Section five, in which the number of the roots of common equations are determined through equipollent canons.

DEFINITION.

Two equations of similar degree and affection [*i. e.* the same lesser powers present] are to be called equipollent [or equivalent] in what follows: in addition, the homogeneous coefficient or coefficients (if they should be plural) of one of these given equations is shown to satisfy a simple inequality, which is also the case for the like coefficient or coefficients of the other equation, indeed each inequality so formed has a larger and a smaller part. Again, the inequality is to be re-interpreted as an equality, if the roots of the equation are equal. Hence, for the equations generated from binomial roots (or from the reduced forms of these), which have been established above in Section Three, and to which we have given the name Canons, the products formed from these canonical forms are to be compared with those formed from common equations, if the above-mentioned conditions for equipollence between the equations are to be in agreement, since the canons can thus distinguish and determine the number of roots in common equations, if there are sure and reliable examples available. Agreement therefore can be established between the coefficients of the common equations and the canons of the given homogeneity. The coefficients of the common equation are to be similarly partitioned with the formal homogeneous parts of the canons ; and similar parts of both are taken to form the inequality. Like parts are be accepted for both sides of the inequality Care must be taken to choose homogeneous parts, in order to simplify [correctly] the homogeneous parts [of the inequality] so obtained; for coefficients and homogeneous parts of a given equation may be by necessity heterogeneous, and from the appearance of such heterogeneous parts, no relation can be assessed between them. [Thus, the analysis can only be performed on terms that are all of the same sign.]

DEFINITIO.

Duae aequationes similiter graduatae & similiter affectae, quarum coefficiens vel coefficienta (si plura sint) & homogeneum datu unius coefficienti vel coefficientibus & homogeneo dato alterius in simplici inaequalitas, maioritatis scilicet & minoritatis habitudine conformia sunt, aequipollentes in sequentibus appellandae sunt. Quod sic rursus interpretandum est, quasi aequali radicum numero pollentes. Hinc est quod aequationibus e radicibus binomiis generatis& earum reductitiis, de quibus in superioribus tribus Sectionibus tractum est, Canonicarum nomen impositum est : quia facta earum ad aequationes communes comparatione, si supradictis aequipollentiae conditionibus inter se conveniant, ad radicum numerum in aequationibus communibus dignoscendum & determinandum canones sive exemplaria certa & solennia sint. In conformitate igitur inter aequationum communium & canonicorum coefficientia & homogenea data instituenda, aequationum communium coefficientia & homogenea formali canonicurum partitioni similiter partienda sunt, & similes utrinque partes sumendae, servata in partium habitudine aestimanda homogeniae lege, per reductionem scilicet procurata homogenia; cum coefficientia & homogenea data necessario heterogenea sint, & de heterogeneorum inter se habitudine nulla possit aestimatio.

Lemma 1.

If a quantity is cut in two unequal parts, then the square from the half of the whole is greater than the product from the two unequal parts.

If the two unequal parts are p & q, [$p > q > 0$] the required inequality is :

$$\left.\begin{array}{c}\frac{p+q}{2}\\ \frac{p+q}{2}\end{array}\right\rfloor > p\,q.$$

For from the three continued proportions $pp, pq, qq,$ of which pp is the maximum, and qq is the minimum,

it follows that : $pp - pq > pq - qq$, [since $p(p - q) > q(p - q)$],

Therefore . . . . . $pp + qq > 2.pq$

And by adding to each $2.pq$

It will be . . . . . $pp + qq + 2.pq > 4.pq$

But . . . . . $pp + qq + 2.pq \;=\; \left.\begin{array}{c}p+q\\ p+q\end{array}\right\rfloor$

Therefore . . . . . $\left.\begin{array}{c}p+q\\ p+q\end{array}\right\rfloor > 4.p\,q.$

Therefore . . . . . $\dfrac{\left.\begin{array}{c}p+q\\ p+q\end{array}\right\rfloor}{4} > p\,q.$

Therefore . . . . . $\left.\begin{array}{c}\frac{p+q}{2}\\ \frac{p+q}{2}\end{array}\right\rfloor > p\,q.$ Which was to be proven.

