The equations of the second canons are deduced from those of the first canons by the removal of some parodic power* [*i. e. one of the lesser powers in the equation is removed by making its coefficient equal to zero*], *by substituting an invariant quantity for the remaining root.

Aequationum canonicarum secundarium a primariis reductio per gradu alicuius parodici sublationem radice supposititia invariata manente.

The reduction of a quadratic canonical equation to one with a single term.

PROBLEM 1.

The binomial equation

$$\begin{array}{lcl} aa - ba & & \\ \quad + ca & = & + bc \end{array}$$

is reduced to the monomial $aa = bb$

That is, by removing the first degree term or step a,

by placing $b = c,$

the binomial equation is reduced by changing c into b.

Thus the equation becomes

$$\begin{array}{lcl} aa - ba & & \\ \quad + ba & = & + bb \end{array}$$

Therefore by taking away contradictory parts, the excess

is . . . . $aa = bb,$ which is the monomial sought.

Therefore the reduction is accomplished for the proposed binomial equation to the monomial, as ordered.

[Note for Problem 1: in modern notation, the equation $(a-b)(a+c)=0$ is given with equal and opposite roots $\pm b$, in which case it is reduced to $a^2 - b^2 = 0$. Only the positive root is considered.]

By reducing the order of a cubic canon.

PROBLEM 2.

The trinomial equation

$$\begin{array}{rrrcl} aaa & -\ baa & +\ bca & & \\ & -\ caa & -\ bda & & \\ & +\ daa & -\ cda & = & -\ bcd, \end{array}$$

is to be reduced to the binomial

$$\begin{array}{rrcl} aaa & -\ bba & & \\ & -\ bca & & \\ & -\ cca & = & -\ bbc \\ & & & -\ bcc \end{array}$$

That is, by removing the second degree or *aa.*

Placing $b + c = d$

And the trinomial equation is reduced by changing *d* into $b + c$.

Thus the equation becomes:

$$\begin{array}{rrrcl} aaa & -\ baa & +\ bca & & \\ & -\ caa & -\ bba & & \\ & +\ baa & -\ bca & & \\ & +\ caa & -\ bca & & \\ & & -\ cca, & = & -\ bbc, \\ & & & & -\ bcc. \end{array}$$

Therefore by rejecting particular contradictory terms , the amount left

is . . . .

$$\begin{array}{rrcl} aaa & -\ bba & & \\ & -\ bca & & \\ & -\ cca & = & -\ bbc \\ & & & -\ bcc \end{array}$$

which is the binomial equation sought.

Therefore the reduction is accomplished for the proposed trinomial equation to the binomial, as prescribed.

[Note for Problem 2: the equation $(a-b)(a-c)(a+d) = 0$ has the square term removed by setting $b + c = d$; in which case the reduced equation becomes $a^3 - (b^2 + bc + c^2)a = -(b+c)bc$.]

PROBLEM 3.

The trinomial equation

$$\begin{array}{l} aaa - baa + bca \\ \quad - caa - bda \\ \quad + daa - cda = - bcd, \end{array}$$

to be reduced to the binomial

$$\begin{array}{l} aaa - bbaa \\ \quad - bcaa \\ \quad - \dfrac{ccaa}{b+c} = - \dfrac{bbcc}{b+c} \end{array}$$

by removing the first power a.

Placing $bc = bd + cd$

Thus in the proposed equation by the cancellation of parts, the first power is removed.

Thus there remains . . . .

$$\begin{array}{l} aaa - baa \\ \quad - caa \\ \quad + daa = - bcd, \end{array}$$

Therefore the remaining part of the equation becomes, on substituting $\dfrac{bc.}{b+c}$ for d ,

thus:

$$\begin{array}{l} aaa - baa \\ \quad - caa \\ \quad + \dfrac{bcaa}{b+c} = - \dfrac{bbcc}{b+c} \end{array}$$

The remaining terms baa & caa are restored with the common divisor $b + c$

$$\begin{array}{l} aaa - bbaa \\ \quad - bcaa \\ \quad - bcaa \\ \quad - ccaa \\ \quad + \dfrac{bcaa}{b+c} = - \dfrac{bbcc}{b+c} \end{array}$$

The contradictory excess is taken away.

