SECTION VI.

Concerning the motion and resistance of simple pendulums.

PROPOSITION XXIV. THEOREM XIX.

The quantities of matter in suspended bodies, the centres of oscillation of which are equidistant from the centre of suspension, are in a ratio compounded from the ratio of the weights and in the square ratio of the times of the oscillations in a vacuum.

[In this proposition and corollaries, the suspended bodies which are to be compared, are supposed to be oscillating on cycloidal arcs or on the arcs of exceedingly large circles. In addition, the acceleration of gravity is not assumed constant, but varies from place to place between the pendulums, and the force hence likewise on account of the gravitational forces differing, while the inertial masses in turn give rise to differing accelerations. (From an *L*. & *J*. note).

This section follows on from Section10, Book I, which the reader may wish to consult. We may also note here the use of the word *funipendulus* by Newton in describing what we call a simple pendulum, which is a made up Latin word, describing a body hanging from a rope or cord, and implied able to swing freely, or in this case for the cord to wrap around cycloidal cusps; the involute of the cycloid described being a similar inverted cycloid. A vertical cycloid arc supports simple harmonic motion of any amplitude for a particle travelling along such a curve under the influence of gravity, as can be worked out from the parametric equations of the curve, found in most older books on mechanics. (We note however, that the motion of a bead on a smooth wire of this shape has different boundary conditions from a pendulum swinging and suspended between the inverted arcs of a cycloid: as in the latter case the length of the pendulum is taken customarily as twice the diameter of the circle that generates the cycloid, so that the bob never comes into contact with the cycloid walls, only the string.) Thus the arc length is proportional to the (– ve) acceleration, and so to the force acting on the mass. These questions were taken up and solved analytically by Euler in Book II of his *Mechanica*, Chapter 3, §545 – §601; see e.g. the translations by this writer. We will give here in addition, the partial analytic solutions of Brougham and Routh and from modern sources; the relevant notes of Leseur & Janquier are used occasionally to augment Newton's arguments.]

For the velocity, that a given force in a given material, can generate in a given time, is as the force and the time directly, and as the [quantity of] matter indirectly. So that the greater the force or the greater the time or the less the matter, from that a greater velocity will be generated. Because that is evident from the second law of motion. Now truly if the pendulums are of the same length, the motive forces in places at equal distances from the perpendicular are as the weights : and thus if two bodies describe equal arcs by oscillating, and these arcs may be divided into equal parts ; since the times in which the bodies describe the corresponding parts of arcs shall be as the times of the whole oscillations, the velocities in turn, in the corresponding parts of the oscillations, shall be as the motive forces and the times of the whole oscillations directly, and inversely as the quantities of matter : and thus the quantities of matter are as the [motive] forces and the times of oscillations directly and inversely as the velocities. But the velocities are inversely as the times, and thus the times directly and the velocities inversely together are as the squares of the times, and therefore the quantities of matter are as the motive forces and the squares of the times, that is, as the weights and the squares of the times. *Q.E.D.*

[Thus, in an obvious notation, $\frac{m}{M} \propto \frac{w}{W} \times \frac{t^2}{T^2}$.]

Corol. 1. Thus if the times shall be equal, the quantities of matter in the individual bodies shall be as the weights.

Corol. 2. If the weight shall be equal, the quantities of matter shall be as the squares of the times.

Corol. 3. If the quantities of matter shall be equal, the weights shall be inversely as the squares of the times.

Corol.4. From which since the squares of the times, with all else being equal, shall be as the lengths of the pendulums, and if both the times and the quantities of matter are equal, the weights shall be as the lengths of the pendulums.

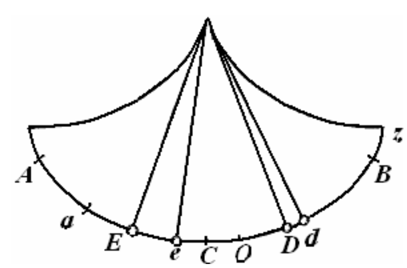
Corol. 5. And generally, the quantity of matter of the pendulum is as the weight and the square of the time directly, and inversely as the length of the pendulum.

Corol. 6. But also in a non resisting medium, the quantity of matter in the pendulum is as the comparative weight, the square of the time directly, and the length of the pendulum inversely. For the comparative weight is the motive force of the body in some heavy medium, as I have explained above; and thus likewise performs in such a non-resisting medium as the absolute weight in a vacuum.

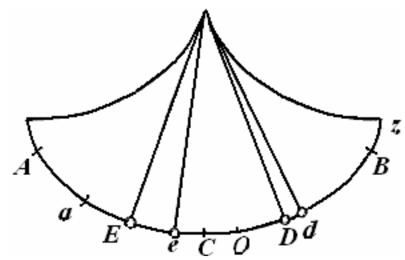
Corol. 7. And hence an account may be clear both of comparing bodies between each other, as far as the quantity of matter in each, as well as comparing the weights of the same bodies in different places, and requiring an understanding of the variation of gravity. But with the most accurate experiments performed I have always found that the quantity of matter in individual bodies are proportional to the weights of these.

PROPOSITIO XXV. THEOREMA XX.

Bodies suspended as simple pendulums in which, with some medium present that resists in the ratio of instants of time, and suspended bodies which may be moving without resistance in a medium of the same specific gravity, complete oscillations on a cycloid in the same time, and describe proportional parts of the arcs in the same time.



Let *AB* be the arc of a cycloid, that the body *D* will describe by oscillating in some time in a non resisting medium. The same may be bisected in *C*, thus so that *C* shall be the lowest point of this ; and the accelerative force by which the body may be urged on if in some place *D* or *d* or *E* shall be as the length of the arc *CD* or *Cd* or *CE*. That force may be shown by the same arc, and since the resistance shall be as a moment of time [which is taken to mean that the resistance is constant], and it may be given thus, the same may be shown by the part *CO* of the given arc of the cycloid, and the arc *Od* may be taken in the ratio to the arc *CD* that the arc *CB* has to the arc *CB*: and the force by which the body may be acted on at *d* in the resisting medium, since it shall be the excess of the force *Cd* over the resistance *CO*, may be shown by the arc *CD*, and thus it will be to the force, by which the body may be acted on in the medium without resistance at the place *D*, as the arc *Od* to the arc *CD*; and therefore also at the place *B* as the arc *OB* to the arc *CB*. Therefore if two bodies, *D*, *d* may depart from the place *B*, and may be acted on by these forces: since the forces from the beginning shall be as the arcs *CB* and *OB*, the first velocities and the first arc will be described in the same ratio. Let these arcs be *BD* and *Bd*, and the remaining arcs *CD* and *Od* will be in the same ratio. Hence the forces will remain proportional to *CD* and *Od* themselves in the same ratio and from the start, and therefore the bodies may traverse the arcs to be likewise described in the same ratio. Therefore the forces and the velocities and the remaining arcs *CD* and *Od* will always be as the whole arcs *CB* and *OB*, and therefore these remaining arcs will be described likewise. Whereby the two bodies *D* and *d* will arrive at the places *C* and *O* at the same time, indeed the one at the place *C* in the non-resisting medium, and the other at the place *O* in the resisting medium. But since the velocities at *C* and *O* shall be as the arcs *CB* and *OB*; these arcs, which the bodies likewise describe by going further, will be in the same ratio. Let these be *CE* and *Oe*. The force by which the body *D* may be retarded in the non-resisting medium at *E* is as *CE*, and the force by which the body *d* may be retarded in the resisting medium at *e* is as the sum of the force *Ce*, and of the resistance *CO*, that is as *Oe*; and thus the forces, by which the bodies may be retarded, are as the arcs *CB* and *OB* proportional to the arcs *CE, Oe*; and hence the velocities, retarded in that given ratio, remain in that same given ratio. Therefore the velocities and the arcs described by the same are always in that given ratio of the arcs *CB* and *OB*; and therefore if the whole arcs *AB* and *aB* may be taken in the same ratio, the bodies *D* and *d* likewise describe these arcs, and all the motion may be let go from the places *A* and *a* at the same time. Therefore all the oscillations are isochronous, and any parts of arcs *BD* and *Bd*, or *BE* and *Be* which are described at the same time shall be proportional to the whole arcs *BA* and *Ba*. *Q.E.D.*



Corol. Therefore the most rapid motion does not fall on the lowest point *C*, but is found at that point *O*, by which the whole arc described *aB* may be bisected. And the body at once by progressing to *a*, may be retarded by the same steps by which before it was accelerated in its descent from *aB* to *O*.

[This proposition follows at once from the isochronous nature of the oscillations, being independent of the amplitude; thus, the role of the resistance is merely to enable the system to pass through successive isochronous oscillations of diminishing amplitudes : on the assumption that there is no dependence of the time of oscillation on the resistance, which is certainly the case for damping of this kind.
For a modern analysis; see, e.g. Chorlton, *Textbook of Dynamics*, p.96, V.N.: Let l be twice the radius of the generating circle, s the arc length from C, v the velocity, and take

the mass of the particle as unity. Let $\frac{g}{2l}f$ be the constant resistance. We note that the intrinsic equation of the cycloid can be written as $s = 2l\,sin\psi$, where ψ is the angle the tangent to the cycloid makes at this point, in the sense of increasing *s*. The motive force on the body can be written as $\frac{d^2s}{dt^2} + \frac{g}{2l}s = \frac{g}{2l}f$ or $\frac{d^2(s-f)}{dt^2} = -\frac{g}{2l}(s-f)$; this is the standard differential equation for S.H.M. about a point distant f to the right of the lowest point, with the angular frequency $\sqrt{\frac{g}{2l}}$, and we can immediately write down $s(t) = (s_0 - f)cos\sqrt{\frac{g}{2l}}t + f$ for the first half cycle of the motion starting from s_0 and reaching the maximum tangential speed at *f*, where the attracting force becomes zero; however, when the particle comes to rest after completing the arc, the sign of the resistance changes, and a new equation can now be set up $\frac{d^2(s+f)}{dt^2} = -\frac{g}{2l}(s+f)$ for the other half of the swing, and solved for these conditions; the outcome being amplitudes of swings decreasing in an arithmetic progression, as an arc *f* is removed in each half swing while the frequency and hence the period of the oscillation is unchanged, as Newton has shown. *B*. & *R*. have incorrectly used *l* rather than 2*l* in their derivations.]

PROPOSITION XXVI. THEOREM XXI.

The oscillations of simple pendulums on a cycloid, which are resisted in the ratio of the velocities, are isochronous.

For if two bodies, at equal distances from the centre of suspension, by oscillating describe unequal arcs, and the velocities in the corresponding parts of the arcs shall be in turn as the total arcs ; the resistances proportional to the velocities, also in turn they will be as the same arcs. Hence if with the motive forces arising from gravity, which shall be as the same arcs, these resistances may be taken away or added on, the differences or the sums in turn will be in the same ratio of the arcs : and since the increments or decrements of the velocities shall be as these differences or sums, the velocities always will be as the whole arcs : Therefore the velocities, if in any case they shall be as the whole arcs, always will remain in the same ratio. But at the start of the motion, when the bodies begin to descend and to describe these whole arcs, the forces, since they shall be proportional to the arcs, will generate velocities proportional to the arcs. Therefore the velocities always will be as the whole arcs described, and therefore these arcs may be described in the same times. *Q.E.D.*

[In this case analytically, the force equation can be written as $\frac{d^2s}{dt^2} + 2k\frac{ds}{dt} + \frac{g}{2l}s = 0$; in the case of under damped motion, the particle oscillates with a reduced frequency, depending on the size of the damping coefficient *k*, and decays exponentially, or the amplitudes are in a geometric progression . Again, this is a standard equation that can be found in the appropriate texts.]

PROPOSITION XXVII. THEOREM XXII.

If for funicular bodies there is resistance in the square ratio of the velocities, the differences between the times of the oscillations in a resisting medium and the times of the oscillations in a non-resisting medium of the same specific gravity, will be nearly proportional to the arcs described.

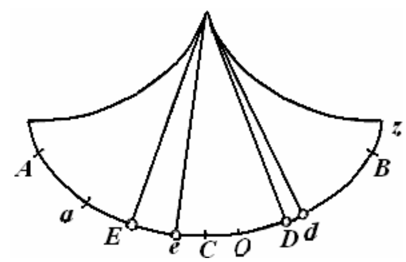
For with equal pendulums in the resisting medium, unequal arcs A and B may be described ; and the resistance of the body in the arc A to the resistance of the body in the corresponding arc B will be in the square ratio of the velocities, that is, as A^2 to B^2, as an approximation. If the resistance in the arc B should be to the resistance in the arc A as AB to A^2 ; the times in the arcs A and B become equal, by the above Proposition. [As the resistances are again in the ratio of B to A.] And thus the resistances A^2 in the arc A, or AB in the arc B, brings about the excess of the time in the arc A above the time in the non-resisting medium, and the resistance B^2 brings about an excess of the time in the arc B above the time in the non-resisting medium. But these excesses are as the effecting forces AB and BB as an approximation, that is, as the arcs A and B. *Q.E.D.*

Corol 1. Hence from the times of the oscillations made, in the resisting medium, in unequal arcs, the times of the oscillations are able to become known in the non-resisting medium of the same specific gravity. For the differences of the times will be to the excess of the time in the smaller arc above the time in the non-resisting medium, as the difference of the arcs to the minor arc.

Corol. 2. The shorter oscillations are more isochronous, and the shortest may be carried out in the same times as in the non-resisting medium, approximately. Truly of these which are made in the greater arcs, the times then become a little greater, and therefore so that the resistance in the descent of the body in which time it may be produced, shall be greater for the ratio of the length to be described in the descent, than the resistance in the subsequent ascent in which the time is shortened. Moreover the time of the oscillations both of the shorter as well as of the longer may be seen never to be produced by the motion of the medium. For with the bodies slowed there is a little less resistance, for the ratio of the velocities, and with the accelerations a little more than from these which are progressing uniformly : and thus because the medium, from that motion it has acquired from the bodies, may be progressing in the same direction, in the first case is disturbed more, in the second less; and hence may agree more or less with the motion of the body. Therefore with descending pendulums the resistance will be greater, and in the ascent less than with the ratio of the velocities, and the time is increased from each cause.

PROPOSITION XXVIII. THEOREM XXIII.

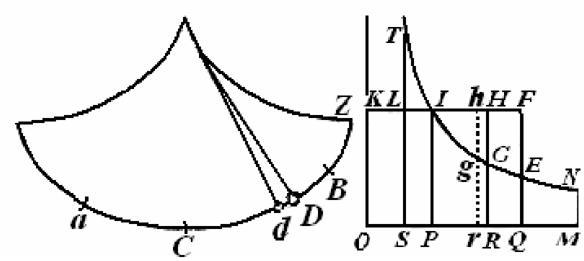
If, with a simple pendulum oscillating in the cycloid, it may be resisted in the ratio of the moments of time [i.e. constant resistive forces], the resistance of this to the force of gravity will be as the excess of the whole descending arc described over the whole ascending arc subsequently described, to the length of the pendulum doubled.



Let *BC* designate the descended arc described, *Ca* the ascended arc described, and *Aa* the difference of the arcs : and with the demonstrations and constructions which were put in place in Proposition XXV, the force will be, by which the oscillating body may be urged at some location *D*, to the force of resistance as the arc *CD* to the arc *CO*, which is half of that difference *Aa*. And thus the force, by which the oscillating body may be acted on in the beginning or at highest point of the cycloid, that is, the force of gravity, will be to the resistance as the arc of the cycloid between the highest point of that and the lowest point *C* to the arc *CO*; that is (if the arcs may be doubled): as the whole arc of the cycloid, or twice the length of the pendulum, to the arc *Aa*. *Q. E. D.*

PROPOSITION XXIX. PROBLEM VI.

For a body put in place oscillating on a cycloid resisting in the square ratio of the velocity, to find the resistance at individual places.



