in order to satisfy their desire.



We are, therefore, to observe in the first place, that the determination of the *maxima* and *minima* is nothing other than the determination of a special case about the tangents of curves ; and to understand this more easily, I shall illustrate it by an example. Let any indeterminate quantity BC be $= x$, which is to determine some *maximum;* let *a* denote some given line BH, and let the *maximum* to be determined be $aax^3 - 2ax^4 + x^5$.

Let this *maximum* be represented by the line FG, which put $=y$; $aax^3-2ax^4+x^5$, will therefore be $=y^5$, which equation expresses the nature of any curve by means of two indeterminate quantities: But now to find this *maximum*, nothing other is required than that the greatest ordinate of this curve BDFH be determined ; but as this ordinate is the line FG, where the line EF, parallel to the axis BH, is the tangent of this curve, there is nothing other necessary than to find the tangent of this curve BDFH in any point generally, and then all the special cases will be likewise determined, and consequently this case too, where the line EF, or the tangent, is parallel to the axis; in the present case therefore, let AB be $=t$; According to the rule I have given for tangents it will be

$$t=\frac{2aax^3-2ax^4}{3aax^2-8ax^3+5x^4}.$$

We are to observe in the second place, as it is very evident, that if the tangent EF be parallel to the axis, the denominator of this fraction is $=0$:Whence we shall have $3aa-8ax+5xx=0$, an equation, by means of which the quantity *x*, and consequently the *maximum*, is determined.

We are to remark in the third place, that in order to determine the same quantity BG, if any *maximum* be given, yet still another equation may always be found by seeking the tangent on the other side of this curve : But as thus two equations may be had including the same unknown quantity, it will be no difficulty for the skillful in these matters to determine what is sought: Thus putting $\mathrm{LH}=z$, the *maximum* will be $a^3zz-3aaz^3+3az^4-z^5=y^5$. And hence according to my rule of tangents HM will be $=\dfrac{3a^3zz-6aaz^3+3az^4}{2az-9aaz^2+12az^3-5z^4}$: Whence now according to the second remark, the denominator of this fraction $2az-9aaz^2+12az^3-5z^4=0$; in which equation, if instead of *z* you substitute $a-x$, which is equal thereto, you will have $-3axx+5x^3$, or $x=\frac{3a}{5}$, the second equation, by means of which the length B G is found.

And thus I have disclosed a principle, from which is derived as expeditious a method of determining the *maxima* and *minima*, as I have hitherto seen ; and it is as follows.

Let BH any given quantity be $=a$, in respect of which any *maximum* or *minimum* is sought; put $\mathrm{BG}=x$, and $\mathrm{GH}=z$:

Now 1. it will be $z=a-x$.

2. Let the *maximum* be formed by means of the quantity *z*, so that *x* does not enter into this expression ; and let, for instance, that *maximum* be $a^3z^5-3aaz^6+3az^7$, in which quantity let 2 be prefixed to the term where *z* has 2 dimensions, 3, where 3, but 4, where 4, *&c.* and thus let the exponent of the power of *z* be always prefixed to the same term; but let the whole aggregate thus produced be $=0$: It will therefore be in the present example $5a^3z^3-18aaz^5+21az^7-8z^9=0$, or $3a^3-18aaz+21azz-8z^3=0.$

3. In this last equation let $z=a-x$ be restored ; and a very simple equation will be found for the quantity *x*, which determines the required *maximum* or *minimum*, as in the present instance, if in the equation, *viz.* $3a^3-18aaz+21azz-8z^3=0$, $z=a-x$ be restored, *x* will be found $=\frac{3a}{8}$, by means of which the *maximum* required is determined.

A method of determining either the Quadrature, or the impossibility thereof, in a given figure, terminated by right lines and a geometrical curve. By D. T. *From the* Acta Eruditorum Lips. *An*.1683. p. 433.

Translated from the LATIN.

THAT, I may briefly, and at the same time with perspicuity, disclose: these things to the skilful in these matters, it is to be noted.

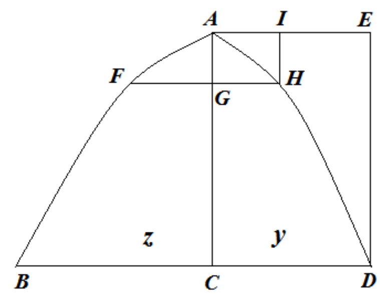


Fig. 1

1. Let the space ACB be terminated by the geometrical curve AFB, by the axis AC, and by the ordinate BC: Let now the space ACB be understood to be equal to the rectangle AEDC, as also the space AGF to be equal to the rectangle AGHG : If now the same thing be everywhere supposed, there will hence arise a curve or some line AHD; but this will be either a geometrical or mechanical curve : If the former, the space ACB will both in the whole, as in all its parts, be geometrically squarable, and therefore I call such a quadrature possible: But if it be any mechanical curve, the space ACB, both as to the whole and all its parts, will be mechanically squarable, or its quadrature cannot be found geometrically; and, therefore, I call such a space impossible

II. Now, let the following equations be formed:

$$\left.\begin{matrix} by+ca \\ +dx \end{matrix}\right\}=0. \qquad \left.\begin{matrix} by^2+cay+ea^2 \\ +dxy+fax \\ +gx^2 \end{matrix}\right\}=0. \qquad \left.\begin{matrix} by^3+cay^2+ea^2y+ha^2 \\ +dxy^2+faxy+iax \\ +gx^2y+kax^2 \\ +lx^3 \end{matrix}\right\}=0.$$

and so on as far as you please.

But we are to observe of these equations, that they exhibit all geometrical curves AHD and all their possible relations to a right line ; that *x* represents all the abscissa's AC, but *y* all the ordinates CD; *b, c, d, e, f, g,* &c. design the known quantities, prefixed to these terms, with what signs so ever they are affected ; but the quantity *a* is instead of unit, in order to make the dimensions equal in all the terms of these equations. But that without a prolix. calculus I may explain the following theorems which demonstrate the quadrature of a given geometrical curve, or the impossibility thereof, let it moreover be

$$c = ca + dx \qquad i = ca + 2dx$$
$$d = ea^2 + fax + gx^2 \quad \text{also} \quad k = 2ea^2 + 3fax + 4gxx$$
$$c = ha^3 + ia^2x + kax^2 + lx^3, \qquad l = 3ba^3 + 4ia^2x + 5kbax^2 + 6lx^3.$$

III. Farther let all the abscissas AC of the geometrical curves AFB be $= x$; but their ordinates $BC = z$; then let it be put

1. $\mathrm{B} = bz + i$ and the first theorem will be $\mathrm{B} = 0$.

2. $\left\{\begin{array}{l} \mathrm{B} = 2bz + i \\ \mathrm{C} = cz + k \end{array}\right\}$ the second theorem will be $\mathrm{BB}d - BCc + CCb = 0.$

$\left\{\begin{array}{l} \mathrm{B} = 3bz + i \\ \mathrm{C} = 2cz + k \\ \mathrm{D} = \mathrm{d}z + l \end{array}\right\}$ the the third theorem will be

3.

$$\left.\begin{array}{c} \mathrm{B}^3e^2 - B^2Cde + BC^2ce - C^3be + C^2Dbd - CD^2bc + D^3b^3 \\ -2B^2Dce + 3BCDbe - 2BD^2bd \\ +B^2Dd - BCDcd + BD^2c^2 \end{array}\right\} = 0.$$

These things I shall further enlarge on in their proper place, which now I am obliged to omit, in order to avoid prolixness.

IV. Now the use of these theorems is twofold. In the first place, we hence very expeditiously determine all the possible quadratures of geometrical curves; for, assuming any geometrical curve AHD, we immediately find, by means of the theorems just now exhibited, the quadrature of the space ACB, terminated by the geometrical curve: And as the equations in Remark II exhibit all the curves and all their possible relations to a right line, even by these theorems all the possible spaces, terminated by a geometrical curve, are squared; which let it suffice to illustrate by one example : Let the curve AHD be a circle, whose nature is expressed by this equation $y^2 = 2ax - x^2$ or $y^2 - 2ax + x^2 = 0$; now in the equation, which is of the same degree in Remark II.

$$\left.\begin{array}{r} by^2+cay+eaa \\ +dxy+fax \\ +gx^2 \end{array}\right\}=0, \text{ it will be } \left.\begin{array}{r} b=1 \\ c,d,e,=0 \\ f=-2 \\ g=1. \end{array}\right\}$$

Whence in the second theorem $\mathrm{BB}d-BCc+CCb=0$ (which is to be chosen, because the assumed equation is of two dimensions) it will be

$$\begin{array}{lll} c=0 & i=0 & \mathrm{B}=2x \\ d=-2ax+x^2 & k=-6ax+xx & \mathrm{C}=-6ax+4xx. \end{array}$$

All which if restored there is obtained $z^2=\frac{9a^2x-12axx+4x^3}{2a-x}$, explaining the nature of the curve AFB; all whose spaces are squared by means of the circle, in regard the space AGF is always equal to the rectangle AIHG. In the same manner a different geometrical curve AHD being assumed, a different geometrical curve is found, and quadrable: Hence thus it appears, in what manner these theorems include all possible quadratures.

V. The other use of these theorems is no less considerable, which is, that a geometrical curve being given, not the quadrature of any other may be found, as was just now explained, but either the quadrature of the given or proposed curve itself be exhibited, or its impossibility demonstrated; which we shall illustrate by two examples; for, in the first place let the nature of the given curve AFB be expressed by this equation

$$zz=\frac{9aax-12axx+4x^3}{2a-x}$$

Now the question is, whether its quadrature may be given or no ? And, indeed, it is plain from Remark IV that it is given : But suppose that this now is unknown, and observe the process in determining what is sought.

1. Because the ordinate *z* rises to two dimensions, the third theorem is to be chose (had it had three dimensions, the third theorem was to have been assumed, and so on) but that theorem is as follows

$$\mathrm{BB}d-BCc+CCb=0.$$

2. In this theorem let the quantities B and C, as also *b*, *c* and *d*, be restored, and it will be

$$\left.\begin{array}{l} bz^2+caz+ea^2+\frac{d^2e+c^2g+f^2b-cdf-4beg+a^2x^2}{4bea^2+4bfax+4bgx^2} \\ \quad +2dxz+2fax \\ \qquad +4gx^2-ca^2-2cdax-d^2x^2 \end{array}\right\}=0.$$

3· Now let all the terms of this theorem be compared with all those

(where *x* in each case has the same dimension) of the equation which explains the nature of the given curve ; and hence there will arise some new equations, by help of which the quantities *b, c, d, e, f,* &c. will be found:
And here *b* is found $=\frac{1}{2},\ c,d,e=0,\ f=-1\ \ g=\frac{1}{2}$; all which as they involve no absurdity, the quadrature of the given curve is given; had they included any impossibility, or were they all equal to nothing, the quadrature would be impossible.
4. If now from the equations of the second remark that of two dimensions be assumed, such as is

$$\left.\begin{matrix} by^2+cay+eaa \\ +dxy+\ fax \\ +\ gx^2 \end{matrix}\right\}=0$$

(because the quantity *z* of the given curve also rises to two dimensions) and in this let the quantities *b, c, d, e, f,* just now found, be restored, it will be $\frac{yy}{2}-ax+\frac{xx}{2}=0$, or $yy=2ax-xx$, the property of the curve AHD, by help of which the given curve is squared; consequently, it is now determined, that the proposed curve admits of quadrature, and to what known space it is equal.
Q.E.F.
For a second example, let the given geometrical curve AFB be a circle, whose property is $zz=2ax-xx$; now the question is, whether its quadrature be possible or no? To discover this, as the quantity *z* rises to two dimensions, we must here use the same theorem, as in the preceding example. And if now all the terms of the given equation (which expresses the nature of the given curve) be compared with all the terms of that theorem, *b, c, d, e, f* will be found $=0$; and consequently, there is given no geometrical curve A H D, by means of which the circle may be squared ; and therefore, the quadrature of the circle, in the sense it is commonly sought for by Mathematicians, is impossible. If the hyperbola be assumed to be squared, the same thing entirely will be found, namely that its quadrature is impossible : But from these things it will abundantly appear, in what manner by help of the like theorems, some of which I have given in Remark IV. either the quadrature of a given figure, terminated by a geometrical curve, may be exhibited,· or its impossibility demonstrated (as the same process may be performed in all geometrical curves in the manner I have explained in these examples) which was my design at this time to disclose. But we are to observe

VI. That what I have just now shown is in no manner contrary to the quadrature of the circle and hyperbola, which *Leibnitz* has exhibited in the *Acta Eruditorum, An.* 1682 ; for, while the geometrical quadrature of any space is not given, in that case the last resource is to express it by a series of numbers decreasing *in infinitum*: And 1 am persuaded, that there can be found no arithmetical expression of the quadrature of the circle or hyperbola mere simple, than that which the illustrious *Leibnitz* has published. But as to mechanical curves AHD, and which seem to be very simple in their nature, that by means thereof the circle and hyperbola, nay all curves, that admit not of a geometrical quadrature, may be mechanically squared, I shall exhibit on a proper occasion the method of determining them, when it shall appear, how great an affinity they have with geometrical curves, as their spaces are always equal to the space terminated by a geometrical curve.

