SECTION VII.

Concerning the motion of fluids & the resistance of projectiles.

PROPOSITION XXXII. THEOREM XXVI.

If two similar systems of bodies may consist of an equal number of particles, and the corresponding particles shall be similar and proportional, with an individual in one system to an individual in the other, and situated similarly to each other, and in turn they may have a given ratio of density to each other, and they may begin to move similarly between each other in proportion to the time (these amongst themselves which are in one system and those amongst themselves which are in the other system), and if those which are in the same system may not touch each other, except at instants of reflection, nor attract each other, nor repel each other, except by accelerative forces which shall be inversely as the corresponding diameters of the particles and directly as the squares of the velocities : I say that these particles of the systems may in proportional times go on moving similarly among themselves.

[In this proposition, Newton sets out his mechanical view of the world, essentially following the geometrical lines of Euclidean geometry : the motions of particles proceed along straight lines in time; groups of particles or bodies may describe geometrical figures under the influence of action at a distance forces ; collisions between particles and bodies also are geometrical in form, in that they have outcomes known from calculation given the in-going boundary conditions ; thus, the whole motion of a system of particles and bodies proceeds like some giant piece of clockwork, for ever; thus, such a system could be traced back to some starting configuration in the past, as Laplace was to observe later. The formulas introduced here in the notes correspond to these in Ch. IV, *B*. & *R*.; these authors have changed the order of presenting Newton's work at this point.]

I say that similar bodies, situated similarly, move similarly amongst themselves in proportional times, the situations of which in turn at the end of these times shall be similar always: for example, the particles of one system may be compared with the corresponding particles of the other system. From which the times shall be proportional, in which the parts of similar and proportional figures will be described by corresponding particles [in the given system]. Therefore if there shall be two systems of this kind, on account of the similitude of the starting motions, they will proceed to move similarly, until finally they may meet each other. For if they are not disturbed by forces, they may be progressing uniformly on right lines by the first law of motion. If they may be mutually disturbed by some forces, and these forces shall be inversely as the diameters of the corresponding particles and directly as the velocity squared, because the situations of the particles are similar and the forces proportional, the whole forces by which the corresponding particles are agitated, composed from individual agitating forces (by the corollary of the second law), will have similar determinations, and likewise as if the forces may be considered acting between the centres of similarly situated particles ; and those total forces in turn will be as the individual composing forces, that is, inversely as the diameters of the corresponding particles and as square of the velocities directly: and therefore the forces act so that the particles go on to describe corresponding figures. Thus these themselves may be had (by Corol.1 & 8, Prop. IV. Book. I.) only if the centres may be at rest. But if the centres may be moving, because on account of the similitude of the translations, the particles of the systems remain in their places ; similar changes may be produced in the figures which the particles describe. Therefore the motions of the corresponding particles are similar until their first meeting, and therefore there are similar meetings and reflections, and from that (by that shown) the motions among themselves again are similar until they next meet, and thus henceforth indefinitely. *Q.E.D.*

*Corol.*1. Hence if any two [large] bodies, which shall be similar and similarly put in place corresponding to the particles of the systems, similarly may begin to move between themselves in proportional times, and the magnitudes of these bodies shall be as the densities and in turn as the magnitudes and densities of the corresponding particles: these bodies may likewise proceed in proportional times and to be moving similarly. For the ratio of each of the greater parts of the system and of the particles is the same.

Corol. 2. And if all similar and similarly placed parts of systems are at rest relative to each other: and two parts of these, which shall be larger than the rest, and may correspond between each other mutually in each system, may begin to move with some motion along similar lines similarly situated : these may excite similar motions in the remaining parts of the systems, and may go on to move similarly between themselves in proportional times ; and thus to describe distances proportional to their own diameters.

PROPOSITION XXXIII. THEOREM XXVII.

With the same in place, I say that the greater parts of the system are resisted in the ratio composed from the square of their velocities and the square ratio of the diameters and in the ratio of the densities of the parts of the system.

For the resistance arises partially from the centripetal or centrifugal forces by which the particles of the systems mutually disturb each other, partially from the coming together and reflections of the particles, and of the greater parts [present in the systems]. Moreover, the resistances of the first kind in turn shall be as the whole motive forces from which they arise, that is, the accelerations and quantities of matter in the corresponding parts ; that is (by hypothesis) as the squares of the velocities directly and inversely as the distances of the corresponding particles, and directly as the quantities of matter in the corresponding parts :
[We may write these resistive forces algebraically, in an obvious notation, in the form: $\frac{v^2}{d} \times d^3 \rho = v^2 \times d^2 \times \rho$.]
and thus since the distances of one system of particles shall be to the corresponding distances of the other particles, as the diameter of the particle or of the parts in the first system to the diameter of the particles, or to the corresponding parts in the other, and the quantities of matter shall be as the densities of the parts and the cubes of the diameters;

the resistances shall be in turn as the squares of the velocities and the squares of the diameters of the parts of the systems. *Q.E.D.*

The resistances of the second kind are as the number of corresponding reflections and the forces taken together. But the number of reflections are in turn as the velocities of the corresponding parts directly, and inversely as the distances between the reflections of these. And the forces of the reflections are as the velocities and the magnitudes and densities of the corresponding parts jointly ; that is, as the velocities and the cubes of the diameters and the densities of the parts. And from all these ratios jointly, the resistance of the corresponding parts are in turn as the squares of the velocities and the squares of the diameters and the densities of the parts jointly. *Q. E. D.*

[In the second case, the resistances can be written as the frequency of the collisions by the momentum change per collision. The frequency is given by $\frac{v}{d}$, where d is the particle separation and v the velocity; the momentum change per collision is proportional to $v\times\rho\times d^3$, giving same result $\frac{v}{d}\times v\times d^3\rho = v^2\times d^2\times\rho$ as above.]

Corol. 1. Therefore if these systems shall be two elastic fluids after the manner of air, and the parts of these may be at rest within themselves : moreover two similar bodies may be projected along certain lines put in place in some manner, similarly put in place among these parts, as long as the magnitude and density shall be proportional, and moreover the accelerative forces, by which the particles of the fluid disturb each other, shall be inversely as the diameters of the projected bodies, and directly as the squares of the velocities: then these bodies will excite motions in the fluids proportional to the times, and describe similar distances with their diameters proportional to these.

Corol. 2. Therefore in the same fluid a swift projectile suffers a resistance, which is in the square ratio of the velocity approximately. For if the forces, by which distant particles mutually agitate each other, may be increased in the square ratio of the velocity, the resistance will be in the same ratio squared accurately ; and thus in a medium, the parts of which in turn are disturbed by no forces with the distances, the resistance is accurately in the square ratio of the velocity. Therefore let there be three mediums *A*, *B*, *C* with similar and equal parts consistently set out regularly along equal distances. The parts of the mediums *A* and *B* may mutually repel each other by forces which shall be in turn as *T* and *V*, and these parts of the medium *C* shall be completely free from forces of this kind. And if four equal bodies *D, E*, *F*, *G* may be moving in these mediums, the first two *D* and *E* in the first two mediums *A* and *B*, and the other two *F* and *G* in the third medium *C*.

[*i.e.* *D* is in medium *A*, *E* is in medium *B*; *F* and *G* are in medium *C*.]

The velocity of the body *D* to the velocity of the body *E*, and the velocity of the body *F* to the velocity of the body *G* shall be in the square root ratio of the forces *T* to the forces *V* [*i.e.* $\frac{\text{vel.of } D}{\text{vel.of } E} = \frac{\text{vel.of } F}{\text{vel.of } G} = \sqrt{\frac{T}{V}}$; recall that the resistances are as the square of the velocities and inversely as the diameters of the particles, which latter are equal in this case, and hence]

: the resistance of the body *D* will be to the resistance of the body *E*, and the resistance of the body *F* to the resistance of the body *G*, in the square ratio of the velocities, and therefore the resistance of the body *D* will be to the resistance of the body *F* as the resistance of the body *E* to the resistance of the body *G*.

[*i.e.* $\frac{\text{res.of } D}{\text{res.of } E} = \frac{\text{res.of } F}{\text{res.of } G} = \frac{T}{V}$ and if $\text{vel.of } D = \text{vel.of } F$ then $\text{vel.of } E = \text{vel.of } G$.]

Let the bodies *D* & *F* have equal velocities, as with the bodies *E* & *G*; and on increasing the velocities of the bodies *D* and *F* in some ratio, and by reducing the forces of the particles of the medium *B* in the same ratio, the medium *B* will approach to the form and condition of the medium *C* as it pleases, and on that account the resistances of the equal bodies and of the bodies moving equally *E* & *G* in these mediums, may always approach to equality, thus so that the difference may emerge finally less than any given amount. Hence since the resistances of the bodies *D* and *F* shall be in turn as the resistances of the bodies *E* and *G*, these similarly approach to the ratio of equality. Therefore the resistances of the bodies *D* and *F* are approximately equal, when they are moving fastest : and therefore since the resistance of the body *F* shall be in the square ratio of the velocity, the resistance of the body *D* will be approximately in the same ratio.
[Thus, if a body is moving much faster than the particles of the medium, they can be considered at rest, and all the collisions occur with particles as if at rest, for which the v^2 formula holds accurately.]
Corol. 3. The resistance of the fastest moving body in some elastic fluid is almost the same as if the parts of the fluid were without their centrifugal forces, and with these not mutually repelling each other : but only if the elastic force of the fluid may arise from the centrifugal forces of the particles, and the velocity shall be so great that the forces have not enough time to act.

Corol. 4. Hence since the resistances of similar and equally swift bodies, in a medium of which the distant parts do not fly apart mutually, shall be as the squares of the diameters ; also the resistances of the fastest and equally speedy bodies in an elastic fluid are as the squares of the diameters approximately.

Corol: 5. And since similar bodies, equal and equally swift, in mediums of the same densities, the particles of which do not mutually fly apart, these particles either shall be more plentiful and smaller, or fewer and larger, may impinge on equal quantities of matter in equal times, may impress an equal quantity of motion, and in turn (by the third law of motion) from the same they may experience an equal reaction, that is, they are resisted equally : it is evident also that the resistances of the same density of elastic fluids shall be approximately the same, when they are moving the quickest, whether that fluid shall consist of grosser particles, or be constituted from the most subtle of all. From the most subtle resistance the speed of the fastest projectiles is not much diminished.

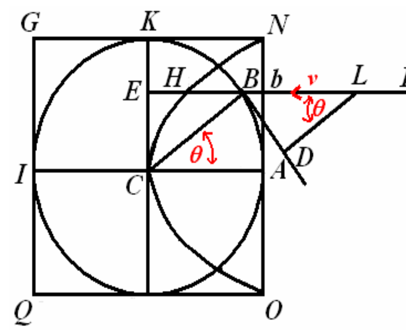
Corol. 6. All these thus may themselves come about in fluids, the elastic force of which has its origin in the centrifugal forces of the particles. But if that may arise otherwise, or as from the expansion of the particles in the manner of wool or the branches of trees, or from any other cause whatever, by which the motions of the particles among themselves are rendered less free, on account of smaller fluidity of the medium, it will be greater than in the above corollaries.

[We now need to consider the effect of the individual shapes of bodies on the resistance. It is necessary to assume that the density of the fluid particles is much rarer than of the solid body, and each fluid particle can deliver its blow independently without interference from other particles of fluid]

PROPOSITION XXXIV. THEOREM XXVIII.

If a sphere and a cylinder be described with equal diameters moving with equal velocity, in a rarefied medium with equal particles, and with these being placed freely at equal distances along the direction of the axis of the cylinder : the resistance of the sphere will be half the resistance of the cylinder.

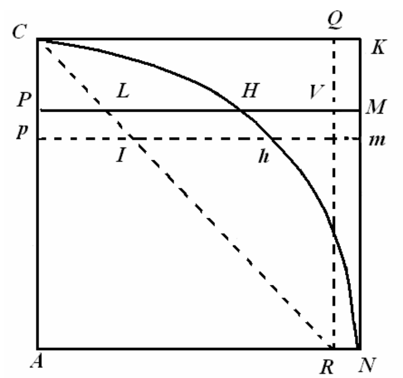
For because the action of the medium on the same body is the same (by corol.5 of the laws) whether the body may be moving in a medium at rest, or the particles of the medium may strike the body at rest with the same velocity : we may consider the body as being at rest, and we may consider by which impetus it may be urged by the moving medium. Therefore *ABKI* may designate the spherical body described with centre *C* and with radius *CA*, and medium particles may be incident on that spherical body with a given velocity, along right lines parallel to *AC* itself : and let *FB* be a right line of this kind. On that *LB* may be taken equal to the radius *CB*, and *BD* may be drawn a tangent to the sphere at *B*. The perpendiculars *BE* and *LD* may be sent to the lines *KC* and *BD*, and the force by which a particle of the medium, by being incident obliquely along the right line *FB*, strikes the sphere at *B*, will be to the force by which the particle may strike the same cylinder *ONGQ* at *b*, with the axis *ACI* described perpendicularly around the sphere, as *LD* to *LB* or as *BE* to *BC*



[*i. e.* as the cosine of the angle θ; thus, the normal component of the force striking the body on a small planar area *A* at that point varies as $Av^2 \times \overline{cos^2\theta}$, where the average value of $cos^2\theta$ must be calculated. Since only the component of this force along the direction of motion is required, the resistance is as $Av^2 \times \overline{cos^3\theta}$. Thus for a spherical surface, the force on an annulus is proportional to $2\pi r^2 \sin\theta \times \overline{cos^3\theta} v^2 d\theta$, which integrates to give $2\pi v^2 r^2 \times \frac{r}{4}$; which is half the value for a great circle; below we show how S. & J. tackle this problem geometrically.]

Again the effectiveness of this force to the movement of the sphere along its direction *FB* or *AC*, is to the same effectiveness of the sphere moving along the direction of its determination, that is, along the direction *BC* by which the sphere may be urged directly as *BE* to *BC*. And with the ratios taken jointly, the effectiveness of a particle on the sphere along the oblique right line *FB*, to the same being moved along the direction of its incidence, is to effectiveness of the same particle incident on the cylinder along the same right line perpendicularly, to that required circle to be moved in the same direction, as *BE* squared to *BC* squared. Whereby if on *bE*, which is perpendicular to the circular base of the cylinder *NAO* and equal to the radius *AC*, *bH* may be taken equal to $\frac{BEquad.}{CB}$; *bH* will be to *bE* as the effectiveness of the force of the particle on the sphere to the effectiveness of the force of the particle on the cylinder. And therefore the solid that may be occupied by all the right lines *bH* will be to the solid which may be occupied by all the right lines *bE*, as the effect of all the particles on the sphere to the effect of all the particles on the cylinder. But the first solid is a paraboloid described with vertex *C*, axis *CA* and with the latus rectum *CA* , and the latter solid is the circumscribed cylinder to the paraboloid : and it is to be noted that the paraboloid shall be half of the circumscribed cylinder. Therefore the total force of the medium on the sphere is twice as small as the same total force on the cylinder. And therefore if the particles of the medium are at rest, and both the cylinder and the sphere may be moving with equal velocities, the resistance of the sphere will be less than the resistance of the cylinder by a factor of two. *Q.E.D.*

[*L*. & *S*. notes : If on all the points of the right line *NA* perpendiculars may be erected as *bH* and *bE*, and let *NHC* be that curve the point *H* always touches, and the line *KC* the locus of all the points *E*; the solid that may be occupied by all the perpendiculars *bH*, drawn through the whole base of the cylinder, will be equal to a conoid or to the solid figure which may be generated by the rotation made of the plane figures *NHCA* about the axis *CA*, and the solid that may be formed from all the right lines *bE* will be the cylinder described by the rotation of the rectangle *AK* made about the same axis.

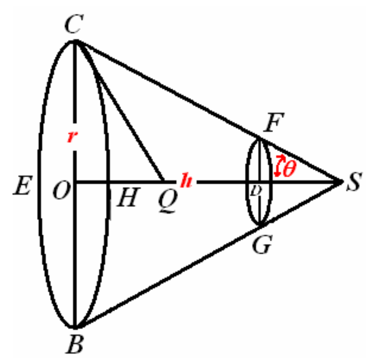


Since, by construction, there shall be $bH = \frac{BE^2}{CB}$ and thus $bH \times CB = BE^2 = BC^2 - CE^2$, and from the nature of the circle, $BC = CA = KC$, and thus $BE^2 = KC^2 - CE^2$ and $bH \times CB$, or $(KC - EH) \times KC$, or $KC^2 - KC \times EH = KC^2 - CE^2$, and thus $KC \times EH = CE^2$; but if the point *H* may be drawn to *CA*, the perpendicular ordinate, this must be equal to *CE*, and from *CA* there may be cut the part equal to *EH*. Whereby the rectangle under the given abscissa and the given line *KC* or *CA*, is equal to the square of the ordinate the perpendicular *CA*; thence the curve *CHN*, by Theorem I (*de parabola* ; *Apollonius*) is a parabola the vertex of which is *C*, the axis *CA*, and the latus rectum *CA*.

The paraboloid or solid generated from the rotation of the parabola *CHN*, about the axis *CA* is half of the circumscribed cylinder, which is produced from the rotation of the rectangle *AK* about the line *CA*. Through the moveable point P, the normal *PM* is erected to the axis *CA*, cutting the parabola in *H*, and the right line *KN* in *M*; and in the rotation of the whole figure about the axis *CA*, the lines *PH* and *PM* describe circles, which will be among themselves as the squares of the radii *PH* and *IM*, or, from the nature of the parabola and on account of $PM = AN$, as the abscissae *CP* and *CA*. Now the point *P* is drawn with the vertical *PHM* through the whole altitude *CA*, and the solid arising from the rotation of the figure *CHN* will be to the cylinder arising from the rotation of the rectangle *CKNA*, as the sum of all the circles which the moving line *PH* will describe by rotating, to the sum of all the circles which the right line *PM* will describe, that is, as the sum of all *CP* , to the sum of all *CA*. On the line *AN* there may be taken *AR* equal to *AC*, *CR* is joined cutting *PH* in *L*, and the perpendicular *RQ* is erected to *AR*, cutting *PM* in *V*; since there shall always be $PL = CP$, and $PC = CA$, the sum of all *CP*, or *PL*, by the whole altitude *CA*, is the isosceles triangle *CRA*, and the sum of all *CA*, or *PV*, by the same altitude *CA*, is the square *CARQ*; therefore since the triangle *CRA*, shall be half of the square *CARQ*, the paraboloid is also half of the circumscribed cylinder.]

Scholium.

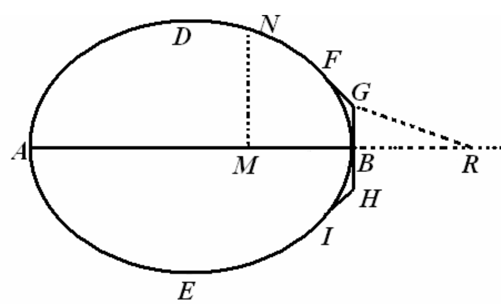
By the same method other figures may be compared between themselves as far as concerns resistance, and these found which are more suited to continue their motion in resisting mediums. So that if to the circular base *CEBH*, which will be described with centre *O*, with radius *OC*, and with the altitude *OD*, the conic frustum *CBGF* shall be constructed, which of all constructed with the same base and height and their axis following towards the direction of progression *D* shall be resisting the least: bisect the height *OD* in *Q* and produce *OQ* to *S* so that there shall be *QS* equal to *QC*, and *S* will be the vertex of the cone of which the frustum is sought.



[Following *B* & *R*, if the left hand base of the cone has radius *r*, the right hand base has radius $r - h\tan\theta$, and the resistance of the curved surface will be proportional to $\sin^2\theta\left(r^2 - \overline{r - h\tan\theta}^2\right)$, while that of front circle will be proportional to $\left(r - h\tan\theta\right)^2$, on omitting other constant factors. The total resistance can be seen to be $r^2 - rh\sin 2\theta + h^2\sin^2\theta$; to find the turning point, differentiate w.r.t. *h* and we find $h\sin\theta\cos\theta = r - 2r\sin^2\theta$; if *x* is the whole height of the cone, then this equation becomes :

$h \times \frac{xr}{r^2+x^2} = r - 2r \times \frac{r^2}{r^2+x^2} \therefore x^2 - hx = r^2$ and $x = \frac{h}{2} + \sqrt{r^2 + \frac{h^2}{4}}$, leading to the above construction.]

From which by the way, since the angle *CSB* shall always be acute, it follows that if the solid *ADBE* made may be generated by the rotation of the elliptic or oval figure *ADBE* about the axis *AB*, and the generating figure may be touched by the three right lines *FG*, *GH*, *HI* at the points *F*, *B* and *I*, by that rule so that *GH* shall be perpendicular to the axis at the point of



contact *B*, and *FG*, *HI* may contain with the same *GH* the angles *FGB* and *BHI* of 135^0 of the solid, that may be generated by rotation of the figure *ADFGHIE* about the same axis *AB*, resists less than the first solid; but only if each may be progressing along the direction of its axis *AB*, and each end *B* goes in front. Indeed which proposition I consider to be of use in the future construction of ships.

But if the figure *DNFG* shall be a curve of this kind, so that if the perpendicular *NM* may be sent from some point *N* of this curve to the axis *AB*, and from some given point *G* the right line *GR* may be drawn which shall be parallel to the tangent of the figure at *N*, and may cut the axis produced at *R*, *MN* will be to *GR* as GR^3 to $4BR \times GB^2$; the solid that is described by the revolution made of the figure about the axis *AB*, in the aforementioned rare medium by moving from *A* towards *B*, will be resisted less than some other encircled solid described with the same length and breadth.

PROPOSITION XXXV. PROBLEM VII.

If a rare medium may consist of the smallest equal particles at rest and in turn to be placed at equal distances freely : to find the resistance of a sphere progressing uniformly in this medium.

Case 1. A cylinder described with the same diameter and altitude may be understood to be progressing with the same velocity along the length of its axis in the same medium. And we may consider that the particles of the medium, on which the sphere or cylinder is incident, may recoil with the greatest force of reflection.

[Thus, elastic collisions offer the greatest resistance to motion, and completely inelastic collisions the lest resistance, in this model. Thus, for a cylinder of length *l*, the time to describe half the axis will be

$\frac{1}{2}\frac{l}{v}$ and the acceleration giving a velocity v in this time is $\frac{2v^2}{l}$; hence the resistance is simply $2Av^2\rho$, where *A* is the base area of the cylinder and ρ the velocity.]

And since the resistance of the sphere (by the latest Proposition) shall be half as great as the resistance of the cylinder, and the [volume of the] sphere shall be to the [volume of the] cylinder as two to three, and the cylinder by being incident on the particles perpendicularly, and these by being maximally reflected, may impart twice their velocity : the cylinder progressing uniformly in that time will describe half the length of its axis, will communicate the motion to the particles, which shall be to the whole motion of the cylinder as the density of the medium to the density of the cylinder ; and the sphere, in which time the whole length of its diameter will be described in progressing uniformly, will share the same motion with the particles ; and in that time in which it will describe a motion two thirds part of its diameter, that it will communicate to the motion of the particles, which shall be to the whole motion of the sphere as the density of the medium to the density of the sphere. And therefore the sphere suffers a resistance, which shall be to the force by which the whole motion of this may be taken away or generated in which time the two thirds part of its diameter will be described by progressing uniformly, as the density of the medium to the density of the sphere.

*Cas*e 2. We may consider that the particles of the medium incident on the sphere or cylinder are not reflected, and by being incident perpendicularly on the particles the cylinder will simple communicate its velocity, and thus the resistance experienced is half as great as in the previous case, and the resistance of the sphere will also be half as great as before.

*Cas*e 3. We may consider that the particles of the medium are reflected neither maximally nor not at all, but they may circle from the sphere in some manner in between, and the resistance of the sphere will be in the same mean ratio between the resistance in the first case and the resistance in the second case. *Q.E.I.*

Corol. 1. Hence if the sphere and the particles shall be indefinitely hard, and therefore destitute of all elasticity and all force of reflection : the resistance of the sphere will be to the force by which the whole of that motion may be removed or generated, in the time that the sphere can describe four thirds parts of its diameter, as the density of the medium to the density of the sphere.

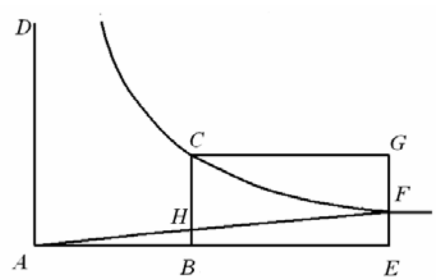
Corol. 2. The resistance of the sphere, with all else being equal, is in the square ratio of the velocity.

Corol. 3. The resistance of the sphere, with all else being equal, is in the square ratio of the diameter.

Corol. 4 The resistance of the sphere, with all else being equal, is as the density of the medium.

Corol. 5 The resistance of the sphere is in a ratio which is composed from the square ratio of the velocity and the square ratio of the diameter and in the ratio of the density of the medium.

Corol. 6. And the motion of the sphere with its resistance may be explained thus. Let *AB* be the time in which the sphere can lose all its motion by a uniformly continued resistance. The perpendiculars *AD*, *BC* may be erected to *AB*. And *BC* shall be that total motion, and through the point *C* a hyperbola *CF* may be described with the asymptotes *AD* and *AB* ; *AB* may be produced to some point *E*. The perpendicular *EF* may be erected crossing the hyperbola at *F*. The parallelogram *CBEG* may be completed, and *AF* may be drawn crossing *BC* at *H*. And if the sphere in some time *BE*, continued uniformly by its own initial motion *BC*, in a non-resisting medium may describe the distance *CBEG* shown by the area of the parallelogram, likewise in the resisting medium it describes the distance *CBEF* shown by the hyperbolic area, and the motion of this at the end of the time may be shown by the ordinate *EF* of the hyperbola, with the part of its motion *EG* lost. And the resistance of this at the end of any time may be expressed by the length *BH*, with the part *CH* lost to resistance. All these are apparent by Corol.1.& 3, Prop. V. Book II.



