PROBLEM 10.

The quadrinomial equation

$$\begin{array}{l} aaaa - baaa + bcaa \\ \quad - caaa + bdaa \\ \quad - daaa + cdaa - bcda \\ \quad + faaa - bfaa + bcfa \\ \qquad\qquad - cfaa + bdfa \\ \qquad\qquad - dfaa + cdfa = +bcdf \end{array}$$

is to be reduced to the trinomial :

$$aaaa \; \frac{\begin{array}{l} - bbaaa \\ - ccaaa \\ - ddaaa \\ - bcaaa \\ - bdaaa \\ - cdaaa \end{array}}{b + c + d} \; \frac{\begin{array}{l} + bbcca \\ + bbcda \\ + ccdda \\ + bcdda \\ + bccda \\ + bbcda \end{array}}{b + c + d} = \frac{+bbccd \; +bbcdd \; +bccdd}{b + c + d}$$

To wit, by removing the second degree or step *aa*.

Placing $bc + bd + cd = bf + cf + df.$

Hence in the proposed equation through the contradiction of the parts the second step is taken away.

Then there remains . . .

$$\begin{array}{l} aaaa - baaa - bcda \\ \quad - caaa + bcfa \\ \quad - daaa + bdfa \\ \quad + faaa + cdfa = +bcdf \end{array}$$

A part of the equation has been reduced.

From substituting $bc + bd + cd = bf + cf + df;\ \dfrac{bc + bd + cd}{b + c + d} = f.$

Therefore in the part of the equation remaining and in the particular parts in which f is to be found, f becomes $\dfrac{bc + dc + cd}{b + c + d}$

Then

$$aaaa \; \frac{\begin{array}{l} -bbaaa \\ - caaa \\ - daaa \\ +bcaaa \\ +bdaaa \\ +cdaaa \end{array}}{b + c + d} \; \frac{\begin{array}{l} + bbcca \\ + bbcda \\ + bbcda \\ + bccda \\ + bbcda \\ + bbdda \\ + bcdda \\ + bccda \\ + bcdda \\ + ccdda \end{array}}{b + c + d} = \frac{+bbccd \; +bbcdd \; +bccdd}{b + c + d}$$

The particular remaining parts left *baaa, caaa, daaa, & bcda,*
are reduced to the common denominator $b + c + d$.
Thus the equation becomes :

$$aaaa \frac{\begin{array}{l} -bbaaa \\ -bcaaa \\ -bdaaa \\ -bcaaa \\ -ccaaa \\ -dcaaa \\ -bdaaa \\ -cdaaa \\ -ddaaa \\ +bcaaa \\ +bdaaa \\ +cdaaa \end{array}}{b+c+d} \frac{\begin{array}{l} +bbcda \\ +bccda \\ -bcdda \\ +bbcca \\ +bbcda \\ +bccda \\ +bbcdc \\ +bbdda \\ +bcdda \\ +bccda \\ +bcdda \\ +ccdda \end{array}}{b+c+d} = \frac{\begin{array}{l} +bbccd \\ +bbcdd \\ +bccdd \end{array}}{b+c+d}$$

The redundant particular parts are rejected as they are of opposite sign.

Thus

$$aaaa \frac{\begin{array}{l} -bbaaa \\ -bcaaa \\ -ccaaa \\ -bdaaa \\ -cdaaa \\ -ddaaa \end{array}}{b+c+d} \frac{\begin{array}{l} +bbcca \\ +bbcda \\ +bccda \\ +bbdda \\ +bcdda \\ +ccdda \end{array}}{b+c+d} = \frac{\begin{array}{l} +bbccd \\ +bbcdd \\ +bccdd \end{array}}{b+c+d}$$

But this itself is the trinomial required
Therefore the reduction of the proposed quadrinomial to the prescribed trinomial has been done.

[Note for Problem 10: the equation $(a-b)(a-c)(a-d)(a+f)=0$ has the square term removed by setting $bc + bd + cd = bf + cf + df$, or $f = \frac{bc+bd+cd}{b+c+d}$; in which case the reduced equation becomes

$$a^4 - a^3(b^2+c^2+d^2+bc+bd+cd)/(b+c+d) + a(b^2c^2+b^2d^2$$
$$+c^2d^2+b^2cd+bc^2d+bcd^2)/(b+c+d) = (bc+bd+cd)bcd/(b+c+d).]$$

PROBLEM 11.

The quadrinomial equation

$$\begin{array}{l} aaaa - baaa + bcaa \\ \quad - caaa + bdaa \\ \quad - daaa + cdaa - bcda \\ \quad + faaa - bfaa + bcfa \\ \qquad - cfaa + bdfa \\ \qquad - dfaa + cdfa \;=\; +bcdf \end{array}$$

is to be reduced to the trinomial :

$$\begin{array}{l} aaaa - bbcaaa \\ \quad - bbdaaa + bbccaa \\ \quad - bccaaa + bbddaa \\ \quad - bddaaa + ccddaa \\ \quad - cddaaa + bcddaa \\ \quad \dfrac{-2.bcdaaa}{bc + cd + db} + \dfrac{bbcdaa}{bc + cd + db} \;=\; \dfrac{+bbccdd}{bc + cd + db.} \end{array}$$

To wit, by removing the first [degree or] step *a*.

