PAPERS ON THE RESISTANCE OF MEDIUMS AND OF GRAVITY ON PROJECTILE MOTION IN A RESISTING MEDIUM.

Acta Erud. Feb. 1689.

When Galileo investigated the rules of motion of projectiles, he set aside the resistance of the medium; Torricelli did the same and those who followed, yet they admit some deficiency in the theory and hence errors arise in practice. Indeed Blondel considers in the book *de Jactu Bomborum* [*The Art of throwing Bombs*], that this consideration can safely be ignored, but the arguments about that are insufficient, nor do the experiments offered bring forwards anything much. Besides the geometrical investigation is of a more difficult matter, than as with these other rules, although expected and hoped by the most learned men to be easy, then with the matter still not resolved nor certainty being known with sufficient aids everywhere to its understanding. And yet the true laws of projectiles and a calculus known from experiment, thus could be considered to depend chiefly as being of great use for ballistics and pyro-ballistics in the future.

Now I communicated some thoughts about this matter some time ago to the celebrated Royal Academy of the Sciences of Paris, when I was active in these matters, and the manner of estimating from the part I treated, and the different kinds I distinguished between. The resistance of a medium therefore is twofold, the one kind of resistance is absolute [*i.e.* intrinsic], the other respective [*i.e.* depending on other circumstances, such as the speed of the body], and which generally are accustomed to act together. The absolute resistance absorbs just as much of the strength of the motion, whether the body may be moving with a small or large velocity, provided it shall be moving, and depends on the stickiness of the medium ; and indeed it is likewise as if the parts [of the medium] were connected to each other by filaments broken by the moving body. The same can be considered for the friction of rough surfaces, across which moving bodies are travelling: for the obstacles are being abraded or perhaps pressed together, according to the example of elastic fibers themselves rising up again afterwards; but for either elastic depression or breaking the threads [or fibers, etc.], the impeding force is the same always, nor does it matter what the velocity of the body acting shall be. The respective resistance arises from the density of the medium, and it is greater for a greater velocity of the moving body, because there the parts of the medium themselves are being agitated by [the moving body] penetrating, moreover to move in some manner is to expend a force, and there it is greater, as a greater motion may be communicated to the parts of the medium, that is, where the motion of penetration is faster. And the resistance of the fluid at rest running towards the body is equal to the force of the fluid running towards the body at rest, which is greater, when the motion of the fluid is faster, as we see bodies moving in wind or water; indeed with water thrown with enough impetuosity for a weight to be sustained, here the respective resistance is permitted to be mixed up as well with the absolute resistances themselves, from which that resistance still can be separated by the mind, since we may estimate the respective resistance, as if the tenacity of the medium were zero. This too is a concern between the two kinds of resistance, because the absolute resistance has to be accounted in some way to the surface of the moving body, or with the respective resistance, to the contact truly of the whole body [*i.e.* volume]. In both cases a paradox arises, because a moving body penetrating into a medium resisting uniformly everywhere, indeed at no time may be reduced to rest from that : yet from the absolute resistance a body, which may be considered to be moved once by a single force nor may be accelerated by another, thus has a certain limit of distance or of penetration into the medium as it may approach the same right line always, yet never may it reach that, which I call the maximum exclusive penetration, or the greatest which none can reach ; truly from the respective resistance a body accelerating uniformly (as with a weight falling) has a certain limit of the velocity or a maximum exclusive velocity to which it approaches always (so that in the end the difference shall be negligible), thus yet so that it may never reach that exactly. And this velocity is that itself, by which the motion of a fluid (*e.g.* the jet of water) may be able to support a weight, lest it may begin to fall. And here we may explain the primary laws of the motion in each case, as much as that may be allowed briefly, for the whole matter to be treated in depth would require a whole treatise.

Absolute Resistance.

Art. I.

If the motion of the body shall itself be uniform [*i.e.* in the sense of being continuous, but not constant] and retarded equally by the medium following equal distances.

1) The decrements of the strengths are proportional to the increments of the distances (which is the hypothesis in the present case).

2) The velocities are proportional to the distances still required to be run through at this point, with the velocities lost proportional to the distances run through. The increments of the distance shall be equal, the decrements of the strengths shall be equal (per prop. I) ; now if the decrements of the same strengths of the moving body shall be equal, also the decrements of the velocity shall be equal, (for the strengths are as the squares of the velocities, but with equal squares present also the lengths of the sides shall be equal [*i.e.* of the squares]); and thus the elements of the lost velocities are as the elements of the distances traversed, so that of the remaining velocity at this point are as that required to be traversed. Therefore the velocities are as the distances. Evidently, if in Fig. 15 the initial velocity shall be AE, the whole distance to be transversed in the medium shall be the right line AB, now with the part of this transversed AM, the part still to be transversed MB, the velocity remaining MC (or AF), with the part FE removed, will be the right line ECB.

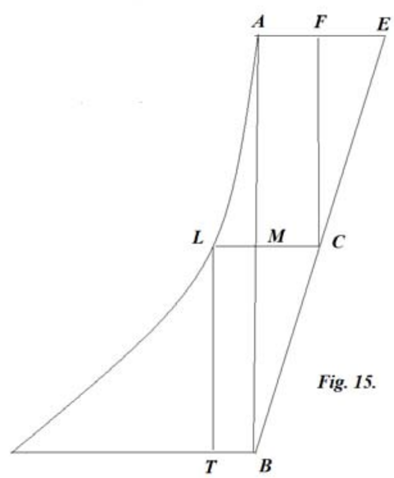

Fig. 15.

3) If the distances remaining (MB or LT) shall be as the numbers, the times spent (ML or BT) will be as the logarithms; [*i.e.* $t_1 : t_2 :: \frac{\log(L_0 - L_2)}{\log(L_0 - L_1)}$] for if the elements of the distances shall be in a geometric progression, the distances remaining shall be of the same progression, therefore (by 2) also the velocities remaining, therefore the increments of the time are equal, and therefore the times themselves shall be in an arithmetical progression.

[For if L_0 shall be the complete length, then according to this statement, the length L at some later time t after the start is given by $L = L_0\left(1 - e^{-\alpha t}\right)$, where the initial value of L is zero, and after an infinite length of time, the mass has transversed the whole length AB, called here L_0 ; at some intermediate time t the velocity is given by $V = \frac{dL}{dt} = \alpha L_0 e^{-\alpha t} = V_0 e^{-\alpha t} = \alpha\left(L_0 - L\right)$; thus the velocity is proportional to the distance still to be transversed, and the velocity lost is given by $V_0 - V = L\frac{V_0}{L_0}$, which is proportional to L, the distance traversed, as claimed above. The acceleration is given by $a = \frac{dV}{dt} = -\alpha V_0 e^{-\alpha t} = \alpha^2\left(L - L_0\right)$. In addition, the kinetic energy of the mobile mass at some time t is given be $\frac{1}{2}mV^2 = \frac{1}{2}mV_0^2 e^{-2\alpha t} = \frac{1}{2}m\alpha^2\left(L_0 - L\right)^2$, and from which it follows as indicated above that the velocity is proportional to the distance to be traversed, and $d\left(\frac{1}{2}mV^2\right) = m\alpha^2\left(L_0 - L\right)dL \propto dL$ for a given $L_0 - L$.