[Note on Lemma 1: This is of course the inequality of the arithmetic and geometric means for two positive numbers p and q : $((p+q)/2)^2 > pq$.]

Lemma 2.

If there are three numbers in continued proportion, the sum of the extremes is greater than twice the middle number.

If the continued proportions are $b, c, d,$ then the required inequality is : $b + d > 2.c.$

For if the maximum is b, then . . . $b - c > c - d.$

Therefore . . . . $b + d > 2.c.$

Or, if d is the maximum, then . . . . . . $d - c > c - b.$

Therefore . . . . $d + b > 2.c.$

Therefore, the sum of the extremes is greater than twice the middle.

[Note on Lemma 2: If $b > c > d$ and r is the ratio $0 < r < 1$, then $c = rb$ and $d = r^2b$; from which $b - c = b(1 - r)$, and $c - d = br(1- r) < b(1 - r) = (b - c)$; from which the result $b + d > 2c$ follows; and likewise for the other case.]

Lemma 3.

If there are four numbers in continued proportion, the sum of the extremes is greater than the sum of the two middle terms.

If the continued proportions are b, c, d, f, then the required inequality is : $b + f > c + d$.
For if the maximum is b, then . . . $b - c > d - f$.
Therefore . . . . $b + f > c + d$.
Or, if f is the maximum, then . . . . . . $f - d > c - b$.
Therefore . . . . $f + b > c + d$.
Therefore, the sum of the extremes is greater than twice the sum of the means, as stated.

[Note on Lemma 3: If $b > c > d > f$, and r is the ratio $0 < r < 1$, then $c = rb$, $d = r^2b$, and $f = r^3b$, from which $b - c = b(1 - r)$, and $d - f = br^2(1 - r) < b(1 - r) = (b - c)$; from which the result $b + f > c + d$ follows; and likewise for the other case.]

PROPOSITION 1.

The common equation $aaa - 3.bba = +2.ccc$, in which $c > b$, is shown to have a single [*i. e.* positive] root.

For the common equation proposed is an example of the Canonical equation with similar powers and affections:

$$aaa - 3.rqa = \begin{array}{l} +rrr \\ +qqq. \end{array}$$

And (by lemma 4, following) in the canonical equation . . . $\left.\begin{array}{c} rq \\ rq \\ rq \end{array}\right| < \dfrac{\left.\begin{array}{c} rrr + qqq \\ rrr + qqq \end{array}\right|}{4}$

And in the proposed equation in which $b < c$ is substituted, the inequality for the canonical equation becomes : $bb.bb.bb < (ccc + ccc)^2/4$.

Therefore the coefficient and the given homogenous term of the proposed equation satisfy the same condition for the positive and negative terms as the coefficient and homogeneous term of the given Canon. The proposed equation and the Canon are therefore equipollent (by definition), and indeed are provided with the same number of roots.
But (by Prop.14. Sect. 4) the Canonical equation has been solved by the single root $q + r$,
[Note that we have to pick out the terms for an inequality from the homogeneous coefficients of the canonical form, namely p^3, q^3, and pq, for which
$((r^3 + q^3)/2)^2 > r^3q^3$].
The common equation proposed therefore can be solved by a single root, as stated.
[Note for Prop. 15 : $a^3 - 0.a^2 - 3bca - b^3 - c^3 = [a - (b + c)[a^2 + (b + c)a + (b^2 - bc + c^2)] = 0.$]

Lemma 4.