METHODUS FIGURARUM LINEIS RECTIS & CURVIS COMPREHENSARUM QUADRATURAS DETERMINANDI;

Authore Johannes Craige.

LONDONI:

Impensis *Mosis Pitt,* ad insigne Angeli in Caemeterio D. *Paulo,* MDCLXXXV.

HONORABILI VIRO

DOMINO

ROBERTO DAWES,

BARONETTO *ANGLO,*

TRACTATUM HUNC,

Benevolentiae & Observantiae ergo,

D.C.Q. **[Dicat consecratque]**

JOHANNIS CRAIGE.

V

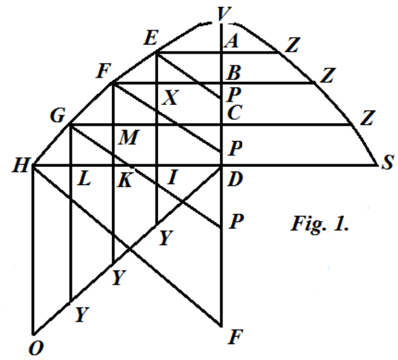
Methodus Figurarum, &c.

Optime nuper observarunt Geometrae quasdam esse Figuras indefinitae Quadraturae capaces, quae tum quoad totas, tum quoad singulas partes sunt Quadrabiles; Alias vero esse, quae, licet hujusmodi Quadraturam indefinitam non admittant, aliquam tamen habent portionem quadrabilem, imo tota figura nonnunquam Quadrari potest, cum quaelibet ejus pars non potest. Nec credibile est ex alio fonte eorum errorem ortum esse, qui Circuli, Hyperbolae, & aliarum quarundam figurarum Quadraturas impossibiles existimarunt, quam quia hanc figurarum distinctionem non considerarunt. Methodis enim utentes quae supponunt figuras esse indefinite Quadrabiles, cum aliquam Quadrandam assumerent, quae eorum Methodos recusabat, statim illius Quadraturam impossibilem esse crediderunt; cum exinde amplius non esset concludendum, quam Methodos quibus usi sunt esse inperfectas, & ad omnes figuras non extendere. Sed cum institutum meum non sit aliorum errores detegere, ast quid in hac materia excogitavi paucis exponere; Methodum hic tradam (non ex Arithmeticis sed Geometricis principiis deductam) quae figurarum utriusque generis quadraturas determinabit. Prioris generis Geometricas, posterioris vero Algebraicas quadraturas per series infinitas exhibebit. Et quia Methodus quae speciales talium figurarum quadraturas determinet, a nemine hactenus vulgata est, speramus praeclarum illum Germanum (qui publice eam promisit, & omnino in potestate sua esse asseruit, in Actis Eruditorum Lipsae publicatis) suam brevi in lucem emissurum.

Theorema 1.

Sit Curva quaevis VH. (cujus axis VD, applicata HD ad VD perpendicularis) item linea VZS talis, ut si a curvae puncto libere sumpto puta E ducatur recta EP ad Curvam, & EAZ ad axem perpendicularis, sit recta AZ, interceptae AP aequalis, erit spatium $\mathrm{VDS} = \frac{\mathrm{DHq}}{2}$.

Demonstratio :



Sit angulus HDO semi-rectus, & aequi secetur UD, indefinite punctis A, B, C per quae ducantur, EAZ, FBZ, GCZ ad HD parallellae, & Curvae occurrentes in E, F, G a quibus ducantur EIY, FKY, GLY, ad UD parallelae, quin & rectae EP, FP, GP, HP sint Curvae VH perpendiculares. Est Triangulum HLG simile triangulo PDH (nam ob indefinitam sectionem Curvula GH pro recta haberi potest) quare est $\mathrm{HL} \cdot \mathrm{LG} :: \mathrm{PD} \cdot \mathrm{DH}$, adeoque $\mathrm{HL} \times \mathrm{DH} = \mathrm{LG} \times \mathrm{PD}$, hoc est, $\mathrm{HL} \times \mathrm{HO} = \mathrm{DC} \times \mathrm{DS}$, & simili discursu monstrabitur, quoniam Triangulum GMF triangulo PCG assimilatur, fore $\mathrm{LK} \times \mathrm{LY} = \mathrm{CB} \times \mathrm{CZ}$ & similiter erit $\mathrm{KI} \times \mathrm{KY} = \mathrm{BA} \times \mathrm{BZ}$, itidem erit $\mathrm{ID} \times \mathrm{IY} = \mathrm{AV} \times \mathrm{AZ}$; unde constat triangulum HDO (quod a rectangulis istis $\mathrm{HL} \times \mathrm{HO} + \mathrm{LK} \times \mathrm{LY} + \mathrm{KI} \times \mathrm{KY} + \mathrm{ID} \times \mathrm{IY}$ minime a differt) aequari Spatio VDS (quod itidem a rectangulis $\mathrm{DC} \times \mathrm{DS} + \mathrm{CB} \times \mathrm{CZ} + \mathrm{BA} \times \mathrm{BZ} + \mathrm{AV} \times \mathrm{AZ}$ minime differt) hoc est: $\mathrm{VDS} = \frac{\mathrm{DHq}}{2}$.

Quod erat Demonstrandum.

Nobile hoc Theorema debetur viro Celeberrimo. D. Doctori *Barrow*, qui innumera habet, & sublimia Theoremata circa linearum Curvarum proprietates : nec mihi quenquam (quorum Scripta edita sunt) vidisse contigit (imo nec aliis contigisse puto) qui tanto judicio, & Successu tanto, abstrusiorem hanc & minus cultam Geometriae partem, tractavit & promovit.

PROB. I.

Data relatione inter PM (*quae distantiam inter Curvae perpendicularem PC & ordinatim applicatam MC designat*) *& abscissam AM* (*quae distantiam inter applicatam & verticem A designat*) *aequationem invenire Curvae lineae AC naturam definientem.*

Ut omnes Curvas sub una Regula generali comprehendam adnoto in quacunque linea Curva AC fore semper $\mathrm{PM} \times \mathrm{MT} = \mathrm{CMq}$ propter angulum rectum PCT. Quare multiplico singulos terminos PM denotantes per terminum AM (prius in diversos numeros incognitos multiplicatum) & produtum pono aequale Quadrato applicatae CM. Ratio hujus regulae colligi potest, ex Methodo inveniendi Tangentes a Clarissimo Slusio edita. in *Actis Philosophicis Reg. Societatis Anglicanae*. Exemplis rem illustrabo.

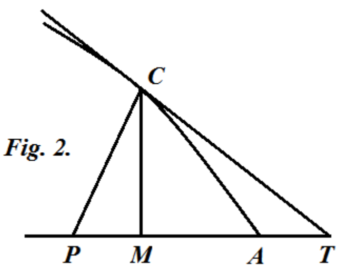


Fig. 2.

*Exemp.*1. *Fig.* 2. Detur $\mathrm{PM} = \frac{1}{2}\mathrm{r}$, & vocetur AM *y*, CM *x*, a, b, c, i, &c. denotent quantitates cognitas & determinatas, item *l, m, n, h. k* &c. numeros incogitos. Jam juxta regulam multiplico $\frac{1}{2}\mathrm{r}$ per ny, & productum $\frac{\mathrm{ny}}{2} = x^2$. quae est aequatio ad parabolam.

Exemp. 2. Fig.2.Sit $\mathrm{PM} = y + \frac{1}{2}r$ & quaerenda sit aequatio illam curvam determinans: procedens secundum regulam multiplico $\frac{1}{2}r + y$ per ny, my & productum $\frac{nry}{2} + my^2$ pono aequale quadrato ab *x*, nempe $\frac{nry}{2} + my^2 = x^2$ quae est aequatio ab curvam quaesitam.

Exemp. 3. Esto $\mathrm{PM} = \frac{y^2}{a} + a$ & quaeratur curva AC, in qua Sit $\mathrm{PM} = \frac{y^2}{a} + a$ multiplico $\frac{y^2}{a} + a$ per *ny*, *my*, eritque productum $\frac{ny^3}{a} + may = x^2$.

Exemp.4. Sit $\mathrm{PM} = \frac{y^4}{aaa} + \frac{y^3}{aa} + \frac{y^2}{a} + y,$ & querenda sit *aequatio* istius curvae naturam definiens, secundum regulam multiplico

$\frac{y^4}{aaa}+\frac{y^3}{aa}+\frac{y^2}{a}+y$ per *ny*, *my*, *ly*, *hy*, eritque

$\frac{ny^5}{a^3}+\frac{my^4}{a^2}+\frac{ly^3}{a}+hy^2=x^2$ qua est aequatio quaestita.

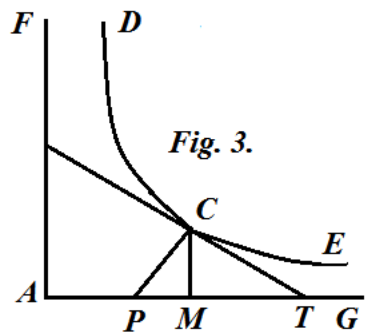


Fig. 3.

Denique sit $\text{PM}=\frac{a^2}{y^2}$, multiplicato $\frac{a^3}{y^2}$ per ny, erit productum $\frac{na^3}{y}=x^2$, vel $na^3=yx^2$.
sed quia *n, m, l, h;* &c. adhuc sint incognita, modum ostendam ea determinandi.

PROB. II.

Quantitates l, m, n. &c. *in praecedenti Problemate usurpatas determinare.*

Per aequationem inventam investigo PM (procedendo secundum vulgarem aliquam methodum inveniendi Tangentes) & ejus valorem compare cum valore dato, nempe singulos hujus cum singulis illius terminis quae comparatio determinabit, *l, m, n,* &c.

Ut in exemplo primo $\frac{nry}{2}=x^2$, invenio $\text{PM}=\frac{3}{4}nr$, hunc valorem comparo cum valore dato sc. $\frac{1}{2}r=\frac{1}{4}nr$ unde post reductionem $n=2$, quare substituto hoc valore pro (*n*) in aequatione $\frac{nxy}{2}=x^2$ erit $ry=x^2$.

Sic in example secundo $\frac{nry}{a}+my^2=x^2$ invenio fore $\text{PM}=\frac{nx}{4}+my$, jam comparo utrosque hos terminos cum correspondentibus terminis valoris dati sc. $\frac{nr}{4}=\frac{r}{2}$ unde $n=2$, deni $my=y$ unde $m=1$. si substituantur hi valores prodibit aequatio quaesita & plene determinata $ry+y^2=x^2$.

