SECTION V.

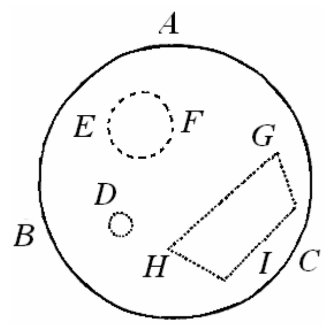
Concerning the hydrostatic density and compression of fluids.

The Definition of a Fluid.

A fluid is any body, the parts of which yield to any force inflicted, and by yielding the parts may move easily among themselves.

PROPOSITION XIX. THEOREM XIV.

All the parts of a homogeneous and motionless fluid, that is confined and pressed together on all sides in some motionless vessel (with the consideration of condensation, gravity, and of all centripetal forces disregarded) are pressed together equally on all sides, and without any motion arising from that pressure, remain in their places.



Case 1. A fluid may be confined to a spherical vessel *ABC* and may be uniformly compressed on all sides : I say that no part of the same will be moved by that compression. For if some part *D* may be moved, it is necessary that all the parts of this kind, in place at the same distance from the centre, likewise may be moved in a similar motion; and thus because this pressure is the same and equal generally, and the motion of the whole may be supposed excluded, unless that which may arise from the pressure. And all the parts cannot approach closer towards the centre, unless the fluid may be condensed at the centre ; contrary to the hypothesis. The parts are unable to recede further from the centre, unless the fluid may be condensing towards the circumference; also contrary to the hypothesis. They are unable to move in any direction that maintains their distance from the centre, for by the same reason parts will be moving in the opposite direction, but the same part cannot be moving in opposite directions at the same time. Therefore no part of the fluid will be moving from its place. *Q.E.D.*

Case 2. I say now, that all the spherical parts of this fluid are compressed equally in every direction. For let *EF* be a spherical part of the fluid, if this may not be pressed equally on all sides, the lesser pressure may be increased, so that then the sphere may be pressed upon equally; and the parts of this, by the first case, will remain in their places. But before the pressure increase they would remain in their own places, by the same first case, and with the addition of the new pressure they will be moved from their places, from the definition of a fluid. Which two explanations contradict each other. Therefore it was said incorrectly that the sphere *EF* may not be pressed on equally in all directions. *Q.E.D.*

Case 3. I say besides that the pressure on the different parts of the spheres shall be equal. For the parts of the spheres in contact press on each other equally at the point of contact, by the third law of motion. And also, by the second case, they are pressed on by the same force on all sides. Therefore any two non contiguous spherical parts, because an intermediate spherical part can touch each, will be pressed on by the same force. *Q.E.D.*

Case 4. Now I say that all parts of a fluid are pressed on equally everywhere. For any two parts can be in contact with spherical parts at any points, and these spherical parts press there equally : by the third case 3. And in turn they are pressed equally by the others, by the third law of motion. *Q.E.D.*

Case 5. Therefore since any part of the fluid *GHI* shall be enclosed by the remaining fluid as in the vessel, and pressed upon equally on all sides; moreover the parts of this shall press mutually on each other equally and shall be at rest among themselves, it is evident that every part *GHI* of the fluid, that is pressed on all sides equally, that all the parts press on each other equally, and are at rest between themselves. *Q.E.D.*

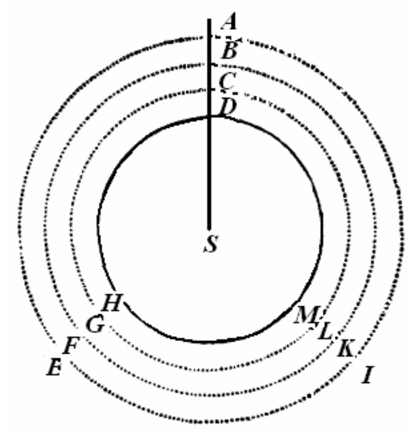
Case 6. Therefore if that fluid may be retained in a non rigid vessel, and may not be pressed on equally on all sides, the same fluid will withdraw from the stronger pressure, by the definition of fluidity.

Case: 7. And thus in a rigid vase the fluid cannot sustain a stronger pressure from one side than from the other, but concedes to the same [pressure], and that in an instant of time, because the side of the rigid vessel does not follow the yielding liquid. But on yielding the opposite side will be acted on, and thus the pressure inclines towards equality in each direction. And because the fluid, that first is trying to recede from the part with the greater pressure, may be prevented by the resistance of the vessel on the opposite side, the pressure will be reduced to equality on both sides, in a moment of time, without movement from the place : and at once the parts of the fluid, by the fifth case, may press on each other equally, and will rest between themselves. *Q.E.D.*

Corol. From which neither can the motions of the parts of the fluids amongst themselves change, by a pressure inflicted somewhere on the external surface, unless in so far as either the shape of the surface itself may be changed somewhere, or else all the parts of the fluid may slide over each other with more or less difficulty on account of pressing on each other more or less intensely.

PROPOSITION XX. THEOREM XV.

If the individual parts of a homogeneous spherical fluid, and at equal distances from the centre, concentric with the bottom sphere, gravitate towards the centre of the incumbent whole ; the innermost sphere may sustain the weight of a cylinder, the base of which is equal to the surface of the bottom sphere, and the height the same as that of the incumbent fluid.



Let *DHM* be the bottom surface, and *AEI* the upper surface of the fluid. The fluid may be separated into innumerable spherical surfaces *BFK*, *CGL* by shells of equal thickness ; and consider the force of gravity to act only on the upper surface of each shell, and the forces to be equal on equal parts of all of the surfaces. Therefore the upper surface *AE* is pressed by the simple force of its own weight, and by which all the parts of the upper shell and the second surface *BFK* (by Prop. XIX.) may be pressed, by its own measure. Besides the second surface *BFK* is pressed by the force of its own weight, which added to the first force makes the pressure double. The third surface *CGL* is acted on by this pressure, according to its measure, and in addition to its own weight, that is, by triple the pressure. And similarly the fourth surface is acted on by four times the pressure, the fifth by five times, and thus so forth. Therefore the pressure by which any surface may be acted on, is not as the solid quantity of the fluid acting, but as the number of shells as far as the largest of the fluid, and is equal to the weight of the lowest shell multiplied by the number of shells : that is, to the weight of the solid the final ratio of which to the cylinder determined becomes one of equality (only if the number of shells may be increased and the thickness may be diminished indefinitely, thus so that the action of gravity from the lowest surface to the highest may be returned continually). Therefore the lowest surface will sustain the weight of the determined cylinder. *Q.E.D.* And by a similar argument it is apparent the proposition, when the force of gravity decreases in some assigned ratio of the distance from the centre, and so that where the fluid rarer upwards, denser downwards. *Q.E.D.*

Corol. 1. Therefore the bottom is not acted on by the whole weight of the incumbent fluid, but only sustains that part of the weight which will be described in the proposition ; with the rest of the weight supported by an arched figure of fluid.

Corol. 2. But the size of the pressure is always the same at equal distances from the centre, whether the surface pressed shall be parallel to the horizontal or the vertical or oblique, or if the fluid, continued upwards from the pressed surface, rises perpendicularly along a right line, or may creep along sideways through turning cavities and channels, and these regular or very irregular, wide or most narrow. And in these circumstances no change in the pressure is to be deduced, by applying the demonstration of this theorem to the individual cases of fluids.

Corol. 3. It is also deduced from the same demonstration (by Prop. XIX.) that no parts of the weight of the fluid, from the pressure of the incumbent weight, acquire any motion between themselves, but only if the motion that may arise from condensation may be excluded.

Corol. 4. And therefore if another body of the same specific gravity, which shall be free from condensation [*i.e*. the immersed body cannot be condensed or change its density by contracting], may be submerged in this fluid, that will acquire no motion from the pressure of the incumbent weight : it will neither descent nor ascent, nor will it be forced to change its shape. If it is spherical, it will remain spherical, not withstanding the pressure; if it is square, it will remain square; and whether it shall be either soft or more fluid ; whether it may swim around freely in the fluid, or rest on the bottom. Indeed any internal part of the fluid has the ratio of the submerged body, and the ratio of every magnitude of this kind, of the figures and of the specific gravities of the submerged bodies, is the same. If a submerged body may be liquefied with the weight preserved and may adopt the form of the fluid ; this, if at first it may have ascended or descended or adopted a new figure from the pressure, also now it may ascend or descend or may be forced to form a new figure : and that is the case because the weight and the other causes of motion remain. But (by case 5. Prop. XIX.) it may now be at rest and it may retain the shape. And therefore as at first.

Corol. 5. Hence a body which has a slightly greater specific gravity than the fluid will subside, and that which has a lesser specific gravity will rise, and both the motion and a change of the shape may follow, as much as the excess or deficiency of the weight can bring about. In as much as that excess or defect ratio may give an impulse, by which the body may be forced into equilibrium with parts of the fluid in place elsewhere, and can be compared with the excess or defect of the weight on either plate of the scales.