Let *Ba* be the whole arc described in an oscillation, and let *C* be the lowest point of the cycloid, and *CZ* the half arc of the whole cycloid, equal to the length of the pendulum ; and the resistance may be sought at some place *D*. The indefinite right line *OQ* may be cut at the points *O, S, P, Q*, by that rule, so that (if the perpendiculars *OK, ST, PI, QE* may be erected and with centre *O* and with the asymptotes *OK, OQ* , the hyperbola *TIGE* may be described cutting the perpendiculars *ST, PI, QE* in *T, I* & *E*, and through the point *I* , *KF* may be drawn parallel to the asymptote *OQ* meeting the asymptote *OK* in *K*, and the perpendiculars *ST* and *QE* in *L* and *F*), the hyperbolic area *PIEQ* to the hyperbolic area *PITS* shall be as the arc *BC* described by the descent of the body to the arc *Ca* described by the ascent, and the area *IEF* to the area *ILT* as *OQ* to *OS*. Then the hyperbolic area *PINM* may be cut by the perpendicular *MN* which shall be to the hyperbolic area *PIEQ* as the arc *CZ* to the arc *BC* described in the descent. And if the hyperbolic area *PIGR* may be cut by the perpendicular *RG*, which shall be to the area *PIEQ* as some arc *CD* to the arc *BC* described in the whole descent, the resistance at the point *D* to the force of gravity, shall be as the area $\frac{OR}{OQ}\times IEF - IGH$ to the area *PINM*.

For since the forces arising from gravity by which the body may be acted on at the places *Z, B, D, a*, shall be as the arcs *CZ, CB, CD, Ca*, and these arcs shall be as the areas *PINM, PIEQ, PIGR, PITS*; then both the arcs as well as the forces may be represented by these areas respectively. Let *Dd* above be as the minimum distance described in the descent of the body, and likewise there may be shown by the area *RGgr* taken as the minimum from the parallels *RG*, *rg*, and *rg* may be produced to *h*, so that *GHhg* and *RGgr* shall be the decrements of the areas *IGH* and *PIGR* during the same time. And the increment $GHhg - \frac{Rr}{OQ}IEF$, or $Rr \times HG - \frac{Rr}{OQ}IEF$ of the area $\frac{OR}{OQ} \times IEF - IGH$, will be to the decrement *RGgr* of the area *PIGR*, or $OR \times HG - \frac{Rr}{OQ}IEF$ to $OR \times GR$ or $OP \times PI$, that is (on account of the equality of $OR \times HG$, $OR \times HR - OR \times GR$, $ORHK - OPIK$, $PIHR$ & $PIGR + IGH$), as $PIGR + IGH - \frac{OR}{OQ} \times IEF$ to $OPIK$. Therefore if the area $\frac{OR}{OQ} \times IEF - IGH$ may be called *Y*, and the decrement *RGgr* of the area *PIGR* may be given, the increment of the area *Y* will be as *PIGR–Y*.

But if *V* may designate the force arising from gravity, proportional to the arc *CD* described, by which the body may be acted on at *V*, and *R* may be put in place for the resistance; $V - R$ will be the whole force by which the body is acted on at D. And thus the increment of the velocity is as $V - R$ and that small element of the time in which it has been made jointly. But also the velocity itself shall be as the increment of the distance described directly and inversely to the same element of the time. From which , since the resistance by hypothesis shall be as the square of the velocity, the increment of the resistance, (by Lemma II) will be as the velocity and the increment of the velocity jointly, that is, as the moment of the distance and $V - R$ together ; and thus, if the moment of the distance may be given, as $V - R$; that is, if for a given force *V* it may be expressed by writing *PIGR*, and the resistance *R* may be expressed by some other area *Z*, as $PIGR - Z$.

Therefore the area *PIGR* uniformly decreasing by the removal of the given moments, the areas *Y* increase in the ratio $PIGR - Y$, and the area *Z* in the ratio $PIGR - Z$. And therefore if the areas *Y* and *Z* may be taken together and they shall be equal from the beginning, these by the addition of equal moments will be able to go on equal, and likewise from the equal moments at once the decreases vanish together. And in turn, if they both begin and vanish together, they will have equal moments and they will always be equal: hence that is the case because if the resistance *Z* may be increased, the velocity together with the that arc *Ca*, which will be described in the ascent of the arc, will be diminished, and at the point at which all the motion together with the resistance may cease by approaching closer to the point *C*, the resistance may vanish more rapidly than the area *Y*. And conversely it will arise when the resistance is diminished.

Now truly the area *Z* begins and is definite when the resistance is zero, that is, in the first place the motion of the arc *CD* may be equal to the motion of the arc *CB* everywhere and the right line *RG* begins on the right line *QE*, & in the end the motion everywhere of the arc *CD* may be equal to the arc *Ca* and *RG* falls on the right line *ST*. And the area *Y* or $\frac{OR}{OQ} \times IEF - IGH$ begins and is defined where it is zero, and thus $\frac{OR}{OQ} \times IEF$ and IGH are equal everywhere : that is (by the construction) where the right line *RG* successively begins on the right lines *QE* and *ST*. And hence these areas begin and vanish together, and therefore they are always equal. Therefore the area $\frac{OR}{OQ}\times IEF - IGH$ is equal to the area *Z*, by which the resistance is expressed, and therefore it is to that area *PINM* through which gravity is expressed, as the resistance to the weight. *Q.E.D.*

Corol. I. Therefore the resistance at the lowest place *C* is to the force of gravity, as the area $\frac{OP}{OQ}\times IEF$ to the area *PINM*.

Corol: 2. But the maximum shall come about, when the area *PIHR* is to the area *IEF* as *OR* to *OQ*. For in that case the moment of this (without doubt $PIGR - Y$) becomes zero.

Corol. 3. Hence also the velocity at individual places may become known, clearly which is in the square root ratio of the resistance, and the motion from the beginning itself may be equal to the velocity of the body on the same cycloid without resistance of oscillation.

Subsequently, on account of the difficult calculation by which the resistance and the velocity are required to be found by this proposition, it has been considered to attach the following proposition.

[It is convenient to give here the Brougham & Routh derivation, starting from $v\frac{dv}{ds} = -\omega^2 s - kv^2$ in an obvious notation, where $\omega^2 = \frac{g}{2l}$; initially the particle is moving in the direction of the increasing arc; when it returns, the damping factor must change sign. *B* & *R* point out that the displacement cannot be written finitely as a function of time; however, it is possible to find the velocity of the particle at any point on the arc using an integrating factor. Put the equation in the form :

$\frac{dv^2}{ds} + 2kv^2 = -2\omega^2 s$ or $e^{2ks}\frac{dv^2}{ds} + 2ke^{2ks}v^2 = -2\omega^2 e^{2ks}s$, giving

$\frac{d\left(v^2e^{2ks}\right)}{ds} = -2\omega^2 e^{2ks}s$ or $v^2e^{2ks} = -2\omega^2\int e^{2ks}sds$, which in turn gives :

$v^2e^{2ks} = -2\omega^2\int e^{2ks}sds = -\frac{\omega^2}{k}\left[e^{2ks}s - \int e^{2ks}ds\right] = C - \frac{\omega^2e^{2ks}}{k}\left[s - \frac{1}{2k}\right]$. The constant *C* may be found by putting $v = V$ when $s = 0$; in which case $C = V^2 - \frac{\omega^2}{2k^2}$; if the damping is small, this finite expression can be expanded out approximately. These writers dismiss Newton's geometrical solution for the velocity as, ' of no value except as a matter of curiosity'. We follow on with *B*. & *R*.'s analysis for deducing the law of the resistance from experiments with a pendulum, before examining Newton's general formulation of the problem in the next two propositions, for a small disturbance produced in the oscillation of any kind whatever.

Let the quantities *s, t,* ω, etc., have the same meaning as before, and let f be some small disturbance acting along the tangent of the motion of the particle. The equation of motion will then be, on adapting the original terminology slightly to more modern :

$$\frac{d^2s}{dt^2}+\omega^2 s= f.$$

If $f=0$, the motion will be given by $s = A\sin(\omega t+\varphi)$; $v = A\omega\cos(\omega t+\varphi)$; we assume that these are the equations of motion when *f* is not zero, and in which case *A* and φ are functions of the time *t* (Camb. Phil. Trans. 1826), were the second equation is the differential of the first: $\frac{dA}{dt}\sin(\omega t+\varphi)+A\cos(\omega t+\varphi)\left(\omega+\frac{d\varphi}{dt}\right)= A\omega\cos(\omega t+\varphi)$; or

$$\frac{dA}{dt}\sin(\omega t+\varphi)+A\cos(\omega t+\varphi)\frac{d\varphi}{dt}=0 ;$$

and since these equations satisfy the original equation of motion, we have
$\frac{dv}{dt}+\omega^2 s= f$, or $\frac{dA}{dt}\omega\cos(\omega t+\varphi)-A\omega\left(\omega+\frac{d\varphi}{dt}\right)\sin(\omega t+\varphi)+\omega^2 A\sin(\omega t+\varphi)= f$;
leading to :

$$\frac{dA}{dt}\cos(\omega t+\varphi)-A\frac{d\varphi}{dt}\sin(\omega t+\varphi)=\frac{f}{\omega}.$$

Solving these equations, we have :

$$\frac{dA}{dt}=\frac{f}{\omega}.\cos(\omega t+\varphi) \text{ and } \frac{d\varphi}{dt}=-\frac{f}{\omega A}.\sin(\omega t+\varphi)]$$

These equations, when solved, will give the changes in the arc and the time, produced by the cause *f*.

If *f* is very small, then so are the variations of the amplitude *A* and of the phase φ, and so can be ignored when multiplied by *f*, as quantities proportional to f^2 arise. Hence, if A' and φ' are the new values of *A* and φ,

$$A'-A=\frac{1}{\omega}\int f\cos(\omega t+\varphi)dt \text{ and } \varphi'-\varphi=-\frac{1}{\omega A}\int f\sin(\omega t+\varphi)dt ;$$

from which we learn that if *f* consists of two disturbing causes, the total disturbance will be equal almost to the sum of the two disturbances separately.

Suppose that $f=kv^m$, i.e. the resistance is proportional to the m^{th} power of the velocity, then the velocity in moving from the lowest point is :

$$v = A\omega\cos(\omega t+\varphi) \text{ and hence } f = kA^m\omega^m\cos^m(\omega t+\varphi);$$

on substituting in the above integrals, and integrating between the limits $\omega t+\varphi=-\frac{\pi}{2}$ and $\omega t+\varphi=\frac{\pi}{2}$, we have

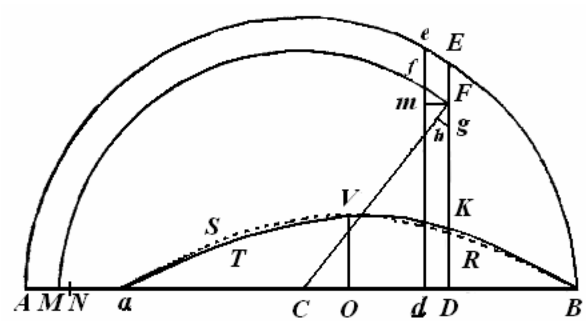
$$A'-A=-k\alpha.\omega^{m-2}A^m.\frac{m(m-2)(m-4)...}{(m+1)(m-1)(m-3)......}, \text{ and } \varphi'-\varphi=0$$

where $\alpha = \pi$ if m is odd, and $\alpha = 2$ if m is even. Hence on ignoring second order quantities, since the phase is preserved, the time of the oscillations is unchanged, and the arcs decrease continually, and the difference between the arc described in the descent and that described on the subsequent ascent will be proportional to the same power of the arc that expresses the power of the velocity for the velocity, all else being unchanged. Thus the law governing the resistance can be found.]

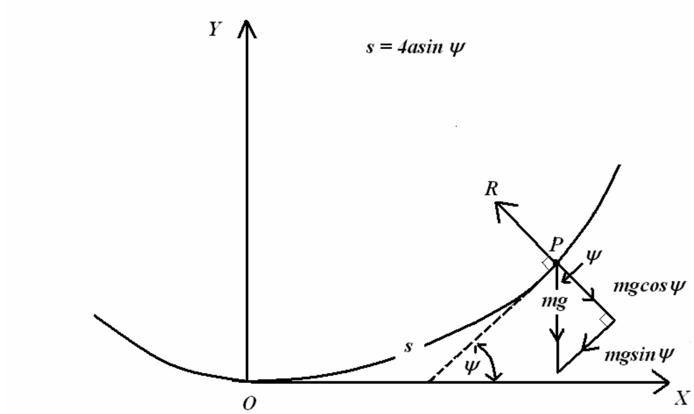
PROPOSITION XXX. THEOREM XXIV.

If the right line aB shall be equal to the arc of the cycloid that the body will describe by oscillating, and at the individual points of this D the perpendiculars DK may be erected, which shall be to the length of the pendulum as the resistance of the body at the corresponding points of the arc to the force of gravity : I say that the difference between the arc described in the whole descent and the arc described in the subsequent whole ascent multiplied by the half sum of the same arcs, will be equal to the area BKa occupied by all the perpendiculars DK.

For both the arc of the cycloid may be expressed in a whole oscillation, described by that right line itself equal to *aB*, as well as the arc which may be described in a vacuum by the length *AB*. *AB* may be bisected in *C*, and the point *C* will represent the lowest point of the cycloid, and *CD* will be the force arising from gravity, by which the body at *D* may be urged along the tangent to the cycloid, and it will have that ratio to the length of the pendulum that the force at *D* had to the force of gravity. Therefore that force may be expressed by the [arc] length *CD*, and the force of gravity expressed by the length of the pendulum, and if *DK* may be taken on *DE*, in that ratio to the length of the pendulum that the resistance has to the weight, *DK* will be expressing the resistance.

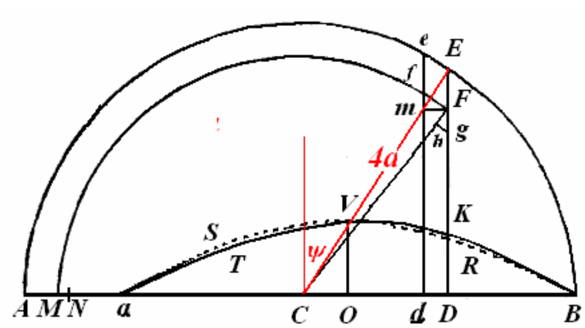


[It is convenient to interrupt Newton's discourse here, by noting that the intrinsic equation of the inverted cycloid is $s = 4a\,sin\psi$, where ψ is the tangent angle to the horizontal at



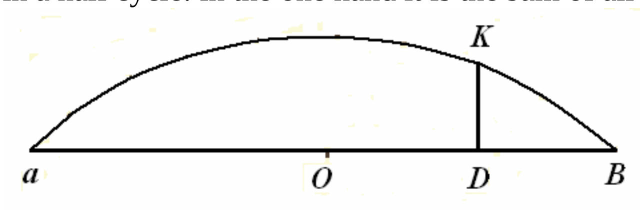
the point *P* in question where the bob is present, and at this point the force along the tangent is $mg\,sin\psi = \frac{mgs}{4a}$, where 4*a* is the length of the pendulum and also the length of the complete arc of the cycloid, where the generating circle has radius *a*. Thus the force along the tangent at this point to the force of gravity is as the arc length to the length of the pendulum, as stated by Newton. In turn, the resistance *f* is to *mg* as the arc *DE* is to the whole arc 4*a*. Thus as above, all the tangential forces are represented by corresponding arcs, while the tangential velocity at this point is $\dot{s} = 4a\dot{\psi}\,cos\psi$. In the diagram below, the radii *CA* and *CB* are of length 4*a*, and ratios are taken for which the quantity $\dot{\psi}$ is the same for the damped and undamped cases.]