We should note however, that the standard notation developed by Euler lay many years in the future, and Leibniz presumably made use of the log function also, although he was aware of the inverse function, as that is apparent from his first calculus paper; thus, each of these exponential equations would have been written in the less familiar form, for us perhaps, as e.g. $L = L_0\left(1 - e^{-\alpha t}\right)$ becomes $t = \frac{1}{\alpha}\log\frac{L_0}{L_0 - L}$, or the proportionality suggested above, etc.

Finally we may ask why Leibniz used the logarithm approach in solving this problem, and part of the answer lies in the origin of logarithms themselves, as established by Napier, who had solved a similar problem numerically under special circumstances, but primarily as an aid to calculation rather than a dynamics problem; also, the logarithm satisfies the area under the section of a hyperbola rule, and the log of a quotient can be extended as a telescoping sum of differences, close to Leibnitz's thinking. In addition, Newton's approach of regarding the solution of every dynamics problem to follow from his laws of motion has not been adopted at all here, at this stage, and the solution which has been asserted without proof has more bearing perhaps for us, on radioactive decay than motion, and this case below also, where the resistance is not intrinsic, but is proportional to the velocity. An interesting account, without mathematics, of the origin of these papers is given in the book *Leibniz* by Antognazza on pp. 295-297, where it appears in fact to be Leibniz's hasty response to Newton's *Principia*, which had been published a few years previously. The standard account of these matters is that by E. J. Aiton, in the *Archives of the Exact Sciences* for Dec.1972. I have done my own translation and notes, though a useful comparison can be made with that of Aiton by the interested reader.]

4) At no time does the moving mass M complete the whole distance required to be transversed (AB), even if always it may approach towards the limit (B), for it is apparent BT is the asymptote of the logarithmic line AL, truly here the number AB is 0, the logarithm of 0 is infinite. Meanwhile in practice the motion shall be indistinguishable at last, as the distance from B; moreover nowhere is the medium given perfectly uniform.

5) If the moving mass may be moved by a motion composed from a uniform motion and equally from a retardation following the distance traversed in the medium, or if the moving mass (M) may be carried in a rigid rod (AB) according to the present hypothesis (as actually thus it can be produced well enough on account of friction, if a globe may be moving on a rigid ruler along a horizontal right line), truly meanwhile itself remaining parallel to that rule (AB) , with a uniform motion, with one end (B) it approaches some other right line (BT), it describes the logarithmic line (AL). For generally, if a moving mass may be carried by a motion composed from a uniform motion and the relation of another law, it will describe a line expressing the relation between the times and distances of the said law from its ordinates and abscissas, which is a remarkable theorem. Hence we have also a physical way of constructing logarithms, which cannot be constructed exactly by common geometry.

Art. II.

If the motion shall be by a weight accelerated and uniformly retarded by a medium following the positions.

I) This is the same as the first hypothesis in place with the single precedent, truly the decrements of the forces (that is here in place of the velocities) made from the absolute resistance, are proportional to the increments of the distances.

2) The increases of the velocities from gravity are proportional to the increments of the time, and the other hypothesis is from the nature of the motion of the weight.

3) The right lines are given proportional to the times spent, from each of which if the right line shall be drawn equal to the corresponding distance traversed by the moving point, the remaining part of the right line will be proportional to the velocity acquired; for the velocities impressed are proportional to the times (by 2), with the distances traversed dismissed (here by I, according to the manner of prop.2 of the preceding article), therefore with different remainders acquired.

4) If the complements of the velocities acquired for the maximum velocity shall be as numbers, the times spent will be as the logarithms [of these numbers]. Truly, with the former figure 15 retained, AB shall be the maximum velocity (as soon it will be apparent to be exclusively such), AM the acquired, BM or TL at this stage requiring to be acquired or the complements of the acquired, BT and ML the time spent ; from prop. 1 and 2 it may be found, with the increments of the time BT supposed equal, the increments of the

velocities AM to be proportional to BM themselves. Therefore if BT will be the logarithms, BM will be the numbers.

[The acc. $\frac{dV}{dt} = ge^{-\alpha t}; V = \frac{g}{\alpha}\left(1-e^{-\alpha t}\right) = V_{\max}\left(1-e^{-\alpha t}\right)$, *i.e.* $t_1 : t_2 :: \frac{\log(V_{\max}-V_2)}{\log(V_{\max}-V_1)}$ etc.]

5) Hence it is apparent (according to the manner of the preceding prop. 4) for the velocity never to reach the maximum AB, or for such to be excluded.

Art. III.

If a weight shall be projected in a medium having absolute resistance, that is, if it may be carried by a motion composed from the motions of two preceding articles. In figure 16 the weigh is put in position in A, trying to fall along or in a line parallel to AG, projected from A in the direction AMB at some angle MAG, and to describe the line AP; AB shall be the exclusive maximum way of the first article ; the parallelograms MAGP and BAGK may be completed.

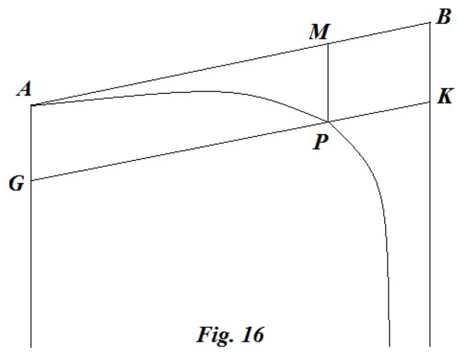

Fig. 16

1) The right line perpendicular to the horizontal (BK) passing through (B) the limit of the penetration (or through the point, to which by the object moving by itself uniformly, in a medium of uniform and absolute resistance, progressing along the right line AM, is unable to reach) is the asymptote of the line of projection, or the two lines, clearly the right line BK and by some curve AP continued indeed always to be approaching each other, yet themselves never touching, because the moving object moves towards the part B on AB and to the parallel line in the same manner from the composite motion, and if following the laws of the first article alone it may not be carried off by gravity, at no time will it arrive at B or some point equivalent to that on the right line BK produced somehow.