So finally

$$\begin{array}{l} aaa - bbaa \\ \quad - bcaa \\ \quad - \dfrac{ccaa}{b+c} = - \dfrac{bbcc}{b+c} \end{array}$$

is the reduced prescribed binomial equation. Thus the required reduction of the proposed equation to the prescribed form has been effected.

[Note for Problem 3: the equation $(a-b)(a-c)(a+d)=0$ has the linear term removed by setting $d=\frac{bc}{b+c}$, $b+c=d$; in which case the reduced equation becomes $a^3-a^2(b^2+bc+c^2)/(b+c)=-b^2c^2/(b+c)$.]

PROBLEM 4.

The trinomial equation

$$\begin{array}{l} aaa + baa + bca \\ + caa - bda \\ - daa - cda = +bcd, \end{array}$$

is to be reduced to the binomial

$$\begin{array}{l} aaa - bba \\ -bca \\ -cca = +bbc \\ \qquad +bcc \end{array}$$

That is, by removing the second degree or *aa* step.

Placing $b+c = d$

And the trinomial equation is reduced by changing d into $b+c$.

Thus the equation becomes :

$$\begin{array}{l} aaa + baa + bca \\ + caa - bba \\ - baa - bca \\ -caa - bca \\ - cca = +bbc, \\ \qquad +bcc. \end{array}$$

Therefore by rejecting a particular part with a contradictory [one] , the excess

becomes . . . .

$$\begin{array}{l} aaa - bba \\ - bca \\ - cca = +bbc, \\ \qquad +bcc. \end{array}$$

Which is the binomial equation sought.

So therefore the reduction of the proposed trinomial equation to the binomial is done, as prescribed.

[Note for Problem 4 : the equation $(a+b)(a+c)(a-d)=0$ has the quadratic term removed by setting $b+c=d$; in which case the reduced equation becomes $a^3-(b^2+bc+c^2)a=(b+c)bc$.]

PROBLEM 5.

The trinomial equation

$$\begin{array}{l} aaa + baa + bca \\ \quad + caa - bda \\ \quad - daa - cda = + bcd, \end{array}$$

is to be reduced to the binomial :

$$\begin{array}{l} aaa + bbaa \\ \quad + bcaa \\ \quad + \dfrac{ccaa}{b+c} = +\dfrac{bbcc}{b+c} \end{array}$$

by removing the first step or power *a*.

Putting $bc = bd + cd$

Thus in the proposed equation, by a contradiction of parts, the first power *a* is removed.

Thus there remains . . . .

$$\begin{array}{l} aaa + baa \\ \quad + caa \\ \quad - daa = + bcd, \end{array}$$

a part of the equation is thus reduced.

By substituting $bc = bd + cd,\quad d = \dfrac{bc.}{b+c}$

in that part of the equation in which *d* belongs, by changing *d* into $\dfrac{bc.}{b+c}$

Hence it becomes

$$\begin{array}{l} aaa + baa \\ \quad + caa \\ \quad - \dfrac{bcaa}{b+c} = + \dfrac{bbcc}{b+c} \end{array}$$

The remainder baa & caa is reduced to the common divisor $b + c$

$$\begin{array}{l} aaa + bbaa \\ \quad + bcaa \\ \quad + bcaa \\ \quad - ccaa \\ \quad - \dfrac{bcaa}{b+c} = +\dfrac{bbcc}{b+c} \end{array}$$

The contradictory excess is taken away.