Placing $bcd = bcf + bdf + cdf.$

Hence in the proposed equation through the contradiction [*i.e* cancellation] of parts, the first degree is taken

Then there remains . . . .

$$\begin{array}{l} aaaa - baaa - bcaa \\ \quad - caaa + bdaa \\ \quad - daaa + cdaa \\ \quad + faaa - bffa \\ \qquad - cfaa \\ \qquad - dfaa \;=\; +bcdf \end{array}$$

A part of the equation is removed

by substituting $bcd = bcf + bdf + cdf, \quad \dfrac{bcd}{bc + bd + cd} = f.$

Therefore in the remaining part of the equation, and in which *f* in particular belongs,

let it be by first changing *f* into $\dfrac{bc + dc + cd}{b + c + d}$

Secondly by the reduction of the remaining parts to the common divisor $bcf + bdf + cdf.$
Thirdly by the rejection of redundant particular parts from cancellation.

With these accomplished (as in Problem 10) the equation is:

$$\begin{array}{l} aaaa - bbcaaa \\ \quad - bbdaaa + bbccaa \\ \quad - bccaaa + bbddaa \\ \quad - bddaaa + ccddaa \\ \quad - ccdaaa + bcddaa \\ \quad - cddaaa + bccdaa \\ \quad \dfrac{-2.bcdaaa}{bc + cd + db} + \dfrac{bbcdaa}{bc + cd + db} \;=\; \dfrac{+bbccdd}{bc + cd + db} \end{array}$$

But this itself is the required trinomial equation, in which the first step or power *a* is removed.

Thus the required reduction is complete

[Note for Problem 11: the equation $(a-b)(a-c)(a-d)(a+f)=0$ has the linear term removed by setting $bcd = bcf + cdf + bdf,$ or $f = \frac{bcd}{bc+cd+bd}$; in which case the reduced equation becomes
$a^4 - a^3(b^2c + b^2d + bc^2 + bd^2 + c^2d + cd^2 + 2bcd)/(bc + cd + bd) + a^2(b^2c^2 + b^2d^2 + c^2d^2 + b^2cd + bc^2d + bcd^2)/(bc + bd + cd) = b^2c^2d^2/(bc + bd + cd)$.]

PROBLEM 12.

The quadrinomial equation

$$\begin{array}{llll} aaaa & +baaa & +bcaa & \\ & +caaa & +bdaa & \\ & +daaa & +cdaa & +bcda \\ & -faaa & -bfaa & -bcfa \\ & & -cfaa & -bdfa \\ & & -dfaa & -cdfa \quad = \quad +bcdf \end{array}$$

is to be reduced to the trinomial:

$$\begin{array}{llll} aaaa & -bbaa & -bbca & \\ & -ccaa & -bbda & \\ & -ddaa & -bcca & \\ & -bcaa & -ccda & \\ & -bdaa & -bdda & \\ & -cdaa & -cdda & \\ & & -2.bcda & = +bbcd \\ & & & \;\;\; +bccd \\ & & & \;\;\; +bcdd \end{array}$$

To wit, by removing the third degree or step *aaa*.

Placing $b + c + d = f$,

the quadrinomial equation is reduced by changing f into $b + c + d$

Then the equation is . . .
. .

$$\begin{array}{llll} aaaa & +baaa & +bcaa & \\ & +caaa & +bdaa & \\ & +daaa & +cdaa & +bcda \\ & -baaa & -bbaa & -bcba \\ & -caaa & -bcaa & -bcca \\ & -daaa & -bdaa & -bcda \\ & & -cbaa & -bdba \\ & & -ccaa & -bdca \\ & & -cdaa & -bdda \\ & & -ddaa & -cdba \\ & & -dcaa & -cdca \\ & & -ddaa & -cdda \quad = +bcdb \\ & & & \qquad\quad\;\; +bcdc \\ & & & \qquad\quad\;\; +bcdd \end{array}$$

Particular redundant [parts] are rejected by cancellation.

hence it becomes . . . .

$$\begin{array}{llll} aaaa & -bbaa & -bbca & \\ & -ccaa & -bbda & \\ & -ddaa & -bcca & \\ & -bcaa & -ccda & \\ & -bdaa & -bdda & \\ & -cdaa & -cdda & \\ & & -2.bcda & = +bbcd \\ & & & \;\;\; +bccd \\ & & & \;\;\; +bcdd, \end{array}$$

which is the quadrinomial equation sought.

Therefore the reduction has been effected for the proposed equation.

[Note for Problem 12: the equation $(a+b)(a+c)(a+d)(a-f)=0$ has the cubic term removed by setting $b + c + d = f$; in which case the reduced equation becomes $a^4-(b^2+c^2+d^2+bc+bd+cd)a^2-(b^2c+bc^2+c^2d+cd^2+b^2d+bd^2+2bcd)a$ $=(b+c+d)bcd$.]

PROBLEM 13.