[Thus, using customary integration, if a body is dropped from rest ,with the resistance proportional to the velocity squared given by $F_r = kmv^2$, and g' is the apparent acceleration of gravity, with u the terminal velocity given by $g' = ku^2$, then at any instant,

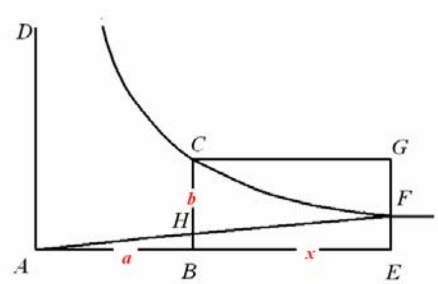
$v\frac{dv}{dx} = g' - \frac{g'}{u^2}v^2$ and $\frac{dv^2}{dx} + \frac{2g'}{u^2}v^2 = 2g'$; this requires the integration factor $e^{\frac{2g'}{u^2}x}$, and hence $\frac{d}{dx}v^2 e^{\frac{2g'}{u^2}x} = 2g' e^{\frac{2g'}{u^2}x}$, giving $v^2 = u^2\left(1 - e^{-\frac{2g'}{u^2}x}\right)$;

again, $\frac{dv}{dt} = k\left(u^2 - v^2\right)$ and $\frac{dv}{u^2 - v^2} = \frac{1}{u-v} + \frac{1}{u+v} = 2ukdt$ giving $ln\frac{u+v}{u-v} = 2ukt + C$; from which in turn $v = u\frac{1-e^{-2kut}}{1+e^{-2kut}} = u\frac{e^{kut}-e^{-kut}}{e^{kut}+e^{-kut}} = u\,tanh\,kut$; hence

$$\frac{dx}{dt} = u\frac{e^{kut}-e^{-kut}}{e^{kut}+e^{-kut}} = \frac{1}{k}\frac{dp}{p} \therefore x = \frac{1}{k}ln\left(e^{kut} + e^{-kut}\right) = \frac{1}{k}ln\,e^{kut}\left(1 + e^{-2kut}\right)$$
$$= ut + \frac{1}{k}ln\left(1 + e^{-2kut}\right) + C = ut + \frac{1}{k}ln\left(1 + e^{-2kut}\right) - \frac{1}{k}ln\,2.]$$

Corol. 7. Hence if the sphere may lose its whole motion *M* in the time *T* by a resistance *R* continued uniformly: the same sphere in the time *t* in the resisting medium, by a resistance *R* decreasing as the square of the speed, may lose a part $\frac{tM}{t+T}$ of its motion *M*, with the part $\frac{TM}{t+T}$ remaining ; and it will describe a distance that shall be to the distance described by the uniform motion *M* in the same time *t*, as the logarithm of the number $\frac{t+T}{t}$ multiplied by the number 2,30258092994 is to the number $\frac{t}{T}$, as that hyperbolic area *BCFE* is in this proportion to the rectangle *BCGE* .

[It seems appropriate to give here the geometrical explanations of *L*. & *J*. for Cor. 7, which although they are rather longer than the modern calculus approach, may yield some clues as to Newton's manner of thinking at the time:



And the resistance at the end of this time, etc. The resistance, at the beginning of the motion when the velocity is BC, may be shown by the same line *BC*, and because the resistances are as the squares of the velocities, and *BC* shall be to *FE* as the velocity from the start of the motion to the velocity at the end of the time *BE*, to FE^2, as *BC* to the line which may show the resistance at the end of the time *BE*, and thus this line is equal to $\frac{FE^2}{BC}$: that is

$\frac{R_0}{R_t} = \frac{BC^2}{FE^2} = \frac{BC}{R_t} \therefore R_t = \frac{FE^2}{BC}$. Now, from the nature of the hyperbola, we have $AB \times BC = AE \times FE$, and in addition from the similar triangles, *ABH* & *AEF*, $\frac{BC}{FE} = \frac{AE}{AB} = \frac{FE}{HB}$, and hence $HB = \frac{FE^2}{BC}$. Whereby the right line *HB* will show the resistance at the end of the time *BE*, and hence the part of the right line *CH* may show the part of the resistance that has been lost since the start of the motion, shown by the line *BC*.
Thus proceeding, the part of the motion *M* remaining at the end of the time *t* may be called *m*, and since $\frac{T}{t} = \frac{AB}{BE}$ and hence $\frac{T+t}{T} = \frac{AE}{AB}$, and besides $\frac{M}{m} = \frac{CB}{FE} = \frac{AE}{AB}$, there will

be $\frac{T+t}{T} = \frac{M}{m}$, from which there may be had $m = \frac{MT}{T+t}$, and thence the part of the motion M lost is $M - \frac{MT}{T+t} = \frac{Mt}{T+t}$ as required.

Proceeding further, the parts of the axis and ordinates of hyperbola may be called $AB = a, BC = b, BE = x, AE = a + x;$ and from the nature of the hyperbola, as $FE = y = \frac{ab}{a+x}$, the element of the area $CFEB$ will be $\frac{abdx}{a+x}$, and the area $CFEB$ itself, is equal to $ab\int \frac{dx}{a+x}$, which integral (fluent) thus is to be summed so that it may vanish when $x = 0$, but the integral $\int \frac{dx}{a+x}$ thus taken is the logarithm (called L. in this work, but which we shall call *ln*) of the number $\frac{a+x}{a}$, chosen from the logistic curve the subtangent of which is unity (see the additional material for Section 1 of Book 2 : essentially the definition of the exponential function, or antilogarithm curve in these days, being that curve, which on finding the gradient at the point *x*, *y*, gives a right angled triangle under the tangent, for which the tangent of the angle with the *x* axis is *y*/1 ; or *dy/dx* = *y*), or what amounts to the same thing, from the hyperbola of which the power is one; if indeed there may be put $x = 0$, the number $\frac{a+x}{a}$ becomes equal to 1, and thus $ln\left(\frac{a+x}{a}\right) = 0$. Whereby the area $BCFE = ab \times ln\left(\frac{a+x}{x}\right)$; truly the rectangle $BCGE = bx$. Therefore the hyperbolic area $BCFE$ is to the rectangle $ACGE$ as $ab \times ln\left(\frac{a+x}{x}\right):bx$, that is , as $ln\left(\frac{a+x}{x}\right):\frac{x}{a}$. Indeed here we have $\frac{a+x}{a} = \frac{T+t}{T}$ and $\frac{x}{a} = \frac{t}{T}$; whereby the hyperbolic area $BCFE$ is to the rectangle $BCGE$, as $ln\left(\frac{T+t}{T}\right)$ to $\frac{t}{T}$. Therefore it remains to find the logarithm of the number $\frac{T+t}{T}$; by the logarithmic curve of which the subtangent is one. Again the logarithms of different kinds of the same number are between themselves in a given ratio, and the number 2,302585092994 is the logarithm of the number ten in the species of logarithms the subtangent of which is unity (i.e. natural logarithms) and the logarithm of the number ten taken in tables is $1,0000000 = 1$; thus as the logarithm of the number $\frac{T+t}{T}$ taken in tables to the logarithm of the same number taken in logarithms the subtangent of which is unity, or in the hyperbola the power of which is 1; therefore the logarithm sought , if the logarithm of the number $\frac{T+t}{T}$ taken from tables may be multiplied by the number 2,302585092994.]

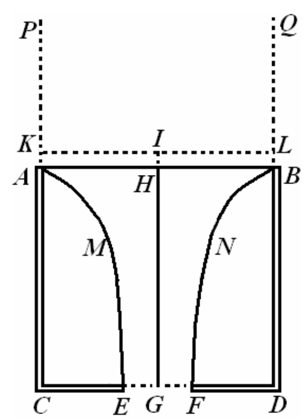
Scholium.

In this proposition I have set out the resistance and retardation of spherical projectiles in non continuous mediums, and I have shown that this resistance shall be to the force by which the whole motion of the sphere shall be removed or generated in the time in which the sphere may describe two thirds parts of the diameter, with a uniformly continued velocity, as the density of the medium to the density of the sphere, but only if the sphere and the particles of the medium shall be completely elastic and they may be influenced by the maximum force of reflection: and that this force shall be half as great when the sphere and the particles of the medium are infinitely hard, and evidently without any force of reflection. But in continuous mediums such as water, hot oil, and quicksilver, in which the sphere is not incident at once on all the fluid particles generating resistance, but presses only to the nearby particles and these press on other particles, and these in turn still others, the resistance is hence twice as small. Certainly a sphere in mediums of this kind of the most fluid, experience a resistance which is to the force by which the whole motion of this may either be removed or generated in the time, with that motion continued uniformly, eight third parts of its diameter will be described, as the density of the medium to the density of the sphere. That which we will try to show in the following medium.

PROPOSITION XXXVI. PROBLEM VIII.

To define the motion of water flowing from a hole made at the bottom of a cylindrical vessel.

[An excellent introduction to the history of this complex problem can be found in : *Die Werke von Daniel Bernoulli* : *Gesammelten Werke der Mathematiker und Physiker der Familie Bernoulli* (Basel; Boston; Birkhauser, 1982-2002; ed. David Speiser.) Vol. I, page 200. *Early studies on the outflow of water from vessels*, by G.K. Mikhailov (tr. from Russian into English by Rainer Radok.); this includes an independent modern view of Newton's contributions.]

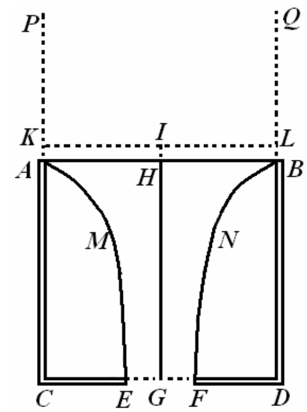


Let *ACDB* be the cylindrical vessel, *AB* the upper opening of this, *CD* the base parallel to the horizontal, *EF* a circular hole in the middle of the base, *G* the centre of the hole, and *GH* the axis of the cylinder perpendicular to the horizontal. And imagine a cylinder of ice *APQB* to be of the same width as the cavity of the vessel, and to have the same axis, and to be descending with a constant uniform motion, and the parts of this that first touch the surface *AB* become liquid, and converted into water to flow by their own weight into the vessel, and the head of flowing water formed by falling *ABNFEM* passes through the hole *EF*, and likewise to be filled equally.

[One might wonder initially why Newton considered this rather odd way of obtaining a constant head of water; perhaps it was just a domestic problem that intrigued him..... In any case, there are fundamental defects in Newton's approach, which does not agree with experiment for the motion of the water within the vessel near the walls; thus the idea of a funnel or an ice funnel through which the water flowed was completely erroneous, if that is what Newton had in mind; however, he may well have realized that the original problem was too hard to solve, so that he decided to solve an easier problem, involving the ice funnel, for which he was able to calculate the shape, and thus have a far better idea of the flow of water through such a shape, as we show below in an *L* & *S* derivation. In addition, in the first edition, the initial contraction or waisting of the stream of water emerging – the *vena contracta* - had been ignored totally – to be fixed up in the second edition after complaints from the Bernoulli camp; otherwise, Newton may have considered water to be far stickier or to have a much greater viscosity than it really does, as his approach might well describe the fall of a sticky liquid such as honey or oil through the hole at the bottom of a vessel. For in truth all the particles of water in the vessel descent slowly with no horizontal motion until the level of the hole is reached, at which stage there is considerable horizontal motion of a complex nature. Did Newton not bother to observe what actually happens, or could not see because he used wooded vessels ? It certainly would be atypical of his methods in investigating natural phenomena, such as the decomposition of light into its spectrum, if this were the case. As it was, his investigations described here provided a foundation for further inquiries, which had actually started some time previously, as the above reference sets out; in particular, his lines drawn in the fluid are essentially the lines of force or flow lines along which the particles travel. The hydrodynamics text of Daniel Bernoulli a little later gave improved explanations of the phenomena involved, following his [*Bernoulli's*] *Principle*, which made use of Leibniz's *vis viva* idea : the missing half was to create havoc in understanding kinetic energy for at least 100 years! So the complete idea of laminar flow had to await the concept of energy conservation before it could be placed on a firmer footing. The question of the shape of the vessel and the nature of its surface more or less dictates whether or not turbulent flow will occur near the hole; thus, this is still a very difficult problem to solve, and Newton may have been content to give approximate answers only by solving another problem.]

As indeed there shall be a constant velocity of the ice descending and with the adjoining water [formed] in that case describing the circle *AB*, so that the water by falling can acquire an altitude *IH*, and both *IH* and *HG* may be placed along the same direction, and through the point *I* the right line *KL* may be drawn parallel to the horizontal, and meeting the sides of the ice at *K* & *L*. And the velocity of the water flowing through the hole *EF*, will be as that the water, by falling from *I*, is able to acquire in its own case by describing the height *IG* . And thus by a theorem of *Galileo*, [really *Torricelli*] *IG* will be to *IH* in the square ratio of the velocity flowing from the orifice to the velocity of the water in the circle *AB*, that is, in the square ratio of the circle *AB* to the circle *EF*; for these circles are inversely as the velocities of the water which through themselves may adequately be passed in the same time and in an equal amount. [Thus, the continuity equation of fluid flow is called upon, and also the conversion of potential into kinetic energy, in modern terms, of any small particle of the fluid.] Here we are concerned with the velocity of the water that is disturbed horizontally : the motion parallel to the horizontal by which the parts of water falling may approach in turn will not be considered here, since it does not arise from gravity, nor may the motion arising from gravity perpendicular to the horizontal change. Indeed we may suppose that the parts of the water adhere just a little, and by it cohesion in falling approach each other through a motion parallel to the horizontal, so that they may form only a single stream downwards and may not be divided into several: but we shall not consider here the motion parallel to the horizontal arising from that cohesion.

Case 1. Consider now the whole cavity in the vessel, in the circulation of the falling water *ABNFEM*, to be filled with ice, so that the water may only pass through the ice as by a funnel. And if the water may hardly touch the ice, or what amounts to the same thing, if yet it may touch and on account of its great smoothness may slide freely and without any resistance, the water may run down through the opening *EF* with the same velocity as at first, and the whole weight of the column of water *ABNFEM* may be pressing on the flow of this being produced as at first, and the bottom of the vessel will sustain the weight of the surrounding column of ice.



Now the ice in the vessel may liquefy; and the efflux of the water will remain the same as at first. It will not be less, because ice dissolved in water is trying to descend : nor greater, because ice dissolved in water cannot descend unless by impeding the descent of water descending equally to itself. The same force must generate the velocity of the flowing water.

But the aperture at the bottom of the vessel, on account of the oblique motion of the particles of water flowing out, must be a little greater than at first. For the particles of water do not all pass through the opening perpendicularly, but flow together from each side of the vessel and converge on the opening, and they pass through with oblique motions, and in the course of their downwards motion they conspire in the stream of water by bursting forth, which is a little smaller below the hole than at the hole itself, with the diameter of this present to the diameter of the hole as 5 to 6, or $5\frac{1}{2}$ to $6\frac{1}{2}$ as an approximation, but only if I have measured the diameters correctly. Certainly I had obtained a thin plane sheet pierced by a hole in the middle, with the diameter of the circular hole present of five eights of an inch. And so that the stream of water bursting forth might not be accelerated by falling and by the acceleration rendered narrower, thus I fastened this sheet not onto the base but onto the side of the vessel, so that that stream may emerge along a line parallel to the horizontal. Then when the vessel should be full of water, I uncovered the hole so that the water could flow out; and the diameter of the stream produced of $\frac{21}{40}^{\text{th}}$ of an inch as accurately as could be measured, at a distance around half an inch from the opening. Therefore the diameter of the circle of this hole to the diameter of the stream was as 25 to 21 approximately. Therefore the water by passing through the hole, converges on all sides, and after it has flowed out from the vessel, is rendered narrower by converging, and it is accelerated by the attenuation on arriving at a distance of half an inch from the hole, and with that narrower at that distance it shall be faster than at the hole itself in the ratio 25×25 to 21×21 or approximately as 17 to 12, that is around the ratio of the square root of 2 to 1.

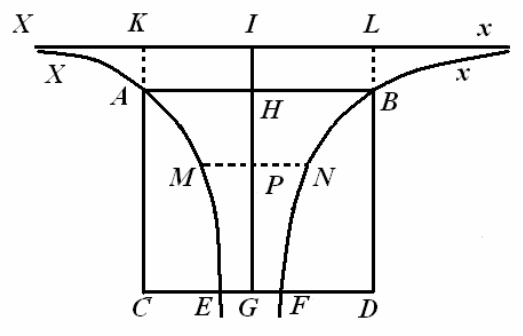
[Thus, from the continuity equation, $A_1 \times v_1 = A_2 \times v_2$, and the ratio of the velocities $\frac{v_2}{v_1} = \frac{A_1}{A_2} = \frac{625}{441} = 1.42... \doteq \sqrt{2}$, where A_1 and A_2 are the areas of cross-section.]

Indeed it is agreed by experiment that the quantity of water, which flows out in a given time through the circular hole made in the bottom of the vessel, shall be with the predicted velocity, not only through that hole, but it must flow also in the same time through the circular hole, the diameter of which is to the diameter of this hole as 21 to 25. And thus that water flowing has a velocity downwards at this [actual] hole describing approximately *by the falling of a weight through half the height of the water at rest in the vessel*. [Translator's italics and underlining here and below.] But after it has escaped from the vessel, it may be accelerated by converging until it arrives at a distance from the opening nearly equal to the diameter of the hole, and it will have acquired a velocity greater nearly in the ratio of the square root of two, than certainly in this case it can acquire a velocity described approximately _by a weight falling the whole height of the water at rest in the vessel._

Therefore in the following the diameter of the stream may be designated by that smaller hole that we have called *EF*. And another superior plane *VW* is understood to be drawn parallel to the plane of the opening *EF* at a distance approximately equal to the diameter and with a greater hole *ST* bored through, through which certainly the stream falls, which completely fills the lower hole *EF*, and thus the diameter of this shall be to the diameter of the lower opening approximately as 25 to 21. For thus the stream will be able to cross over perpendicularly from the lower opening ; and the quantity of water flowing out, for the size of this hole, will be as the solution of the problem postulated approximately. Indeed the distance, that is enclosed by the two planes and by the stream falling, can be considered as the bottom of the vessel. But so that the solution of the problem shall be simpler and more mathematical, it may be agreed to take only the lower plane for the base of the vessel, and to imagine that the water either flowing through the ice or through the funnel, and escaping from the vessel at the opening made in the lower plane *EF* , perpetually maintains its motion, and the ice remains at rest. In the following therefore let *ST* be the diameter of the circular opening *Z* described through which the stream flowed from the vessel when all the water in the vessel is fluid. And *EF* shall be the diameter of the opening through which by falling the stream may adequately pass through, either the water may exit from the vessel through that upper opening *ST*, or it may fall through the middle of the ice in the vessel as if it were through a funnel. And let the diameter of the upper opening *ST* be to the diameter of the lower *EF* as around 25 to 21, and the distance of the perpendicular between the planes of the openings shall be equal to the diameter of the smaller opening *EF*. And the velocity of the water escaping from the vessel by the opening *ST* there will be in the opening itself as the velocity that a body can acquire by falling from a height of half *IZ* : but the velocity of each stream by falling in the opening *EF* there will be as a body would acquire in falling from the whole height *IG*.

[Thus we have $v^2 = u^2 + 2gs$, which essentially is an energy conservation equation where unit mass is falling ; Newton considers that the motion of the water through *EF* is not vertically downwards, and the average velocity downwards corresponds to the water falling a height equal to half the height of the column; however, the column narrows, really by surface tension forces, so that the area of cross-section diminishes by $\sqrt{2}$, and at this point the water is considered to be flowing downwards only, and a drop can be considered to have fallen the whole height of the column, approximately; the extra distance being ignored. It is convenient here to consider the actual shape of the ice funnel proposed by Newton: as developed by L. & J. in Note 272 :

With these items in place, the geometrical figure of the cataract is readily defined. Let *MN* cut the axis *IG* at *P*; and because the altitude *IP* is in the square ratio of the velocity at *P*, truly this velocity is inversely as the circle *MN*, and thence the circle *MN* is in the square ratio of the radius *MP*, and thus *IP* or the abscissa is in the inverse fourth power of the radius or of the ordinate *MP*, or $IP \propto \frac{1}{MP^4}$, and $MP^4 \times IP$ a given quantity. (Thus, the Galilean proportionality $h \propto v^2$ is combined with the continuity equation $A \propto \frac{1}{v}$ to give $h \propto v^2 \propto \frac{1}{A^2} \propto \frac{1}{r^4}$)

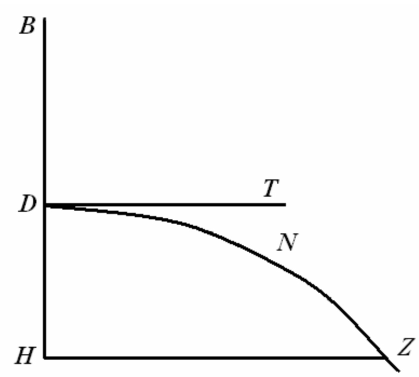


Therefore the curve EMA is a hyperbola of the fourth order, having the asymptotes *IG, IK*, to which convexity it turns towards. The arc *EMA* and the asymptotes *IK* may be produced indefinitely towards the parts *X*, and the figure *EAXXIG* describes the cataract, rotated around the asymptote or the axis *IG*, produced indefinitely to the parts *X*, *x* ; truly the figure *EMAHG* will be generated, that part of the cataract which is contained within the vessel *ABDC*.]

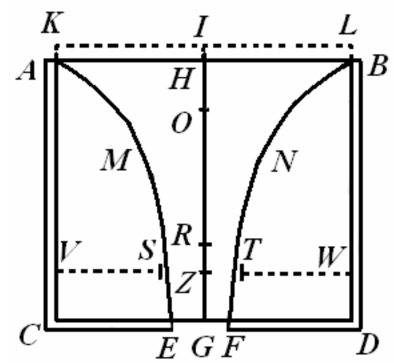
Case 2. If the opening *EF* shall not be in the middle of the base of the vessel, but the base may be perforated in some manner : the water may flow out with the same velocity as before, but only if the hole shall have the same size. For indeed a weight descends in a longer time through the same depth along an oblique line than along the perpendicular line, but in falling it acquires the same velocity in each case, as *Galileo* has shown.

Case 3. The velocity of water is the same flowing out from a hole in the side of the vessel. For if the opening shall be small, so that the interval between the surfaces *AB* and *KL* may be considered to vanish, and the jet of water springing forth horizontally will form the a parabolic figure: from the latus rectum of this parabola it may be deduced, what that velocity of the water flowing out from the water at rest in the vessel shall be, as a body may be able to acquire by falling with a height *HG* or *IG*. Certainly with that experiment done I found that, if the height of the water at rest above the opening should be twenty inches and the height of the opening above a plane parallel to the horizontal should also be twenty inches, the stream of water streaming out would fall on that plane at a distance of around 37 inches from the perpendicular that may be taken in that plane from the opening. For without resistance, the jet would have been incident in that plane at a distance of 40 inches, with the latus rectum of the parabolic jet arising of 80 inches.

[Note from *L* & *S* : A drop of water from the location *D*, may gush out along some horizontal direction *DT* with that velocity which it can acquire by falling through the (half the height of the vessel) *BD*, and being borne by the resistance of the medium, may describe the parabola *DNZ*, the vertex of which *D*, the tangent *DT*, and diameter DH or with the vertical line *BD* produced, the abscissa *DH* may be taken equal to the height *BD*, and the ordinate *HZ* may be drawn, which will be parallel to the tangent *DT*; and in that time taken for the drop of water to fall through the height *BD* or *DH* under gravity, it will describe the length *HZ* of *BD* or *DH* squared. The latus rectum *DNZ* of the parabola pertaining to the diameter *DH*



is $\frac{HZ^2}{DH}$, and thus since there shall be $HZ = 2DH = 2BD$, the latus rectum is 4*BD*:(*i.e.* $y^2 = 4ax$). Therefore the height *BD* that the water must describe by falling as the velocity it may acquire with that bursting out from the place *D*, is the fourth part of the latus rectum pertaining to the diameter *DH* of the parabola *DNZ*.]



Case 4. Truly the water flowing out, also if it were carried upwards, emerges with the same velocity. For the small jet of water streaming out rises in a perpendicular motion to the height *GH* or *GI* of the water at rest in the vessel, except in as far as its ascent may be impeded a little by air resistance ; and hence that flows out with the velocity that it may be able to acquire by falling from that height. Each particle of the water at rest is pressed equally on all sides (by Prop. XIX, Book. 2.) and by conceding to the pressure to all the parts there is imparted an equal impetus, either it may fall through a hole in the bottom of the vessel, or may flow out horizontally through a hole in the side of this, or it may go out along a pipe and thence ascent through a small hole made in the upper part of the pipe. An the velocity by which the water flows is that, as we have assigned in this proposition, not only is it deduced from reasoning, but also it is shown by the well known experiments now described.

Case 5. The velocity of the water emerging is the same whether the shape of the opening *D* shall be circular or square or triangular or some other equal to the circular shape. For the velocity of the water emerging does not depend on the figure of the opening but arises from the height of this below the plane *KL*.

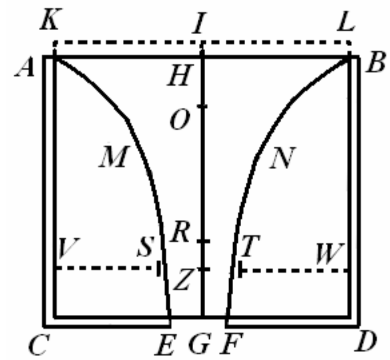
Case 6. If the lower part of the vessel *ABDC* may be immersed in still water, and the height of the still water above the bottom of the vessel shall be *GR*: the velocity with which the water in the vessel may flow out through the hole *EF* into the still water, will be as that which the water can acquire describing by falling in that case the height *IR*. For the weight of all the water in the vessel which is below the surface of the still water, will be sustained in equilibrium by the weight of the still water, and thus the motion of the water descending in the vessel will be accelerated less. In this case it will also be apparent by experiments, evidently by measuring the time in which the water flowed out.

Corol. 1. Hence if the height of the water *CA* may be produced to *K*, so that there shall be *AK* to *CK* in the square ratio of the area of the opening made in some part of the bottom, to the area of the circle *AB* : the velocity of the water flowing out will be equal to the velocity that the water is able to acquire by falling and by describing the height *KC* in that case.

[Thus, we apply the continuity equation to a circle of water at *AB*, and to the water at the opening *EF*: The speed acquired by the water falling twice the height AK may be called v_A , and the speed at *EF* may be called v_E, then

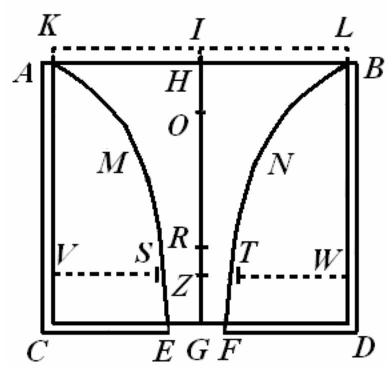
$v_A \times AB^2 = v_E \times EF^2$ and $AK \times AB^4 = GI \times EF^4$.]

Corol. 2. And the force, by which the whole motion of the water streaming forth can be generated, is equal to the weight of the cylindrical column of water, the base of which is the opening *EF*, and the height 2*GI* or 2*CK*. For the water rushing out is able to acquire its emergence velocity, to which in time that column may be equated, by its own weight falling from a height of *GI*.



[Thus, the force is taken to be this weight of water in modern terms $\pi EF^2 \times \rho \times g \times 2GI$. The actual mechanism for the acceleration of the water in the opening to form the jet emerging is not explained, but we are to accept the final speed of emergence in modern terms, correcting for the *vena contracta*, as being $\sqrt{2g \times GI}$.]