The semicircle *BEeA* may be put in place with centre *C* and with the radius *CA* or *CB*. Moreover, the body may describe the distance *Dd* in a minimum time, and with the perpendiculars *DE* and *de* erected, meeting the circumference at *E* and *e*, these will be as the velocities which the body in a vacuum, by descending from the point *B*, may acquire at the places *D* and *d*. This is apparent (by Prop. LII. Book I.) And thus these velocities may be expressed by these perpendiculars *DE* and *de*; and the velocity *DF* as that acquired at *D* by [the body] falling from *B* in the resisting medium.



And if, with centre *C* and with radius *CF*, a circle *FfM* may be described crossing the right lines *de* and *AB* at *f* and *M*, *M* will be the place to which it would ascend henceforth without further resistance, and *df* the velocity that it may acquire at *d*. From which also if *Fg* may indicate the moment of the velocity that the body *D*, by describing the distance taken as minimum *Dd*, loses from the resistance of the medium ; and *CN* may be taken equal to *Cg*: *N* will be the place to which the body henceforth would ascend without resistance, and *MN* will be the decrease in the ascent arising from the loss of that velocity. The perpendicular *Fm* may be sent to *df*, and the decrement *F*g of the velocity *DF* arising from the resistance *DK*, will be to the increment of this same velocity *fm* arising from the force *CD*, as the generating force *DK* to the generated force *CD*. But also on account of the similar triangles *Fmf*, *Fhg*, *FDC*: *fm* is to *Fm* or *Dd* as *CD* is to *DF* ; and from the equality, *Fg* is to *Dd* as *DK* to *DF* . Likewise *Fh* is to *Fg* as *DF* to *CF*; and from the rearranged equation, *Fh* or *MN* to *Dd* is as *DK* to *CF* or *CM*; and thus the sum of all the terms $MN \times CM$ will be equal to the sum of all the terms $Dd \times DK$. At the moveable point *M* a rectangular coordinate may always be understood to be erected equal to the indeterminate *CM*, which may be drawn by a continued motion along the whole length *Aa*; and the trapezium described by that motion or equal to this rectangle $Aa \times \frac{1}{2} aB$ will be equal to the sum of all the products $MN \times CM$, and thus to the sum of all $Dd \times DK$, that is, to the area *BKVTa*. *Q.E.D.*

[We see now that Newton's diagram above considers the arc length lost due to the friction in a half cycle: in the one hand it is the sum of all the decrements of the arc given by the sum of $Dd \times DK$, and on the other hand it is the difference of the initial and final arcs in the half oscillation, given by the sum of $MN \times CM$. We return now to *B*. & *R*., who give a simple explanation of the final formulas produced by Newton geometrically in the following corollary : Let the straight line *aB* be drawn equal to the arc of the cycloid described by the oscillating body, and at each point *D* draw the perpendicular *DK* equal to the fraction $\frac{1}{\omega^2}$ of the resistance at *D*.



Let a_0 and a_1 be the arcs described in the descent and subsequent ascent , then the area under the curve *aKB* is $\left(a_1 - a_0\right)\frac{a_1 + a_0}{2}$. This can be proven readily, for the equation of motion is $v\frac{dv}{ds} + \omega^2 s = f$, for some resistance *f*; and we have

$v\frac{dv}{ds} + \omega^2 s = R$, then on integrating, $\left[v^2 + \omega^2 s^2\right]_{-a_0}^{+a_1} = 2\int_{-a_0}^{+a_1} f\mathrm{d}s$. Now, at the limits of

integration, the speed *v* is zero, and hence $\omega^2\left(a_1^2 - a_0^2\right) = 2\int_{-a_0}^{+a_1} fds$, or $\frac{1}{2}\left(a_1^2 - a_0^2\right) = \int_{-a_0}^{+a_1} \frac{fds}{\omega^2}$

as required.

We have now to find the nature of the resistance that describes the curve *aKB*. As previously, we have $s = A\sin\left(\omega t + \varphi\right)$ and $v = A\omega\cos\left(\omega t + \varphi\right)$. Now, if *y* is the ordinate and the resistance is of the form kv^m , then *DK* or *y* becomes equal to

$y = \frac{k}{\omega^2} A^m \omega^m \cos^m\left(\omega t + \varphi\right) = kA^m \omega^{m-2} \cos^m\left(\omega t + \varphi\right)$, while the distance *x* from the initial geometric centre + half the differences of the ascending and descending arcs, $\frac{a_1 - a_0}{2}$, gives an average location about which the oscillation takes place : thus, at least approximately, $x+\frac{a_1-a_0}{2}=A\sin(\omega t+\varphi)$.

Now, if the resistance varies as the velocity, then $x+\frac{a_1-a_0}{2}=A\sin(\omega t+\varphi)$ and $y=\frac{k}{\omega}A\cos(\omega t+\varphi)$, and on eliminating *t*, we have approximately, $\left(x+\frac{a_1-a_0}{2}\right)^2+\left(\frac{y\omega}{k}\right)^2=A^2$, which is the equation of an ellipse, if we consider *A* as being time independent. (There is a misprint in the original equation here.)

Again, if the resistance varies with the velocity squared, we have $x+\frac{a_1-a_0}{2}=A\sin(\omega t+\varphi)$ as before, while $y=kA^2\cos^2(\omega t+\varphi)$, and on eliminating *t*, we have : $\left(x+\frac{a_1-a_0}{2}\right)^2+\frac{y}{k}=A^2$, which is the equation of a parabola, again on treating *A* as constant.]

Corol. Hence from the law of the resistance and the difference of the arcs *Ca* and *CB* the difference *Aa* can always be deduced for the proportion of the resistance to the weight approximately.

For if the resistance *DK* shall be uniform, the rectangular figure *BKTa* will be under *Ba* and *DK*; and thence the rectangle under $\frac{1}{2}Ba$ and *Aa* will be equal to the rectangle under *Ba* and *DK*, and *DK* will be equal to $\frac{1}{2}Aa$. Whereby since *DK* shall be expressing the resistance, and the length of the pendulum an expression of the weight, the resistance to the weight will be as $\frac{1}{2}Aa$ to the length of the pendulum ; everything has been shown as in Prop. XXVIII.

If the resistance shall be as the velocity, the figure *BKTa* will approximate to an ellipse. For if the body, in the non-resisting medium, may describe by oscillating the whole length *BA*, the velocity at some place *D* may be as the applied ordinate *DE* of the circle with diameter *AB*. Therefore with *Ba* in the resisting medium, and *BA* in the non resisting medium, they may be described in around equal times ; and thus the velocities at the individual points of *Ba*, shall be approximately to the velocities at the corresponding points of the length *BA*, as *Ba* is to *BA*; the velocity at the point *D* in the resisting medium will be as the applied ordinate *Ba* of the circle or ellipse described on the diameter ; and thus the figure *BKVTa* will be approximately an ellipse. Since the resistance may be supposed proportional to the velocity, *OV* shall be an expression of the resistance at the middle point *O*; and the ellipse *BRVSa*, with centre *O*, described with semi-axes *OB* and *OV*, and the figure *BKVTa*, equal to the rectangle $Aa\times BO$, will be approximately equal. Therefore $Aa\times BO$ is to $OV\times BO$ as the area of this ellipse to $OV\times BO$: that is, *Aa* to *OV* as the area of the semicircle to the square of the radius, or approximately as 11 to 7: And therefore $\frac{7}{11}Aa$ shall be to the length of the pendulum as the resistance of the oscillating body at *O* to the weight of the same.

But if the resistance *DK* shall be in the square ratio of the velocity, the figure *BKVTa* will be nearly a parabola having the vertex *V* and the axis *OV*, and thus will be approximately equal to the rectangle under $\frac{2}{3}Ba$ and *OV*. Therefore the rectangle under $\frac{1}{2}Ba$ and *Aa* is equal to the rectangle under $\frac{2}{3}Ba$ and *OV*, and thus *OV* equals $\frac{1}{4}Aa$: and therefore the resistance of the oscillating body at *O* to the weight of this is as $\frac{3}{4}Aa$ to the length of the pendulum. And I think that these conclusions have been taken care of abundantly enough in practical matters. For since the ellipse or parabola *BRVSa* may agree with the figure *BKVTa* at the mid-point *V*, this if either the part *BRV* or *VSa* exceeds that figure, it will be deficient from that figure by the same for the other part, and thus it will be approximately equal to the same.

PROPOSITION XXXI. THEOREM XXV.

If the resistance in the individual proportional parts of the arcs of an oscillating body described may be augmented or diminished in a given ratio ; the difference between the arc described in the descent and the arc subsequently described in the ascent, will be increased or diminished in the same ratio.

For that difference arises from the retardation of the pendulum by the resistance of the medium, and thus is proportional to that retarding resistance. In the above proposition the rectangle under the right line $\frac{1}{2}aB$ and the difference *Aa* of these arcs *CB* and *Ca* was equal to the area *BKTa*. And that area, if the length *aB* may remain, may be increased or decreased in the ratio of the applied ordinates *DK* [*i.e.* the *y* ordinate]; that is, in the ratio of the resistance, and thus is as the length *aB* and the resistance jointly. And hence the rectangle under *Aa* and $\frac{1}{2}aB$ is as *aB* and the resistance jointly, and therefore *Aa* is as the resistance. *Q.E.D.*

Corol. I. From which if the resistance shall be as the velocity, the difference of the arcs in the same medium will be as the total arc described , and conversely.

Corol. 2. If the resistance shall be in the square ratio of the velocity, that difference will be in the square ratio of the whole arc, and conversely.

Corol. 3. And generally, if the resistance shall be in the cubic or some other power of the velocity, the difference will be in that same ratio of the total arc, and conversely.

Corol. 4. And if the resistance shall be partially in the simple ratio of the velocity, and partially in the same square ratio, the difference will be partially in the ratio of the whole arc and partially in the ratio of the whole arc squared, and conversely. The law will be the same both for the ratio of the resistance for the velocity, and which is also the law of that difference for the length of the arc.

Corol. 5. And thus if with a pendulum successively describing unequal arcs, the ratio of the increment or decrement of this difference for the length of the described arc can be found and also the ratio may be come upon of the increment and decrement of the resistance for a larger or smaller velocity.

General Scholium.

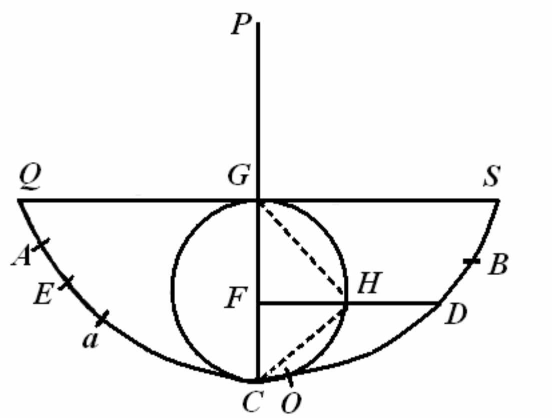
From these Propositions, through the oscillations of pendulums in mediums of any kinds, it is permitted to find the resistance of the mediums. Indeed I have investigated the resistance of the air by the following experiments. A wooden sphere weighing $57\frac{7}{22}$ ounces Avoirdupois, made with a diameter of $6\frac{7}{8}$ London inches, I have hung from a hook securely enough by a thin thread, thus so that between the hook and the centre of oscillation of the sphere should be $10\frac{1}{2}$ feet. I noted a point on the thread ten feet and one inch distance from the centre of suspension, and in the region of that point I have put in place a ruler separated into inches, with the aid of which I could observe the lengths of the arcs described by the pendulum. Then I have counted the oscillations in which the sphere lost an eighth part of its motion. If the pendulum may be drawn away from the perpendicular to a distance of two inches, and then it may be sent off, thus so that in its whole descent it may describe an arc of two inches, and in the first whole oscillation, composed from the descent and the subsequent ascent, an arc of almost four inches : the same lost an eighth part of its motion in 164 oscillations, thus so that in its final ascent it described an arc of one and three quarter inches. If the first arc described an arc of four inches; it lost an eighth part of the motion in 121 oscillations, thus so that in the final ascent it described an arc of $3\frac{1}{2}$ inches. If the first descent described an arc of eight, sixteen, thirty two or of sixty four inches , then an eighth part of the motion was lost in $69, 35\frac{1}{2}, 18\frac{1}{2}, 9\frac{2}{3}$, oscillations respectively. Therefore the difference between the first arc descended and the final arc ascended, in the first, second, third, forth, fifth and sixth case was of $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$ inches respectively. These differences may be divided in each case by the number of oscillations, and into one mean oscillation, by which the arc of $3\frac{1}{4}, 7\frac{1}{2}, 15, 30, 60, 120$ inches were described, the difference of the arcs described in the descent and subsequent ascent [*i.e.* the difference in the amplitude per half oscillation in modern terms] will be $\frac{1}{656}, \frac{1}{242}, \frac{1}{69}, \frac{4}{71}, \frac{8}{37}, \frac{24}{29}$ parts of an inch respectively. But these are approximately in the square ratio of the arcs described in the larger oscillations, truly in the smaller a little greater than in this ratio ; and therefore (by Corol. 2. Prop. XXXI. of this book) the resistance of the sphere, when it may be moving faster, is in the square ratio of the velocity as an approximation; when it moves slower, a little greater than in this ratio.

Now *V* may designate the maximum velocity in some oscillation, and let *A, B, C* be given quantities, and we may suppose that the difference of the arcs shall be $AV + BV^{\frac{3}{2}} + CV^2$. Since the maximum velocities in the cycloid shall be as half of the arcs described in the oscillation, truly in the circle as half chords of these arcs, and thus with equal arcs the velocities shall be greater in the cycloid than in the circle, in the ratio of half the arcs to the same chords ; but the times in the circle shall be greater than in the cycloid in the inverse ratio of the velocity, it is apparent the differences of the arcs (which are as the resistance and the square of the time jointly) to be approximately the same, in each curve : for these differences must be increased in the cycloid, together with the resistance, in around the square ratio of the arc to the chord, on account of the velocity increased in that simple ratio ; and to be diminished, together with the square of the time, in the same square ratio. And thus so that it comes about by reduction to the cycloid, that the same differences of the arcs are required to be taken which were observed in the circle, indeed the greatest velocities are required to be put in place analogous either with the halve or whole arcs, that is to the numbers $\frac{1}{2}, 1, 2, 4, 8, 16$. Therefore we may write in the second, fourth and sixth case the numbers 1, 4 and 16 for *V*; and in the second case there will be produced the difference of the arcs $\frac{\frac{1}{2}}{121} = A + B + C$; $\frac{2}{35\frac{1}{2}} = 4A + 8B + 16C$ in the fourth case ; and $\frac{8}{9\frac{2}{3}} = 16A + 64B + 256\,C$ in the sixth case. And from these equations, by placing together and reducing analytically, there shall be $A = 0{,}0000916,\ B = 0{,}0010847,\ \&\ C = 0{,}0029558$. Therefore the difference of the arcs is as $0{,}0000916V + 0{,}0010847V^{\frac{3}{2}} + 0{,}0029558V^2$; and therefore since (by the corollary of Proposition XXX applied to this case) the resistance of the sphere in the average arc described in the oscillation, where the velocity is *V*, shall be to the weight of this as $\frac{7}{11}AV + \frac{7}{10}BV^{\frac{3}{2}} + \frac{3}{4}CV^2$ to the length of the pendulum ;

[for this theorem asserts that at any point in the oscillation, the resistive force *f* shall be to the weight of the bob mg as the ordinate *DK* to the length of the pendulum *l* : $\frac{DK}{l} = \frac{f}{mg}$; we have shown above that the area under the curve is $(a_1 - a_0)\frac{a_1 + a_0}{2}$; and for the case of an approximate ellipse, half the area of such an ellipse is $\frac{\pi ab}{2} \doteq \frac{11}{7}ab$, where the semi-minor axis *b* is the maximum resistance f_m, and the semi-major axis *a* is $\frac{aB}{2}$; hence $(a_1 - a_0)\frac{a_1 + a_0}{2} \doteq \frac{11}{7}\frac{aB}{2}f_m$ or $f_m \doteq \frac{7}{11}(a_1 - a_0)\frac{a_1 + a_0}{aB} = \frac{7}{11}(a_1 - a_0) = \frac{7}{11}AV$; Newton gives a similar argument in the corollary above for the elliptic case and also for the parabolic case depending on the square of the velocity, while the middle term is an approximate average of both cases.]