The quadrinomial equation

$$\begin{array}{l} aaaa + baaa + bcaa \\ \quad + caaa + bdaa \\ \quad + daaa + cdaa + bcda \\ \quad - faaa - bfaa - bcfa \\ \qquad\quad - cfaa - bdfa \\ \qquad\quad - dfaa - cdfa \end{array} = +bcdf$$

is to be reduced to the trinomial

$$\begin{array}{l} aaaa + bbaaa - bbcca \\ \quad + ccaaa - bbdda \\ \quad + ddaaa - ccdda \\ \quad + bcaaa - bcdda \\ \quad + bdaaa - bccda \\ \quad + \underline{cdaaa} - \underline{bbcda} \\ \quad b+c+d \quad b+c+d \end{array} = \frac{+bbccd + bbcdd + bccdd}{b+c+d}$$

To wit, by removing the second degree or power *aa*.

Placing $bc + bd + cd = bf + cf + df$

And thus the second step is removed in the proposed quadrinomial equation by the cancellation of particular parts.

Then there remains . . . .

$$\begin{array}{l} aaaa + baaa \\ \quad + caaa - bcfa \\ \quad + daaa - bafa \\ \quad - faaa - cdfa \end{array} = +bcdf$$

The part of the equation thus is reduced.

by substituting $bc + bd + cd = bf + cf + df.$

That is $\frac{bc + bd + cd}{b + c + d} = f$

in the first place is put in that part of the remaining equation where f is present to become:

$$\frac{bc + bd + cd}{b + c + d}$$

Secondly by the reduction of the particular parts to a common divisor.

Thirdly particular redundant [parts] are rejected by cancellation.

With this carried out (as in Prob. 10) there shall be

$$\begin{array}{l} aaaa + bbaaa - bbcca \\ \quad + ccaaa - bbdda \\ \quad + ddaaa - ccdda \\ \quad + bcaaa - bcdda \\ \quad + bdaaa - bccda \\ \quad + \underline{cdaaa} - \underline{bbcda} \\ \quad b+c+d \quad b+c+d \end{array} = \frac{+bbccd + bbcdd + bccdd}{b+c+d}$$

But this itself is the trinomial equation sought.
So therefore the prescribed reduction is complete.

[Note for Problem 13: the equation $(a+b)(a+c)(a+d)(a-f)=0$ has the square term removed by setting $bc + bd + cd = bf + cf + df$, or $f = \frac{bc+bd+cd}{b+c+d}$; in which case the reduced equation becomes

$$a^4 + a^3(b^2 + c^2 + d^2 + bc + bd + cd)/(b+c+d) - a(b^2c^2 + b^2d^2 + c^2d^2 + b^2cd + bc^2d + bcd^2)/(b+c+d) = (bc+bd+cd)bcd/(b+c+d).]$$

PROBLEM 14.

The quadrinomial equation

$$\begin{array}{llll} aaaa & + baaa & + bcaa & \\ & + caaa & + bdaa & \\ & + daaa & + cdaa & + bcda \\ & - faaa & - bfaa & - bcfa \\ & & - cfaa & - bdfa \\ & & - dfaa & - cdfa \; = \; +bcdf \end{array}$$

is to be reduced to the trinomial

$$\begin{array}{lll} aaaa & + bbcaaa & \\ & + bbdaaa & + bbccaa \\ & + bccaaa & + bbddaa \\ & + ccdaaa & + ccddaa \\ & + bddaaa & + bcddaa \\ & + cddaaa & + bccdaa \\ & + \dfrac{2.\ bcdaaa}{bc + bd + cd} & + \dfrac{bbcdaa}{bc + bd + cd} \; = \; \dfrac{bbccdd}{bc + bd + cd} \end{array}$$

To wit, by subtracting the first degree step *a*.

Placing $bcd = bcf + bdf + cdf$

And thus the first step *a* is removed in the proposed quadrinomial by the cancellation of particular parts.

Then there remains . . . .

$$\begin{array}{lll} aaaa & + baaa & + bcaa \\ & + caaa & + bdaa \\ & + daaa & + cdaa \\ & - faaa & - bfaa \\ & & - cfaa \\ & & - dfaa \; = \; +bcdf \end{array}$$

A part of the equation thus is reduced.

By substituting $bcd = bcf + bdf + cdf$

or $\dfrac{bcd}{bc + bd + cd} = f.$

In the remaining part of the equation, in which *f* is present, and which in the first place is changed into $\dfrac{bcd.}{bc + bd + cd}$

Secondly by the reduction of the particular parts to a common divisor.

Thirdly particular redundant [parts] are rejected by cancellation.

With this carried out (as in Prob. 10) there remains

$$\begin{array}{lll} aaaa & + bbcaaa & \\ & + bbdaaa & + bbccaa \\ & + bccaaa & + bbddaa \\ & + ccdaaa & + ccddaa \\ & + bddaaa & + bcddaa \\ & + cddaaa & + bccdaa \\ & + \dfrac{2.\ bcdaaa}{bc + bd + cd} & + \dfrac{bbcdaa}{bc + bd + cd} \; = \; \dfrac{+bbccdd}{bc + bd + cd} \end{array}$$

But this itself is the trinomial equation sought in which the first step has been taken away.

So therefore the prescribed reduction is complete.

[Note for Problem 14: the equation $(a+b)(a+c)(a+d)(a-f)=0$ has the linear term removed by setting $bcd = bcf + cdf + bdf$, or $f = \frac{bcd}{bc+cd+bd}$; in which case the reduced equation becomes

$$a^4 + a^3(b^2c + b^2d + bc^2 + bd^2 + c^2d + cd^2 + 2bcd)/(bc + cd + bd) + a^2(b^2c^2 + b^2d^2 + c^2d^2 + b^2cd + bc^2d + bcd^2)/(bc + bd + cd) = b^2c^2d^2/(bc + bd + cd).]$$

PROBLEM 15.