2) The curve of projection is not derived from any of the conic sections, certainly not from the parabola, the circle or ellipse, for these are without asymptotes, but indeed here as in a hyperbola through some point a right line BK drawn on both sides of the point can be assumed to be the asymptote for some other curve.

3) A certain reliable simple curve is given (this is parabolic or hyperbolic in nature), of which the abscissas if they shall be proportional to the remaining distances (BM) at the end penetration (AB) of the prescribed projection, the ordinates are proportional to the velocity deficiencies at this stage according to acquiring the prescribed limit of the velocity of descent: I understand here a simple curve, whose ordinates are in some multiple or fraction of the abscissas. Therefore it is considered, the velocities for the descent at this stage to be deficient by reason of the failing to reach the limit of penetration, requiring to be multiplied by some constant number. This may be shown from

that, because both can be understood to be geometric progressions, if the times spent shall be arithmetical progressions by art.1 prop. 3 and art.2 prop. 4, and everywhere the logarithm of the maximum number is 0, that of the minimum infinite, by art. I prop. 4 and art. 2 prop. 5. When the number multiplying the ratio is rational, some parabolic or hyperbolic line arises of common geometry. Again here whatever the number, it can be found from experiment.
[Thus Leibniz has resorted to curve fitting.]
4) The line of projection AP can be found, or the relation between the coordinates AG of the distance of descent and of the distance AM of the progression uniform by itself. For by art. 2 prop. 3 a simple relation is given between the time spent, the distance traversed in the descent AG, and the velocity acquired in the descent at G. In this relation AM may be substituted for the time, with the aid of the relation between the given articles themselves of the given art. l prop. 3, a relation remains therefore between AG and AM, which even if it shall be transcending, yet it may be considered as nothing other than a logarithm.

[Aiton expands on this case, making use of his access to the original writings on microfilm in the Hanover Library, and later developments by Leibniz following work by the Bernoulli brothers, to arrive finally at a solution to the problem. I suggest you view Aiton's original paper for these developments; the present work here relates to what was published in the Acta.]

Art. IV.

Concerning the respective resistance of a medium, if the motion uniform by itself shall be retarded by the medium uniformly in proportion to the speed, just as only the density of the medium shall be considered, with no account had of the tenacity [*i.e.* the intrinsic resistance or viscosity is ignored, and the motion is horizontal].

1) The diminutions of the velocities are in a ratio composed from the velocity present and of the increments of the distance. Which is the hypothesis of the present case.

2) If the velocities remaining (as MB or LT in Fig. 15) shall be as the numbers, the distance traversed (BT or ML) shall be as the logarithms. In the same manner as was shown in art. 1, prop. 3, if for the distances put there you may put the velocities, and for the times the distances [*i.e.* $V_1 : V_2 :: \frac{\log(L_{\max} - L_2)}{\log(L_{\max} - L_1)}$; in modern terms we can write, for a body of unit mass moving horizontally with the initial speed v_0 in a medium resisting as the velocity v :

$\frac{dv}{dt} = -kv; \frac{dv}{v} = -kdt; v = v_0 e^{-kt}$ or $\log\left(\frac{v_0}{v}\right) = kt;$ hence

$\frac{dx}{dt} = v_0 e^{-kt}$ and $x = x_0 - \frac{v_0}{k} e^{-kt}$ or $\left(x_0 - x\right) = \frac{v_0}{k} e^{-kt} = \frac{v}{k};$

hence the velocity decreases linarly with the distance, in contradiction to Lieibniz's assertion : *i.e.* $V_1 : V_2 :: \frac{\log(L_{\max} - L_2)}{\log(L_{\max} - L_1)}$ should be $V_1 : V_2 :: \frac{(L_{\max} - L_1)}{(L_{\max} - L_2)}$.

Aiton here goes along with Leibniz's reasoning, and presents the same incorrect result.]

3) If the times spent, increased by a certain constant quantity, shall be as the numbers, the distances traversed are as the logarithms. For with equal distances of the element present, the elements of the time are inversely as the velocities, that is they increase in a geometric progression (by the preceding), therefore (from the quadrature of the logarithms) the times increase by a constant amount also are in a geometric progression.

$$[i.e. :\ T_1 : T_2 :: \frac{\log(L_{\max} - L_2)}{\log(L_{\max} - L_1)}\ .]$$

4) Hence also the times increased by a constant amount are inversely as the velocities remaining. To be apparent from the preceding considerations. But that constant is the finite time, in which the infinite distance may be traversed, if the first velocity may be increased in that proportion, by which now it may be diminished by the resistance of the medium. And this quantity can be found from two experiments, from the distances and times brought together, indeed by a single experiment, in which the time and the velocity may be considered.

Art. V.

If a motion accelerated by gravity may be retarded by a medium in the proportion of the speed. [This resistance turns out to be as the square of the speed in the calculation, as it is compounded from a resistance proportional to the speed, and an extra proportional due to the distance traversed in an element of time, also proportional to the speed.]

1) This is in place of hypothesis I, as the same single hypothesis of the preceding article.
2) And hypothesis 2 is as the same hypothesis 2 of the second article.
3) The resistance is to the new impressed force, made by the weight in the same element of time (or to the diminution of the velocity for ascending) as the square of the excess maximum velocity acquired above to the square of the maximum. [A better description is to be found in the correspondence with Huygens : 'The resistance is to the impressed weight as the square of the velocity acquired to the square of the maximum velocity.' Noted on p.141 by Gerhardt. This results in an equation of the form : $\frac{\text{resistance}}{mg} = \frac{kv^2}{v_{\max}^2}$.

Hence the increase in the velocity downwards in time t is given by :

$$dv = gdt - \frac{kv^2}{v_{\max}^2}dt; \text{ or } dt = \frac{v_{\max}^2 dv}{gv_{\max}^2 - kv^2} = \frac{v_{\max}}{2\sqrt{g}}\left\{\frac{dv}{\sqrt{g}v_{\max}+\sqrt{k}v} + \frac{dv}{\sqrt{g}v_{\max}-\sqrt{k}v}\right\}$$

$$t = \frac{v_{\max}}{\sqrt{kg}}\log\sqrt{\left(\frac{\sqrt{g}v_{\max}+\sqrt{k}v}{\sqrt{g}v_{\max}-\sqrt{k}v}\right)}.]$$

For from prop. 4 it follows here the resistances are to be in a composite ratio of the elements of time and the squares of the velocities [note that Leibniz takes the acc. due to gravity as equal to 1, as was often the case at this time]; but the new forces are as the elements of time by prop. 2, and in the case of the maximum velocity the diminution and the increase of the velocity are equal. From which the proposition is concluded easily.