Corol. 3. The weight of all the water in the vessel *ABDC* is to the part of the weight, which is used in the out flowing of the water, as the sum of the circles *AB* and *EF* to the double of the circle *EF*. For if *IO* is the mean proportional between *IH* and *IG*; and the water emerging through the opening *EF*, in the time that a drop falling from *I* may be able to describe the height *IG*, will be equal to a cylinder the base of which is the circle *EF* and the height is 2*IG*, that is, to a cylinder whose base is the circle *AB* and height is 2*IO*, for the circle *EF* is to the circle *AB* in the square root ratio of the height *IH* to the height *IG*, that is, in the simple ratio of the mean proportional *IO* to the height *IG*: and in which time the drop by falling from *I* can describe the height *IH*, the water passing out will be equal to the cylinder of which the base is the circle *AB* and the height is $2IH$: in which time the drop by falling from *I* through *H* to *G* will describe the difference of the heights *HG*, the water emerging, that is, all the water in the figure *ABNFEM* will be equal to the difference of the cylinders, that is, to a cylinder of which the base is *AB* and the height 2*HO*. And therefore the total water in the vessel *ABDC* is to the total water fallen in the figure *ABNFEM* as *HG* to 2*HO*, that is, as $HO + OG$ to 2*HO*, or $IH + IO$ to 2*IH*. But the weight of all the water in the figure *ABNFEM* is expended in the out flowing of the water: and hence the weight of all the water in the vessel is to the part of the weight which is implemented in the out flowing of the water, as $IH + IO$ to 2*IH*, and thus as the sum of the circles *EF* and *AB* to twice the circle *EF*.



[Enlarged note from *L*. & *J*. : Thus, the same velocity of the efflux arises from the whole circle *AB* and the height 2*IO* as from the circle *EF* and the height 2*IG*, or what amounts to the same, cylinders which have these equal volumes. Again, the same amount of water passes through the circles *AF* and *EF* in the same time, and the amount of water passing through *AB* will be in the time that a drop falls through the height *IH*, equal to the volume of a cylinder of water of which the base is the circle *AB* and the height 2*IH*. We may add to this, inverted, that $\frac{Area\ AB}{Area\ EF} = \frac{v_E}{v_A} = \frac{\sqrt{IG}}{\sqrt{IH}}$.

Now, if $IO = \sqrt{IG} \times \sqrt{IH}$ or $\frac{IG}{IO} = \frac{IO}{IH}$ then $\frac{Area\ AB}{Area\ EF} = \frac{v_E}{v_A} = \frac{\sqrt{IG}}{\sqrt{IH}} = \frac{IG}{IO} = \frac{IO}{IH}$. Thus, in the time a drop falls the distance *IH*, the jet emits a volume of water equal to *the area* $AB \times 2IH$, due to the waist narrowing; then by simple proportion, in the time a drop falling from *I* to *G* via *H* describes the difference of the heights *GH*, that is the whole volume of water in the ice funnel *ABNFEM*, the jet will emit a volume equal to the difference of the (whole) cylinders, that is, a cylinder with base *AB* and height 2*HO*. Hence, the total amount of water in the vessel is to the water escaped through the ice funnel, as *HG* to *2HO*. The volume of water contained in the vessel *ABDC* is equal to the capacity of the vessel or cylinder the base of which is the circle *AB*, and the height *HG*; and therefore the total water in the vessel *ABCE*, is to the total water falling in the solid *ABNFEM*, as *HG* to 2*HO*, that is, as $\frac{HO+OG}{2HO}$, and because by hypothesis:

$$\frac{Area\ EF}{Area\ AB} = \frac{IH}{IO} = \frac{IO}{IG} = \frac{IO-IH}{IG-IO} = \frac{HO}{OG},$$

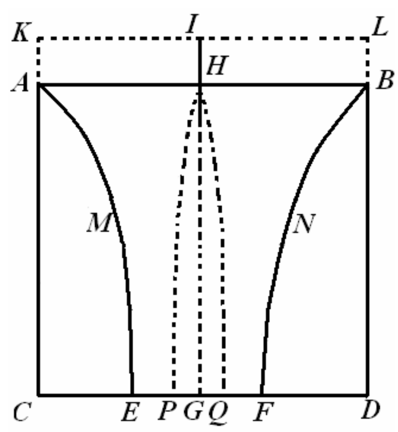
there is $\frac{HG}{2HO} = \frac{HO+OG}{2HO} = \frac{IH+IO}{2IH} = \frac{Area\ AB + Area\ EF}{2 \times Area\ EF} = \frac{AB^2+EF^2}{2 \times EF^2}$.]

Corol. 4. And hence the weight of all the water in the vessel *ABDC* is to the part of the weight that the base of the vessel sustains, as the sum of the circles *AB* and *EF* to the difference of the same circles.

[The weight of all the water in the vessel *ABCD* shall be *P*, the part of that weight which is involved in the efflux of the water shall be *p*, and hence the part $P-p$ of the whole weight or clearly equal to the difference of the circles *CD* and *EF* which is sustained by the bottom of vessel and is not involved in the efflux. And, by Cor. 3, there will be $\frac{P}{p} = \frac{AB^2+EF^2}{2EF^2}$ and hence $\frac{P}{P-p} = \frac{AB^2+EF^2}{AB^2-EF^2}$.]

Corol. 5. And the part of the weight that the base of the vessel sustains, is to the other part of the weight, which is implemented in the out flowing of the water, as the difference of the circles *AB* and *EF* to twice the smaller circle *EF*, or as the area of the base to twice the aperture.

[Since $\frac{P}{p} = \frac{AB^2+EF^2}{2EF^2}$, also $\frac{P-p}{p} = \frac{AB^2-EF^2}{2EF^2}$.]



Corol. 6. But the part of the weight, by which the base only may be acted on, is to the whole weight of the water, which will press perpendicularly on the base, as the circle *AB* to the sum of the circles *AB* and *EF*, or as the circle *AB* to the excess of twice the circle *AB* above the base. For the part of the weight, by which the base may be acted on alone, is to the weight of the whole water in the vessel, as the difference of the circles *AB* and *EF* to the sum of the same circles by Cor. 4 :

[As above, $\frac{P}{p} = \frac{AB^2+EF^2}{2EF^2}$ and hence $\frac{P-p}{p} \times \frac{p}{P} = \frac{AB^2-EF^2}{2 \times EF^2} \times \frac{2 \times EF^2}{AB^2+EF^2}$, giving

$\frac{P-p}{P} = \frac{AB^2-EF^2}{AB^2+EF^2}$ and $\frac{P}{P-p} = \frac{AB^2+EF^2}{AB^2-EF^2}$ or $\frac{P-\frac{p}{2}}{P-p} = \frac{AB^2}{AB^2-EF^2}$ and $\frac{P-\frac{p}{2}}{P-p} \times \frac{P-p}{P} = \frac{P-\frac{p}{2}}{P} = \frac{AB^2}{AB^2+EF^2}$]

and the weight of all the water in the vessel is to the weight of all the water which presses normally on the base, as the circle *AB* to the difference of the circles *AB* and *EF*. And thus from the rearranged equation, the part of the weight, by which the base alone is acted on, is to the weight of all the water, which presses normally on the base, as the circle *AB* to the sum of the circles *AB* and *EF*, or the excess of twice the circle *AB* above the base.

Corol. 7. If a small circle *PQ* may be located in the middle of the hole *EF* described with centre *G* and parallel to the horizontal : the weight of water which that circle shall sustain, is greater than the weight of a third part of the cylinder of water the base of which that circle and the height is *GH*. For let *ABNFEM* be the cataract or column of water falling having the axis *GH* as above, and as it is understood all the water in the vessel has become frozen, as long as the most immediate and fastest may not be required in the circulation of the cataract above the circle. And let *PHQ* be the column of water frozen above the circle, having the vertex *H* and the height *GH*. And consider the cataract both to fall by its own total weight, and not in the least to encumber or press upon *PHQ*, but to slide past freely and without friction, except perhaps at the vertex of the ice itself from which the cataract itself may begin to fall in the cavity. And in whatever manner the frozen water in the circuit of the cataract *AMEC*, *BNFD* is convex on the internal surface *AME, BNF* falling towards the cataract, thus also this column *PHQ* will be convex towards the stream, and therefore greater than for a cone the base of which is that circle *PQ* and with the height *GH*, that is, greater than a third part of the cylinder with the same base and with the height described. But the circle may sustain the weight of that column, that is, the weight which is greater than the weight of the cone or a third part of the cylinder.

[Thus, the weight of water sustained by the small circle is greater than the weight of the cone, given by $\frac{1}{3}Ch.\rho g$ in an obvious notation with *C* the area of the small circle, while in the following, the weight is again less than the weight of a semi-spheroid, given by $\frac{2}{3}Ch.\rho g$. If the circle is very small, then the weight supported can be taken as he arithmetic mean, that is $\frac{1}{2}Ch.\rho g$ (*Cor*. 8 & 9 following.) ; this leads on in the following to an examination of the frictional force on a horizontal circle in a stream.]

Corol. 8. The weight of water that a very small circle *PQ* can sustain, may be seen to be less than two thirds of the cylinder of water the base of which is that circle and the altitude is *HG*. For with everything now remaining in place, it may be understood to have described half of a spheroid, the base of which is that circle and the semi axis or height is *HG*. And this figure will be equal to two third parts of that cylinder and it may be understood that the circle may sustain the weight of this frozen column of water *PHQ*. For as the motion of the water shall be mainly straight [down], the surface of that column may meet externally with the base *PQ* in some acute angle, therefore as with the water falling it always will be accelerated, and on account of the acceleration it shall become narrower, and since that angle shall be less than a right angle, this column will be laid within the half spheroid in the lower parts of this. Truly the same upwards will be acute or pointed, lest the horizontal motion of the water at the vertex of the spheroid shall be infinitely faster than the motion of this towards the horizontal. And so that the smaller the circle *PQ* becomes the more acute the vertex of the column becomes to that ; and with the tiny circle diminished indefinitely, the angle *PHQ* will be diminished indefinitely, and therefore that column will be placed within the half spheroid. Therefore that column is less than the half spheroid, or than two thirds parts of the cylinder of which the base is that small circle, and with the altitude *GH*. But the small circle sustains the force equal to the weight of the water of this column, since the weight of the water around is employed in the outflow of this water.

Corol 9. The weight of water that a very small circle *PQ* may sustain, is equal to the weight of the cylinder of water the base of which is that circle and the height is $\frac{1}{2}GH$ approximately. For this weight is the arithmetical mean between the weights of the cone and of the aforementioned hemisphere. But if that circle shall not be very small, but may be increased until it may equal the opening *EF* ; here it will be the weight of all the water itself overhanging perpendicularly, that is, the weight of the cylinder of water of which the base is that circle and the height is *GH*.

Corol. 10. And (as far as I know) the weight that the little circle sustains is always to the weight of the cylinder of water, the base of which is that little circle and of which the height is $\frac{1}{2}GH$, as EF^2 to $EF^2 - \frac{1}{2}PQ^2$, or as the area of the circle *EF* to the excess of this circle above half the area of this circle of this circle *PQ*, as an approximation.

[*L* & *S* note : For this supposition satisfies the above requirements : For if *p* shall be the weight of water sustained by the small circle, and *P* the weight of the cylinder of water sustained by the small circle and with the height *GH*; and if there is put $p : \frac{1}{2}P = EF^2 : EF^2 - \frac{1}{2}PQ^2$ then $p = \frac{\frac{1}{2}P \times EF^2}{EF^2 - \frac{1}{2}PQ^2}$. But the quantity $\frac{P \times EF^2}{2EF^2 - PQ^2}$ is always greater than $\frac{1}{3}P$, as it may satisfy Cor. 7, and likewise it can be shown always to be less than $\frac{2}{3}P$.

B. & *R.* proceed here as follows: the weight of the water on the little circle is $P = \frac{1}{2}Ch.\rho g$, from the above average, if the area of the circle is very small; however, it is comparable to the cross-sectional area *B* of the pipe *EF*, we may assume this weight takes the form $P = \frac{\beta}{1-\alpha\frac{C}{B}}Ch\rho g$, then when *C* is very small with respect to *B*, this reverts to the above result, so that $\beta = \frac{1}{2}$, and when $C = B$, the weight supported is that of a cylinder with base *C* and height *h*, so that also $\alpha = \frac{1}{2}$; hence, while *C* is less than $\frac{B}{2}$, the expression $P = \frac{B}{B-\frac{1}{2}C} \times \frac{1}{2}Ch\rho g$ makes the weight lie between the given limits above.

There is an analogy between the force due to water running out through a hole, and the resistance experienced by a body travelling at a constant rate through a medium. The resistance Newton has in mind here is of the dynamic kind, due to the particles of the medium rebounding from the moving surface, rather than due to the adhering nature of the medium. If *u* is the velocity at the surface, *v* that at the orifice, *A* the area of the cross-section of the base, *B* and *C* that of the orifice and the little circle *PQ*, and *h* the height of the cylinder. Then $v^2 = u^2 + 2gh$ and $v(B-C) = uA$; the former we may now think of as an energy conservation equation, while the latter is a continuity equation. Now if *A* is

much greater than *B*, then $v^2 = 2gh$ and the weight acting on the small circle is $P = \alpha \times \frac{1}{2} Ch\rho g$, where $\alpha \to 1$ when $\frac{C}{B} \to 0$, in any manner. $\therefore P = \alpha \times \frac{Cv^2}{4} \times \rho$.
Now let the opening of the pipe *EFST* be closed, and let the small circle ascend with such a velocity that the relative motion of the circle and fluid compelled to flow past it is the same as before; hence the weight or force acting on the small circle will be the same as before; now the velocity of the fluid will be $\frac{Cv}{B-C}$ and that of the plane moving through it $\frac{Bv}{B-C}$. If we imagine that *B* is infinitely greater than C, then the resistance of a plane moving in still water with a velocity *v* will be $\therefore P = \frac{Cv^2}{4} \times \rho$. The resistance depends only on the area of the great circle, and according to this, a sphere. a spheroid, and a cylinder will offer the same resistance. As many commentators have already indicated, there are many flaws in these arguments; the principal one being that water does not flow out of a hole in a vessel as Newton had envisaged! Nevertheless, the arguments are intriguing, and there may be situations where a fluid obeys these rules. In any case, Newton went about this analysis in order that he could examine the resistance to the motions of projectiles, falling bodies, and pendulums, in resisting mediums, that follow.]

LEMMA IV.

The resistance of a cylinder, which is progressing along its own length uniformly, does not change if the length of this may be increased or diminished ; and thus it is the same as the resistance of a circle described with the same diameter, and with the same velocity of progression along a right line perpendicular to its plane.

For the sides of the cylinder by the motion of this are minimally opposed : and the cylinder, with the length of this diminished indefinitely, is turned into a circle.

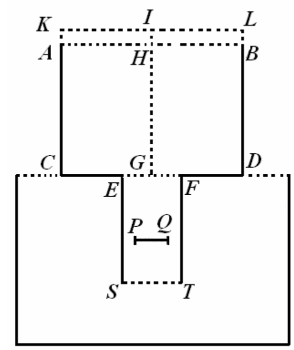
[Thus Newton solves the simpler problem where the resistance is purely dynamic, and ignores viscous effects in mediums, where the length would be important.]

PROPOSITION XXXVII. THEOREM XXIX.

The resistance of a cylinder which arises from the magnitude of the transverse section, which is progressing uniformly along its length in a fluid compressed infinitely and non-elastic, is to the force by which the whole motion of this, while meanwhile it describes four times its length may be either removed or generated, as the density of the medium to the density of the cylinder approximately.

For if the vessel *ABDC* may touch the surface of the still water with its base *CD*, and water may flow from this vessel by the cylindrical pipe *EFTS* perpendicular to the horizontal into still water, and moreover the small circle *PQ* may be located parallel to the horizontal somewhere in the middle of the pipe, and *CA* may be produced to *K*, so that *AK* shall be to *CK* in the squared ratio that the excess of the opening of the pipe *EF* over the circle *PQ* has to the circle *AB*: it is evident (by Cases 5 and 6, & Cor. 1. Prop XXXVI) that the velocity of the water passing through the annular space between the small circle and the side of the vessel, that will be as that velocity the water can acquire by falling, and in that case by describing the height *KC* or *IG*.

And (by Corol. X, Prop. XXXVI.) if the width of the vessel is infinite, so that the small line *HI* may become vanishing and the altitudes *IG* and *HG* are equal: the force of the water flowing down on the circle will be to the weight of the cylinder whose base is that small circle and the height is $\frac{1}{2}IG$, as EF^2 to $EF^2 - \frac{1}{2}PQ^2$ approximately. For the force of the water, by flowing with a uniform velocity through the whole pipe, will be the same at the location of the circle *PQ* as in any part of the pipe.



Now the openings of the pipe *EF* and *ST* may be closed, and the small circle may ascend in the fluid compressed on all sides and by itself ascending it may force the water above to fall through the annular space between the small circle and the side of the pipe : and the velocity of the ascending small circle will be to the velocity of the water descending as the difference of the circles *EF* and *PQ* to the circle *PQ*, and the velocity of the ascending little circle to the sum of the velocities, that is, to the relative velocity of the water descending which flows past the ascending circle, as the difference of the circles *EF* and *PQ* to the circle *EF*, or as $EF^2 - PQ^2$ to EF^2.

[For : $v_{asc.} \times PQ^2 = v_{desc.} \times \left(EF^2 - PQ^2\right)$ and

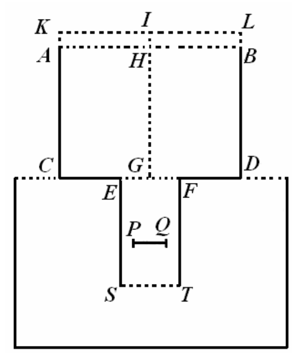
$$\frac{v_{asc.}}{v_{desc.}} = \frac{EF^2 - PQ^2}{PQ^2},\ \frac{v_{desc.} + v_{asc.}}{v_{desc.}} = \frac{v_{rel.}}{v_{desc.}} = \frac{EF^2}{PQ^2} \text{ and } \frac{v_{desc.}}{v_{rel.}} \times \frac{v_{asc.}}{v_{desc.}} = \frac{v_{asc.}}{v_{rel.}} = \frac{EF^2 - PQ^2}{PQ^2} \times \frac{PQ^2}{EF^2} = \frac{EF^2 - PQ^2}{EF^2}$$]

Let that relative velocity be equal to the velocity, which has been shown above to pass through the same annular space while the small circle meanwhile may remain at rest, that is, to the velocity that the water can acquire by falling and in that case by describing the altitude *1G* : and the force of the water in the ascending small circle will be the same as before (by the rule of Corol. V.) that is, the resistance of the ascending circle will be to the weight of the cylinder of water whose base is that small circle and the height is $\frac{1}{2}IG$, as EF^2 to $EF^2 - \frac{1}{2}PQ^2$ approximately. But the velocity of the small circle will be to the velocity that the water acquires by falling and in that case be describing the altitude *IG*, as $EF^2 - PQ^2$ to EF^2.

The size of the pipe may be increased indefinitely : and these ratios between $EF^2 - PQ^2$ and EF^2, and between EF^2 and $EF^2 - \frac{1}{2}PQ^2$ finally will approach to ratios of equality [, as *EF* >>*PQ*]. And therefore the velocity of the small circle will now be that as can be acquired by the water descending in that case from the altitude described *IG*, and truly the resistance of this will emerge equal to the weight of the cylinder whose base is that small circle and the altitude is half of the altitude *IG*, from which the cylinder must fall as the velocity of the ascending small circle may acquire ; and the cylinder with this velocity, in the time of falling, will describe four times its own length. [As the force is proportional to the velocity squared, then the cylinder must fall a distance

$2 \times IG = 4 \times \left(\frac{1}{2} \times IG\right)$.] But the resistance of the cylinder progressing along its length with this velocity, is the same as the resistance of the small circle (by Lemma IV.) and thus is equal to the force by which the motion of this, as long as the quadruple of its length shall be described, is able to be generated approximately.

If the length of the cylinder may be increased or diminished : so that the motion of this and the time in which it will describe four times its length, will be increased or diminished in the same ratio; and thus that force will not be changed, by which the increased or diminished motion, equally increased or decreased in the time, may be able to be generated or taken away, and thus even now is equal to the resistance of the cylinder, and since that too remains unchanged by Lemma IV.



If the density of the cylinder may be increased or diminished : so that the motion of this, and the force by which the motion can be generated or removed in the same time, will be increased or diminished in the same ratio. And thus the resistance of the cylinder of any kind will be to the force by which the whole of its motion, as the fourfold of its length meanwhile will be described, may be generated or removed, as the density of the medium to the density of the cylinder approximately. *Q.E.D.*

But a fluid must be pressed together so that it shall be continuous, truly it must be continuous and not elastic so that all the pressure, which arises from the compression of this, may be propagated in an instant, and in the motion all the parts of the body acted on equally may not change the resistance. Certainly the pressure, which arises from the motion of the body, is impeded in the motion of the parts of the fluid being generated and may create the resistance. But the pressure which arises from the compression of the fluid, however strong it may be, if it may be propagated instantaneously, shall generate no motion in the parts of the fluid, and in general it will lead to no change of the motion ; and thus the resistance shall neither be increased or diminished. Certainly the reaction of the fluid, which arises from the compression of this, cannot be stronger in the rear parts of the of the moving body than in the front parts, and thus the resistance described in this proposition cannot be diminished : and cannot be stronger in the front parts than in the latter parts, but only if the propagation of this may be infinitely faster than the motion of the compressed body. But it will only be infinitely faster and propagate instantaneously if the fluid shall be continuous and not elastic.

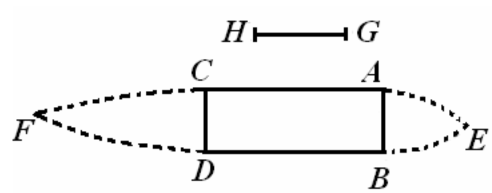
Corol. 1. The resistances of cylinders, which are progressing uniformly along their lengths in infinitely continuous mediums, are in a ratio composed from the square of the ratio of the velocities and in the square ratio of the diameters and in the ratio of the densities of the mediums.

Corol. 2. If the width of the pipe may not be increased indefinitely, but the cylinder may be progressing along its length in an enclosed medium at rest, and meanwhile its axis may coincide with the axis of the pipe : the resistance of this will be to the force by which the whole motion of this, in the time it will describe the quadruple of its length, either generated or removed, in a ratio which is composed from EF^2 to $EF^2 - \frac{1}{2}PQ^2$ once [this factor as explained above, see Cor. 10, expressed the force on the small circle compared to the force on the cylinder with the same small circular base and height $\frac{1}{2}IG$ as equal to the ratio EF^2 to $EF^2 - \frac{1}{2}PQ^2$, and to the ratio EF^2 and $EF^2 - PQ^2$ squared [*i.e.* the velocity squared], and in the ratio of density of the medium to the density of the cylinder.]

Corol. 3. With the same in place, and since the length L shall be to the quadruple of the length of the cylinder in a ratio which is composed from the ratio $EF^2 - \frac{1}{2}PQ^2$ to EF^2 once, and in the ratio $EF^2 - PQ^2$ to EF^2 squared : the resistance of the cylinder will be to the force, either to be taken away or generated, by which the whole motion of this, while the length L meanwhile will be described, as the density of the medium to the density of the cylinder.

Scholium.

In this proposition we have investigated the resistance which arises from the magnitude of the transverse section of the cylinder only, with the part of the resistance ignored which may arise from the obliquity of the motions. For just as in the first case of Proposition XXXVI, the obliquity of the motions by which the parts of the water in the vessel certainly converged on the hole *EF*, impeding the efflux of this water through the hole : thus in this proposition, obliquity of the motion, in which the parts of the water compressed by the front part of the cylinder concede to the compression and diverge on every side, retards the transition of these from the places at its anterior end by circulating to the posterior parts of the cylinder, and effects that the fluid will be moved together at a greater distance and will increase the resistance, and that almost in the ratio by which the efflux of the water from the vessel may be diminished, that is, in around the ratio 25 to 21. squared. And in the same manner, as in the first case of that proposition, we effected that the parts of the water be transferred perpendicularly and with maximum abundance through the hole *EF*, by putting all that water in a vessel in which the circulation of the stream was frozen, and the oblique motion of this was not used and remained in place without motion : thus in this proposition, so that the obliquity of the motion may be removed, and the parts of the water may allow the passage proceeding most easily by the cylinder in the shortest time with the motion directed mainly forwards, and only the resistance may remain, which arises from the magnitude of the transverse section, and which cannot be diminished except by diminishing the diameter of the cylinder. It is considered that the parts of the fluid, the motions of which are both useless and create resistance, may remain at rest among themselves at each end of the cylinder, and may stick together and be joined to the cylinder. Let *ABCD* be a rectangle, and both *AE* and *BE* shall be two parabolic arcs described with the axis *AB*, moreover with the latus rectum which shall be to the distance *HG*, with a cylinder being described by falling while it acquired its velocity, as *HG* to $\frac{1}{2}AB$. Also *CF* and *DF* shall be two other parabolic arcs, with the axis *CD* and with the latus rectum which shall be four times the first latus rectum described ; and by the rotation of the figure around the axes *EF* a solid may be generated the middle part of which shall



be the cylinder *ABDC* on which we are acting, and the end parts *ABE* and *CDF* may contain parts of the fluid at rest amongst themselves and acting on the body as two rigid solid parts, which adhere to the cylinder at each ends as head and tail. And the resistance of the solid *EACFDB*, following the length of its axis *FE* progressing in the direction of *E,* will be as nearly as we have described in this proposition, that is, which has that ratio to the force which the whole motion of the cylinder, while the whole length 4*AC* meanwhile may be described by that motion continued uniformly, that may be taken away or generated, that the density of the fluid has to the density of the cylinder approximately. And the resistance of this force cannot be less than in the ratio 2 to 3, by Corol. 7, Prop. XXXVI.

LEMMA V.

If a cylinder, a sphere, and a spheroid, the widths of which are equal, thus may be placed successively in the middle of a cylindrical pipe so that their axes may coincide with the axis of the pipe: these bodies may equally impede the flow of the water through the pipe.

For the spaces through which the water passes between the pipe and the cylinder, the sphere, and the spheroid are equal: and the water passes through equal spaces equally.

Thus these themselves are had by hypothesis, because all the water above the cylinder, sphere, or spheroid may be frozen, of which the fluidity is not required for the swiftest passage of the water, as I have explained in Corol. VII, Prop. XXXVl.

LEMMA VI.

With the same in place, the aforementioned bodies may be acted on equally by the water flowing in the pipe.

It is apparent by Lemma V and the third law of motion. Certainly the water and the bodies act on each equally and mutually.

LEMMA VII.

If the water may be at rest in the pipe, and these bodies may be carried in opposite directions with equal velocities, then the resistances of these are equal to each other.

This is agreed upon from the above lemma, for the relative motions remain between each other.

Scholium.