So finally

$$\begin{array}{l} aaa + bbaa \\ \quad + bcaa \\ \quad + \dfrac{ccaa}{b+c} = + \dfrac{bbcc}{b+c} \end{array}$$

is reduced to the prescribed binomial equation

And thus the reduction of the prescribed equation is made , as required.

[Note for Problem 5: the equation $(a+b)(a+c)(a-d)=0$ has the linear term removed by setting $d=\frac{bc}{b+c}$, $b+c=d$; in which case the reduced equation becomes $a^3+a^2(b^2+bc+c^2)/(b+c)=b^2c^2/(b+c)$.]

PROBLEM 6.

The trinomial equation $aaa - 3.baa + 3.bba = bbb - ccc$
is reduced to the binomial $aaa + 3.bca = bbb - ccc$

That is, by removing the second degree or second power *aa.*

The equation is generated by setting $b - a = c$

Thus . . . . $\begin{array}{r|}-a+b\\ \hline 3.ba\end{array} = \begin{array}{r|}+c\\ \hline 3.ba\end{array}$

But . . . . $\begin{array}{r|}-a+b\\ \hline 3.ba\end{array} = -3.baa + 3.bba$

And . . . . $\begin{array}{r|}+c\\ \hline 3.ba\end{array} = +3.bca.$

Thus . . . . $-3.baa + 3.bba = +3.bca.$

Thus . . . . $aaa - 3.baa + 3.bba = aaa + 3.bca.$

But . . . . $aaa - 3.baa + 3.bba = bbb - ccc$

For this is the binomial equation itself proposed.

Thus . . . . $aaa + 3.bca. = bbb - ccc,$ which is the binomial sought.

Therefore by placing $b - a = c,$

the prescribed reduction shall be performed for the proposed trinomial equation to the binomial.

[Note for Problem 6: the equation $(a-b)^3=-c^3$ is the same as $a^3-3ab(a-b)-b^3=-c^3$; this has the quadratic term removed by setting $a - b =- c$; in which case the reduced equation becomes $a^3+3abc=b^3-c^3$ as required.]

PROBLEM 7.

The trinomial equation $aaa + 3.baa + 3.bba = - bbb + ccc$

is reduced to the binomial $aaa + 3.bca = - bbb + ccc$

That is, by removing the second degree or step aa.

The root equation is established again, by setting $a + b = + c$

Thus . . . . $\left.\begin{array}{r} +a+b \\ \underline{3.ba} \end{array}\right| = \left.\begin{array}{r} +c \\ \underline{3.ba} \end{array}\right|$

But . . . . $\left.\begin{array}{r} +a+b \\ \underline{3.ba} \end{array}\right| = +3.baa + 3.bba$

And . . . . $\left.\begin{array}{r} +c \\ \underline{3.ba} \end{array}\right| = + 3.bca.$

Thus . . . . $+ 3.baa + 3.bba = + 3.bca.$

Thus . . . .$aaa + 3.baa + 3.bba = aaa + 3.bca.$

But . . . . $aaa + 3.baa + 3.bba = - bbb + ccc$

For this is the binomial equation itself proposed.

Thus . . . . $aaa + 3.bca. = - bbb + ccc,$ which is the binomial sought.

Therefore by placing $a + b = c,$

the prescribed reduction is accomplished for the proposed trinomial equation to the binomial.

[Note for Problem 7: the equation $(a+b)^3 = c^3$ is the same as $a^3 + 3ab(a+b) + b^3 = c^3$; this has the quadratic term removed by setting $a+ b =- c$; in which case the reduced equation becomes $a^3 + 3abc = -b^3 + c^3$ as required.]

PROBLEM 8.

The trinomial $aaa - 3.baa + 3.bba = + bbb + ccc$

is reduced to the binomial $aaa - 3.bca = + bbb + ccc$

That is, by removing the second degree or step aa.

