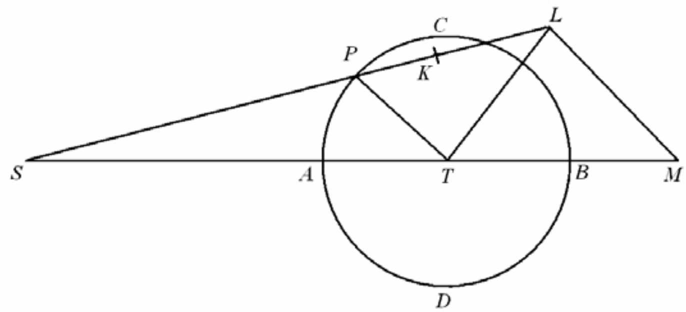
PROPOSITION XXXVI. PROBLEM XVII.
To find the force of the sun required to move the waters of the sea.

The force [per unit mass] *ML* or *PT* of the sun required to perturb the lunar motion acting at the lunar quadratures, was to the force of gravity with us on earth (by Prop. XXV. of this Book;), as 1 to 638092,6. And the force $TM - LM$ or $2PK$ at the lunar



syzygies is twice as great. [see the notes at the start of the previous section, from which the results presented here follow; see also Chandrasekhar, p. 412.] But these forces, if one descends to the surface of the earth, are diminished in the ratio of the distances from the centre of the earth, that is, in the ratio $60\frac{1}{2}$ to 1; and thus the first force on the surface of the earth is to the force of gravity as 1 to 38604600. It is by this force that the sea is lowered in places, which stand apart at 90 degrees from the sun. The sea is raised both under the sun and in the region opposite the sun by the other force, which is greater by twice as much,. The sum of the forces is to the force of gravity as 1 to 12868200. And because the same force produces the same motions, that either lowers the water in regions which are set at 90 degrees to the sun, or raises the same in regions under the sun and in regions opposite to the sun, thus sum will be the total force of the sun required to set the seas in motion ; and the same effect will be had, if the whole sea may be raised in regions under the sun and in regions opposite the sun, but in regions which are set at 90 degrees to the sun it gives rise to no effect.

This is the force of the sun required to set the sea in motion at some given place, both when the sun is situated overhead as well as at its mean distance from the earth. In other places the force required to raise the sea from the position of the sun is directly as the versed sine of twice the altitude of the sun above the horizon of the place, and inversely as the cube of the distance of the sun from the earth.

Corol. Since the centrifugal force of the parts of the earth arises from the diurnal motion of the earth, which is to the force of gravity as 1 to 289, it has the effect that the height of the water at the equator exceeds the height of this at the poles by an amount of 85472 Parisian feet, as in Prop. XIX above ; the force of the sun by which we are driven, since it shall be to the force of gravity as 1 to 12868200, and thus to that force as 289 to 12868200 or 1 to 44527, effects that the height of water in regions under the sun and opposite to the sun shall surpass the height of this in places, which are set at 90 degrees to the sun, by a measure of one Parisian foot and $11\frac{1}{30}$ inches. For this measure is to the measure of 85472 feet as 1 to 44527.

PROPOSITION XXXVII. PROBLEM XVIII.
To find the force of the moon required to move the waters of the sea.

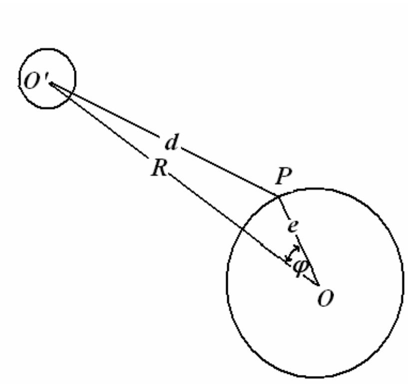
The force of the moon required to move the sea can be deduced from the proportion of its force to the force of the sun, and this proportion is to be deduced from the proportion of the motions of the seas, which arise from these forces.
[At the time of writing, the mass of the moon was not known precisely: Newton could hardly go to the moon and measure the acceleration of gravity there, and as there was no small satellite on hand on which measurements could be made, so he adopted this rather imprecise method, the only one available to him at the time.]

Before the mouth of the river *Avon* at the third milestone below *Bristol*, at spring time and in autumn the whole rise of the water at the conjunction and opposition of the moon, according to the observation of *Samuel Sturmy*, is around 45 feet, but in the quadratures it is only 25 feet. The first height arises from the sum of the forces, and the last from the difference of the same. Therefore let *S* and *L* be the forces of the sun and moon situated above the equator and at their mean distances from the earth, and $L+S$ to $L-S$ will be as 45 to 25, or as 9 to 5.

[These tide-producing forces, due to the sun and moon individually as viewed from the earth, act on a unit mass at some place in question; this mass is situated a whole earth-radius from the centre of the mean earth-sun distance and the mean earth-moon distances respectively at the meridian, by which the force is increased by the inverse square law above the average value at the centre of the earth ; each force in turn sweeping through 180^0, from acting initially along one tangent to the sea through the normal at the meridian and finally acting along the tangent in the opposite direction, in a time of 12 hours, due to the earth's diurnal rotation, while the sun or moon can be considered as almost at rest. These actions are simultaneously performed, by the forces of the sun and the moon acting through the earth on the water on the far side of the earth, and now diminished by the inverse square law, giving rise on the whole to corresponding tidal bulges on opposite sides of the earth.

This sum or difference of the forces arises from the simple addition of the vectors *L* and *S* at the various places : at the new moon syzygies these forces are simply added $L+S$, allowing 6 hours for the earth to rotate into the tangential position of the forces; the full moon case requiring both forces to act in the opposite direction, this being the case for the sun as the force is now less than the average at the centre of the earth by the same amount *S*; An elementary derivation of part of this argument is given, e.g. in *Ohanian, Gravitation and Spacetime*, p.26. At the quadrature positions, the tangential moon's force and that of the sun after a quarter rotation of the earth, are considered to act in opposite directions, giving the magnitude of the force to be now $L-S$ in this simplified scheme. See the notes at the start of the last section.

For a concise modern analysis, you may consult p.355 onwards of Volume 7 of the *Encyclopaedic Dictionary of Physics*, where the whole matter of tide-generating forces is addressed from the point of view of potentials, introduced some hundred years later by Laplace. This is one section of such a work that does not get dated! Thus, the potential energy of a unit mass due to the moon, situated on the earth's surface at a point P a distance d from the centre of the moon, is given by GM_m / d , where M_m is the mass of the moon, G the constant of universal gravitation; a negative sign may be attached to give the gradient forces the correct sign for the direction in which they act. Hence, if R is the distance between the centres of the earth and moon, and e the radius of the earth making an angle φ to R, then $d^2 = e^2 + R^2 - 2eR\cos\varphi$; in which case the potential becomes



$$GM_m / d = \frac{GM_m}{R\sqrt{1+\frac{e^2}{R^2}-2\frac{e}{R}\cos\varphi}} = \frac{GM_m}{R}\sum_n \frac{e^n}{R^n}P_n\left(\cos\varphi\right),$$

where the coefficients $P_n\left(\cos\varphi\right)$ are the Legendre polynomials defined by

$$P_0\left(\cos\varphi\right) = 1; P_1\left(\cos\varphi\right) = \cos\varphi;$$

$$P_2\left(\cos\varphi\right) = \tfrac{1}{2}\left(3\cos^2\varphi - 1\right); P_3\left(\cos\varphi\right) = \tfrac{1}{2}\left(5\cos^3\varphi - 3\cos\varphi\right); \text{etc.}$$

Now, the mean value of $\frac{e}{R}$ is 0.01659, and due to its small magnitude, only a few terms need be considered; on differentiation of the potential function to obtain the forces, the first term is constant and disappears; the second term can be shown to give rise to a constant force along the line of the centres, and hence only the derivative of the third term involving P_2 need be considered in most cases to cause relative motion. Since $g = \frac{GM_E}{\bar{e}^2}$, where M_E is the mass of the earth and $\bar{e}$ the mean radius of the earth , then the potential generating the lunar tides has the form $V = gV_0\left(\cos^2\varphi - \frac{1}{3}\right)\frac{e^2}{\bar{e}}$ and $V_0 = \frac{3}{2}\frac{M_m}{M_E}\left(\frac{\bar{e}}{R}\right)^3$; from which we see that the tangential component $\frac{dV}{ed\varphi} = -gV_0\sin 2\varphi$ is the only large component, generating tides over large distances on the earth's surface, since the vertical component $\frac{dV}{de} = 2gV_0\left(\cos^2\varphi - 1\right)$ is negligible in comparison with the local acceleration of gravity g, since V_0 has a mean value of 5.6×10^{-8} . From the historical viewpoint, we may note that Newton worked with forces in his analysis, a hard task, while Laplace on introducing the idea of potential, was able to perform a simpler analysis; but of course Laplace could not have seen the easier method without Newton's great labours.]

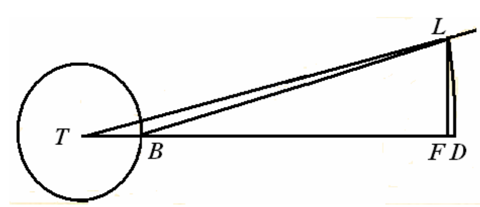
In the port of *Plymouth* the tides of the sea from the observations of *Samuel Colepress* rise around 16 feet from the average, and at spring time and autumn the height of the tide in syzygies can surpass the height at quadrature by 7 or 8 feet. If the maximum difference of these heights shall be 9 feet, then $L+S$ to $L-S$ will be as $20\frac{1}{2}$ to $11\frac{1}{2}$ or 41 to 23.

Which proportion agrees well enough with the previous. On account of the magnitude of the tide at the port of *Bristol*, it may appear that the observations of *Sturmy* to be more reliable, and thus we will use the proportion 9 to 5, as it is a little more certain.