The ratio is the same of all convex and rounded bodies, the axes of which coincide with the axis of the pipe. Any differences can arise from the greater or lesser resistance, but in these lemmas we suppose the bodies to be very smooth, and the stickiness and friction of the medium to be zero, and because the parts of the fluid, which by their oblique motions and superfluous flow of water through the pipe, are able to disturb, impede, and to retard, are at rest among themselves as if they were restricted by ice, and may be attached to the front and rear parts of bodies, just as I have shown in the scholium of the preceding proposition. For in the following the smallest resistance of all that round bodies can have is acting, with the greatest transverse cross section described.

Bodies swimming on fluids, where they may be moving in a straight direction, effect that the fluid may ascend at the front parts and subside at the back parts, especially if the figures shall be obtuse , and therefore they may feel the resistance a little greater than if they were with sharp heads and tails. And bodies moving in elastic fluids, if they shall be obtuse before and aft, they may compress the fluid a little more at the front part and be a more relaxed at the rear part ; and therefore they experience a little more resistance than if they were sharp at the head and tail. But we do not work with elastic fluids in these lemmas and propositions, but with non elastic ones ; not with sitting on the fluid, but with deeply immersed in it. And when the resistance of bodies in non elastic fluids becomes known, this resistance will be increased by a small amount in elastic fluids, such as air, as on the surfaces of fluids at rest, such as seas and marshes.

PROPOSITION XXXVIII. THEOREM XXX.

The resistance of a sphere progressing in an infinitely compressed and non elastic fluid is approximately to the force by which the whole motion, in which time a three eights part of its diameter will be described, either taken away or generated, as the density of the fluid to the density of the sphere.

For the sphere is to the circumscribed cylinder as two is to three ; and therefore that force, which may be able to remove all the motion, while the cylinder meanwhile may describe a length of four diameters, all the motion of the sphere meanwhile may be taken away while the sphere describes two third parts of this length, that is, eight third parts of its own diameter. But the resistance of the cylinder is to this force approximately as the density of the fluid to the density of the cylinder or sphere by Prop. XXXVII, and the resistance of the sphere is equal to the resistance of the cylinder by Lem. V, VI, VII. *Q.E.D.*

Corol. I. The resistances of spheres, in infinitely compressed mediums, are in a ratio that is composed from the square ratio of the velocities, and in the square ratios of the diameters, and in the ratio of the densities of the mediums.

Corol. 2. The maximum velocity by which a sphere, by a force to be compared with its own weight, is able to descend in a resisting fluid, is that which the sphere likewise can acquire, by the same weight, by falling without resistance and in that case by describing a distance which shall be to four thirds parts of its diameter as the density of the sphere to the density of the fluid. For the sphere in the time of its own case, with the velocity of falling acquired, describes a distance which will be as eight thirds of its diameter, as the density of the sphere to the density of the fluid ; and the force of this weight generating this motion, will be to the force which may generate the same motion, in that time the sphere will describe eight thirds of its diameter with the same velocity, as the density of the fluid to the density of the sphere : and thus by this proposition, the force of the weight will be equal to the force of the resistance, and therefore the sphere cannot accelerate.

Corol. 3. With both the density of the sphere and its velocity at the beginning of the motion given, and as with the density of the compressed fluid at rest in which the sphere is moving; both the velocity of the sphere and the resistance of this sphere may be given at any time, as well as the distance described by that, by Corol. VII. Prop. XXXV.

Corol. 4. A sphere in a compressed fluid at rest with the same density by moving, the half part of its motion will be lost as it describes a length of two of its diameters, by the same Corol. VII.

PROPOSITION XXXIX. THEOREM XXXI.

The resistance of a sphere, progressing uniformly through the compressed fluid in a closed pipe, is to the force, by which the whole of this motion, while it will describe meanwhile the eight third parts of its diameter, either may be generated or removed, in a ratio which is composed from the ratio of the openings of the pipe to the excess of the opening over half the great circle of the sphere, and in the ratio doubled of the opening to the excess of this ratio over the great circle of the sphere, and with the density of the fluid to the density of the sphere approximately.

This is apparent by Corol. 2. Prop. XXXVII, and indeed the demonstration proceeds as in the preceding proposition.

Scholium.

In the two most recent demonstrations (in the same manner as in Lem. V.) I suppose that all the water which precedes the sphere may be turned to ice, and its fluidity increases the resistance of the sphere. If all that water may become liquid, the resistance will be increased a small amount. But that increase will be small in these propositions and can be ignored, provided that the whole convex surface of the sphere almost serves to be made of ice.

PROPOSITION XL. PROBLEM IX.

To find the resistance by phenomena, of a sphere progressing in a most compressed fluid medium.

Let A be the weight of the sphere in a vacuum, B its weight in a resisting medium, D the diameter of the sphere, F the distance which shall be to $\frac{4}{3}D$ as the density of the sphere to the density of the medium, that is, as A to $A-B$, G the time by which the sphere with the weight B by falling without resistance will describe the distance F, and H the velocity that the sphere acquires in this case itself. And H will be the maximum velocity by which the sphere, by its weight B, can fall in the resisting medium, by Corol. 2. Prop. XXXVIII, and the resistance that the sphere is allowed falling with that velocity, will be equal to the weight B of this: truly the resistance that is experienced with any velocity, will be to the weight B in the square ratio of the velocity of this to that maximum velocity H, by Corol, I. Prop. XXXVIII.

This is the resistance that arose from the inertia of the matter of the fluid. Indeed that which arises from the elasticity, the tenacity, and from the friction of the parts of this, thus will be investigated.

A sphere may be sent off so that it may descend in the fluid by its own weight B ; and P shall be the descent time, and that may be had in seconds, if the time G may be given in seconds. The absolute number N may be found which agrees with the logarithm $0{,}4342944819\frac{2P}{G}$, and let L be the logarithm of the number $\frac{N+1}{N}$; and the velocity acquired by falling will be $\frac{N-1}{N+1}H$, but the height described will be $\frac{2PF}{G}-1{,}3862943611F+4{,}605170186LF$. If the fluid may be of sufficient depth, the term $4{,}605170186LF$ can be ignored; and $\frac{2PF}{G}-1{,}3862943611F$ will describe the height approximately. These are shown by ninth proposition of the second book and its corollary, from the hypothesis that the sphere experiences no other resistance except for that which arises from the inertia of the matter. For if another resistance may be experienced above, the descent will be slower, and from the retardation the amount of this resistance may become known.

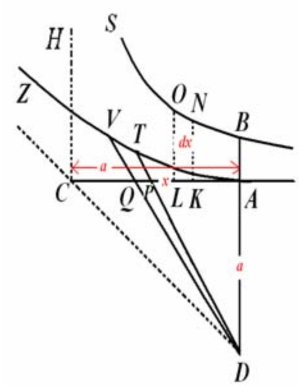
In order that the velocities of a body falling in a medium may become known more easily, I have composed the following table, the first column of which may denote the times of the descent, the second shows the velocities acquired by falling with the maximum velocity present 100000000, the third shows the distances described by falling in these times, with the distance $2F$ that the body will describe in the time G with the maximum velocity, and the fourth shows the distances in the same times described with the maximum velocity. The numbers in the fourth column are $\frac{2P}{G}$, and by subtracting the number $1{,}3862944-4{,}6051702\,L$, the numbers in the third column are found, and these numbers are multiplied by the distance F so that the distances described by falling may be obtained. To these above a fifth column is added, which contains the distances described in the same times by a body falling in a vacuum, to be compared with the force of its own weight B.

.

Times P.	*Velocities falling in fluid.*	*Distances described falling in fluid.*	*Distances described at the maximum velocity.*	*Distances described falling in a vacuum.*
0,001G	$99999\frac{29}{30}$	0,000001F	0,002F	0,000001F
0,01G	999967	0,0001F	0,02F	0,0001F
0,1G	9966799	0,0099834F	0,2F	0,01F
0,2G	19737532	0,0397361F	0,4F	0,04F
0,3G	29131261	0,0886815F	0,6F	0,09F
0,4G	37994896	0,1559070F	0,8F	0,16F
0,5G	46211716	0,2402290F	1,0F	0,25F
0,6G	53704957	0,3402706F	1,2F	0,36F
0,7G	60436778	0,4545405F	1,4F	0,49F
0,8G	66403677	0,5815071F	1,6F	0,64 F
0,9G	71629787	0,7196609F	1,8F	0,81F
1G	7615 9416	0,8675617F	2F	1F
2G	96402758	2,65 00055F	4F	4F
3G	99505475	4,6186570F	6F	9F
4G	99932930	6,6143765F	8F	16F
5G	99990920	8,6137964F	10F	25F

6*G*	99998771	10,6137179*F*	12*F*	36*F*
7*G*	99999834	12,6137073*F*	14*F*	49*F*
8*G*	99999980	14,6137059*F*	16*F*	64*F*
9*G*	99999997	16,6137057*F*	18*F*	81*F*
10*G*	99999999 $\frac{3}{5}$	18,6137056*F*	20*F*	100*F*

[Note 284. L. & J. p. 699.
So that the demonstration of these things which Newton has presented may be understood easily, some of the matters which he demonstrated in Propositions VIII & IX in Section 2 are to be recalled. Let *CH* and *AB* be perpendicular right lines to a given right line *AC*, indeed with *CH* infinite, and with $BA = \frac{1}{4}AC$. With centre *C* and with the asymptotes *CH, CA*, a hyperbola *BNS* may be described through the point B taking *AC, AP*, *AK* in continued proportion, and the right line *KN* is drawn parallel to *AB*. And if a heavy body may fall from rest in a medium that resists in the square ratio of the velocity, the area *ABNK* may represent the distance described by the body in falling; and the velocity of the body acquired in this case will be able to be represented by the line *AP*, and its maximum velocity by the given line *AC* (by Cor.1 & 2 Prop. VIII). Now *BA* may be produced to *D* so that $AD = AC$, DC may be joined, and with centre *D*, with the asymptote *DC*, and with the the principal vertex *A*, another hyperbola *ATZ* may be described, which the line *DP* produced may cut in *T*, and the line *DQ* becomes infinitely near to the line *DP* itself at *V*; and the vanishing sector $DTV = \frac{PDQ\times AC}{CK}$, and the sector *ATD* represents the time in which the falling body describes the distance *ABNK*, and by which it acquires the velocity *AP* (by Case 2, Prop. IX). Truly the distance that the body will describe falling in some time *ATD*, will be to the distance that the body can describe by progressing uniformly in the same time with the maximum velocity *AC*, as the area *ABNK* to the area *ATD* (by Cor.1. Prop. IX), and the time in which the body by falling in the resisting medium acquires the velocity *AP*, will be to the time in which the maximum velocity *AC* in non-resisting medium by the force of its weight by falling in comparison may acquire, as the sector *ATD* to the triangle *ADC* (by Cor.5. Prop. IX).



Note 285. L. & J. With these presumed, there may be called $AC = AD = a, AB = \frac{1}{4}a, AP = x, PQ = dx$. The hyperbola *SNB* with the origin at *A*, is given by $y = \frac{a}{4}\cdot\frac{1}{x+a}$; while the hyperbola *ZT*A, with the origin at *D*, is given by $y^2 - x^2 = a^2$.
Because $\frac{AC}{AP} = \frac{AP}{AK}$, $AK = \frac{x^2}{a}, CK = \frac{a^2-x^2}{a}$, the triangle $PDQ = \frac{1}{2}adx$, and the sector $DTV = \frac{PDQ\times AC}{CK} = \frac{\frac{1}{2}a^3dx}{a^2-x^2} = \frac{\frac{1}{4}a^2dx}{a+x} + \frac{\frac{1}{4}a^2dx}{a-x}$;

[Thus, in modern terms, from the equation $\frac{dv}{dt} = k\left(v_0^2 - v^2\right)$, where v_0 represents the terminal velocity from $g = kv_0^2$, on separating the terms and integrating, we have using different variables, $kdt = \frac{dv}{v_0^2 - v^2}$; $2kv_0 t = \int_0^v \frac{dv}{v_0+v} + \frac{dv}{v_0-v} = ln\left(\frac{v_0+v}{v_0-v}\right)$.] ; hence, on taking the integral, the sector *ATD* representing the time is obtained $ATD = \frac{1}{4}a^2 L.\overline{a+x} - \frac{1}{4}a^2 L.\overline{a-x} = \frac{1}{4}a^2 L.\frac{a+x}{a-x}$, to which quantity nothing is required to be added or subtracted, because there shall be $ATD = 0$, and $x = 0$, vanishing as required.

LO may be drawn parallel to *KN* and infinitely close to it; and since there shall be $AK = \frac{x^2}{a}$, and (by Theorem IV by hypothesis) $\frac{CA \times AB}{CK} = \frac{\frac{1}{2}a^3}{a^2 - x^2}$ and $KL = \frac{2xdx}{a}$, the differential of the area *ABNK* will be $= \frac{\frac{1}{2}a^2 xdx}{a^2 - x^2}$, and with the integrations made, the area corresponding to the distance $ABNK = Q - \frac{1}{4}a^2\ L.\overline{a^2 - x^2}$; indeed because the area *ABNK* vanishes when $x = 0$, the constant *Q* will become $\frac{1}{4}a^2\ L.a^2$, and the final area $ABNK = \frac{1}{4}a^2 L.a^2 - \frac{1}{4}a^2\ L.\overline{a^2 - x^2} = \frac{1}{4}a^2 \times L.\frac{a^2}{a^2 - x^2}$.

[In this case, from the equation $v\frac{dv}{ds} = k\left(v_0^2 - v^2\right)$ on taking the downwards direction as positive, this gives $\int \frac{2vdv}{v_0^2 - v^2} = \int \frac{dv}{v_0 - v} - \int \frac{dv}{v_0 + v} = 2k\int ds;$ and $s = \frac{1}{2k} ln \frac{v_0^2}{v_0^2 - v^2}$, with an obvious change of variables, recalling that $kv_0^2 = g$, where the terminal velocity is v_0, so that *k* has the dimensions of time. Back to L. & J.]

Again the time *P* in which the body, by falling in the resisting medium, acquires the velocity of the line *AP*, or the proportional *x* is to the time *G* in which the maximum velocity *H* can be acquired by the force of its weight, by the weight B falling in comparison without resistance, as the sector *ATD* to the triangle *ADC*, that is, $\frac{P}{G} = \frac{\frac{1}{4}a^2 L.\frac{a+x}{a-x}}{\frac{1}{2}a^2} = \frac{1}{2}L.\frac{a+x}{a-x}$.

[For without resistance, the weight *B* falls under gravity *g* alone, and in time *T* it will reach the terminal velocity v_0, so that $v_0 = gT$ and $kv_0^2 = g = \frac{v_0}{T}$; hence $ka^2 = g = \frac{a}{G}$ or $G = \frac{1}{ka}$; $t = \frac{1}{2kv_0} ln\left(\frac{v_0+v}{v_0-v}\right) \rightarrow P = \frac{1}{2ka} ln\left(\frac{a+v}{a-v}\right)$; hence $\frac{2P}{G} = L.\frac{a+v}{a-v} = L.\frac{a+x}{a-x}$.]

Whereby there will be $\frac{2P}{G} = L.\frac{a+x}{a-x}$, with this logarithm taken with the logistic of which the subtangent is one. On account of which if the logarithm of the numbers $\frac{a+x}{a-x}$ may be taken from tables, it is done by multiplying by the number 2,302585093 [$= ln10$], as in Cor. 7 Prop. XXXV, and there will be had $\frac{2P}{G} = 2{,}302585093 L.\frac{a+x}{a-x}$, and thus on dividing 1. by 2,3025..... the number $0{,}4342944819 \times \frac{2P}{G}$ is the logarithm of the number $\frac{a+x}{a-x}$ [to base 10]. And thus if the absolute number *N* is sought from tables which agree with the

logarithm $0{,}4342944819\times\frac{2P}{G}$, there will be $N=\frac{a+x}{a-x}$, and thus $x=\frac{a(N-1)}{N+1}$. But *AC* to *AP* or *a* to *x*, shall be as the maximum velocity *H* to the velocity acquired by falling. Whereby this velocity will be $\frac{xH}{a}=\frac{N-1}{N+1}\times H$, just as Newton found. The distance the sphere will describe in the time *P*, progressing uniformly with the maximum velocity *H*, is to the distance 2*F* that it can traverse with the same velocity *H* in the time *G*, as the time *P* to the time *G* (5. Lib. I), and therefore that distance is $\frac{2PF}{G}$. The height *S* that the sphere will describe in the time P by falling in a resisting medium, is to the distance $\frac{2PF}{G}$, as the area *ABNK* to the sector *ATD*, that is, as
$\frac{1}{4}a^2\times L.\frac{a^2}{a^2-x^2}$ to $\frac{1}{4}a^2\times L.\frac{a+x}{a-x}$ or $L.\frac{a^2-x^2}{a^2}$ to $L.\frac{a+x}{a-x}$, but from above, $N=\frac{a+x}{a-x}$, and $x=\frac{a(N-1)}{N+1}$, and hence $\frac{a^2}{a^2-x^2}=\frac{(N+1)^2}{4N}=\frac{N\times(N+1)^2}{4N^2}$, and if logarithms are taken in the logistic of which the subtangent is one (*i.e*. natural logs)
$\frac{2P}{G}=L.\frac{a+x}{a-x}=L.N+2L.\frac{N+1}{N}-L.4$; and hence

$$L.\frac{a^2}{a^2-x^2}:L.\frac{a+x}{a-x}=L.N+2L.\frac{N+1}{N}-L.4:L.N$$

$$=1+\frac{2L.\frac{N+1}{N}-L.4}{L.N}:1=1+\frac{G}{P}L.\frac{N+1}{N}-\frac{G}{2P}L.4:1=S:\frac{2PF}{G}.$$

Whereby the altitude $S=\frac{2PF}{G}-FL.4+2FL.\frac{N+1}{N}$
But if we wish to use log. tables, these are multiplied by the number 2,302585092994 or 2,302585093. Here the number may be called M, the logarithm of the number 4 taken from the tables *Q*, and the logarithm of the number $\frac{N+1}{N}$ shall be *L*; and there will be $S=\frac{2PF}{G}-MQF+2MLF$. But there is $2M=4{,}605170186$, and *Q* in common tables of logarithms is 0,60206; or more accurately 0,602059991333, and thus $MQ=1{,}3862943611$ approximately. Whereby the height *S*, that the sphere describes in the time *P* by falling in the resisting medium, is $\frac{2PF}{G}-1{,}3862943911F+4{,}605170186LF$, as Newton defined.

If the distance *S* that the sphere may fall were so great, that the term $4{,}605170186LF$ could be ignored; then *L* shall be the logarithm of the number $\frac{N+1}{N}$, where *N* shall be a number so large, or where the number $\frac{N+1}{N}$ may be almost equal to one, the logarithm *L* vanishes approximately. But, if the maximum velocity may be called *H*, and the velocity *V* of the sphere is acquired in that time *P* , there is $\frac{H}{V}=\frac{a}{x}$ and thus $\frac{H+V}{H-V}=\frac{a+x}{a-x}=N$, and when the distance *S* is large enough, there becomes $V=H$ approximately, and hence $\frac{H+V}{H-V}$ or *N* a number large enough, as is evident from the above table; hence the proposition is shown.]

Scholium.

In order that I might investigate the resistance of fluids by experiments, I prepared a square wooden vessel, with length and width of nine inches inside of English feet, with a depth of nine and a half feet, the same I filled with rainwater; and with spheres formed from wax with lead enclosed, I noted the times of descent of the spheres, with the descent in the height being 112 inches. A volume of an English cubic foot contains 76 pounds *Avoirdupois* of rainwater, and of this a cubic inch contains $\frac{19}{36}$ ounces of weight or $253\frac{1}{3}$ grains ; and a sphere of water of diameter one inch contains 132,645 grains in the medium of air, or 132,8 grains *Avoirdupois* in a vacuum ; and any other sphere is as the excess of this weight in a vacuum over its weight in water.

Expt. 1. A sphere, the weight of which was $156\frac{1}{4}$ grains in air and 77 grains in water, described the whole height of 112 digits in a time of 4 seconds. And with the experiment repeated, the sphere again fell in the same time of 4 seconds.

The weight of the sphere in a vacuum is $156\frac{13}{38}$ grains and the excess of this over the weight of the sphere in water is $79\frac{13}{38}$ grains. From which the diameter of the sphere produced is 0,84224 parts of an inch. But as that excess is to the weight of the sphere in a vacuum, hence the density of the water to the density of the sphere, and thus 8/3 parts of the diameter of the sphere (*viz*. 2,24597 inches) to the distance 2*F*, that hence will be 4,4156 inches. The sphere by falling in a vacuum in the time of one second with its whole weight of $156\frac{13}{38}$ grains, will describe $193\frac{1}{3}$ inches; and with a weight of 77 grains, in the same time, by falling without resistance in water will describe 95,219 inches; and in the time *G*, which shall be to one second in the square root ratio of the distance *F*, or 2,2128 inches to 95,219 inches, it will describe 2,2128 inches, and it will be able to acquire that maximum velocity *H* to descend in water. Therefore the time *G* is 0,15244 seconds. And in this time *G*, with that maximum velocity *H*, the sphere will describe a distance 2*F* of 4,4256 inches; and thus in the time of four seconds it will describe a distance of 1l6,1245 inches. The distance 1,3862944*F* or 3,0676 inches may be taken away, and there will remain a distance of 113,0569 inches that the sphere by falling in water, in the widest vessel, will describe in a time of four seconds. This distance, on account of the aforementioned narrow wooden vessel, ought to be lessened in a ratio that is composed from the square root ratio of the opening of the vessel to the excess of this opening over the greatest semicircle of the sphere, and from the simple of the same opening to the excess of this over the great circle of the sphere, that is, in the ratio 1 to 0,9914. With which done, it will give a distance of 112,08 inches, which the sphere by falling in water in this wooden vessel in a time of four seconds must describe approximately. Indeed by experiment it has described 112 inches.

Expt. 2. Three equal spheres, of which the weights themselves were $76\frac{1}{3}$ grains in air and $5\frac{1}{16}$ grains in water, were released successively ; and each one fell in a time of 15 seconds, in each case by describing a height of 112 inches.

By entering into the computation they produced a weight of the sphere in a vacuum of $76\frac{5}{12}$ grains, the excess of the weight of $71\frac{17}{48}$ grains over the weight in water, of a sphere of diameter 0,81296 inches, $\frac{8}{3}$ parts of this diameter 2,16789 inches; the distance 2*F* 2,3217 inches; the distance that the sphere with a weight of $5\frac{1}{16}$ grains in a time of 1 second may describe by falling 11,808 inches without resistance, and the time *G* 0,301056 seconds. Therefore the sphere, with that maximum velocity it can describe in water by the weight of the force $5\frac{1}{16}$ grains, in the time 0,301056 seconds will describe the distance 2,3217 inches and in the time 15 a distance of 115,678 inches. The distance 1,3862944F or 1,609 inches may be subtracted and the distance114,069 inches will remain which the sphere must be able to describe by falling in the same time in the widest vessel. Therefore the narrowness of our vessel must take away a distance of around 0,895 inches. And thus there will remain a distance of 113,174 inches which the sphere by falling in this vessel, ought to describe in 15 seconds by the theorem approximately. Truly it describes 112 inches by experiment. The difference is insignificant.

Expt. 3. Three equal spheres, whose weights were separately 121 grains in air and 1 grain in water, were successively dropped ; and they were falling in water describing heights of 112 inches in the times 46, 47, and 50 seconds.

By the theorem these spheres should fall in a time around 40 seconds. Because they have fallen slower, whether for a smaller part of the resistance arising from the force of inertia in slowing the motions, or it is required to attribute a resistance that arises to other causes ; perhaps to some bubbles adhering to the sphere, or to the evaporation of the wax either by the heat or warmth of the season or by dropping the sphere by hand, or even by unknown errors in weighing the spheres in water, I am unsure. And thus the weight of the sphere in water must be of several grains, so that the experiment may be rendered certain and trustworthy.

Expt. 4. The experiments described so far I had began so that I could investigate the resistance of fluids, before the theory in the nearby preceding propositions set out by me was known. Afterwards, so that I could examine the theory found, I prepares a wooden vessel with an internal width of $8\frac{2}{3}$ inches, with a depth of $15\frac{1}{3}$ feet. Then I made four spheres from wax with lead inside, the individual ones weighing $139\frac{1}{4}$ grains in air and $7\frac{1}{8}$ grains in water. And these I released so that I could measure the falling times in water by a pendulum, oscillating in half seconds. The spheres, when they were being weighed and afterwards were cold, and they remained cold for some time ; because heat evaporated the wax, and by the evaporation diminished the weight of the sphere in water, and the evaporated wax is not at once restored to the former density by cold. Before they fell, they were thoroughly immersed in water; lest with the weight from some parts standing clear from the water might accelerate the descent from the start. And when immersed they become completely still, they were being dropped most cautiously, lest they might accept some impulse from the hand on being dropped. Moreover they fell in the successive times of oscillation $47\frac{1}{2}$, $48\frac{1}{2}$, 50 and 51, describing a height of 15 feet and 2 inches. But the weather was now a little colder than when the spheres were weighed, and thus I repeated the experiment on another day, and the spheres were falling in the times of 49, $49\frac{1}{2}$, 50 and 53 oscillations, and on a third attempt with the times of $49\frac{1}{2}$, 50, 51 and 53 oscillations. And with the experiment taken more often, the spheres fell mainly from the times of the oscillations $49\frac{1}{2}$ and 50. When falling slower, I suspect to be retarded by striking with the sides of the vessel.

Now the computation by the theorem being entered into, they produce the weight of the sphere in a vacuum of $139\frac{2}{5}$ grains. An excess of this weight over the weight of the sphere in water of $132\frac{11}{40}$ grains. The diameter of the sphere is 0,99868 inches. The $\frac{8}{3}$ parts of the diameter 2,66315 inches. The interval $2F$ becomes 2,8066 inches. The distance which a sphere with a weight of $7\frac{1}{8}$ grains describes in a time of one second by falling without resistance 9,88164 inches. And the time G 0, 376843 seconds. Therefore the sphere, with the maximum velocity by which it can descent in water by a force of a weight of $7\frac{1}{8}$, in a time 0,376843 seconds will describe a distance 2,8066 inches, and in a time of 1 second, a distance of 7,44766 inches, and in the time of 25 seconds or of 50 oscillations a distance of 186,1915 inches [in these days an oscillation was the motion of a pendulum from one side to the other, or half the modern period]. The distance 1,386294F, or 1,9454 inches may be taken away, and there will remain 184,2461 inches which the sphere in the same time in the widest vessel. On account of the narrowness of our vessel, this distance may be diminished in a ratio which is composed from the square root ratio of the opening of the vessel and the excess of this opening over the great semicircle of the sphere, and to the simple ratio of this same orifice to its excess over a great circle of the sphere ; and the distance 181,86 inches will be had, which the sphere ought to describe in this vessel in the time of 50 approximately by the theorem. In truth it may describe a distance of 182 inches in a time of $49\frac{1}{2}$ or 50 oscillations by experiment.