4) If the ratios between the sum and difference of the maximum and minimum velocities assumed shall be as the numbers, the times, in which the assumed velocities are acquired, shall be as the logarithms.
[From this we gather, without regard to constant quantities, as these are taken as 1:

$$\log\frac{v_{\max}+v_{\min}}{v_{\max}-v_{\min}} = \frac{t_{\max}}{t_{\min}}, \text{ as indicated above.]}$$

For since the increment of the velocity shall be the difference between the impressed force and the resistance, hence (from the preceding) it follows immediately the following impressed force to be to the increment of the velocity, as the square of the maximum velocity to the excess of its square over the square of the present velocity assumed. From which we know by quadrature, the sum of the impressed forces thence from the beginning, which is proportional to the time spent, to be as the logarithm, if the number shall be such as we have enunciated to be as in this proposition.

5) The maximum velocity of such is excluded, or cannot be attained, even if it may approach to that unassignable interval. For when the ratio is one of equality, or when the velocity is just beginning or infinitely small, the time (and thus the logarithm) is 0, and hence since the ratio shall be infinite, that is when the velocity assumed is by itself a maximum, the logarithm of the ratio is infinite. And thus there shall be a need for an infinite time for that velocity being acquired. Moreover the maximum velocity can be found by two experiments, with the times and the velocities likewise deduced by prop. 3.

6) If the velocities acquired (AV fig. 17) shall be as the sine (of the arcs HK of the part of the quadrant of the circle HKBJ, the distances ran through (AS) will be as the logarithms of the complements of the sines (VK), with the radius or the maximum sine (AB) to be as the maximum velocity. For from hypothesis 2. it follows the increments of the distance to be in the ratio composed of the velocities acquired and force of gravity, but the forces are as the increments of the velocity, as has been set out in the demonstration prop. 4. Hence it follows the increments of the distance to be in the composite ratio of the increments of the velocity directly, and inversely in the ratio of the excess of the square of the maximum speed on the assumed square. From which we know through the quadratures following the proposition. Hence it is apparent the logarithm of the total sine to be 0 (since the velocity

is 0), but the logarithm of the sine of the complement vanishing (when the velocity is a maximum) or the distance shall be infinite, from which at last it is evident the maximum velocity can never be reached.

[For on setting $v = v_{\max} \sin\phi$ and y the distance fallen, then as previously above,

$dv = gdt - \frac{kv^2}{v_{\max}^2} dt$; here we may set

$g = k = 1$ and $dt = \frac{v_{\max}^2}{v_{\max}^2 - v^2} dv$, in which case

$$dt = \frac{v_{\max}^2}{v_{\max}^2 - v^2} dv = \frac{v_{\max}}{2}\left[\frac{1}{v_{\max} - v} + \frac{1}{v_{\max} + v}\right], \text{ and}$$

$$t = \frac{v_{\max}}{2}\left[\log\left(v_{\max} + v\right) - \log\left(v_{\max} - v\right)\right]$$

$$= v_{\max} \log\sqrt{\frac{v_{\max}+v}{v_{\max}-v}}.]$$

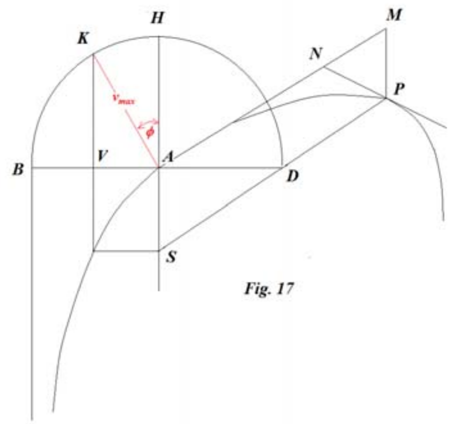

Fig. 17

7) If the distances traversed (AS fig. 17) shall be as the logarithms of the sines (KV of the arc BK), the times spent shall be as the logarithms of the ratios, which are between the versed sine (BV) and its complement (VD) to (BD) the diameter or twice the total sine (AB). This is apparent from propositions 4 and 6 taken together.

[As Aiton shows, the integration for this and the following article can be represented geometrically as in Fig. 17. For from $dt = \frac{v_{\max}^2}{v_{\max}^2 - v^2} dv$ and assuming the distance fallen in the time dt to be dy, we have $dy = vdt \propto \frac{v}{v_{\max}^2 - v^2} dv$ and $y \propto \log \frac{v}{\sqrt{v_{\max}^2 - v^2}}$; on taking $v = v_{\max} \sin\phi,$ then $\sqrt{v_{\max}^2 - v^2} = v_{\max} \cos\phi,$ giving Leibniz's geometric interpretation, etc.

For $t = \frac{v_{\max}}{2}\left[\log\left(v_{\max} + v\right) - \log\left(v_{\max} - v\right) \propto \log\left(\frac{v_{\max}+v}{v_{\max}-v}\right) = \log\left(\frac{1+\cos\phi}{1-\cos\phi}\right)\right]$

where $= v_{\max} \cos\phi$.]

Art. VI.

If a weight may be projected in a medium having a uniform respective resistance, or may be carried by the composite motion of the two preceding articles. The projection shall be along and parallel to AM (fig. 17), the descent shall be along and parallel to AS, for some angle MAS; the locus of the composite motion P may be had from the complete parallelogram MASP.

1) The curve of the projection can be found (or the relation between AS and AM). From the distance AS the velocity of descent AV (by art.5 prop. 6) at S or at P can be found. From this (by prop. 7) the time spent is given. From this (by art. 4 prop. 3) the distance

AM or SP is given. Therefore from the given line of the abscissa AS the ordinates SP are given, and hence the points of the line can be found.

2) The tangent of the curve can be found, of the direction of the moving weight in that direction. MN may be taken at AM, which shall be to MP, as the velocity at M, found by the time spent (art. 4. prop. 4*)* to the velocity at S, found through the same time (art. 5 prop. 4), and NP joined is a tangent to the curve at the point P. And since the velocity of descent shall be the same at P, which it is at S, the velocity of the projection being pursued at P, shall be the same at M, it is apparent that will be such an amount at the point P ; it is apparent also by which way the moving mass may be carried on that line, for the velocity on the curve is to the velocity of descent, as NP to MP.

We may be able also to place together as one the absolute resistance from articles 1, 2, 3, and the respective resistance from articles 4, 5, 6, surely as they concur actually in nature, but prolixity here is required to be avoided. Many things in practice can be deduced conveniently from these, but for us now the geometrical foundation to have been thrown together must suffice, in which the greatest difficulty lies. And perhaps by diligent consideration we may see certain new ways or at least enough impediments will have been uncovered before progressing further. But everything corresponds to our analysis of infinite quantities, that is the calculation of sums and of differences (certain elements of this we have given in these Actis), as long as it is possible to express these in common words.