For the remainder, on account of the reciprocal motion of the waters, the maximum tides do not occur at the syzygies of the luminary bodies themselves, but are the third tides after the syzygies as it has been said, either nearest to the third passage of the moon through the meridian of the place, or rather, (as it has been noted by *Sturmy*) they are the third tides to arrive after the day of the new moon or the full moon, or within an [additional] interval of [half of] 12 hours of the new or full moon, and thus are incident a little more or less than the 43rd hour from the new or full moon. [Such situations may arise when the moon acts as a forcing oscillator on a body of water in a confining basin, which builds up a resonance, as it were.] Truly these arrive at this port at around the seventh hour from the approach of the moon at the place of the meridian ; and thus the tides follow the passage of the moon through the meridian very closely, when the moon either stands apart from the sun, or is in opposition by 18 or 19 degrees as a consequence. [In fact, the tidal forces at the time act along the tangent rather than normally, as Chandrasakher points out; also, this is a place where Mott translates Newton's *octodecim vel novemdecim* as 18 or 19, which would seem to be correct – though this is not the usual way of writing these numbers, and which is translated by *Madame du Chatelet* as 80 or 90.] The summer and winter tides are especially vigorous, not in the solstices themselves, but when the sun stands at around a twentieth part of the whole circuit, or around 36 or 37 degrees. And similarly the maximum tide of the sea arises from the approach of the moon to the meridian of the place, when the moon stands away from the sun by a tenth part of the whole motion from tide to tide. That distance shall be of around $18\frac{1}{2}$ degrees. And the force of the sun at this distance of the moon from syzygies and quadratures, will have a smaller value towards increasing or decreasing the motion of the seas arising from the force of the moon, than at the syzygies and the quadratures, in the ratio of the radius to the sine of twice the complement of the angle $18\frac{1}{2}$, *i.e.* of 37^0, that is, in the ratio 10000000 to 7986355. And thus in the above analogy 0,7986355 S must be written for S.

And thus the force of the moon in quadrature, on account of the declination of the moon from the equator, must be diminished. For the moon in quadrature, or rather at $18\frac{1}{2}^0$ after quadrature, is present at a declination of around $22^0.13'$. And the force required to move the sea is diminished by the decline from the equator in the square ratio approximately as the square of the complement of the sine of the declination.

[Note from *L* & *S*. p. 110, *De Mund. Syst.*: Let *TBD* be the plane of the equator, *T* the centre of the earth, and the moon shall be at *L*, the angle *LBD* will measure the declination from the equator, or on account of the very small angle *TLB*, that declination will be approximately equal to the angle *LTD*, the cosine of which angle is *TF*, by taking *TL* for the radius. Now the force which pulls on the water from the centre *T*, at the location *B* of



the equator, when the moon is present in the equatorial plane at *D*, is to force that pulls the same water directly from the centre, when the moon is at *L*, as *TL* to *TF*, that is, as the radius to the sine of the compliment of the declination *LDT*, distinct from the centripetal force of the water towards *T*. But with that centripetal force increased, the other force drawing the water from the centre is diminished in the same ratio; whereby, on compounding the effects, the force of the moon at the location *D*, is to the force of this at *L*, as the square of the whole sine *TL*, to the square of the complement of the sine, *TF*, of the declination of the moon *LTD*.]

And thus the force of the moon in these quadratures is only 0,857327 L. Therefore there is $L+0{,}7986355S$ to $0{,}857032.7\text{ L} - 0{,}79863558$ as 9 to 5.

Besides the diameters of the orbit, in which the moon must be moving without eccentricity [as in the Horrox sense], are in turn as 69 to 70; and therefore the distance of the moon at syzygies is to its distance at the quadratures as 69 ad 70, with all else being equal. And the distances of this at $18\frac{1}{2}^0$ from syzygies, when the maximum tide is generated, and at $18\frac{1}{2}^0$ from quadratures, when the minimum tide is generated, are to the mean distance of this as 69,098747 and 69,897345 to $69\frac{1}{2}$. But the lunar forces requiring to move the sea are inversely in the cubic ratio of the distances, and thus the forces at the maximum and minimum of these distances are to the force at the mean distance as 0,9830427 and 1,017522 to 1. From which there arises $1{,}017522L + 0{,}7986355S$ to $0{,}9830427\times 0{,}8570327L - 0{,}7986355S$ as 9 to 5. And *S* to *L* is as 1 to 4,4815. And thus since the force of the sun shall be to the force of the gravity 1 to 12868200, the force of the moon shall be to the force of gravity as 1 ad 12871400.

[Because this value of *S* to *L* adopted is even further than the original estimate of 3.5 from the now accepted value of 2.34, Chandrasekhar wisely decides not to investigate the following Corollaries.]

Corol. 1. Since the water disturbed by the force of the sun rises to a height of $1\text{ft.},11\frac{1}{30}''$, by the force of the moon the same will rise to a height of $8\text{ft.},7\frac{5}{22}''$ and the force with each acting shall be to a height of $10\frac{1}{2}\text{ft.}$, and when the moon is in perigee to a height of $12\frac{1}{2}\text{ft.}$ and more, especially when the tide is helped by the winds blowing. Moreover so much force suffices in abundance for all the excited motions of the sea, and corresponds properly to the quantity of the motions. For in seas which extend out widely from east to west, as in the *Pacific*, and in the parts of the *Atlantic* & *Ethiopic* [i.e. *Indian*] *Oceans* outside the tropics, the water is accustomed to be raised to a height of 6, 9, 12, or 15 feet. But in the *Pacific Ocean*, because it is deeper and extends wider, the tides are said to be greater than in the *Atlantic* and *Indian Oceans*. And indeed so that the tide shall be full, the width of the sea from east to west cannot be less than 90^0. In the *Indian Ocean* the rise of the water within the tropics is less than in the temperate zone, on account of the narrow seas between the *African* and the southern part of *America*. The water is unable to rise in the middle of the sea unless at it may fall on each eastern and western shore at the same time: since yet with our narrow seas it must fall alternately in turn on these shores. For that reason the flow and ebb for islands which are farthest from the shores, is usually very small. In certain ports, where water is trying to flow in and out with great force through the shallow places, alternately filling and emptying the bays, the flow and ebb must be customarily greater, as at *Plymouth* & at *Chepstowe bridge* in *England* ; at mount *St. Michael* and at *Avranches* in *Normandy* ; at *Cambaia* and *Pegu* in the *East Indies*. In these places the sea, by approaching and receding with great speed, now inundates the shore and again now leaves it dry for many miles. Neither the impetus of the inflow or of the return of the first can be broken, as the water is raised or lowered by 30, 40, or 50 feet and more. And equal is the account of long narrow and shallow seas, as the *Magellanic Straits*, and of that by which England is surrounded. The tide in ports of this kind and in narrow channels is augmented by the impetuosity of the flow and ebb above the normal. Truly, the magnitude of the tide corresponds to the forces of the sun and moon for shores which descend to the depths of the abyss and which are seen to be open, and where water without precipitation can flow and ebb by rising and falling without an impetuous motion.

Corol. 2. Since the force of the moon requiring to move the seas shall be to the force of gravity as 1 to 2871400, it is evident that that force shall be much less than those considered in any experiments with pendulums, or in statics or hydrostatics. This force is able to produce a sensible effect only in the tides in the sea.

Corol. 3. Because the force of the moon required to move the sea compared to the force of the sun is as 4,4815 to 1 [recall that this is an overestimate roughly by a factor of 2, so that the following values are not numerically correct and the conclusion is wrong], and these forces (by corol. 14 Prop. LXVI, Book I.) are as the densities of the bodies of the moon and of the sun and the cubes of the apparent diameters conjointly ; the density of the moon will be to the density of the sun as 4,4815 to 1 directly, and the cube of the diameter of the moon to the cube of the diameter of the sun inversely : that is (since the mean apparent diameters of the moon and sun shall be $31'.16''\frac{1}{2}$, and $32'.12''$) as 4891 to 1000. But the [mean] density of the sun was to the density of the earth as 1000 to 4000; and therefore the density of the moon is to the density of the earth as 4891 to 4000 or 11 to 9. Therefore the body of the moon is more dense that our earth.

Corol. 4. And since the true diameter of the moon from astronomical observations shall be to the true diameter of the earth as 100 to 365 the mass of the moon to the mass of the earth shall be as 1 to 39,788.

Corol. 5. And the gravitational acceleration on the surface of the moon will be as if three times smaller than the acceleration of gravity on the surface of the earth.

*Corol.*6. And the distance of the centre of the moon from the centre of the earth will be to the distance of the centre of the moon from the common centre of gravity of the earth and the moon, as 40,788 to 39,788.

Corol. 7. And the mean distance of the centre of the moon from the centre of the earth in the lunar octants will be approximately $60\frac{2}{5}$ of the maximum earth radii. For the maximum radius of the earth became 19658600 Parisian feet, and the mean distance of the centres of the earth and the moon from $60\frac{2}{5}$ constant radii [these are called diameters in the original] of this kind, is equal to 1187379440 feet. And this distance (by the above corollary) is to the distance of the centre of the moon from the common centre of gravity of the earth and the moon, as 40,788 to 39,788: and thus the latter distance is 158268534 feet. And since the moon is revolving with respect to the fixed stars, in 27 days, 7 hours, an $43\frac{4}{9}$ minutes; the versed sine of the angle, that the moon will describe in a time of one minute, is 12752341 parts, for a radius of 1000,000000,000000 parts. And as the radius is to this versed sine, thus as 1158268534 feet is to 14,7706353 feet. Therefore the moon, by that force by which it is held in orbit, by falling towards the earth, describes a distance of 14,7706353 feet in a time of one minute. And by increasing this force in the ratio $178\frac{29}{40}$ to $177\frac{29}{40}$, the total force of gravity of the moon in orbit will be had, by the corol. of Prop. III. And by this force the moon by falling for a time of one minute describes a distance of 14,8538067 feet. And to the sixtieth part of the distance of the moon from the centre of the earth, that is to the distance of 197896573 feet from the centre of the earth, a heavy body in a time of one minute will also describe a distance of 14,853067. And thus at the distance of 19615800 feet, which is the mean radius of the earth, a weight by falling describes 15,11175 feet, or 15 feet, 1 inch, and lines $4\frac{1}{11}$ [a line was the twelfth part of an inch]. This will be the descent of bodies at 45^0 latitude. And by the table described in the preceding Prop. *XX*, the descent will be a little greater arising at the latitude of *Paris* by an excess of as much as the $\frac{2}{3}$ parts of a line. Therefore a weight through this computation by falling in a vacuum at the latitude of Paris, describes around 15ft,, 1inch, and $4\frac{25}{33}$ lines in a time of one second. And if the weight may be diminished by taking away the centrifugal force, which arises from the diurnal motion of the earth at that latitude, then weights by falling there describe15 feet,1 inch, and $1\frac{1}{2}$ lines in a time of one second. And a weight had been shown above in Prop. IV and XIX to be falling with this speed at the latitude of Paris.