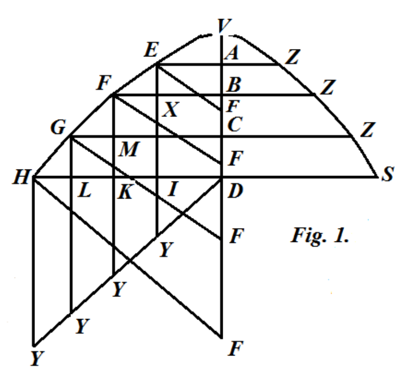


Fig. 1.

Similiter in exemplo 3 : $\frac{ny^3}{a}+may=x^2$ inveni $\text{PM}=\frac{3ny^3}{2a}+\frac{ma}{2}$, facta igitur debita comparatione *sc.* $\frac{3ny^2}{2}+\frac{y^2}{a}$ erit $n=\frac{2}{3}$,& ex $\frac{ma}{2}=a$, erit $m=2$,& substitutis his valoribus erit $\frac{2y^3}{3}+2ay=x^2$.

Et in exemplo 4. erit

$$\mathrm{PM}=\frac{5ny^4}{2a^3}+\frac{2my^3}{a^2}+\frac{2ly^2}{2}+by.$$

& facta comparatione horum terminorum cum terminis dati erit

$\frac{5ny^4}{2a^3}=\frac{y^4}{a^3}$, inde $n=\frac{2}{5}$, & ex $\frac{2my^3}{a^2}=\frac{y^3}{2a}$ erit $m=\frac{1}{2}$, ex $\frac{3ly^2}{2a}=\frac{y^2}{a}$ erit $l=\frac{2}{3}$,
& ex $hy=y$ erit $h=1$. Et substituendo hos Valores, erit equatio

$$\frac{2ny^5}{5a^3}+\frac{y^4}{2a^2}+\frac{2y^3}{3}+y^3=x.$$

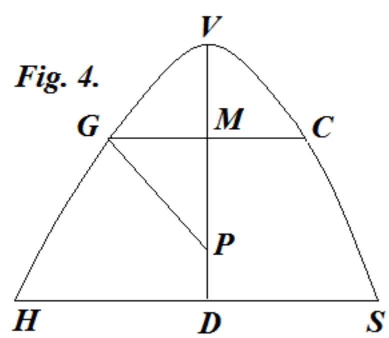
Denique in *Exemp.* 5. Est $\mathrm{PM}=\frac{ny^3}{2y^2}=\frac{a^3}{y^2}$ unde $n=2$ unde n = 2, [*Fig.* 3]
adeoque $2a^3=yx^2$, quae est ad Hyperboliformem DCE.
Haec duo Problemata (vel potius duas partes unius Problematis) fusius sum profecutus, eo quod a nemine adhuc tractata sint, saltem quorum scripta ad manus meas pervenerunt; tum maxime, quod horum ope Figurarum Quadraturas sim determinaturus.

PROB. III.

Parabolae Quadraturam determinare.

Esto parabola *VCS* cujus latus rectum *Ftg.* 4 sit *r.* VM vocetur, *y*, MC, *z.* unde ex natura parabolae $\sqrt{ry}=z=\mathrm{MC}$ tum per problema primum inveniatur curva VH, talis ut sit $\mathrm{PM}=\sqrt{ry}$ (per PG hic & in sequentibus intelligenda est Curvae quaestiae perpendicularis) sed per methodum jam traditam invenio Curvam quaesitam definire per hanc equationem $nry^3=x^4$ (per *x* designo applicatas GM, HD Curvae quaesitae) & determinando

Fig. 4.



n per Prob. 2, invenies $n=\frac{16}{9}$ unde $\frac{16}{9}ry^3=x^4$, ac proinde $\sqrt{\frac{4}{9}ry^3}=\frac{x^2}{2}=\frac{GM}{2}q=VMC$, ut constat ex Theoremate jam praemisso.

PROB. IV.

Paraboloidis Cubicalis Quadraturam determinare.

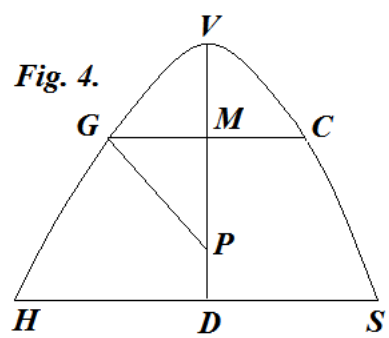
Esto VCS parabolois Cubicalis, VD axis & latus rectum *r*, & $\text{VM} = y$, unde eae natura Curvae istius erit $rry = z^3$ adeoque $\sqrt[3]{rry} = z$, propterea pro determinatione Quadraturae areae VMC invenienda est Curva (per Prob. 1.) VH talis ut sit semper $\text{PM} = \sqrt[3]{rry}$, & procedens secundum regulam ibi propositam invenio Curvam VH definiri per hanc aequationem $nr^2y^4 = x^6$ &. determinando *n* (per Prob. 2.) inveniens $n = \frac{27}{8}$, adeoque aequationem quaesitam esse $\frac{27}{8}r^2y^4 = x^6$ unde $\frac{3}{4}\sqrt[3]{r^2y^4} = \frac{x^2}{2} = \frac{\text{GM}}{2}q = \text{VMC}.$ Et in hunc modum Quadrantur infinitae paraboloides quae definiuntur per $r^3y = z^4;\ r^4y = z^5,\ r^5y = z^6,$ &c.

PROB. V.

Paraboloidis Semicubicalis Quadraturam invenire.

Sit VCS parabolois semicubicalis, cujus haec est proprietas $ry^2 = z^3$, unde $\sqrt[3]{ry^2} = z^1$, invenienda igitur est Curva VH in qua $\text{PM} = \sqrt[3]{ry^2} = \text{MC}$; & per problema primum invenio illam definiri per hanc aequationem $nry^3 = x^6$; & ut determinetur *n* procedo in hunc modum per Prob. 2. quaero PM ex aequatione inventa $ny^5r = x^6$ & invenio $\text{PM} = \dfrac{5nry^4}{\sqrt[3]{216\, n^2r^2y^{10}}}$ & facta comparatione cum dato Valore erit $\dfrac{5nry^4}{\sqrt[3]{216\, n^2r^2y^{10}}} = \sqrt[3]{ry^2}$, unde

Fig. 4.



provenit $n = \frac{216}{125}$ post debitam reductionem, adeoque Curva VH definitur per $\frac{216ry^5}{125} = x^6$, unde erit $\frac{3}{5}\sqrt[3]{ry^5} = \frac{x^2}{2} = \frac{\text{GM}}{2}q = \text{VMC}.$ Et eodem modo Quadrari possunt paraboliformes infinitae, quae definiuntur per $ry^3 = z^4;\ ry^4 = z^5,\ ry^5 = z^6,$ &c.

PROB. VI.

In Hyperbola OCN *quadranda sit Area interminata* OCMVL.

Esto Hyperbolae potentia $=a^2$, unde $\frac{a^2}{y}=z$, inquirenda igitur est Curva VH talis ut in ea sit semper $\mathrm{PM}=\frac{a^2}{y}$ at nullam esse hujusmodi Methodus jam tradita statim deprehendit : nam juxta regulam multiplicanda est $\frac{a^2}{y}$ per *ny*, & productum na^2 non potest poni aequale x^2, Quadratum determinatum nequit aequari Quadrato indeterminato, ac proinde concludendum est Spatium interminatum non esse Quadrabile : nam si daretur illius Quadratura, darentur etiam Curva quaedam VH in qua esset semper $\mathrm{PM}=\mathrm{MC}$.

PROB. VII.

Hyperboliformis OCN *cujus haec sit proprietas* $yz^2=a^3$.
Quadraturam determinare Areae Interminatae OCMVL.

Quoniam ex natura Curva $\sqrt{\frac{a^3}{y}}=z=\mathrm{MC}$ *Fig.*5 . quaerenda est curva VH in qua semper sit $\mathrm{PM}=\sqrt{\frac{a^3}{y}}$, atque per *Prob.* 1, invenio Curvam VH definiri per $na^3y=x^4$ & determinando *n* per *Prob.* 2. Invenies $n=16$, adeoque $16a^3y=x^4$ unde Spatium interminatum $\mathrm{OCMVL}=2\sqrt{a^3y}$.

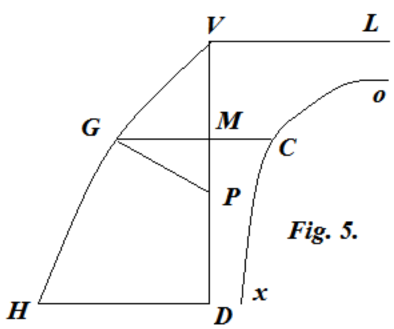


Fig. 5.

PROB. VIII.

Sit natura Hyperboliformis difinita hac aequatione $yz^3 = a^4$, *& Quadranda sit Area interminata OCMVL.*

Ex natura curvae $\sqrt[3]{\frac{a^4}{y}} = z$, & invenietur curva VH in qua $\mathrm{PM} = \sqrt[3]{\frac{a^4}{y}}$ definiri per $27a^4y^2 = x^6$, Unde $\frac{3}{2}\sqrt[3]{a^4y^2} = \frac{x^2}{2} = \frac{GM}{2}q = \mathrm{OCMVL}$. Et sic quadrantur Hyperboliformes infinitae definitae per $yz^4 = x^5$, $yz^5 = x^6$, $yz^6 = x^7$, &c.,

PROB. IX.

In Hyperboliformi OCK cujus haec sit proprietas $y^2z = a^3$ *Quadranda sit aerae interminata OCMVL.*

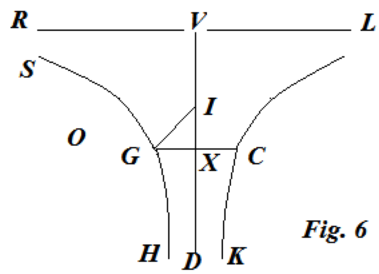


Fig. 6

Ex hujus curvae natura manifestum est $z = \frac{a^3}{y^2} = \mathrm{MC}$ quare Quaerenda est curva aliqua, ut distantia inter ejus perpendicularem & applicatam sit aequalis $\frac{a^3}{y^2}$ & procedendo secundum usitatam Methodum invenio Curvam quaesitam definire per $yx^2 = na^3$ & determinando (n) per Prob. Secundum erit $n = 2$ adeoque aequatio est $2a^3 = yx^2$, quae itidem est aequatio ad Hyberboliformem (sed alterius naturae) SGH) & quoniam illius perpendicularis GP, inter verticem & applicatam cadit, vel sursum tendit ideo $\frac{a^3}{y} = \frac{x^2}{2} =$ KCMD. Et Aream OCMVL esse earum numero quas Geometrae vocant plusquam infinitas, jam monuit clarissimus Nostras *D.David Gregorius* in pulcherrimo suo Tractatu, de Dimensione Figurarum.

PROB. X.

Sit ACD curva talis ut *ducta ut cunque MC ad AD normali, sit potestas quaevis ipsius AD ad similem potestatem partis AM, ut potestas quaevis partis DM ad similem potestatem applicatae MC, & determinanda sit Quadratura Areae AMC.*

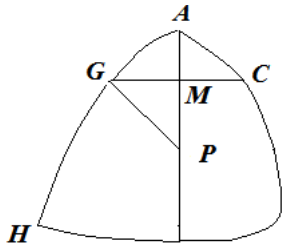


Fig. 7

Esto AD = b, Fig. 7, & exponens illius potestatis 2, AM vocetur y unde etiam exponens illius potestatis est, 2 esto praeterea exponens potestatis lineae DM seu $b-y$, (1) adeoque exponens applicatae MC seu z est 1, tum ex naturae lineae curva. $b^2 y^2 :: b-y \cdot z$ unde $z=\frac{by^2-y^3}{b^2}$, Quaeratur ergo curva AH in qua sit $\mathrm{PM}=\frac{by^2-y^3}{b^2}$, invenieturque per Prob. 1. & 2. illam definiri per $\frac{2}{3}by^3-1y^4=b^2x^2$, unde $\frac{y^3}{3b}-\frac{y^4}{4bb}=\frac{x^2}{2}=\mathrm{AMC}$. Atque haec eadem est curva de qua loquitur D. *Cartesius* in *tom.*3. Epist. pag. 219. quam praeferendam putat (ob constructionis facilicatem) curvae quam Galli vocant *la Galande.*

PROB. XI.