Corol. 6. Therefore the weight of bodies constituted in fluids is two-fold : the one the true and absolute weight, the other the apparent, common and comparative weight. The absolute weight is the whole force by which the body tends downwards: the relative and common weight is the excess of the weight by which the body tends more downwards than the ambient fluid. The parts of all fluids and bodies of the first kind gravitate by weight in their own places : and thus the weights joined together compose the weight of the whole body. For any weight, it is the whole that one is allowed to experience, as in vessels full of liquids, and the weight of the whole is equal to the weight of all the parts, and thus may be composed from the same. With weights of the other kind, bodies do not gravitate at their places, that is, they are not weighed down gathered among themselves, but mutually endeavour to impede the descending of bodies, remaining in their own places, and thus as if there were no gravity. Which weights present in air and that do not weigh down, may not be concluded to be the common [or apparent] weight. Which weights that do weigh down together may be concluded to be the common weights, as long as they may not be sustained by the weight of the air. Common weights are nothing other than the excess of the true weight over the weight of the air. And with the commonly called levities [*i.e*. bodies that rise under everyday conditions; thus, at this time according to *L*. & *J*., the terms *specific weight* or *gravity* designated the ratio of weight to volume for a substance that sinks in a given medium, usually water, whereas *specific levity* or lightness similarly designates a substance that rises in a given medium], which are negative weights, and by yielding to the weightiness of the air, they desire to rise higher. They are comparative levities, not true ones, because they descent in a vacuum. And thus in water, bodies which on account of greater or lesser weight descent or ascend, are comparatively and apparently weights or levities, and the apparent and comparative weight or levity of these is the excess or deficiency by which the true weight either exceeds the weight of the water or by that is exceeded. Which truly neither descent by weighing down, nor ascending by not yielding to weighing down, even if they may increase the total weight by their true weight, yet comparatively in water and in the common sense they may not gravitate in water. For the demonstration of these cases is similar.

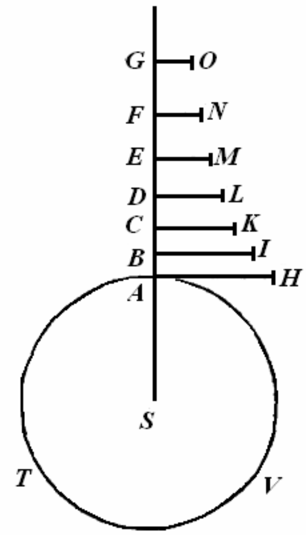
*Corol.*7. These things which are demonstrated from weights may be obtained from any other centripetal forces.

Corol. 8. Hence if a medium, in which some body may be moving, may be acted on either by gravity proper, or by some other centripetal force, and the body may be acted on more strongly by that same force, the difference of the forces is that motive force, that we have considered in the preceding propositions as the centripetal force. But if the body may be acted on by that lighter force, the difference of the forces must be had for the centrifugal force.

Corol. 9. But since fluids pressing included bodies shall not change the external shapes of these figures, it is apparent above (by corollary of Prop. XIX) that they will not change the position of the internal parts between themselves : and hence, if animals may be immersed, and the sensation of all the parts may arise from the motion ; neither will fluids harm these immersed bodies, nor will they excite any sensation, unless at this point they are able to condense these bodies by compression. And the account is the same of any system of bodies pressed together by the surrounding fluid. All the parts of the system will be agitated by the same motion, as if they were put in place in a vacuum, and may only retain their comparative weight, unless as far as either the fluid may resist the motion of these parts a little, or according to some sticking together that may come about by compression.

PROPOSITION XXI. THEOREM XVI.

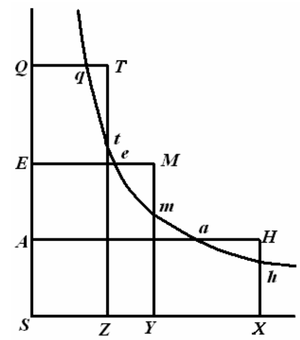
Let the density of a certain fluid be proportional to the compression, and the parts of this may be drawn downwards by a centripetal force inversely proportional to the distances of these from the centre : I say that, if the distances may be taken in continued proportion, the densities of the fluid at the same distances also will be in continued proportion.



ATV may describe the spherical bottom on which the fluid will lie, *S* the centre, *SA, SB, SC, SD, SE, SF,* &c. the distances in continued proportion. The perpendiculars *AH, BI*, *CK*, *DL*, *EM*, *FN*, &c. may be erected which shall be as the densities of the medium at the places *A, B, C, D, E, F* ; and the specific gravities at the same places will be as $\frac{AH}{AS}, \frac{BI}{BS}, \frac{CK}{CS}$, &c.
[*i.e.* the local specific gravity $\propto$ local density $\times$ acc. of gravity $\propto \frac{\text{density}}{\text{distance from } S}$,]

or, which likewise is thus, as $\frac{AH}{AB}, \frac{BI}{BC}, \frac{CK}{CD}$,&c. Consider first these weights to be uniformly continued well enough from *A* to *B*, from *B* to *C*, from *C* to *D*, &c. by the steps of the decrease at the points *B, C, D*, &c. And these weights [or specific gravities] multiplied by the heights *AB, BC, CD*, &c. will make the pressures *AH*, *BI*, *CK*, etc. by which the bottom *ATV* (as in Theorem XV.) is acted on. Therefore the small element *A* will sustain all the pressures *AB, BI, CK, DL*, going of indefinitely, and also the element *B* all the pressures except *AH*; and the element *C* all except the first two *AB*, *BI* ; and thus henceforth : and thus the density *AH* of the first element *A* is to the density *B* of the second element as the sum of all $AH + BI + CK + DL$ to infinity, to the sum of all $BI + CK + DL$, &c. And the second density *BI* of the element *B* is to the density *CK* of the third *C*, as the sum of all $BI + CK + DL$, &c. to the sum of all $CK + DL$, &c. Therefore these sums are proportional to their differences *AH*, *BI*, *CK*, &c., and thus are continued proportionals (by Lem. I of this section) and hence the proportional differences *AH, BI, CK*, &c. from the sums, are also continued proportionals. Whereby since the densities at the places *A*, *B*, *C*, &c. shall be as *AH*, *BI*, *CK*, &c. these are also continued proportions. It may advance by a leap, and from the equation with the distances *SA*, *SC*, *SE* in continued proportions, the densities *AH, CK, EM* will be in continued proportions. And by the same argument, with the distances in some continued proportions *SA, SD, SG*, the densities *AH*, *DL*, *GO* will be in continued proportions. Now the points *A*, *B*, *C*, *D*, *E*, &c. may coalesce, so that therefore the progression of specific gravities from the bottom *A* to the top of the fluid may be returned continually, and in some continued proportions in the distances *SA*, *SD*, *SG*, the densities *AH*, *DL*, *GO*, also present as continued proportions, will also remain continued proportionals. *Q.E.D.*

Corol. Hence if the density of the fluid may be given at two places, for example, *A* and *E*, it is possible to deduce its density at some other place *Q*. With centre *S*, with the rectangular asymptotes *SQ* and *SX* a hyperbola may be described cutting the perpendiculars *AH, EM, QT* at *a, e, q*, and so the perpendiculars *HX, MY, TZ*, sent to the asymptote *SX*, at *h, m* and *t*. The area *YmtZ* is made to the given area *YmhX* as the given area *EeqQ* to the given area *EeaA*; and the line *Zt* produced will cut the proportional line of the density *QT*. For if the lines *SA, SE, SQ* are continued proportionals, the areas *EeqQ* and *EeaA* are equal, and thence the areas *YmtZ* and *XhmY* are also equal from these proportionals, and the lines *SX, SY, SZ*, that is *AH, EM, QT* are continued proportionals, as required. And if the lines *SA, SE, SQ* maintain some other order in the continued series of proportions, the lines *AH, EM, QT*, on account of the proportional hyperbolic areas, will maintain the same order in another series of quantities in continued proportion.

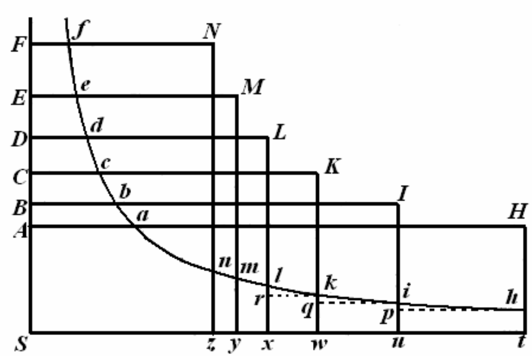


[Note from *L*. & *J*. : Indeed the hyperbolic areas *EeaA* and *QqaA* are the logarithms of the lines *SE* and *SQ*, and equally the areas *YmtZ* and *XhtZ* are the logarithms of the lines *SY* and *SX*; but since the areas *YmtZ* and *XhtZ* shall be by construction proportional to the areas *EeaA* and *QqaA* , these areas *YmtZ* and *XhtZ* by the theory of logarithms can become the logarithms of the lines *SE* and *SQ*; therefore since the same quantities shall be able to be the logarithms both of the quantities *SE* and *SQ*, as well as of the quantities *SY* and *SX*, it is required that these quantities *SE, SY* and *SQ, SX* will occupy corresponding places in the geometric progressions to which they belong.]