*Corol.*8. The mean distance between the centres of the earth and of the moon at syzygies is 60 of the earth's maximum radius, with around a 30th part of the radius taken away. And in quadrature the mean distance of the moon from the same centres is $60\frac{5}{6}$ earth radii. For these two distances are to the mean distance of the moon in the octants as 69 and 70 to $69\frac{1}{2}$ by Prop. XXVIII.

*Corol.*9. The mean distance between the centres of the earth and the moon in the lunar syzygies is $60\frac{1}{10}$ radii of the mean earth radius. And in the lunar quadratures the mean distance of the same centres is 61 mean radii of the earth, with the 30th part of a radius taken away.

*Corol.*10. In the lunar syzygies, its mean horizontal parallax at the latitudes of 0^0, 30^0, 38^0, 45^0, 52^0, 60^0, 90^0, is
$57'.\,20''$, $57'.16''$, $57'.\,14''$, $57'.\,12,''$, $57'.\,10''$, $57'.\,8''$, $57'.\,4''$ respective.

[This quantity of almost 1^0 can be considered as the difference in the angle the moon subtends against the fixed stars, by viewing the moon vertically at one place on the earth, and horizontally at another.]

In these computations I have not considered the magnetic attraction of the earth, as its quantity is very little and it has been ignored. Indeed if whenever this attraction will be able to be investigated, then all the measurements of degrees at the meridian, the lengths of isochronous pendulums at different places, the laws of motion of the seas, the parallax of the moon, the different apparent diameters of the sun and moon, will have to be determined more carefully from the phenomena: then one would be permitted to repeat all these calculations more accurately.

PROPOSITION XXXVIII. PROBLEM XIX.
To find the figure of the body of the moon.

If the lunar body were fluid just like our seas, the force of the earth required to raise that fluid into parts both nearest and furthest from the earth, would be to the force by which our sea is raised in parts both under the moon and opposite the moon, as the acceleration of gravity of the moon on the earth is to the gravitational acceleration of the earth on the moon, and the diameter of the moon to the diameter of the earth conjointly : that is, as 39,788 to 1 and 100 to 365, or as 1081 to 100. From which since our sea is raised to $8\frac{3}{5}$ feet by the force of the moon, the lunar fluid by the force of the earth must be raised to 93 feet. And for that reason the figure of the moon must be a spheroid, of which the maximum diameter produced passes through the centre of the earth, and exceeds the perpendiculars by an excess of 186 feet. Therefore the moon affects such a figure, and the same must endure from the beginning. *Q.E.I.*

Corol. Hence indeed it shall be for this reason that the same face of the moon is always turned towards the earth. For the moon cannot be at rest in any other situation, but by oscillating always returns to that situation. Yet the oscillations, on account of the smallness of the forces of agitation, have to be the slowest, acting over a long period of time : and thus so that that face, which must always look towards the earth, can look towards the other focus of the moon's orbit (on account of the reason referred to in Prop. XVII), and cannot at once be turned to look away from the earth.

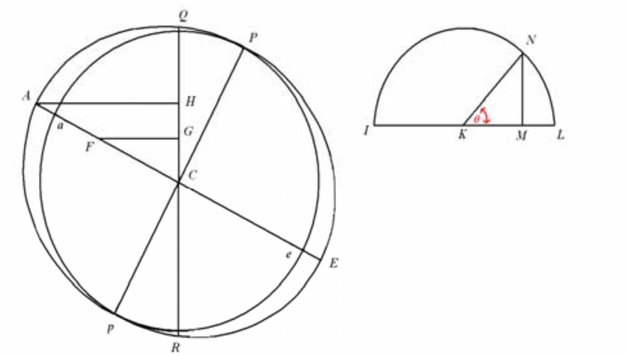
LEMMA I.

If APEp may designate an earth of uniform density, with centre C and with the poles P, p and with the equator delineated by the line AE; and furthermore from the centre C and with the [smallest] radius CP, the sphere Pape is understood to be described ; moreover QR shall be a plane, to which the right line drawn from the centre of the sun to the centre of the earth stands normally; and the individual particles of all the exterior part of the earth PapAPepE, only those particles which have been described beyond the sphere will try to recede thence from the plane QR, and the attempt of each particle to recede shall be as its distance from the plane.

[*This Lemma starts the calculation of the torque exerted on the earth by the sun due to the equatorial bulge of the earth, as the earth rotates about an axis at an angle of* 23.5^0 *to the normal to the plane of the ecliptic. The forces acting away from the plane QR towards the sun arise from the inverse square law applied at the slightly shorter distance than average at the centre of the earth, while those in the opposite direction away from the sun arise from the slightly smaller than average inverse square forces. Such forces are proportional to their distance from the plane QR, and exert a torque along an axis formed by the intersection of the plane of the equator with the plane QR.* There is an uncertainty about this proposition, as the particles within the inner sphere are not supposed to contribute to the bulge by having an unbalanced force acting on them]

I say in the first place, that the total force and effectiveness [*i.e. the torque*] *in making the earth rotate around its centre [i.e. precess], of all the particles that have been placed in the plane of the equator AE, and which are arranged regularly around the globe in the form of a ring, will be to the whole force exerted by an equivalent number of particles placed at the point A of the equator, which shall be maximally distant from the plane QR, and which constitute a similar force and effectiveness for the circular motion [i.e. precession] of the earth moving around its centre, in the ratio one to two. And this circular motion will be carried out around the axis lying in the common section of the equator and the plane QR.*

[The diagram below is actually in three dimensions : and represents the earth with its axis *Pp* tilted to the plane of the ecliptic, which is itself normal to the plane *QR*, and both planes can be imagined to extend out of the plane of the diagram (note that Newton calls *QR* a plane and not a line); the line *AE* is then the outermost part of the permanent semicircular equatorial bulge as viewed normally, and the lines *AH, FG*, etc. are the perpendicular distances of points on this bulge to the plane *QR* of the average distance to the sun, passing through the centre of the earth *C*. Thus, the axis normal to the plane of the diagram passing through *C* is the one about which the torques act (according to the right-hand thumb rule, with the thumb pointing out of the plane of the page), due to the equal and opposite attractions of the sun on the equatorial bulges in front of and behind the earth w.r.t. the sun, leading to an unbalanced torque acting on the earth, which thus precesses slowly about its angular momentum vector direction *Pp*. The component of the torque described normal to *Pp*, performs this task ; and which is proportional to the whole torque acting, because of the constant angle. The first Lemma shows that the torque acting on all the points on the bulge at the equator added together is half the torque required to produce the same motion with all the equatorial mass of the bulge placed at *A*, directly under the sun, at the maximum separation from the plane *QR* ; note that the mass of a line or curve can be taken as proportional to its length, as detailed in Lemma III below. We may visualise the arch starting at *C* out of the plane, passing through *A* and going behind the plane of the diagram. Note that we should use phrases such as *moment of inertia* and *angular momentum* with due caution, as these notions came later from Euler's work on mechanics. This then is Newton's excursion into the realm of extended bodies acted on by torques, rather than particles being acted on by forces. It is then perhaps inappropriate to consider these lemmas from the point of view of vector calculus, as Chandrasekhar does. The approach adopted by the old faithful Le Seur and Jacquier is probably more useful.]



For with centre *K* and diameter *IL* the semicircle *INLK* is described. It may be understood that the semi circumference *INL* has been divided into innumerable equal parts, and from each of the individual parts *N* the sine *NM* may be sent to the diameter *IL*. And the sum of the squares from all the sines *NM* will be equal to the sum of the squares from all the [co]sines *KM*, and both sums will be equal to the sum of the squares from the whole radius *KN*; and thus the sum of the squares from all *NM* is half as large as the sum of the squares from the whole radius *KN*.
[Thus, from a more modern viewpoint, the moment of inertia of the semicircular arch through *K* normal to the diagram is twice as great as the moment of inertia about either diameter *IL* or the vertical one through *K* in the plane of the diagram. Physically, the torque required to generate the same angular speed is twice as great in the first case than in the following two equal cases.]

Now the perimeter of the circle *AE* may be divided into just as many equal parts, and from any of these *F* a perpendicular *FG* is sent to the plane *QR*, just as the perpendicular *AH* is sent from the point *A*. And the force, by which a small particle *F* tries to recede from the plane *QR*, will be by hypothesis as that perpendicular *FG*, and this force multiplied by the distance *CG* will be the effectiveness [*i.e.* the word Newton uses here for torque] of the small particle *F* in turning the earth about its centre [*i.e.* about an axis normal to the plane of the diagram, passing through *C*]. And thus the effectiveness of the particle at the place *F*, will be to the effectiveness of the particle at the place *A*, as $FG \times GC$ to $AH \times HC$, that is, as FC^2 to AC^2 ;

[For the ratio $FG \times GC$ to $AH \times HC$ or $\frac{FG\times GC}{AH\times HC} = \frac{FC}{AC} \times \frac{FC}{AC}$ by the similar triangles *AHC* and *FGC*. Thus the torques are as the equivalent moments of inertia. For, the sum of all the terms $FC^2.dm = KM^2.dm = AC^2\sigma d\theta \cos^2\theta$, and the summations give $\sum_i (FC)_i^2.\Delta M = AC^2 \frac{M}{2\pi.AC}\int d\theta \cos^2\theta = \frac{1}{2}AC^2 \times \frac{M}{AC}$, where $AC = KN$, and where $AC.\Delta M = M$; while $\sum_i (AC)^2.\Delta M = \frac{M}{AC} \times AC^2$, thus giving the required ratio. Thus, the torque produced by the annulus is half the torque exerted by the whole mass placed at *A* required to produce the same motion, where both the force and distance are a maximum.]

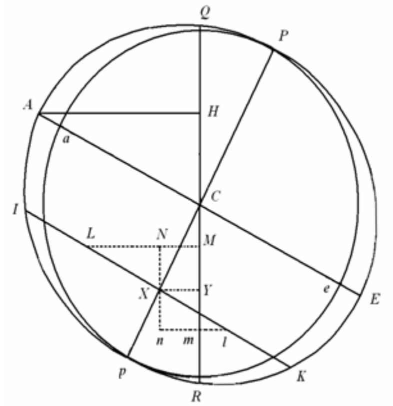
and therefore the total effectiveness of all the particles in the positions of *F* will be to the effectiveness of an equivalent number of particles at the place *A*, as the sum of all FC^2 to the sum of just as many AC^2, that is (as now shown) as one to two.
Q. E. D.

And because the particles exert forces by receding perpendicularly from the plane *QR*, and that equally from each side of the plane: the same will cause the circumference of the equatorial circle to turn, and with that holding fast to the earth, around both the axis lying in that plane *QR* as well as that in the plane of the equator.