The quadrinomial equation

$$\begin{array}{llll} aaaa & -\,baaa & +\,bcaa & \\ & -\,caaa & -\,bdaa & \\ & +\,daaa & -\,cdaa & +\,bcda \\ & +\,faaa & -\,bfaa & +\,bcfa \\ & & -\,cfaa & -\,bdfa \\ & & +\,dfaa & -\,cdfa \; = \; -\,bcdf \end{array}$$

is to be reduced to the trinomial

$$\begin{array}{lll} aaaa & +\,bdaa & +\,bbca \\ & +\,cdaa & +\,bcca \\ & -\,bbaa & +\,bdda \\ & -\,bcaa & +\,cdda \\ & -\,ccaa & -\,bbda \\ & -\,ddaa & -\,ccda \\ & & -\,2.bcda \; = \; \begin{array}{l} -\,bbcd \\ -\,bccd \\ +\,bcdd \end{array} \end{array}$$

To wit, by removing the third [degree or] step *aaa*.

If $b + c = d + f$, then $b + c - d = f$.

Therefore placing $b + c - d = f$.

And the quadrinomial equation is reduced first by changing *f* into $b + c - d$

Secondly by rejecting redundant terms by cancellation.

With these accomplished (as in Prob. 10) then . . .

$$\begin{array}{lll} aaaa & +\,bdaa & +\,bbca \\ & +\,cdaa & +\,bcca \\ & -\,bbaa & +\,bdda \\ & -\,bcaa & +\,cdda \\ & -\,ccaa & -\,bbda \\ & -\,ddaa & -\,ccda \\ & & -\,2.bcda \; = \; \begin{array}{l} -\,bbcd \\ -\,bccd \\ +\,bcdd \end{array} \end{array}$$

which is the trinomial equation sought, in which the third power *aaa* has been removed.

So therefore the reduction has been done for the proposed equation to be in the required form, as prescribed.

[Note for Problem 15: the equation $(a-b)(a-c)(a+d)(a+f)=0$ has the cubic term removed by setting $b + c - d = f$; in which case the reduced equation becomes
$$a^4-(b^2+c^2+d^2+bc-bd-cd)a^2+(b^2c+bc^2-c^2d+cd^2-b^2d+bd^2-2bcd)a$$
$$=-(b+c-d)bcd.$$]

Problem 16.

The quadrinomial equation

$$\begin{array}{llll} aaaa & -baaa & +bcaa & \\ & -caaa & -bdaa & \\ & +daaa & -cdaa & +bcda \\ & +faaa & -bfaa & +bcfa \\ & & -cfaa & -bdfa \\ & & +dfaa & -cdfa \end{array} = -bcdf$$

is to be reduced to the trinomial

$$aaaa \begin{array}{l} -bbaaa \\ -bcaaa \\ -ccaaa \\ -ddaaa \\ +bdaaa \\ +cdaaa \\ \hline b+c-d \end{array} \begin{array}{l} +bbcca \\ +bbdda \\ +bcdda \\ +ccdda \\ -bbcda \\ -bccda \\ \hline b+c-d \end{array} = \begin{array}{l} -bbccd \\ +bbcdd \\ +bccdd \\ \hline b+c-d \end{array}$$

To wit, by subtracting the second [degree or] step aa.

If $bc + df = bd + cd + bf + + cf.$

That is $bc - bd - cd = bf + cf - df,$ then $\dfrac{bc - bd - cd}{b + c - d} = f.$

Therefore put $\dfrac{bc - bd - cd}{b + c - d} = f.$

Therefore in the quadrinomial equation which it is proposed to reduce,

this is done by first changing f to $\dfrac{bc - bd - cd}{b + c - d}$

Secondly by the reduction of the remaining parts to a common divisor $b + c - d$

Thirdly by the rejection of redundancies by cancellations.

With these being accomplished (as in Prob. 9),

there remains . .

$$aaaa \begin{array}{l} -bbaaa \\ -bcaaa \\ -ccaaa \\ -ddaaa \\ +bdaaa \\ +cdaaa \\ \hline b+c-d \end{array} \begin{array}{l} +bbcca \\ +bbdda \\ +bcdda \\ +ccdda \\ -bbcda \\ -bccda \\ \hline b+c-d \end{array} = \begin{array}{l} -bbccd \\ +bbcdd \\ +bccdd \\ \hline b+c-d \end{array}$$

But this itself is the trinomial equation sought in which the second step has been taken away.

So therefore the prescribed reduction is complete.

[Note for Problem 16: the equation $(a+b)(a+c)(a-d)(a-f)=0$ has the square term removed by setting $bc - bd - cd = bf + cf - df,$ or $f = \frac{bc-bd-cd}{b+c-d}$; in which case the reduced equation becomes
$a^4 - a^3(b^2 + c^2 + d^2 + bc - bd - cd)/(b+c-d) + a(b^2c^2 + b^2d^2$
$+ c^2d^2 - b^2cd - bc^2d + bcd^2)/(b+c-d) = (-bc + bd + cd)bcd/(b+c-d).$]

PROBLEM 17.