Determinanda sit Quadratura Areae AMC, & definiatur natura Curvae per
$$y^5+ay^4+a^2y^3+a^3y^2+a=a^4z.$$

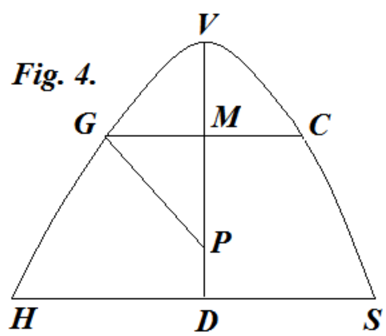


Fig. 4.

Quaerenda est Curva VH, Fig. 4, talis ut in ea sit semper

$$\mathrm{PM{=}MC}=z=\frac{y^5}{a^4}+\frac{y^4}{a^3}+\frac{y^3}{a^2}+\frac{y^2}{a}+a.$$

Et per Prob. 1. definietur per

$$ny^6+may^5+la^2y^4+ka^3y^3+ha^3y=a^4x^2$$

& determinando n, m, l, k, h (per *Prob. 2.)* erit $n=\frac{1}{3},\ m=\frac{2}{5},\ l=\frac{1}{2},\ k=\frac{2}{3},\ h=2$; adeoque aequatio quasita est $\frac{1}{3}y^6+\frac{2}{5}ay^5+\frac{1}{2}a^2y^4+\frac{2}{3}a^3y^3+2a^3y=a^4x^2$, adeoque

$$\frac{y^6}{6a^4}+\frac{y^5}{5a^3}+\frac{y^4}{4a^2}+\frac{y^3}{3a}+ay=\frac{x^2}{2}=\frac{GMq}{2}=\mathrm{AMC}.$$

Atque hactenus solas illas figuras tractavi quae sunt indefinite Quadrabiles, & quantillo labore earum Quadraturae, per hanc Methodum determinentur, aliis judicandum relinquo : Ad illas jam progredior quae hujusmodi Quadraturam respuunt *:* & expresse moneo me Quadraturas quas hic exhibiturus sum per series infinitas, non pro Geometris sed Algebraicis vel Arithmeticis habere.

PROB. XII.

Circuli Quadraturam determinare.

A Circulo initium faciam, qui omnium linearum curvarum simplicissima est, si curvae simplicitas non ex aequationis, sed descriptionis (ut re vera debet) simplicitate aestimetur. Sit itaque Circuli Quadrans ASD in quo AM vocetur *y*, & ordinata MC *z*, & radius $\mathrm{AL} = r$, tum ex Circuli natura erit $z^2 = r^2 - y^2$,

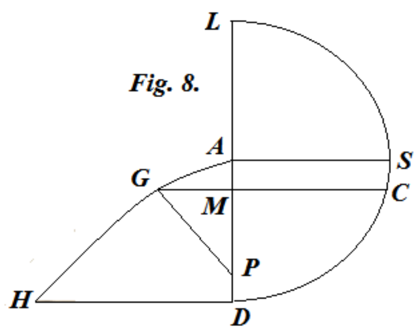


Fig. 8.

ac proinde $z = \sqrt{r^2 - y^2}$ hunc valorem resolvo in seriem secundum Methodum celeberrimi D. *Isaaci Newtoni*, & invenio $z = r - \frac{y^2}{2r} - \frac{y^4}{8r^3} - \frac{y^6}{16r^5} -$ etc. quaerenda igitur Curva AGH in qua

$\mathrm{PM} = r - \frac{y^2}{2r} - \frac{y^4}{8r^3} - \frac{y^6}{16r^5} - .....$etc.

& invenietur per Prob. primum Curvam quaesitam definiri per hanc aequationem.

$nry - \frac{my^3}{2r} - \frac{ly^5}{8r^3} - \frac{ky^7}{16r^5} = x^2$; & determinando Quantitates, *n, m, l, k,* per Prob. secundum invenietur $n = 2,\ m = 3,\ l = \frac{2}{7}$, & substituendo hoc valores, aequatio erit

$2ry - \frac{y^3}{3r} - \frac{y^5}{20r^3} - \frac{y^7}{56r^5} = x^2$. Unde $ry - \frac{y^3}{6r} - \frac{y^5}{40r^3} - \frac{y^7}{112r^5} = \frac{x^2}{2} = \frac{\mathrm{GMq}}{2} = \mathrm{AMCS}$.

Vel si quaereretur Quadratura totius Quadrantis , erit $\mathrm{ASD} = r^2 - \frac{1}{6}rr - \frac{1}{40}rr - \frac{1}{112}rr - ...$etc. Unde $4rr - \frac{2}{3}rr - \frac{1}{8}r^2 - \frac{1}{28}r^2 - ... =$ toto Circulo. Et si haec series per numeros exprimatur, ponendo $r = \frac{1}{2}$ erit area Circuli $1 - \frac{1}{4} - \frac{1}{32} - \frac{1}{112} -$ etc. in infinitum Notatu dignum arbitror hinc elici posse dimensionem Zonae Circularis, quam a celeberrimo Geometra D. *Isaaco Newtono* inventam refert clariss. *David Gregorius* in memorato tractatu. Esto ABCD Zona cujus latitud. $\mathrm{VL} = y$, & Circuli radius $= r$ per praecedentem Quadraturam …..

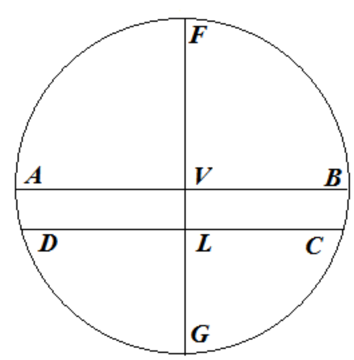


Fig. 9

$\text{VBCL} = ry - \frac{y^3}{6r} - \frac{y^5}{40r^3} - \frac{y^7}{112r^5} - \ldots\ldots$ Adeoque

$2\text{VBCL} = \text{ABCD} = 2ry - \frac{y^3}{3r} - \frac{y^5}{20r^3} - \frac{y^7}{56r^5}.$

PROB. XIII.

Hyperbola Quadraturum determinare.

SIT LSC Hyperbola cujus Asymptoti VD, VP, & in qua $\text{VE} = \text{EL} = a$, atque $\text{VA} = c$, Vocetur abscissa. A*My,* & ordinatim applicata *z*, sed ex natura Hyperbola $\text{VE}x\text{EL} = \text{VM} = \text{VM}\ x\text{MC}$, id est $a^2 = yz + cz$; adeoque est $z = \frac{a^2}{c+y}$, & facta divisione, secundum jam receptam Methodum, erit $z = \frac{a^2}{c} - \frac{a^2 y}{c^2} + \frac{a^2 y^2}{c^3} \cdots$ etc. Quaerenda igitur est Curva AGH, in qua sit $\text{PM} = \frac{a^2}{c} - \frac{a^2 y}{c^2} + \frac{a^2 y^2}{c^3} \cdots$ etc.

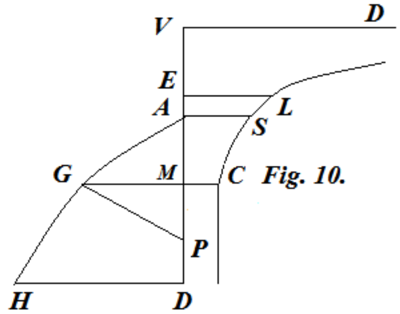


Fig. 10.

per Prob. primum invenietur illam definiri hac aequatione $\frac{na^2 y}{c} - \frac{ma^2 y^2}{c^2} + \frac{la^2 y^3}{c^3} = x^2$ & determinando, *n, m, l,* per Prob. 2. erit $n = 2,\ m = 1,\ l = \frac{2}{3}$, ac proinde aequatio quaesita est $\frac{2a^2 y}{c} - \frac{a^2 y^2}{c^2} + \frac{2a^2 y^3}{3c^3} = x^2$ unde $\frac{a^2 y}{c} - \frac{a^2 y^2}{2c^2} + \frac{a^2 y^3}{3c^3} = \text{ASCM} = \frac{x^2}{2} = \frac{\text{GMq}}{2}$. Haec eadem est Hyperbolae Quadratura quam exhibuit celebris vir *Nicolaus Mercator* in sua Logarithmo-technia, quamvis methodo usus sum ab illius plane diversa.

Considerando aliam Hyperbolae proprietatem; aliam etiam illius Quadraturam inveniemus. Sit ergo in apposito schemate SCL Hyperbola aequilatera cujus centrum A & latus transversum RS, ponatur $\text{AM} = y = \text{KC},\ \text{MC} = z,\ \text{AR} = \text{AS} = r$, unde eae natura Hyperbolae $rr + yy = zz$, adeoque $z = \sqrt{rr + yy}$, extrahendo radicem Quadraticam ex $rr + yy$ erit $z = r + \frac{y^2}{2r} - \frac{y^4}{8r^3} + \frac{y^6}{16r^5} \cdots$ etc.

Quaerenda ergo est Curva AH in qua sit $\text{PM} = z = r + \frac{y^2}{2r} - \frac{y^4}{8r^3} + \frac{y^6}{16r^5} \cdots$ etc. & procedendo per Prob. primum invenietur illam definiri hac aequatione $nry + \frac{my^3}{2r} - \frac{ly^5}{8r^3} + \frac{ky^7}{16r^5} = x^2$, & determinando, *n, m, l, k,* per Prob. secundum $n = 2,\ m = \frac{2}{3},\ l = \frac{2}{5},\ k = \frac{2}{7}$, erit, substitutis his

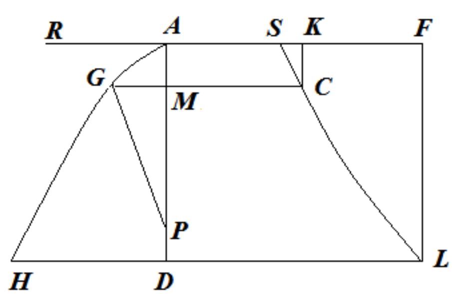


Fig. 11

valoribus erit aequatio ad Curvam quaesitam plene determinata.

$2ry + \frac{my^3}{3r} - \frac{y^5}{20r^3} + \frac{y^7}{56r^5} = x^2$, adeoque erit

$$2ry + \frac{y^3}{6r} - \frac{y^5}{40r^3} + \frac{y^7}{112r^5} = \frac{x^2}{2} = \frac{GMq}{2} = ASCM.$$

Ex hac Hyperbolae Quadratura facile est Zonae Hyperbolicae Quadraturam determinare. Sint EDA, GCB Hyperbolae oppositae, quarum centrum K & Vertices A, B, Zona ABCD cujus latitudo KL = y, semiaxis transversus AK, vel $\text{KB} = r$, unde per praecedentem Quadraturam

$\text{KLCB} = ry + \frac{y^3}{6r} - \frac{y^5}{40r^3} + \frac{y^7}{112r^5}$ ac proinde erit

$\text{ABCD} = 2ry + \frac{y^3}{3r} - \frac{y^5}{20r^3} + \frac{y^7}{56r^5} \cdots$ etc.

