

***The equations of the second canons are deduced from those of the first canons by the removal of some parodic power [i. e. one of the lesser powers in the equation is removed by making its coefficient equal to zero], by substituting an invariant quantity for the remaining root.***

*Aequationum canonicarum secundarium a primariis reductio per gradu alicuius parodici sublationem radice supposititia invariata manente.*

***The reduction of a quadratic canonical equation to one with a single term.***

**PROBLEM 1.**

The binomial equation  $aa - ba$   
 $+ ca \quad \text{=====} \quad + bc$

is reduced to the monomial  $aa \quad \text{=====} \quad bb$

That is, by removing the first degree term or step  $a$ ,

by placing  $b \quad \text{=====} \quad c$ ,

the binomial equation is reduced by changing  $c$  into  $b$ .

Thus the equation becomes  $aa - ba$   
 $+ ba \quad \text{=====} \quad + bb$

Therefore by taking away contradictory parts, the excess

is . . . .  $aa \quad \text{=====} \quad bb$ , which is the monomial sought.

Therefore the reduction is accomplished for the proposed binomial equation to the monomial, as ordered.

[Note for Problem 1: in modern notation, the equation  $(a - b)(a + c) = 0$  is given with equal and opposite roots  $\pm b$ , in which case it is reduced to  $a^2 - b^2 = 0$ . Only the positive root is considered.]

**By reducing the order of a cubic canon.**

**PROBLEM 2.**

The trinomial equation  $aaa - baa + bca$

$$- caa - bda$$

$$+ daa - cda \quad \text{=====} - bcd,$$

is to be reduced to the binomial  $aaa - bba$

$$- bca$$

$$- cca \quad \text{=====} - bbc$$

$$- bcc$$

That is, by removing the second degree or  $aa$ .

Placing  $b + c \quad \text{=====} \quad d$

And the trinomial equation is reduced by changing  $d$  into  $b + c$ .

Thus the equation becomes:  $aaa - baa + bca$

$$- caa - bba$$

$$+ baa - bca$$

$$+ caa - bca$$

$$- cca, \quad \text{=====} - bbc,$$

$$- bcc.$$

Therefore by rejecting particular contradictory terms, the amount left

is . . . .  $aaa - bba$

$$- bca$$

$$- cca \quad \text{=====} - bbc$$

$$- bcc$$

which is the binomial equation sought.

Therefore the reduction is accomplished for the proposed trinomial equation to the binomial, as prescribed.

[Note for Problem 2: the equation  $(a - b)(a - c)(a + d) = 0$  has the square term removed by setting  $b + c = d$ ; in which case the reduced equation becomes  $a^3 - (b^2 + bc + c^2)a = -(b + c)bc$  .]

**PROBLEM 3.**

The trinomial equation  $aaa - baa + bca$   
 $- caa - bda$   
 $+ daa - cda \quad \text{=====} \quad - bcd,$

to be reduced to the binomial  $aaa - bbaa$   
 $- bcaa$   
 $- \frac{cca}{b+c} \quad \text{=====} \quad - \frac{bhcc}{b+c}$

by removing the first power  $a$ .

Placing  $bc \quad \text{=====} \quad bd + cd$

Thus in the proposed equation by the cancellation of parts, the first power is removed.

Thus there remains . . . .  $aaa - baa$   
 $- caa$   
 $+ daa \quad \text{=====} \quad - bcd,$

Therefore the remaining part of the equation becomes, on substituting

$\frac{bc}{b+c}$  for  $d$ ,

thus:  $aaa - baa$   
 $- caa$   
 $+ \frac{bca}{b+c} \quad \text{=====} \quad - \frac{bhcc}{b+c}$

The remaining terms  $baa$  &  $caa$  are restored with the common divisor  $b + c$

$aaa - bbaa$   
 $- bcaa$   
 $- bcaa$   
 $- ccaa$   
 $+ \frac{bca}{b+c} \quad \text{=====} \quad - \frac{bhcc}{b+c}$

The contradictory excess is taken away.