If the numbers found may be written with *A*, *B* and *C*, the resistance of the sphere to its weight becomes as $0{,}0000583V + 0{,}0007593V^{\frac{3}{2}} + 0{,}0022169V^2$ to the length of the pendulum between the centre of suspension and the ruler, that is, to 121 inches. From which since *V* may be appointed equal to 1 in the second case, 4 in the fourth, and 16 in the sixth: the resistance will be to the weight in the second case as 0,0030345 to 121, in the fourth as 0,041748 to 121, and in the sixth as 0,61705 to 121.

The arc that the noted point on the string described in the sixth case was $120-\frac{8}{9\frac{2}{3}}$ or $119\frac{5}{29}$ inches. And therefore since the radius must be 121 inches, and the length of the pendulum between the point of suspension and the centre of the sphere must be 126 inches, the arc that the centre of sphere described was $124\frac{3}{31}$ inches. Because the maximum velocity of the body, on account of the resistance of the air, does not fall on the lowest point of the arc described, but is situated almost at the centre of the arc : this motion will be almost the same if the sphere described the half arc of $62\frac{3}{62}$ inches in a non-resisting in the whole descent in a cycloid, to which we have reduced the motion of the pendulum above: and therefore the velocity will be equal to that velocity that the sphere may acquire, by falling perpendicularly, and in that case describing an arc equal to the versed sine of that height.



[Note from *L.* & *J.* : The body by oscillating may describe the arc *Ba* in the resisting medium, and the arc *BA* in the non-resisting medium; let *C* be the lowest point of the cycloid; *O*, the midpoint of the arc *Ba*; and the arc *CD* shall be equal to the arc *BO* : the maximum velocity acquired in the descent of the body in the resisting medium in the arc *BO* is to the maximum velocity acquired in the arc *BC* in the resisting medium as the arc *BO*, to the arc *BC*. But if the body by falling from the location *D* in the non-resisting medium may describe the arc *DC*, also its velocity at *C* acquired by the descent through the arc *DC*, in the same place in the descent through the arc *BC* will be as the arc *CD*, or equally, *BO*, to the arc *BC*. Therefore the velocity in the resisting medium acquired by the resisting medium at *O* is equal to the velocity that the body falling in the non-resisting medium by the arc $DC = BO$ may have at *C*; and therefore that velocity is equal to that velocity that the body may be able to acquire by falling perpendicularly in the non-resisting medium, and in that case by describing its own height *FC* equal to the versed sine of the arc *CH*. Now *P* shall be the point of suspension, *PC* the length of the pendulum; *SDC* the half cycloid; *SG* and *DF* normals to *PC*, and *CHGC* the circle with the diameter *GC* described cutting the secant *DF* in *H*. The chord *CH* may be joined, and the arc of the cycloid $SD = 2GC - 2CH$, and the arc $SG = 2GC$ and thus the arc $DC = 2CH$. But moreover, from the nature of the circle, $\frac{CF}{CH} = \frac{CH}{CG}$ and hence $\frac{CF}{2CH(\text{or } DC)} = \frac{2CH}{4CG}$, or $\frac{DC}{2PC}$; that is, the versed sine *CF* is to the arc *CD*, as the same arc to double the length of the pendulum.]

But that versed sine in the cycloid is to the arc itself $62\frac{3}{62}$ as the same arc to a length double the length of the pendulum 252, and therefore equal to 15,278 inches. Whereby that velocity is what a body itself may be able to acquire, and in its own case by falling through a distance of 15,278 inches. Therefore with such a velocity the sphere is subject to a resistance, which shall be to its weight as 0,61705 to 121, or (if that part of the resistance only may be considered which is as the square ratio of the velocity) as 0,56752 to 121.

But I have found by a hydrostatic experiment that the weight of this wooden sphere to be equal to a water sphere of the same size as 55 to 97: and therefore since 121 shall be in the same ratio to 213,4, the resistance of the prepared sphere with the velocity of progression to the weight itself is as 0,56752 to 213,4 that is, as 1 to $376\frac{1}{50}$. From which since the weight of the water sphere, in which time the sphere continued uniformly with the velocity may describe a length of 30,556 inches, may be able to generate all that velocity by the sphere falling, it is evident that the force of the resistance continued uniformly in the same time may be able to remove a velocity in the smaller ratio 1 to $376\frac{1}{50}$, that is, the part $\frac{1}{376\frac{1}{50}}$ of the whole velocity. And therefore in which time the sphere, with that uniform velocity continued, may be able to describe a length of half its own diameter, or of $3\frac{1}{16}$ inches, and it may lose the $\frac{1}{3542}$ part of its motion.

I was also counting the number of oscillations in which the pendulum lost the fourth part of its motion. In the following table the top numbers indicate the lengths of the arcs described in the first arc, expressed in inches and parts of inches : the middle numbers indicate the length of the arc described in the final ascent ; and at the lowest level stand the number of oscillations. I have described the experiment as more accurate than in which only the eighth part was lost. Anyone who wishes may test the calculation.

First descent	2	4	8	16	32	64
Final ascent	$1\frac{1}{2}$	3	6	12	24	48
Number of osc.	374	272	$162\frac{1}{2}$	$83\frac{1}{2}$	$41\frac{2}{3}$	$22\frac{2}{3}$

Later I suspended a leaden sphere with a diameter of 2 inches, and with a weight of $26\frac{1}{4}$ ounces Avoirdupois by the same thread, thus so that the interval between the centre of the sphere and the point of suspension should be $10\frac{1}{2}$, and I counted the number of oscillations in which a given part of the motion was lost. The beginning of the following tables shows the number of oscillations in which an eighth part of the whole motion had ceased; the second the number of oscillations in which a quarter part of the same oscillations had been lost.

First descent	1	2	4	8	16	32	64
Final ascent	$\frac{7}{8}$	$\frac{7}{4}$	$3\frac{1}{2}$	7	14	28	56
Number of osc.	226	228	193	140	$90\frac{1}{2}$	53	30

First descent	1	2	4	8	16	32	64
Final ascent	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24	48
Number of osc.	510	518	420	318	204	121	70

In the first table by selecting from the third, fifth, and seventh observations, and by expressing the maximum velocity for these observations particularly by the numbers 1, 4, 16 respectively, and generally by the quantity V as above: from the third observation $\frac{\frac{1}{2}}{193} = A + B + C$ will arise, in the fifth $\frac{2}{90\frac{1}{2}} = 4A + 8B + 16C$, and in the seventh $\frac{8}{30} = 16A + 64B + 256C$. Truly these reduced equations give $A = 0{,}0014,\ B = 0{,}000297,\ C = 0{,}000879$. From thence the resistance of motion of the sphere arises with the velocity V in that ratio to its weight of $26\frac{1}{4}$ ounces, that it has $0{,}0009V + 0{,}000208\ V^{\frac{3}{2}} + 0{,}000659V^2$ to the length of the pendulum of 121 inches. And if we may consider only that part of the resistance which is in the square ratio of the velocity, this will be to the weight of the sphere as $0{,}000659V^2$ to 121 inches. But this part of the resistance in the first experiment was to the weight of the wooden sphere of $57\frac{7}{22}$ ounces as $0{,}002217V^2$ to 121: and thence the resistance of the wooden sphere to the resistance of the leaden sphere (with the same velocities of these), as $57\frac{7}{22}$ by $0{,}002217$ to $26\frac{1}{4}$ by $0{,}000659$, that is, as $7\frac{1}{3}$ to 1. The diameters of the two sphere were of $6\frac{7}{8}$ and 2 inches, and the squares of these are in turn as $47\frac{1}{4}$ and 4, or $11\frac{15}{16}$ and 1 approximately. Therefore the resistances of these equally moving spheres were in a smaller ratio than the double of the diameters. But we have not yet considered the resistance of the thread, which certainly by the size it was, and that ought to be taken away from the resistance of the pendulums found. This I was not able to define accurately, but yet I found it to be greater than the third part of the resistance of the smaller pendulum; and then I have learned that the resistance of the spheres, without the resistance of the thread, are almost in the square ratio of the diameters. For the ratio $7\frac{1}{3} - \frac{1}{3}$ to $1 - \frac{1}{3}$, or $10\frac{1}{2}$ to 1 is not far removed from the ratio of the diameters $11\frac{15}{16}$ squared to 1.

Since the resistance of the thread shall be of less concern in the larger spheres, I have also tested an experiment with a sphere the diameter of which was $18\frac{3}{4}$ inches. The length of the pendulum between the point of suspension and the centre of oscillation was $122\frac{1}{2}$ inches, between the point of suspension and the knot on the string $109\frac{1}{2}$ inches. The first arc of the pendulum in the descent from the node described 32 inches. The arc in the ascend after five oscillations from the same knot described 28 dig. The sum of the arcs or the whole arc described in the oscillation from the centre was 60 inches. The difference of the arcs was 4 inches. The tenth part of this or the mean difference between the descent and the ascent the oscillation was $\frac{2}{5}$ of an inch. So that as the radius $109\frac{1}{2}$ to the radius

$122\frac{1}{2}$, thus the whole arc 60 inches described by the knot in the oscillation from the centre, to the whole arc $67\frac{1}{8}$ inches described in the oscillation from the centre of the sphere, and thus the difference $\frac{2}{5}$ in. to the new difference 0,4475 in. If the length of the pendulum, with the length of the arc described remaining, were increased in the ratio 126 to $122\frac{1}{2}$; the time of the oscillation may be increased and the velocity of the pendulum may be diminished in that square root ratio, the true difference of the descent and subsequent ascent of the arcs 0,4475 may remain. Then if the arc described may be increased in the ratio $124\frac{3}{31}$ to $67\frac{1}{8}$, the difference 0,4475 itself may be increased in that ratio squared, and thus 1,5295 will arise. Thus these may themselves be considered, under the hypothesis that the resistance of the pendulum may be in the square ratio of the velocity. Hence if the pendulum may describe the whole arc of $124\frac{3}{31}$ inches, and the length of this between the point of suspension and the centre of oscillation should be126 inches, the difference of the arcs described in the descent and subsequent ascent should be 1,5295 inches. And this difference taken by the weight of the pendulum, which was 208 inches, gives 318,136. Again when the above pendulum constructed from the wooden sphere with the centre of oscillation, that was 126 inches from the point of suspension, described a whole arc of $124\frac{3}{31}$ inches, the difference of the descending and ascending arcs was $\frac{126}{121}$ by $\frac{8}{9\frac{2}{3}}$, which taken by the weight of the sphere, which was $57\frac{7}{22}$ ounces, produced 49,396. Moreover I multiplied these differences into the weights of the spheres so that I could find the resistances of these. For the differences are arising from the resistances, and they are as the resistance directly and the weights inversely. Therefore the resistances are as the numbers 318,136 and 49,396. But the part of the resistance of the small sphere, which is in the square ratio of the velocity, was to the whole resistance as 0,56752 to 0,61675, that is, as 45,453 to 49,396 ; and the part of the resistance of the greater sphere may be almost equal to the whole resistance itself, and thus these parts are as 318,136 to 45,453 approximately, that is, as 7 to 1. But the diameters of the spheres $18\frac{3}{4}$ and $6\frac{7}{8}$; and the squares of these $351\frac{9}{16}$ and $47\frac{17}{64}$ are as 7,438 to 1, that is, approximately as the resistances of the spheres 7 to 1. The difference of the ratios in not much greater, than what can arise from the resistance of the string. Therefore these parts which are of the resistances, which are with equal spheres, as the squares of the velocities; are also, with equal velocities, as the squares of the diameters of the spheres.

Of the other spheres, with which I have used in these experiments, the greatest was not perfectly spherical, and therefore in the calculation reported here I have ignored certain details for the sake of brevity ; not being too concerned in the accuracy of the calculation in a not very satisfactory experiment. And thus I might have chosen, since the demonstration of empty space may depend on these, so that experiments might be tried with several greater and more accurate spheres. If spheres may be taken in geometric proportion, e.g. the diameters of which shall be 4, 8, 16, 32 inches ; from the progression of the experiments it may be deduced what must come about from still larger spheres.

Now indeed for comparing the resistances of different fluids between each other I have tried the following. I have prepared a wooden box four feet long, with width and height of one foot. This I filled with spring water without the lid, and I have arranged so that immersed pendulums may be moving by oscillating in the water medium. Moreover a leaden sphere with a weight of $166\frac{1}{6}$ ounces, and with a diameter of $3\frac{5}{8}$ inches is moved as we have described in the following table, it may be seen with the length of the pendulum from the point of suspension to a certain marked point on the string of 126 inches, but to the centre of the oscillation of $134\frac{3}{8}$ inches.

First descent arc described by point marked on the string, in inches	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
Final ascent arc described, in inches	48	24	12	6	3	$1\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{5}{16}$
Difference of the arcs, proportional to the motion lost, in inches.	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
Number of oscillations in water.			$\frac{29}{60}$	$1\frac{1}{3}$	3	7	$11\frac{1}{4}$	$12\frac{2}{3}$	$13\frac{1}{3}$
Number of oscillations in air.	$85\frac{1}{2}$		287	535					

In the experiment of the fourth column, an equal motion with 535 oscillations in air, and $1\frac{1}{3}$ in water were lost. The oscillations in air were indeed a little faster than in water. But if the oscillations in water were accelerated in that ratio so that the motion of the pendulums in each medium were made with equal velocities, the same number$1\frac{1}{3}$ of oscillations would remain in water, from which the same motion was lost as at first, on account of the increased resistance and likewise the square of the time diminished in the same ratio squared. Therefore with equal velocities of the pendulums equal motions have been lost, with 535 oscillations in the air and with $1\frac{1}{3}$ oscillations in water; and thus the resistance of the pendulum in water is to its resistance in air as 535 to $1\frac{1}{3}$. This is the proportion of the whole resistance in the case of the fourth column.

Now $AV+CV^2$ may designate the difference of the arcs described in the descent and in the subsequent ascent by the sphere in air I set in motion with the maximum velocity V ; and since the maximum velocity in the case of the fourth column shall be to the maximum velocity in the case of the first column, as 1 to 8; and that difference of the arcs in the case of the fourth column to the difference in the case of the first column as $\frac{2}{535}$ to $\frac{16}{85\frac{1}{2}}$ or as $85\frac{1}{2}$ to 4280: we may write in these cases 1 and 8 for the velocities, and $85\frac{1}{2}$ and 4280 for the differences of the arcs, and there becomes $A+C=85\frac{1}{2}$; and $8A+64C=4280$ or $A+8C=535$; and thence by the reduction of the equation there comes about $7C=449\frac{1}{2}$, $C=64\frac{3}{4}$, and $A=21\frac{2}{7}$: and thus the resistance since it shall b as $\frac{7}{11}AV+\frac{3}{4}CV^2$, it will become as $13\frac{6}{11}V+48\frac{9}{56}V^2$. Whereby in the case of the fourth column when the velocity was 1, the whole resistance is to its proportional part with the square of the velocity, as $13\frac{6}{11}+48\frac{9}{56}$ or $61\frac{12}{17}$ to $48\frac{9}{56}$; and thus the resistance of the pendulum in water is to that part of the resistance in air, which is proportional to the square of the velocity, and which alone in the more rapid motions come to be considered, as $61\frac{12}{17}$ to $48\frac{9}{56}$ and 535 to $1\frac{1}{5}$ taken jointly, that is, as 571 to 1. If the whole string of the pendulum should be immersed in water, its resistance would be greater still ; and thus that resistance of the pendulum oscillating in water, which is proportional to the square of the velocity, and which alone comes to be considered in more rapidly moving bodies, shall be to the resistance of the same whole pendulum, oscillating in air with the same velocity, as around 850 to 1, that is, as the density of water to the density of air approximately.