The quadrinomial of the above proposition can also be reduced to this trinomial :

$$aaaa + \frac{\begin{array}{l} + bbaaa \\ + bcaaa \\ + ccaaa \\ + ddaaa \\ - bdaaa \\ - cdaaa \end{array}}{d - b - c} \begin{array}{l} - bbcca \\ - bcdda \\ - bbdda \\ - ccdda \\ + bbcda \\ + bccda \end{array} \Big/ (d - b - c) = \frac{\begin{array}{l} - bbccd \\ - bccdd \\ + bccdd \end{array}}{d - b - c}$$

To wit, by subtracting the second [degree or] step *aa*

by placing $f = \frac{bd + cd - bc,}{d - b - c}$

by changing f into $\frac{bd + cd - bc}{d - b - c}$ (as above),

by reduction to a common divisor $d - b - c,$

& by the rejection of redundancies from cancellations, according to the example of reduction in Problem 9.

[Note for Problem 17: the equation $(a+b)(a+c)(a-d)(a-f)=0$ also has the square term removed on a sign change by setting $bc - bd - cd = bf + cf - df,$ or $f = \frac{bd+cd-bc}{d-b-c}$; in which case the reduced equation becomes

$$a^4 + a^3(b^2 + c^2 + d^2 + bc - bd - cd)/(d - b - c) - a(b^2c^2 + b^2d^2 + c^2d^2 - b^2cd - bc^2d + bcd^2)/(d - b + c) = (bc - bd - cd)bcd/(d - b - c).$$]

PROBLEM 18.

The quadrinomial equation

$$\begin{array}{l} aaaa - baaa + bcaa \\ \quad - caaa - bdaa \\ \quad + daaa - cdaa + bcda \\ \quad + faaa - bfaa + bcfa \\ \qquad - cfaa - bdfa \\ \qquad + dfaa - cdfa = - bcdf \end{array}$$

to be reduced to the trinomial

$$\begin{array}{l} aaaa + bbaaa \\ \quad + bccaaa - bbccaa \\ \quad + bddaaa - bbddaa \\ \quad + ccdaaa - bcddaa \\ \quad - bddaaa - bcddaa \\ \quad - ccdaaa + bccdaa \\ \quad \dfrac{+2.\,bcdaaa}{bd + cd - bc} + \dfrac{bccdaa}{bd + cd - bc} = \dfrac{+bbccdd}{bd + cd - bc} \end{array}$$

To wit, by subtracting the first [degree or] step *a*.

If $bcd + bcf = bdf + cdf,$

or $bcd = bdf + cdf - bcf.$

Then $\dfrac{bcd}{bd + cd - bc} = f.$

Therefore setting $\dfrac{bcd}{bd + cd - bc} = f.$

And in the quadrinomial equation which is proposed to reduce, and in the particular parts in which *f* is present, first by changing *f* into $\dfrac{bcd}{bd + cd - bc}$

Secondly by the reduction of the remaining parts to the common divisor $bd + cd - bc$

Thirdly by the rejection of redundancies by cancellation.

With these executed, (as in Prob. 9), then

$$\begin{array}{l} aaaa + bbcaaa \\ \quad + bccaaa - bbccaa \\ \quad + bddaaa - bbddaa \\ \quad + ccdaaa - bcddaa \\ \quad - bddaaa - ccddaa \\ \quad - ccdaaa + bbcdaa \\ \quad \dfrac{- 2.bcdaaa}{bd + cd - bc} + \dfrac{bccdaa}{bd + cd - bc} = \dfrac{+bbccdd}{bd + cd - bc} \end{array}$$

But this itself is the trinomial equation sought in which the first step has been taken away.

So therefore the prescribed reduction is complete.

[Note for Problem 18: the equation $(a-b)(a-c)(a+d)(a+f)=0$ has the linear term removed by setting $bcd + bcf = bdf + cdf$, or $f = \frac{bcd}{bd+cd-bc}$, in which case the reduced equation becomes

$$a^4 + a^3(b^2c + b^2d + bc^2 - bd^2 - c^2d + cd^2 - 2bcd)/(bd + cd - bc) - a^2(b^2c^2 + b^2d^2 + c^2d^2 - b^2cd - bc^2d + bcd^2)/(bc + cd - bc) = b^2c^2d^2/(bd + cd - bc).]$$

PROBLEM 19.

The quadrinomial equation

$$\begin{array}{lllll} aaaa & -\,baaa & +\,bcaa & & \\ & -\,caaa & -\,bdaa & & \\ & +\,daaa & -\,cdaa & +\,bcda & \\ & +\,faaa & -\,bfaa & +\,bcfa & \\ & & -\,cfaa & -\,bdfa & \\ & & +\,dfaa & -\,cdfa & = -\,bcdf \end{array}$$

is to be reduced to the binomial :

$$\begin{array}{lll} aaaa & +\,bbba & \\ & -\,bbca & \\ & -\,bcca & \\ & -\,ccca & = -\,bbbc \\ & & \ -\,bbcc \\ & & \ -\,bccc \end{array}$$

by placing $b + c = d + f$

To wit, with the steps *aa* & *aaa* taken away.