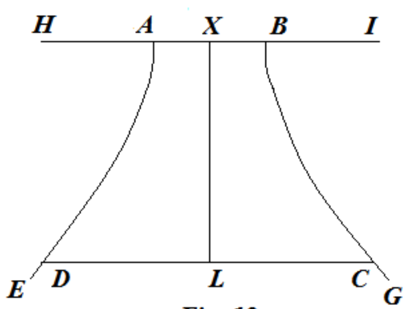


Fig. 12.

PROB. XIV.

Ellipseos Quadraturam determinare.

In semi-ellipsi LSCD sit semiaxis transversus $\text{AS} = b$ & semiaxis conjugatus $\text{AL} = a$, & ponatur abscissa $\text{AM} = y$, ordinatim applicata $\text{MC} = z$, unde eae natura Ellipseos $z = \frac{b}{a}\sqrt{a^2 - y^2}$, ut ergo determinetur Area AMCS, primo resolvenda est $\frac{b}{a}\sqrt{a^2 - y^2}$, in seriem extrahendo radicem ex $\sqrt{a^2 - y^2}$, unde invenietur $z = b - \frac{by^2}{2a^2} + \frac{by^4}{8a^4} - \frac{by^6}{16a^6} \cdots$ etc.. Quaerenda igitur est Curva aliquae AH in qua semper $\text{PM} = b - \frac{by^2}{2a^2} - \frac{by^4}{8a^4} - \frac{by^6}{16a^6} \cdots$ etc.

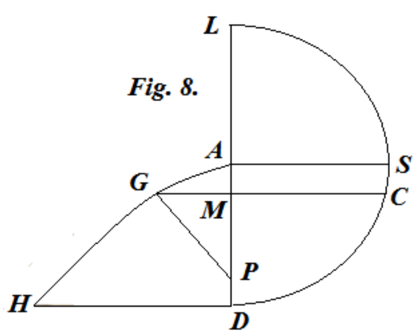


Fig. 8.

& invenietur per Prob.1 illam definiri hac aequatione $nby - \frac{mby^3}{2a^2} - \frac{lby^5}{8a^4} - \frac{kby^7}{16a^6} = x$ & determinando quantitates, $n, m, l, k.$ per Prob. 2. erit $n = 2,\ m = \frac{2}{3},\ l = \frac{2}{5},\ k = \frac{2}{7}$, adeoque

$2by - \frac{mby^3}{3a^2} - \frac{lby^5}{20a^4} - \frac{kby^7}{56a^6} = x^2$ unde $by - \frac{mby^3}{6a^2} - \frac{lby^5}{40a^4} - \frac{by^7}{112a^6} = \frac{x^2}{2} = \frac{\text{GMq}}{2} = \text{AMSC}.$

Inde facile eruitur Zonae Ellipticae dimensio, ut si latitudo Zonae sit $\mathrm{AM} = y$, caeteris positis ut prius erit Zona. $2\mathrm{AMCS} = 2by - \frac{mby^3}{3a^2} - \frac{lby^5}{20a^4} - \frac{by^7}{56a^6} \cdots$etc.

PROB. XV.

Sit $AD(=d)$ positione & magnitudine data, & Curvae SCD talis ut ea ducta utcunque recta $MC(=z)$ ad AD perpendicularis sit $d^3 = z^3 + y^3$ determinanda sit Areae AMCS Quadratura.

Quoniam eae natura curvae $z = \sqrt[3]{d^3 - y^3}$; extrahenda est radix Cubica ex $d^3 - y^3$, & invenietur fore
$z = d - \frac{y^3}{3d^2} - \frac{y^6}{9d^5} - \cdots$etc.

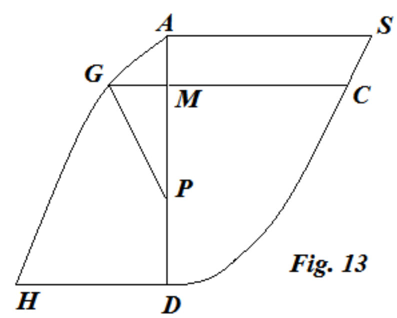


Fig. 13

Quaerenda est linea Curva AGH in qua
$\mathrm{PM} = d - \frac{y^3}{3d^2} - \frac{y^6}{9d^5} - \cdots$etc.
definietur Curva quasita AH hac aequatione
$ndy - \frac{my^4}{3d^2} - \frac{ly^7}{9d^5} = x^2$ & determinando
$n, m, l,$ per problema secundum erit $n = 2, m = \frac{1}{2},\ l = \frac{2}{7}$, adeoquae $2dy - \frac{y^4}{6d^2} - \frac{2y^7}{63d^5} = x^2$,
unde $dy - \frac{y^4}{12d^2} - \frac{y^7}{63d^5} = \frac{x^2}{2} = \frac{\mathrm{GMq}}{2} = \mathrm{AMCS}.$
Et sic Quadrantur Cycliformes infinitae quae definiuntur per

$$d^4 - y^4 = z^4,\ d^5 - y^5 = z^5 \cdots \text{etc.}$$

PROB. XVI.

Esto $AD(=d)$ linea recta positione & magnitudine data, & SCD linea Curva talis ut ducta utcunque $MC(=z)$ ad AD normali sit Cubus ex AD cum Cubo ex $AM(=y)$ aequales Cubo ex MC sc. $d^3 + y^3 = z^3$ & determinanda sit Quadratum Areae AMC. [*Fig.13.*]

Quoniam $z = \sqrt[3]{d^3 + y^3}$ resolvenda est $\sqrt[3]{d^3 + y^3}$ in seriem radicem cubicam extrahendo, & inventetur $z = d + \frac{y^3}{3d^2} - \frac{y^6}{9d^5} + \cdots$etc. Quaerende est : Curva AH in qua sit

semper $\mathrm{PM} = d + \frac{y^3}{3d^2} - \frac{y^6}{9d^5} + \cdots$etc. & per Prob. 1. & 2. erit aequatio ad Curvam quaesitam $\cdots 2dy + \frac{y^4}{6d^2} - \frac{2y^7}{63d^5} = x^2$ unde $dy + \frac{y^4}{12d^2} - \frac{y^7}{63d^5} = \frac{x^2}{2} = \mathrm{AMCS}.$
Et sic Quadrari possunt Hyperboliformes infinitae quae definiuntur per

$$d^4 + y^4 = z^4,\ d^5 + y^5 = z^5,\ \text{etc.}$$

PROB. XVII.

Sit $AD = a,\ AS = b$, *& sit Curva SCD* [*Fig.13.*] *talis ut ducta a quavis* $MC(= z)$ *ad AD perpendiculari sit* $z^3.a^3 - y^3 :: b^3.a^3$, *& determinanda sit Area AMCS.*

Quoniam ex natura Curvae $z = \frac{b}{a}\sqrt[3]{a^3 - y^3}$ extrahenda est radix ex $a^3 - y^3$ invenieturque $z = b - \frac{by^3}{3a^3} - \frac{by^6}{9b^6} - \cdots$etc. Quaerenda est Curva AH in qua $\mathrm{PM} = \mathrm{MC} = b - \frac{by^3}{3a^3} - \frac{by^6}{9b^6}$, etc. & per Prob. 1 & 2. invenietur Curvam quaesitam definiri hac aequatione $2by - \frac{by^4}{6a^3} - \frac{2by^7}{63a^6} = x^2$, Propterea $by - \frac{by^4}{12a^3} - \frac{2by^7}{126a^6} = \frac{x^2}{2} = \frac{\mathrm{GMq}}{2} = \mathrm{AMCS}.$
Et sic quadraentur Ellipsiformes infinitae, quae definiuntur aequationibus

$$\frac{b}{a}\sqrt[4]{a^4 - y^4} = z,\ \frac{b}{a}\sqrt[5]{a^5 - y^5} = z, \text{etc.}$$

PROB. XVIII.

Esto $AD(= d)$ *linea recta positione* [*Fig.13.*] *& magnitudine data & SCD Curva talis ut ducta utcunque MC ad AD normali sit* $d^2z + y^2z = r^3$, & *Quadranda sit Area AMCS.*

Quoniam $z = \frac{r^3}{d^2+y^2}$; fiat divisio, &, invenietur $z = \frac{r^3}{d^2} - \frac{r^3y^2}{d^4} + \frac{r^3y^4}{d^6}$; eritque $\frac{2r^3y}{d^2} - \frac{2r^3y^3}{3d^4} + \frac{2r^3y^5}{5d^6} = x^2$ aequatio ad Curvam AH in qua $\mathrm{PM} = \frac{r^3}{d^2} - \frac{r^3y^2}{d^4} + \frac{r^3y^4}{d^6}$ unde

$$\frac{r^3y}{d^2} - \frac{r^3y^3}{3d^4} + \frac{r^3y^5}{5d^6} = \frac{x^2}{2} = \frac{\mathrm{GMq}}{2} = \mathrm{AMCS}.$$

PROB. XIX.

Cujusvis Figurae Quadraturas infinitas invenire.

Sint duae qaelibet Curvae ACE, BRF, & a quovis puncto C in curva ACE ducatur tangens CT, & CP ad EP, & CR ad AB parallela, si fiat $\mathrm{TP.P\,C :: DR.PX}$, erit $\mathrm{ABZY = BEF,\ APXY = DBR.}$ Eximium hoc Theorema debetur etiam viro celeberrimo D. Doctori *Barrow*.

Quaerenda sint, exempli gratia, Quadratura infinitae Paraboloidis Cubicalis BRF. Assumatur pro arbitrio quaelibet Curva ACE puta parabola communis cujus parameter sit *r* (quod idem sit cum parametro paraboloidis) & ponatur $\mathrm{AP} = y,\ \mathrm{PX} = z,$ eritque $\mathrm{TP} = 2y,\ \mathrm{PC} = \sqrt{ry}$ & ex natura paraboloidis $\mathrm{DR} = \sqrt[6]{ry}$ adeoque Analogia erit $2y.\sqrt{ry} :: \sqrt[6]{ry}.z$ unde $z = \sqrt[6]{\frac{r^8}{64y^2}}$ qaea est aequatio ad Curvam YXZ, cujus Quadratura invenietur per methodum jam traditam sc. $\frac{1}{4}\sqrt[6]{r^8y^4} = \mathrm{APYZ = DBR.}$ Quae est Quadratura Paraboloidis diversa ab illa quam dedi in Prob. 4. Et eodem modo inveniri potest alia atque alia Quadratura, assumendo aliam atque aliam Curvam ACE. Et sic tracturi possunt omnes aliae Curvae, quemadmodum parabolidem hic tractavi.

Exhinc etiam manifestum est Figuras deprimi posse ad simpliciores & Quadratu faciliores, nam in figura ABZY Curva YXZ definita hac aequatione $z^6 = \frac{r^8}{64y^2}$ magis est composita quam Curva BRF. Adeoque non parum Geometriam promoveret, qui methodum daret figuras ad simplicissimas reducendi.

PROB. XX.

Curvam invenire cujus Area per datam quamlibet aequationem designetur.

Designetur Area hac aequatione $\sqrt{r^3y} = \mathrm{VMC}$ (concipiendo VCS esse Curvam quaesitam.) Tum ex ostensis patet $\sqrt{r^3y} = \frac{x^2}{2}$ esse aequationem ad Curvam aliam VGH in qua $\mathrm{PM = MC}$ (quae est ordinata Curva quaesitae) investigetur ergo valor lineae PM, & invenietur $\mathrm{PM} = \frac{r^3}{4y} = z$ seu $4z^2y = r^3$, quae est aequatio ad Curvam quaesitam VCS cujus area $= \sqrt{r^3y}$. Notandum quod hic (ut prius) *y* denotat abscissas VM, *z* ordinatas MC, & *x* ordinatas GM.

PROB. XXI.

Curvas infinitas invenire quarum Areae per unam datam aequationem designentur.

Solutio hujus problematis a duobus praecedentibus pendet; inveniatur una Curva cujus Area per datam aequationem exprimatur (per problema 20) & sic infinitae inveniri possunt per Prob. 19.