The inequality is :

$$\dfrac{\left.\begin{array}{r} rrr + qqq \\ rrr + qqq \end{array}\right|}{4} > \left.\begin{array}{r} rq \\ rq \\ rq \end{array}\right|$$

for the three continued proportions : *rrrrrr, rrrqqq, qqqqqq*

Therefore (by Lem.2) . . . *rrrrrr + qqqqqq > 2.rrrqqq*

And by adding to each *2.rrrqqq*

the inequality becomes : *rrrrrr + 2.rrrqqq + qqqqqq > 4.rrrqqq*

But . . . . . *rrrrrr + qqqqqq + 2.rrrqqq* $=\!=$ $\left.\begin{array}{r} rrr + qqq \\ rrr + qqq \end{array}\right|$

And . . . . . *4.rrrqqq* $=\!=$ $\left.\begin{array}{r} 4.rq \\ rq \\ rq \end{array}\right|$

Therefore . . . . . $\left.\begin{array}{r} rrr + qqq \\ rrr + qqq \end{array}\right| > \left.\begin{array}{r} 4.rq \\ rq \\ rq \end{array}\right|$

Therefore . . . . .

Therefore . . . . . $\dfrac{\left.\begin{array}{r} rrr + qqq \\ rrr + qqq \end{array}\right|}{4} > \left.\begin{array}{r} rq \\ rq \\ rq \end{array}\right|$ Which is to be shown.

[Note on Lemma 4: If $r > q$ then let the three proportions be r^6, r^3q^3, and q^6; it follows from Lemma 2 that $r^6 + q^6 > 2.r^3q^3$; and hence $(r^3 + q^3)^2 > 4.r^3q^3$; $(r^3 + q^3)^2/4 > r^3q^3$, or $(r^3 + q^3)/2 > \sqrt{r^3q^3}$ as required. These inequalities are of course variations on the arithmetic and geometric mean inequality, applied to special cases.]

PROPOSITION 2.

The common equation *aaa - 3.bba = +2.ccc,* in which $c < b$, is shown to have a single [*i. e.* positive] root.

For the common equation proposed, the Canonical equation with similar powers and affections is

$$\begin{array}{ll} aaa - qqa & \\ \quad\;\, - qra & \\ \quad\;\, - rra & = +qqr \\ & \quad\, + qrr \end{array}$$

And (by Lemma 5, following) the inequality in the canonical equation is . . .

$$\dfrac{\left.\begin{array}{r} + qqr + qrr \\ + qqr + qrr \end{array}\right|}{4} < \dfrac{\left.\begin{array}{r} qq + qr + rr \\ qq + qr + rr \\ qq + qr + rr \end{array}\right|}{27}$$

And in the proposed equation in which $c < b$ is substituted, the inequality for the canonical equation becomes: *ccc.ccc < bb.bb.bb*

Therefore the coefficient and the given homogenous term of the proposed equation satisfy the same condition for the positive and negative terms as the coefficient and homogeneous term of the given Canon.

The proposed equation and the Canon are therefore equipollent (by definition), and indeed are provided with the same number of roots.
But (by Prop. 7. Sect. 4) the Canonical equation has been solved by the single root $q + r$,
The common equation proposed has therefore been solved by a single root, as stated.