Expt. 5. Four spheres with a weight of $154\frac{1}{8}$ grains in air and $21\frac{1}{2}$ grains in water are dropped often, falling in a time of $28\frac{1}{2}$, 29, $29\frac{1}{2}$ and 30 oscillations, and occasionally of 31, 32 and 33 oscillations, describing heights of 15 feet and 2 inches.
By the theorem they ought to fall in a time of 29 approximately.

Expt. 6. Five spheres with a weight of $212\frac{3}{8}$ grains in air and $79\frac{1}{2}$ grains in water were dropped a number of times, they were falling in the times of $15\frac{1}{2}$, 16, 17 and 18 oscillations, describing heights of 15 feet and 2 inches.

By the theorem they ought to fall in a time of approximately15 oscillations.

Expt.7. Four spheres weigh $293\frac{3}{8}$ grains in air and $35\frac{7}{8}$ grains in water were dropped a number of times, they were falling in the times $29\frac{1}{2}$, 30, $30\frac{1}{2}$, 31, 32 and 33 oscillations, describing heights of 15 feet and 2 inches.

By the theorem they ought to fall in a time of approximately 28 oscillations.

The cause requiring to be investigated why of spheres of the same weight and magnitude, some may fall faster or slower, I fell upon this ; because the spheres, when they were being first released and they were beginning to fall, were turning about the centres, with the side that was perhaps the heavier to be the first to descend, and by generating a motion of oscillation. For by its oscillations the sphere could communicate more motion to the water, than if it were descending without oscillations ; and by communicating, it lost a part of its proper motion by which it ought to descend: and by a greater or smaller oscillation, it may be retarded more or less. Truly indeed the sphere always departed from its side that descended by the oscillation, and by receding approached the sides of vessel and occasionally struck the sides. And this oscillation was stronger in heavier spheres, and with the larger disturbed more water. On which account, so that the oscillation of the spheres could be made less, I constructed nine spheres from wax and lead, I put in place the lead on some side of the sphere close to the surface of this ; and the sphere thus dropped, so that the heavier side, as long as that could be done, should be the lowest from the beginning of the descent. Thus the oscillations were made much less than at first, and the spheres fell in less unequal times, as in the following experiments.

Expt. 8. Four spheres, with a weight of 139 grains in air and $6\frac{1}{2}$ in water, were dropped a number of times, they fell in times of not more than 52 oscillations, not many less than 50, and the most from a time of around 51 oscillations, describing a height of 182 inches. By the theorem they ought to fall in a time of approximately 52 oscillations.

Expt. 9. Four spheres, with a weight of $273\frac{1}{4}$ in air and $140\frac{1}{4}$ in water, were dropped a number of times, they fell in times of not fewer than 12 oscillations, not of much more than 13, describing a height of 182 inches.

By the theorem they ought to fall in a time of approximately $11\frac{1}{3}$ oscillations.

Expt. 10. Four spheres, with a weight of 384 in air and $119\frac{1}{2}$ in water, were dropped a number of times, they fell in the times of $17\frac{3}{4}$, 18, $18\frac{1}{2}$, and 19, oscillations, describing a height of $181\frac{1}{2}$ inches. And when they fell in the time of 19 oscillations, I heard only a few strike the side of the vessel before they arrived at the bottom.

By the theorem they ought to fall in a time of approximately $15\frac{5}{8}$ oscillations.

Expt. 11. Three equal spheres, with weights of 48 grains in air and $3\frac{29}{30}$ grains in water, were dropped often, and they fell in times of $43\frac{1}{2}$, 44, $44\frac{1}{2}$, 45 and 46 oscillations, and for the greater part, from 44 and 45 , describing a height of $182\frac{1}{2}$ approximately.

By the theorem they ought to fall in a time of approximately $46\frac{5}{9}$ oscillations.

Expt. 12. Three equal spheres, with weights of 141 grains in air and $4\frac{3}{8}$ grains in water, were dropped a number of times, they dropped in times of 61, 62, 63, 64 and 65 oscillations, describing a height of 182 inches.

By the theorem they ought to fall in a time of approximately $64\frac{3}{4}$ oscillations.

By these experiments it is clear that, when the spheres fell slowly, as in the second, fourth, fifth, eighth, eleventh and twelfth experiments, the falling times were correctly shown by theory; but when the spheres fell faster, as in the sixth, ninth, and tenth experiments, the resistance stood out a little more than in the square of the velocity. For the spheres during falling oscillate a little, and this oscillation in the lighter and slower falling spheres quickly ceases, on account of the lightness of the motion ; but in the heavier and greater, on account of the strength the motion the oscillations may endure a long time, and cannot be confined until after several oscillations in the surrounding water. Truly the swifter spheres, there may be pressed on less by the fluid on their rear parts ; and if the velocity may be constantly increased, they will leave finally a vacuum in the space behind, unless likewise the compression of the fluid may be increased. But the compression of the fluid must be increased in the square ratio of the velocity (by Prop. XXXII. & XXXIII.), so that the resistance shall be in the same square ratio. Because this may not be, the faster spheres are pressed a little less from behind, and from the deficiency of this pressure, the resistance of these shall be a little greater than in the square ratio of the velocities.

Therefore the theory agrees with the phenomena of bodies falling in water, it remains that we examine the phenomena of bodies falling in air.

Expt.13. From the top of St. Paul's Church, in the town of London, in the month of June, 1710, two glass spheres were dropped simultaneously, the one full of mercury, and the other air ; and falling they described a height of 220 English feet. A wooden table was suspended at one end by an iron rod, at the other it rested on a wooden peg, and the two spheres set on this table were dropped at the same time, by removing the peg with the help of an iron wire sent as far as the ground so that the table supported only by the iron rod could rotate about the same, and at the same instant by pulling on that wire a pendulum could start oscillating in seconds. The diameters and weights of the spheres and the times of falling are shown in the following table.

The spheres full of mercury.			*The spheres full of air.*		
Weights in grains.	*Diameters in inches*	*Times of falling in seconds*	*Weights in grains.*	*Diameters in inches.*	*Times of falling in seconds*
908	0,8.	4	510	5,1	$8\frac{1}{2}$
983	0,8	4 –	642	5,2	8
866	0,8	4	599	5,1	8
747	0,75	4+	5I5	5,0	$8\frac{1}{4}$
808	0,75	4	483	5,0	$8\frac{1}{2}$
784	0,75	4+	641	5,2	8

Besides the observed times must be corrected. For the mercury spheres (from *Galilio's* theory) describe 257 English feet in four seconds, and 220 feet in only 3'' 41''' . [We will use Newton's notation henceforth; thus $3'' \;\, 41'''$ means $3\frac{41}{60}$seconds]. Certainly the wooden table, with the peg removed, was turning slower than suitable, and by its slowness in rotation impeded the descent of the spheres from the start. For the spheres were resting on the table near its middle, and indeed they were a little nearer to the axis of this than to the peg. And hence the falling times were prolonged around 18''' , and now must be corrected by taking that small amount, especially with the larger spheres which were resting a little longer on the rotating table on account of the size of the diameters. With which done the times, in which the 6 larger spheres fell, became 8'' 12''' , 7'' 42''' , 7'' 42''', 7'' 57''', 8'' 12''' , and 7'' 42'''.

Therefore the fifth of the spheres full of air, constructed with a diameter of five inches and with a weight of 483 grains, fell in a time of 8'' 12''' , in describing a height of 220 feet. The weight of water equal to this sphere is 16600 grains; and the weight of air equal to the same [volume] is $\frac{16600}{800}$ grains or $19\frac{3}{10}$ grains and thus the weight of the sphere in a vacuum is $502\frac{3}{10}$ grains and this weight is to the weight of the air in the sphere, as $502\frac{3}{10}$ to $19\frac{3}{10}$, and thus there shall be $2F$ to $\frac{8}{3}$ parts of the diameter of the sphere, that is, to $13\frac{1}{3}$ inches. From which $2F$ produces 28 feet 11 inches. The sphere by falling in a vacuum, with its whole weight $502\frac{3}{10}$ grains, will describe in a time of one second $193\frac{1}{3}$ inches as above, and with a weight of 483 grains it will describe 185,905 inches, and with the same weight of 483 grains also in a vacuum it will describe a distance *F* or 14 feet $5\frac{1}{2}$ inches in a time of 57" 58"" , and with that maximum velocity it can acquire by falling in air. With this velocity the sphere, in the time 8" 12"', describes the distance 245feet and $5\frac{1}{3}$ inches. Take away 1,3863*F* or 20 feet $0\frac{1}{2}$ inches and 225 feet 5 inches shall remain. Therefore the sphere, in the time 8" 12"' must describe this distance by falling according to the theory. Truly the distance described will be 220 feet by experiment. The difference is negligible.

By similar computations applied also to the remaining spheres filled with air, I have put together the following table.

Weights of the spheres (grains).	*Diameters (inches).*	*Time to fall 220 feet. (sec.)*	*Distance describe by theory.*	*Excess.*
510	5,1	8" 12"'	226 *ft.* 11 *in.*	6 *ft.* 11 *in.*
642	5,2	7 42	230 9	7 10
599	5,1	7 42	227 10	7 10
515	5	7 57	224 5	4 5
483	5	8 12	225 5	5 5
641	5,2	7 42	230 7	10 7

Expt. 14. In the year 1719. in the month of July, *Dr. Desaguliers* took in hand experiments of this kind again, by forming pigs bladders into spherical shapes with the aid of a concave wooden sphere, which wetted were forced to be filled with air ; and these were then dried and removed. By dropping from a higher place in the same holy place from the arch of the copula, namely from a height of 272 feet; and at the same moment of time by dropping also a leaden sphere, whose weight was around two pounds avoirdupois. And meanwhile someone standing in the upper part of the dome, when the spheres were dropped, was noting the whole time of falling, and others standing on the ground were noting the difference of the times between the case of the leaden sphere and of the bladder. Moreover the times were measured by pendulums oscillating at the half second. And of these who were standing on the ground one had a clock which vibrated with a sound in a individual quarter seconds, another had a different machine skillfully constructed also with a pendulum that vibrated four times per second. And one of those present who were at the top of the church had a similar machine. And these instruments thus were formed, so that the motion of these could either be started or stopped as it pleased. Moreover the leaden sphere fell in a time of around $4\frac{1}{4}$ seconds. And by adding this time to the aforementioned difference of the times, the total time could be deduced in which the bladder fell. The times, in which the five bladders fell after the case of the first leaden sphere, were $14\frac{3}{4}'',12\frac{3}{4}'',14\frac{5}{8}'',17\frac{3}{4}'',$ and $16\frac{3}{4}''$, and the following in turn $14\frac{1}{2}'',12\frac{1}{4}'',14'',19'',$ and $16\frac{3}{4}''$. There may be added $4\frac{1}{4}''$, certainly the time in which the leaden sphere fell, and the total time, in which the five bladders fell, were in the first place $19'',17'',18\frac{7}{8}'',22'',$ and $21\frac{1}{8}''$; and in the second place, $18\frac{3}{4}'',18\frac{1}{2}'',18\frac{1}{4}'',23\frac{1}{4}'',$ and $21''$. Moreover the times noted from the top of the church , were in the first turn $19\frac{3}{4}'',17\frac{1}{4}'',18\frac{3}{4}'',22\frac{1}{8}'',$ and $21\frac{5}{8}''$; and in the second turn $19'',18\frac{5}{8}'',18\frac{3}{8}'',24'',$ and $21\frac{1}{4}''$.

The rest of the bladders did not always fall straight down, but sometimes were flying about, and hence thus they were moving to and fro while falling. And from these motions the times of falling were extended and sometimes increased by as much as half a second, sometimes by a whole second. But the second and fourth fell more straight in the first turn; and the first and third in the second turn. The fifth bladder was more wrinkled and by it wrinkles somewhat retarded. The diameters of the bladders I deduced from their circumferences measured twice by a fine thread passed around. And I have brought together the theory with the experiments in the following table, by assuming the density of air to be to the density of water as 1 to 860, and by computing the distances which the spheres must describe by falling by theory.

The weights of the bladders (grains).	*Diameters (inches).*	*Times required to fall from a height of 272 feet.*	*Distance described in the same times by theory.*	*Difference between theory & expt.*
128	5, 28	19"	271*ft.* 11 *in.*	– 0*ft.* 1*in.*
156	5, 19	17	272 $0\frac{1}{2}$	+ 0 $0\frac{1}{2}$
$137\frac{1}{2}$	5, 3	$18\frac{1}{2}$	272 7	+ 0 7
$97\frac{1}{2}$	5 26	22	277 4	+ 5 4
$99\frac{1}{8}$	5	$21\frac{1}{8}$	282 0	+ 10 0

Therefore the resistances of nearly all the spheres moving both in air and in water are shown correctly from our theory, and is proportional to the densities of the fluids, with equal velocities and magnitudes of the spheres.

In the scholium, which has been added to the sixth section, we have shown by experiments with pendulums that the resistances of the motions of equal and equal moving spheres in air, water, and in quicksilver are as the densities of the fluids. Here we have shown the same more accurately from experiments with bodies falling in air and in water. For the individual oscillations of pendulums always move the fluid in a direction opposite to the direction of the returning swing, and the resistance that arises from that motion, and as the resistance of the thread by which it was being suspended, the total resistance of the pendulum were rendered greater than the resistance produced by a body falling. And also by the experiments with pendulums set out there in the scholium, a sphere of the same density as water, by describing a length of half its diameter in air, ought to lose the $\frac{1}{3342}$ th part of its motion. But by the theory I have set out in this seventh section and I have confirmed from the experiments with falling bodies, the same sphere by describing the same length, ought to lose only the $\frac{1}{4586}$ th part, supposing the density of water shall be put to the density of air as 860 to 1. Therefore greater resistances were produced by the experiments with pendulums (on account of the reasons now described) than by the experiments with falling spheres, and that approximately in the ratio of 4 to 3. Yet since the resistance of pendulums in air, water, and in quicksilver may be likewise increased by like causes, the proportion of the resistance in these mediums, both by the experiments with pendulums, as well as by the experiments with falling spheres, may be demonstrated well enough. And thence it can be concluded that the resistances of the motions of bodies in any of the most free of fluids, with all else being equal, are as the densities of the fluids.

Thus with these established, now it is permitted [to find] what part of its motion will be lost by any sphere, projected in some fluid, in some given time approximately. D shall be the diameter of the sphere, and V its initial velocity, and T the time, in which the sphere with the velocity V in a vacuum may describe a distance, which shall be to the distance $\frac{8}{3}D$ as the density of the sphere to the density of the fluid : and the sphere projected into the fluid, at some other time t, will lose the $\frac{tV}{T+t}$ part of its velocity, with the part $\frac{TV}{T+t}$ remaining, and a distance described, which shall be to the distance described with the uniform speed V in the same time, as the logarithm of the number $\frac{T+t}{T}$ multiplies by the number 2,302585093 is to the number $\frac{t}{T}$ by Corol. VII, Prop. XXXV. In the slower motions the resistance can be a little less, because that figure of the sphere shall be a little more suited to the motion than the figure of a cylinder described of the same diameter. In motions with greater velocities the resistance can be a little greater, because since the elasticity and the compression of the fluid may not be increased in the square ratio of the velocity. But I will not dwell on trifling details of this kind here.

And although air, water, quicksilver and like fluids, by the indefinite division of parts, may become more subtle and be made infinitely fluid mediums ; yet they may offer no less resistance to projected spheres. For the resistance, by which it was acted on in the preceding propositions, arises from the inertia of the matter, and the essential inertia of matter in bodies is always in proportion to the quantity of matter. By the division of the parts of the fluid, the resistance which arises from the tenacity and the friction of the parts can indeed be diminished : but the quantity of matter through the divisions of the parts of this is not diminished; and with the quantity of matter remaining, the inertial force of this remains, to which the resistance, by which this is acted on, is always proportional. In order that this resistance may be diminished, the quantity of matter must be diminished in the interval through which the body is moving. And because the celestial spaces, through which the spheres of the planets and comets to all parts freely and without any diminution of the motion may be considered to be moving perpetually, they are free of all corporal fluid, if perhaps rare vapours and the trajectories of light ray be excepted.

Certainly projectiles excite motions in fluids by passing through them, and this motion arises from the excess of the pressure of the fluid on the anterior parts of the projectile over the pressure on the posterior parts of this, and cannot be less in infinite fluid mediums than in air, water, and quicksilver for the density of the matter in each. But this excess pressure, from its amount, not only may excite motion in the fluid, but also act on the projectile and to retard its motion : and therefore the resistance in any fluid is to the motion excited in the fluid by a projectile, cannot be less in the most subtle aether than to the density of that aether, as it is in air, water, and quicksilver to the densities of these fluids.

SECTIO VII.

De motu fluidorum & resistentia projectilium.

PROPOSITIO XXXII. THEOREMA XXVI.

Si corporum systemata duo similia ex aequali particularum numero constent, & particulae correspondentes similes sint & proportionales, singule in uno systemate singalis in altero, & similiter sitae inter se, ac datam habeant rationem densitatis ad invicem, & inter se temporibus proportionalibus similiter moveri incipiant (eae inter se quae in uno sunt systemate & inter se quae sunt in altero) & si non tangant se mutuo quae in eodem sunt systemate, nisi in momentis reflexionum, neque attrahant, vel fugent se mutuo, nisi viribus acceleratricibus quae sint ut particularum correspondentium diametri inverse & quadrata velocitatum directae: dico quod systematum particulae illae pergent inter se temporibus proportionalibus similiter moveri.

Corpora similia & similiter sita temporibus proportionalibus inter se similiter moveri dico, quorum situs ad invicem in fine temporum illorum semper sunt similes: puta si particulae unius sysematis cum alterius particulis correspondentibus conferantur. Unde tempora erunt proportionalia, in quibus similes & proportionales figurarum similium partes a particulis correspondentibus describuntur. Igitur st duo sint eiusmodi systemata, particulae correspondentes, ob similitudinem incoeptorum motuum, pergent similiter moveri, usque donec sibi mutuo occurrant. Nam si nullis agitantur viribus, progredientur uniformiter in lineis rectis per motus leg. I. Si viribus aliquibus se mutuo agitant, & vires illae sint ut particularum correspondentium diametri inverse & quadrata velocitotum directe , quoniam particularum situs sunt similes & vires proportionales, vires totae quibus particulae correspondentes agitantur, ex viribus singulis agitantibus (per legum corollarium secundum) compositae, similes habebunt determinationes, perinde ac si centra inter particulas similiter sita respicerent; & erunt vires illae totae ad invicem ut vires singulae componentes, hoc est, ut correspondentium particularum diametri inverse, & quadrata velocitotum directe: & propterea efficient ut correspondentes particulae figuras similes describere pergant. Haec ita se habebunt (per corol.1 & 8, Prop. IV. Lib. I.) si modo centra illa quiescant. Sin moveantur, quoniam ob translationum similitudinem, similes manent eorum situs inter systematum particulas ; similes inducentur mutationes in figuris quas particulae describunt. Similes igitur erunt correspondentium & similium particularum motus usque ad occursus suos primos, & propterea similes occursus, & similes reflexiones, & subinde (per iam ostensa) similes motus inter se donec iterum in se mutuo inciderint, & sic deinceps in infinitum.
Q.E.D.

*Corol.*1. Hinc si corpora duo quaevis, quae similia sint & ad systematum particulas correspondentes similiter sita, inter ipsas temporibus proportionalibus similiter moveri incipiant, sintque eorum magnitudines ac densitates ad invicem ut magnitudines ac densitates correspondentium particularum: haec pergent temporibus proportionalibus similiter moveri. Est enim eadem ratio partium maiorum systematis utriusque atque particularum.

Corol. 2. Et si similes & similiter positae systematum partes omnes quiescant inter se: & earum duae, quae caeteris maiores sint, & sibi mutuo in utroque systemate correspondeant, secundum lineas similiter sitas simili cum motu utcunque moveri incipiant : hae similes in reliquis systematum partibus excitabunt motus, & pergent inter ipsas temporibus proportionalibus similiter moveri; atque ideo spatia diametris suis proportionalia describere.

PROPOSITIO XXXIII. THEOREMA XXVII.

Eidem positis, dico quod systematum partes maiores resistuntur in ratione composita ex duplicata ratione velocitatum suaram & duplicata ratione diametrorum & ratione densitatis partium systematum.

Nam resistentia oritur partim ex viribus centripetis vel centrifugis quibus particulae systematum se mutuo agitant, partim ex occursibus & reflexionibus particularum & partium maiorum. Prioris autem generis resistentiae sunt ad invicem ut vires totae motrices a quibus oriuntur, id est, ut vires totae acceleratrices & quantitates materiae in partibus correspondentibus; hoc est (per hypothesin) ut quadrata velocitatum directae & distantiae particularum correspondentium inverse & quantitates materiae in partibus correspondentibus directae : ideoque cum distantiae particularum systematis unius sint ad distantias correspondentes particularum alterius, ut diameter particulae vel partis in systemate priore ad diametrum particulae vel partis correspondentis in altero, & quantitates materiae sint ut densitates partium & cubi diametrorum; resistentiae sunt ad invicem ut quadrata velocitatum & quadrata diametrorum & densitates partium systematum. *Q.E.D.* Posterioris generis restitentiae sunt ut reflexionum correspondentium numeri & vires coniunctim. Numeri autem reflexionum sunt ad invicem ut velocitates partium correspondentium directe, & spatia inter earum reflexiones inverse. Et vires reflexionum sunt ut velocitates & magnitudines & densitates partium correspondentium coniunctim; id est, ut velocitates & diametrorum cubi & densitates partium. Et coniunctis his omnibus rationibus, resistentiae partium correspondentium sunt ad invicem ut quadrata velocitatum & quadrata diametrorum & densitates partium coniunctim. *Q. E. D*.

Corol. 1. Igitur si systemata illa sint fluida duo elastica ad modum aeris, & partes eorum quiescant inter se: corpora autem duo similia & partibus fluidorum quoad magnitudinem & densitatem proportionalia, & inter partes illas similiter posita, secundum lineas similiter positas utcunque proiiciantur, vires autem acceleratrices, quibus particulae fluidorum se mutuo agitant, sint ut corporum proiectorum diametri inverse, & quadrata velocitatum directe : corpora illa temporibus proportionalibus similes excitabunt motus in fluidis, & spatia similia ac diametris suis proportionalia describent.

Corol. 2. Proinde in eodem fluido proiectile velox resistentiam patitur, quae est in duplicata ratione velocitatis quam proxime. Nam st vires, quibus particulae distantes se mutuo agitant, augetentur in duplicata ratione velocitatis, resistentia foret in eadem ratione duplicata accurate; ideoque in medio, cuius partes ab invicem distantes sese viribus nullis agitant, resistenaea est in duplicata ratio velocitatis accurate. Sunto igitur media tria *A*, *B*, *C* ex partibus similibus & aequalibus & secundum distantias aequales regulariter dispositis constantia. Partes mediorum *A* & *B* fugiant se mutuo viribus quae sint ad invicem ut *T* & *V*, illae medii *C* eiusmodi viribus omnino destituantur. Et si corpora quatuor aequalia *D, E, F, G* in his mediis moveantur, priora duo *D* & *E* in prioribus duobus *A* & *B,* & altera duo *F* & *G* in tertio *C*; sitque velocitas corporis *D* ad velocitatem corporis *E,* & velocitas corporis *F* ad velocitatem corporis *G* in subduplicata ratione virium *T* ad vires *V*: resistentia corporis *D* erit ad resistentiam corporis *E,* & resistentia corporis *F* ad resistentiam corporis *G*, in velocitatum ratione duplicata, & propterea resistentia corporis *D* erit ad resistentiam corporis *F* ut resistentia corporis *E* ad resistentiam corporis *G*. Sunto corpora *D* & *F* aequivelocia ut & corpora *E* & *G*; & augendo velocitates corporum *D* & *F* in ratione quacunque, ac diminuendo vires particularum medii *B* in eadem ratione duplicate, accedet medium *B* ad formam & conditionem medii *C* pro libitu, & idcirco resistentiae corporum aequalium & aequivelocium *E* & *G* in his mediis, perpetuo accedent ad aequalitatem, ita ut earum differentia evadat tandem minor quam data quaevis. Proinde cum resistentiae corporum *D* & *F* sint ad invicem ut resistentiae corporum *E* & *G*, accedent etiam hae similiter ad rationem aequalitatis. Corporum igitur *D* & *F,* ubi velocissime moventur, resistentiae sunt aequales quam proxime: & propterea cum resistentia corporis *F* sit in duplicata ratione velocitatis, erit resistentia corporis *D* in eadem ratione quam proxime.

Corol. 3. Corporis in fluido quovis elastico velocissime moti eadem fere est resistentia ac si partes fluidi viribus suis centrifugis destituerentur, seque mutuo non fugerent : si modo fluidi vis elastica ex particularum viribus centrifugis oriatur, & velocitas adeo magna sit ut vires non habeant satis temporis ad agendum.

Corol. 4. Proinde cum resistentia similium & aequivelocium corporum, in medio cuius partes distantes se mutua non fugiunt, sint ut quadrats diametrorum; sunt etiam aequivelocium & celerrime motorum corporum resistentiae in fluido elastico ut quadrato diametrorum quam proxime.

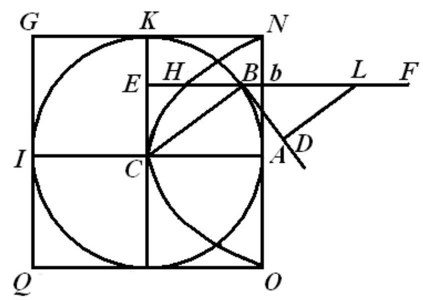
Corol: 5. Et cum corpora similia, aequalia & aequivelocia, in mediis eiusdem densitatis, quorum particulae se mutuo non fugiunt, sive particulae illae sint plures & minores, sive pauciores & maiores, in aaequalem materiae quantitatem temporibus aequalibus impingant, eique aaequalem motus quantitatem imprimant, & vicissim (per motus legem tertiam) aaequalem ab eadem reactionem patiantur, hoc est, aequaliter resistantur : manifestum est etiam quod in eiusdem densitatis fluidis elasticis, ubi velocissime moventur, aequales sint eorum resistentiae quam proxime , sive fluida illa ex particulis crassioribus constent, sive ex omnium subtilissimis constituantur. Ex medii subtilitate resistentia proiectilium celerrime motorum non multum diminuitur.

Corol. 6. Haec omnia ita se habent in fluidis, quorum vis elastica ex particularum viribus centrifugis originem ducit. Quod si vis illa aliunde oriatur, vel uti ex particularum expansione ad instar lanae vel ramorum arborum, aut ex alia quavis causa, qua motus particularum inter se redduntur minus liberi: resistentia, ob minorem medii fluiditatem, erit maior quam in superioribus corollariis.

PROPOSITIO XXXIV. THEOREMA XXVIII.

Si globus & cylindrus aequalibus diametris descripti, in media raro ex pariticulis aequalibus & ad aequales ab invicem distantias libere dispositis constante, secundum plagam axis cylindri, aequali cum velocitate moveantur: erit resistentia globi duplo minor quam resistentia cylindri.