PROB. XXII.

Data qualibet Curva AHD Curvam aliam AFB invenire cujus area AGF aequetur rectangulo contento sub ordinata GH & abscissa AG Curvae datae.

In Curva AHD sit $\mathrm{AG}=y$, $\mathrm{GH}=x$, & exprimatur illius natura hac aequatione $2ay-yy=x^2$, unde $\sqrt{2ay-yy}=x$ adeoque $\sqrt{2ay^3-y^4}=xy=\mathrm{AH}$. Habetur ergo Area figurae AGF, unde facile Curvae AFB definitur per problema 20. sc.

$$z^2=\frac{9a^2y-12ay^2+4y^3}{2a-y}$$

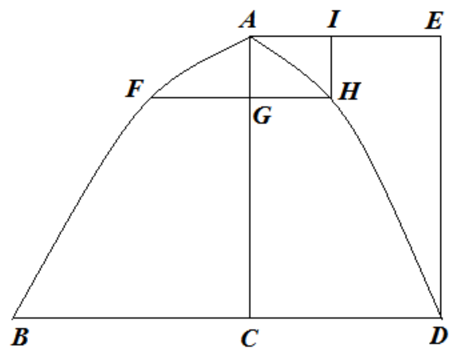


Fig. 20.

PROB. XXIII.

Data qualibet Curva AHD, aliam Curvam AFB invenire cujus area AGF aequatur rectangulo contento sub ordinata GH Curvae AHD, [Fig. 20.] _& constanti aliqua data recta_ (a)

Definiatur Curva AHD ut prius $\sqrt{2ay-yy}=x$ unde $\sqrt{2a^3y-a^2y^2}=ax=\mathrm{AGF}$, habetur ergo natura Curvae AFB per Prob. 20. sc.

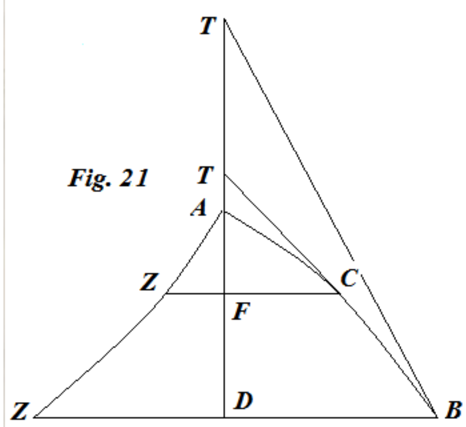
$$z^2=\frac{a^4-2a^2y^2+a^2y^2}{2ay-y^2}$$

hic & in precedenti z denotat ordinatas Curvae quaesitae AFB.

Et quidem aliis modis infinitis (praeter duos jam traditos) inveniri potest curva cujus area, ope alterius curvae datae sit quadrabilis, per Prob.20. Quod fieri posse asseruit jam laudatus Germanus, sed quo modo faciendum sit nequaquam ostendit.

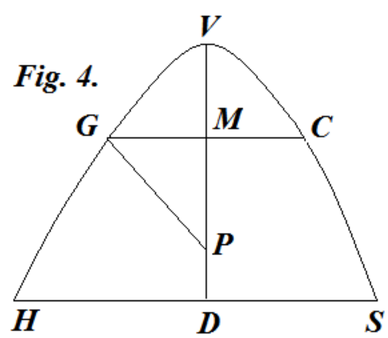
Alia solutio problematis praecedentis.

Sit Curva data ACB, CT tangens in puncto quolibet C, ordinata CF; fiatq; $\mathrm{TF.FC} :: a.\mathrm{FZ}$, orietur hinc Curva AZZ talis ut $a + \mathrm{FC} = \mathrm{AFZ}$; ut demonstratum est ab illustrissimo D. Doctori *Barrow*.



Nec quidquam jam deest ut Methodus quam tradidi Figurarum Quadraturas determinandi, ad omnes figuras extendatur (exceptis iis quae a Curvis transcendentibus terminantur, quas nulla hactenus vulgata Methodus comprehendit) nisi ut difficultates duas amoveam ; quae in quibusdam casibus contingere possunt;
quarum prior accidit cum figuram aliquam Quadrando nec esse sit & Radicem ex aequatione effecta (& supra Quadraticas ascendente) extrahere, in quo casu unicum remedium mihi cognitum est: radicem istius aequationis in seriem infinitam (jucta Methodum clarissimi viri D. Isaaci Newtoni Geometrae non minus quam Analystae praestantissimi) resolvere, quam praelo commissam esse a clariss. Wallisio audimus, quamque insignis ipse D. Newtonus mihi in manuscriptis pro summa sua humanitate communicavit : Nam Methodus Generalis aequationum radices Analytice determinandi (in Actis Eruditorum Lipsiae publicaxis A$^{\mathrm{o}}$ 1683, Mense Maio a praeclaro ilia Germano edita) huic negotio parum vel nihil inservit; ut de insuperabili in ea calculi molestia nihil dicam. Sed nihilominus inventum est inter praecipua Artis Analyticae merito numerandum.

Secunda difficultas est cum valor ordinatim applicatae constat terminis asymmetris, nam res esset immensi laboris aequationem ab asymmetria liberare, si plures sint quam quatuor termini signis radicalibus affecti, ut satis norunt Analyseos periti. Sed huic difficultati remediun optimum suppeditavit insignis Geometra G. G. Leebnitius in nova sua Methodo Tangentes inveniendi in Actis Eruditorum Anni superioris publicata, ibi enim praeclarus vir viam expeditam ostendit, Tangentes inveniendi, quamvis aequatio curvae naturam exprimens terminis irrationalibus quam maxime sit implicita, non ablatis irrationalibus. Quomodo ista methodums ad praesens negotium sit applicanda exemplo ostendam.



Esto VCS circuli Quadrans cujus Diameter sit (r) & VM vocetur y, item ordinata MC [Fig. 4.], tum ex natura circuli $z = \sqrt{ry - y^2}$ & resolvendo $ry - y^2$ in seriem per extractionem radicis, invenietur $z = \sqrt{ry}\left[1 - \frac{1}{2}\frac{y}{r} - \frac{1}{8}\frac{y^2}{r^2} - \right] = \sqrt{ry} - \sqrt{\frac{y^3}{4r}} - \sqrt{\frac{y^5}{4r^3}} \cdots \text{etc.}$.
Ut determinetur Quadratura areae VMC, invenienda

est curva VH, in qua $\text{PM}=\sqrt{ry}-\sqrt{\frac{y^3}{4r}}-\sqrt{\frac{y^5}{4r^3}}$, eritq; per Prob. 1. aequatio ad curvam quasitam $\text{VH}=\sqrt{nry^3}-\sqrt{\frac{my^5}{4r}}-\sqrt{\frac{ly^7}{4r^3}}=x^2$; & auferendo quantitates fractas (quod tamen absolute necesse non est, sed hic sit ob majorem facilitatem) multiplicando per $\sqrt{4r^3}$: erit $\sqrt{4nr^4y^3}-\sqrt{mr^2y^5}-\sqrt{ly^7}=x^2\sqrt{4r^3}$; & determinando *n, m, l*, (per Prob. 2) quae sola est difficultas fic procedo: compendii causa pono $p=4nr^4y^3,\ q=mr^2y^5,\ s=ly^7$; eritq; $\sqrt{p}-\sqrt{q}-\sqrt{s}=x^2\sqrt{4r^3}$; sed per calculum ibi explicatum invenietur $\sqrt{p}=\frac{dp}{\sqrt{4p}},\ \sqrt{q}=\frac{dq}{\sqrt{4q}},\ \sqrt{s}=\frac{ds}{\sqrt{4s}}$, atque $x^2\sqrt{4r^3}=2x\sqrt{4r^3}dx$, & substitutis his valoribus erit $\frac{dp}{\sqrt{4p}}-\frac{dq}{\sqrt{4q}}-\frac{ds}{\sqrt{4s}}=2x\sqrt{4r^3}dx$, sed per eundem, calculum erit $dp=12nr^4y^2dy,\ dq=5mr^2y^4dy$ & deniq; $ds=7ly^6dy$, & substituendo hos valores cum valoribus quantitatum $\sqrt{p},\ \sqrt{q},\ \sqrt{s}$, aequatio erit

$$\frac{12nr^4y^2dy}{\sqrt{16nr^4y^3}}-\frac{5mr^2y^4dy}{\sqrt{4mr^2y^5}}-\frac{7ly^6dy}{\sqrt{4ly^7}}=2x\sqrt{4r^3}dx,$$

Quam clarissimus Author aequationem differentialem appellat : & haec aequatio in Analogiam resoluta dat

$$dy.dx :: \sqrt{16r^3}.\left[\frac{12nr^4y^2}{\sqrt{16nr^4y^3}}-\frac{5mr^2y^4}{\sqrt{4mr^2y^5}}-\frac{7ly^6}{\sqrt{4ly^7}}\right] :: x.\text{PM},$$

ut ex eodem calculo est manifestum, adeoq; erit

$$\text{PM}=\frac{12nr^4y^2}{\sqrt{256nr^7y^3}}-\frac{5mr^2y^4}{\sqrt{64mr^5y^5}}-\frac{7ly^6}{\sqrt{64lr^3y^7}}=\text{etc.}$$

Et facta comparatione horum terminorum cum terminis prioribus PM denotantibus, juxta cognitas comparationis leges sc. $\frac{48nr^4y^2}{\sqrt{4096nr^7y^3}}=\sqrt{ry}$, inde $n=\frac{16}{9}$; similiter $m=\frac{16}{25}$;& $l=\frac{16}{49}$; quibus substitutis erit

$$\frac{\sqrt{16ry}}{9}-\frac{\sqrt{16y^5}}{100r}-\frac{\sqrt{y^7}}{496r^3}=x^2=GMq,$$

adeoque

$$\frac{\sqrt{4ry}}{9}-\frac{\sqrt{16y^5}}{400r}-\frac{\sqrt{y^7}}{196r^3}=\frac{x^2}{2}=\frac{CMq}{2}=\text{VMC}.$$

Adeoq; etiam hoc modo habetur circuli Quadratura. Et similem discursum in aliis adhibere non erit difficile, cuivis in singulari hoc calculi genere versato, ita ut superfluum duae duxi praestantissimae hujus Methodi usum pluribus exemplis illustrare. Unum tamen est quod hic obiter notandum puto, posse ex hac Tangentium methodo breviter demonstrari veritatem Regulae quam dedi pro solutione problematis primi.
Namq; $dy.dx :: \text{TM.MC}$ (ut ex ista Methodo est manifestum) sed $\text{TM.MC} :: \text{MC.PM.}$ ob angulum rectum TMP. Ergo $dy.dx :: \text{MC.PM}$ (vel posito x pro MC) erit $dy.dx :: x.\text{PM}$. Unde $\text{PM}xdy = xdx$, & substituendo y & x pro earum differentiis dy, dx erit $\text{PM}xy = x^2$. Quod demonstrandum erat.

Jamq; concludo, si nulla sit Curva. in qua distantia inter illius perpendicularem & ordinatam sit aequalis correspondenti ordinatae in Curva Figuram (cum recta. vel rectis) comprehendentem, illam Figuram non esse indefinite Quadrabilem ; nam si daretur illius quadratura indefinita, daretur etiam hujusmodi Curva ut patet ex Prob. 20. Et nullam esse talem Curvam pro Circulo & Hyperbola, facile possum demonstrare, sed demonstationem ob nimiam prolixitatem hic omitto.

De Linearum Curvarum Rectificatione.