So finally  $aaa - bbaa$   
 $- bcaa$   
 $- \frac{cca}{b+c} \quad \text{=====} \quad - \frac{bhcc}{b+c}$

is the reduced prescribed binomial equation. Thus the required reduction of the proposed equation to the prescribed form has been effected.

[Note for Problem 3: the equation  $(a - b)(a - c)(a + d) = 0$  has the linear term removed by setting  $d = \frac{bc}{b+c}$ ,  $b + c = d$ ; in which case the reduced equation becomes

$$a^3 - a^2(b^2 + bc + c^2)/(b + c) = -b^2c^2/(b + c).]$$

**PROBLEM 4.**

The trinomial equation  $aaa + baa + bca$

$$+ caa - bda$$

$$- daa - cda \quad \underline{\quad} + bcd,$$

is to be reduced to the binomial

$$aaa - bba$$

$$- bca$$

$$- cca \quad \underline{\quad} + bbc$$

$$+ bcc$$

That is, by removing the second degree or  $aa$  step.

Placing  $b + c \quad \underline{\quad} d$

And the trinomial equation is reduced by changing  $d$  into  $b + c$ .

Thus the equation becomes :  $aaa + baa + bca$

$$+ caa - bba$$

$$- baa - bca$$

$$- caa - bca$$

$$- cca \quad \underline{\quad} + bbc,$$

$$+ bcc.$$

Therefore by rejecting a particular part with a contradictory [one], the excess

becomes . . . .  $aaa - bba$

$$- bca$$

$$- cca \quad \underline{\quad} + bbc,$$

$$+ bcc.$$

Which is the binomial equation sought.

So therefore the reduction of the proposed trinomial equation to the binomial is done, as prescribed.

[Note for Problem 4 : the equation  $(a + b)(a + c)(a - d) = 0$  has the quadratic term removed by setting  $b + c = d$ ; in which case the reduced equation becomes

$$a^3 - (b^2 + bc + c^2)a = (b + c)bc .]$$

**PROBLEM 5.**

The trinomial equation  $aaa + baa + bca$

$$+ caa - bda$$

$$- daa - cda \quad \text{=====} + bcd,$$

is to be reduced to the binomial :  $aaa + bbaa$

$$+ bcaa$$

$$+ \frac{cca}{b+c} \quad \text{=====} \quad + \frac{bbcc}{b+c}$$

by removing the first step or power  $a$ .

Putting  $bc \quad \text{=====} \quad bd + cd$

Thus in the proposed equation, by a contradiction of parts, the first power  $a$  is removed.

Thus there remains . . . .  $aaa + baa$

$$+ caa$$

$$- daa \quad \text{=====} \quad + bcd, \quad \text{a part of the equation is thus reduced.}$$

By substituting  $bc \quad \text{=====} \quad bd + cd, \quad d \quad \text{=====} \quad \frac{bc}{b+c}$

in that part of the equation in which  $d$  belongs, by changing  $d$  into  $\frac{bc}{b+c}$

Hence it becomes  $aaa + baa$

$$+ caa$$

$$- \frac{bcaa}{b+c} \quad \text{=====} \quad + \frac{bbcc}{b+c}$$

The remainder  $baa$  &  $caa$  is reduced to the common divisor  $b + c$

$$aaa + bbaa$$

$$+ bcaa$$

$$+ bcaa$$

$$- ccaa$$

$$- \frac{bcaa}{b+c} \quad \text{=====} \quad + \frac{bbcc}{b+c}$$

The contradictory excess is taken away.

So finally  $aaa + bbaa$

$$+ bcaa$$

$$+ \frac{cca}{b+c} \quad \text{=====} \quad + \frac{bbcc}{b+c}$$

is reduced to the prescribed binomial equation

And thus the reduction of the prescribed equation is made , as required.