LEMMA II.

With these in place: in the second place I say that the force and the effectiveness of all the individual particles situated on both sides outside the globe, for making the earth rotate [*i.e. precess*] *around the same axis, shall be as the total force of just as many particles set out uniformly on the equator of the circle AE in the manner of a ring, to set the earth moving in a similar circular motion, as two to five.*

[Thus, if we sum over the whole bulge, it will exert a torque equal to $\frac{2}{5}$ th of the torque provided by a uniform ring with the same number of particles in the bulge over the equator at the maximum distance.]



Let *IK* be some smaller circle parallel to the equator *AE*, and *L*, *l* shall be any two equal particles placed on this circle beyond the globe *Pape*. And if perpendiculars *LM*, *lm* are sent to the plane *QR*, that has been drawn perpendicularly to a ray from the sun: the total forces, by which these particles flee from the plane *QR*, will be proportional to these perpendiculars *LM* and *lm*. But the right line *Ll* [lying under the arc *Ll*] shall be parallel to the plane *Pape*, and is bisected by the same at *X* and through the point *X* there is drawn *Nn*, which shall be parallel to the plane *QR* and crosses the perpendiculars *LM*, *lm* at *N* and *n*, and the perpendicular *XY* may be sent to the plane *QR*. And the opposite forces of the particles *L* and *l*, rotating in opposite directions to the earth, are as the forces

[here and subsequently, Newton uses the word *force*, but clearly as the quantities are forces times perpendicular distances from the vertical axis through *C*, these are moments or torques]

$LM \times MC$ and $lm \times mC$, that is, as $LN \times MC + NM \times MC$ and $ln \times mC - nm \times mC$, or $LN \times MC + NM \times MC$ and $LN \times mC - NM \times mC$: and the difference of these $LN \times Mm - NM \times \overline{MC + mC}$, is the force [*i.e.* resultant torque] of both the particles taken together required to rotate the earth [*i.e.* precess].

The positive part of this difference : $LN \times Mm$ or $2LN \times NX$ is to force of two particles of the same magnitude put in place at A, $2AH \times HC$, as LX^2 to AC^2.

And the negative part : $NM \times \overline{MC + mC}$ or $2XY \times CY$ is to the force of the same two particles put in place at *A*, $2AH \times HC$, as CX^2 to AC^2.

And hence the difference of the parts, that is, the force of the two particles L & *l* taken together at the place *A* required to turn the earth is to the force of the same two equal particles situated in the same place turning the earth in the same manner, as $LX^2 - CX^2$ to AC^2. [By part of Lemma I.]

But if the circumference *IK* of the circle may be divided into innumerable equal parts *L*, all the terms LX^2 will be to a comparable number of terms IX^2 as 1 to 2 (by applying the same argument as in Lemma I.) and as a consequence to a like number particles in AC^2, as IX^2 to $2AC^2$; and the equivalent number of particles in CX^2 to just as many in AC^2 is as $2CX^2$ to $2AC^2$ [*i.e.* this ratio is maintained, and written as shown for later convenience in subtracting.].

[Thus, $\frac{\sum_i m_i (LX)_i^2}{IX^2 \sum m_i} = \frac{\sum_i m_i (LX)_i^2}{M \times IX^2} = \frac{1}{2}$, and $\frac{\sum_i m_i (LX)_i^2}{AC^2 \sum m_i} = \frac{IX^2}{2AC^2}$; while $\frac{CX^2}{AC^2} = \frac{2CX^2}{2AC^2}$; hence the total torques acting at *A* are as $IX^2 - 2CX^2$ to $2AC^2$.]

Whereby the forces of all the particles taken together in the course of the circle *IK* are to the forces taken together of just as many particles at the place *A*, as $IX^2 - 2CX^2$ to $2AC^2$: and therefore (by Lem. 1.) to the forces taken together of an equivalent number of the particles on the circumference of the circle *AE*, as $IX^2 - 2CX^2$ to AC^2.

[Thus, a circuit of the bulge has been replaced by a mass at *A*, and then this mass is distributed about the equatorial ring *AE*, and as the distance of the masses is now variable to the plane *QR*, the torque is decreased to half its maximum value, according to Lemma I.]

Now truly if the diameter of the sphere *Pp* may be divided into innumerable equal parts, in which there are present just as many circles *IK*; the matter in the perimeter of each circle *IK* will be as IX^2 : [Recall that all the matter in the ring *IK* was put at this position above, where it exerted the maximum torque for that ring; and which matter is proportional to the length of the *IX*, so that the quantity of motion is taken as IX^2 .], and thus the force of that matter requiring to rotate the earth will be as IX^2 into $IX^2 - 2CX^2$. But the force of the same matter, if it shall be present in the circumference of the circle *AE*, shall be as IX^2 into AC^2. And therefore the force [*torque*] of all the particles of all the matter, outside the globe present in the perimeters of all the circles [constituting the bulge], is to the force of just as many particles present in the perimeter of the great circle *AE*, as all IX^2 by $IX^2 - 2CX^2$ to just as many IX^2 by AC^2, that is, as all $AC^2 - CX^2$ by $AC^2 - 3CX^2$ to just as many $AC^2 - CX^2$ into AC^2,

[For from the centre C, to the point *I*, the right line *CI* is supposed to be drawn, and there will be $IX^2 = CI^2 - CX^2$: but $CI = AC$, whereby $IX^2 = AC^2 - CX^2$, and hence the total number of particles by their appropriate torques,
$IX^2 \times \left(IX^2 - 2CX^2\right) = \left(AC^2 - CX^2\right) \times \left(AC^2 - 3CX^2\right)$.]

that is, as the whole of $AC^4 - 4AC^2 \times CX^2 + 3CX^4$ to just as many $AC^4 - AC^2 \times CX^2$, that is, as the total fluent quantity, the fluxion of which is $AC^4 - AC^2 \times CX^2 + 3CX^4$, to the total fluent quantity , the fluxion of which is $AC^4 - AC^2 \times CX^2$;
and hence by the method of fluxions, as $AC^4 \times CX - \frac{4}{3} AC^2 \times CX^3 + \frac{3}{5} CX^5$ to $AC^4 \times CX - \frac{1}{3} AC^2 \times CX^3$, that is, if the total *Cp* or *AC* may be written for *CX*, as $\frac{4}{15} AC^5$ to $\frac{2}{3} AC^5$, that is, as two is to five. *Q.E.D.*

[The quantities $AC^4 - 4AC^2 \times CX^2 + 3CX^4$ and $AC^4 - AC^2 \times CX^2$, taken multiplied together by the fluxion of the line *CX*, and with the fluents taken, will be the fluent of the first quantity $AC^4 \times CX - \frac{4}{3}AC^2 \times CX^3 + \frac{3}{5}CX^5$; but the fluent of the second quantity becomes $AC^4 \times CX - \frac{1}{3}AC^2 \times CX^3$, and from that the total effectiveness may be obtained, for *CX* there is written *Cp* or *AC*, the first fluent to the second fluent will be as $\frac{4}{15}AC^5$ to $\frac{2}{3}AC^5$, giving the ratio quoted. (A note from L. & S.). Basically, the torque required to rotate the extended shell around earth is $\frac{2}{5}$ th of the torque required to rotate a uniform ring of the same mass about the earth with the same angular acceleration.]

[Chandrasekhar establishes a result similar to this by modern methods, but he appears to have solved a different problem, involving the moment of inertia of the whole earth and solid discs. The results as stated here are also shown by Cohen.]

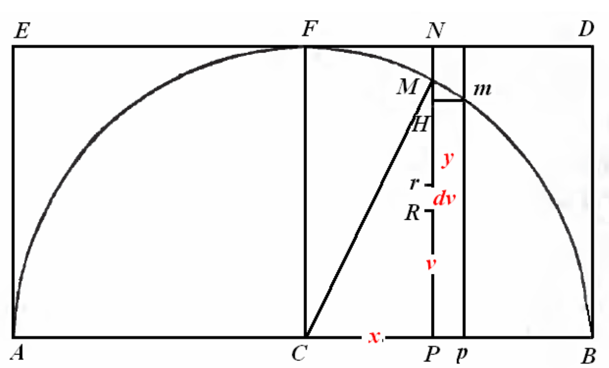
LEMMA III.

With the same in place: I say in the third place that the motion of the whole earth about the axis already described, composed from the sum of all the motions of the particles, will be to the motion of the aforesaid ring around the same axis in a ratio, which is composed from the ratio of the matter in the earth to the matter in the ring, and in the ratio of three times the square of the arc of the quadrant of some circle to twice the square of the diameter ; that is, in the ratio of the matter in the ring to the matter in the earth, and of number 925275 to the number 1000000.

[This ratio amounts to $\frac{3\pi^2}{32}$]

For the motion of a cylinder revolving about its fixed axis is to the motion of the inscribed sphere and likewise rotating, as any four equal squares are to three circles inscribed in these squares ; and the motion of the cylinder to the motion of a very thin ring, going around with the sphere and the cylinder at their common point of contact, as twice the matter in the cylinder to three times the matter in the ring ; and the continued uniform motion of this ring around the axis of the cylinder is to its uniform motion around its own, made in the same periodic time, as the circumference of the circle to twice the diameter.

[L. & J. Note 126: Lemma III is demonstrated. By rotating the semicircle *AFB*, and the circumscribed rectangle *AEDB* of the same, a sphere and a circumscribed cylinder may be described. Let the radius $CB = 1$, the periphery of the circle described in this ratio $= n, (\text{or } 2\pi)$, the abscissa $CP = x$, the ordinate $PM = y$, some part of this



$PR = v$, $rR = dv$; the periphery of the circle described with the radius *PR* is $2\pi v$; the circular ring from the revolution of the line increment $rR = 2\pi v dv$; the velocity of the point $R = v$; (taking the angular velocity as 1 for convenience and setting the density as 1, so that the mass and volume are given by the same variable ; thus the angular momentum considered – here called the motion, for a given increment is the volume times the radius) the motion of the aforementioned ring $= 2\pi v^2 dv$, the motion of the whole circle described (*i.e.* disc with unit thickness) with the radius $PR = \frac{2\pi}{3}v^3$; the motion of the whole disc with unit thickness described with the radius $PM = \frac{2\pi}{3}y^3$; the motion of the unit disc described with the radius $PN = \frac{2\pi}{3}$, and the motion of the whole cylinder $= \frac{4\pi}{3}$.