[Note for Problem 19: the equation $(a-b)(a-c)(a+d)(a+f)=0$ has the cubic term removed by setting $b + c = d + f$; the square term is also removed for $bc+df=bd+cd+bf+cf=(b+c)(d+f)=(b+c)^2$, from which it follows that $df=b^2+bc+c^2$, if the second power is also removed; in which case the linear term in the reduced equation becomes:

$bcd+bcf-bdf-cdf=bc(d+f)-(b+c)df=bc(b+c)-(b^2+bc+c^2)bc$

$=-(b^3+b^2c+bc^2+c^3)$;

while the constant term becomes: $bcdf=bc(b^2+bc+c^2)$ as required.]

Problem 20.

The quadrinomial

$$\begin{array}{llll} aaaa & -baaa & +bcaa & \\ & -caaa & -bdaa & \\ & +daaa & -cdaa & +bcda \\ & +faaa & -bfaa & +bcfa \\ & & -cfaa & -bdfa \\ & & +dfaa & -cdfa \;=\; -bcdf \end{array}$$

is reduced to the binomial

$$\frac{\begin{array}{l} aaaa - bbbaaa \\ \quad - bbcaaa \\ \quad - bccaaa \\ \quad - cccaaa \end{array}}{bb + bc + cc} = \frac{-bbbccc}{bb + bc + cc}$$

by placing $bc + df = bd + cd + bf + cf.$

To wit, with the steps a and aa taken away

[Note for Problem 20: the equation $(a-b)(a-c)(a+d)(a+f)=0$ has the square term removed by setting $bc + df = (b + c)(d + f)$; the linear term is also removed by setting $bc(d+f)=(b+c)df$, from which it follows after some working that $df = \frac{b^2c^2}{b^2+bc+c^2}$ and $d+f = \frac{bc(b+c)}{b^2+bc+c^2}$. Thus, the sum of the roots $-b - c + d + f$ is equal to $-\frac{(b^2+c^2)(b+c)}{b^2+bc+c^2}$ while the product $bcdf = \frac{b^3c^3}{b^2+bc+c^2}$, as required.]

Problem 21.

The quadrinomial equation

$$\begin{array}{llll} aaaa & -baaa & +bcaa & \\ & -caaa & -bdaa & \\ & +daaa & -cdaa & +bcda \\ & +faaa & -bfaa & +bcfa \\ & & -cfaa & -bdfa \\ & & +dfaa & -cdfa \;=\; -bcdf \end{array}$$

to be reduced to the binomial

$$\begin{array}{l} aaaa - bbaa \\ \quad\;\; - ccaa \end{array} = -bbcc,$$

by placing $d + f = b + c$

To wit, with the steps a & aaa taken away.

[Note for Problem 21: the equation $(a-b)(a-c)(a+d)(a+f)=0$ has the linear term removed by setting $b + c = d + f$; the cubic term is also removed by setting $bc(d+f)=(b+c)df$, from which it follows after some working that $bc = df$ and the coefficient of the square term $bc + df - (b + c)(d + f) = - b^2 - c^2$; and the constant term $bcdf = b^2c^2$; from which the result follows.]

Notes.

The three preceding binomials reductions of problems 19, 20, 21 can be established in terms of the same roots *b* and *c*, as the above three trinomials of problems 16, 17, and 18 may be suitably reduced from the same quadrinomial here proposed, as shown in propositions 32, 33, 34, & 35, 36, 37 of Section Four. However since the reduction of these are given in more obscure handwriting, they have been referred to a better inquiry. [It is unclear whether this refers to writing by Harriot which is more difficult to read, or if it indicates a transcription of the original; in any case the editor has made the decision not to proceed with the explanations given.

Nota.

Tres antecedentes binomiae reductitiae Problematum 19. 20. 21 licet in iisdem radicibus explicatoriis *b. c*. cum trinomiis tribus superioribus problematum 16.17.18. ab eadem quadrinomia hic proposita reductis conveniant, ut in propositonibus 32.33.34. & 35.36.37 Sectionis quartae demonstrandum est : reductiones tamen earum cum in autographis obscurius traditae sint, ad meliorem inquisitionem referendae sunt.

General Corollary.

In the reductions which are made through the Problems of this Third Section, neither is it shown how to go about finding of the roots *a ;* nor of the rest of the powers, nor of any changes made to the given elements *b*, *c*, *d*.

Corollarium generale.

In reductionibus quae per Problemata tertiae huius Sectionis fiunt, nec radicis quaesititiae *a.* aut reliquorum graduum, nec elementorum datorum *b. c. d*. ullam factam mutationem, manifestum est.

A recapitulation of the reduction of the canonical equations expounded in Section Three

QUADRATICS.

1\. . . . . $aa = bb$

CUBICS.

1\. . . . .

$$\begin{aligned} aaa &- bba \\ &- bca \\ &- cca = - bbc, \\ &\qquad\quad - bcc. \end{aligned}$$

2\. . . . .

$$\begin{aligned} aaa &- bbaa \\ &- bcaa \\ &- \frac{ccaa}{b + c} = - \frac{bbcc}{b + c} \end{aligned}$$

3\. . . . .

$$\begin{aligned} aaa &- bba \\ &- bca \\ &- cca = + bbc, \\ &\qquad\quad + bcc, \end{aligned}$$

4\. . . . .

$$\begin{aligned} aaa &+ bbaa \\ &+ bcaa \\ &+ \frac{ccaa}{b + c} = + \frac{bbcc}{b + c} \end{aligned}$$

5\. . . . . $aaa + 3.bca = bbb - ccc$

6\. . . . . $aaa + 3.bca = - bbb + ccc$

7\. . . . . $aaa - 3.bca = + bbb + ccc$

BIQUADRATICS.