[Note for Prop. 7 : $a^3 - (0)a^2 + (-b^2 - bc - c^2)a - bc(b + c) = (a + b)(a + c)(a - (b + c)) = 0$.
Note for Lemma 5: Proposition 1 is concerned with a reduced equation, while Proposition 2 is concerned with the case where q and r are considered different, and the canonical equation has not been reduced: a more complicated inequality is hence needed. This inequality links the sum of the homogeneous coefficients q^2, qr, and r^2 in the linear term of the equation with the sum of the constant coefficients q^2r and qr^2 .
The inequality to be demonstrated is $((q^2 + qr + r^2)/3)^3 > ((q^2r + qr^2)/2)^2$, and it is established here in two ways.
In the first way, we write $t = r/q < 1$, then the inequality becomes : $[\frac{1+t+t^2}{3}]^3 > [\frac{t(1+t^2)}{2}]^2$. Setting $t = 1$ gives equality of course, and we are concerned with the values of t that satisfy $0 < t < 1$. The inequality can be expanded out to give to give: $4 + 12t + 24t^2 + 28t^3 + 24t^4 + 12t^5 + 4t^6 > 27t^2 + 54t^4 + 27t^6$.
and factorised to give: $(1 - t)(4 + 16t + 13t^2 + 41t^3 + 11t^4 + 23t^5) > 0$.
As both factors are positive for $0 < t < 1$, it follows that the inequality is true in this range.
Next we proceed according to the *Praxis*, as set out below in Lemma 5, but written with powers in modern notation, and with the inequality inverted:
Consider the set of proportional numbers :
$q^6 > q^5r^1 > q^4r^2 > q^3r^3 > q^2r^4 > q^1r^5 > r^6$; formed from $q > r > 0$.
By Lem.3: $\mathbf{3(q^6 + r^6) > 3(q^4r^2 + q^2r^4)}$ for the four even powers;
(*i. e.* the sum of the extremes > sum of middle terms);
and by Lem.2 : $\mathbf{12(q^5r^1 + q^1r^5) > 24q^3r^3}$ for the three odd powers;
(*i. e*. the sum of the extremes > twice the middle term);
Also by Lem.2 : $\mathbf{q^6 + r^6 > 2q^3r^3}$. Hence, on adding these three inequalities together :
$\mathbf{4q^6 + 12q^5r^1 + ... + 12q^1r^5 + 4r^6 > 3q^4r^2 + 26q^3r^3 + 3q^2r^4}$.
Thus, we have successfully generated the four terms $4 + 12t + ...+ 12t^5 + 4t^6$ on the left of the above inequality (with t in place of p/q). We are now required to obtain the terms corresponding to $24t^2 + 28t^3 + 24t^4$: this is done simply by adding $\mathbf{24q^4r^2 + 28q^3r^3 + 24q^2r^4}$ to both sides of the current inequality. This gives: $\mathbf{4q^6 + 12q^5r^1 + 24q^4r^2 + 28q^3r^3 + 24q^2r^4 + 12q^1r^5 + 4r^6 > 27q^4r^2 + 54q^3r^3 + 27q^2r^4}$, as required.]

Lemma 5.

The required inequality is: $$\frac{\left.\begin{array}{l} +qqr+qrr \\ +qqr+qrr \end{array}\right|}{4} < \frac{\left.\begin{array}{l} qq+qr+rr \\ qq+qr+rr \\ qq+qr+rr \end{array}\right|}{27}$$

For, (by Lem.3) . . . . . $3.qqqqrr + 3.qqrrrr < 3.qqqqqq + 3.rrrrrr$

And, (by Lem.2) . . . . . $12.qqqrrr + 12.qqqrrr < 12.qqqqqr + 12.qrrrrr.$

And, (by Lem.2) . . . . . $qqqrrr + qqqrrr < qqqqqq + rrrrrr.$

Therefore . . . . . $$\begin{array}{l} +3.qqqqrr \\ +26.qqqrrr \\ +3.qqrrrr \\ \end{array} < \begin{array}{l} +4.qqqqqq \\ +12.qqqqqr \\ +12.qrrrrr \\ +4.rrrrrr \end{array}$$

Therefore by adding to both $24.qqqqrr + 28.qqqrrr + 24.qqrrrr.$

The inequality is . . . . .

$$\begin{array}{l} +27.qqqqrr \\ +54.qqqrrr \\ +27.qqrrrr \\ \\ \\ \\ \\ \end{array} < \begin{array}{l} +4.qqqqqq \\ +12.qqqqqr \\ +12.qrrrrr \\ +4.rrrrrr \\ +24.qqqqrr \\ +28.qqqrrr \\ +24.qqrrrr. \end{array}$$

Therefore by dividing both parts commonly by 4 and again commonly by 27

The inequality is . . . . . $$\frac{\begin{array}{l} +1.qqqqrr \\ +2.qqqrrr \\ +1.qqrrrr \end{array}}{4} < \frac{\begin{array}{l} +1.qqqqqq \\ +3.qqqqqr \\ +3.qrrrrr \\ +1.rrrrrr \\ +6.qqqqrr \\ +7.qqqrrr \\ +8.qqrrrr. \end{array}}{27}$$