PROPOSITION XXII. THEOREM XVII.

Let the density of a certain fluid be proportional to the compression, and the parts of this may be drawn downwards by a weight inversely proportional to the squares of the distances from the centre: I say that, if the distances are taken in a harmonic progression, the density of the fluid at these distances are in a geometric progression.

Let *S* designate the centre, and *SA, SB, SC, SD, SE* the distances in a geometric progression. The perpendiculars *AH, BI, CK*, etc. may be erected which shall be as the densities of the fluid at the places *A, B, C, D, E*, etc. and the specific gravities of this at the same places will be as $\frac{AH}{SA^2}, \frac{BI}{SB^2}, \frac{CK}{SC^2}$, &c. Suppose these gravities to continue uniformly, the first from *A* to *B*, the second from *B* to *C*, the third from *C* to *D*, etc. And these multiplied by the heights *AB, BC, CD, DE*, etc. or, as it is likewise, by the distances *SA, SB, SC*, etc. proportional to these heights, make the exponents of the pressure



$\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC}$ etc. Whereby since the densities shall be as the sum of these pressures, the differences of the densities $AH - BI$, $BI - CK$, etc. will be as the sum of the differences $\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC}$, etc. With centre *S*, with the asymptotes *SA* and *Sx* some hyperbola may be described, which will cut the perpendiculars *AH, BI, CK,* etc. at *a, b, c,* etc. and so also the perpendiculars sent to the asymptote *Sx*, *Ht, Iu, Kw* in *h, i, k*; and the differences of the densities *tu, uw*, etc. will be as $\frac{AH}{SA}, \frac{BI}{SB}$, etc. And the rectangles $tu \times th, uw \times ui$, etc., or *tp, uq*, etc. as $\frac{AH \times th}{SA}, \frac{BI \times ui}{SB}$, etc. That is , as *Aa, Bb*, etc. For there is, from the nature of the hyperbola, *SA* to *AH* or *St*, as *th* to *Aa*, and thus $\frac{AH \times th}{SA}$ is equal to *Aa*. And by similar reasoning, $\frac{BI \times ui}{SB}$ is equal to *Bb*, etc. Moreover *Aa, Bb, Cc*, etc. are in continued proportion, and therefore proportional to the differences of these : $Aa - Bb$, $Bb - Cc$, etc.; and thus the rectangles *tp, uq*, etc. are also proportional to these differences, and as with the sums of the differences $Aa - Cc$ or $Aa - Dd$, the sums of the rectangles $tp + uq$ or $tp + uq + wr$. Let there be a number of terms of this kind, and the sum of all the differences, such as $Aa - Ff$, will be proportional to the sum of all the rectangles, such as *zthn*. The number of terms may be increased and the distances between the points *A, B, C*, etc. diminished indefinitely, and these rectangles emerge equal to the area of the hyperbola *zthn*, and thus the difference $Aa - Ff$ is proportional to this area. Now there may be assumed some distances, such as *SA, SD, SF* in a harmonic progression, and the differences $Aa - Dd$, $Dd - Ff$ will be equal; and therefore with these differences proportional to the area , *thlx* and *xlnz* will be equal to each other, and the densities *St, Sx, Sz*, that is, *AH, DL, FN*, continued proportionals. *Q.E.D.*

Corol. Hence if some two densities of the fluid may be given, for example *AH* and *BI*, the area *thiu* will be given, corresponding to the difference *tu* of these ; and thence the density *FN* may be found at some height *SF*, on taking the area *thnz* to that given area *thiu*, the difference $Aa - Ff$ is to the difference $Aa - Bb$.

[L. & J. Note : Truly since the area *thiu* is to the area *thnz* as the logarithm of the line *St* or *AH* to the logarithm of the line *Sz* or *FN*, the density *FN* can be found from a table of logarithms. And conversely, with the density *FN* given, the height *SF* may be found : for by the above proposition, $Aa - Ff$ will be given, and thence *Ff* will be given, from which $FS = \frac{SA \times An}{Ff}$.]

Scholium.

By a similar argument it is possible to prove, that if the particular gravity of a fluid be diminished in the triple ratio of the distances from the centre, and the reciprocals of the squares of the distances *SA, SB, SC*, etc. (clearly $\frac{SA^3}{SA^2}, \frac{SA^3}{SB^2}, \frac{SA^3}{SC^2}$) may be assumed in an arithmetic progression ; the densities *AH, BI, CK*, etc. will be in a geometric progression. And if the gravity may be diminished in the quadruple ratio of the distances, and the inverse cubes of the distances are taken in an arithmetic progression (such as

$\frac{SA^4}{SA^3}, \frac{SA^4}{SB^3}, \frac{SA^4}{SC^3}$, etc.) ; the densities *AH, BI, CK,* etc. will be in a geometric progression. And thus *ad infinitum*. Again if the gravity of a particular fluid shall be the same at all distances, and the distances shall be in an arithmetic progression, the densities will be in a geometric progression, as *Edmund Halley,* the most distinguished of men, has found. If the gravity shall be as the distance, and the squares of the distances shall be in an arithmetic progression, the densities will be in a geometric progression. And thus indefinitely. These thus are had themselves where with the compression of the fluid the density of condensation is as the force of compression, or just as because the distance occupied by a fluid is inversely as this force. Other laws of condensation can be devised, so that the cube of the compressing force shall be as the fourth power of the density, or the same triplicate ratio of the force with the quadruple ratio of the density. In which case, if the gravity is inversely as the square of the distance : from the centre, the density will be inversely as the cube of the distance. It may be devised that the cube of the compressing force shall be as the fifth of the density, and if gravity is inversely as the square of the distance, the density shall be inversely as the three on two ratio of the distance. [See the final *L* & *J*. notes 177 & 178 below.] It may be devised that the compressing force shall be in the duplicate ratio of the density, and gravity inversely in the duplicate ratio of the distance as the distance. It would be tedious to run through all the cases. Finally it is agreed by experiment that the density of the air shall be as the compressing force, either accurately or at least approximately : and therefore the density of the air in the earth's atmosphere is as the weight of all the incumbent air, that is, as the height of mercury in a barometer.

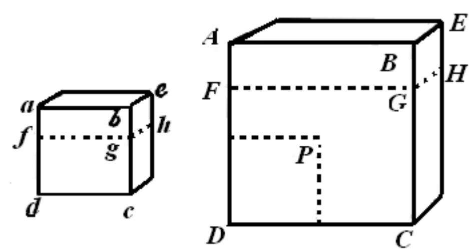
[Extended note 177 from L. & J. : Let the centripetal force of some fluid be as some power of the distance, the index of which is *n*; *S* may designate the centre, and *SA, SB, SC, CD, SE* may designate the distances in a geometric progression. The perpendiculars *AH, BI, CK*, etc., may be erected which shall be as the fluid densities at the places *A, B, C, D, E,* etc. And the specific gravities of this at the same places will be $\frac{AH}{SA^n}, \frac{BI}{SB^n}, \frac{CK}{SC^n}$, etc. Suppose these gravities to be continued uniformly first from *A* to *B*, second from *B* to *C*, third for *C* to *D*, etc. And thus multiplied by the heights *AB, BC, CD, DE*, etc., or what amounts to the same, by the distances *AS, SB, SC*, etc. , by these proportional heights, puts in place the terms of the pressure $\frac{AH}{SA^{n-1}}, \frac{BI}{SB^{n-1}}, \frac{CK}{SC^{n-1}}$, etc. Whereby since the densities shall be as the sum of these pressures, the differences of the densities $AH-BI$, $BI-CK$, etc. will be as the differences of the sums $\frac{AH}{SA^{n-1}}, \frac{BI}{SB^{n-1}}, \frac{CK}{SC^{n-1}}$, etc. made by the same constructions, as above in Prop. XXII., and the difference of the densities *tv, uw*, etc. will be as $\frac{AH}{SA^{n-1}}, \frac{BI}{SB^{n-1}}, \frac{CK}{SC^{n-1}}$, etc. and the rectangles $tv \times th, uw \times ui$, etc., or *tp, uq*, etc. as $\frac{AH \times th}{SA^{n-1}}, \frac{BI \times ui}{SB^{n-1}}$, etc., that is, as Aa^{n-1}, Bb^{n-1}, etc. For indeed, by Theorem IV by Hypothesis, $AH \times th$, equals $SA \times An$, and *Aa* inversely as *SA*, or directly as $\frac{1}{SA}$, and thus $\frac{AH \times th}{SA^{n-1}}$ as $SA \times Aa \times Aa^{n-1}$, or as Aa^{n-1} since there shall be $SA \times Aa = 1$, and by a similar argument $\frac{BI\times ui}{SB^{n-1}}$ as Bb^{n-1},etc.; but *Aa, Bb, Cc*, etc., and thus $Aa^{n-1}, Bb^{n-1}, Cc^{n-1}$,etc. are continued proportionals, and therefore with the differences of these $Aa^{n-1}-Bb^{n-1}, Bb^{n-1}-Cc^{n-1}$,etc. proportionals, and thus with these differences the proportionals are the rectangles *tp, uq*, etc. and as with the sum of the differences $Aa^{n-1}-Cc^{n-1}$,or $Aa^{n-1}-Dd^{n-1}$ the sums of the rectangles $tp+uq$, or $tp+uq+wr$. There will be several terms of this kind, and the sum of all the differences; for example $Aa^{n-1}-Ff^{n-1}$,will be the sum of all the rectangles, for example *zthn*, with the proportionals. The number of terms may be increased, and the separations of the points *A, B, C*, etc., diminished indefinitely, and these rectangles become equal to the hyperbolic areas *zthn*, and thus the difference $Aa^{n-1}-Ff^{n-1}$ is proportional to this area.