The radix equation is reconstituted, by setting $a - b = + c$

Thus . . . . $\left.\begin{array}{r} -a+b \\ \underline{3.ba} \end{array}\right| = \left.\begin{array}{r} -c \\ \underline{3.ba} \end{array}\right|$

But . . . . $\left.\begin{array}{r} -a+b \\ \underline{3.ba} \end{array}\right| = -3.baa + 3.bba$

And . . . . $\left.\begin{array}{r} -c \\ \underline{3.ba} \end{array}\right| = -3.bca.$

Thus . . . . $-3.baa + 3.bba = -3.bca.$

Thus . . . . $aaa - 3.baa + 3.bba = aaa - 3.bca.$

But . . . . $aaa - 3.baa + 3.bba = + bbb + ccc$

is indeed the trinomial equation itself proposed.

Thus . . . . $aaa - 3.bca. = + bbb + ccc,$ which is indeed the binomial sought.

Therefore by placing $a - b = c,$

the reduction of the proposed trinomial equation to the binomial has been accomplished, as required.

[Note for Problem 8: the equation $(a-b)^3 = c^3$ is the same as $a^3 - 3ab(a-b) - b^3 = c^3$; this has the quadratic term removed by setting $a - b = c$; in which case the reduced equation becomes $a^3 - 3abc = b^3 + c^3$ as required.]

Reducing the order of the biquadratic canon.
PROBLEM 9.

The quadrinomial equation

$$\begin{array}{l} aaaa - baaa + bcaa \\ \quad - caaa + bdaa \\ \quad - daaa + cdaa - bcda \\ \quad + faaa - bfaa + bcfa \\ \qquad - cfaa + bdfa \\ \qquad - dfaa + cdfa = +bcdf \end{array}$$

is to be reduced to the trinomial

$$\begin{array}{l} aaaa - bbaa + bbca \\ \quad - ccaa + bbda \\ \quad - ddaa + bcca \\ \quad - bcaa + ccda \\ \quad - bdaa + bdda \\ \quad - cdaa + cdda \\ \qquad + 2.bcda = +bbcd \\ \qquad\qquad +bccd \\ \qquad\qquad +bcdd \end{array}$$

To wit, by removing the third [degree or] step *aaa.*

Placing $b + c + d = f$

And the quadrinomial equation is reduced by changing *f* into $b + c + d$

Then the eqn. becomes :

$$\begin{array}{l} aaaa - baaa + bcaa \\ \quad - caaa + bdaa \\ \quad - daaa + cdaa - bcda \\ \quad + baaa - bbaa + bcba \\ \quad + caaa - bcaa + bcca \\ \quad + daaa - bdaa + bcda \\ \qquad - cbaa + bdba \\ \qquad - ccaa + bdca \\ \qquad - cdaa + bdda \\ \qquad - ddaa + cdba \\ \qquad - dcaa + cdca \\ \qquad - ddaa + cdda = +bcdb \\ \qquad\qquad +bcdc \\ \qquad\qquad +bcdd \end{array}$$

Therefore by rejecting particular redundant [parts] by contradiction [cancellation],

hence . . . .

$$\begin{array}{l} aaaa - bbaa + bbca \\ \quad - ccaa + bbda \\ \quad - ddaa + bcca \\ \quad - bcaa + ccda \\ \quad - bdaa + bdda \\ \quad - cdaa + cdda \\ \qquad + 2.bcda = +bbcd \\ \qquad\qquad +bccd \\ \qquad\qquad +bcdd. \end{array}$$

which is the trinomial equation sought.

So therefore the reduction has been done for the proposed equation to the required. as prescribed.

[Note for Problem 9: the equation $(a-b)(a-c)(a-d)(a+f)=0$ has the cubic term removed by setting $b + c + d = f$; in which case the reduced equation becomes $a^4-(b^2+c^2+d^2+bc+bd+cd)a^2+(b^2c+bc^2+c^2d+cd^2+b^2d+bd^2+2bcd)a$ $=(b+c+d)bcd$.]