Let $Pp = dx$; the motion of the solid ring described by the revolution of the figure *PMmp* is equal to $\frac{2\pi}{3}y^3 dx = \frac{2\pi}{3}dx \times \left(1 - x^2\right)^{\frac{3}{2}} = \frac{2\pi}{3} \times -\sin\theta d\theta \times \sin^3\theta$, on setting $x = \cos\theta$. From which the motion of the solid figure of revolution of figure described *CFMB* is equal to $\frac{2\pi}{3}\int \sin^4\theta d\theta = \frac{\pi^2}{8}$. Therefore the motion of the half cylinder to the motion of the hemisphere is as $\frac{2\pi}{3}$ to $\frac{\pi^2}{8}$ or 16 to 3π; that is, as some four equal squares $4 \times 2^2 = 16$ to three circles with $\pi \times 1^2$; and so for the whole cylinder and sphere.

The most tenuous matter of the ring going around touching the sphere and the cylinder adjacent to the common point *F* shall be at the height *m*, and the velocity shall be as *CF*, or as 1; and thus the motion $= m$, and thus the motion (or angular momentum) of the cylinder to the motion of this ring is $\frac{4\pi}{3}$ to m, or as 4π to $3m$, that is, as twice the matter in the cylinder (taken as its volume) to three times the matter in the ring; for the base of the cylinder is the circle $\pi \times 1^2$ and the diameter height $AF = 2$, and thus the cylinder $= 2\pi$. The matter of the aforesaid ring shall be $a^2.2\pi$ (where a^2 is the line density), and thus the motion around the axis of the cylinder itself $= a^2.2\pi$. Now likewise the ring may be revolving around closer to the axis that the diameter *AB* may be showing, and the particles of the matter of the arc corresponding to the infinitesimal *Mm*, will be $a^2 \times Mm$ and the motion of this $a^2 y \times Mm = a^2 dx$, on account of the proportion $\frac{arc\ Mm}{mH(or\ dx)} = \frac{CM(i.e.1)}{PM(i.e.\ y)}$. Whereby the motion of the part *FM*, of the ring is $a^2 x$, and by making $x = 1$, the motion of the quadrant of the ring $= a^2$; and the motion of the whole ring nearer the axis is $= 4a^2$. Therefore the motion of the ring about the axis of the cylinder to its motion about the nearer axis is as $a^2.2\pi$ to $4a^2$, or as 2π to 4 ; that is, as the circumference of the circle 2π to twice the diameter 4. On account of which we have the ratios :

the motion of the cylinder to the motion of the sphere is as 16 to 3π ;
the motion of the ring around the axis of the cylinder is to the motion of the cylinder as m to $\frac{4\pi}{3}$;

and the motion of the ring around the nearer axis is to its motion around the axis of the cylinder as 4 to 2π.
Whereby, by the composition of the ratios and from the equation, the motion of the sphere around the closer axis is to the motion of the ring as $(2\pi)^3$ to $64m$. But $\frac{8\pi^3}{64m} = \frac{\frac{4\pi}{3}\times\frac{12\pi^2}{16}}{8\times m}$, and $\frac{4\pi}{3}$ is the quantity of matter in the earth of radius 1; *m*, the quantity of matter in the ring; $\frac{12\pi^2}{16}$ is the sum of three squares from the arc of the circle *AFB*, and 8 is the sum of two squares from the diameter *AB*. Whereby the motion of the whole earth around the axis now described, composed from the motions of all the particles, will be to the motion of the aforesaid ring around the same axis, in the ratio that is composed from the ratio of the matter of the earth to the matter in the ring, and from the ratio of three squares from the arc of the quadrant of some circle to two squares from the diameter, that is, in the ratio of the matter to the matter and the number 925275 to the number 1000000 or $\frac{3\pi^2}{32}$, with the ratio of the diameter to the periphery taken approximately as 1 to 3.141 approximately. Q.e.d.]

HYPOTHESIS II.

If the aforesaid ring, with all of the remaining earth removed, alone may be carried in orbit around the sun in its annual motion, and meanwhile revolving in a diurnal motion around its axis inclined to the plane of the ecliptic at an angle of $23\frac{1}{2}$ *degrees : likewise the motion of the equinoctial points shall be the same, whether the annulus shall be fluid or constructed from rigid and firm matter.*

PROPOSITION XXXIX. PROBLEM XX.
To find the precession of the equinoxes.

The mean hourly motion of the lunar nodes in a circular orbit, when the nodes are in quadrature, was $16''.35'''.16^{\text{iv}}.36^{\text{v}}$, and the mean hourly motion of the nodes in such an orbit is half of this $8''.17'''.38^{\text{iv}}.18^{\text{v}}$ (on account of the reasons explained above); and in a sidereal year in total it shall be $20^{\text{gr}}.11'.\,46''$. Therefore because in the preceding the lunar nodes in such an orbit put such an annual amount in place $20^{\text{gr}}.11'.\,46''$; and if there were several moons, the motion of the nodes of each (by Corol. 16. Prop. LXVI. Book I.) would become as the periodic times ; if a moon were revolving next to the surface of the earth in the space of a sidereal day, the annual motion of the nodes would be to $20^{\text{gr}}.11'.\,46''$ as the sidereal day of 23. 56' hours to the periodic time of the moon of 27 days, 7 hours, 43minutes ; that is, as 1436 to 39343. And the ratio of the nodes of a ring of moons around the earth is the same ; whether these moons do not affect each other, or they merge together and form a continuous ring, or in fact that ring may be rendered rigid and inflexible.

Therefore we may consider that this ring as a quantity of matter that shall be equal to all the earth *PapAPepE* which is above the globe *Pape*; (See the figure for Lemma II) and because such a globe is to the earth above as aC^2 to $AC^2 - aC^2$, that is (since the smaller radius of the earth *PC* or *aC* shall be to the greater radius *AC* as 229 to 230) as 52441 to 459; if this ring should surround the earth around the equator and each were revolved around the diameter of the ring, the motion of the ring would be to the motion of the interior globe (by Lem. III of this part) as 459 to 51441 and 1000000 to 925175 conjointly, that is, as 4590 to 485223 ; and thus the motion of the ring would be to the sum of the motions of the ring and the globe, as 4590 to 489813. From which if the ring were attached to the globe, and its motion, by which nodes of this or the equinoctial points were regressing, since it may share with the globe: the motion which will remain in the ring will be to the first motion, as 4590 to 489813; and therefore the motion of all the equinoctial points will be diminished in the same ratio. Therefore the annual equinoctial motion of all the points of the body composed from the ring and the globe will be to the motion $20^{\text{gr}}.11'.\,46''$, as 1436 to 39343 and 4590 to 489813 conjointly, that is, as 100 to 292369. But the forces by which the lunar nodes (as I have set out above) and thus by which the equinoctial points of the ring are regressing (that is the forces $3IT$ in the figure to Prop. 30) are by the individual particles as the distances of the particles from the plane *QR*, and by these forces these particles flee from that plane; and therefore (by Lem. II.) if the matter of the ring may be spread out over the whole globe in the manner of the figure *PapAPepE* above that part of the earth put in place, the force and the total turning effect of all the particles rotating around the equator of the earth in some manner, and thus to the movement of the equinoctial points, will emerge smaller than in that first ratio of 2 to 5. And thus the annual regression of the equinoxes now will be to $20^{\text{gr}}.11'.\,46''$ as 10 to 73092 : and hence it becomes $9''.\,56'''.\,50^{\text{iv}}$.

Further this motion on account of the inclination of the plane of the equator to the plane of the ecliptic is to be diminished, and that in the ratio of the sine 91706 (which is the complement of the sine of $23\frac{1}{2}$ degrees) to the radius 100000. On which account this motion now becomes $9''.\,7'''.\,20^{\text{iv}}$. This is the annual precession of the equinoxes arising from the force of the sun.

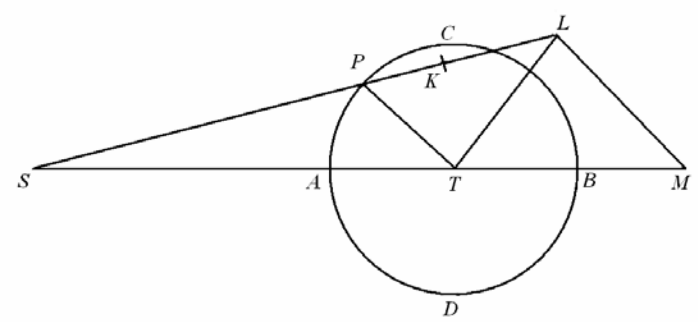
Moreover the force of the moon requiring to move the sea was to the force of the sun as 4,4815 to 1 approximately. And the force of the moon required to move the equinoxes is in the same proportion to the force of the sun. And thence the annual precession of the equinoxes by the force of the moon arises : $40''.\,52'''.\,52^{\text{iv}}$, and thus the total annual precession arising from both forces will be $50''.\,00'''.\,12^{\text{iv}}$. And this motion agrees with the phenomena. For the precession of the equinoxes from astronomical observations is annually a little more or less than fifty minutes.

If the height of the earth at the equator should exceed that at the poles, by more than $17\frac{1}{6}$ miles, its matter will be rarer at the circumference than at the centre : and the precession of the equinoxes on account of that height will be increased, on account of the rareness it must be diminished.

Now we have described the system of the sun, of the earth, moon and planets; it remains that something may be added concerning comets.

PROPOSITIO XXXVI. PROBLEMA XVII.
Invenire vim solis ad mare movendum.

Solis vis *ML* seu *PT*, in quadraturis lunaribus, ad perturbandos motus lunares erat (per Prop. XXV. huius) ad vim gravitatis apud nos, ut 1 ad 638092,6. Et vis $TM - LM$ seu $2PK$ in syzygiis lunaribus est duplo major. Hae autem vires, si descendatur ad superficiem



terrae, diminuuntur in ratione distantiarum a centro terrae, id est, in ratione $60\frac{1}{2}$ ad 1; ideoque vis prior in superficie terrae est ad vim gravitatis ut 1 ad 38604600. Hac vi mare deprimitur in locis, quae 90 gradibus distant a sole. Vi altera, quae duplo major est, mare elevatur & sub sole & in regione soli opposita. Summa virium est ad vim gravitatis ut 1 ad 12868200. Et quoniam vis eadem eundem ciet motum, sive ea deprimat aquam in regionibus quae 90 gradibus distant a sole, sive elevet eandem in regionibus sub sole & soli oppositis, haec summa erit tota solis vis ad mare agitandum ; & eundem habebit effectum, ac si tota in regionibus sub sole & soli oppositis mare elevaret, in regionibus autem quae 90 gradibus distant a sole nil ageret.