I have seen the use of these in machines concerned with the use of gunpowder both in the Acta of Leipzig as well as the New Acta of Rotterdam, I may say first the Most Cel. Thevenot, because he agrees with me, to be cognizant regarding these matters applied to hydraulics, from which and by me again some matters has been raised and required to be considered about *Cleanthes* [$\Pi\varepsilon\rho\iota\alpha\nu\delta\rho o\upsilon$ $K\lambda\varepsilon\alpha\nu\theta\eta s$]. Which I would have wished to add along the way.

Addition.

After I had published certain considerations about the resistances of media in these Acta, there came to hand what the most distinguished Huygens and Newton have considered in the latest works concerning the same argument of the nature of force in mathematics. But I have noted that these only to have touched on respective resistance, as I call it, evidently as far as a body is considered in a liquid without perceptible tenacity [i.e. viscosity], or as in air, truly not absolute, which resistance arises from the tenacity of the medium or from the friction of the surface of contact effectively by rubbing, about which now moreover I have shown much concern, since the respective resistance may be had with respect to the speed of the body, and from that increased the resistance may increase, but not likewise the absolute resistance. I see concerning the respective resistance that it has not been built from these foundations, even if in the first place from the front it may seem otherwise. For the resistances themselves have been established in the square ratio of the speeds, truly I, in talking about absolute resistances (which I judge from the decrease of the velocity to have arisen from the density of the medium) to have said in a ratio composed from the velocity and of the elements of the distance which evidently they have began to traverse with the corresponding velocities ; from which now with equal elements of time taken (in which case the elements of distance being traversed are with velocities proportional to the

velocities) so that the resistances are in the square ratio of the velocities, which also I have made a note of under art. 5, prop. 3. Nor does the conclusion differ concerning the relation between times and velocities for a weight falling in a medium. For this Newton reduced to the sector of a hyperbola, according to the infinite series of Huygens, which he found to depend on the quadrature of the hyperbola, we ourselves presenting according to logarithms in art. 6 prop. 4 as it were the most perfect way of presenting such experiments.
Evidently if the maximum velocity shall be a, the present being v, the time t becomes $= \int dv.aa : \overline{aa - vv}$, with which put in place t shall be as the logarithms in the ratio $a + v$ to $a - v$; also there becomes $t = \frac{1}{2}v + \frac{1}{3}v^3 + \frac{1}{5}v^5 + \frac{1}{7}v^7 \ldots$ etc. on putting one for a.
The most cel. Huygens has advised concerning the composition of motion in a medium with resistance acting vertically, that thus it may not be so simple as in free motion, and that as I have set out in art. 3 and 6 thus is to be taken as an example, and if some body may be moving in a medium according to one law of composition, and for that body itself (such as a ship) to include a medium of the same kind as of the first kind, in which again another body may be carried, of which now the motion from the common motion of the ship and as well as the motion it may make by its own projection thus itself may be had as we have described.
[Thus, e.g. the motion, including the resistance, of a ball dropped vertically on a moving ship may be observed as a curved trajectory by someone on the shore.]

VII.
SCHEDIASMA DE RESISTENTIA MEDII ET MOTU PROJECTORUM GRAVIUM IN MEDIO RESISTENTE.

Galilaeus cum regulas motus projectorum investigavit, resistentiam medii seposuit; fecere idem Torricellius et qui secuti sunt, fatentur tamen aliqui defectum doctrinae atque hinc orientes in praxi errores. Blondellus quidem in libro de Jactu Bomborum putat, impune posse negligi hanc considerationem, sed argumenta ejus non sufficient, nec experimenta affert in magno sumta. Caeterum difficilior est rei Geometrica investigatio, quam ut ab illi, doctissimis licet viris, expectari facile et sperari potuerit, nondum inventis tunc aut certe non satis passim notis subsidiis. Et tamen leges projectorum verae et calculus experimentis consentiens, magno in balistica et pyrobolicis usui futurus, hinc potissimum pendere videntur.

Ego jam dudum inclytae Academiae Scientiarum Regiae Parisinae, cum apud illos agerem, de hoc argumento ratiocinationes communicavi et modum aestimandi ex parte tradidi, speciesque distinxi. Duplex igitur medii resistentia est, una absoluta, altera respectiva, quae plerumque concurrere solent. Absoluta resistentia est, quae tantundem virium mobilis absorbet, sive id parva sive magna velocitate moveatur, dummodo moveatur, et pendet a medii glutinositate; perinde enim est ac si partes filamentis motu mobilis perrumpendis connexae essent inter se. Eadem locum habet in frictionibus superficierum asperarum, in quibus mobilla decurrunt: nam obstacula sunt abradenda vel saltem deprimenda, ad instar pilorum elasticorum sese postea rursus erigentium; ad elastrum autem deprimendum vel ad filum rumpendum eadem semper vis impendenda est, nec refert quae sit agentis velocitas. Resistentia respectiva oritur ex medii densitate, et major est pro majori mobilis velocitate, eo ipso quod partes medii agitandae sunt a penetrante, movere autem aliquid est vim impendere, et eo majorem, quo major communicatur motus medii partibus, hoc est, quo celerior est motus penetrantis. Et resistentia fluidi quiescentis erga corpus incurrens est aequalis vi fluidi incurrentis in corpus quiescens, quae major est, cum celerior est motus fluidi, ut videmus corpora vento et aqua moveri, imo jactu aquae satis impetuoso gravia sustineri, licet hic quoque sese absoluta resistentia immisceat, a qua tamen abstrahendus est animus, cum respectivam aestimamus, quasi nulla esset medii tenacitas. Hoc quoque interest inter duas resistentiarum species, quod absoluta habet quodammodo rationem superficiei mobilis sive contactus, respectiva vero soliditatis. Utrobique paradoxum occurrit, quod mobile penetrans in medium uniforme ubique resistens, nunquam quidem ab eo redigetur ad quietem : a resistentia tamen absoluta corpus, quod vi semel concepta movetur neque aliunde acceleratur, certum habet limitem spatii sive penetrationis in medium ita ut semper ad ipsum recta accedat, nunquam tamen eo perveniat, quam voco penetrationem maximam exclusivam, seu maximam quae non; a resistentia vero respectiva corpus uniformiter acceleratum (ut grave descendens) habet certum limitem velocitatis, seu maximam velocitatem exclusivam, ad quam semper accedit (ut postremo differentia sit insensibilis), ita tamen ut eam nunquam perfecte attingat. Et haec velocitas est illa ipsa, qua motum fluidum (ad instar jactus aquae) posset grave sustincre, ne descendere incipiat. Utriusque motus leges primarias hic exponemus, quantum ista brevitas patitur, nam cuncta distincte tradere res integri tractatus foret.