Now there may be taken powers of any distances you please, such as *SA, SD, SF*, $SA^{n-1}, SD^{n-1}, SF^{n-1}$,etc. in a harmonic progression, and thus the reciprocals of these $\frac{1}{SA^{n-1}}, \frac{1}{SD^{n-1}}, \frac{1}{SF^{n-1}}$,etc. or $Aa^{n-1}, Dd^{n-1}, Ff^{n-1}$ are in an arithmetic progression, and the differences $Aa^{n-1}-Dd^{n-1}, Dd^{n-1}-Ff^{n-1}$ will be equal; and therefore the differences from these proportional areas thlx, xlnz are equal to each other, and the densities *St, Sx, Sz*, that is, *AH, DL, FN* are continued proportionals. Whereby if the weights of the particles of fluid may be diminished in some multiple of the distances, the exponent of which shall be *n*, and of the powers $SA^{n-1}, SB^{n-1}, SC^{n-1}$,etc., the reciprocals (evidently $\frac{SA^{n}}{SA^{n-1}}, \frac{SA^{n}}{SB^{n-1}}, \frac{SA^{n}}{SC^{n-1}}$,etc., in which *SA* has been given) may be taken in an arithmetic progression; the densities *AH, BI, CK*, etc., will be in a geometric progression.

And if therefore in place of *n* there may be written the numbers 3,4,5,6, etc., indefinitely; and again there may be written 0,-1, -2, -3, etc., indefinitely, the truth of the scholii in the hypothesis of the proportional density for the compressing force. But when $n=0$, or when the gravity of a particular fluid is the same at all distances, there is $\frac{SA^{n}}{SA^{n-1}}=SA, \frac{SA^{n}}{SB^{n-1}}=SB$, and thus if the distances may be taken in an arithmetical progression, the densities will be in a geometric progression, and thus the distances are as the logarithms of the densities, because with increased distances in the arithmetical progression, the densities will decrease in a geometric progression. Because truly it is agreed by experiment, that the density of air, with all else being equal and mainly with the same degree of heat remaining, shall be as the force of compression either accurately or perhaps as an approximation in air that we are able to expose by experiments, but the force on the air being pressed below, with all else being equal, shall be equal to the weight of all the incumbent air, and thus proportional to the height of mercury in a barometer, and besides the weight of particles of air, perhaps at small distances from the surface of the earth, will be agreed upon to be constant, it is apparent that, with all else equal, the density of the air at small distances of this kind, we are able to measure with logarithms. Consult the work of Wolfe, *Elementis Aerometriae*, Book II of *Phoronomiae* by Hermann, and section 10 of *Hydrodynamicae* by Daniel Bernoulli.]

PROPOSITION XXIII. THEOREM XVIII.

If the density of a fluid composed from particles mutually repelling each other shall be as the compression, the centrifugal forces of the particles are inversely proportional to their distances from their centres. And in turn, the small particles with the forces which are inversely proportional to the distances of their centres mutually repelling each other comprise an elastic fluid, the density of which is proportional to the compression.



The fluid may be understood to be enclosed within the cubical space *ACE*, then on compression to be reduced into a smaller cubical volume *ace*; and of the particles, maintaining a similar position in each case between each other, the distances will be as the cube roots of the sides *AB*, *ab* ; and the densities of the mediums inversely as the containing volumes AB^3 and ab^3. On the side of the greater cube in the plane *ABCD*, the square *DP* may be taken equal to the side in the lesser cube *db*; and from hypothesis, the pressure, by which the square *DP* presses on the enclosed fluid, will be to the pressure, by which that square *db* presses on the enclosed fluid, as the medium densities in turn, that is, as ab^3 to AB^3. But the pressure, by which the square *DB* presses on the enclosed fluid, is to the pressure, by which *DP* likewise presses on the fluid, as the square *DB* to the square *DP*, that is, as AB^2 to ab^2. Therefore, from the equality, the pressure by which the square *DB* presses on the fluid, is to the pressure by which the square *db* presses on the fluid, as *ab* to *AB*. With the planes *FGH* and *fgh* drawn through the middles of the cubes, the fluid may be separated into two parts, and these press on each other mutually by the same forces, by which they were pressed upon by the planes *AC* and *ac*, that is, in the proportion *ab* to *AB*: and thus the centrifugal forces, by which these pressures may be sustained, are in the same ratio. On account of the same number of particles and the similar situation in each cube, the forces which all the particles exercise on the planes *FGH*, *fgh* generally, are as the forces which the individual particles exert on the individual particles. Therefore the forces, which the individuals exert on the individuals following the plane *FGH* in the larger cube, are to the forces, which the individuals exert on the individuals following the plane *fgh* in the minor cube, *ab* to *AB*, that is, inversely as the separations of the particles in turn. *Q.E.D.*

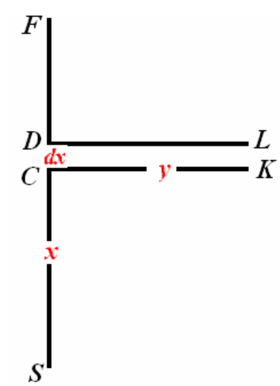
And vice versa, if the forces of the individual particles are inversely as the separations, that is, inversely as the cube root of the sides *AB*, *ab* ; the sum of the forces will be in the same ratio, and the pressures of the sides *DB*, *db* will be as the sum of the forces ; and the pressure on the square *DP* will be to the pressure of the side *DB* as ab^2 to AB^2. And, from the equation, the pressure of the square *DP* to the pressure of the side *db* as ab^3 to AB^3, that is, the force of compression to the force of compression as the density to the density. *Q.E.D.*

[*L*. & *J*. Note : *D* and *d* shall be the separations of the particles in the volumes of the cubes *ACE* and *ace* which are as *AB* and *ab*, the centrifugal forces of the same inversely as D^n and d^n, the densities of the fluids *E* and *e*, and the compressing forces shall be as $E^{\frac{n+2}{3}}$ and $e^{\frac{n+2}{3}}$.

For since the sum of the forces which all the forces exert at the same time on the sides *DB* and *db* shall be as the forces of the individual particles, the sum of these forces will be inversely as D^n and d^n, or as ab^n and AB^n directly; and the pressure of the square *DP* to the pressure of the square *db* , as ab^2 to AB^2 ; from which equation the pressure of the square *DP* to the pressure of the square *db*, that is, the compressing force in the space *ACE* to the compressing force in the space *ace*, as ab^{n+2} to AB^{n+2}. But the densities, either are as *E* to *e*, or as ab^3 to AB^3, and thus $E^{\frac{n+2}{3}}$ to $e^{\frac{n+2}{3}}$ as ab^{n+2} to AB^{n+2}. Whereby the compressing forces are as $E^{\frac{n+2}{3}}$ and $e^{\frac{n+2}{3}}$. And conversely, etc.]

Scholium.

By a similar argument, if the centrifugal forces of the particles shall be in the inverse square ratio of the distances between the centres, the cubes of the compressing forces are as the fourth powers of the densities. If the centrifugal forces shall be in the triplicate or quadruplicate ratio of the distances, the cubes of the compressing forces will be as the squares cubes or the cubed cubes of the densities. And generally, if *D* may be put for the distance, and *E* for the density of the compressed fluid, and the centrifugal forces shall be inversely as some power of the distance *D*, the index of which is the number *n*, the compressing forces will be as the cube roots of the power E^{n+2}, the index of which is $n+2$: and conversely. Truly all these are to be understood concerning the centrifugal forces of the particles which are restricted to nearby particles, but do not spread out far beyond. We have an example with magnetic bodies. The attractive force of these may be restricted almost to nearby bodies of the same kind close to themselves. The magnetic force may be drawn together by interposing a sheet of iron, and may be almost be restricted to the iron. For bodies beyond are not attracted by the magnet as much as the iron. In the same manner, if particles repel other particles of the same kind close to themselves, but exert no force on more remote particles, fluids may be composed from particles of this kind as has been acted on in this proposition. Because if the force of particles of this kind may be propagated to infinity, there would be a need for a greater force to equal the condensation of a greater quantity of fluid. Or indeed the physical question is, that it may be agreed that an elastic fluid may consist of a particles mutually repelling each other. We have shown our property of fluids mathematically from particles of this kind agreeing mathematically, so that we may offer an opening to philosophers towards treating that question.