1\. . . . . .

$$\begin{array}{l} aaaa - bbaaa + bbcca \\ \qquad - ccaa + bbda \\ \qquad - ddaa + bcca \\ \qquad - bcaa + ccda \\ \qquad - bdaa + bdda \\ \qquad - cdaa + cdda \\ \qquad\qquad + 2.bcda \end{array} = \begin{array}{l} +bbcd \\ +bccd \\ +bcdd \end{array}$$

2\. . . . . .

$$aaaa + \frac{\begin{array}{l} - bbaaa \\ - ccaaa \\ - ddaaa \\ + bcaaa \\ - bdaaa \\ - cdaaa \end{array}}{b + c + d} + \frac{\begin{array}{l} + bbcca \\ + bbdda \\ + ccdda \\ + bcdda \\ + bccda \\ + bbcda \end{array}}{b + c + d} = \frac{\begin{array}{l} +bbccd \\ +bbcdd \\ +bccdd \end{array}}{b + c + d}$$

3\. . . . . .

$$aaaa + \frac{\begin{array}{l} - bbcaaa \\ - bbdaaa \\ - bccaaa \\ - bddaaa \\ - cddaaa \\ -2.bcdaaa \end{array}}{bc + cd + da} + \frac{\begin{array}{l} + bbccaa \\ + bbddaa \\ + ccddaa \\ + bcddaa \\ + bbcdaa \end{array}}{bc + cd + da} = \frac{+bbccdd}{bc + cd + da}$$

4\. . . . . .

$$\begin{array}{l} aaaa - bbaa - bbca \\ \qquad - ccaa - bbda \\ \qquad - ddaa - bcca \\ \qquad - bcaa - ccda \\ \qquad - bdaa - bdda \\ \qquad - cdaa - cdda \\ \qquad\qquad - 2.bcda \end{array} = \begin{array}{l} +bbcd \\ +bccd \\ +bcdd \end{array}$$

5\. . . . . .

$$aaaa + \frac{\begin{array}{l} + bbaaa \\ + ccaaa \\ + ddaaa \\ + bcaaa \\ + bdaaa \\ + cdaaa \end{array}}{b + c + d} + \frac{\begin{array}{l} - bbcca \\ - bbdda \\ - ccdda \\ - bcdda \\ - bccda \\ - bbcda \end{array}}{b + c + d} = \frac{\begin{array}{l} +bbccd \\ +bbcdd \\ +bccdd \end{array}}{b + c + d}$$

6. . . . .

$$\begin{array}{llcl}
aaaa + bbaaa & & & \\
+ bbdaaa & + bbccaa & & \\
+ bccaaa & + bbddaa & & \\
+ ccdaaa & + ccddaa & & \\
+ bddaaa & + bcddaa & & \\
+ cddaaa & + bccdaa & & \\
+\underline{2.\ bcdaaa} & + \underline{bbcdaa} & = & +\underline{bbccdd} \\
bc + bd + cd & bc + bd + cd & & bc + bd + cd
\end{array}$$

7. . . . .

$$\begin{array}{llcl}
aaaa + bdaa & + bbca & & \\
+ cdaa & + bcca & & \\
- bbaa & + bdda & & \\
- bcaa & + cdda & & \\
- ccaa & - bbda & & \\
- ddaa & - ccda & & \\
 & - 2.bcda & = & - bbcd \\
 & & & - bccd \\
 & & & + bcdd
\end{array}$$

8. . . . .

$$\begin{array}{llcl}
aaaa - bbaaa & + bbcca & & \\
- bcaaa & + bbdda & & \\
- ccaaa & + bcdda & & \\
- ddaaa & + ccdda & & \\
+ bdaaa & - bbcda & & \\
\underline{+ cdaaa} & - \underline{bccda} & = & - bbccd \\
b + c - d & b + c - d & & + bbcdd \\
 & & & \underline{+ bccdd} \\
 & & & b + c - d
\end{array}$$

9. . . . .

$$\begin{array}{llcl}
aaaa + bbaaa & - bbcca & & \\
+ bcaaa & - bcdda & & \\
+ ccaaa & - bbdda & & \\
+ ddaaa & - ccdda & & \\
- bdaaa & + bbcda & & \\
\underline{- cdaaa} & + \underline{bccda} & = & - bbccd \\
d - b - c & d - b - c & & - bccdd \\
 & & & + \underline{bccdd} \\
 & & & d - b - c
\end{array}$$

10. . . . .

$$\begin{array}{llcl}
aaaa + bbcaaa & & & \\
+ bccaaa & - bbccaa & & \\
+ bddaaa & - bbddaa & & \\
+ cddaaa & - bcddaa & & \\
- bbdaaa & - ccddaa & & \\
- ccdaaa & + bccdaa & & \\
\underline{- 2.\ bcdaaa} & + \underline{bccdaa} & = & - \underline{bbccdd} \\
bd + cd - bc & bd + cd - bc & & bd + cd - bc
\end{array}$$