But . . . . . $$\frac{\begin{array}{l} +1.qqqqrr \\ +2.qqqrrr \\ +1.qqrrrr \end{array}}{4} = \frac{\left.\begin{array}{l} +qqr+qrr \\ +qqr+qrr \end{array}\right|}{4}$$

And . . . . . $$\frac{\begin{array}{l} +1.qqqqqq \\ +3.qqqqqr \\ +3.qrrrrr \\ +1.rrrrrr \\ +6.qqqqrr \\ +7.qqqrrr \\ +8.qqrrrr. \end{array}}{27} = \frac{\left.\begin{array}{l} +qq+qr+rr \\ +qq+qr+rr \\ +qq+qr+rr \end{array}\right|}{27}$$

Therefore . . . . . $$\frac{\left.\begin{array}{l} +qqr+qrr \\ +qqr+qrr \end{array}\right|}{4} < \frac{\left.\begin{array}{l} +qq+qr+rr \\ +qq+qr+rr \\ +qq+qr+rr \end{array}\right|}{27}$$

Which was to be proven.

PROPOSITION 3.

The common equation *aaa - 3.bba = +2.ccc,* in which $c = b$, is shown to have a single root.

For the common equation proposed, the Canonical equation
with similar steps and affections is *aaa - 3.qqa = + 2.qqq*

And in the canonical equation it is $q = q$.
And in the proposed equation in which $b = c$ is substituted.
Therefore the coefficient and the given homogenous term of the proposed equation satisfy the same condition for the positive and negative terms as the coefficient and homogeneous term of the given Canon.
The proposed equation and the Canon are therefore equipollent (by definition), and indeed are provided with the same number of roots.
But (by Prop. 17. Sect. 4) the Canonical equation has been solved by the single root *2q*.
The common equation proposed has therefore been solved by a single root, as was stated.

PROPOSITION 4.

The common equation *aaa - 3.bba = - 2.ccc,* in which $b < c$, is shown to have two [positive] roots.

For the common equation proposed the Canonical equation
with similar steps and affections is

$$\begin{array}{ll} aaa - qqa & \\ \quad - qra & \\ \quad - rra & = - qqr \\ & \quad - qrr \end{array}$$

And in the proposed equation, in which $b > c$ is substituted, the inequality becomes

(by Lemma 5 to Prop. 2), in the canonical equation . . .

$$\frac{\left.\begin{array}{c} qq + qr + rr \\ qq + qr + rr \\ qq + qr + rr \end{array}\right|}{27} > \frac{\left.\begin{array}{c} + qqr + qrr \\ + qqr + qrr \end{array}\right|}{4}$$

bb.bb.bb > ccc.ccc.

Therefore the coefficient and the given homogenous term of the proposed equation satisfy the same condition for the positive and negative terms as the coefficient and homogeneous term of the given Canon.
The proposed equation and the Canon are therefore equipollent (by definition), and indeed are provided with the same number of roots.
But, (by Prop.6, Sect. 4) the canonical equation is shown to have the two roots *q & r.*
The common equation proposed is therefore shown to have two root, as stated.
[Note for Prop. 6 : $a^3 - (0)a^2 + (-b^2 - bc - c^2)a + bc(b + c) = (a - b)(a - c)(a + b + c) = 0$.]

PROPOSITION 5.

The common equation *aaa - 3.bba + 3.cca= + ddd,* in which $b > c$, and $b > d$ is shown to have three [positive] roots.