Quisnam fuerit qui primo Curvae rectam aequalem invenit diu multumq; Anglos inter & Batavos disputatum fuit ; & qui plenius de ea re sibi satisfieri volunt, totam disputationem videre possunt, in eximio libello de Cycloide a Clariss. Wallisio Edito pag. 91, 92, 93, &c. itemq; in Horologio Oscillatorio illustrissimi Hugenii pag. 72, 73, & deniq; in Epistola Wallisii in Actis Philosophicis Reg.Societ. publicata Num. 98 res enim tanti non est ut ulteriori disquisitione digna videatur, mihi praesertim qui nec Anglus sum nec Batavus. Ea tamen, quae, re bene perpensa, utrinq; manifesta videntur, breviter annotabo:

1. Quod Guliel. Nelius Equitis Angli Filius omnium primus rectam Curvae aequalem invenerit.
2. Quod non datam Curvam rectificaverit sed Curvam rectificationis capacem exhibuerit.
3. Quod dignissimus & Geometra peritissimus D. Christoph. Wren primo oblatae Curvae (sc. Cycloidi) rectam aequalem determinaverit.
4. Quod Heuratius primo ostenderit quamlibet datam Curvam rectificare, suppositis Figurarum Quadraturis. Et in eo Heuratii Methodi non parum est conspicua quod statim indicat quaenam illa Figura sit cujus Quadratura Curvam datam rectificaret; Adeoq; cum jam Methodum generalem praemisi Figurarum Quadraturas determinandi ; facile erit Curvam aliquam in rectam transmutare; Et recta, illa vel per aequationem finitam (cum nempe Figura est indefinite Quadrabilis) vel per seriem infinitam exprimetur. Heuratius enim tali methodo destitutus, non potuit methodum suam Curvas rectificandi, ad omnes illas Curvas extendere, Quarum recticationes a figuris indefinite Quadribilibus dependent; multoq; minus cum a figura specialis tantum Quadraturae capaci dependerent.

THEOREM 2.

[Fig.14.] Sint duae lineae Curvae ACE, GIL, & recta AF ejus naturae ut (ducta ex puncto M libere sumpto perpendiculari MI secante Curvas in C & I, uti & CP perpendiculari ad Curvam ACE) sit .... MC.CP:: R.MI (R hic est quaelibet linea recta data vel assumpta) erit $\text{AGILEF} = \text{R}x\text{ACE}.$ Demonstratio hujus Theorematis habetur in Epistola Heuratii ad Schotenium.

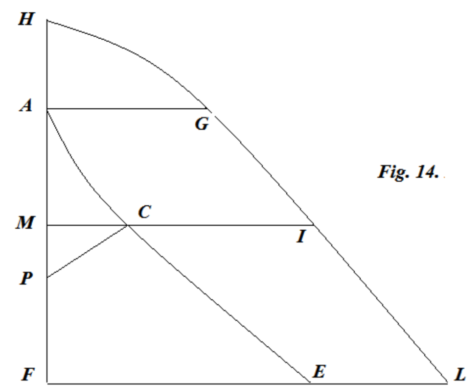


Fig. 14.

PROB. I.

Determinare Longitudinem Parabolae. ACE.

Sit parabolae vertex A, ipsius axis AG & parameter (*a*) *AM* vocetur *x* & *MC* vocetur *y* [Fig. 14.]; unde ex natura parabolae $x^2 = ay$; per methodum, aliquam vulgarem Tangentes inveniendi, constabit fore $\text{PM} = \frac{2x^3}{a^2}$, adeoq;

$\text{PMq} = \frac{4x^6}{a^4}$, unde $\text{PC} = \sqrt{\frac{4x^6}{a^4} + \frac{x^4}{a}}$ jam quia $\text{CM.CP} :: a.\text{MI}$; vel in terminis Analytricis $\frac{x^4}{a}.\sqrt{\frac{4x^6}{a^4} + \frac{x^4}{a}} :: a.z$ (posito nimirum $MI = z$) unde $z = \sqrt{a^2 + 4x^2}$ quae est aequatio ad Hyperbolam ; adeoq; pro determinatione longitudinis lineae paraboliae ACE, Quadranda est Area Hyperbolica AGILEF (ut in Prob.13 .) eritque,

$\text{AGILEF} = ax + \frac{2x^3}{3a} - \frac{2x^5}{3a^3} + \frac{4x^7}{3a^5} - \text{etc.}$

Unde $ax\text{ACE} = ax + \frac{2x^3}{3a} - \frac{2x^5}{3a^3} + \frac{4x^7}{3a^5} - \text{etc.}$ per Theor. 2.

Adeoq. $\text{ACE} = x + \frac{2x^3}{3a^2} - \frac{2x^5}{5a^4} + \frac{4x^7}{7a^6} - \text{etc.}$

PROB. II.

Circuli Peripheriae rectam aequalem exhibere.

Sit ACF circuli Quadrans cujus radius sit d & vocerur PM y, MC x; & MI z, [Fig. 15.] sitq; GIL talis curva ut ducta utcunq; normali CMI ad rectam PF sit $\text{MC.PC} :: d.(\text{recta libere sumpta}).\text{MI}$. id est $\sqrt{dd-yy}.\, d :: d.z.$, unde $z=\frac{dd}{\sqrt{dd-yy}}$ quae est aequatio ad curvam GIL ; adeoq;

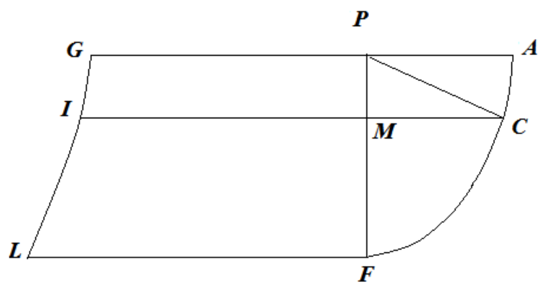


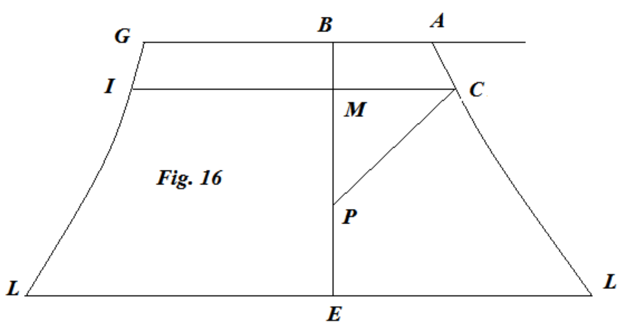
Fig. 15.

$\text{PM} = dy + \frac{y^3}{6d} + \frac{3y^5}{40d^3} \cdots \text{etc.}$ Sed per Theor. 2 est

$dx\text{AC} = dy + \frac{y^3}{6d} + \frac{3y^5}{40d^3} + \frac{5y^7}{112d^5} \cdots \text{etc.}$ Ergo $\text{AC} = y + \frac{y^3}{6d^2} + \frac{3y^5}{40d^4} + \frac{5y^7}{112d^4} \cdots \text{etc.}$.

PROB. III.
Hyperbolae rectam aequalem exhibere.

Sit ACE Hyperbola aequilatera cujus semiaxis $\text{BA} = a$ & centrum B ; & BM vocetur y, AC x, unde ex natura Hyperbolae $a^2 + y^2 = x^2$; ponatur PC Hyperbolae in C perpendicularis [Fig. 16.] ; invenietur $\text{PM} = y$ adeoq; $\text{PC} = \sqrt{a^2+2y^2}$ si fiat $\text{MC.CP} :: a.\text{M}$ id est, $\sqrt{a^2+y^2}.\sqrt{a^2+2y^2} :: a.\, z$; erit $z=\frac{\sqrt{a^4+2a^2y^2}}{\sqrt{a^2+y^2}}$, quae est aequatio ad Curvam GIL.



Sed $\sqrt{a^4+2a^2y^2} = a^2 + y^2 - \frac{y^4}{2a^2} - \cdots \text{etc.}$

Et $\sqrt{a^2+y^2} = a + \frac{y^2}{2a} - \frac{5y^4}{8a^3} \cdots \text{etc.}$

Unde $\mathrm{BELG} = ay + \frac{y^3}{6a} + \frac{5y^5}{40a^3} \cdots \text{etc.}$

Atq; $\mathrm{ACE} = y + \frac{y^3}{6a^2} - \frac{5y^5}{40a^4} \cdots \text{etc.}$

De Curvarum superficierum dimensione.

Quemadmodum linearum Curvarum longitudines, sic etiam superficierum, quae ab illarum rotatione generantur, dimensio ex quarundam Figurarum Quadraturis dependet, ut ex sequenti Theoremate constat.

THEOREMA 3.

Sit MP Curvae AMB perpendicularis & linea KZL talis ut (ducta MFZ ad axem AD normali) sit MP correspondenti FZ aequalis ; erit superficies producta a rotatione Curvae AMB circa axem AD, ad spatium ADLK, ut Circumferentia Circuli ad suum radium.

Hoc etiam unum est ex innumeris & praeclaris Theorematis viri celeberrimi D. Isaaci Barrow.

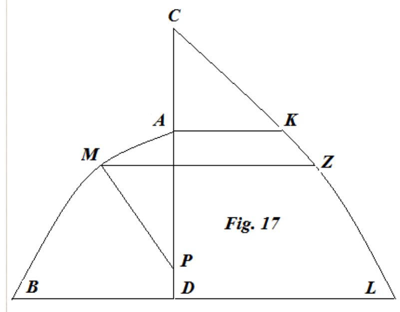


Fig. 17

PROB. I.

Superficiem Sphaerae determinare.

Sit AMB Semicirculus a cujus rotatione data sphaera producitur: & designet r radium & c circumferentiam cujuslibet circuli ; & sic AB (diameter Semicirculi AMB) $= 2d$ jam quoniam omnes lineae Circulo perpendiculares MP perveniunt ad Circuli centrum P; ideo erit KZL parallelogramium rectangulum cuius longitudo diameter AB & altitudo $\mathrm{AK} = d$ radius Semicirculi AMB; unde $\mathrm{AL} = 2d^2$ (per literam s ubiq; designo superficiem Curvam;) ideo per Theorem tertium $s.2d^2 :: c.r.$ unde $s = \frac{2d^2c}{r}$ vel ponendo $r = d$ erit $s = 2dc$; ac proptera Superficies. Sphaerae aequatur rectangulo cujus longitudo est Circumferentia & Latitudo Diameter Circuli in Sphaera maximi.

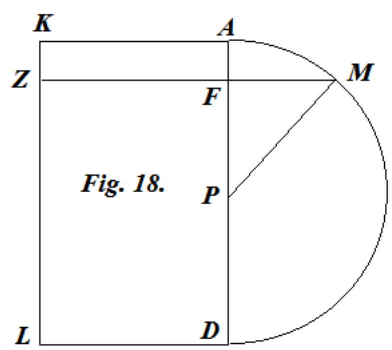


Fig. 18.

Notatu dignum arbitror hinc consequi omnium Theorematum longe nobilissimum quo aeternam sibi famam acquisivit Geometrarum Princeps *Archimedes*; Quod scilicet superficies Sphaerae sit aequalis quatuor maximis in ea Circlis. Sit enim $\mathrm{Q} =$ maximo in Sphaera Circulo ; at qui $\mathrm{Q} = \frac{dc}{2}$ ut ab Archimede demonstratum est ; Ergo $2\mathrm{Q} = dc$, & $4\mathrm{Q} = 2dc$; sed jam inventum est: $s = 2dc$; Ergo $4\mathrm{Q} = s$ .Quod erat demonstrandum.

PROB. II.

Superficiem Conoidis Parabolici determinare.