11. . . . .

$$\begin{array}{lcl}
aaaa + bbba & & \\
- bbca & & \\
- bcca & & \\
- ccca & = & - bbbc \\
 & & - bbcc \\
 & & - bccc
\end{array}$$

12. . . . .

$$\begin{array}{lcl}
aaaa - bbbaaaa & & \\
- bbcaaa & & \\
- bccaaa & & \\
\underline{- cccaaa} & = & \underline{- bbbccc} \\
bb + bc + cc & & bb + bc + cc
\end{array}$$

13.. . . . . . .

$$\begin{array}{lcl}
aaaa - bbaa & & \\
- ccaa & = & - bcdf.
\end{array}$$

A collection of other canonical equations of such an kind
that the generation of these other higher steps [or powers] is readily apparent.

$$+bc = +ba + ca - aa$$

$$+bbc + bcc = +bba + bca + cca - aaa$$

$$+bbbc + bbcc + bccc = +bbba + bbca + bcca + ccca - aaaa$$

$$+bbbbc + bbbcc + bbccc + bcccc = +bbbba + bbbca + bbcca + bccca + cccca - aaaaa$$

And in the same way indefinitely.

$$\frac{+bbcc}{b+c} = \frac{+bbaa + bcaa + ccaa}{b+c} - aaa$$

$$\frac{+bbbcc + bbccc}{b+c} = \frac{+bbbaa + bbcaa + cccaa}{b+c} - aaaa$$

$$\frac{+bbbbcc + bbbccc + bbcccc}{b+c} = \frac{+bbbbaa + bbbcaa + bbccaa + bcccaa + ccccaa}{b+c} - aaaaa$$

And in the same way indefinitely.

$$\frac{+bbbccc}{bb+bc+cc} = \frac{+bbbaaa + bbcaaa + bccaaa + cccaaa}{bb+bc+cc} - aaaaa$$

$$\frac{+bbbbccc + bbbcccc}{bb+bc+cc} = \frac{+bbbbaaa + bbbcaaa + bbccaaa + bcccaaa + ccccaaa}{bb+bc+cc} - aaaaa$$

And thus for the rest in the same way indefinitely.

$$\frac{+bbbbccc}{bbb+bbc+bcc+ccc} = \frac{+bbbbaaaa + bbbcaaaa + bbccaaaa + bcccaaaa + ccccaaaa - aaaaa}{bbb+bbc+bcc+ccc}$$

And thus for the rest in the same way indefinitely.

Another collection and series of canons.

$$+bcd \;=\; \begin{array}{l} +\,bca - baa \\ +\,bda - caa \\ +\,cda - daa + aaa \end{array}$$

$$\begin{array}{l} +bbcd \\ +cbcd \\ +dbcd \end{array} \;=\; \begin{array}{l} +\,bbca \\ +\,bbda - bbaa \\ +\,ccba - ccaa \\ +\,ccda - ddaa \\ +\,ddba - bcaa \\ +\,ddca - bdaa \\ +2.bcda - cdaa + aaaa. \end{array}$$

$$\frac{\begin{array}{l} +\,bcbcd \\ +\,bdbcd \\ +\,cdbcd \end{array}}{b+c+d} \;=\; \frac{\begin{array}{l} +\,bbcca \\ +\,bbdda \\ +\,ccdda \\ +\,bbcda \\ +\,cbcda \\ +\,dbcda \end{array}}{b+c+d} \; \frac{\begin{array}{l} -\,bbaaa \\ -\,ccaaa \\ -\,ddaaa \\ -\,bcaaa \\ -\,bdaaa \\ -\,cdaaa + aaaa. \end{array}}{b+c+d}$$

$$\frac{+\,bcdbcd}{bc+bd+cd} \;=\; \begin{array}{l} +\,bbccaa - bbcaaa \\ +\,bbddaa - bbdaaa \\ +\,ccddaa - ccbaaa \\ +\,bbcdaa - ccdaaa \\ +\,cbcdaa - dcbaaa \\ +\,dbcdaa - ddcaaa \end{array}$$

$$bc+bd+cd \quad - \frac{2.bcdaaa}{bc+bd+cd} + aaaa.$$

$$\frac{\begin{array}{l} +\,bbcbcd \\ +\,bbdbcd \\ +\,ccbbcd \\ +\,ccdbcd \\ +\,ddbbcd \\ +\,ddcbcd \\ +2.bcdbcd \end{array}}{b+c+d} \;=\; \frac{\begin{array}{l} +\,bbccca \\ +\,bbbdda \\ +\,cccbba \\ +\,bbbcda \\ +\,cccdda \\ +\,cccbda \\ +\,dddbba \\ +\,dddcca \\ +\,dddbca \\ +2.bcbcba \\ +2.bdbcba \\ +2.bdbcda \end{array}}{b+c+d} \; \frac{\begin{array}{l} \\ \\ -\,bbbaaa \\ -\,cccaaa \\ -\,dddaaa \\ -\,bbcaaa \\ -\,bbdaaa \\ -\,ccbaaa \\ -\,ccdaaa \\ -\,ddbaaa \\ -\,ddcaaa \\ -\,bcdaaa \end{array}}{b+c+d} + aaaa.$$