Haec est vis solis ad mare ciendum in loco quovis data, ubi sol tam in vertice loci versatur quam in mediocri sua distantia a terra. In aliis solis positionibus vis ad mare attollendum est ut sinus versus duplae altitudinis solis supra horizontem loci directe & cubus distantia: solis a terra inverse.

Corol. Cum vis centrifuga partium terrae a diurno terrae motu oriunda, quae est ad vim gravitatis ut 1 ad 289, efficiat ut altitudo aquae sub aequatore superet eius altitudinem sub polis mensura pedum *Parisiensium* 85472, ut supra in Prop. XIX; vis solaris de qua egimus, cum sit ad vim gravitatis ut 1 ad 12868200, atque ideo ad vim illam centrifugam ut 289 ad 12868200 seu 1 ad 44527, efficiet ut altitudo aquae in regionibus sub sole & soli oppositis superet altitudinem eius in locis, quae 90 gradibus distant a sole, mensura tantum pedis unius *Parisiensis* & digitorum undecim cum tricesima parte digiti. Est enim haec mensura ad mensuram pedum 85472 ut 1 ad 44527.

PROPOSITIO XXXVII. PROBLEMA XVIII.

Invenire vim lunae ad mare movendum.

Vis lunae ad mare movendum colligenda est ex eius proportione ad vim solis, & haec proportio colligenda est ex proportione motuum maris, qui ab his viribus oriuntur. Ante ostium fluvii *Avonae* ad lapidem tertium infra *Bristolium*, tempore verno & autumnali totus aquae ascensus in conjunctione & oppositione luminarium, observante *Samuele Sturmio*, est pedum plus minus 45, in quadraturis autem est pedum tantum 25. Altitudo prior ex summa virium, posterior ex earundem differentia oritur. Solis igitur & lunae in aequatore versantium & mediocriter a terra distantium sunto vires S & L, & erit $L+S$ ad $L-S$ ut 45 ad 25, seu 9 ad 5.

In portu *Plymuthi* aestus maris ex observatione *Samuele Colepressi* ad pedes plus minus sexdecim altitudine mediocri attollitur, ac tempore verno & autumnali altitudo aestus in syzygiis superare potest altitudinem eius in quadraturis pedibus plus septem vel octo. Si maxima harum altitudinum differentia sit pedum novem, erit $L+S$ ad $L-S$ ut $20\frac{1}{2}$ ad $11\frac{1}{2}$ seu 41 ad 23. Quae proportio satis congruit cum priore. Ob magnitudinem aestus in portu *Bristoliae*, observationibus *Sturmii* magis fidendum esse videtur, ideoque donec aliquid certius constiterit, proportionem 9 ad 5 usurpabimus.

Caeterum ob aquarum reciprocos motus, aestus maximi non incidunt in ipsas luminarium syzygias, sed sunt tertii a syzygiis ut dictum fuit, seu proxime sequuntur tertium lunae post syzygias appulsum ad meridianum loci, vel potius (ut a *Sturmio* notatur) sunt tertii post diem novilunii vel plenilunii, seu post horam a novilunio vel plenilunio plus minus duodecimam, ideoque incidunt in horam a novilunio vel plenilunio plus minus quadragesimam tertiam. Incidunt vero in hoc portu in horam septimam circiter ab appulsu lunae ad meridianum loci; ideoque proxime sequuntur appulsum lunae ad meridianum, ubi luna distat a sole vel ab oppositione solis gradibus plus minus octodecim vel novemdecim in consequentia. Aestas & hyems maxime vigent, non in ipsis solstitiis, sed ubi sol distat a solstitiis decima circiter parte totius circuitus, seu gradibus plus minus 36 vel 37. Et similiter maximus aestus maris oritur ab appulsu lunae ad meridianum loci, ubi luna distat a sole decima circiter parte motus totius ab aestu ad aestum. Sit distantia illa graduum plus minus $18\frac{1}{2}$. Et vis solis in hac distantia lunae a syzygiis & quadraturis, minor erit ad augendum & ad minuendum motum maris a vi lunae oriundum, quam in ipsis syzygiis & quadraturis, in ratione radii ad sinum complementi distantiae huius duplicatae seu anguli graduum 37, hoc est. in ratione 10000000 ad 7986355. Ideoque in analogia superiore pro S scribi debet 0,7986355 S.

Sed & vis lunae in quadraturis, ob declinationem lunae ab aequatore, diminui debet. Nam luna in quadraturis, vel potius in gradu $18\frac{1}{2}$ post quadraturas, in declinatione graduum plus minus 22.13' versatur. Et luminaris ab aequatore declinantis vis ad mare movendum vendum diminuitur in duplicata ratione sinus complementi declinationis quamproxime. Et propterea vis lunae in his quadraturis est tantum 0,857327 L. Est igitur $L+0{,}7986355S$ ad 0,857032.7 L $-$ 0,79863558 ut 9 ad 5.

Praeterea diametri orbis, in quo luna sine eccentricitate moveri deberet, sunt ad invicem ut 69 ad 70; ideoque distantia lunae a terra in syzygiis est ad distantiam eius in quadraturis ut 69 ad 70, caeteris paribus. Et distantiae eius in gradu $18\frac{1}{2}$ a syzygiis, ubi aestus maximus generatur, & in gradu $18\frac{1}{2}$ a quadraturis, ubi aestus minimus generatur, sunt ad mediocrem eius distantiam ut 69,098747 & 69,897345 ad $69\frac{1}{2}$. Vires autem lunae ad mare movendum sunt in triplicara ratione distantiarum inverse, ideoque vires in maxima & minima harum distantiarum sunt ad vim in mediocri distantia ut 0,9830427 & 1,017522 ad 1. Unde fit $1{,}017522L\ +0{,}7986355S$ ad $0{,}9830427\times 0{,}8570327L - 0{,}7986355S$ ut 9 ad 5. Et S ad L ut 1 ad 4,4815. Itaque cum vis solis sit ad vim gravitatis ut 1 ad 12868200, vis lunae erit ad vim gravitatis ut 1 ad 2871400.

Corol. 1. Cum aqua vi solis agitata ascendat ad altitudinem pedis unius & undecim digitorum cum tricesima parte digiti, eadem vi lunae ascendet ad altitudinem octo pedum & digitorum $7\frac{5}{22}$ & vi utraque ad altitudinem pedum decem cum semisse, & ubi luna est in perigaeo ad altitudinem pedum duodecim cum semisse & ultra, praesertim ubi aestus ventis spirantibus adjuvatur. Tanta autem vis ad omnes maris motus excitandos abunde sufficit, & quantitati motuum probe respondet. Nam in maribus quae ab oriente in occidentem late patent, uti in mari *Pacifico*, & maris *Atlantici* & *Aethiopici* partibus extra tropicos, aqua attolli solet ad altitudinem pedum sex, novem, duodecim vel quindecim. In mari autem *Pacifico*, quod profundius est & latius patet, aestus dicuntur esse majores quam in *Atlantico* & *Aethiopico*. Etenim ut plenus sit aestus, latitudo maris ab oriente in occidentem non minor esse debet quam graduum nonaginta. In mari *Aethiopico* ascensus aquae intra tropicos minor est quam in zonis temperatis, propter angustiam maris inter *Africam* & australem partem *Americae*. In medio mari aqua nequit ascendere nisi ad littus utrumque & orientale & occidentale simul descendat: cum tamen vicibus alternis ad littora illa in maribus nostris angustis descendere debeat. Ea de causa fluxus & refluxus in insulis, quae a littoribus longissime absunt, perexiguus esset solet. In portubus quibusdam, ubi aqua cum impetu magno per loca vadosa, ad sinus alternis vicibus implendos & evacuandos, influere & effluere cogitur, fluxus & refluxus debent esse solito majores, uti ad *Plymuthum* & pontem *Chepstowae* in *Anglia*; ad montes *S. Michaelis* & urbem *Abrincatuorum* (vulgo *Auranchies*) in *Normannia*; ad *Cambaiam* & *Pegu* in *India* orientali. His in locis mare, magna cum velocitate accedendo & recedendo, littora nunc inundat nunc arida relinquit ad multa millaria. Neque impetus influendi & remeandi prius frangi potest, quam aqua attollitur vel deprimitur ad pedes 30, 40, vel 50 & amplius. Et par est ratio fretorum oblongorum & vadosorum, uti *Magellanici* & eius quo *Anglia* circundatur. Aestus in huiusmodi portubus & fretis per impetum cursus & recursus supra modum augetur. Ad littora vero quae descensu praecipiti ad mare profundum & apertum spectant, ubi aqua sine impetu effuendi & remeandi attolli & subsidere potest, magnitudo aestus respondet viribus solis & lunae.

Corol. 2. Cum vis lunae ad mare movendum sit ad vim gravitatis ut 1 ad 2871400, perspicuum est quod vis illa sit longe minor quam quae vel in experimentis pendulorum, vel in staticis aut hydrostaticis quibuscunque sentiri possit. In aestu solo marino haec vis sensibilem edit effectum.

Corol. 3. Quoniam vis lunae ad mare movendum est ad solis vim consimilem ut 4,4815 ad 1, & vires illae (per corol. 14 Prop. LXVI, Lib. I.) sunt ut densitates corporum lunae & solis & cubi diametrorum apparentium conjunctim; densitas lunae erit ad densitatem solis ut 4,4815 ad 1 directe, & cubus diametri lunae ad cubum diametri solis inverse: id est (cum diametri mediocres apparentes lunae & solis sint $31'.\,16''\frac{1}{2}$, & $32'.12''$) ut 4891 ad 1000. Densitas autem solis erat ad densitatem terrae ut 1000 ad 4000; & propterea densitas lunae est ad densitatem terrae ut 4891 ad 4000 seu 11 ad 9. Est igitur corpus lunae densius & magis terrestre quam terra nostra.

Corol. 4. Et cum vera diameter lunae ex observationibus astronomicis sit ad veram diametrum terrae ut 100 ad 365 erit massa lunae ad massam terrae ut 1 ad 39,788.

Corol. 5. Et gravitas acceleratrix in superficie lunae erit quasi triplo minor quam gravitas acceleratrix in superficie terrae.

*Corol.*6. Et distantia centri lunae a centro terrae erit ad distantiam centri lunae a communi gravitatis centro terrae & lunae, ut 40,788 ad 39,788.