[Note for Problem 5: the equation  $(a + b)(a + c)(a - d) = 0$  has the linear term removed by setting  $d = \frac{bc}{b+c}$ ,  $b + c = d$ ; in which case the reduced equation becomes

$$a^3 + a^2(b^2 + bc + c^2)/(b + c) = b^2c^2/(b + c).]$$

**PROBLEM 6.**

The trinomial equation  $aaa - 3.baa + 3.bba \equiv bbb - ccc$   
 is reduced to the binomial  $aaa + 3.bca \equiv bbb - ccc$

That is, by removing the second degree or second power  $aa$ .

The equation is generated by setting  $b - a \equiv c$

Thus . . . .  $\left. \begin{array}{l} - a + b \\ \hline 3.ba \end{array} \right| \equiv \left. \begin{array}{l} + c \\ \hline 3.ba \end{array} \right|$

But . . . .  $\left. \begin{array}{l} - a + b \\ \hline 3.ba \end{array} \right| \equiv - 3.baa + 3.bba$

And . . . .  $\left. \begin{array}{l} + c \\ \hline 3.ba \end{array} \right| \equiv + 3.bca.$

Thus . . . .  $- 3.baa + 3.bba \equiv + 3.bca.$

Thus . . . .  $aaa - 3.baa + 3.bba \equiv aaa + 3.bca.$

But . . . .  $aaa - 3.baa + 3.bba \equiv bbb - ccc$

For this is the binomial equation itself proposed.

Thus . . . .  $aaa + 3.bca. \equiv bbb - ccc$ , which is the binomial sought.

Therefore by placing  $b - a \equiv c$ ,

the prescribed reduction shall be performed for the proposed trinomial equation to the binomial.

[Note for Problem 6: the equation  $(a - b)^3 = -c^3$  is the same as  $a^3 - 3ab(a - b) - b^3 = -c^3$  ; this has the quadratic term removed by setting  $a - b = -c$ ; in which case the reduced equation becomes  $a^3 + 3abc = b^3 - c^3$  as required.]

**PROBLEM 7.**

The trinomial equation  $aaa + 3.baa + 3.bba \text{ ———— } - bbb + ccc$   
 is reduced to the binomial  $aaa + 3.bca \text{ ———— } - bbb + ccc$

That is, by removing the second degree or step  $aa$ .

The root equation is established again, by setting  $a + b \text{ ———— } + c$

Thus . . . . .  $\begin{array}{r} + a + b \\ \hline 3.ba \end{array} \Bigg| \text{ ———— } \begin{array}{r} + c \\ \hline 3.ba \end{array} \Bigg|$

But . . . . .  $\begin{array}{r} + a + b \\ \hline 3.ba \end{array} \Bigg| \text{ ———— } + 3.baa + 3.bba$

And . . . . .  $\begin{array}{r} + c \\ \hline 3.ba \end{array} \Bigg| \text{ ———— } + 3.bca.$

Thus . . . . .  $+ 3.baa + 3.bba \text{ ———— } + 3.bca.$

Thus . . . . .  $.aaa + 3.baa + 3.bba \text{ ———— } aaa + 3.bca.$

But . . . . .  $aaa + 3.baa + 3.bba \text{ ———— } - bbb + ccc$

For this is the binomial equation itself proposed.

Thus . . . . .  $aaa + 3.bca. \text{ ———— } - bbb + ccc,$  which is the binomial sought.

Therefore by placing  $a + b \text{ ———— } c,$   
 the prescribed reduction is accomplished for the proposed trinomial equation to the binomial.

[Note for Problem 7: the equation  $(a + b)^3 = c^3$  is the same as  $a^3 + 3ab(a + b) + b^3 = c^3$  ;  
 this has the quadratic term removed by setting  $a + b = -c$ ; in which case the reduced  
 equation becomes  $a^3 + 3abc = -b^3 + c^3$  as required.]

**PROBLEM 8.**

The trinomial  $aaa - 3.baa + 3.bba \quad \text{=====} + bbb + ccc$   
 is reduced to the binomial  $aaa - 3.bca \quad \text{=====} + bbb + ccc$

That is, by removing the second degree or step  $aa$ .