De resistentia absoluta.

Artic. I.

Si motus mobilis sit per se uniformis et a medio aequaliter secundum spatia retardatus.
1) Decrementa virium sunt proportionalia incrementis spatiorum (quae est hypothesis casus praesentis).

2) Velocitates sunt proportionales spatiis, perditae percursis, residuae adhuc percurrendis. Ponantur incrementa spatii esse aequalia, erunt decrementa virium aequalia (per proposit. I) ; jam si ejusdem mobilis decrementa virium sint aequalia, etiam decrementa velocitatum sunt aequali a*) (sunt enim vires ut quadrata velocitatum, aequalibus autem existentibus quadratis etiam aequalia sunt latera); itaque elementa velocitatum amissarum sunt ut elementa spatiorum percursorum, residuarum ut adhuc percurrendorum. Ergo velocitates sunt ut spatia. Nempe si in Fig. 15 velocitas initio sit AE, spatium integrum in medio percurrendum sit recta AB, ejus pars jam percursa AM, adhuc percurrenda MB, velocitas residua MC (vel AF), amissa FE, erit ECB recta.

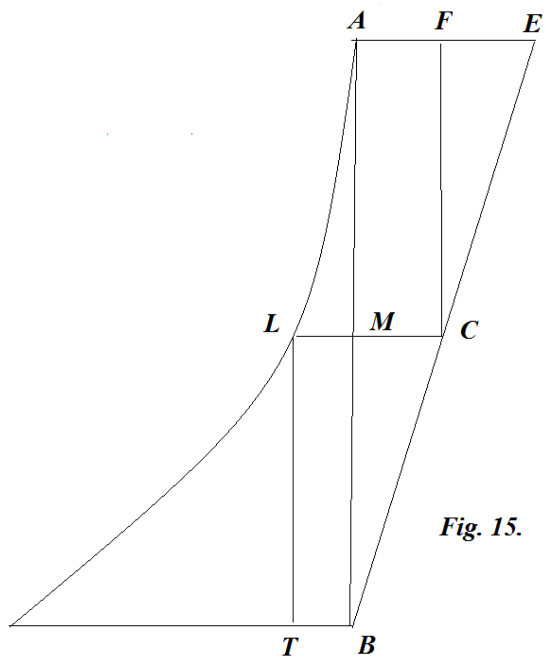

Fig. 15.

3) Si spatia residua (MB vel LT) sint ut numeri, tempora insumta (ML vel BT) erunt ut logarithmi; nam si elements spatii sint progressionis Geometricae, erunt spatia residua ejusdem progressionis Geometricae, ergo (per 2) etiam velocitates residuae, ergo incrementa temporis sunt aequalia, ergo tempora ipsa progressionis Arithmeticae.

4) Mobile M nunquam absolvit spatium percurrendum integrum (AB), etsi semper accedat ad limitem (B), patet enim BT esse asymptoton lineae logarithmicae AL, scilicet ipsius AB numerus hic est 0, ipsius 0 logarithmus est infinitus. Interim in praxi motus fit tandem insensibilis, ut et distantia a B; praeterea nullibi datur medium perfecte uniforme.

5) Si mobile moveatur motu composito ex uniformi et aequaliter a medio secundum spatia retardato, seu si mobile (M) feratur in regula rigida (AB) secundum hypothesin praesentem (uti revera sic satis contingit ob frictionem, si globus in regula rigida horizontali recta moveatur), ipsa vero interim regula (AB) sibi parallela manens, uniformiter mota, uno extremo (B) incedat in aliqua recta (BT), describet lineam

logarithmicam (AL). Generaliter enim, si mobile feratur motu composito ex uniformi et alterius legis describet lineam ordinatis suis et abscissis relationem inter tempora et spatia dictae legis exprimentem, quod est memorabile Theorema. Habemus etiam hinc modum Physicum construendi Logarithmos, quos Geometria communis exacte construere non potest.

Artic. 11.

Si motus sit a gravitate acceleratus et a medio aequabiliter secundum loca retardatos.

I) Est hoc loco hypothesis prima eadem cum hypothesi unica praecedenti, nempe decrementa virium (id est hoc loco velocitatum) facta a resistentia absoluta, sunt proportionalia incrementis spatiorum.

2) Accessiones velocitatum a gravitate sunt proportionales incrementis temporis, estque hypothesis altera ex natura motus gravium.

3) Dantur rectae proportionales temporibus insumtis, a quarum unaquaque si detrahatur recta aequalis respondenti spatio percurso a puncto mobili, residua recta erit proportionalis velocitati acquisitae; nam velocitates impressae sunt proportionales temporibus (per 2), amissae spatiis percursis (per I hic, ad modum proposit.2 articuli praecedentis), ergo residuae acquisitae differentiis.

4) Si velocitatum acquisitarum complementa ad maximam sint ut numeri, tempora insumta erunt ut logarithmi. Nempe, retenta figura priore 15, sit AB velocitas maxima (quam mox patebit esse talem exclusive), AM acquisita, BM vel TL adhuc acquirenda seu complementum acquisitae BT, et ML tempus impensum; ex prop. 1 et 2 reperietur, temporum BT incrementis sumtis aequalibus, velocitatum AM incrementa esse ipsis BM proportionalia. Ergo si BT logarithmi, erunt BM numeri.

5) Hinc patet (ad modum proposit. 4 articuli praecedentis) ad velocitatem maximam AB nunquam perveniri, seu esse talem exclusive.

Artic. III.

Si grave projiciatur in medio resistentiam habente absolutam, hoc est si feratur motu composito ex motibus duorum articulorum praecedentium. In figura 16 ponator grave in A positum, conans descendere in AG et parallelis, projici ex A directione AMB angulo quocunque MAG, et describere lineam AP; sit AB via maxima exclusiva articuli primi; compleantur parallelogramma MAGP, BAGK.

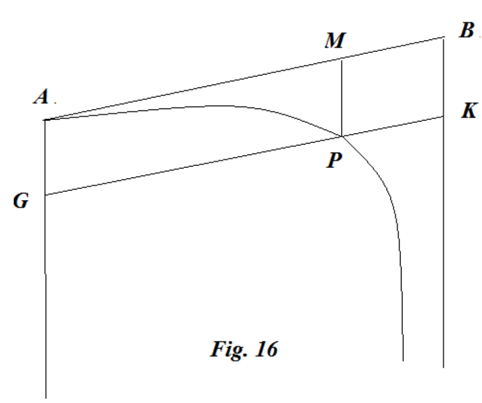

Fig. 16

1) Recta horizonti perpendicularis (BK) per (B) limitem penetrationis (seu per punctum, ad quod mobile motu per se uniformi, in medio uniformi et absolute resistente, in recta AM progrediens, penetrare non potest) est lineae projectionis asymptotos, seu lineae duae, videlicet recta BK et curva AP utcunque continuatae sibi quidem semper accedunt, sese tamen nunquam attingunt, quia mobile ad partes B in AB et parallelis eodem modo tendit motu composito, ae si secundum solius articuli primi leges sine gravitate ferretur, nunquam ergo pervenit ad B vel aliquod ei aequivalens punctum in recta BK utcunque producta.