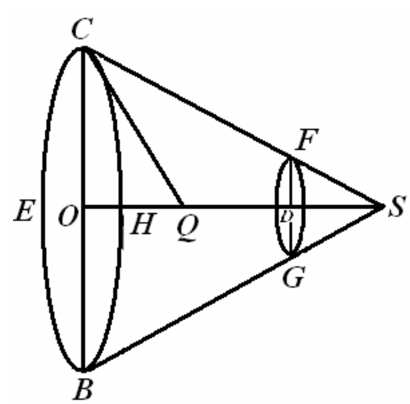
Nam quoniam actio medii in corpus eadem est (per legum corol.5) sive corpus in medio quiescente moveatur, sive medii particulae eadem cum velocitate impingant in corpus quiescens: consideremus corpus tanquam quiescens, & videamus quo impetu urgebitur a media movente. Designet igitur *ABKI* corpus sphaericum centro *C* semidiametro *CA* descriptum, & incidant particulae medii data cum velocitate in corpus illud sphaericum, secundum rectas ipsi *AC* parallelas: sitque *FB* eiusmodi recta. In ea capiatur *LB* semidiametro *CB* aequalis, & ducatur *BD* quae sphaeram tangat in *B*. In *KC* & *BD* demittantur perpendiculares *BE*, *LD*, & vis qua particula medii, secundum rectam *FB* oblique incidendo, globum ferit in *B*, erit ad vim qua particula eadem cylindrum *ONGQ* axe *ACI* circa globum descriptum perpendiculariter feriret in *b*, ut *LD* ad *LB* vel *BE* ad *BC*. Rursus efficacia huius vis ad movendum globum secundum incidentiae suae plagam *FB* vel *AC*, est ad eiusdem efficaciam ad movendum globum secundum plagam determinationis suae, id est,



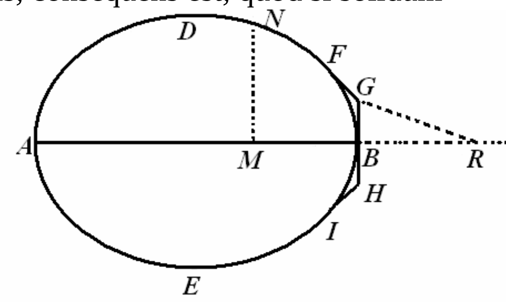
secundum plagam rectae *BC* qua globum directe urget ut *BE* ad *BC*. Et coniunctis rationibus, efficacia particulae in globum secundum rectam *FB* oblique incidentis, ad movendum eundem secundum plagam incidentiae suae, est ad efficaciam particulae eiusdem secundum eandem rectam in cylindrum perpendiculariter incidentis, ad ipsum movendum in plagam eandem, ut *BE* quadratum ad *BC* quadratum. Quare si in *bE*, quae perpendiculuris est ad cylindri basem circularem *NAO* & aequalis radio *AC*, sumatur *bH* aequalis $\frac{BEquad.}{CB}$: erit *bH* ad *bE* ut effectus particulae in globum ad effectum particulae in cylindrum. Et propterea solidum quod a rectis omnibus *bH* occupatur erit ad solidum quod a rectis omnibus *bE* occupatur, ut effectus particularum omnium in globum ad effedum particularum omnium in cylindrum. Sed solidum prius est parabolois vertice *C*, axe *CA* & latere recto *CA* descriptum, & solidum posterius est cylindrus paraboloidi circumscriptus: & notum est quod parabolois sit semissis cylindri circumscripti. Ergo vis tota medii in globum est duplo minor quam eiusdem vis tota in cylindrum. Et propterea si particulae medii quiescerent, & cylindrus ac globus aequali cum velocitate moverentur, foret resistentia globi duplo minor quam resistentia cylindri. *Q.E.D.*

Scholium.

Eadem methodo figurae aliae inter se quoad resistentiam comparari possint, aeque inveniri quae ad motus suos in mediis resistentibus continuandos aptiores sunt. Ut si base circulari *CEBH*, quae centro *O*, radio *OC* describitur, & altitudine *OD*, construendum sit frustum coni *CBGF*, quod omnium eadem basi & altitudine constructorum & secundum plagam axis sui versus *D* progredientium frustorum minime resistatur : biseca altitudinem *OD* in *Q* & produc *OQ* ad *S* ut sit *QS* aequalis *QC*, & erit *S* vertex coni cuius frustum queritur.



Unde obiter, cum angulus *CSB* semper sit acutus, consequens est, quod si solidum *ADBE* convolutione figurae ellipticae vel ovalis *ADBE* circa axem *AB* facta generetur, & tangatur figura generans a rectis tribus *FG, GH, HI* in punctis *F, B* & *I*, ea lege ut *GH* sit perpendicularis ad axem in puncto contactus *B*, & *FG*, *HI* cum eadem *GH* contineant angulos *FGB, BHI* graduum 135, folidum, quod convolutione figurae *ADFGHIE* circa axem eundem *AB* generatur, minus resistitur quam solidum prius; si modo utrumque secundum plagam axis sui *AB* progrediatur, & utriusque terminus B praecedat. Quam quidem propositionem in construendis navibus non inutilem futuram esse censeo.



Quod si figura *DNFG* eiusmodi sit curva, ut, si ab eius puncto quovis *N* ad axem *AB* demittatur perpendiculum *NM*, & a puncto dato *G* ducatur rect *GR* quae parallela sit rectae figuram tangenti in *N*, & axem productum secet in *R*, fuerit *MN* ad *GR* ut *GR cub*. ad $4BR \times GBq$; solidum quod figurae huius revolutione a circa axem *AB* facta describitur, in medio raro praedicto ab *A* verae versus *B* movendo, minus resistetur quam aliud quodvis eadem longitudine & latitudine descriptum solidum circulare.

PROPOSITIO XXXV. PROBLEMA VII.

Si medium rarum ex particulis quam minimis quiescentibus aequalibus & ad aequales ab invicem distantias libere dispositis constet: invenire resistentiam globi in hoc medio uniformiter progredientis.

Cas: 1. Cylindrus eadem diametro & altitudine descriptus progredi intelligatur eadem velocitate secundum longitudinem axis sui in eodem medio. Et ponamus quod particulae medii, in quas globus vel cylindrus incidit, vi reflexionis quam maxima resiliant. Et cum resistentia globi (per propositionem novissimam) sit duplo minor quam resistentia cylindri, & globus sit ad cylindrum ut duo ad tria, & cylindrus incidendo perpendiculariter in particulas, ipsas que quam maxime reflectendo, duplam sui ipsius velocitatem ipsis communicet: cylindrus quo tempore dimidiam longitudinem axis sui uniformiter progrediendo describit, communicabit motum particulis, qui sit ad totum cylindri motum ut densitas medii ad densitatem cylindri ; & globus, quo tempore totam longitudinem diametri suae uniformiter pregrediendo describit, communicabit motum eundem particulis; & quo tempore duas tertias partes diametri suae describit, communicabit motum particulis, qui sit ad totum globi motum ut densitas medii ad densitarem globi. Et propterea globus resistentiam patitur, quae sit ad vim qua totus eius motus vel auferri possit vel generari quo tempore duas tertias partes diametri suae uniformiter progrediendo describit, ut densitas medii ad densitatem globi.

Cas. 2. Ponamus quod particulae medii in globum vel cylindrum incidentes non reflectantur, & cylindrus incidendo perpendiculariter in particulas simplicem suam velocitatem ipsis communicabit, ideoque resistentiam patitur duplo minorem quam in priore casu, &: resistentia globi erit etiam duplo minor quam prius.

Cas. 3. Ponamus quod particula medii vi reflexionis neque maxima neque nulla, sed mediocri aliqua resiliant a globo , & resistentia globi erit in eadem ratione mediocri inter resistentiam in primo casu & resistemiam in secundo. *Q.E.I.*

Corol. 1. Hinc si globus & particulae sint infinite dura, & vi omni elastica & propterea etiam vi omni reflexionis destituta : resistentia globi erit ad vim qua totus eius motus vel auferri possit vel generari, quo tempore globus quatuor tertias partes diametri suae describit, ut densitas medii ad densitatem globi.

Corol. 2. Resistentia globi, caeteris paribus, est in duplicata ratione velocitatis.

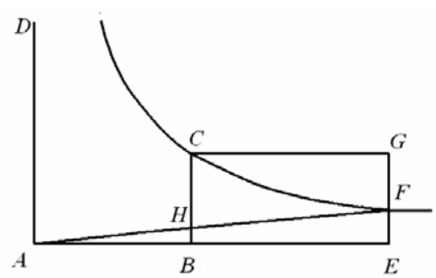
Corol. 3. Resistentia globi, caeteris paribus, est in duplicata ratione diametri.

Corol. 4 Resistentia globi, caeteris paribus, est ut densitas medii.

Corol. 5 Resistentia globi est in ratione quae componitur ex duplicata ratione velocitatis & duplicata ratione diametri & ratione densitatis medii.

Corol. 6. Et motus globi cum eius resistentia sic exponi potest. Sit *AB* tempus quo globus per resistentiam suam uniformiter continuaram totum suum motum amittere potest, Ad *AB* erigantur perpendicula *AD*, *BC*. Sitque *BC* motus ille totus, & per punctum *C* asymptotis *AD*, *AB* describatur hyperbola *CF*. Producatur *AB* ad punctum quodvis *E*. Erigatur perpendiculum *EF* hyperbolae occurrens in *F*. Compleatur parallelogrammum *CBEG*, & agatur *AF* ipsi *BC* occurrens in *H*. Et si globus tempore quovis *BE*, motu suo primo *BC* uniformiter continuato, in medio non resistente describat spatium *CBEG* per aream parallelogrammi expositum, idem in medio resistente describet spatium *CBEF* per aream hyperbole expositum, & motus eius in fine temporis illius exponetur per hyperbole ordinatam *EF*, amissa motus eius parte *EG*. Et resistentia eius in fine temporis eiusdem exponetur per longitudinem *BH*, amissa resistentia parte *CH*. Patent haec omnia per Corol.1.& 3. Prop. V. lib. II.

Corol. 7. Hinc si globus tempore *T* per resistentiam *R* uniformiter continuatam amittat motum suum totum *M*: idem globus tempore *t* in media resistente, per resistentiam *R* in duplicate vclocitatis decrescentem, amittet motus sui *M* partem $\frac{tM}{t+T}$, manente parte $\frac{TM}{t+T}$; & describet spatium quod sit ad spatium motu uniformi *M* eodem tempore *t* descriptum, ut logarithmus numeri $\frac{t+T}{t}$ multiplicatus per numerum 2,30258092994 est ad numerum $\frac{t}{T}$ propterea quod area hyperbolica *BCFE* est ad rectangulum *BCGE* in hac proportione.



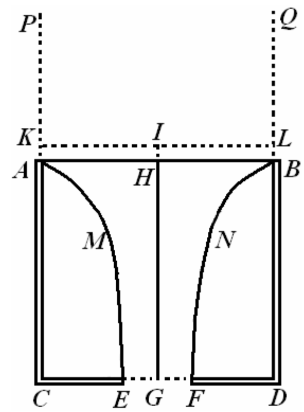
Scholium.

In hac propositione exposui resistentiam & retardationem proiectilium sphaericorum in mediis non continuis, & ostendi quod haec resistentia sit ad vim qua totus globi motus vel tolli possit vel generari quo tempore globus duas tertias diametri suae partes velocitate uniformiter continuata describat, ut densitas medii ad densitatem globi, si modo globus & particulae medii sint summe elastica & vi maxima reflectendi polleant: quodque haec vis sit duplo minor ubi globus & particulae medii sunt infinite dura & vi reflectendi prorsus destituta. In mediis autem continuis qualia sunt aqua, oleum calidum, & argentum vivum, in quibus globus non incidit immediate in omnes fluidi particulas resistentiam generantes, sed premit tantum proximas particulas & hae premunt alias & hae alias, resistentia est adhuc duplo minor. Globus utique in huiusmodi mediis fluidissimis resistentiam patitur quae est ad vim qua totus eius motus vel tolli possit vel generari quo tempore, motu illo uniformiter continuato, partes octo tertias diametri suae describat, ut densitas medii ad densitatem globi. Id quod in sequentibus conabimur ostendare.

PROPOSITIO XXXVI. PROBLEMA VIII.

Aquae de vase cylindrico per foramen in fundo factum effluentis definire motum.

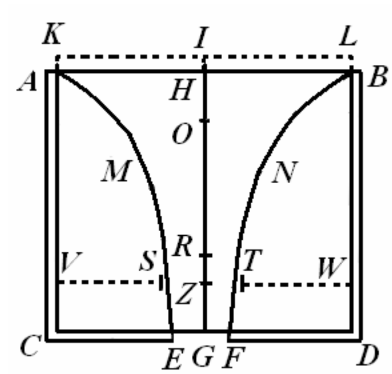
Sit *ACDB* vas cylindricum, *AB* eius orificium superius, *CD* fundum horizonti parallelum, *EF* foramen circulare in medio fundi, *G* centrum foraminis, & *GH* axis cylindri horizonti perpendicularis. Et finge cylindrum glaciei *APQB* eiusdem esse longitudinis cum cavitate vasis, & axem eundem habere, & uniformi cum motu perpetuo descendere, & partes eius quam primum attingunt superficiem, *AB* liquescere, & in aquam conversas gravitate sua defluere in vas, & cataractam vel columnam aquae *ABNFEM* cadendo formare, & per foramen *EF* transire, idemque adaequate implere. Ea vero sit uniformis velocitas glaciei descendentis ut & aquae contiguae in circulo *AB*, quam aqua cadendo & casu suo describendo altitudinem *IH* acquirere potest , & iaceant *IH* & *HG* in directum, & per punctum *I* ducatur recta *KL* horizonti parallela. Et lateribus glaciei occurrens in *K* & *L*. Et velocitas aquae effluentis per foramen *EF* ea erit quam aqua cadendo ab *I* & casu suo describendo altitudinem *IG* acquirere potest. Ideoque per theoremata *Galilaei* erit *IG* ad *IH* in duplicata ratione velocitatis aquae per foramen effluentis ad velocitatem aquae in circulo *AB*, hoc est, in duplicata ratione circuli *AB* ad circulum *EF*; nam hi circuli sunt reciproce ut velocitates aquarum quae per ipsos, eodem tempore & aequali quantitate, adequate transeunt. De velocitate aquae horizontem versus hic agitur. Et motus horizonti parallelus quo partes aquae cadentis ad invicem accedunt, cum non oriatur a gravitate, nec motum horizonti perpendicularem a gravitate oriundum mutet, hic non consideratur. Supponimus quidem quod partes aquae aliquantulum coherent, & per cohaessonem suam inter cadendum accedant ad invicem per motus horizonti parallelos, ut unicam tantum efforment cataractam & non in plures cataractas dividantur: sed motum horizonti parallelum, a cohaesione illa oriundum, hic non consideramus.



Cas 1. Concipe iam cavitatem totam in vase, in circuitu aquae cadentis *ABNFEM*, glacie plenam esse, ut aqua per glaciem tanquam per infundibulum transeat. Et si aqua glaciem tantum non tangat, vel, quod perinde est, si tangat & per glaciem propter summam eius polituram quam liberrime & sine omni resistentia labatur, haec defluet per foramen *EF* eadem velocitate ac prius, & pondus totum columna aquae *ABNFEM* impendetur in defluxum eius generandum uti prius, & fundum valis sustinebit pondus glaciei columnam ambientis.

Liquescat iam glacies in vase; & effuxus aquae, quoad velocitatem, idem manebit ac prius. Non minor erit, quia glacies in aquam resoluta conabitur descendere: non maior, quia glacies in aquam resoluta non potest descendere nisi impediendo descensum aquae alterius descensui suo aaequalem. Eadem vis eandem aquae effiuentis velocitatem generare debet.

Sed foramen in fundo vasis, propter obliquos motus particularum aquae effluentis, paulo maius esse debet quam prius. Nam particulae aquae iam non transeunt omnes per foramen perpendiculatiter; sed a lateribus vasis undique confluentes & in foramen convergentes, obliquis transeunt motibus, & cursum suum deorsum flectentes in venam aquae exilientis conspirant, quae exilior est paulo infra foramen quam in ipso foramine, existente eius diametro ad diametrum foraminis ut 5 ad 6, vel $5\frac{1}{2}$ ad $6\frac{1}{2}$ quam proxime, si modo diametros recte dimensus sum. Parabam utique laminam planam pertenuem in media perforatam, existente circularis foraminis diametro partium quinque octavarum digiti. Et ne vena aqua exilientis cadendo accelararetur & acceleratione redderetur angustior, hanc laminam non fundo sed lateri vasis affixi sic, ut vena illa egredere tur secundum lineam horizonti parallelam. Dein ubi vas aqua plenum esset, aperui foramen ut aqua effluereet; & venae diameter, ad distantiam quasi dimidii digiti a foramine quam accuritissime mensurata, prodiit partium viginti & unius quadragesimarum digiti.



Erat igitur diameter foraminis huius circularis ad diametrum venae ut 25 ad 21 quamproxime. Aqua igitur transeundo per foramen, convergit undique, & postquam effuxit ex vase, tenuior redditur convergendo, & per attenuationem acceleratur donec ad distantiam semissis digiti a foramine pervenerit, & ad distantiam illam tenuior & celerior sit quam in ipso foramine in ratione 25×25 ad 21×21 seu 17 ad 12 quamproxime, id est in subduplicata ratione binarii ad unitatem circiter. Per experimenta vero constat quod quantitas aquae, quae per foramen circulare in fundo valis factum, dato tempore effluit, ea sit quae cum velocitate praedicta, non per foramen illud, sed per foramen circulare, cuius diameter est ad diametrum foraminis illius ut 21 ad 25, eodem tempore effluere debet. Ideoque aqua illa effluens velocitatem habet deorsum in ipso foramine quam grave cadendo & casu suo describendo dimidiam altitudinem aquae in vase stagnantis acquirere potest quamproxime. Sed postquam exivit ex vase, acceleratur convergendo donec ad distantiam a foramine diametro foraminis prope aaequalem pervenerit, & velocitatem acquisiverit maiorem in ratione subduplicata binarii ad unitatem circiter, quam utique grave cadendo, & casu suo describendo totam altitudinem aquae in vase stagnantis, acquirere potest quamproxime.

In sequentibus igitur diameter venae designetur per foramen illud minus quod vocavimus *EF*. Et plano foraminis *EF* parallelum duci intelligatur planum aliud superius *VW* ad distantiam diametro foraminis aequalem circiter & foramine maiore *ST* pertusum, per quod utique vena cadat, que adaequate impleat foramen inferius *EF*, atque ideo cuius diameter sit ad diametrum foraminis inferioris ut 25 ad 21 circiter. Sic enim vena per foramen inferius perpendiculariter transibit ; & quantitas aquae effluentis, pro magnitudine foraminis huius, ea erit quam solutio problematis postulat quamproxime. Spatium vera, quod planis duobus & vena cadente clauditur, pro fundo vasis haberi potest. Sed ut solutio problematis simplicior sit & magis mathematica, praestat adhibere planum solum inferius pro fundo vasis, & fingere quod aqua quae per glaciem ceu per infundibulum defluebat, & e vase per foramen *EF* in plano inferiore factum egrediebatur, motum suum perpetuo servet, & glacies quietem suam. In sequentibus igitur sit *ST* diameter foraminis circularis centro Z descripti per quod cataracta effluit ex vase ubi aqua tota in vase fluida est. Et sit *EF* diameter foraminis per quod cataracta cadendo adequate transit, sive aqua exeat ex vase per foramen illud superius *ST*, sive cadat per medium glaciei in vase tanquam per infundibulum. Et sit diameter foraminis superioris *ST* ad diametrum inferioris *EF* ut 25 ad 21 circiter, & distantia perpendicularis inter planae foraminum aequalis sit diametro foraminis minoris *EF*. Et velocitas aquae vase per foramen *ST* exeuntis ea erit in ipso foramine deorsum quam corpus cadendo a dimidio altitudinis *IZ* acquirere potest velocitas autem cataractae utriusque cadentis ea erit in foramine *EF*, quam corpus cadendo ab altitudine tota *IG* acquiret,

Cas. 2. Si foramen *EF* non sit in medio fundi vasis, sed fundum alibi perforetur : aqua effluet eadem cum velocitate ac prius, si modo eadem sit foraminis magnitudo. Nam grave maiori quidem tempore descendit ad eandem profunditatem per lineam obliquam quam per lineam perpendicularem, sed descendendo eandem velocitatem acquirit in utroque casu, ut *Galileus* demonstravit.

Cas. 3. Eadem est aquae velocitas effluentis per foramenin latere vasis. Nam si foramen parvum sit, ut intervallum inter superficies *AB* & *KL* quoad sensum evanescat, & vena aquae horizontaliter exilientis figuram parabolicam efformet: ex latere recto huius parabolae colligetur, quod velocitas aquae effluentis ea sit quam corpus ab aquae in vase stagnantis altitudine *HG* vel *IG* cadendo acquirere potuisset. Facto utique experimento inveni quod, si altitudo aquae stagnantis supra foramen esset viginti digitorum & altitudo foraminis supra planum horizonti parallelum esset quoque viginti digitorum, vena aquae prosilientis incideret in planum illud ad distantiam digitorum 37 circiter a perpendiculo quod in planum illud a foramine demittebatur captam. Nam sine resistentia, vena incidere debuisset in planum illud ad distantiam digitorum 40, existente venae parabolicae latere recto digitorum 80.

Cas. 4. Quinetiam aqua effluens, si sursum feratur, eadem egreditur cum velocitate. Ascendit enim aquae exilientis vena parva motu perpendiculari ad aquae in vase stagnantis altitudinem *GH* vel *GI*, nisi quatenus ascensus eius ab aeris resistentia aliquantulum impediatur ; ac proinde ea effluit cum velocitate quam ab altitudine illa cadendo acquirere potuisset. Aquae stagnantis particula unaquaeque undique premitur aequaliter (per Prop. XIX. Lib. 2.) & pressioni cedendo aequali impetu in omnes partes fertur, sive descendat per foramen in fundo vasis, sive horizontaliter effluat per foramen in eius latere, sive egrediatur in canalem & inde ascendat per foramen parvum in superiore canalis parte factum. Et velocitatem qua aqua effluit eam esse, quam in hac propositione assignavimus, non solum ratione colligitur, sed etiam per experimenta notissima iam descripta manifestum est.

Cas. 5 Eadem est aquae effluentis velocitas sive figura foraminis *D* sit circularis sive quadrata vel triangularis aut alia quaecunque circulari aequalis. Nam velocitas aquae effluentis non pendet a figura foraminis sed oritur ab eius altitudine infra planum *KL*.

Cas. 6. Si vasis *ABDC* pars inferior in aquam stagnantem immergatur, & altitudo aquae stagnantis supra fundum vasis sit *GR*: velocitas quacum aqua quae in vase est, effluet per foramen *EF* in aquam stagnantem, ea erit quam aqua eadendo & casu suo describendo altitudinem *IR* acquirere potest, Nam pondus aquae omnis in vase quae inferior est superficie aquae stagnantis, sustinebitur in aequilibrio per pondus aquae stagnantis, ideoque motum aquae descendentis in vase minime accelerabit. Patebit etiam & hic casus per experimenta, mensurando scilicet tempora quibus aqua effluit.

Corol. 1. Hinc si aquae altitudo *CA* producatur ad *K*, ut sit *AK* ad *CK* in duplicata ratione areae foraminis in quavis fundi parte facti, ad aream circuli *AB* : velocitas aquae effluentis aequalis erit velocitati quam aqua cadendo & casu suo describendo altitudinem *KC* aequirere potest,

Corol. 2. Et vis, qua totus aquae exilientis motus generari potest, aequalis est ponderi cylindricae columnae aquae, cuius basis est foramen *EF*, & altitudo 2*GI* vel 2*CK*. Nam aqua exiliens, quo tempore hanc columnam aequat, pondere suo ab altitudine *GI* cadendo velocitatem suam, qua exilit, acquirere potest,

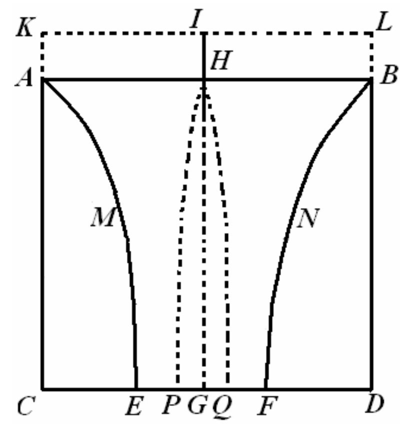
Corol. 3. Pondus aquae totius in vase *ABDC* est ad ponderis partem, quae in defluxum aquae impenditur, ut summa circulorum *AB* & *EF* ad duplum circulum *EF*.
Sit enim *IO* media proportionalis inter *IH* & *IG*; & aqua per foramen *EF* egrediens, quo tempore gutta cadendo ab *I* describere posset altitudinem *IG*, aequalis erit cylindro cuius basis est circulus *EF* & altitudo est 2*IG*, id est, cylindro cuius basis est circulus *AB* & altitudo est 2*IO*, nam circulus *EF* est ad circulum *AB* in subduplicata ratione altitudinis *IH* ad altitudinem *IG*, hoc est, in simplici ratione mediae proportionalis *IO* ad altitudinem *IG*: & quo tempore gutta cadendo ab *I* describere potest altitudinem *IH*, aqua egrediens aequalis erit cylindro cuius basis est circulus *AB* & altitudo est 2*IH* : & quo tempore gutta cadendo ab *I* per *H* ad *G* describit altitudinum differentiam *HG*, aqua egrediens, id est, aqua tota in solido *ABNFEM* aequalis erit differentiae cylindrorum, id est, cylindro cuius basis est *AB* & altitudo 2*HO*. Et propterea aqua tota in vase *ABDC* est ad aquam totam cadentem in solido *ABNFEM* ut *HG* ad 2*HO*, id est, ut $HO+OG$ ad 2*HO*, seu $IH+IO$ ad 2*IH*. Sed pondus aquae totius in solido *ABNFEM* in aquae defluxum impenditur: ac proinde pondus aquae totius in vase est ad ponderis partem quae in defluxum aquae impenditur, ut $IH+IO$ ad 2*IH*, atque ideo ut summa circulorum *EF* & *AB* ad duplum circulum *EF*.

Corol. 4. Et hinc pondus aquae totius in vare *ABDC* est ad ponderis partem alteram quam fundum vasis sustinet, ut summa circulorum *AB* & *EF* ad differentiam eorundem circulorum.

Corol. 5. Et ponderis pars, quam fundum vasis sustinet, est ad ponderis partem alteram, quae in defluxum aquae impenditur, ut differentia circulorum *AB* & *EF* ad duplum circulum minorem *EF*, sive ut area fundi ad duplum foramen.