For the common equation proposed, the Canonical equation with similar steps and affections is

$$\begin{array}{l} aaa - paa + pqa \\ \quad\; - qaa + pra \\ \quad\; - raa + qra = + pqr \end{array}$$

And in the canonical equation , (by Lemma 6 following) . .

$$\left.\begin{array}{c} \frac{p+q+r}{3} \\ \frac{p+q+r}{3} \end{array}\right| > \frac{pq+pr+qr}{3}$$

And (by Lemma 7 following) . . .

$$\left.\begin{array}{c} \frac{p+q+r}{3} \\ \frac{p+q+r}{3} \\ \frac{p+q+r}{3} \end{array}\right| > pqr$$

And in the proposed equation, in which $b > c$ and $b > d$ are substituted, the inequalities for the canonical equation become: $bb > cc$, and $bbb > ccc$.
Therefore the coefficient and the given homogenous term of the proposed equation satisfy the same condition for the positive and negative terms as the coefficient and homogeneous term of the given Canon.
The proposed equation and the Canon are therefore equipollent (by definition), and indeed are provided with the same number of roots.
But, (by Prop.5, Sect. 4) the canonical equation has been expounded from three roots *p, q & r.*
The common equation proposed has therefore been solved from three root, as was stated.
[Note for Prop. 5 : $a^3 - (b+c+d)a^2 + (bc+cd+db)a - bcd = (a-b)(a-c)(a-d) = 0$.]

Lemma 6.

If a quantity should be cut into three unequal parts the square of the third part of the whole is greater than the third part of the sum of the binary products of the individual parts.

If the three unequal parts are *p, q, r,* the inequality is :

$$\left.\begin{array}{c} \frac{p+q+r}{3} \\ \frac{p+q+r}{3} \end{array}\right| > \frac{pq+pr+qr.}{3}$$

But, (by Lemma 2) . . . $pp + qq > 2.pq$,

and . . . . $qq + rr > 2.qr$,

and . . . . $pp + rr > 2.pr$.

Therefore : $2.pp + 2.qq + 2.rr > 2.pq + 2.qr + 2.pr$

Therefore . . $2.pp + 2.qq + 2.\,rr > 2.pq + 2.qr + 2.pr$

Therefore . . $pp + qq + rr > pq + qr + pr$

Therefore by adding to each . . . . $2.pq + 2.qr + 2.pr$

we have . .

$$\begin{array}{c} pp + qq + rr \\ + 2.pq + 2.qr + 2.pr \end{array} > 3.pq + 3.qr + 3.pr$$

But . . .

$$\begin{array}{c} pp + qq + rr \\ + 2.pq + 2.qr + 2.pr \end{array} = \begin{array}{c|} p + q + r \\ p + q + r \\ \hline \end{array}$$

Therefore . .

$$\begin{array}{c|} p + q + r \\ p + q + r \\ \hline \end{array} > 3.pq + 3.qr + 3.pr$$

That is . .

$$\begin{array}{c|} p + q + r \\ p + q + r \\ \hline \end{array} \, \begin{array}{c} \\ \\ 3 \end{array} > pq + qr + pr$$

Therefore

$$\begin{array}{c|} \frac{p + q + r}{3} \\ \frac{p + q + r}{3} \\ \hline \end{array} > \frac{pq + pr + qr}{3}$$

Which is to be shown.

Note : In modern notation, this inequality is : $\left(\frac{p+q+r}{3}\right)^2 > \frac{pq+qr+pr}{3}$.

Lemma 7.

If a quantity is cut into three unequal parts, the cube of the third part of the whole is the greater than the product of the three parts.

If the three unequal parts are *p, q, & r*, then the inequality is :

$$\begin{array}{c|} \frac{p + q + r}{3} \\ \frac{p + q + r}{3} \\ \frac{p + q + r}{3} \\ \hline \end{array} > pqr$$

For, (by Lemma 2) . . . $pp + qq > 2.pq,$

and . . . . $qq + rr > 2.qr,$

and . . . . $pp + rr > 2.pr.$

Therefore . . $ppr + qqr > 2.pqr,$

and . . . . $pqq + prr > 2.pqr,$

and . . . . $ppq + qrr > 2.pqr,$

Therefore . .
$$\begin{array}{l} ppr + qqr \\ pqq + prr \\ ppq + qrr \end{array} > 6.pqr.$$