[Extended note 178 from *L.* & *J.* : It will be better to treat the general formula, from which the individual cases will be elicited in a more pleasing manner. Therefore with the same which have been put in place above, let the variable distance $SC = x$, the height $CD = dx$, the density $CK = y$, the total compressing force at the place $C = v$, the force of gravity at the same place $= g$; and the specific gravity at the same place will be as *gy*, and with this multiplied by the vanishing height *CD* or *dx* there may be prepared the moment of the pressure $gydx = -dv$. But take the fluxion to be negative, because by increasing the distance *x*, the incumbent weight *v* may decrease.

Let the gravity *g* be as $\frac{1}{x^m}$, the density *y* as the compressing force to the power v^n, and thus $y^{\frac{1}{n}}$ as v, and on taking the fluxions $\frac{1}{n}y^{\frac{1-n}{n}}dy$ as dv. In place of *g* and *dv* these values may be substituted into the equation $gydx = -dv$, and there arises $\frac{ydx}{x^m} = -\frac{1}{n}y^{\frac{1-n}{n}}dy$ or $\frac{dx}{x^m} = -\frac{1}{n}y^{\frac{1-2n}{n}}dy$. Truly in these equations we have set out no equalities, but only proportionalities, and thus we have ignored the given coefficients.

If there is put $n = 1$ in the final equation, that is, the density of the proportional compressing force, will be $\frac{dy}{y} = -\frac{dx}{x^m}$. The quantities $\frac{1}{x^{m-1}}$ may be taken in arithmetic progression; and the fluxions of these, or the differentials arising $-\frac{(m-1)dx}{x^m}$, and thus $-\frac{dx}{x^m}$ also will be constants, and therefore the quantities $\frac{dy}{y}$ also given; and hence the proportional densities *y* from its differentials *dy*, will be continued proportionals, by Lemma II, Book II. If under the same hypothesis there may be put $m = 1$, there shall be $\frac{dy}{y} = -\frac{dx}{x}$; hence if there may be taken the constant quantities $\frac{dx}{x}$, or the distances *x* in a geometric progression, the quantities $\frac{dy}{y}$ will also be constant, and thus the densities *y* are in a geometric progression. In short as has been demonstrated in Prop. XXI and Prop. XXII, and the first scholium of this section. On taking the fluents, the equation $\frac{1}{n}y^{\frac{1-2n}{n}}dy = -\frac{dx}{x^m}$ will become $\frac{1}{1-n}y^{\frac{1-n}{n}} = -\frac{1}{m-1}x^{1-m} + Q$, in which it is apparent that that there cannot be $m = 1$; $n = 1$; or $n = 0$, and *Q* is a constant. But in order that the value of the constant *Q* may be determined, initially the height *SF* must be defined, where the density y vanishes; Now if that height is finite and it may be said to be $= a$, on putting $y = 0$, there will be had $Q = -\frac{1}{m-1}a^{1-m}$, and hence $\frac{1}{1-n} \times y^{\frac{1-n}{n}} = -\frac{x^{1-m}-a^{1-m}}{m-1} = \frac{a^{1-m}-x^{1-m}}{m-1}$, in which equation $\frac{1-n}{n}$ must be a positive number less than one, so that with increasing distances *x*, the densities *y* decrease, and conversely. If the height *SF* at which the density *y* vanishes may be supposed to be infinite, the constant *Q* will be equal to zero, and hence the equation $\frac{1}{1-n}y^{\frac{1-n}{n}} = \frac{1}{m-1}x^{1-m}$. For if in the equation $\frac{1}{1-n} \times y^{\frac{1-n}{n}} = \frac{x^{1-m}-a^{1-m}}{m-1}$, *y* may be put equal to zero and *x* becomes infinite, the constant quantity *a* will be infinite, contrary to the hypothesis.

Now indeed if gravity is inversely as the square of the distance, that is if $m = 2$, the equation $\frac{1}{1-n}y^{\frac{1-n}{n}} = \frac{1}{m-1}x^{1-m}$ changes into $\frac{1}{1-n}y^{\frac{1-n}{n}} = \frac{1}{x}$, from which $y \propto \frac{1}{x^{\frac{n}{1-n}}}$. It may be supposed that the cube of the compressing force shall be as the quadruple power of the density, or $y^4 \propto v^3$ and thus $y \propto v^{\frac{3}{4}}$, and hence $n = \frac{3}{4}$, and there will be $y \propto \frac{1}{x^{\frac{n}{1-n}}} = \frac{1}{x^3}$ [there is a misprint here in the L. & *J*. text], and hence the density *y* varies inversely as x^3, or the density varies inversely as the cube of the distance. Again, the cases mentioned by Newton, $y^5 \propto v^3$ and thus $y \propto v^{\frac{3}{5}}$, and hence $n = \frac{3}{5}$, and there will be $y \propto \frac{1}{x^{\frac{3}{2}}}$; again, $y^2 \propto v$ and thus $y \propto v^{\frac{1}{2}}$, and hence $n = \frac{1}{2}$, gives *y* inversely proportional to *x*. See *Monumenta Academiae Regiae Scientiarum* (1716), where P. Varignon has treated this material.]

SECTIO V.

De densitate & compressione fluidorum, deque hydrostatica.

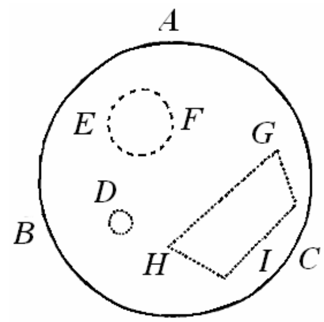
Definitio Fluidi.

Fluidum est corpus omne, cuius partes cedunt vi cuicunque illatae, ac cedendo facile moventur inter se.

PROPOSITIO XIX. THEOREMA XIV.

Fluidi homogenei & immoti, quod in vase quocunque immoto clauditur & undique comprimitur, partes omnes (seposita condensationis, gravitatis, & virium omnium centripetarum consideratione) aequaliter premuntur undique, & sine omni motu a pressione illa orto permanent in locis suis

Cas. I. In vase sphaerico *ABC* claudatur & uniformiter comprimatur fluidum undique: dico quod eiusdem pars nulla ex illa pressione movebitur. Nam si pars aliqua *D* moveatur, necesse est ut omnes huiusmodi partes, ad eandem a centro distantiam undique consistentes, simili motu simul moveantur; atque hoc ideo quia similis & aequalis est omnium pressio, & motus omnis exclusus supponitur, nisi qui a pressione illa oriatur. Atqui non possunt omnes ad centrum propius accedere, nisi fluidum ad centrum condensetur ; contra hypothesin. Non possunt longius ab eo recedere, nisi fluidum ad circumferentiam condensetur; etiam contra hypothesin. Non possunt servata sua a centro distantia moveri in plagam quamcunque, quia pari ratione movebuntur in plagam contrariam , in plagas autem contrarias non potest pars eadem, eodem tempore, moveri. Ergo fluidi pars nulla de loco suo movebitur.



Q.E.D.