Esto r latus rectum Parabolae AMB a cujus rotatione conoides producitur, sit axis AD, vertex A & vocetur AF, y ; FM x; per methodum aliquam tangentium invenietur $\mathrm{PMq} = \frac{1}{4}r^2 + ry;$ vel ponendo $\mathrm{FZ} = z$, quia supponitur $\mathrm{PM} = \mathrm{FZ}$, aut $\frac{1}{4}r^2 + ry = Z^2$ quae est aequatio ad parabolam cujus axis idem est cum axe parabolae datae AMB ; cujus vertex est C, existente $\mathrm{AC} = \frac{1}{4}r^2$; & latus illius rectum etiam r, invenietur $\mathrm{AKLD} = \sqrt{\frac{4}{9}rv^3} - \frac{1}{12}r^2$ existente $\mathrm{CD} = v$; sed: $s.\sqrt{\frac{4}{9}rv^3} - \frac{1}{12}r^2 :: c.r$. per Theorema 3. Ergo $s = \sqrt{\frac{4c^2v^3}{9r}} - \frac{1}{12}rc.$

In hunc modum mensurantur non modo superficies Conoidis Hyperbolici , & Sphaeroidis, sed Quaevis alia Curva superficies quae generatur a rotatione lineae Curvae & haec duo exempla satis ostendunt quomodo eadem Methodus ad omnos alias superficies Curvas sit Applicanda.

ANIMADVERSIO

In Methodum Figuras dimetiendi,

A clarissimo Quodam Germano editam in Actis Eruditorum Lipsiae publicatis.

METHODUM hanc proposuerat Doctissimus illius Author Anno 1683, Mense *Octob*. quam adeo perfectam credebat ; ut vel Quadraturam Figurae, vel ejusdem Quadraturae impossibilitatem determinaret ; & ex ea Circuli & Hyperbolae Quadraturam Geometricam impossibilem esse concluserat. Postea vero perspexit clarissimus vir tanta perfectione praeditam non esse, ut exinde Circuli, Hyperbolae aut alterius Figurae Quadraturae impossibilitas probari possit, ut ingenue ipse fatetur in iisdem actis Anni sequentis, ubi ait fe amore veritatis coactum hoc unum monere. Unde existimat quasdam esse Figuras quae indefinitae Quadraturae non sunt capaces & exemplum Figuraea scribit in qua succedere ait Quadraturam specialem sine generali: in hoc tamen hallucinatus est claris. vir quod ex sua Methodo Figuram aliquam, Quadraturam indefinitam recusare conclusit ; priusquam demonstrasset Methodum suam ad omnes figuras indefinite quadrabiles extendere ; quod demonstratu est impossibile, cum unam e millesimis non comprehendat ; ut postea patebit. Dantur enim infinitae figurae indefinitae quadrabiles, quae nullo modo per istam Methodum sunt Quadribiles ; & quarum exempla postea apponam; Et ut non modo errorem sed erroris fontem detegam, nec esse videtur breve illius Methodi compendium adjungere.

Adhibet aequationes Curvarum Generales, quarum unaquaeque omnes Curvas ejusdem gradus exprimere existimat : Et talis Curvae generalis consideratae tanquam Quadratricis quaerit Quadrandam Generalem. Et oblatae Quadrandae specialis aequationem comparat cum aliqua ex formulis generalibus Quadrandaram naturam exprimentibus ; unde deducit Quadratricem specialem Quadrandae speciali, convenientem ; exemplo res erit manifesta.

Sit ABC figura, rectis AC, CB & Curva AB comprehensa, sitq; $\mathrm{ACDE} = \mathrm{ABC}$, $\mathrm{AGF} = \mathrm{AGHL}$, &. idem concipiatur ubique, proveniet hinc Curva aliqua AHD, quam Quadratricem appellat, quia illus ope Quadratur area ABC, jam aequationem assumit ad Quadratricem generalem, AHD, & ex ea deducit Quadrandam generalem ABC : ut si notentur abscissae AG, AC per *x*, & ordinatae Quadratricis CD, GH per *y*, & deniq; ordinatae in Quadranda per *z*, ponitq; aequationem ad Quadratricem generalem in qua ordinata *x* est: duarum dimensionum hujusmodi,

$$\left.\begin{matrix} by^2 + cay + ea^2 \\ + xy + fax \\ + gx \end{matrix}\right\} = 0,$$

ex qua deducit aequationem ad Quadrandam generalem, in qua ordinata *z* est etiam duarum dimensionum,

$$\left.\begin{matrix} bz^2 + caz + ea^2 + \frac{d^2e + c^2g + f^2b - cdf - 4beg + a^2x^2}{4bea^2 + 4bfax + 4bgx^2} \\ + 2dxz + 2fax \\ + 4gx^2 - ca^2 - 2cdax - dx^2 \end{matrix}\right\} = 0$$

Et similiter pro reliquis Quadratricibus generalibus Quadrandas generales investigat. Proponatur jam Figura aliqua Quadranda specialis ABC, & exprimatur natura Curvae AFB hac aequatione $z^2 = \frac{9a^2x - 12axx + 4x^3}{2a - x}$; hanc aequationem comparat cum aequatione generalis Quadrandae jam positae, (quia ordinata *z* in Quadranda speciali ad duas tantum dimensiones ascendit) nempe singulos hujus cum singulis illius terminis (ubi *x* eandem utrobiq; compositronem obtinet) eritque ex hac comparatione $c,\ d,\ e = 0$, & $b = \frac{1}{2},\ f = -1$ *ac* $g = \frac{1}{2}$; & hos valores substituit in aequatione ad Quadratricem supra positam in qua ordinate *x* est duarum dimensionum, (quia hic ordinata *z* ad duas quoque dimensiones ascendit) eritque $\frac{y^2}{2} - ax + \frac{x^2}{2} = 0$ seu $y^2 = 2ax - x^2$, proprietas Quadratricis specialis AHD in qua $\mathrm{AGF} = \mathrm{AGHL}$ adeoq; habeur Figurae propositae Quadratura.

In eo tamen latet ratiocinationis & ipsius Methodi defectus, quod omnes Curvas in quibus z ad duas dimensiones (nec ultra) ascendit comparet cum una & eadem Quadranda generali, in quibus z non ultra duas dimensiones ascendit; & quod concludat Figuram non esse indefinite Quadrabilem si haec comparatio Quadratricem non determinet. Infinitae enim sunt Quadrandae generales (ex ipsius etiam Methodo deducibiles) in quibus z non ultra duas dimensiones ascendit, & non nunquam aequatio Curvae propositae, cum prima, secunda, tertia, &c. comparata. Quadraticem non habebit, & tamen comparatio cum Millesima quadratricem determinabit. Si enim ab aequatione tertia (quam posuit pro Quadratrice generali in qua x esset trium dimensionum primum terminum $by^3 + dxy^2$ auferat, ex reliquo Quadrandam generalem deducere potest, in qua z non ultra duas dimensions ascendit, & quae Quadratricem determinet, cum illa quam ille statuit generalem non succedit: Et sic ex aequatione quarta, quinta (quas ille poneret pro Quadratricibus altiorum graduum) &c. ablatis iis terminis in quibus y ultra duas dimensiones ascendit, ex reliquo habere potest Quadrandae generalis aequatio, quae Quadratricem determinabit, cum nec ejus Quadranda, nec illa quam dixi esse deducibilem ex aequatione tertia, deteminare potest, ita ut casu non arte incidimus in Quadrandam Generalem requisitam. Sed quia dixi istas aequationes ad Quadrandas generales ex ipsius Methodo esse deductibiles ; volo hic paucis ostendere, Quomodo clariss. hic vir aequationes ad Quadrandas generales invenerit, vel saltem facile invenire potuisset.

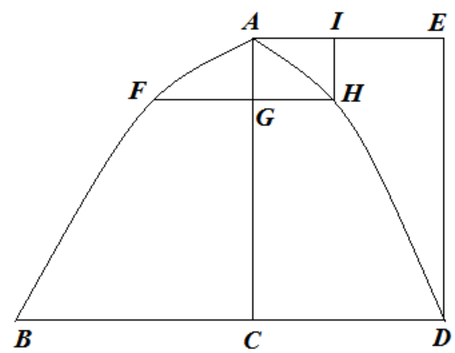


Fig. 20.

Ex *Prob*. 22. constat, quomodo data aequatione [Fig. 20] ad Curvam aliquam AHD, alia curva AFB sit invenienda cujus area AGF aequatur rectangulo comprehenso sub ordinata GA & abscissa AG; id est Quomodo data Quadratrice, invenienda sit Quadranda; adeoq; assumpta aequatione ad Quadratricem generalem (qualem hic sub initio ascripsi) proveniet aequatio ad Quadrandam generalem. Jamq; exemplum unum aut alterum Figurae hic ascribam, in qua Quadratriae secundum hanc Methodum est impossibilis, & tamen alio modo determinabilis. Sit aequatio naturam curvae AFB exprimens $z^2 = \frac{\overline{m^2 + x + x^2}}{p^2}$

in qua x denotat abscissas AC, AG, & z ordinatas BC, GF, m & p quantitates datas & determinatas ; jam si Quadranda sit Area AGF, comparanda est haec aequatio cum aequatione ad Quadrandam generalem jam tradita, quia in hac proposita aequatione z ad duas dimensiones ascendit; sed manifestum est comparationem non suceedere (ut ipse alibi argumentatur) si vel solus numerator fractionis utrobique existens comparetur, deberet enim $m^2 + x^2$ coincidere cum $\overline{-d^2 e + ag + bf^2 - cdf - 4beg + a^2}$ indeterminatum cum determinato, quod fieri nequit, itaq; figura hoc modo Quadratricem non habet, & tamen ipsa haec figura est indefinite quadrabilis, scil. $\mathrm{AGF} = \sqrt{\frac{a^6+3a^4x^2+3a^2x^4+x^6}{9pp}}$. Et non una tantum sed infinitae possunt inveniri Figurae indefinite Quadrabiles, quarum Quadratrices hoc modo sunt impossibiles per Prob. 23. [Fig. 20] Definiatur AHD hac aequatione $x^7 = a^7 y^7$, & per *Prob*. 23. inveniatur curva AFB cujus Area AGF aequetur rectangulo contento sub ordinata GH & data qualibet recta puta (*a*), & definietur AFB hac aequatione $z^2 = \frac{81x^7}{4a^3}$; jamq; si Quadranda sit area AGF secundum hanc Methodum, comparanda est haec aequatio cum aequatione ad Quadrandam generalem Jam tradita, quia in proposita aequatione *z* non ultra duas dimensiones ascendit ; sed comparatio est impossibilis, quia in proposita Curva *x* ad septimam potestatem ascendit; & in ejus aequatione ad Quadrandam generalem ultra quartam ascenderc non potest ; sed terminus in quo *x* est septimae, non potest comparari cum termino in quo *x* est quartae potestatis; nam secundum ipsius Regulam, comparatio est sic instituenda ut *x* utrobiq; eandem obtineat compositionem; adeoq; Quadratrix hoc modo haberi non potest, & tamen Quadratricem habet AHD definita hac aequatione $x^9 = a^7 y^2$, in qua. $\mathrm{GH} + a = \mathrm{AGF}$; id est $\sqrt{\frac{x^9}{a^7}} = \mathrm{AGF}$. Unde abunde constat hanc Methodum omnes Figuras indefinite Quadrabiles non comprehendere ; & infinitas posse inveniri quarun Areae hoc modo non sunt quadrabiles ; assumatur enim quaelibet aequatio in qua *z* non ultra duas, & *x* non infra quatuor dimensiones reperitur; & habetur aequatio naturam Curvae exprimens cujus area per hanc Methodum non sunt quadrabiles; ut in his exemplis $z^2 = \frac{x^9}{a^7}$, $z^2 = \frac{x^{11}}{a^9}$, $z^2 = \frac{x^{13}}{a^{11}}$, etc. quae sunt aequationes Curvarum naturas definientes quarum Areae facile determinantur & tamen nullo modo per hanc Methodum possunt inveniri. Sed nolo in hac materia ulterius digredi sperans clariss. virum boni consulturum quicquid dixerim; quia praecipua ratio quae me impulit ut haec scriberem, non alia esset, quam ut (errores ejus ostendendo) illum extimulem ad publicanda illa, quibus Geometriam in immensum ultra terminos a Vieta & Cartesio positos se promovere posse asseruit.

FINIS.