Corol. 7. Et mediocris distantia centri lunae a centro terrae in octantibus lunae erit semidiametrorum maximarum terrae $60\frac{2}{5}$ quamproxime. Nam terrae semidiameter maxima fuit pedum *Parisiensium* 19658600, & mediocris distantia centrorum terrae & lunae, ex huiusmodi diametris $60\frac{2}{5}$ constans, aequalis est pedibus 1187379440. Et haec distantia (per corollarium superius) est ad distantiam centri lunae a communi gravitatis centro terrae & lunae, ut 40,788 ad 39,788: ideoque distantia posterior est pedum 1158268534. Et cum luna revolvatur respectu fixarum, diebus 27, horis 7, & minutis primis $43\frac{4}{9}$; sinus versus anguli, quem luna tempore minuti unius primi describit, est 12752341, existente radio 1000,000000,000000. Et ut radius est ad hunc sinum versum, ita sunt pedes 1158268534 ad pedes 14,7706353. Luna igitur vi illa, qua retinetur in orbe, cadendo in terram, tempore minuti unius primi describet pedes 14,7706353. Et augendo hanc vim in ratione $178\frac{29}{40}$ ad $177\frac{29}{40}$, habebitur vis tota gravitatis in orbe lunae per Corol. Prop. III. Et hac vi luna cadendo tempore minuti unius primi describet pedes 14,8538067. Et ad sexagesimam partem distantia lunae a centro terrae, id est ad distantiam pedum 197896573 a centro terrae, corpus grave tempore minuti unius secundi cadendo describet etiam pedes 14,853067. Ideoque ad distantiam pedum 19615800, quae sunt terrae semidiameter mediocris, grave cadendo describet pedes 15,11175, seu pedes 15, dig. 1, & lin. $4\frac{1}{11}$. Hic erit descensus corporum in latitudine graduum 45. Et per tabulam praecedentem in Prop. XX. descriptam, descensus erit paulo major in latitudine *Lutetitae Parisiorum* existente excessu quasi $\frac{2}{3}$ partium lineae. Gravia igitur per hoc computum in latitudine *Lutetitae* cadendo in vacuo describent tempore unius secundi pedes *Parisienses* 15, dig. 1, & lin. $4\frac{25}{33}$ circiter. Et si gravitas minuatur auferendo vim centrifugam, quae oritur a motu diurno terrae in illa latitudine; gravia ibi cadendo describent tempore minuti unius secundi pedes 15, dig. 1, & lin. $1\frac{1}{2}$. Et hac velocitate gravia cadere in latitudine *Lutetitae* supra ostensum est ad Prop. IV, & XIX.

*Corol.*8. Distantia mediocris centrorum terrae & lunae in syzygiis lunae est sexaginta semidiametrorum maximarum terrae, dempta tricesima parte semidiametri circiter. Et in quadraturis lunae distantia mediocris eorundem centrorum est $60\frac{5}{6}$ semidiametrorum terrae. Nam hae duae distantiae sunt ad distantiam mediocrem lunae in octantibus ut 69 & 70 ad $69\frac{1}{2}$ per Prop. XXVIII.

*Corol.*9. Distantia mediocris centrorum terrae & lunae in syzygiis, lunae est sexaginta semidiametrorum mediocrium terrae cum decima parte semidiametri. Et in quadraturis lunae distantia mediocris eorundem centrorum est sexaginta & unius semidiametrorum mediocrium terrae, dempta tricesima parte semidiametri.

*Corol.*10. In syzygiis lunae parallaxis eius horizontalis mediocris in latitudinibus graduum $0, 30, 38, 45, 52, 60, 90$, est
$57'.20'', 57'.16'', 57'.14'', 57'.12'', 57'.10'', 57'.8'', 57'.4''$ respective.
In his computationibus attractionem magneticam terrae non consideravi, cuius utique quantitas perparva est & ignoratur. Siquando vero haec attractio investigati poterit, & mensurae graduum in meridiano, ac longitudines pendulorum isochronorum in diversis parallelis, legesque motuum maris, & parallaxis lunae cum diametris apparentibus solis & lunae ex phaenomenis accuratius determinare fuerint: licebit calculum hunc omnem accuratius repetere.

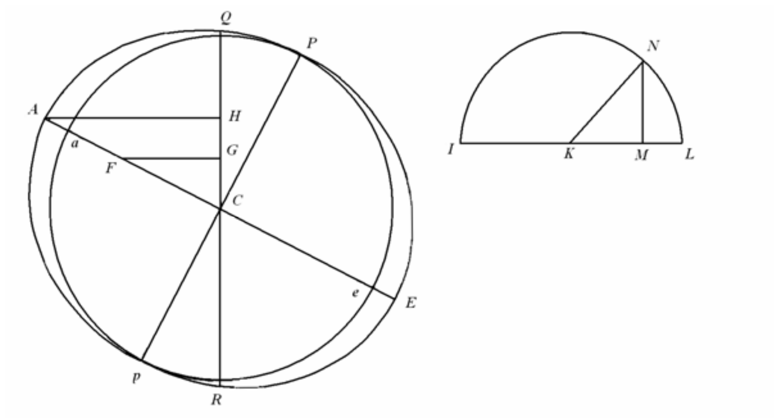
PROPOSITIO XXXVIII. PROBLEMA XIX.
Invenire figuram corporis lunae.

Si corpus lunare fluidum esset ad instar maris nostri, vis terrae ad fluidum illud in partibus & citimis & ultimis elevandum esset ad vim lunae, qua mare nostrum in partibus & sub luna & lunae oppositis attollitur, ut gravitas acceleratrix lunae in terram ad gravitatem acceleratricem terrae in lunam, & diameter lunae ad diametrum terrae conjunctim; id est, ut 39,788 ad 1 & 100 ad 365, seu 1081 ad 100. Unde cum mare nostrum vi lunae attollatur ad pedes $8\frac{3}{5}$, fluidum lunare vi terrae attolli deberet ad pedes 93. Eaque de causa figura lunae sphaerois esset, cuius maxima diameter producta transiret per centrum terrae, & superaret diametros perpendiculares excessu pedum 186. Talem igitur figuram luna affectat, eamque sub initio induere debuit. *Q.E.I.*

Corol. Inde vero sit ut eadem semper lunae facies in terram obvertatur. In alio enim situ corpus lunare quiescere non potest, sed ad hunc situm oscillando semper redibit. Attamen oscillationes, ob parvitatem virium agitantium, essent longe tardissimae: adeo ut facies illa, quae terram semper respicere deberet, possit alterum orbis lunaris umbilicum (ob rationem in Prop. XVII allatam) respicere, neque statim abinde retrahi & in terram converti.

LEMMA I.

Si APEp terram designet uniformiter densam, centroque C & polis P, p & aequatore AE delineatam; & si centro C radio CP describi intelligatur sphaera Pape; sit autem QR



planum, cui recta a centro solis ad centrum terrae ducta normaliter insistit; & terrae totius exterioris PapAPepE, quae sphaera modo descripta altior est, particulae singulae conentur recedere hinc inde a plano QR, sitque conatus particulae cuiusque ut eiusdem distantia a plano: Dico primo, quod tota particularum omnium in aequatoris circulo A E, extra globum uniformiter per totum circuitum in morem annuli dispositarum, vis & efficacia ad terram circum centrum eius rotandam, sit ad totam particularum totidem in aequatoris puncto A, quod a plano QR maxime distat, consistentium vim & efficaciam, ad terram consimili motu circular; circum centrum eius movendam, ut unum ad duo. Et motus iste circularis circum axem, in communi sectione atequatoris & plani QR jacentem, peragetur.

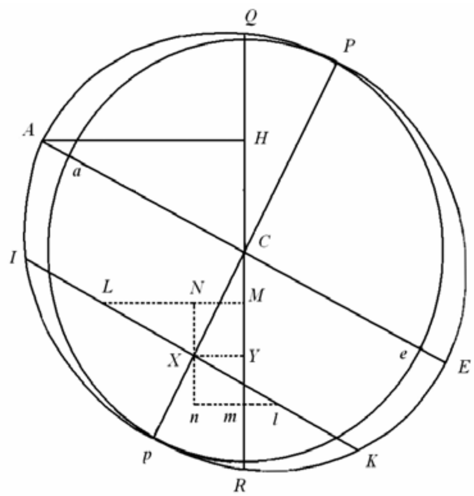
Nam centro *K* diametro *IL* describatur semicirculus *INLK*. Dividi intelligatur semicircumferentia *INL* in partes innumeras aequales, & a partibus singulis *N* ad diametrum *IL* demittantur sinus *NM*. Et summa quadratorum ex sinibus omnibus *NM* aequalis erit summae quadratorum ex sinibus *KM*, & summa utraque aequalis erit summae quadratorum ex totidem semidiametris *KN*; ideoque summa quadratorum ex omnibus *NM* erit duplo minor quam summa quadratorum ex totidem semidiametris *KN*. Jam dividatur perimeter circuli *AE* in particulas totidem aequales, & ab earum unaquaque *F* ad planum *QR* demittatur perpendiculum *FG*, ut & a puncta *A* perpendiculum *AH*. Et vis, qua particula *F* recedit a plano *QR*, erit ut perpendiculum illud *FG* per hypothesin, & haec vis ducta in distantiam *CG* erit efficacia particulae *F* ad terram circum centrum eius convertendam. Ideoque efficacia particulae in loco *F*, erit ad efficaciam particulae in loco *A*, ut $FG \times GC$ ad $AH \times HC$, hoc est, ut FC^2 ad AC^2; & propterea efficacia tota particularum omnium in locis suis *F* erit ad efficaciam particularum totidem in loco *A*, ut summa omnium FC^2 ad summam totidem AC^2, hoc est (per jam demonstrata) ut unum ad duo.
Q. E. D.

Et quoniam particulae agunt recedendo perpendiculariter a plano *QR*, idque aequaliter ab utraque parte huius plani: eaedem convertent circumferentiam circuli aequatoris, eique inhaerentem terram, circum axem tam in plano illo *QR* quam in plano aequatoris jacentem.

LEMMA II.

Iisdem positis: dico secundo quod vis & efficacia tota particularum omnium extra globum undique sitarum, ad terram circum axem eundem rotandam, sit ad vim totam particularum totidem, in aequatoris circulo AE uniformiter per totum circuitum in morem annuli dispositarum, ad terram consimili motu circular; movendam, ut duo ad quinque.