Cas: 2. Dico iam, quod fluidi huius partes omnes sphaericae aequaliter premuntur undique. Sit enim *EF* pars sphaerica fluidi, & si haec undique non premitur aequaliter, augeatur pressio minor, utque dum ipsa undique prematur aequaliter; & partes eius, per casum primum, permanebunt in locis suis. Sed ante auctam pressionem permanebunt in locis suis, per casum eundem primum, & additione pressionis novae movebuntur de locis suis, per definitionem fluidi. Quae duo repugnant. Ergo falso dicebatur quod sphaera *EF* non undique premebatur aequaliter. *Q.E.D.*

Cas: 3. Dico praeterea quod diversarum partium sphaericarum aequalis sit pressio. Nam partes sphaericae contiguae se mutuo premunt aequaliter in puncto contactus, per motus legem III. Sed &, per casum secundum, undique premuntur eadem vi. Partes igitur duae quavis sphaericae non contiguae, quia pars sphaerica intermedia tangere potest utramque, prementur eadem vi. *Q.E.D.*

Cas: 4. Dico iam quod fluidi partes omnes ubique premuntur aequaliter. Nam partes duae quaevis tangi possunt a partibus sphaericis in punctis quibuscunque, & ibi partes illas sphericas aequaliter : premunt, per casum 3. & vicissim ab illis aequaliter premuntur, per motus legem tertiam. *Q.E.D.*

Cas: 5. Cum igitur fluidi pars quaelibet *GHI* in fluido reliquo tanquam in vase claudatur, & undique prematur aequaliter, partes autem eius se mutuo aequaliter premant & quiescant inter se, manifestum est quod fluidi cuiuscunque *GHI*, quod undique premimur, aequaliter, partes omnes se mutuo premunt aequaliter, & quiescunt inter se. *Q.E.D.*

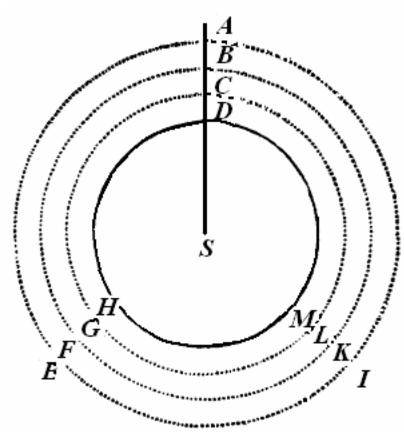
Cas: 6. Igitur si fluidum illud in vase non rigido claudatur, & undique non prematur aequaliter; cedet idem pressioni fortiori, per definitionem fluiditatis.

Cas: 7. Ideoque in vase rigido fluidum non sustinebit pressionem fortiorem ex uno latere quam ex alio, sed eidem cedet, idque in momenta temporis, quia latus vasis rigidum non persequitur liquorem cedentem. Cedendo autem urgebit latus oppositum, & sic pressio undique ad aequalitatem verget. Et quoniam fluidum, quam primum a parte magis pressa recedere conatur, inhibetur per resistentiam vasis ad latus oppositum , reducetur pressio undique ad aequalitatem, in momenta temporis, sine motu locali: & subinde partes fluidi, per casum quintum, se mutuo prement aequaliter, & quiescent inter se. *Q.E.D.*

Corol. Unde nec motus partium fluidi inter se, per pressionem fluido ubivis in externa superficie illatam, mutari possunt, nisi quatenus aut figura superficiei alicubi mutatur, aut omnes fluidi partes intensius vel remissius sese premendo difficilius vel facilius labuntur inter se.

PROPOSITIO XX. THEOREMA XV.

Si fluidi sphaerici, & in aequalibus a centro distantiis homogenei, fundo sphaerico concentrico incumbentis partes singulae versus centrum totius gravitent; sustinet fundum pondus cylindri, cuius basis aequalis est superficiei fundi, & altitudo eadem quae fluidi incumbentis.



Sit DHM superficies fundi, & *AEI* superficies superior fluidi. Superficiebus sphericis innumeris *BFK*, *CGL* distinguatur fluidum in orbes concentricos aequaliter crassos; & concipe vim gravitatis agere solummodo in superficiem superiorem orbis cuiusque, & aequales esse actiones in aequales partes superficierum omnium. Premitur ergo superficies.suprema *AE* vi simplici gravitatis propriae, qua & omnes orbis supremi partes & superficies secunda *BFK* (per prop. XIX.) pro mensura sua aequaliter premuntur. Premitur preterea superficies secunda *BFK* vi propria gravitatis, quae addita vi priori facit pressionem duplam. Hac pressione, pro mensura sua, & insuper vi propria gravitatis, id est, pressione tripla, urgetur superficies tertia *CGL*. Et similiter pressione quadrupla urgetur superficies quarta, quintupla quinta, & sic deinceps. Pressio igitur qua superficies unaquaeque urgetur, non est ut quantitas solida Fluidi incumbentis, sed ut numerus orbium ad usque summitatem fluidi , & aequatur gravitati orbis infimi multiplicatae per numerum orbium: hoc est, gravitati solidi cuius ultima ratio ad cylindrum praefinitum (si modo orbium augeatur numerus & minuatur crassitudo in infinitum, sic ut actio gravitatis a superficie infima ad supremam continua reddatur) fiet ratio aequalitatis. Sustinet ergo superficies infima pondus cylindri praefiniti. *Q.E.D.* Et simili argumentatione patet propositio, ubi gravitas decrescit in ratione quavis assignata distantiae a centro, ut & ubi fluidum sursum rarius est, deorsum densius. *Q.E.D.*

Corol. 1. Igitur fundum non urgetur a toto fluidi incumbentis pondere, sed eam solummodo ponderis partem sustinet que in propositione describitur; pondere reliquo a fluidi figura fornicata sustentato.

Corol. 2. In aequalibus autem a centro distantiis eadem semper est pressionis quantitas, sive superficies pressa sit horizonti parallela vel perpendicularis vel obliqua , sive fluidum, a superficie pressa sursum continuatum, surgat perpendiculariter secundum lineam rectam, vel serpit oblique per tortas cavitates & canales, easque regulares vel maxime irregulares, amplas vel angustissimas. Hisce circumstantiis pressionem nil mutari colligitur, applicando demonstrationem theorematis huius ad casus singulos fluidorum.

Corol. 3. Eadem demonstratione colligitur etiam (per prop. XIX.) quod fluidi gravis partes nullum, ex pressione ponderis incumbentis, acquisunt motum inter se; si modo excludatur motus qui ex condensatione oriatur.

Corol. 4. Et propterea si aliud eiusdem gravitatis specificae corpus, quod sit condensationis expers, submergatur in hoc fluido, id ex pressione ponderis incumbentis nullum acquiret motum: non descendet, non ascendet, non cogetur figuram suam mutare, Si sphaericum est manebit sphericum, non obstante pressione; si quadratum est manebit quadratum; idque sive molle sit, sive fluidissimum; sive fluido libere innatet, sive fundo incumbat. Habet enim fluidi pars quaelibet interna rationem corporis submersi, & par est rario omnium eiusdem magnitudinis, figurae & gravitatis specificae submersorum corporum. Si corpus submersum servato pondere liquesceret & indueret formam fluidi; hoc, si prius ascenderet vel descenderet vel ex pressione figuram novam induret, etiam nunc ascenderet vel descenderet vel figuram novam induere cogeretur : id adeo quia gravitas eius caeteraeque motuum causae permanent. Atqui (per cas.5. Prop. XIX.) iam quiesceret & figuram retineret. Ergo & prius.

Corol. 5. Proinde corpus quod specifice gravius est quam fluidum sibi contiguum subsidebit, & quod specifice levius est ascendet, motumque & figurae mutationem consequetur, quantum excessus ille vel defectus gravitatis efficere possit. Namque excessus ille vel defectus rationem habet impulsus, quo corpus, alias in aequilibrio cum fluidi partibus constitutum, urgetur , & comparari potest cum excessu vel defectu ponderis in lance alterutra librae.

Corol. 6. Corporum igitur in fluidis constitutorum duplex est gravitas : altera vera & absoluta, altera apparens, vulgaris & comparativa. Gravitas absoluta est vis tota qua corpus deorsum tendit : relativa & vulgaris est excessus gravitatis quo corpus magis tendit deorsum quam fluidum ambiens. Prioris generis gravitate partes fluidorum & corporum omnium gravitant in locis suis: ideoque coniunctis ponderibus componunt pondus totius, Nam totum omne grave est, ut in vasis liquorum plenis experiri licet , & pondus totius aequale est ponderibus omnium partium, ideoque ex iisdem componitur. Alterius generis gravitate corpora non gravitant in locis suis, id est, inter se collata non praegravant, sed mutuos ad descendendum conatus impedientia permanent in locis suis, perinde ac si gravia non essent. Quae in aere sunt & non praegravant, vulgus gravia non iudicat. Quae praegravant vulgus gravia iudicat, quatenus ab aeris pondere non sustinentur. Pondera vulgi nihil aliud sunt quam excessus verorum ponderum supra pondus aeris. Unde & vulgo dicuntur levia, quae sunt minus gravia, aerique praegravanti cedendo superiora petunt. Comparative levia sunt, non vere, quia descendunt in vacuo. Sic & in aqua corpora, quae ob maiorem vel minorem gravitatem descendunt vel ascendunt, sunt comparative & apparentei gravia vel levia, & eorum gravitas vel levitas comparativa & apparens est excessus vel defectus quo vera eorum gravitas vel superat gravitatem aquae vel ab ea superatur. Quae vero nec pregravando descendunt, nec praegravanti cedendo ascendunt, etiamsi veris suis ponderibus adaugeant pondus totius, comparative tamen & in sensu vulgi non gravitant in aqua. Nam similis est horum casuum demonstratio.

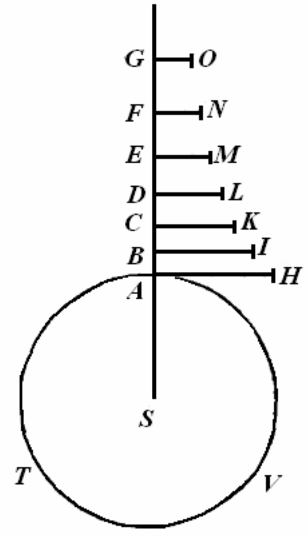
*Corol.*7 Quae de gravitate demonstrantur, obtinent in aliis quibuscunque viribus centripetis.