The radix equation is reconstituted, by setting  $a - b \quad \text{=====} + c$

Thus . . . . .  $- a + b \quad \text{=====} \quad - c$   
 $\quad \quad \quad \underline{3.ba} \quad \quad \quad \underline{3.ba}$

But . . . . .  $- a + b \quad \text{=====} \quad - 3.baa + 3.bba$   
 $\quad \quad \quad \underline{3.ba}$

And . . . . .  $- c \quad \text{=====} \quad - 3.bca.$   
 $\quad \quad \quad \underline{3.ba}$

Thus . . . . .  $- 3.baa + 3.bba \quad \text{=====} \quad - 3.bca.$

Thus . . . . .  $aaa - 3.baa + 3.bba \quad \text{=====} \quad aaa - 3.bca.$

But . . . . .  $aaa - 3.baa + 3.bba \quad \text{=====} + bbb + ccc$

is indeed the trinomial equation itself proposed.

Thus . . . . .  $aaa - 3.bca. \quad \text{=====} + bbb + ccc$ , which is indeed the binomial sought.

Therefore by placing  $a - b \quad \text{=====} \quad c$ ,

the reduction of the proposed trinomial equation to the binomial has been accomplished, as required.

[Note for Problem 8: the equation  $(a - b)^3 = c^3$  is the same as  $a^3 - 3ab(a - b) - b^3 = c^3$  ;  
 this has the quadratic term removed by setting  $a - b = c$ ; in which case the reduced  
 equation becomes  $a^3 - 3abc = b^3 + c^3$  as required.]



*Reducing the order of the biquadratic canon.*

**PROBLEM 9.**

The quadrinomial equation

$$\begin{array}{r}
 aaaa - baaa + bcaa \\
 - caaa + bdaa \\
 - daaa + cdaa - bcda \\
 + faaa - bfaa + bcfa \\
 - cfaa + bdfa \\
 - dfaa + cdfa = +bcdf
 \end{array}$$

is to be reduced to the trinomial

$$\begin{array}{r}
 aaaa - bbaa + bbca \\
 - ccaa + bbda \\
 - ddaa + bcca \\
 - bcaa + ccda \\
 - bdaa + bdda \\
 - cdaa + cdda \\
 + 2.bcda = +bbcd \\
 +bccd \\
 +bccd
 \end{array}$$

To wit, by removing the third [degree or] step  $aaa$ .

Placing  $b + c + d = f$

And the quadrinomial equation is reduced by changing  $f$  into  $b + c + d$

Then the eqn. becomes :

$$\begin{array}{r}
 aaaa - baaa + bcaa \\
 - caaa + bdaa \\
 - daaa + cdaa - bcda \\
 + baaa - bbaa + bcba \\
 + caaa - bcaa + bcca \\
 + daaa - bdaa + bcda \\
 - cbaa + bdba \\
 - ccaa + bdca \\
 - cdaa + bdda \\
 - ddaa + cdca \\
 - dca a + cdca \\
 - ddaa + cdda = +bcdb \\
 +bccd \\
 +bccd
 \end{array}$$

Therefore by rejecting particular redundant [parts] by contradiction [cancellation],

hence . . . .

$$\begin{array}{r}
 aaaa - bbaa + bbca \\
 - ccaa + bbda \\
 - ddaa + bcca \\
 - bcaa + ccda \\
 - bdaa + bdda \\
 - cdaa + cdda \\
 + 2.bcda = +bbcd \\
 +bccd \\
 +bccd.
 \end{array}$$

which is the trinomial equation sought.

So therefore the reduction has been done for the proposed equation to the required. as prescribed.

[Note for Problem 9: the equation  $(a - b)(a - c)(a - d)(a + f) = 0$  has the cubic term removed by setting  $b + c + d = f$ ; in which case the reduced equation becomes  $a^4 - (b^2 + c^2 + d^2 + bc + bd + cd)a^2 + (b^2c + bc^2 + c^2d + cd^2 + b^2d + bd^2 + 2bcd)a = (b + c + d)bcd$ .]