In this calculation that part of the resistance of the pendulum in the water must also be taken, which might be as the square of the velocity, but (which may be considered as a surprise) the resistance in water may be increased in a ratio greater than the square. With the cause of this matter requiring to be investigated, I came upon this [explanation], that the area should be exceedingly narrow for the size of the sphere of the pendulum, and the motion of the water going before was being impeded exceedingly by its own narrowness. For if the sphere of a pendulum, the diameter of which was one inch, were immersed, the resistance was increased approximately in the square ratio of the velocity. I tested that by the construction of a pendulum with two spheres, the lower and smaller of which could oscillate in water, the upper and greater being attached just above the water, and by oscillating in the air, might aid the motion of the pendulum and render it more long lasting. Moreover the experiments set up in this manner may themselves be had as described in the following table.

Arc described in the first descent.	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
Arc described in the final ascent.	12	6	3	$1\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{5}{16}$
Diff. of the arcs, prop.to motion lost.	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
Number of oscillations.	$3\frac{3}{8}$	$6\frac{1}{2}$	$12\frac{1}{12}$	$21\frac{1}{5}$	34	53	$62\frac{1}{5}$

By requiring the resistances of mediums to be compared between each other, I also arranged that iron pendulums could oscillate in quicksilver. With the length of the string of the iron pendulum to be nearly three feet, and the diameter of the sphere of the pendulum to be a third of an inch. But another leaden sphere was fixed to the string near the surface of the mercury of such a size that the motion of the pendulum could continue for some time. Then a vessel, that could hold almost three pounds of quicksilver, I filled alternately with quicksilver and ordinary water, so that with a pendulum oscillating successively in each fluid, I was able to find the proportion of the resistances : and the resistance of quicksilver to water was produced as around 13 or 14 to 1 : that is, as the density of quicksilver to the density of water. When I used an iron sphere a little larger, for example with a diameter that should be around $\frac{1}{3}$ or $\frac{2}{3}$ of an inch, the resistance of the quick silver was produced in that ratio to the resistance of water, that the number 12 or 10 has to 1 roughly. But the first experiment is more trustworthy, because that vessel in these final experiments was exceedingly narrow for the size of the sphere immersed. With the sphere enlarged, the vessel also is required to be enlarged. I set up certain experiments of this kind in larger vessels and with liquids both of melted metals as well as with certain others to be repeated with hot as well as with cold liquids: but there was not time to prove everything, and now from the description it may be clear enough how the resistance of bodies moving rapidly may be approximately proportional to the density of the fluids in which they are moving. I do not say accurately. For more tenacious fluids, on a par with the density, without doubt are more resistant than the more liquid, such as cold oil rather than hot oil, cold rather than hot rainwater, water rather than spirit of wine [*i.e.* brandy]. Truly with liquids, which are fluid enough to the senses, as with air, water either sweet or salty, with spirits , spirits of wine, of turpentine and salts, with oil freed from its dregs by distillation and by heating, and with oil of vitriol and with mercury, and with liquefied metals, and others which there may be, which are both fluid so that they may conserve a long time impressed motion of agitation, and on being poured and running away freely may be resolved into drops, I have no reason to doubt that the rule already laid down may prevail accurately enough : especially if experiments with larger pendulous bodies with faster motions may be put in place.

Finally since it shall be the opinion of some, of a certain must subtle ethereal medium extending afar, that may permeate all the pores of all bodies freely; moreover the resistance must originate from such a medium flowing through the pores of bodies : So that I can test the resistance that we experience from the motions of bodies, whether the whole shall be on the external surface of these, or the resistance may be perceived also from the various internal parts near the surfaces, I have thought out a test for such. With a string eleven feet long firmly attached by a steel hook, I hung up a round wooden box by means of a steel ring, towards constructing a pendulum of the aforestated length. The upper concave edge of the hook was sharpened, so that the ring resting on its upper arc could move freely on the sharp edge. Moreover the string was attached to the lower arc. The pendulum thus constituted I drew away from the vertical to a distance of around six feet, and that along a plane perpendicular to the plane of the sharpened hook, lest the ring, with the pendulum oscillating, might slip beyond the side of the sharpened edge of the hook. For the point of the suspension, in which the ring touches the hook, must remain motionless. Therefore I marked the place accurately, to which I drew the pendulum aside, then with the pendulum sent off I noted the three other places to which it returned at the end of the first, second, and third oscillation. I repeated this more often, so that I could find these places most accurately. Then I filled the box with lead and with metal weights which were at hand. But first I weighed the empty box, together with the part of the string that was tied around the box and with half of the remaining part which was being stretched between the hook and the pendulum box. For the extended string always acts with half its weight on the pendulum when drawn aside from the perpendicular. To this weight I added the weight of the air that the box contained. And the whole weight was as if a 78th part of the weight full of the metal. Then because with the box filled with metal, with the weight itself by stretching the string, the length of the pendulum must be increased, I shortened the string so that now the length of the pendulum oscillating should be the same as at first. Then with the pendulum drawn aside and released from the first marked place, I counted around 77 oscillations, then the box returned to the second marked place, and just as many one after the other until the box returned to the third marked place, and again with just as many until the box returned to reach its fourth marked place. From which I conclude that the whole resistance of the full box did not have a greater proportion to the resistance of the empty box than 78 to 77. For if the resistance of both should be equal, the full box on account of its inertial force , which was

78 times greater than the inertial force of the empty box, the motion of its oscillations must be maintained for such a longer time, and thus always with 78 oscillations completed to have returned to that marked place. But it returned to the same after 77 complete oscillations.

Therefore *A* may designate the resistance of the box on the external surface, and *B* the resistance of the empty box from the inside parts; and if the resistances of the bodies of the same velocities in the inside parts shall be as the matter, or the number of small parts by which it is resisted : 78 *B* will be the resistance of the full box from the inside parts : and thus the whole resistance $A+B$ of the empty box will be to the total resistance of the full box $A+78B$ as 77 to 78, and separately $A+B$ to 77*B*, as 77 to 1, and thence $A+B$ to *B* as 77×77 to 1, and separately *A* to *B* as 5918 to 1. Therefore the resistance of the empty box from the interior parts is more than 5000 times less than the same resistance of this from the external surface. Thus truly we may question that hypothesis that the greater part of the resistance of the full box may arise from no other hidden cause than from the action alone of some subtle fluid enclosed in the metal.

This experiment I have recalled from memory. For the page, in which I have described that a little, has become lost. From which certain fractional parts of the numbers, which have disappeared from memory, I have been compelled to omit.

For there is not time to try everything anew. In the first place, since I was using an infirm hook, the full box was being slowed down more. On seeking the reason, to be found that a weak hook yielded to the weight of the box, and in the oscillations of this was being bent by all the parts giving way. Therefore I prepared a strong hook, so that the point of suspension might remain fixed, and then everything thus came about as we have described above.

[It seems likely that the extra friction at the loaded knife edge gave the slightly different results.]

SECTIO VI.
De motu & resistentia corporum funependulorum.

PROPOSITIO XXIV. THEOREMA XIX.

Quantitates materiae in corporibus funependulis, quorum centra oscillationum a centro suspensionis aequaliter distant, sunt in ratione composita ex ratione ponderum & ratione duplicata temporum oscillationum in vacuo.

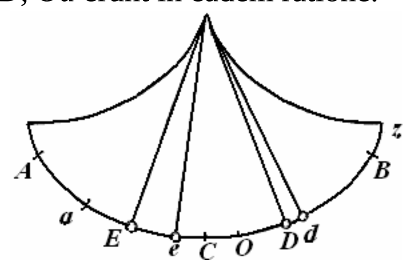
Nam velocitas, quam data vis in data materia, dato tempore generare potest, est ut vis & tempus directe, & materia inverse. Quo maior est vis vel maius tempus vel minor materia, eo maior generabitur velocitas. Id quod per motus legem secundam manifestum est. Iam vero si pendula eiusdem sint longitudinis, vires motrices in locis a perpendiculo aequaliter distantibus sunt ut pondera: ideoque si corpora duo oscillando describant arcus aequales, & arcus illi dividantur in partes aequales ; cum tempora quibus corpora describant singulas arcuum partes correspondentes sint ut tempora oscillationum totarum, erunt velocitates ad invicem in correspondentibus oscil1ationum partibus, ut vires motrices & tota oscillationum tempora directe & quantitates materiae reciproce : ideoque quantitates materia ut vires & oscillationum tempora directe & velocitates reciproce. Sed velocitates reciproce sunt ut tempora, atque ideo tempora directe & velocitates reciproce sunt ut quadrata temporum, & propterea quantitates materiae sunt ut vires motrices & quadrata temporum, id est, ut pondera & quadrata temporum. *Q.E.D.*

Corol. I. Ideoque si tempora sunt aequalia, quantitates materiae in singulis corporibus erunt ut pondera.
Corol. 2. Si pondera sunt aequalia, quantitates materiae erunt ut quadrata temporum.
Corol. 3. Si quantitates materiae aequantur, pondera erunt reciproce ut quadrata temporum.
Corol.4. Unde cum quadrata temporum, caeteris paribus, sint ut longitudines pendulorum, si & tempora & quantitates materiae aequalia sunt, pondera erunt ut longitudines pendulorum.
Corol. 5. Et universaliter, quantitas materiae pendulae est ut pondus & quadratum temporis directe, & longitude penduli inverse.
Corol. 6. Sed & in media non resistente quantitas materiae pendulae est ut pondus comparativum & quadratam temporis directe & longitudo penduli inverse. Nam pondus comparativum est vis motrix corporis in medio quovis gravi, ut supra explicui; ideoque idem praestat in tali medio non resistente atque pondus absolutum in vacuo.
Corol. 7. Et hinc liquet ratio tum comparandi corpora inter se, quoad quantitatem materiae in singulis; tum comparandi pondera eiusdem corporis in diversis locis, ad cognoscendam variationem gravitatis. Factis autem experimentis quam accuratissimis inveni semper quantitatem materiae in corporibus singulis eorum ponderi proportionalem esse.

PROPOSITIO XXV. THEOREMA XX.

Corpora Funependula quibus, in medio quovis, resistitur in ratione momentorum temporis, &. corpora funependula quae in eiusdem gravitatis specificae medio non resistente moventur, oscillationes in cycloide eodem tempore peragunt, & arcuum partes proportionales simul describunt.

Sit *AB* cycloidis arcus, quem corpus *D* tempore quovis in medio non resistente oscillando describit. Bisecetur idem in *C*, ita ut *C* sit infimum eius punctum; & erit vis acceleratrix qua corpus urgetur si in loco quovis *D* vel *d* vel *E* ut longitudo arcus *CD* vel *Cd* vel *CE*. Exponatur vis illa per eundem arcum , & cum resistentia sit ut momentum temporis, ideoque detur, exponatur eadem per datam arcus cycloidis partem *CO*, & sumatur arcus *Od* in ratione ad arcum *CD* quam habet arcus *CB* ad arcum *CB*: & vis qua corpus in *d* urgetur in media resistente, cum sit excessus vis *Cd* supra resistentiam *CO*, exponetur per arcum *CD*, ideoque erit ad vim, qua corpus est urgetur in media non resistente in loco *D*, ut arcus *Od* ad arcum *CD*; & propterea etiam in loco *B* ut arcus *OB* ad arcum *CB*. Proinde si corpora duo, *D, d* exeant de loco *B*, & his viribus urgeantur: cum vires sub initio sint ut arcus *CB* & *OB*, erunt velocitates primae & arcus primo descripti in eadem ratione. Sunto arcus illi *BD* & *Bd*, & arcus reliqui *CD, Od* erunt in eadem ratione. Proinde vires, ipsis *CD, Od* proportionales manebunt in eadem ratione ac sub initio, & propterea corpora pergent arcus in eadem ratione simul describere. Igitur vires & velocitates & arcus reliqui *CD, Od* semper erunt ut arcus toti *CB, OB*, & propterea arcus illi reliqui simul describentur. Quare corpora duo *D, d* simul pervenient ad loca *C* & *O*, alterum quidem in media non resistente ad locum *C*, & alterum in media resistente ad locum *O*.



Cum autem velocitates in *C* & *O* sint ut arcus *CB, OB*; erunt arcus, quos corpora ulterius pergendo simul describunt, in eadem ratione. Sunto illi *CE* & *Oe*. Vis qua corpus *D* in media non resistente retardatur in *E* est ut *CE*, & vis qua corpus *d* in media resistente retardatur in *e* est ut summa vis *Ce*, & resistentiae *CO*, id est ut *Oe*; ideoque vires, quibus corpora retardantur, sunt ut arcus *CE, Oe* proportionales arcus *CB, OB*; proindeque velocitates, in data illa ratione retardatae, manent in eadem illa data ratione. Velocitates igitur & arcus iisdem descripti semper sunt ad invicem in data illa ratione arcuum *CB* & *OB*; & propterea si sumantur arcus toti *AB, aB* in eadem ratione, corpora *D, d* simul describent hos arcus, & in locis *A* & *a* motum omnem simul amittent. Isochronae sunt igitur oscillationes totae, & arcubus totis *BA, Ba* proportionales sunt arcuum partes quelibet *BD, Bd* vel *BE, Be* quae simul describuntur. *Q.E.D.*

Corol. Igitur motus velocissimus in medio resistente non incidit in punctum infimum C. sed reperitur in puncto illo *O*, quo arcus totus descriptus *aB* bisecatur. Et corpus subinde pergendo ad *a*, iisdem gradibus retardatur quibus antea accelerabatur in descensu suo *aB* ad *O*.

PROPOSITIO XXVI. THEOREMA XXI.

Corporum funependulorum, quibus resistitur in ratione velocitatatum, oscillationes in cycloide sunt Isochronae.

Nam si corpora duo, a centris suspensionum aequaliter distantia, oscillando describant arcus inaequales, & velocitates in arcuum partibus correspondentibus sint ad invicem ut arcus toti; resistentiae velocitatibus proportionales, erunt etiam ad invicem ut iidem arcus. Proinde si viribus motricibus a gravitate oriundis, quae sint ut iidem arcus, auferantur vel addantur hae resisteniae, erunt differentiae vel summe ad invicem in eadem arcuum ratione: cumque velocitotum incrementa vel decrementa sint ut hae differeniae vel summae, velocitates semper erunt ut arcus toti : Igitur velocitates, si sint in aliquo casu ut arcus toti, manebunt semper in eadem ratione. Sed in principio motus; ubi corpora incipiunt descendere & arcus illos describere, vires, cum sint arcubus proportionales, generabunt velocitates arcubus proportionales. Ergo velocitates semper erunt ut arcus toti describendi, & propterea arcus illi simul describentur. *Q.E.D.*

PROPOSITIO XXVII. THEOREMA XXII.

Si corporibus funependulis resistitur in duplicata ratione velocitatum, differentiae inter tempora oscillationum in medio resistente ac tempora oscillationum in eiusdem gravitatis specificae medio non resistente, erunt arcubus oscillando descriptis proportionales quam proxime.