Corol.8 Proinde si medium, in quo corpus aliquod movetur, urgeatur vel a gravitate propria, vel ab alia quacunque vi centripeta, & corpus ab eadem vi urgeatur fortius, differentia virium est vis illa motrix, quam in praecedentibus propositionibus ut vim centripetam consideravimus. Sin corpus a vi illa urgeatur levius, differentia virium pro vi centrifuga haberi debet.

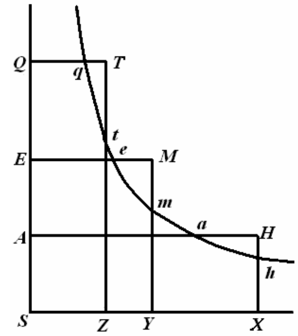
Corol. 9 Cum autem fluida premendo corpora inclusa non mutent eorum figuras externas, patet insuper (per corollarium Prop.XIX) quod non mutabunt situm partium internarum inter se : proindeque, si animalia immergantur, & sensatio omnis a motu partium oriatur ; nec laedent corpora immersa, nec sensationem ullam excitabunt, nisi quatenus haec corpora a compressione condensari possunt. Et par est ratio cuiuscunque corporum systematis fluido comprimente circundati. Systematis partes omnes iisdem agitabuntur motibus, ac si vacuo constituerentur, ac solam retinerent gravitatem suam comparativam, nisi quatenus fluidum vel motibus earum nonnihil resistat, vel ad easdem compressione conglutinandas requiratur.

PROPOSITIO XXI. THEOREMA XVI.

Sit fluidi cuiusdam densitas compressioni proportionalis, & partes eius a vi centripeta distantiis suis a centro reciproce proportionali deorsum trahantur : dico quod, si distantiae sumantur continue proportionales, densitates fluidi in iisdem distantiis erunt etiam continue proportionales.



Designet *ATV* fundum sphaericum cui fluidum incumbit, *S* centrum, *SA, SB, SC, SD, SE, SF,* &c. distantias continue proportionales. Erigantur perpendicula *AH, BI, CK, DL, EM, FN,* &c. qui sint ut densitates medii in locis *A, B, C, D, E, F*; & specificae gravitates in iisdem locus erunt ut $\frac{AH}{AS}, \frac{BI}{BS}, \frac{CK}{CS}$, &c. vel, quod perinde est, ut $\frac{AH}{AB}, \frac{BI}{BC}, \frac{CK}{CD}$, &c. Finge primum has gravitates uniformiter continuari ab *A* ad *B*, a *B* ad *C*, a *C* ad *D*, &c. satis per gradus decrementis in punctis *B, C, D*, &c. Et hae gravitates ductae in altitudines *AB, BC, CD*, &c. conficient pressiones *AH, BI, CK*, &c. quibus fundum *ATV* (iuxta theorema XV.) urgetur. Sustinet ergo particula *A* pressiones omnes *AB, BI, CK, DL*, pergendo in infinitum , & particula *B* ac pressiones omnes praeter primam *AH*; & particula *C* omnes praeter duas primas *AB, BI*;& sic deinceps : ideoque particulae primae *A* densitas *AH* est ad particulae secundae *B* densitatem ut summa omnium $AH+BI+CK+DL$, in infinitum, ad summam omnium $BI+CK+DL$, &c. Et *BI* densitas secundae *B* est ad *CK* densitatem tertae *C*, ut summa omnium $BI+CK+DL$, &c. ad summam omnium $CK+DL$, &c. Sunt igitur summae illae differentiis suis *AH, BI, CK*, &c. proportionales, atque ideo continue proportionales (per huius Lem. I.) proindeque differentiae *AH, BI, CK*, &c. summis proportionales, sunt etiam continue proportionales. Quare cum densitates in locis *A, B, C*, &c. sint ut *AH, BI, CK*, &c. erunt etiam hae continue proportionales. Pergatur per saltum, & ex aequo in distantiis *SA, SC, SE* continue proportionalibus, erunt densitates *AH, CK, EM* continue proportionales. Et eodem argumento, in distantiis quibusvis continue proportionalibus *SA, SD, SG*, densitates *AH, DL, GO* erunt continue proportionales. Coeant iam puncta *A, B, C, D, E*, &c. eo ut progressio gravitatum specificarum a fundo *A* ad summitatem fuidi continua reddatur, & in distantiis quibusvis continue proportionalibus *SA, SD, SG*, densitates *AH, DL, GO*, semper existentes continue proportionales, manebunt etiamnum continue proportionales. *Q.E.D.*

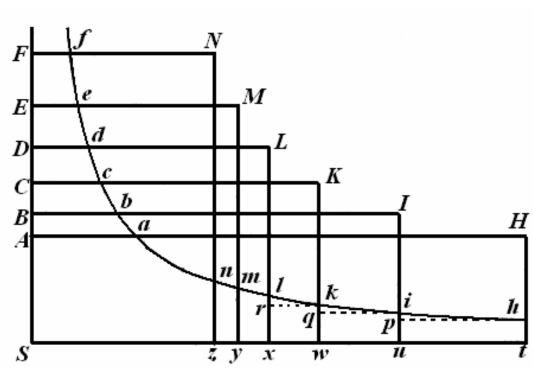


Corol. Hinc si detur densitas fluidi in duobus locis, puta *A* & *E*, colligi potest eius densitas in alia quovis loco *Q*. Centro *S*, asymptotis rectangulis *SQ, SX* describatur hyperbola secans perpendicula *AH, EM, QT* in *a, e, q*, ut & perpendicula *HX, MY, TZ*, ad asymptoton *SX* demissa, in *h, m* & *t*. Fiat area *YmtZ* ad aream datam *YmhX* ut area data *EeqQ* ad aream datam *EeaA*; & linea *Zt* producta abscindet lineam *QT* densitati proportionalem. Namque si linae *SA, SE, SQ* sunt continue proportionales, erunt areae *EeqQ, EeaA* aequales, & inde areae his proportionales *YmtZ, XhmY* etiam aequales, & linae *SX, SY, SZ*, id est, *AH, EM, QT* continue proportionales, ut oportet. Et si lineae *SA, SE, SQ* obtinent alium quemvis ordinem in serie continue proportionalium, lineae *AH, EM, QT*, ob proportionales areas hyperbolicas, obtinebunt eundem ordinem in alia serie quantitatum continue proportionalium.

PROPOSITIO XXII. THEOREMA XVII.

Sit fluidi cuiusdam densitas compressioni proportionalis, & partes eius a gravitate quadratis distantiarum suarum a centro reciproce proportionali deorsum trahantur: dico quod, si distantiae sumantur in progressione musica, densitates fluidi in his distantiis erunt in progressione geometrica.

Designet *S* centrum, & *SA, SB, SC, SD, SE* distantias in progressione geometrica. Erigantur perpendicula *AH, BI, CK*, &c. quae sint ut fluidi densitates in locis *A, B, C, D, E*, &c. & ipsius gravitates specificae in iisdem locis erunt $\frac{AH}{SAq}, \frac{BI}{SBq}, \frac{CK}{SCq}$, &c. Finge has gravitates uniformiter continuari, primam ab *A* ad *B*, secundam a *B* ad *C*, tertiam a *C* ad *D*, &c. Et hae ductae in altitudines *AB, BC, CD, DE*, &c. vel, quod perinde est, in distantias *SA, SB, SC*, &c. altitudinibus illis proportionales, conficient exponentes pressionum $\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC}$ &c. Quare cum densitates sint ut harum pressionum summae, differentia densitorum $AH-BI$, $BI-CK$, &c. erunt ut summarum differentiae $\frac{AH}{SA}, \frac{BI}{SB}, \frac{CK}{SC}$, &c. Centro *S*, asymptotis *SA, Sx* describatur hyperbola quaevis, quae secet perpendicula *AH, BI, CK*, &c. in *a, b, c*, &c. ut & perpendicula ad asymptoton *Sx* demissa *Ht, Iu, Kw* in *h, i, k*; & densitatum differentiae *tu, uw*, &c. erunt ut $\frac{AH}{SA}, \frac{BI}{SB}$, &c. Et rectangula $tu \times th, uw \times ui$, &c. seu *tp, uq*, &c. ut $\frac{AH \times th}{SA}, \frac{BI \times ui}{SB}$, &c. Id est, ut *Aa, Bb*, &c. Est enim, ex natura hyperbolae, *SA* ad *AH* vel *St*, ut *th* ad *Aa*, ideoque $\frac{AH \times th}{SA}$ aequale *Aa*. Et simili argumento est $\frac{BI \times ui}{SB}$ aequale *Bb*, &c. Sunt autem *Aa, Bb, Cc*, &c. continue proportionales, & propterea differentiis suis $Aa-Bb$, $Bb-Cc$, &c. proportionales; ideoque differentiis hisce proportionalia sunt rectangula *tp, uq*, &c. ut & summis differentiarum $Aa-Cc$ vel $Aa-Dd$ summae rectangulorum $tp+uq$ vel $tp+uq+wr$. Sunto eiusmodi termini quam plurimi, & summa omnium differentiarum, puta $Aa-Ff$, erit summae omnium rectangulorum, puta *zthn*, proportionalis. Augeatur numerus terminorum & minuantur distantia punctorum *A, B, C*, &c. in infinitum, & rectangula illa evadent aequalia areae hyperbolicae *zthn*, ideoque huic area proportionalis est differentia