Corol. 6. Ponderis autem pars, qua sola fundum urgetur, est ad pondus aquae totius, quae fundo perpendiculariter incumbit, ut circulus *AB* ad summam circulorum *AB* & *EF*, sive ut circulus *AB* ad excessum dupli circuli *AB* supra fundum. Nam ponderis pars, qua sola fundum urgetur, est ad pondus aquae totius in vase, ut differentia circulorum *AB* & *EF* ad summam eorundem circulorum per Cor. 4 : & pondus aquae totius in vase est ad pondus aquae totius quae fundo perpendiculariter incumbit, ut circulus *AB* ad differentiam circulorum *AB* & *EF*. Itaquae ex aequo perturbate, ponderis pars, qua sola fundum urgetur, est ad pondus aquae totius, quae fundo perpendiculariter incumbit, ut circulus *AB* ad summam circulorum *AB* & *EF* vel excessum dupli circuli *AB* supra fundum.

Corol. 7. Si in medio foraminis *EF* locetur circellus *PQ* centro *G* descriptus & horizonti parallelus : pondus aquae quam circellus ille sustinet, maius est pondere tertiae partis cylindri aquae cuius basis est circellus ille & altitude est *GH*. Sit enim *ABNFEM* cataracta vel columna aquae cadentis axem habens *GH* ut supra, & congelari intelligatur aqua omnis in vase, tam in circuitu cataractae quam supra circellum, cuius fluiditas ad promptissimum & celerrimum aquae descensum non requiritur. Et sit *PHQ* columna aquae supra circellum congelata, verticem habens *H* & altitudinem *GH*. Et finge cataractam hancce pondere suo toto cadere, & non incumbere in *PHQ* nec eandem premere, sed libere & sine frictione praeterlabi, nisi forte in ipso glaciei vertice quo cataracta ipso cadendi initio incipiat esse cava. Et quemadmodum aqua in circuitu cataractae congelata *AMEC*, *BNFD* convexa est in superficie interna *AME, BNF* versus cataractam cadentem, sic etiam haec columna *PHQ* convexa erit versus cataractam, & propterea maior cono cuius basis est circellus ille *PQ* & altitudo *GH*, id est, maior tertia parte cylindri eadem base & altitudine descripti. Sustinet autem circellus ille pondus huius columnae, id est, pondus quod pondere coni seu teriae partis cylindri illius maius est.



Corol. 8. Pondus aquae quam circellus valde parvus *PQ* sustinet, minor esse videtur pondere duarum tertiarum partium cylindri aquae cuius basis est circellus ille & altitude est *HG*. Nam stantibus iam positis, describi intelligatur dimidium sphaeroidis cuius basis est circellus ille & semiaxis sive altitude est *HG*. Et haec figura aequalis erit duabus tertiis partibus cylindri illius & comprehendet columnam aquae congelatae *PHQ* cuius pondus circellus ille sustinet. Nam ut motus aquae sit maxime directus, columnae illius superficies externa concurret cum basi *PQ* in angulo nonnihil acuto, propterea quod aqua cadendo perpetuo acceleratur & propter accelerationem sit tenuior, & cum angulus ille sit recto minor, haec columna ad inferiores eius partes iacebit intra dimidium sphaeroidis. Eadem vera sursum acuta erit seu cuspidata, ne horizontalis motus aquae ad verticem sphaeroidis sit infinite velocior quam eius motus horizontem versus, Et quo minor est circellus *PQ* eo acutior erit vertex columnae; & circello in infinitum diminuto, angulus *PHQ* in infinitum diminuetur, & propterea columna iacebit intra dimidium sphaeroidis. Est igitur columna illa minor dimidio spheroidis, seu duabus tertiis partibus cylindri cuius basis est circellus ille & altitudo *GH*. Sustinet autem circellus vim aquae ponderi huius columae aequalem, cum pondus aquae ambientis in defluxum eius impendatur.

Corol 9. Pondus aquae quam circellus valde parvus *PQ* sustinet, aequale est ponderi cylindri aquaes cuius basis est circellus ille & altitudo est $\frac{1}{2}GH$ quamproxime. Nam pondus hocce est medium arithmeticum inter pondera coni & hemispheroidis praedicta. At si circellus ille non sit valde parvus, sed augeatur donec aequet foramen *EF* ; hic sustinebit pondus aquae totius sibi perpendiculariter imminentis, id est, pondus cylindri aquae, cuius basis est circellus ille & altitudo est *GH*.

Corol. 10. Et (quantum sentio) pondus quod circellus sustinet, est semper ad pondus cylindri aquae, cuius basis est circellus ille & altitudo est $\frac{1}{2}GH$, ut EFq ad $EFq-\frac{1}{2}PQq$, sive ut circulus *EF* ad excessum circuli huius supra semissem circelli *PQ* quamproxime.

LEMMA IV.

Cylindri, qui secundum longitudinem suam uniformiter progreditur, resistentia ex aucta vel diminuta eius longitudine non mutatur; ideoque eadem est cum resistentia circuli eadem diametro descripti & eadem velocitate secundum lineam rectam plano ipsius perpendicularem progredientis.

Nam latera cylindri motui eius minime opponuntur: & cylindrus, longitudine eius infinitum diminuta, in circulum vertitur.

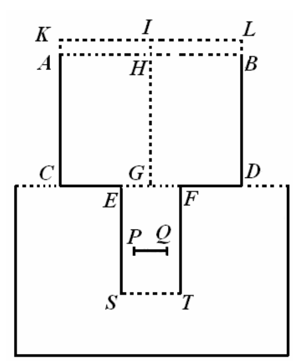
PROPOSITIO XXXVII. THEOREMA XXIX.

Cylindri, qui in fluido compresso infinito & non elastico secundum longitudinem suam uniformiter progreditur, resistentia, quae oritur a magnitudine sectionis tranversae, est ad vim qua totus eius motus, interea dum quadruplum longitudinis suae describit, vel tollit possit vel generari, ut densitas medii ad densitatem cylindi quamproxime.

Nam si vas *ABDC* fundo suo *CD* superficiem aquae stagnantis tangat, & aqua ex hoc vase per canalem cylindricum *EFTS* horizonti perpendicularem in aquam stagnantem effluat, locetur autem circellus *PQ* horizoni parallelus ubivis in medo canalis, & producatur *CA* ad *K*, ut sit *AK* ad *CK* in duplicata ratione quam habet excessus orificii canalis *EF* supra circellum *PQ* ad circulum *AB*: manifestum est (per Cas. 5. Cas.6. & Cor. I. Prop XXXVI) quod, velocitas aquae transeuntis per spatium annulare inter circellum & latera vasis, ea erit quam aqua cadendo & casu suo describendo altitudinem *KC* vel *IG* acquirere potest.

Et (per Corol. X, Prop. XXXVI.) si vasis latitude sit infinita, ut lineola *HI* evanescat & altitudines *IG, HG* aequentur: vis aquae defluentis in circellum erit ad pondus cylindri cuius basis est circellus ille & altitudo est $\frac{1}{2}IG$, ut EFq ad $EFq-\frac{1}{2}PQq$ quam proxime. Nam vis aquae, uniformi motu defluentis per totum canalem, eadem erit in circellum *PQ* in quacunque canalis parte locatum,

Claudantur iam canalis orificia *EF, ST,* & ascendat circellus in fluido undique compresso & ascensu suo cogat aquam superiorem descendere per spatium annulare inter circellum & latera canalis: & velocitas circelli ascendentis erit ad velocitatem aquae descendentis ut differentia circulorum *EF* & *PQ* ad circulum *PQ* , & velocitas circelli ascendentis ad summam velocitatum, hoc est, ad velocitatem relativam aquae descendentis qua praeterfluit circellum ascendentem, ut differentia circulorum *EF* & *PQ* ad circulum *EF*, sive ut $EFq-PQq$ ad EFq. Sit illa velocitas relativa aequalis velocitati, qua supra ostensum est aquam transire per idem spatium annulare dum circellus interea immotus manet, id est, velocitati quam aqua cadendo & casu suo describendo altitudinem *1G* acquirere potest



: & vis aquae in circellum ascendentem eadem erit ac prius (per legum Corol. V.) id est, resistentia circelli ascendentis erit ad pondus cylindri aquae cuius basis est circellus ille & altitudo est : $\frac{1}{2}IG$, ut $EFq-\frac{1}{2}PQq$ quamproxime. Velocitas autem circelli erit ad velocitatem, quam aqua cadendo & casu suo describendo altitudinem *IG* acquirit, ut $EFq-PQq$ ad EFq.

Augeatur amplitudo canalis in infinitum: & rationes illae inter $EFq-PQq$ & EFq, interque EFq & $EFq-\frac{1}{2}PQq$, accedent ultimo ad rationes aequalitatis. Et propterea velocitas circelli ea nunc erit quam aqua cadendo & casu suo describendo altitudinem *IG* acquirere potest, resistentia vero eius aequalis evadet ponderi cylindri cuius basis est circellus ille & altitudo dimidium est altitudinis *IG*, a qua cylindrus cadere debet ut velocitatem circelli ascendentis aequirat; & hac velocitate cylindrus, tempore cadendi, quadruplum longitudinis suae describet. Restitentia autem cylindri, hac velocitate secundum longitudinem suam progredientis, eadem est cum resistentia circelli (per Lemma IV.) ideoque aequalis est vi qua motus eius, interea dum quadruplum longitudinis suae describit, generari potest quamproxime.

Si longitudo cylindri augeatur vel minuatur: motus eius ut & tempus, quo quadruplum longitudinis suae describit, augebitur vel minuetur in eadem ratione; ideoque vis illa, qua motus auctus vel diminutus, tempore pariter aucto vel diminuto, generari vel tolli possit, non mutabitur , ac proinde etiamnum aequalis est resistentiae cylindri, nam & haec quoque immutara manet per Lemma IV.

Si densitas cylindri augeatur vel minuatur : motus eius ut & vis qua motus eodem tempore generari vel tolli potest, in eadem ratione augebitur vel minuetur. Resistentia itaquae cylindri cuiuscunque erit ad vim qua totus eius motus, interea dum quadruplum longitudinis suae describit, vel generari possit vel tolli, ut densitas medii ad densitatem cylindri quamproxime. *Q.E.D.*

Fluidum autem comprimi debet ut sit continuum, continuum vero esse debet & non elasticum ut pressio omnis, quae ab eius compressione oritur, propagetur in instanti, & in omnes moti corporis partes aequaliter agenda resistentiam non mutet. Pressio utique, quae a motu corporis oritur, impenditur in motum partium fluidi generandum & resistentiam creat. Presso autem quae oritur a compressione fluidi, utcunque fortis sit, si propagetur in instanti, nullum generat motum in partibus fluidi continui, nullam omnino inducit motus mutationem ; ideoque resistentiam nec auget nec minuit. Certe actio fluidi, quae ab eius compressione oritur; fortior esse non potest in partes posticas corporis moti quam in eius partes anticas, ideoque resistentiam in hac propositione descriptam minuere non potest : & fortior non erit in partes anticas quam in posticas, si modo propagatio eius infinite velocior sit quam motus corporis pressi. Infinite autem velocior erit & propagabitur in instanti, si modo fluidum sit continuum & non elasticum.

Corol. I. Cylindrorum, qui secundum longitudines suas in mediis continuis infinitis uniformiter progrediuntur, resistentiae sunt in ratione quae componitur ex duplicata ratione velocitarum & duplicata ratione diametrorum & ratione densitatis mediorum.

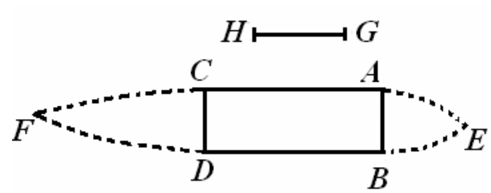
Corol. 2. Si amplitudo canalis non augeatur in infinitum, sed cylindrus in medio quiescente incluso secundum longitudinem suam progrediatur, & interea axis eius cum axe canalis coincidat: resistentia eius erit ad vim qua totus eius motus, quo tempore quadruplum longitudinis suae describit, vel generari possit vel tolli, in ratione quae componitur ex ratione EFq & $EFq-\frac{1}{2}PQq$ semel, & ratione EFq & $EFq-PQq$ bis, & ratione densitatis medii ad densitatem cylindri.

Corol. 3. Iisdem positis, & quod longitudo L sit ad quadruplum longitudinis cylindri in ratione qua componitur ex ratione $EFq-\frac{1}{2}PQq$ ad EFq semel, & ratione $EFq-PQq$ ad EFq bis: resistentia cylindri erit ad vim qua totus eius motus, interea dum longitudinem L describit, vel tolli possit vel generari, ut densitas medii ad densitatem cylindri.

Scholium.

In hac propositione resistentiam investigavimus quae oritur a sola magnitudine transverse sectionis cylindri, neglecta resistentiae parte quae ab obliquitate motuum oriri possit. Nam quemadmodum in casu primo Propositionis XXXVI. obliquitas motuum, quibus partes aquae in vase, undique convergebant in foramen *EF*, impedivit effluxum aquae illius per foramen: sic in hac propositione, obliquitas motuum, quibus partes aquae ab anteriore cylindri termino pressae, cedunt pressioni & undique divergunt, retardat eorum transitum per loca in circuitu termini illius antecedentis versus posteriores partes cylindri, efficitque ut fluidum ad maiorem distantiam commoveatur & resistentiam auget, idque in ea fere ratione qua effluxum aquae e vase diminuit, id est, in ratione duplicata 25 ad 21 circiter.

Et quemadmodum, in propositionis illius casu primo, effecimus ut partes aquae perpendiculariter & maxima copia tranfirent per foramen *EF*, ponendo quod aqua omnis in vase quae in circuitu cataractae congelata fuerat, & cuius motus obliquus erat & inutilis, maneret sine motu: sic in hac propositione, ut obliquitas motuum tollatur, & partes aquae motu maxime directo & brevissimo cedentes facillimum praebeant transitum cylindro, & sola maneat resistentia, quae oritur a magnitudine sectionis transversae, quaeque diminui non potest nisi diminuendo diametrum cylindri, concipiendum est quod partes fluidi, quarum motus sunt obliqui & inutiles & resistentiam creant, quiescant inter se ad utrumque cylindri terminum, & cohaereant & cylindro iungantur. Sit *ABCD* rectangulum, & sint *AE* & *BE* arcus duo parabolici axe *AB* descripti, latere autem recto quod sit ad spatium *HG*, describendum a cylindro cadente dum velocitatem suam acquirit, ut *HG* ad $\frac{1}{2}AB$. Sint etiam *CF* & *DF* arcus alii duo parabolici, axe *CD* & latere recto quod sit prioris lateris recti quadruplum descripti ; & convolutione figurae circum axem *EF* generetur solidum cuius media pars *ABDC* sit cylindrus de quo agimuss, & partes extremae *ABE* & *CDF* contineant partes fluidi inter se quiescentes & in corpora duo rigida concretas, quae cylindro utrinque tanquam caput & cauda adhaereant. Et solidi *EACFDB*, secundum longitudimem axis sui *FE* in partes versus *E* progredientis, resistentia ea erit quamproxime quam in hac propositione descripsimus, id est, quae rationem illam habet ad vim qua totus cylindri motus, interea dum longitudo 4*AC* motu illo uniformiter continuato describatur, vel tolli possit vel generari, quam densitas fluidi habet ad densitatem cylindri quamproxime. Et hac vi resistentia minor esse non potest quam in ratione 2 ad 3, per Corol. 7. Prop. XXXVI.



LEMMA V.

Si cylindrus, sphaera & sphaerois, quorum latitudines sunt aequales, in medio canalis cylindrici ita locentur successive ut eorum axes cum axe canalis coincidant: haec corpora fluxum aquae per canalem aequaliter impedient.

Nam spatia inter canalem & cylindrum, sphaeram, & spheroidem per quae aqua transit, sunt aequalia: & aqua per aequalia spatia aequaliter transit.

Haec ita se habent ex hypothesi, quod aqua omnis supra cylindrum sphaeram vel sphaeroidem congelatur, cuius fluiditas ad celerrimum aquae transitum non requiritur, ut in Corol. VII, Prop. XXXVl. explicui.

LEMMA VI.

Iisdem positis, corpora praedicta aequaliter urgentur ab aqua per canalem fluente.

Patet per lemma V. & motus legem tertiam. Aqua utique & corpora in se mutua aequaliter agunt.

LEMMA VII.

Si aqua quiestat in canali, & haec corpora in partes contrarias aequali velocitate per canalem ferantur : aequales erunt eorum resistentiae inter se.

Constat ex lemmate superiore, nam motus relativi iidem inter se manent.

Scholium.

Eadem est ratio corporum omnium convexorum & rotundorum, quorum axes cum axe canalis coincidunt. Differentia aliqua ex maiore vel minore frictione oriri potest, sed in his lemmatis corpora esse politissima supponimus, & medii tenacitatem & frictionem esse nullam, & quod partes fluidi, quae motibus suis obliquis & superfluis fluxum aquae per canalem perturbare, impedire, & retardare possunt, quiescant inter se tanquam gelu constrictae, & corporibus ad ipsorum partes anticas & posticas adhaereant, perinde ut in scholio propositionis praecedentis exposui. Agitur enim in sequentibus de resistentia omnium minima quam corpora rotunda; datis maximis sectionibus transversis descripta, habere possunt.

Corpora fluidis innatantia, ubi moventur in directum, efficiunt ut fluidum ad partem anticam ascendat, ad posticam subsidat, praesertim si figura sint obtusa , & inde resistentiam paulo maiorem sentiunt quam si capite & cauda sint acutis. Et corpora in fluidis elasticis mota, si ante & post obtusa sint, fluidum paulo magis condensant ad anticam partem & paulo magis relaxant ad posticam ; & inde resistentiam paulo maiorem sentiunt quam si capite & cauda sint acutis. Sed nos in his lemmatis & propositionibus non agimus de fluidis elasticis, sed de non elasticis; non de insidentibus fluido, sed de alte immersis. Et ubi resistentia corporum in fluidis non elasticis innotescit, augenda erit haec resistentia aliquantulum tam in fluidis elasticis, qualis est aer, quam in superficiebus fluidorum stagnantium, qualia sunt maria & paludes.

PROPOSITIO XXXVIII. THEOREMA XXX.

Globi, in fluido compresso infinito & non elastico uniformiter progredientis, resistentia est ad vim qua totus eius motus, quo tempore octo tertias partes diametri suae describit, vel tolli possit vel generari, ut densitas fluidi ad densitatem globi quamproxime.

Nam globus est ad cylindrum circumscriptum ut duo ad tria; & propterea vis illa, quae tollere possit motum omnem cylindri interea dum cylindrus describat longitudinem quatuor diametrorum, globi motum omnem tollet interea dum globus describat duas tertias partes huius longitudinis, id est, octo tertias partes diametri propriae. Resistentia autem cylindri est ad hanc vim quamproxime ut densitas fluidi ad densitatem cylindri vel globi per Prop. XXXVII. & resistentia globi aequalis est resistentiae cylindri per Lem. V, VI, VII. *Q.E.D.*

Corol. I. Globorum, in mediis compressis infinitis, resistentiae sunt in ratione quae componitur ex duplicata ratione velocitatis, & duplicata ratione diametri, & ratione densitatis mediorum.

Corol. 2. Velocitas maxima quacum globus, vi ponderis sui comparativi, in fluido resistente potest descendere, ea est quam acquirere potest globus idem, eodem pondere, sine resistentia cadendo & casu suo describendo spatium quod sit ad quatuor tertias partes diametri sua ut densitas globi ad densitatem fluidi. Nam globus tempore casus sui, cum velocitate cadendo acquisita, describet spatium quod erit ad octo tertias diametri suae, ut densitas globi ad densitatem fluidi; & vis ponderis motum hunc generans, erit ad vim quae motum eundem generare possit, quo tempore globus octo tertias diametri suae eadem velocitate describit, ut densitas fluidi ad densitatem globi: ideoque per hanc propositionem, vis ponderis aequalis erit vi resistentiae, & propterea globum accelerare non potest.

Corol. 3. Data & densitate globi & velocitate eius sub initio motus, ut & densitate fluidi compressi quiescentis in qua globus movetur ; datur ad omne tempus & velocitas globi & eius resistentia & spatium ab eo descriptum, per Corol. VII. Prop. XXXV.

Corol. 4. Globus in fluido compresso quiescente eiusdem secum densitatis movendo, dimidiam motus sui partem prius amittet quam longitudinem duarum ipsius diametrorum descripserit, per idem Corol. VII.

PROPOSlTIO XXXIX. THEOREMA XXXI.

Globi, per fluidum in canali cylindrico clausum & compressum uniformiter progredientis, resistentia est ad vim, qua totas eius motus, interea dum octo tertia partes diametri suae describit, vel generari possit vel tolli, in ratione quae componitur ex ratione orifici canalis ad excessum huius orificii supra dimidium circuli maximi globi, & ratione duplicata orificii canalis ad excessum huius orifici supra circulum maximum globi, & ratione densitatis fluidi ad densitatem globi quamproxime.

Patet per Corol. 2. Prop. XXXVII. procedit vero demonstratio quemadmodum in propositione precedente.

Scholium.

In propositionibus duabus novissimis (perinde ut in Lem. V.) suppono quod aqua omnis congelatur quae globum praecedit, & cuius fluiditas auget resistentiam globi. Si aqua illa omnis liquescat, augebitur resistentia aliquantulum. Sed augmentum illud in his propositionibus parvum erit & negligi potest, propterea quod convexa superficies globi totum fere officium glaciei faciat.

PROPOSITIO XL. PROBLEMA IX.

Globi, in medio fluidissimo compresso progredientis, invenire resistentiam per phaenomena.

Sit A pondus globi in vacuo, B pondus eius in medio resistente, D diameter globi, F spatium quod sit ad $\frac{4}{3}D$ ut densitas globi ad densitatem medii, id est, ut A ad $A-B$, G tempus quo globus pondere B sine resistentia cadendo describit spatium F, & H velocitas quam globus hocce casu suo acquirit. Et erit H velocitas maxima quacum globus, pondere suo B, in medio resistente potest descendere, per Corol. 2. Prop. XXXVIII. & resistentia, quam globus ea cum velocitate descendens patitur, aequalis erit eius ponderi B: resistentia vero, quam patitur in alia quacunque velocitate, erit ad pondus B in duplicata ratione velocitatis huius ad velocitatem illam maximam H, per Corol, I. Prop. XXXVIII.

Haec est resistentia quae oritur ab inertia materiae fluidi. Ea vero qua oritur ab elasticitate, tenacitate, & frictione partium eius, sic investigabitur.

Demittatur globus ut pondere suo B in fluido descendat; & sit P tempus cadendi, idque in minutis secundis si tempus G in minutis secundis habeatur, Inveniatur numerus absolutus N qui congruit logarithmo $0{,}4342944819\frac{2P}{G}$, sitque L logarithmus numeri $\frac{N+1}{N}$; & velocitas cadendo acquisita erit $\frac{N-1}{N+1}H$, altitudo autem descripta erit $\frac{2PF}{G}-1{,}3862943611F+4{,}605170186LF$. Si fluidum satis profundum sit, negligi potest terminus $4{,}605170186LF$; & erit $\frac{2PF}{G}-1{,}3862943611F$ altititudo descripta quamproxime. Patent haec per libri secundi propositionem nonam & eius corollaria, ex hypothesi quod globus nullam aliam patiatur resistentiam nisi quae oritur ab inertia materiae. Si vero aliam insuper resistentiam patiatur, descensus erit tardior, & ex retardatione innotescet quantitas huius resistentiae.

Ut corporis in fluido cadentis velocitas & descensus facilius innotescant, composui tabulam sequentem, cuius columna prima denotat tempora descensus, secunda exhibet velocitates cadendo acquisitas existente velocitate maxima 100000000, tertia exhibet spatia temporibus illis cadendo descripta, existente $2F$ spatio quod corpus tempore G cum velocitate maxima describit, & quarta exhibet spatia iisdem temporibus cum velocitate maxima descripta.

Numeri in quarta columna sunt $\frac{2P}{G}$, & subducendo numerum $1{,}3862944-4{,}6051702\,L$, inveniuntur numeri in tertia columna, & multiplicandi sunt hi numeri per spatium F ut habeantur spatia cadendo descripta. Quinta his insuper adieda est columna, que continet spatia descripta iisdem temporibus a corpore, vi ponderis sui comparativi B, in vacuo cadente.

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Tempora	*Velocitates*	*Spatia cadendo*	*Spatia motu*	*Spatia cadendo*

P	*cadentis in fluido*	*descripta in fluido*	*maximo descripta*	*descripta in vacuo*
0,001G	99999$\frac{29}{30}$	0,000001*F*	0,002*F*	0,000001*F*
0,01G	999967	0,0001*F*	0,02*F*	0,0001*F*
0,1*G*	9966799	0,0099834*F*	0,2*F*	0,01*F*
0,2*G*	19737532	0,0397361*F*	0,4*F*	0,04*F*
0,3*G*	29131261	0,0886815*F*	0,6*F*	0,09*F*
0,4*G*	37994896	0,1559070*F*	0,8*F*	0,16*F*
0,5*G*	46211716	0,2402290*F*	1,0*F*	0,25*F*
0,6*G*	53704957	0,3402706*F*	1,2*F*	0,36*F*
0,7*G*	60436778	0,4545405*F*	1,4*F*	0,49*F*
0,8*G*	66403677	0,5815071*F*	1,6*F*	0,64 *F*
0,9*G*	71629787	0,7196609*F*	1,8*F*	0,81*F*
1*G*	7615 9416	0,8675617*F*	2*F*	1*F*
2*G*	96402758	2,65 00055*F*	4*F*	4*F*
3*G*	99505475	4,6186570*F*	6*F*	9*F*
4*G*	99932930	6,6143765*F*	8*F*	16*F*
5*G*	99990920	8,6137964*F*	10*F*	25*F*
6*G*	99998771	10,6137179*F*	12*F*	36*F*
7*G*	99999834	12,6137073*F*	14*F*	49*F*
8*G*	99999980	14,6137059*F*	16*F*	64*F*
9*G*	99999997	16,6137057*F*	18*F*	81*F*
10*G*	99999999$\frac{3}{5}$	18,6137056*F*	20*F*	100*F*

Scholium.

Ut resistentias fluidorum investigarem per experimenta, paravi vas ligneum quadratum, longitudine & latitudine interna digitorum novem pedis *Londinensis*, profunditate pedum novem cum semisse, idemque implevi aqua pluviali; & globis ex cera & plumbo incluso formatis, notavi tempora descensus globorum, existente descensus altitudine in digitorum pedis. Pes solidus cubicue *Londinensis* continet 76 libras *Romanas* aquae pluvialis, & pedis huius digitus solidus continet $\frac{19}{36}$ uncias librae huius seu grana $253\frac{1}{3}$; & globus aquaeus diametro digiti unius descriptus continet grana 132,645 in medio aeris, vel grana 132,8 in vacuo; & globus quilibet alius est ut excessus ponderis eius in vacuo supra pondus eius in aqua.