Nam pendulis aequalibus in medio resistente describantur arcus inaequales *A*, *B*; & resistentia corporis in arcu *A*, erit ad resistentiam corporis in parte correspondente arcus *B*, in duplicata ratione, velocitatum, id est, ut *AA* ad *BB*, quam proxime. Si resistentia in arci *B* esset ad resistentiam in arcu *A* ut *AB* ad *AA* ; tempora in arcubus *A* & *B* forent aequalia, per propositionem superiorem. Ideoque resistentia *AA* in arcu *A*, vel AB in arcu B, efficit excessum temporis in arcu *A* supra tempus in medio non resistente, & resistentia *BB* efficit excessum temporis in arcu *B* supra tempus in medio non resistente. Sunt autem excessus illi ut vires efficientes *AB* & *BB* quam proxime, id est, ut arcus *A* & *B*. *Q.E.D.*

Corol 1. Hinc ex oscillationum temporibus, in media resistente, in arcubus inaequalibus factarum, cognosci possunt tempora oscillationum in eiusdem gravitatis specifice medio non resistente. Nam differentia temporum erit ad excessum temporis in arcu minore supra tempus in media non resistente, ut differentia arcuum ad arcum minorem.

Corol. 2. Oscillationes breviores sunt magis isochronae, & brevissimae iisdem temporibus peraguntur ac in medio non resistente, quam proxime. Earum vero quae in maioribus arcubus fiunt, tempora tum paulo maiora, propterea quod resistentia in descensu corporis qua tempus producitur, maior sit pro ratione longitudinis in descensu descriptae, quam resistentia in ascensu subsequente qua tempus contrahitur. Sed & tempus oscillationum tam brevium quam longarum nonnihil produci videtur per motum medii. Nam corporibus tardescentibus paulo minus resistitur, pro ratione velocitatis, & corporibus acceleratis paulo magis quam iis quae uniformiter progrediuntur : idque quia medium, eo quem a corporibus accepit motu, in eandem plagam pergendo, in priore casu magis agitatur, in posteriore minus; ac proinde magis vel minus cum corporibus motis conspirat, Pendulis igitur in descensu magis resistit, in ascensu minus quam pro ratione velocitatis, & ex utraque causa tempus producitur.

PROPOSITIO XXVIII. THEOREMA XXIII.

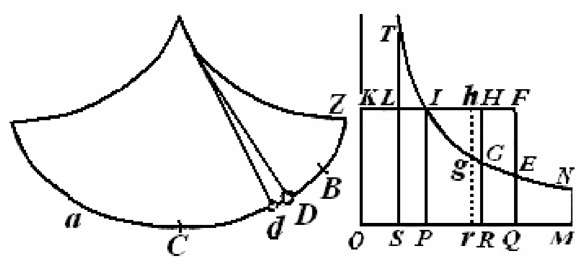
Si corpori funependulo in cycloide oscillanti resistitur in ratione momentotam temporis, erit eius resistentia ad vim gravitatis ut excessus arcus descensu toto descripti supra arcum ascensu subsequente descriptum, ad penduli longitudinem duplicatam.

Designet *BC* arcum descensu descriptum, *Ca* arcum ascensu descriptum, & *Aa* differentiam arcuum: & stantibus quae in propositione *xxv*. constructa & demonstrata sunt, erit vis, qua corpus oscillans urgetur in loco quovis *D*, ad vim resistentiae ut arcus *CD* ad arcum *CO*, qui semissis est differentiae illius *Aa*. Ideoque vis, qua corpus oscillans urgetur in cycloidis principio seu puncto altissimo, id est, vis gravitatis, erit ad resistentiam ut arcus cycloidis inter punctum illud supremum & punctum infimum *C* ad arcum *CO*; id est (si arcus duplicentur): ut cycloidis totius arcus, seu dupla penduli longitude, ad arcum *Aa*. *Q. E. D.*

PROPOSITIO XXIX. PROBLEMA VI.

Posito quod corpori in cycloide oscillanti resistitur in duplicata ratione velocitatis : invenire resistentiam in locis singulis.

Sit *Ba* arcus oscillatione integra descriptus, sitque *C* infimum cycloidis punctum, & *CZ* semissis arcus cycloidis totius, longitudini penduli aequalis; & quaeratur resistentia corporis in loco quovis *D*. Secetur recta infinita *OQ* in punctis *O, S, P, Q*, ea lege, ut (si erigantur perpendicula *OK, ST, PI, QE* centroque *O* & asymptotis *OK, OQ* describatur hyperbola TIGE secans perpendicula *ST, PI, QE* in *T, I* & *E*, & per punctum *I* agatur *KF* parallela asymptoto *OQ* occurrens asymptoto *OK* in *K*, & perpendiculis *ST* & *QE* in *L* & *F*), fuerit area hyperbolica *PIEQ* ad aream hyperbolicam *PITS* ut arcus *BC* descensu corporis descriptus ad arcum *Ca* ascensu descriptum, & area *IEF* ad aream *ILT* ut *OQ* ad *OS*. Dein perpendiculo *MN* abscindatur area hyperbolica *PINM* quae sit ad aream hyperbolicam *PIEQ* ut arcus *CZ* ad arcum *BC* descensu descriprum. Et si perpendiculo *RG* abscindatur area hyperbolica *PIGR*, quae sit ad aream *PIEQ* ut arcus quilibet *CD* ad arcum *BC* descensu toto descriptum, erit resistentia in loco *D* ad vim gravitatis, ut area $\frac{OR}{OQ} \times IEF - IGH$ ad aream *PINM*.



Nam cum vires a gravitate oriundae quibus corpus in locis *Z,B,D,a* urgetur, sint ut arcus *CZ, CB, CD, Ca*, & arcus illi sint ut areae *PINM, PIEQ, PIGR, PITS*; exponantur tum arcus tum vires per has areas respective. Sit insuper *Dd* spatium quam minimum a corpore descendente descriptum, & exponatur idem per aream quam minimam *RGgr* parallelis *RG, rg* comprehensam , & producatur *rg* ad *h*, ut sint *GHhg*, & *RGgr* contemporanea arearum *IGH, PIGR* decrementa. Et areae $\frac{OR}{OQ}\times IEF-IGH$ incrementum $GHhg\ -\frac{Rr}{OQ}IEF$, seu $Rr\times HG-\frac{Rr}{OQ}IEF$, erit ad areae *PIGR* decrementum *RGgr*, seu $OR\times HG-\frac{Rr}{OQ}IEF$ ad $OR\times GR$ seu $OP\times PI$, hoc est (ob aequalia $OR\times HG$, $OR\times HR-OR\times GR$, $ORHK-OPIK$, $PIHR$ & $PIGR+IGH$), ut $PIGR+IGH-\frac{OR}{OQ}IEF$ ad $OPIK$. Igitur si area $\frac{OR}{OQ}IEF-IGH$ dicatur *Y*, atque areae *PIGR* decrementum *RGgr* detur, erit incrementum areae *Y* ut *PIGR–Y*.

Quod si *V* designet vim a gravitate oriundam, arcui describendo *CD* proportionalem, qua corpus urgetur in *V*, & *R* pro resistentia ponatur; erit $V-R$ vis tota qua corpus urgetur in *D*. Est itaque incrementum velocitatis ut $V-R$ & particula illa temporis in qua factum est coniunctim: Sed & velocitas ipsa est ut incrementum contemporaneum spatii descripti directe & particula eadem temporis inverse. Unde, cum resistentia per hypothesin sit ut quadratum velocitatis, incrementum resistentiae (per lem.II.) erit ut velocitas & incrementum velocitatis coniundim, id est, ut momentum spatii & $V-R$ coniunctim ; atque ideo, si momentum spatii detur, ut $V-R$; id est, si pro vi *V* scribatur eius exponens *PIGR*, & resistentia *R* exponatur per aliam aliquam aream *Z*, ut $PIGR-Z$.

Igitur area *PIGR* per datorum momentorum subductionem uniformiter decrescente, crescunt area *Y* in ratione $PIGR-Y$, & area *Z* in ratione $PIGR-Z$. Et propterea si areae *Y* & *Z* simul incipiant & sub initio aequales sint, hae per additionem aequalium momentorum pergent esse aequales, & aequalibus itidem momentis subinde decrescentes simul evanescent. Et vicissim, si simul incipiunt & simul evanescunt, aequalia habebunt momenta & semper erunt aequales: id adeo quia si resistentia *Z* augeatur, velocitas una cum arcu illo *Ca*, qui in ascensu corporis describitur, diminuetur, & puncto in quo motus omnis una cum resistentia cessat propius accedente ad punctum *C*, resistentia citius evanescet quam area *Y*. Et contrarium eveniet ubi resistentia diminuitur.

Iam vero area *Z* incipit definitque ubi resistentia nulla est, hoc est, in principio motus ubi arcus *CD* arcui *CB* aequatur & recta *RG* incidit in rectam *QE*, & in fine motus ubi arcus *CD* arcui *Ca* aequatur & *RG* incidit in rectam *ST*. Et area *Y* seu $\frac{OR}{OQ}\times IEF-IGH$ incipit definitque ubi nulla est, ideoque ubi $\frac{OR}{OQ}\times IEF-IGH$ aequalia sunt: hoc est (per constructionem) ubi recta *RG* incidit successive in rectas *QE* & *ST*. Proindeque areae ille simul incipiunt & simul evanescunt, & propterea semper sunt aequales. Igitur area $\frac{OR}{OQ}\times IEF-IGH$ aequalis est areae *Z*, per quam resistentia exponitur, & propterea est ad aream *PINM* per quam gravitas exponitur, ut resistentia ad gravitatem. *Q.E.D.*

Corol. I. Est igitur resistentia in loco infimo *C* ad vim gravitatis, ut area $\frac{OR}{OQ}IEF$ ad aream *PINM*.

Corol: 2. Fit autem maxima, ubi area *PIHR* est ad aream *IEF* ut *OR* ad *OQ*. Eo enim in casu momentum eius (nimirum $PIGR - Y$ evadit nullum.

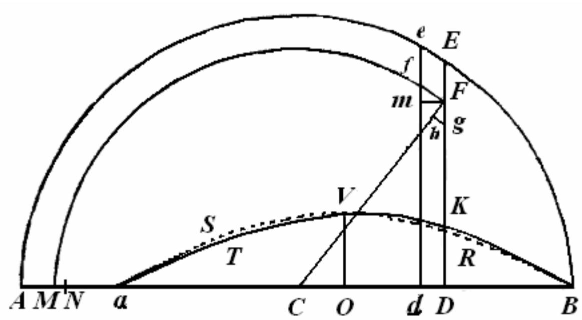
Corol. 3. Hinc etiam innotescit velocitas in locis singulis quippe quae est in subduplicata ratione resistentiae, & ipso motus initio aequatur velocitati corporis in eadem cycloide sine omni resistentia oscillantis.

Caeterum ob difficilem calculum quo resistentia & velocitas per hanc propositionem inveniendae sunt, visum est propositionem sequentem subiungere.

PROPOSITIO XXX. THEOREMA XXIV.

Si recta aB aequalis sit cycloidis arcui quem corpus oscillando describit, & ad singula eius puncta D erigantur perpendicula DK, quae sint ad longitudinem penduli ut resistentia corporis in arcus punctis correspondentibus ad vim gravitatis: dico quod differentia inter arcum descensu toto descriptum & arcem ascensu toto subsequente descriptum ducta in arcuum eorundem semisummam, aequalis erit areae BKa a perpendiculis omnibus DK occupatae.

Exponatur enim tum cycloidis arcus, oscillatione integra descriptus, per rectam illam sibi aequalem *aB*, tum arcus qui describeretur in vacuo per longitudinem *AB*. Bisecetur *AB* in *C*, & punctum *C* representabit infimum cycloidis punctum, & erit *CD* ut vis a gravitate oriunda, qua corpus in *D* secundum tangentem cycloidis urgetur, eamque habebit rationem ad longitudinem penduli quam habet vis in *D* ad vim gravitatis. Exponatur igitur vis illa per longitudinem *CD*, & vis gravitatis per longitudinem penduli. & si in *DE* capiatur *DK* in ea ratione ad longitudinem penduli quam habet resistentia ad gravitatem, erit *DK* exponens resistentiae. Centro *C* & intervallo *CA* vel *CB* construatur semicirculus *BEeA*. Describat autem corpus tempore quam minimo spatium *Dd*, & erectis perpendiculis *DE, de* circumferentiae occurrentibus in *E* & *e*, erunt haec ut velocitates quas corpus in vacuo, descendendo a puncto *B*, acquireret in locis *D* & *d*. Patet hoc (per Prop. LII. Lib. I.) Exponantur itaque hae velocitates per perpendicula illa *DE, de*; sitque *DF* velocitas quam acquirit in *D* cadendo de *B* in media resistente. Et si centro *C* & intervallo *CF* describatur circulus *FfM* occurrens rectis *de* & *AB* in *f* & *M*, erit *M* locus ad quem deinceps sine ulteriore resistentia ascenderet,



& *df* velocitas quam acquireret in *d*. Unde etiam si *Fg* designet velocitatis momentum quod corpus D, describendo spatium quam minimum *Dd*, ex resistentia medii amittit; & sumatur *CN* aequalis *Cg*: erit *N* locus ad quem corpus deinceps sine ulteriore resistentia ascenderet, & *MN* erit decrementum ascensus ex velocitatis illius amissione oriundum. Ad *df* demittatur perpendiculum *Fm*, & velocitatis *DF* decrementum *F*g a resistentia *DK* genitum, erit ad velocitatis eiusdem incrementum *fm* a vi *CD* genitum, ut vis generans *DK* ad vim generantem *CD*. Sed & ob similia triangula *Fmf, Fhg, FDC*, est *fm* ad *Fm* seu *Dd* ut *CD* ad *DF*; & ex aequo *Fg* ad *Dd* ut *DK* ad *DF* . Item *Fh* ad *Fg* ut *DF* ad *CF*; & ex aequo perturbate, *Fh* seu *MN* ad *Dd* ut *DK* ad *CF* seu *CM*; ideoque summa omnium $MN \times CM$ aequalis erit summae omnium $Dd \times DK$. Ad punctum mobile *M* erigi semper intelligatur ordinata rectangula aequalis indeterminate *CM*, quae motu continuo ducatur in totam longitudinem *Aa*; & trapezium ex illo motu descriptum sive huic aequale rectangulum $Aa \times \frac{1}{2}aB$ aequabitur summae omnium $MN \times CM$, ideoque summae omnium $Dd \times DK$, id est, areae *BKVTa*. *Q.E.D.*

Corol. Hinc ex lege resistentiae & arcuum *Ca, CB* differentia *Aa* colligi potest proportio resistentiae ad gravitatem quam proxime.

Nam si uniformis sit resistentia *DK*, figura *BKTa* rectangulum erit sub *Ba* & *DK*; & inde rectangulum sub $\frac{1}{2}Ba$ & *Aa* erit aequale rectangulo sub *Ba* & *DK,* & *DK* aequalis erit $\frac{1}{2}Aa$. Quare cum *DK* sit exponens resistentias, & longitudo penduli exponens gravitatis, erit resistentia ad gravitatem ut $\frac{1}{2}Aa$ ad longitudinem penduli; omnino ut in Prop. XXVIII. demonstratum est.

Si resistentia sit ut velocitas, figura *BKTa* ellipsis erit quam proxime. Nam si corpus, in medio non resistente, oscillatione integra describeret longitudinem *BA*, velocitas in loco quovis *D* foret ut circuli diametro *AB* descripti ordinatim applicata *DE*. Proinde cum *Ba* in medio resistente, & *BA* in medio non resistente, aequalibus circiter temporibus describantur ; ideoque velocitates in singulis ipsius *Ba* punctis, sint quam proxime ad velocitates in punctis correspondentibus longitudinis *BA*, ut est *Ba* ad *BA*; erit velocitas in puncto *D* in medio resistente ut circuli vel ellipseos super diametro *Ba* descripti ordinatim applicata; ideoque figura *BKVTa* ellipsis erit quam proxime. Cum resistentia velocitati proportionalis supponatur, sit *OV* exponens resistentiae in puncto medio *O*, & ellipsis *BRVSa*, centro *O*, semiaxibus *OB*, *OV* descripta, figuram *BKVTa*, eique aequate rectangulum $Aa \times BO$, aequabit quamproxime. Est igitur $Aa \times BO$ ad $OV \times BO$ ut area ellipseos huius ad $OV \times BO$: id est, *Aa* ad *OV* ut area semicirculi ad quadratum radii, sive ut 11 ad 7 circiter: Et propterea $\frac{7}{11}Aa$ ad longitudinem penduli ut corporis oscillantis resistentia in *O* ad eiusdem gravitatem.