$Aa-Ff$. Sumantur iam distantiae quaelibet, puta *SA, SD, SF* progressione musica, & differentiae $Aa-Dd$, $Dd-Ff$ erunt aequaeles; & propterea differentiis hisce proportionaIes areae *thlx*, *xlnz* aequales erunt inter se, & densitates *St, Sx, Sz*, id est, *AH, DL, FN*, continue proportionales. *Q.E.D.*

Corol. Hinc si dentur fluidi densitates duae quaevis, puta *AH* & *BI*, dabitur area *thiu*, harum differenrie *tu* respondens; & inde invenietur densitas *FN* in altitudine quacunque *SF*, sumendo aream *thnz* ad aream illam datam *thiu* est differentia $Aa-Ff$ ad differentiam $Aa-Bb$.

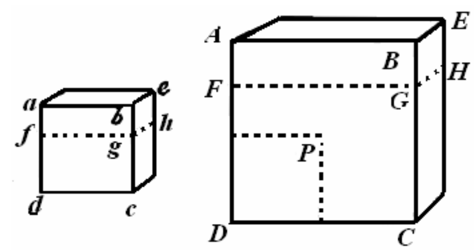
Scholium.

Simili argumentatione probari potest, quod si gravitas particularum fluidi diminuatur in tripliicata ratione distantiarum a centro, & quadratorum distantiarum *SA, SB, SC,* &c. reciproca (nempc $\frac{SAcub.}{SAq}, \frac{SAcub.}{SBq}, \frac{SAcub.}{SCq}$) sumantur in progressione arithmetica; densitates *AH, BI, CK*, &c. erunt in progressione geometrica. Et si gravitas diminuatur in quadruplicata ratione distantiarum, & cuborum distantiarum reciproca (puta $\frac{SAqq}{SAcub.}, \frac{SAqq}{SBcub.}, \frac{SAqq}{SCcub.}$, &c.) sumantur in progressione arithmetica; densitates *AH, BI, CK*, &c. erunt in progressione geometrica. Et sic in infinitum. Rursus si gravitas particularum fluidi in omnibus distantiis eadem sit, & distantiae sint in progressione arithmetica, densitates erunt in progressione geometrica, uti Vir CI. *Edmundsus Hallius* invenit. Si gravitas sit ut distantia, & quadrata distantiarum sint in progressione arithmetica, densitates erunt in progressione geometrica. Et sic in infinitum. Haec ita se habent ubi fluidi compressione condensati densitas est ut vis compressionis, vel, quod perinde est, spatium a fluido occupatum reciproce ut haec vis. Fingi possunt aliae condensatonis leges, ut quod cubus vis comprimentis sit ut quadrato-quadratum densitatis, seu triplicata ratio vis eadem cum quadruplicata ratione densitatis. Quo in casu, si gravitas est reciproce ut quadratum distantia: a centro, densitas erit reciproce ut cubus distaniae. Fingatur quod cubus vis comprimentis sit ut quadrato-cubus densitatis, & si gravitas est reciproce ut quadratum distantiae, densitas erit reciproce in sesquiplicata ratione distantiae, Fingatur quod vis comprimens sit in duplicata ratione densitatis, & gravitas reciproce in ratione duplicata distaniae, & densitas erit reciproce ut distantia. Casus omnes percurrere longum esset. Caeterum per experimenta constat quod densitas aeris sit ut vis comprimens vel accurate vel saltem quam proxime: & propterea densitas aeris in atmosphaera terrae est ut pondus aeris totius incumbentis, id est, ut altitudo mercurii in barometro.

PROPOSITIO XXIII. THEOREMA XVIII.

Si fluidi ex particulis se mutuo fugientibus compositi densitas sit ut compressio, vires centrifugae particularum sunt reciproce proportionales distantiis centrorum suorum. Et vice versa, particulae viribus quae sunt reciproce proportionales distantiis centrorum suorum se mutuo fugientes componunt fluidum elasticum, cuius densitas est compressioni proportionalis.

Includi intelligatur fluidum in spatio cubico *ACE*, dein compressione redigi in spatium cubicum minus *ace*; & particularum, similem situm inter se in utroque spatio obtinentium, distantiae erunt ut cuborum latera *AB, ab* ; & mediorum densitates reciproce ut spatia continentia *AB cub*. & *ab cub*. In cubi maioris latere plano *ABCD* capiatur quadratum *DP* aequale lateri plano cubi minoris *db*; & ex hypothesi, pressio, qua quadratum *DP* urget fluidum inclusum, erit ad pressionem, qua illud quadratum *db* urget fluidum inclusum, ut medii densitates ad invicem, hoc est, ut *ab cub*. ad *AB cub*. Sed pressio, qua quadratum *DB* urget fluidum inclusum, est ad pressionem, qua quadratum *DP* urget idem fluidum, ut quadratum *DB* ad quadratum *DP*, hoc est, ut *AB quad*. ad *ab* quad. Ergo, ex aequo, pressio qua quadratum *DB* urget fluidum, est ad pressionem qua quadratum *db* urget fluidum, ut *ab* ad *AB*. Planis *FGH, fgh*, per media cuborum ductis, distinguatur fluidum in duas partes, & hae se mutuo prement iisdem viribus,quibus premuntur a planis *AC,ac*, hoc est, in proportione a*b* ad *AB*: ideoque vires centrifuge, quibus hae pressiones sustinentur, sunt in eadem ratione. Ob eundem particularum numerum similemque situm in utroque cubo, vires quas particulae omnes secundum plana *FGH, fgh* exercent in omnes, sunt ut vires quas singulae exercent in singulas. Ergo vires, quas singulae exercent in singulas secundum planum *FGH* in cubo maiore, sunt ad vires, quas singulae exercent in singulas secundum planum *fgh* in cubo minore, ut *ab* ad *AB*, hoc est, reciproce ut distantiae particularum ad invicem. *Q.E.D*.



Et vice versa, si vires particularum singularum sunt reciproce ut distantiae, id est, reciproce ut cuborum latera *AB, ab* ; summae virium erunt in eadem ratione, & pressiones laterum *DB, db* ut summae virium; & pressio quadrati *DP* ad pressionem lateris *DB* ut *ab quad*. ad *AB quad*. Et, ex aequo, pressio quadrati *DP* ad pressionem lateris *db* ut *ab cub.* ad *AB cub*. id est, vis compressionis ad vim compressionis ut densitas ad densitatem. *Q.E.D*.

Scholium.

Simili argumento, si particularum vires centrifugae sint reciproce in duplicata ratione distantiarum inter centra, cubi virium comprimentium erunt ut quadrato-quadrata densitorum. Si vires centrifugae sint reciproce in triplicata vel quadruplicata ratione distantiarum, cubi virium comprimentium erunt ut quadrato-cubi vel cubo-cubi densitorum. Et universaliter, si *D* ponatur pro distantia,& *E* pro densitate fluidi compressi, & vires centrifugae sint reciproce ut distantiae dignitas quaelibet *D*, cuius index est numerus n, vires comprimentes erunt ut latera cubica dignitatis E^{n+2}, cuius index est numerus $n+2$: & contra. Intelligenda vero sunt haec omnia de particularum viribus centrifugis quae terminantur in particulis proximis, aut non longe ultra diffunduntur, Exemplum habemus in corporibus magneticis. Horum virtus attractiva terminatur fere in sui generis corporibus sibi proximus. Magnetis virtus per interpositam laminam ferri contrahitur, & in lamina sere terminatur. Nam corpora ulteriora non tam a magnete quam a lamina trahuntur. Ad eundem modum si particulae fugant alias sui generis particulas sibi proximas, in particulas autem remotiores virtutem nullam exerceant, ex huiusmodi particulis componentur fluida de quibus actum est in hac propositione. Quod si particulae cuiusque virtus in infinitum propagetur, opus erit vi maiori ad equalem condensationem maioris quantitatis fluidi. An vero fluida elastica ex particulis se mutuo fugantibus constent, quaestio physica est. Nos proprietatem fluidorum ex eiusmodi particulis constantium mathematice demonstravimus, ut philosophis ansam praebeamus quaestionem illam tractandi.