Exper. 1. Globus, cuius pondus erat $156\frac{1}{4}$ granorum in aere & 77 granorum in aqua, altitudinem totam digitorum 112 tempore minutorum quatuor secundorum descripsit. Et experimento repetito, globus iterum cecidit eodem tempore minutorum quatuor secundorum.

Pondus globi in vacuo est $156\frac{13}{38}$ *gran*. & excessus huius ponderis supra pondus globi in aqua est $79\frac{13}{38}$ *gran*. Unde prodit globi diameter 0,84224 partium digiti. Est autem ut excessus ille ad pondus globi in vacuo, ita densitas aquae ad densitatem globi, & ita partes octo tertiae diametri globi (viz. 2,24597 *dig*.) ad spatium 2*F*, quod proinde erit 4,4156 *dig*. Globus tempore minuti unius secundi, toto suo pondere granorum $156\frac{13}{38}$, cadendo in vacuo describet digitos $193\frac{1}{3}$ & pondere granorum 77, eodem tempore, sine resistentia cadendo in aqua describet digitos 95,219; & tempore *G*, quod sit ad minutum unum secundum in subduplicata ratione spatii *F* seu 2,2128 *dig*. ad 95,219 *dig*. describet 2,2128 *dig*. & velocitatem maximam *H* acquiret quacum potest in aqua descendere. Est igitur tempus *G* 0",15244. Et hoc tempore *G*, cum velocitate illa maxima *H*, globus describet spatium 2*F* digitorum 4,4256; ideoque tempore minutorum quatuor secundorum describet spatium digitorum 1l6,1245 . Subducatur spatium 1,3862944*F* seu 3,0676 *dig*. & manebit spatium 113,0569 digitorum quod globus cadendo in aqua, in vase amplissimo, tempore minutorum quatuor secundorum describet. Hoc spatium, ob angustiam valis lignei praedicti, minui debet in ratione quae componitur ex subduplicate ratione orificii vasis ad excessum orificii huius supra semicirculum maximum globi & ex simplici ratione orificii eiusdem ad excessum eius supra circulum maximum globi, id est, in ratione I ad 0,9914. Quo facto, habebitur spatium 112,08 digitorum, quod globus cadendo in aqua in hoc vase ligneo tempore minutorum quatuor secundorum per theoriam describere debuit quamproxime. Descripsit vero digitos 112 per experimentum.

Exper. 2. Tres globi aequales, quodrum pondera seorsim erant $76\frac{1}{3}$ granorum in aere & $5\frac{1}{16}$ granorum in aqua, successive demittebantur ; & unusquisque cecidit in aqua tempore minutorum secundorum quindecim, casu suo describens altitudinem digitorum 112.

Computum ineundo prodeunt pondus globi in vacuo $76\frac{5}{12}$ *gran*., excessus huius ponderis supra pondus in aqua $71\frac{17}{48}$ *gran*. diameter globi 0,81296 *dig*, octo tertiae partes huius diametri 2,16789 *dig*.; spatium 2*F* 2,3217 *dig*. spatium quod globus pondere $5\frac{1}{16}$ *gran*. tempore 1" sine resistentia cadendo describat 11,808 dig., & tempus *G* 0",301056. Globus igitur, velocitate maxima quacum potest in aqua vi ponderis, $5\frac{1}{16}$ *gran*. descendere, tempore 0",301056 describet spatium 2,3217 *dig*., & tempore 15" spatium 115,678 *dig*. Subducatur spatium 1,3862944*F* seu 1,609 *dig*. & manebit spatium 114,069 *dig*., quod proinde globus eadem tempore in vase latissimo cadendo describere debet. Propter angustiam vasis nostri detrahi debet spatium 0,895 *dig*. circiter. Et sic manebit spatium 113,174 *dig*., quod globus cadendo in hoc vase, tempore 15" describere debuit per theoriam quamproxime. Descripsit vero digitos 112 per experimentum. Differentia eft insensibilis.

Exper. 3. Globi tres aequales, quorum pondera seorsim erant 121 *gran*. in aere & 1 *gran*. in aqua, successive demittebantur ; & cadebant in aqua temporibus 46", 47", & 50" describentes altitudinem digitorum 112.

Per theoriam hi globi cadere debuerunt tempore 40" circiter,
Quod tardius ceciderunt , utrum minori proportioni resistentia, quae a vi inertiae in tardis motibus oritur, ad resistentiam quae oritur aliis causis tribuendum sit; an potius bullulis nonnullis globo adhaerentibus, vel rarefactioni cerae ad calorem vel tempestatis vel manus globum demittentis, vel etiam erroribus insensibilibus in ponderandis globis in aqua, incertum esse puto. Ideoque pondus globi in aqua debet esse plurium granorum, ut experimentum certum & fide dignum reddatur.

Exper. 4. Experimenta hactenus descripta coepi, ut investigarem resistentias fluidorum, antequam theoria in propositionibus proxime pracedentibus exposita mihi innotesceret. Postea, ut theoriam inventam examinarem, paravi vas ligneum latitudine interna digitorum $8\frac{2}{3}$, profunditate pedum quindecim cum triente. Deinde ex cera & plumbo incluso globos quatuor formavi, singulos pondere $139\frac{1}{4}$ granorum in aere & $7\frac{1}{8}$ granorum in aqua. Et hos demisi ut tempora cadendi in aqua per pendulum, ad semi-minuta secunda oscillans, mensurarem. Globi, ubi ponderabantur & possea cadebant, frigidi erant & aliquamdiu frigidi manserant; quia calor ceram rarefacit, & per rarefactionem diminuit pondus globi in aqua, & cera rarefacta non statim ad densitatem pristinam per frigus reducitur. Antequam caderent, immergebantur penitus in aquam; ne pondere partis alicuius ex aqua extantis descensus eorum sub initio acceleraretur. Et ubi penitus immersi quiescebant, demittebantur quam cautissime, ne impulsum aliquem a manu demittente acciperent. Ceciderunt autem successive temporibus oscillationum $47\frac{1}{2}$, $48\frac{1}{2}$, 50 & 51, describentes altitudinem pedum quindecim & digitorum duorum. Sed tempestas iam paulo frigidior erat quam cum globi ponderabantur, ideoque iteravi experimentum alia die, & globi ceciderunt temporibus oscillationum 49, $49\frac{1}{2}$,50 & 53, ac tertio temporibus oscillationum $49\frac{1}{2}$, 50, 51 & 53. Et experimento saepius capto, globi ceciderunt maxima ex parte temporibus oscillationum $49\frac{1}{2}$ & 50. Ubi tardius cecidere, suspicor eosdem retardatos fuisse impingendo in latera vasis.

Iam computum per theoriam ineundo, prodeunt pondus globi in vacuo $139\frac{2}{5}$ granorum. Excessus huius ponderis supra pondus globi in aqua $132\frac{11}{40}$ *gran*. Diameter globi 0,99868 *dig*. Octo tertiae partes diametri 2,66315 *dig*. Spatium 2*F* 2,8066 *dig*. Spatium quod globus pondere $7\frac{1}{8}$ granorum, tempore minuti unius secundi, sine resistentia cadendo describit 9,88164 *dig*. Et tempus G 0", 376843. Globus igitur, velocitate maxima, quacum potest in aqua vi ponderis $7\frac{1}{8}$ granorum descendere, tempore 0",376843 describit spatium 2,8066 digitorum, & tempore 1" spatium 7,44766 digitorum, & tempore 25" seu oscillationum 50 spatium 186,1915 *dig*. Subducatur spatium 1,386294*F*, seu 1,9454 *dig*. & manebit spatium 184,2461 dig. quod globus eodem tempore in vase latissimo describet, Ob angustiam valis nostri, minuatur hoc spatium in ratione quae componitur ex subduplicata ratione orificii vasis ad excessum huius orificii supra semicirculum maximum globi, & simplici ratione eiusdem orificii ad excessum eius supra circulum maximum globi; & habebitur spatium 181,86 digitorum, quod globus in hoc vase tempore oscillationum 50 describere debuit per theoriam quamproxime. Descripsit vera spatium 182 digitorum tempore oscillationum $49\frac{1}{2}$ vel 50 per experimentum.

Exper. 5 . Globi quatuor pondere $154\frac{1}{8}$ *gran*. in aere & $21\frac{1}{2}$ *gran*. in aqua saepe demissi, cadebant tempore oscillationum $28\frac{1}{2}$, 29, $29\frac{1}{2}$ & 30, & nonnunquam 31, 32 & 33, describentes altitudinem pedum quindecim & digitorum duorum.
Per theoriam cadere debuerunt tempore oscillationum 29 quamproxime.

Exper. 6. Globi quinque pondere $212\frac{3}{8}$ *gran*. in aere & $79\frac{1}{2}$ *gran*. in aqua saepe demissi, cadebant tempore oscillationum $15\frac{1}{2}$, 16, 17 & 18, describentes altitudinem pedum quindecim & digitorum duorum.

Per theoriam cadere debuerunt tempore oscillationum 15 quamproxime, *Exper*.7. Globi quatuor pondere $293\frac{3}{8}$ *gran*. in aere & $35\frac{7}{8}$ *gran*. in aqua saepe demissi, cadebant tempore oscillationum $29\frac{1}{2}$, 30, $30\frac{1}{2}$, 31, 32 & 33, describentes altitudinem pedum quindecim & digiti unius cum semisse.

Per theoriam cadere debuerunt tempore oscillationum 28 quamproxime.

Causuam investigando cur globorum, eiusdem ponderis & magnitudinis, aliqui citius alii tardius caderent, in hanc incidi; quod globi, ubi primum demittebantur & cadere incipiebant, oscillarent circum centra, latere illo quod forte gravius esset primum descendente, & motum oscillatorium generante. Nam per oscillationes suas globus maiorem motum communicat aquae, quam si sine oscillationibus descenderet; & communicando, amittit partem motus proprii quo descendere deberet: & pro maiore vel minore oscillatione, magis vel minus retardatur. Quinetiam globus recedit semper a latere suo quod per oscillationem descendit, & recedest, do appropinquat lateribus vasis & in latera nonnunquam impingitur. Et haec oscillatio in globis gravioribus fortior est, & in maioribus aquam magis agitat. Quapropter, ut oscillatio globorum minor redderetur, globos novos ex cera & plumbo construxi, infigendo plumbum in latus aliquod globi prope superficiem eius; & globum ita demisi, ut latus gravius, quoad fieri potuit, esset infimum ab initio descensus. Sic oscillationes factae sunt multo minores quam prius, & globi temporibus minus inequalibus ceciderunt, ut in experimentis sequentibus.

Exper. 8. Globi quatuor, pondere granorum 139 in aere & $6\frac{1}{2}$ in aqua, saepe demissi, ceciderunt temporibus oscillationum non plurium quam 52, non pauciorum quam 50, & maxima ex parte tempore oscillationum 51 circiter, describentes altitudinem digitorum 182.

Per theoriam cadere debuerunt tempore oscillationum 52 circiter.

Exper. 9. Globi quatuor, pondere granorum $273\frac{1}{4}$ in aere & $140\frac{1}{4}$ in aqua, saepius demissi, ceciderunt temporibus oscillationum non pauciorum quam 12, non plurium quam 13, describentes altitudinem digitorum 182.

Per theoriam vero hi globi cadere debuerunt tempore oscillationum $11\frac{1}{3}$ quamproxime.

Exper. 10. Globi quatuor, pondere granorum 384 in aere & $119\frac{1}{2}$ in aqua, saepe demissi, cadebant temporibus oscillationum $17\frac{3}{4}$, 18, $18\frac{1}{2}$ & 19, describentes altitudinem $181\frac{1}{2}$ digitorum. Et ubi ceciderunt tempore oscillationum 19, nonnunquam audivi impulsum eorum in latera vasis antequam ad fundum pervenerunt.

Per theoriam vero cadere debuerunt tempore oscillationum $15\frac{5}{8}$ quamproxime.

Exper. 11. Globi tres aequales, pondere granorum 48 in aere & $3\frac{29}{30}$ in aqua, saepe demissi, ceciderunt temporibus oscillationum $43\frac{1}{2}$, 44, $44\frac{1}{2}$, 45 & 46, & maxima ex parte 44 & 45 , describentes altitudinem digitorum $182\frac{1}{2}$ quamproxime.

Per theoriam cadere debuerunt tempore oscillationum $46\frac{5}{9}$ circiter.

Exper. 12. Globi tres aequales, pondere granorum 141 in aere & $4\frac{3}{8}$ in aqua, aliquoties demissi, ceciderunt temporibus oscillationum 61, 62, 63, 64 & 65, describentes altitudinem digitorum 182.

Et per theoriam cadere debuerunt tempore oscillationum $64\frac{3}{4}$ quamproxime.

Per haec experimenta manifestum est quod, ubi globi tarde ceciderunt, ut in experimentis secundis, quartis, quintis, octavis, undecimis ac duodecimis, tempora cadendi recte exhibentur per theoriam: at ubi globi velocius ceciderunt, ut in experimentis sextis, nonis ac decimis, resistentia paulo maior extitit quam in duplicata ratione velocitatis. Nam globi inter cadendum oscillant aliquantulum, & haec oscillatio in globis levioribus & tardius cadentibus, ob motus languorem cito cessat , in gravioribus autem & maioribus, ob motus fortitudinem diutius durat, & non nisi post plures oscillationes ab aqua ambiente cohiberi potest. Quinetiam globi, quo velociores sunt, eo minus premuntur a fluido ad posticas suas partes; & si velocitas perpetuo augeatur, spatium vacuum tandem a tergo relinquent, nisi compressio fluidi simul augeatur. Debet autem compressio fluidi (per Prop. XXXII. & XXXIII.) augeri in duplicata ratione velocitatis, ut resistentia sit in eadem duplicata ratione. Quoniam hoc non sit, globi velociores paulo minus premuntur a tergo, & defcctu pressionis huius, resistentia eorum sit paulo maior quam in duplicata ratione velocitatis.

Congruit igitur theoria cum phaenomenis corporum cadentium in aqua, reliquum est ut examinemus phaenomena cadentium in aere.

Exper.13. A culmine ecclesiae Sancti *Pauli*, in urbe *Londini*, mense Iunio 1710, globi duo vitaei simul demittebantur, unus argenti vivi plenus, alter aeris ; & cadendo describebant altitudinem pedum *Londinensium* 220. Tabula lignea ad unum eius terminum polis ferreis suspendebatur, ad alterum pessulo ligneo incumbebat , & globi duo huic tabulae impositi simul demittebantur, subtrabendo pessulum ope fili ferrei ad terram usque demissi ut tabula polis ferreis solummocto innixa super iisdem devolveretur, & eodcm temporis momento pendulum ad minuta secunda oscillans, per filum illud ferreum tractum demitteretur & oscillare inciperet. Diametri & pondera globorum ac tempora cadendi exhibentur in tabula sequente.

Globorum mercurio plenorum			*Globorum aere plenorum.*		
Pondera	*Diametri*	*Tempora cadendi.*	*Pondera*	*Diametri*	*Tempora cadendi.*
908 *gran*.	0,8 *digit*	4''	510 *gran*.	5,1 *digit*.	8'' $\frac{1}{2}$
983	0,8	4 –	642	5,2	8
866	0,8	4	599	5,1	8
747	0,75	4+	5I5	5,0	8 $\frac{1}{4}$
808	0,75	4	483	5,0	8 $\frac{1}{2}$
784	0,75	4+	641	5,2	8

Caeterum tempora observata corrigi debent. Nam globi mercuriales (per theoriam *Galilaei*) minutis quatuor secundis describent pedes Londinensis 257, & pedes 220 minutis tantum 3'' 41''' . Tabula lignea utique, detracto pessulo, tardius devolvebatur quam par erat, & tarda sua devolutione impediebat descensum globorum sub initio. Nam globi incumbebant tabule prope medium eius, & paulo quidem propiores erant axi eius quam pessulo. Et hinc tempora cadendi prorogata fuerunt minutis tertiis octodecim circiter, & iam corrigi debent detrahendo illa minuta, praesertim in globis maioribus qui tabulae devolventi paulo diutius incumbebant propter magnitudinem diametrorum. Quo facto tempora, quibus globi sex maiores cecidere, evadent 8'' 12''' , 7'' 42''' , 7'' 42''', 7'' 57''', 8" 12''' , & 7" 42'''.

Globorum igitur aere plenorum quintus, diametro digitorum quinque pondere granorum 483 constructus, cecidit tempore 8" 12''' , describendo altitudinem pedum 220. Pondus aquae huic globo aequalis est 16600 granorum; & pondus aeris eidem aequalis est $\frac{16600}{800}$ *gran*. seu $19\frac{3}{10}$ *gran*. ideoque pondus globi in vacuo est $502\frac{3}{10}$ *gran*. & hoc pondus est ad pondus aeris globo aequalis, ut $502\frac{3}{10}$ ad $19\frac{3}{10}$, & ita sunt $2F$ ad octo tertias partes diametri globi, id est, ad $13\frac{1}{3}$ digitos. Unde $2F$ prodeunt 28 pede 11 *dig*. Globus cadendo in vacuo, toto suo pondere $502\frac{3}{10}$ granorum, tempore minuti unius secundi describit digitos $193\frac{1}{3}$ ut supra, & pondere 483 *gran*. describit digitos 185,905 , & eodem pondere 483 *gran*. etiam in vacuo describit spatium F seu 14 *ped*. $5\frac{1}{2}$ *dig*. tempore 57" 58'''' , & velocitatem maximam acquirit quacum possit in aere descendere. Hac velocitate globus, tempore 8" 12''', describet spatium pedum 245 & digitorum $5\frac{1}{3}$. Aufer $1{,}3863F$ seu 20 *ped*. $0\frac{1}{2}$ *dig*. & manebunt 225 *ped*. 5 *dig*. Hoc spatium igitur globus, tempore 8" 12''', cadendo describere debuit per theoriam. Descripsit vero spatium 220 pedum per experimentum. Differentia insensibilis est.

Similibus computis ad reliquos etiam globos aere plenos applicatis, confeci tabulam sequentem.

Globorum pondera.	*Diametri.*	*Tempora cadendi ab altitudine pedum 220.*	*Spatia describinda per theoriam.*	*Excessus.*
510 *gran*.	5,1 *dig*.	8" 12'''	226*ped*. 11 *dig*.	6 *ped*. 11 *dig*.
642	5,2	7 42	230 9	7 10
599	5,1	7 42	227 10	7 10
515	5	7 57	224 5	4 5
483	5	8 12	225 5	5 5
641	5,2	7 42	230 7	10 7

Exper. 14. Anno 1719. mense Iulio, D Desaguliers huiusmodi experimenta iterum cepit, formando vesicas porcorum in orbem sphaericum ope sphaerae lignese concavae, quam madefactae implere cogebantur inflando aerem; & hasce arefactas & exemptas demittendo ab altiore loco in templi eiusdem turri rotunda fornicata, nempe ab altitudine pedum 272; & eodem temporis momento demittendo etiam globum plumbeum cuius pondus erat duarum librarum Romanarum circiter, Et interea aliqui stantes in suprema parte templi, ubi globi demittebantur, notabant tempora tota cadendi, & alii stantes in terra notabant differentiam temporum inter casum globi plumbei & casum vesicae. Tempora autem mensurabantur pendulis ad dimidia minuta secunda oscillantibus. Et eorum qui in terra stabant unus habebat horologium cum elatere ad singula minuta secunda quater vibrante, alius habebat machinam aliam affabre constructam cum pendulo etiam ad singula minuta secunda quater vibrante. Et similem machinam habebat unus eorum qui habant in summitate templi. Et haec instrumenta ita formabantur, ut motus eorum pro lubitu vel inciperent vel sisterentur. Globus autem plumbeus cadebat tempore minutorum secundorum quaruor cum quadrante circiter. Et addendo hoc tempus ad praedictam temporis differentiam, colligebamr tempus totum quo vesica cecidit. Tempora, quibus vesicae quinque post casum globi plumbei prima vice ceciderunt, erant $14\frac{3}{4}'', 12\frac{3}{4}'', 14\frac{5}{8}'', 17\frac{3}{4}'', \& 16\frac{3}{4}''$, & secunda vice $14\frac{1}{2}'', 12\frac{1}{4}'', 14'', 19'', \& 16\frac{3}{4}''$. Addantur $4\frac{1}{4}''$, tempus utique quo globus plumbeus cecidit, & tempora tota, quibus vesicae quinque ceciderunt, erant prima vice $19'', 17'', 18\frac{7}{8}'', 22'', \& 21\frac{1}{8}''$; & secunda vice, $18\frac{3}{4}'', 18\frac{1}{2}'', 18\frac{1}{4}'', 23\frac{1}{4}'', \& 21''$.. Tempora autem in summitate templi notata, erant prima vice $19\frac{3}{4}'', 17\frac{1}{4}'', 18\frac{3}{4}'', 22\frac{1}{8}'', \& 21\frac{5}{8}''$.; & secunda vice $19'', 18\frac{5}{8}'', 18\frac{3}{8}'', 24'', \& 21\frac{1}{4}''$. . Caeterum vesicae non semper recta cadebant, sed nonnunquam volitabant, & hinc inde oscillabantur inter cadendum. Et his motibus tempora cadendi prorogata sunt & aucta nonnunquam dimidio minuti unius secundi, nonnunquam minuto secundo toto. Cadebant autem rectius vesica secunda & quarta prima vice; & prima ac tertia secunda vice. Vesica quinta rugosa erat & per rugas suas nonnihil retardabatur. Diametros vesicarum deducebam ex earum circumferentiis filo tenuissimo bis circundato mensuratis. Et theoriam contuli cum experimentis in tabula sequente, assumendo densitatem aeris esse ad densitatem aquae pluvialis ut 1 ad 860, & computando spatia quae globi per theoriam describere debuerunt cadendo.

Vesicarum pondera.	*Diametri.*	*Tempora cadendi ab altitudine pedum 272.*	*Spatia iisdem temporibus describenda per theoriam.*		*Differentia inter theor. & exper.*	
128 *gran.*	5, 28 *dig.*	19"	271*ped.*	11 *dig.*	– 0*ped.*	1*dig.*
156	5, 19	17	272	$0\frac{1}{2}$	+ 0	$0\frac{1}{2}$
$137\frac{1}{2}$	5, 3	$18\frac{1}{2}$	272	7	+ 0	7
$97\frac{1}{2}$,	5 26	22	277	4	+ 5	4
$99\frac{1}{8}$	5	$21\frac{1}{8}$	282	0	+ 10	0

Globorum igitur tam in aere quam in aqua motorum resistentia prope omnis per theoriam nostram recte exhibetur, ac densitati fluidorum, paribus globorum velocitatibus ac magnitudinibus, proportionalis est.

In scholio, quod sectioni sextae subiunctum est, ostendimus per experimenta pendulorum quod globorum aequalium & aequivelocium in aere, aqua, & argento vivo motorum resistentiae sunt ut fluidorum densitates. Idem hic ostendimus magis accurate per experimenta corporum cadenuum in aere & aqua. Nam pendula singulis oscillationibus motum cient in fluido motui penduli redeuntis semper contrarium, & resistentia ab hoc motu oriunda, ut & resistentia fili quo pendulum suspendebatur, totam penduli resistentiam maiorem reddiderunt quam resistentia quae per experimenta corporum cadentium prodiit. Etenim per experimenta pendulorum in scholio illo exposita, globus eiudem densitatis cum aqua, describendo longitudinem semidiametri suae in aere. amittere deberet motus sui partem $\frac{1}{3342}$. At per theoriam in hac septima sectione expositam & experimentis cadentium confirmatam, globus idem describendo longitudinem eandem, amittere deberet motus sui partem tantum $\frac{1}{4586}$ posito quod densitas aquae sit ad densitatem aeris ut 860 ad 1. Resistentiae igitur per experimenta pendulorum maiores prodiere (ob causas iam descriptas) quam per experimenta globorum cadentium, idque in ratione 4 ad 3 circiter. Attamen cum pendulorum in aere, aqua & argento vivo oscillantium resistentiae a causis similibus similiter augeantur, proportio resistentiarum in his mediis, tam per experimenta pendulorum, quam per experimenta corporum cadentium, satis recte exhibebitur. Et inde concludi potest quod corporum in fluidis quibuscunque fluidissimis motorum resistentiae, caeteris paribus, sunt ut densitates fluidorum.

His ita stabilitis, dicere iam licet quamnam motus sui partem globus quilibet, in fluido quocunque proiectus, data tempore amittet quamproxime. Sit D diameter globi, & V velocitas eius sub initio motus, & T tempus, quo globus velocitate V in vacuo describet spatium, quod sit ad spatium $\frac{8}{3}D$ ut densitas globi ad densitatem fluidi : & globus in fluido illo proiectus, tempore quovis alio t, amittet velocitatis suae partem $\frac{tV}{T+t}$ manente parte $\frac{TV}{T+t}$ & describet spatium, quod sit ad spatium uniformi velocitate V eadem tempore descriptum in vacuo, ut logarithmus numeri $\frac{T+t}{T}$ multiplicatus per numerum 2,302585093 est ad numerum $\frac{t}{T}$ per Corol. VII. Prop. XXXV. In motibus tardis resistentia potest esse paulo minor, propterea quod figura globi paulo aptior sit ad motum quam figura cylindri eadem diametro descripti. In motibus velocibus resistentia potest esse paulo maior, propterea quod elasticitas & compressio fluidi. non augeantur in duplicata ratione velocitatis. Sed huiusmodi minutias hic non expendo.

Et quamvis aer, aqua, argentum vivum & similia fluida, per divisionem partium in infinitum, subtiliarentur & fierent media infinite fluida ; tamen globis proiectis haud minus resisterent. Nam resistentia, de qua agitur in propositionibus praecedentibus, oritur ab inertia materiae , & inertia materiae corporibus essentialis est & quantitati materiae semper proportionalis. Per divisionem partium fluidi, resistentia quae oritur a tenacitate & frictione partium diminui quidem potest : sed quantitas materiae per divisionem partium eius non diminuitur; & manente quantitate materiae, manet eius vis inertae, cui resistentia, de qua hic agitur, semper proportionalis est. Ut hac resistentia diminuatur, diminui debet quantitas materiae in spatiis per quae corpora moventur. Et propterea spatia coelestia, per quae globi planetarum & cometarum in omnes partes liberrime & sine omni motus diminutione sensibili perpetuo moventur, fluido omni corporeo destituuntur, si forte vapores longe tenuissimos & traiectos lucis radios excipias.

Proiectilia utique motum cient in fluidis progrediendo, & hic motus oritur ab excessu pressionis fluidi ad proiectilis partes anticas supra pressionem ad eius partes posticas, & non minor esse potest in mediis infinite fluidis quam in aere, aqua & argento vivo pro densitate materiae in singulis. Hic autem pressionis excessus, pro quantitate sua, non tantum motum ciet in fluido, sed etiam agit in proiectile ad motum eius retardandum : & propterea resistentia in omni fluido est ut motus in fluido a proiectili excitatus, nec minor esse potest in aethere subtilissimo pro densitate aetheris, quam in aere, aqua & argento vivo pro densitatibus horum fluidorum.