Quod si resistentia *DK* sit in duplicata ratione velocitatis, figura *BKVTa* fere parabola erit verticem habens V & axem *OV*, ideoque aequalis erit rectangulo sub $\frac{2}{3}Ba$ & *OV* quam proxime. Est igitur rectangulum sub $\frac{1}{2}Ba$ & *Aa* aequale rectangulo sub $\frac{2}{3}Ba$ & *OV*, ideoque *OV* aequalis $\frac{1}{4}Aa$: & propterea corporis oscillantis resistentia in *O* ad ipsius gravitatem ut $\frac{3}{4}Aa$ ad longitudinem penduli. Atque has conclusiones in rebus practicis abunde satis accuratas esse censeo. Nam cum ellipsis vel parabola *BRVSa* congruat cum figura *BKVTa* in puncto medio *V*, haec si ad partem alterutram *BRV* vel *VSa* excedit figuram illam, deficiet ab eadem ad partem alteram, & sic eidem aequabitur quam proxime.

PROPOSITIO XXXI. THEOREMA XXV.

Si corporis oscillantis resistentia in singulis arcuum descriptorum partibus proportionalibus augeatur vel minuatur in data ratione ; differentia inter arcum descensu descriptum & arcum subsequente ascensu descriptum, augebitur vel diminuetur in eadem ratione.

Oritur enim differentia illa ex retardatione penduli per resistendam medii, ideoque est ut retardatio tota eique proportionalis resistentia retardans. In superiore propositione rectangulum sub recta $\frac{1}{2}aB$ & arcuum illorum *CB, Ca* differentia *Aa* equalis erat areae *BKTa*. et area illa, si maneat longitudo *aB*, augetur vel diminuitur ratione ordinatim applicatarum *DK*; hoc est, in ratione resistentiae, ideoque est ut longitudo *aB* & resistentia coniunctim, Proindeque rectangulum sub *Aa* & $\frac{1}{2}aB$ est ut *aB* & resistentia coniunctim, & propterea *Aa* ut resistentia. *Q.E.D.*

Corol. I. Unde si resistentia sit ut velocitas, differentia arcuum in eodem media erit ut arcus totus descriptus : & contra.
Corol. 2. Si resistentia sit in duplicata ratione velocitatis, differentia illa erit in duplicata ratione arcus totius : & contra.
Corol. 3. Et universaliter, si resistentia sit in triplicata vel aliaquavis ratione velocitatis, differentia erit in eadem ratione arcus totius : & contra.
Corol. 4. Et si resistentia sit partim in ratione simplici velocitatis, partim in eiusdem ratione duplicata, differentia erit partim in ratione arcus totius & partim in eius ratione duplicata: & contra. Eadem erit lex & ratio resistentiae pro velocitate, quae est differentiae illius pro longitudine arcus.
Corol. 5. Ideoque si, pendulo inaequales arcus successive describente, inveniri potest ratio incrementi ac decrementi differeniae huius pro longitudine arcus descripti ac habebitur etiam ratio incrementi ac decrementi resistentiae pro velocitate maiore vel minore.

Scholium Generale.

Ex his propositionibus, per oscillationes pendulorum in mediis quibuscunque, invenire licet resistentiam mediorum. Aeris vero resistentiam investigavi per experimenta sequentia. Globum ligneum pondere unciarum *Romarum* $57\frac{7}{22}$, diametro digitorum Londinensium $6\frac{7}{8}$ fabricatum, filo tenui ab unco satis firmo suspendi, ita ut inter uncum & centrum oscillationis globi distantia esset pedum $10\frac{1}{2}$. In filo punctum notavi pedibus decem & uncia una a centro suspensionis distans , & e regione puncti illius collocavi regulam in digitos distinctam, quorum ope notarem longitudines arcuum a pendulo descriptas. Deinde numeravi oscillationes quibus globus octavam motus sui partem amitteret. Si pendulum deducebatur a perpendiculo ad distantium duorum digitorum, & inde demittebatur, ita ut toto suo descensu describeret arcum duorum digitorum, totaque oscillatione prima, ex descensu & ascensu subsequente composita, arcum digitorum sere quatuor : idem oscillationibus 164 amisit octavam motus sui partem, sic ut ultimo suo ascensu describeret arcum digiti unius cum tribus partibus quartis digiti. Si primo descensu descripsit arcum digitorum quatuor; amisit octavam motus partem oscillationibus 121, ita ut ascensu ultimo describeret arcum digitorum $3\frac{1}{2}$. Si primo descensu descripsit arcum digitorum octo, sexdecim, triginta duorum vel sexaginta quatuor; amisit octavam motus partem oscillationibus $69, 35\frac{1}{2}, 18\frac{1}{2}, 9\frac{2}{3}$, respective. Igitur differentia inter arcus descensu primo & ascensu ultimo descriptos, erat in casu primo, secundo, tertio, quarto, quinto, sexto, digitorum $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$ respective. Dividantur eae differentiae per numerum oscillationum in casu unoquoque, & in oscillatione una mediocri, ne qua arcus digitorum $3\frac{1}{4}, 7\frac{1}{2}, 15, 30, 60, 120$ descriptus fuit, differentia arcuum descensu & subsequente ascensu descriptorum, erit $\frac{1}{656}, \frac{1}{242}, \frac{1}{69}, \frac{4}{71}, \frac{8}{37}, \frac{24}{29}$ partes digiti respective. Hae autem in maioribus oscillationibus sunt in duplicata ratione arcuum descriptorum quam proxime, in minoribus vero paulo maiores quam in ea ratione; & propterea (per Corol. 2. Prop. XXXI. libri huius) resistentia globi, ubi celerius movetur, est in duplicata ratione velocitatis quam proxime; ubi tardius, paulo maior quam in ea ratione.

Designet iam *V* velocitatem maximam in oscillatione quavis, sintque *A, B, C* quantitates datae, & fingamus quod differentia arcuum sit $AV + BV^{\frac{3}{2}} + CV^2$. Cum velocitates maximae sint in cycloide ut semisses arcuum oscillando descriptorum, in circulo vero ut semissium arcuum illorum chordae , ideoque paribus arcubus maiores sint in cycloide quam in circulo, in ratione semissium arcuum ad eorundem chordas; tempora autem in circulo sint maiora quam in cycloide in velocitatis ratione reciproca , patet arcuum differentias (quae sunt ut resistentia & quadratum temporis coniundim) easdem fore, quamproxime, in utraque curva : deberent enim differentiae illae in cycloide augeri, una cum resistentia, in duplicata circiter ratione arcus ad chordam, ob velocitatem in ratione illa simplici auctam; & diminui, una cum quadrato temporis, in eadem duplicate ratione. Itaque ut reductio fiat ad cycloidem, eodem sumendae sunt arcuum differentae quae fuerunt in circulo observatae, velocitates vero maximae ponendae sunt arcubus vel dimidiatis vel integris, hoc est, numeris $\frac{1}{2}, 1, 2, 4, 8, 16$ analogae. Scribamus ergo in casu secundo, quarto & sexto numeros I, 4 & 16 pro *V*; & prodibit arcuum differentia $\frac{\frac{1}{2}}{121} = A + B + C$ in casu secundo; $\frac{2}{35\frac{1}{2}} = 4A + 8B + 16C$ in casu quarto; & $\frac{8}{9\frac{2}{3}} = 16A + 64B + 256C$ in casu sexto. Et ex his aequationibus, per debitam collationem & reductionem analyticam, sit $A = 0{,}0000916$, $B = 0{,}0010847$, & $C = 0{,}0029558$. Est igitur differentia arcuum ut $0{,}0000916V + 0{,}0010847V^{\frac{3}{2}} + 0{,}0029558V^2$; & propterea cum (per corollarium Propositionis XXX. applicatum ad hunc casum) resistentia globi in medio arcus oscillando descripti, ubi velocitas est *V*, sit ad ipsius pondus ut $\frac{7}{11}AV + \frac{7}{10}BV^{\frac{3}{2}} + \frac{3}{4}CV^2$ & ad longitudinem penduli ; si pro *A*, *B* & *C* scribantur numeri inventi, fiet resistentia globi ad eius pondus, ut $0{,}0000583V + 0{,}0007593V^{\frac{3}{2}} + 0{,}0022169V^2$ ad longitudinem penduli inter centrum suspensionis & regulam, id est, ad 121 digitos, Unde cum *V* in casu secundo designet 1, in quarto 4, in sexto 16.: erit resistentia ad pondus globi in casu secundo ut 0,0030345 ad 121, in quarto ut 0,041748 ad 121, in sexto ut 0,61705 ad 121.

Arcus quem punctum in filo nototum in casu sexto descripsit, erat $120-\frac{8}{9\frac{2}{3}}$ seu $119\frac{5}{29}$ digitorum. Et propterea cum radius esset 121 digitorum, & longitudo penduli inter punctum suspensionis & centrum globi esset 126 digitorum, arcus quem centrum globi descripsit erat $124\frac{3}{31}$ digitorum. Quoniam corporis oscillantis velocitas maxima, ob resistentiam aeris, non incidit in punctum infimum arcus descripti, sed in medio fere loco arcus totius versatur: haec eadem erit circiter ac si globus descensu suo toto in medio non resistente describeret arcus illius partem dimidiam digitorum $62\frac{3}{62}$, idque in cycloide, ad quam motum penduli supra reduximus: & propterea velocitas illa aequalis erit velocitati quam globus, perpendiculariter cadendo & casu suo describendo altitudinem arcus illius finui verso aequalem, acquirere posset. Est autem sinus ille versus in cycloide ad arcum istum $62\frac{3}{62}$ ut arcus idem ad penduli longitudinem duplam 252, & propterea aequalis digitis 15,278. Quare velocitas ea ipsa est quam corpus cadendo & casu suo spatium 15,278 digitorum describendo acquirere posset, Tali igitur cum velocitate globus resistentiam patitur, quae sit ad eius pondus ut 0,61705 ad 121, vel (si resistentiae pars illa sola spectetur quae est in velocitatis ratione duplicata) ut 0,56752 ad 121.

Experimento autem hydrostatico inveni quod pondus globi huius lignei esset ad pondus globi aquei magnitudinis eiusdem ut 55 ad 97: & propterea cum 121 sit ad 213,4 in eadem ratione, erit resistentia globi aquei praefata cum velocitate progredientis ad ipsius ondus ut 0,56752 ad 213,4 id est, ut 1 ad $376\frac{1}{50}$. Unde cum pondus globi aquei, quo tempore globus cum velocitate uniformiter continuata describat longitudinem digitorum 30,556, velocitatem illam omnem in globo cadente generare posset , manifestum est quod vis resistentiae eodem tempore uniformiter continuata tollere posset velocitatem minorem in ratione 1 ad $376\frac{1}{50}$, hoc est, velocitatis totius partem $\frac{1}{376\frac{1}{50}}$. Et propterea quo tempore globus, ea cum velocitate uniformiter continuata, longitudinem semidiametri suae, seu digitorum $3\frac{1}{16}$ describere posset, eodem amitteret motus sui partem $\frac{1}{3542}$.

Numerabam etiam oscillationes quibus pendulum quartam motus sui partem amisit. In sequente tabula numeri supremi denotant longitudinem arcus descensu primo descripti, in digitis & partibus digita exprectam : numeri medii significant longitudinem arcus ascensu ultimo descripti; & loco infimo stant numeri oscillationum. Experimentum descripsi tanquam magis accuratum quam cum motus pars tantum octava amitteretur. Calculum tentet qui volet.

descensus primus	2	4	8	16	32	64
Ascensus ultimus	$1\frac{1}{2}$	3	6	12	24	48
Numerus oscillat.	374	272	$162\frac{1}{2}$	$83\frac{1}{2}$	$41\frac{2}{3}$	$22\frac{2}{3}$

Postea globum plumbeum diametro digitorum 2, & pondere unciarum Romanarum $26\frac{1}{4}$ suspendi filo eodem, sic ut inter centrum globi & punctum suspensionis intervallum esset pedum $10\frac{1}{2}$, & numerabam oscillationes quibus data motus pars amitteretur. Tabularum subsequentium prior exhibet numerum oscillationum quibus pars octava motus totius cessavit; secunda numerum oscillationum quibus eiusdem pars quarta amissa fuit.

descensus primus	1	2	4	8	16	32	64
Ascensus ultimus	$\frac{7}{8}$	$\frac{7}{4}$	$3\frac{1}{2}$	7	14	28	56
Numerus oscillat.	226	228	193	140	$90\frac{1}{2}$	53	30

descensus primus	1	2	4	8	16	32	64
Ascensus ultimus	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24	48
Numerus oscillat.	510	518	420	318	204	121	70

In tabula priore seligendo ex observationibus tertiam, quintam & septimam, & exponendo velocitates maximas in his observationibus particulatim per numeros 1, 4, 16 respective, & generaliter per quantitatem V ut supra: emerget in observatione tertia $\frac{\frac{1}{2}}{193} = A + B + C$, in quinta $\frac{2}{90\frac{1}{2}} = 4A + 8B + 16C$, in septima $\frac{8}{30} = 16A + 64B + 256C$. Hae vero aequationes reductae dant $A = 0{,}0014,\ B = 0{,}000297,\ C = 0{,}000879$. Et inde prodit resistentia globi cum velocitate V moti in ea ratione ad pondus suum unciarum $26\frac{1}{4}$, quam habet $0{,}0009V + 0{,}000208\ V^{\frac{3}{2}} + 0{,}000659V^2$ ad penduli longitudinem 121 digitorum. Et si spectemus eam solummodo resistentiae partem quae est in duplicata ratione velocitatis, haec erit ad pondus globi ut $0{,}000659V^2$ ad 121 digitos. Erat autem haec pars resistentiae in experimento primo ad pondus globi lignei unciarum $57\frac{7}{22}$ ut $0{,}002217V^2$ ad 121: & inde sit resistentia globi lignei ad resistentiam globi plumbei (paribus eorum velocitatibus) ut $57\frac{7}{22}$ in $0{,}002217$ ad $26\frac{1}{4}$ in $0{,}000659$, id est, ut $7\frac{1}{3}$ ad 1. Diametri globorum duorum erant $6\frac{7}{8}$ & 2 digitorum, & harum quadrata sunt ad invicem ut $47\frac{1}{4}$ & 4 seu $11\frac{15}{16}$ & 1 quamproxime. Ergo resisteniae globorum aequivelocium erant in minore ratione quam duplicata diametrorum. At nondum consideravimus resistentiam fili, quae certe per magna erat, ac de pendulorum inventa resistentia subduci debet. Hanc accurate definire non potui, sed maiorem tamen inveni quam partem tertiam resistentiae totius minoris penduli; & inde didici quod resistentiae globorum, dempta fili resistentia, sunt quam proxime in duplicata ratione diametrorum. Nam ratio $7\frac{1}{3} - \frac{1}{3}$ ad $1 - \frac{1}{3}$, seu $10\frac{1}{2}$ ad 1 non longe abest a diametrorum ratione $11\frac{15}{16}$ duplicata ad 1.