But, (by Lemma 3) . . . $ppp + qqq > ppq + pqq,$

and . . . . $qqq + rrr > qqr + qrr,$

and . . . . $ppp + rrr > ppr + prr.$

Therefore . . $ppp + qqq + rrr > 3.\ pqr,$

and . . . .
$$\begin{array}{l} 3.ppr + 3.qqr \\ 3.pqq + 3.\ prr \\ 3.ppq + 3.qrr \end{array} > 18.pqr.$$

Therefore . .
$$\begin{array}{l} ppp + qqq + rrr \\ + 3.ppr + 3.qqr \\ + 3.pqq + 3.\ prr \\ + 3.ppq + 3.qrr \end{array} > 21.pqr$$

And on adding to each $6.pqr$:

The inequality becomes :
$$\begin{array}{l} ppp + qqq + rrr \\ + 3.ppr + 3.qqr \\ + 3.pqq + 3.\ prr \\ + 3.ppq + 3.qrr \\ + 6.pqr \end{array} > 27.pqr$$

But . . . . . .
$$\begin{array}{l} ppp + qqq + rrr \\ + 3.ppr + 3.qqr \\ + 3.pqq + 3.\ prr \\ + 3.ppq + 3.qrr \\ + 6.pqr \end{array} = \begin{array}{c|} p + q + r \\ p + q + r \\ p + q + r \\ \hline \end{array}$$

Therefore . .
$$\begin{array}{c|} p + q + r \\ p + q + r \\ p + q + r \\ \hline \end{array} > 27.pqr$$

That is . .
$$\dfrac{\begin{array}{c|} p + q + r \\ p + q + r \\ p + q + r \\ \hline \end{array}}{27} > pqr$$

$$\begin{array}{c|} \dfrac{p + q + r}{3} \\ \dfrac{p + q + r}{3} \\ \dfrac{p + q + r}{3} \\ \hline \end{array} > pqr$$

Q.e.d.

Note : This last inequality is a masterful way of demonstrating the inequality of the arithmetic and geometric means for three positive numbers.

PROPOSITION 6.

The common equation *aaaa - 4.bbba* = - *3. cccc,* in which $b > c$, is shown to have two [positive] roots.

For the common equation proposed, the Canonical equation with similar steps and affections is

$$\begin{aligned} aaaa &- bbba \\ &- bbca \\ &- bcca \\ &- ccca = - bbbc \\ &\qquad\quad - bbcc \\ &\qquad\quad - bccc. \end{aligned}$$

And in the canonical equation, the biquadratic of $\dfrac{bbb + bbc + bcc + ccc}{4}$ is greater than the cube of $\dfrac{bbbc + bbcc + bccc}{3}$.

[This is similar to the inequality of Lemma 5: $\left[\dfrac{p^3 + p^2q + pq^2 + q^3}{4}\right]^4 > \left[\dfrac{p^3q + p^2q^2 + pq^3 + q^4}{3}\right]^3$;

perhaps the proof has been omitted in the *Praxis* as is rather long; a modern style proof can be constructed as above - this has been left as an exercise for the interested reader.]

And in the proposed equation in which $b > c$ is substituted, the biquadratic of $\dfrac{4.bbb}{4}$ is greater than the cube of $\dfrac{3.cccc}{3}$.

Therefore the coefficient and the given homogenous term of the proposed equation satisfy the same condition for the positive and negative terms as the coefficient and homogeneous term of the given Canon. The proposed equation and the Canon are therefore equipollent (by definition), and indeed are provided with the same number of roots.

But, (by Prop.35, Sect. 4) the canonical equation has been shown to have the two roots *b* or *c.*

The common equation proposed has therefore been solved for two roots, as was stated.

[Note for Prop. 35 :

$$a^4 - (0)a^3 - (0)a^2 - (b^3 + b^2c + bc^2 + c^3)a + b^3c + b^2c^2 + bc^3$$
$$= (a-b)(a-c)(a^2 + (b+c)a + b^2 + bc + c^2) = 0.]$$