2) Linea projectionis non est ex numero conicarum, non utique parabola, circulus, aut ellipsis, hae enim carent asymptotis, non hyperbola, neque enim hic ut in hyperbola per punctum aliquod in recta utrinque indefinita BK sumtum duci potest adhuc alia linea asymptotos.

3) Datur certa quaedam linea simplex (hoc est parabolocides aut hyperbolocides), cujus abscissae si sint proportionales spatiis (BM) residuis ad limitem penetrationis (AB) projectioni praescriptum, ordinatae sunt proportionals velocitatibus adhuc deficientibus ad acquirendum limitem velocitatis descensui praescriptum: lineam simplicem hic intelligo, cujus ordinatae sunt in ratione quacunque multiplicata aut submultiplicata abscissarum. Itaque sensus est, velocitates descensui adhuc deficientes esse in ratione spatiorum adhuc limiti penetrationis deficientium, secundum certum aliquem numerum constantem multiplicata. Hoc ex eo demonstrator, quod ambo possunt intelligi progressionis Geometricae, si tempora insumta sint progressionis Arithmeticae per art.1 prop. 3 et art.2 prop. 4, et utrobique numeri maximi logarithmus est 0, minimi infinitus, per art. I prop. 4 et art. 2 prop. 5. Cum numerus rationem multiplicans est rationalis, oritur aliqua linea parabolocides aut hyperbolocides Geometriae communis. Porro hic numerus quibusdam experimentis inveniri potest.

4) Inveniri potest linea projectionis AP, seu relatio inter coordinatas AG spatium descensus et AM spatium progressionis per se uniformis. Nam art. 2 propos. 3 datur relatio simplex inter tempus insumtum, spatium descensu percursum AG, et velocitatem descensu acquisitam in G. In hac relatione pro tempore substituatur AM, ope relationis inter ipsa datae artic. l propos. 3, restat ergo relatio inter AG et AM, quae etsi sit transcendens, tamen nihil aliud supponit quam logarithmos.

Artic. IV.

De Resistentia Medii respectiva, si motus per se uniformis a medio uniformi retardatur proportione velocitatis, quemadmodum sit considerata tantum medii densitate, nulla habita ratione tenacitatis.

1) Diminutiones velocilatum sunt in ratione composita velocitatum praesentium et incrementorum spatii. Quae est hypothesis casus praesentis.

2) Si velocitates residuae (ut MB seu LT fig. 15) sint ut numeri, spatia percursa (BT seu ML) sunt ut logarithmi. Eodem modo demonstratur ut art. I prop. 3, si pro spatiis illic positis ponas velocitales, et pro temporibus spatia.

3) Si tempora insumta, certa quantitate constanti aucta, sint ut numeri, spatia percursa sunt ut logarithmi. Nam spatii elementis existentibus aequalibus, temporis elementa sunt reciproce ut velocitates, hoc est crescunt progressione Geometrica (per praeced.), ergo (ex quadratura logarithmicae) tempora constanti quantitate aucta etiam sunt progressionis Geometricae.

4) Hinc etiam tempora constanti quantitate aucta sunt reciproce ut velocitates residuae. Patet ex consideratione praecedentis. Constans autem illa quantitas est tempus finitum, quo percurreretur spatium infinitum , si prima velocitas ea proportione cresceret, qua nunc a resistentia medii diminuitur. Et potest inveniri haec quantitas duobus experimentis, ex collatis spatiis et temporibus, imo unico experimento, in quo considerantur tempus et velocitas.

Artic. V.

Si motus a gravitate acceleratus a medio uniformi retardetur proportione velocitatis.

1) Est hoc loco hypothesis I. eadem cum hypothesi unica articuli praecedentis.
2) Et hypothesis 2. est eadem cum hypothesi 2. articuli secundi.
3) Resistentia est ad impressionem novam, a gravitate eodem temporis elemento factam (seu diminutio velocitatis ad accessionem) ut quadratum excessus velocitatis maximae super acquisitam est ad quadratum maximae. Nam ex prop. 4 (hic) sequitur resistentias esse in composita ratione elementorum temporis et quadratorum velocitatum; at impressiones novae sunt ut elementa temporis per prop. 2, et in casu maximae velocitalis diminutio et accession velocitatis sunt aequales. Unde facile concluditur propositum.

4) Si rationes inter summam et differentium velocitasis maximae et minoris assumtae sint ut numeri, tempora, quibus assumtae velositates sunt acquisitae, erunt ut logarithmi. Cum enim incrementum velocitatis sit differentia inter impressionem et resistentiam, hinc (ex praecedenti) statim sequitur impressionem esse ad incrementum velocitatis, ut quadratum velocitatis maximae ad excessum hujus quadrati super quadratum praesentis velocitatis assumtae. Ex quo scimus per quadraturas, summam impressionum inde ab initio, quae est proportionalis insumto tempori, esse ut logarithmum, si numerus sit qualem in propositione hac enuntiavimus.

5) Velocitas maxima est talis exclusive, seu nunquam attingi potest, etsi ad eam intervallo inassignabili accedatur. Nam cum ratio est aequalitatis, seu cum velocitas assumta est incipiens sive infinite parva, tempus (adeoque logar.) est 0, et proinde cum fit ratio infinita, hoc est cum velocitas assumta est ipsamet maxima, logarithmus rationis est infinitus. Itaque ad eam velocitatem acquirendam infinito tempore opus foret. Inveniri autem potest maxima velocitas per duo experimenta, collatis temporibus et velocitatibus,
item per prop. 3.

6) Si velocitates acquisitae (AV fig. 17) sint ut sinus (arcuum HK portionum quadrantis circularis HKBJ, erunt spatia percursa (AS) ut logarithmi sinuum complementi (VK),

posito radium seu sinum totum (AB) esse ut velocitatem maximam. Nam ex hypothesi 2. sequitur incrementa spatii esse in ratione composita velocitatum acquisitarum et impressionum gravitatis, sed impressiones sunt ad incrementa velocitatis, ut enunciatum est in demonstratione prop. 4. Hinc sequitur incrementa spatii esse in ratione composita incrementorum velocitatis et velocitatum directa, et reciproca ratione excessus quadrati maximae velocitatis super quadratum assumtae. Unde scimus per quadraturas sequi propositum. Patet hinc logarithmum sinus totius esse 0 (cum velocitas est 0), at evanescentis sinus complementi (cum velocitas est maxima) logarithmum seu spatium esse infinitum, unde rursus patet velocitatem maximam nusquam attingi.