Cum resistentia fili in globis maioribus minoris sit momenti, tentavi etiam experimentum in globo cuius diameter erat $18\frac{3}{4}$ digitosum. Longitudo penduli inter punctum suspensionis & centrum oscillationis erat digitorum $122\frac{1}{2}$, inter punctum suspensionis & nodum in filo $109\frac{1}{2}$ dig. Arcus primo penduli descensu a nodo descriptus 32 dig. Arcus ascensu ultimo post oscillationes quinque ab eodem nodo descriptus 28 dig. Summa arcuum seu arcus totus oscillatione mediocri descriptus 60 dig. Differentia arcuum 4 dig. ; Eius pars decima seu differentia inter descensum & ascensum in oscillatione mediocri $\frac{2}{5}$ dig. Ut radius $109\frac{1}{2}$ ad radium $122\frac{1}{2}$ ita arcus totus 60 dig. oscillatione mediocri a nodo descriptus ad arcum totum $67\frac{1}{8}$ dig. oscillatione mediocri a centro globi descriptum , & ita differentia $\frac{2}{5}$ ad differentiam novam 0,4475. Si longitudo penduli, manente Iongitudine arcus descripti, augeretur in ratione 126 ad $122\frac{1}{2}$; tempus oscillationis augeretur & velocitas penduli diminueretur in ratione illa subduplicata, maneret vera arcuum descensu & subsequente ascensu descriptorum differentia 0,4475. Deinde si arcus descriptus augeretur in ratione $124\frac{3}{31}$ ad $67\frac{1}{8}$, differentia ista 0,4475 augeretur in duplicata illa ratione, ideoque evaderet 1,5295. Haec ita se haberent, ex hypothesi quod resistentia penduli esset in duplicata ratione velocitatis. Ergo si pendulum describeret arcum totum $124\frac{3}{31}$ digitorum, & longitudo eius inter punctum suspensionis & centrum oscillationis esset 126 digitorum, differentia arcuum descensu & subsequente ascensu descriptorum foret 1,5295 digitorum. Et haec differentia ducta in pondus globi penduli, quod erat unciarum 208, producit 318,136. Rursus ubi pendulum superius ex globe ligneo constructum centro oscillationis, quod a puncto suspensionis digitos 126 distabat, describebat arcum totum $124\frac{3}{31}$ digitorum, differentia arcuum descensu & ascensu descriptum fuit $\frac{126}{121}$ in $\frac{8}{9\frac{2}{3}}$, quae ducta in pondus globi, quod erat unciarum $57\frac{7}{22}$, producit 49,396. Duxi autem differentias hasce in pondera globorum, ut invenirem eorum resistentias. Nam differentiae oriuntur ex resistentiis, suntque ut resisteniae directe & pondera inverse. Sunt igitur resistentia ut numeri 318,136 & 49,396. Pars autem resistentiae globi minoris, quae est in duplicata ratione velocitatis, erat ad resistentiam totam ut 0,56752 ad 0,61675 id est, ut 45,453 ad 49,396 ; & pars resistentiae globi maioris propemodum aequatur ipsius resisteniae toti , ideoque partes illae sunt ut 318,136 & 45,453 quamproxime, id est, ut 7 et 1. Sunt autem globorum diametri $18\frac{3}{4}$ et $6\frac{7}{8}$; & harum quadrata $351\frac{9}{16}$ et $47\frac{17}{64}$ sunt ut 7,438 et 1, id est, ut globorum resisteniae 7 & 1 quamproxime. Differentia rationum haud maior est, quam quae ex fili resistentia oriri potuit. Igitur resistentiarum partes illae quae sunt, paribus globis, ut quadrata velocitotum, sunt etiam, paribus velocitatibus, ut quadrata diametrorum globorum.

Caeterum globorum, quibus usus sum in his experimentis, maximus non erat perfecte sphaericus,& propterea in calculo hic allato minutias quasdam brevitatis gratia neglexi; de calculo accurato in experimento non satis accurato minime sollicitus, Optarim itaque, cum demonstratio vacui ex his dependeat, ut experimenta cum globis & pluribus & maioribus & magis accuratis tentarentur. Si globi sumantur in proportione geometrica, puta quorum diametri sint digitorum 4, 8, 16, 32 ; ex progressione experimentorum colligetur quid in globis adhuc maioribus evenire debeat.

Iam vero conferendo resistentias diversorum fluidorum inter se, tentavi sequentia. Arcam ligneam paravi longitudine pedum quatuor, latitudine & altitudine pedis unius. Hanc operculo nudatam implevi aqua fontana, fecique ut immersa pendula in medio aquae oscillando moverentur. Globus autem plumbeus pondere $166\frac{1}{6}$ unciarum, diametro $3\frac{5}{8}$ digitorum movebatur ut in tabula sequente descripsimus, existente videlicet longitudine penduli a puncto suspensionis ad punctum quoddam in fila nototum 126 digitorum ad oscillationis autem centrum $134\frac{3}{8}$ digitorum.

Arcus descensu primo a puncto in filo notato descriptus, digitorum	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
Arcus ascensu ultimo descriptus, dignitoru ultimus	48	24	12	6	3	$1\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{5}{16}$
Arcuum differentia motui amisso proportionalis, digitorumt.	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
Numerus oscillationum in aqua			$\frac{29}{60}$	$1\frac{1}{3}$	3	7	$11\frac{1}{4}$	$12\frac{2}{3}$	$13\frac{1}{3}$
Numerus oscillationum in aere	$85\frac{1}{2}$		287	535					

In experimento columnae quartae, motus aequales oscillationibus 535 in aere, & $1\frac{1}{3}$ in aqua amissi sunt. Erant quidem oscillationes in aere paulo celeriores quam in aqua. At si oscillationes in aqua in ea ratione accelerarentur ut motus pendulorum in medio utroque fierent aequiveloces, maneret numerus idem oscillationum $1\frac{1}{3}$ in aqua, quibus motus idem ac prius amitteretur , ob resistentiam auctam & simul quadratum temporis diminutum in eadem ratione illa duplicata. Paribus igitur pendulorum velocitatibus motus aequales in aere oscillationibus 535 & in aqua oscillationibus $1\frac{1}{3}$ amissi sunt; ideoque resistentia penduli in aqua est ad eius resistentiam in aere ut 535 ad $1\frac{1}{3}$. Haec est proportio resistentiarum totarum in casu columnae quartae.

Designet iam $AV+CV^2$ differentiam arcuum in descensu & subsequente ascensu descriptorum a globo in aere cum velocitate maxima V moto; & cum velocitas maxima in casu columnae quartae sit ad velocitatem maximam in casu columnae primae, ut 1 ad 8; & differentia illa arcuum in casu columnae quartae ad differentiam in casu columnae primae ut $\frac{2}{535}$ ad $\frac{16}{85\frac{1}{2}}$ ad seu ut $85\frac{1}{2}$ ad 4280: scribamus in his casibus 1 & 8 pro velocitatibus, atque $85\frac{1}{2}$ & 4280 pro differentiis arcuum, & fiet $A+C=85\frac{1}{2}$; et $8A+64C=4280$ seu $A+8C=535$; indeque per reductionem aequationum proveniet $7C=449\frac{1}{2}$ & $C=64\frac{3}{4}$ et $A=21\frac{2}{7}$: atque ideo resistentia cum sit ut $\frac{7}{11}AV+\frac{3}{4}CV^2$, erit ut $13\frac{6}{11}V+48\frac{9}{56}V^2$. Quare in casu columnae quartae ubi velocitas erat 1, resistentia tota est ad partem suam quadrato velocitatis proportionalem, ut

$13\frac{6}{11}+48\frac{9}{56}$ seu $61\frac{12}{17}$ ad $48\frac{9}{56}$; & idcirco resistentia penduli in aqua est ad resistentiae partem illam in aere, quae quadrato velocitatis proportionalis est, quaeque sola in motibus velocioribus consideranda venit, ut $61\frac{12}{17}$ ad $48\frac{9}{56}$ & 535 ad $1\frac{1}{5}$ coniundim, id est, ut 571 ad 1. Si penduli in aqua oscillantis filum totum fuisset immersum, resistentia eius fuisset adhuc maior; adeo ut penduli in aqua oscillantis resistentia illa, quae velocitatis quadrato proportionalis est, quaeque sola in corporibus velocioribus consideranda venit, sit ad resistentiam eiusdem penduli totius, eadem cum velocitate in aere oscillantis, ut 850 ad 1 circiter, hoc est, ut densitas aquae ad densitatem aeris quamproxime.

In hoc calculo sumi quoque deberet pars illa resistentiae penduli in aqua, quae esset ut quadratum velocitatis, sed (quod mirum forte videatur) resistentia in aqua augebatur in ratione velocitatis plus quam duplicata. Eius rei causam investigando, in hanc incidi, quod area nimis angusta esset pro magnitudine globi penduli, & motum aquae cedentis prae angustia sua nimis impediebat. Nam si globus pendulus, cuius diameter erat digiti unius, immergeretur ; resistentia augebatur in duplicata ratione velocitatis quam proxime. Id tentabam construendo pendulum ex globis duobus, quorum inferior & minor oscillaretur in aqua, superior & maior proxime supra aquam filo affixus esset, et in aere oscillando, adiuvaret motum penduli eumque diuturniorem redderet. Experimenta autem hoc modo instituta se habebant ut in tabula sequente describitur.

Arcus descensu primo descritus	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
Arcus ascensu ultimo descriptus	12	6	3	$1\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{5}{16}$
Arcuum diff. motui amisso proport.	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
Numerus oscillationum	$3\frac{3}{8}$	$6\frac{1}{2}$	$12\frac{1}{12}$	$21\frac{1}{5}$	34	53	$62\frac{1}{5}$

Conferendo resistentias mediorum inter se, effeci etiam ut pendula ferrea oscillarentur in argento vivo. Longitude fili ferrei erat pedum quasi trium, & diameter globi penduli quasi tertia pars digiti, Ad filum autem proxime supra mercurium affixus erat globus alius plumbeus satis magnus ad motum penduli diutius continuandum. Tum vasculum, quod capiebat quart libras tres argenti vivi, implebam vicibus alternis argento vivo & aqua communi, ut pendulo in fluido utroque successive oscillante, invenirem proportionem resistentiarum: & prodiit resistentia argenti vivi ad resistentiam aquae ut 13 vel 14 ad 1 circiter: id est, ut densitas argenti vivi ad densitatem aquae. Ubi globum pendulum paulo maiorem adhibebam, puta cuius diameter esset quasi $\frac{1}{3}$ vel $\frac{2}{3}$ partes digiti, prodibat resistentia argenti vivi in ea ratione ad resistentiam aquae, quam habet numerus 12 vel 10 ad 1 circiter. Sed experimento priori magis fidendum est, propterea quod in his ultimis vas nimis angustum fuit pro magnitudine globi immersi. Ampliato globo, deberet etiam vas ampliari. Constitueram quidem huiusmodi experimenta in valis maioribus & in liquoribus tum metallorum fusorum, tum aliis quibusdam tam calidis quam frigidis repetere: sed omnia experiti non vacat, & ex iam descriptis satis liquet resistentiam corporum celeriter motorum densitati fluidorum in quibus moventur proportionalem esse quam proxime. Non dico accurate. Nam fluida tenaciora, pari densitate, proculdubio magis resistunt quam liquidiora, ut oleum frigidum quam calidum, calidum quam aqua pluvialis, aqua quam spiritus vini. Verum in liquoribus, qui ad sensum satis fluidi sunt, ut in aere, in aqua seu dulci seu salsa, in spiritibus vini, terebinthi & salium, in oleo a faecibus per destillationem liberato & calefacto, oleoque vitrioli & mercurio, ac metallis liquefactis, & si qui sint alii, qui tam fluidi sunt ut in vasis agitati motum impressum diutius conservent, effusique liberrime in guttas decurrendo resolvantur, nullus dubito quin regula allata satis accurate obtineat : praesertim si experimenta in corporibus pendulis & maioribus & velocius motis instituantur.

Denique cum nonnullorum opinio sit, medium quoddam aethereum et longe subtilissimum extare, quod omnes omnium corporum poros & meatus liberrime permeet; a tali autem medio per corporum poros fluente resistentia oriri debeat : ut tentarem an resistentia, quamin motis corporibus experimur, tota sit in eorum externa superficie, an vero partes etiam internae in superficiebus propriis resistentiam notabilem sentiant, excogitavi experimentum tale. Filo pedum undecim longitudinis ab unco chalybeo satis firmo, mediante annulo chalybeo, suspendebam pyxidem abiegnam rotundam, ad constituendum pendulum longitudinis praedictae. Uncus sursum preacutus erat acie concave, ut annulus arcu suo superiore aciei innixus liberrime moveretur. Arcui autem inferiori annectebatur filum. Pendulum ita constitutum deducebam a perpendiculo ad distantiam quasi pedum sex, idque secundum planum aciei unci perpendiculare, ne annulus, oscillante pendulo, supra aciem unci ultro citroque laberetur. Nam punctum suspensionis, in quo annulus uncum tangit, immotum manere debet. Locum igitur accurate notabam, ad quem deduxeram pendulum, dein pendulo demisso notabam alia tria loca ad que redibat in fine oscillationis primae, secundae ac tertiae. Hoc repetebam sepius, ut loca illa quam potui accuratissime invenirem. Tum pyxidem plumbo & gravioribus, quae ad manus erant, metallis implebam. Sed prius ponderabam pyxidem vacuam, una cum parte fili quae circum pyxidem volvebatur ac dimidio partis reliquae quae inter uncum & pyxidem pendulam tendebatur. Nam filum tensum dimidio ponderis sui pendulum a perpendiculo digressum semper urget. Huic ponderi addebam pondus aeris quem pyxis capiebat. Et pondus totum erat quasi pars septuagesima octava pyxidis metallorum plene. Tum quoniam pyxis metallorum plenae, pondere suo tendendo filum, augebat longitudinem penduli, contrahebam filum ut penduli iam oscillantis eadem esset longitudo ac prius. Dein pendulo ad locum primo notarum retracto ac dimisso, numerabam oscillationes quasi septuaginta & septem, donec pyxis ad locum secundo nototum rediret, totidemque subinde donec pyxis ad locum tertio nototum rediret, atque rursus totidem donec pyxis reditu suo attingeret locum quartum. Unde concludo quod resistentia tota pyxidis plenae non maiorem habebat proportionem ad resistentiam pyxidis vacuae quam 78 ad 77. Nam si aequales essent ambarum resisteniae, pyxis plena ob vim suam insitam septuagies & octies maiorem vi insita pyxidis vacue, motum suum oscillarorium tanto diutius conservare deberet, atque ideo completis semper oscillationibus 78 ad loca illa notata redire. Rediit autem ad eadem completis oscillationibus 77.

Designet igitur A resistentiam pyxidis in ipsius superficie externa, & B resistentiam pyxidis vacua in partibus internis; & si resistentiae corporum aequivelocium in partibus internis sint ut materia, seu numerus particularum quibus resistitur: erit 78 B resistentia pyxidis plenae in ipsius partibus internis: ideoque pyxidis vacue resistentia tota $A+B$ erit ad pyxidis plenae resistentiam totam $A+78B$ ut 77 ad 78, & divisim $A+B$ ad 77*B*, ut 77 ad 1, indeque $A+B$ ad B ut 77×77 ad 1, & divisim *A* ad *B* ut 5918 ad1. Est igitur resistentia pyxidis vacue in partibus internis quinquies millies minor quam eiusdem resistentia in externa superficie, & amplius. Sic vero disputamus ex hypothesi quod maior illa resistentia pyxidis plenae, non ab alia aliqua causa latente oriatur, sed ab actione sola fluidi alicuius subtilis in metallum inclusum.

Hoc experimentum recitavi memoriter. Nam charta, in qua illud aliquando descripseram, intercidit. Unde fractas quasdam numerorum partes, quae memoria exciderunt, omittere compulsus sum.

Nam omnia denuo tentare non vacat. Prima vice, cum unco infirmo usus essem, pyxis plena citius retardabatur. Causam querendo, reperi quod uncus infirmus cedebat ponderi pyxidis, & eius oscillationibus obsequendo in partes omnes flectebatur. Parabam igitur uncum firmum, ut punctum suspensionis immotum maneret, & tunc omnia ita evenerunt uti supra descripsimus.