Sit enim *IK* circulus quilibet minor aequatori *AE* parallelus, sintque *L*, *l* particulae duae quaevis aequales in hoc circulo extra globum *Pape* sitae. Et si in planum *QR*, quod radio in solem ducto perpendiculare est, demittantur perpendicula *LM*, *lm*: vires totae, quibus particulae illae fugiunt planum *QR*, proportionales erunt perpendiculis illis *LM*, *lm*. Sit autem recta *Ll* plano *Pape* parallela & bisecetur eadem in *X* & per punctum X agatur *Nn*, quae parallela sit plano *QR* & perpendiculis *LM*, *lm* occurrat in *N* ac *n*, & in planum *QR* demittatur perpendiculum *XY*. Et particularum *L* & *l* vires contrariae, ad terram in



contrarias partes rotandam, sunt ut $LM \times MC$ & $lm \times mC$, hoc est, ut $LN \times MC + NM \times MC$ & $ln \times mC - nm \times mC$, seu $LN \times MC + NM \times MC$ & $LN \times mC - NM \times mC$: & harum differentia $LN \times Mm - NM \times \overline{MC + mC}$ est vis particularum ambarum simul sumptarum ad terram rotandam. Huius differentia: pars affirmativa $LN \times Mm$ seu $2LN \times NX$ est ad particularum duarum eiusdem magnitudinis in *A* consistentium vim $2AH \times HC$, ut LX^2 ad AC^2. Et pars negativa $NM \times \overline{MC + mC}$ seu $2\ XY \times CY$ ad particularum earundem in *A* consistentium vim $2AH \times HC$, ut CX^2ad AC^2. Ac proinde partium differentia, id est, particularum duarum *L* & *l* simul sumptarum vis ad terram rotandam est ad vim particularum duarum iisdem aequalium & in loco *A* consistentium ad terram itidem rotandam, ut $LX^2 - CX^2$ ad AC^2. Sed si circuli *IK* circumferentia *IK* dividatur in particulas innumeras aequales *L*, erunt omnes LX^2 ad totidem *IX*2 ut 1 ad 2. (per lem.1.) atque ad totidem AC^2, ut IX^2 ad $2AC^2$; &

totidem CX^2 ad totidem A est ut $2CX^2$ ad $2AC^2$. Quare vires coniunctas particularum omnium in circuitu circuli IK sunt ad vires conjunctas particularum totidem in loco A, ut $IX^2 - 2CX^2$ ad $2AC^2$: & propterea (per lem. 1.) ad vires conjunctas particularum totidem in circuitu circuli AE, ut $IX^2 - 2CX^2$ ad AC^2.

Jam vero si sphaerae diameter Pp dividatur in partes innumeras aequales, quibus insistant circuli totidem IK; materia in perimetros circuli cuiusque IK erit ut IX^2: ideoque vis materiae illius ad terram rotandam, erit ut IX^2 in $IX^2 - 2CX^2$. Et vis materiae ejusdem, si in circuli AE perimetro consisteret, esset ut IX^2 in AC^2. Et propterea vis particularum omnium materiae totius, extra globum in perimetris circulorum omnium consistentis, est ad vim particularum totidem in perimetro circuli maximi AE consistentis, ut omnia IX^2 in $IX^2 - 2CX^2$ ad totidem IX^2 in AC^2, hoc est, ut omnia $AC^2 - CX^2$ in $AC^2 - 3CX^2$ ad totidem $AC^2 - CX^2$ in AC^2, id est, ut omnia $AC^4 - 4AC^2 \times CX^2 + 3CX^4$ ad totidem $AC^4 - AC^2 \times CX^2$, hoc est, ut tota quantitas fluens, cuius fluxio est $AC^4 - AC^2 \times CX^2 + 3CX^4$, ad totam quantitatem fluentem, cuius fluxio est $AC^4 - AC^2 \times CX^2$; ac proinde per methodum fluxionum, ut $AC^4 \times CX - \frac{4}{3}AC^2 \times CX^3 + \frac{3}{5}CX^5$ ad $AC^4 \times CX - \frac{1}{3}AC^2 \times CX^3$, id est, si pro CX scribatur tota Cp vel AC, ut $\frac{4}{15}AC^5$ *ad* $\frac{2}{3}AC^5$. hoc est, ut duo ad quinque. *Q.E.D.*

LEMMA III.

Iidem positis: dico tertio quod motus terrae totius circum axem jam ante descriptum, ex motibus particularum omnium compositus, erit ad motum annuli praedicti circum axem eundem in ratione, quae componiiur ex ratione materiae in terra ad materiam in annulo, & ratione trium quadratorum ex arcu quadrantali circuli cuiuscunque ad duo quadrata ex diametro; id est, in ratione materiae ad materiam & numeri 925275 ad numerum 1000000.

Est enim motus cylindri circum axem suum immotum revolventis ad motum sphaerae inscriptae & simul revolventis, ut quaelibet quatuor aequalia quadrata ad tres ex circulis sibi inscriptis; & motus cylindri ad motum annuli tenuissimi, sphaeram & cylindrum ad communem eorum contactum ambientis, ut duplum materiae in cylindro ad triplum materiae in annulo; & annuli motus iste circum axem cylindri uniformiter continuatus, ad eiusdem motum uniformem circum diametrum propriam, eodem tempore periodico factum, ut cirumferentia circuli ad duplum diametri.

HYPOTHESIS II.

Si annulus praedictus terra omni reliqua sublata, solus in orbe terrae, motu annuo circa solem ferretur, & interea circa axem suum, ad planum eclipticae in angulo graduum $23\frac{1}{2}$ *inclinatum, motu diurno revolveretur : idem foret motus punctarum aequinoctialium, sive annulus iste fluidus esset, sive is ex materia rigida & firma constaret.*

PROPOSITIO XXXIX. PROBLEMA XX.
Invenire praecessionem aequinoctiorum.

Motus mediocris horarius nodorum lunae in orbe circulari, ubi nodi sunt in quadraturis, erat $16''.35'''.16^{iv}.36^{v}$, & huius dimidium $8''.17'''.38^{iv}.18^{v}$. (ob rationes supra explicatas) est motus medius horarius nodorum in tali orbe; sitque anno toto sidereo $20^{gr}.11'.46''$. Quoniam igitur nodi lunae in tali orbe conficerent annuatim $20^{gr}.11'.46''$ in anticedentia; & si plures essent lunae, motus nodorum cuiusque (per Corol. 16. Prop. LXVI. Lib. I.) forent ut tempora periodica; si luna spatio diei siderei iuxta superficiem terrae revolveretur, motus annuus nodorum foret ad $20^{gr}.11'.46''$ ut dies sidereus horarum 23. 56' ad tempus periodicum lunae dierum 27. 7 hor. 43'; id est, ut 1436 ad 39343. Et par est ratio nodorum annuli lunarum terram ambientis; sive lunae illae se mutuo non contingant, sive liquescant & in annulum continuum formentur, sive denique annulus ille rigescat & inflexibilis reddatur.

Fingamus igitur quod annulus iste, quoad quantitatem materiae, aequalis sit terrae omni *PapAPepE* quae globo *Pape* superior est; (Vid.fig. pag. 474,) & quoniam globus iste est ad terram illam superiorem ut aC^2 ad $AC^2 - aC^2$, id est (cum terrae semidiameter minor *PC* vel *aC* sit ad semidiametrum majorem *AC* ut 229 ad 230) ut 52441 ad 459; si annulus iste terram secundum aequatorem cingeret & uterque simul circa diametrum annuli revolveretur, motus annuli esset ad motum globi interioris (per huius Lem. III.) ut 459 ad 51441 & 1000000 ad 925175 conjunctim, hoc est, ut 4590 ad 485223 ; ideoque motus annuli esset ad summam motuum annuli ac globi, ut 4590 ad 489813. Unde si annulus globo adhaereat, & motum suum, quo ipsius nodi seu puncta aequinoctialia regrediuntur, cum globo communicet: motus qui restabit in annulo erit ad ipsius motum priorem, ut 4590 ad 489813; & propterea motus punctorum aequinoctialium diminuetur in eadem ratione. Erit igitur motus annuus punctorum aequinoctialium corporis ex annulo & globo compositi ad motum $20^{gr}.11'.46''$, ut 1436 ad 39343 & 4590 ad 489813 coniunctim, id est, ut 100 ad 292369. Vires autem quibus nodi lunarum (ut supra explicui) atque ideo quibus puncta aequinoctialia annuli regrediuntur (id est vires 3*IT* in fig. pag. 437 & 438) sunt in singulis particulis ut distantiae particularum a plano *QR*, & his viribus particulae illae planum fugiunt; & propterea (per Lem. II.) si materia annuli per totam globi superficiem in morem figurae *PapAPepE* ad superiorem illam terrae partem constituendam spargeretur, vis & efficacia tota particularum omnium ad terram circa quamvis aequatoris diametrum rotandam, atque ideo ad movenda puncta aequinoctialia, evaderet minor quam prius in ratione 2 ad 5. ldeoque annuus aequinoctiorum regressus jam esset ad $20^{gr}.11'.46''$. ut 10 ad 73092 : ac proinde fieret $9''.56'''.50^{iv}$.

Caeterum hic motus ob inclinationem plani aequatoris ad planum eclipticae minuendus est, idque in ratione sinus 91706 (qui sinus est complementi graduum $23\frac{1}{2}$) ad radium 100000. Qua ratione motus iste jam fiet $9''.7'''.20^{iv}$. Haec est annua praecessio aequinoctiorum a vi solis oriunda.

Vis autem lunae ad mare movendum erat ad vim solis, ut 4,4815 ad 1 circiter. Et vis lunae ad aequinoctia movenda est ad vim solis in eadem proportione. Indeque prodit annua aequinoctiorum praecessio a vi luna: oriunda $40''.\,52'''.\,52^{\text{iv}}$, ac tota praecessio annua a vi utraque oriunda $50''.\,00'''.\,12^{\text{iv}}$. Et hic motus cum phaenomenis congruit. Nam praecessio aequinoctiorum ex observationibus astronimicis est annuatim minutorum secundorum plus minus quinquaginta.

Si altitudo terrae ad aequatorem superet altitudinem eius ad polos, milliaribus pluribus quam $17\frac{1}{6}$, materia eius rarior erit ad circumferentiam quam ad centrum: & praecessio aequinoctiorum ob altitudinem illam augeri, ob raritatem diminui debet.

Descripsimus jam systema solis, terrae, lunae, & planetarum: superest ut de cometis nonnulla adjiciantur.