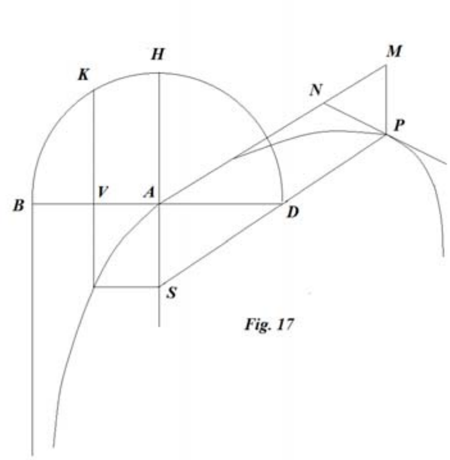

Fig. 17

7) Si spatia percursa (AS fig. 17) sint ut logarithmi sinuum (KV arcuum BK), tempora insumta sunt ut logarithmi rationum, quae sunt inter sinum versum (BV) et (VD) complementum ejus
ad (BD) diametrum seu duplum sinus totius (AB). Patet ex collatis propositionibus 4 et 6.

Artic. VI.

Si grave projiciatur in medio uniformi resistentiam habente respectivam, seu feratur motu composito ex motibus duorum articulorum praecedentium. Sit (fig. 17) projectio in AM et parallelis, descensus in AS et parallelis, angulo MAS quocunque; locus motus compositi P habetur completo parallelogrammo MASP.

1) Inveniri potest linea projectionis (seu relatio inter AS et AM). Ex spatio AS datur (per artic.5 prop. 6) AV velocitas descendendi in S seu in P. Ex hac (per prop. 7) datur tempus insumtum. Ex. hoc (per artic. 4 prop. 3) datur spatium AM seu SP. Ex datis igitur lineis abscissis AS dantur ordinatae SP, ae proinde lineae puncta inveniri possunt.

2) Inveniri potest lineae tangens, seu ipsius mobilis in ea directio. In AM sumatur MN, quae sit ad MP, ut velocitas in M, inventa per tempus insumtum (artic. 4. prop. 4*)* ad velocitatem in S, inventam per idem tempus (artic. 5 prop. 4), et juncta NP tanget curvam in puncto P. Et cum eadem sit velocitas descendendi in P, quae in S, itemque velocitas persequendi projectionis directionem in P, quae in M, patet qualis illa sit in puncto P ; patet etiam qua vi mobile in ipsa linea projectionis feratur, velocitas enim in linea est ad velocitatem descensus, nt NP ad MP.

Possemus etiam in unum componere resistentiam absolutam ex articulis 1, 2, 3, et respectivam ex artic. 4, 5, 6, uti certe revera concurrunt in natura, sed prolixitas hic vitanda est. Multa ex his deduci possent praxi accommodata, sed nobis nunc fundamenta

Geometrica jecisse suffecerit, in quibus maxima consistebat difficultas. Et fortassis attente consideranti vias quasdam novas vel certe satis antea impeditas aperuisse videbimur. Omnia autem respondent nostrae Analysi infinitorum, hoc est calculo summarum et differentiarum (cujus elementa quaedam iu his Actis dedimus), communibus quoad licuit verbis his expresso.

Occasione eorum quae de usu pulveris pyrii mechanico in Lipsiensibus pariter Actis ac Roterodamensibus Novellis vidi, dicam primum celeberrimum Thevenotium , quod mihi constet, de tali re cogitasse ad Hydraulica negotia, unde et mihi aliqua porro meditandi materia nata est, quam comprehendat $\Pi\varepsilon\rho\iota\alpha\nu\delta\rho o\upsilon\ K\lambda\varepsilon\alpha\nu\theta\eta s$. Quod obiter hic adjicere volui.

Additio.

Postquam Meditationes quasdam de Medii resistentia in his Actis publicavi, venere in manus meas , quae Viri in Mathematica naturae cognitione praecellentissimi Hugenius et Newtonus in novissimis operibus de eodem argumento sunt commentati. Animadverti autem eos respectivam tantum (quam voco) resistentiam attigisse, qualem scilicet sentit corpus in liquido tenacitate notabili carente, vel ut in aere, non vero absolutam, quae oritur a tenacitate medii aut asperitate superficiei contactus attritum efficiente inter quas multum interesse jam tum ostendi, cum respectiva habeat respectum ad celeritatem mobilis, eaque aucta crescat, absoluta non item. Circa respectivam video nos iisdem fundamentis inaedificasse, etsi prima fronte aliud videri possit. Ipsi enim statuunt resistentias in duplicata ratione velocitatum, ego vero absolute loquendo resistentias (quas decrementis velocitatis a medii densitate ortis existimo) esse dixi in ratione composita velocitatum et elementorum spatii, quae scilicet velocitatibus respondentibus decurri inchoantur ; unde jam elementis temporis sumtis aequalibus (quo. casu elementa spatii decurrenda velocitatibus proportionalia sunt) utique resistentiae erunt in duplicata ratione velocitatum, quod etiam annotaveram sub art. 5 prop. 3. Nec dissentit conclusio circa relationem inter tempora et velocitates in gravi per medium descendente. Hanc enim ad sectorem hyperbolicum reduxit Newtonus, ad seriem infinitam Hugenius, quam invenit pendere a quadratura hyperbolae, nos ad logarithmos artic. 6 prop. 4 tanquam perfectissimum talia exprimendi modum praebentes.
Nempe sit velocitas maxima a, praesens v, tempus t fiet $=\int dv.aa:\overline{aa-vv}$, quo posito t sunt ut logarithmi rationum $a+v$ ad $a-v$; fiet etiam $t=\frac{1}{2}v+\frac{1}{3}v^3+\frac{1}{5}v^5+\frac{1}{7}v^7 \ldots$ etc. posita a unitate. Circa compositionem motus in medio resistente rectissime monuit celeberrimus Hugenius, eam non ita simpliciter locum habere, ut in motu libero, itaque ea quam exposui articulo 3 et 6 ita accipienda est verbi gratia, ac si corpus aliquod moveatur in medio secundum unam legem motus compositi,
et huic ipsi corpori (veluti navi) sit inclusum medium ejusdem cum priori naturae, in quo iterum aliud corpus feratur, cujus jam motus ex communi navis motu et ipsius proprio velut projectionem faciet ita se habentem ut descripsimus.