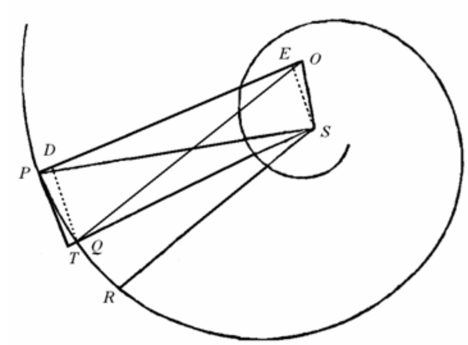
SECTION IV.

Concerning the circular motion of bodies in resisting media.

LEMMA III.

Let PQR be a spiral which may cut all the radii SP, SQ, SR, &c. at equal angles. The right line PT may be drawn touching the same at some point P, and may cut the radius SQ at T; and with the perpendiculars PO and QO erected concurrent at O, SO may be joined. I say that if the points P & Q approach each other in turn an coincide, the angle PSO becomes right, and the final ratio of the rectangle $TQ\times 2PS$ *to* PQ^2 *will be the ratio of equality*.



For indeed from the right angles *OPQ* and *OQR* the equal angles *SPQ* and *SQR* may be taken away, and the equal angles *OPS* and *OQS* remain. Therefore the circle which passes through the points *O*, *S*, *P* will also pass through the point *Q*. The points *P* and *Q* may coalesce and this circle at the point of coalescing *PQ* will touch the spiral, and thus it will cut the right line *OP* perpendicularly. Therefore *OP* becomes the diameter of this circle, and the angle *OSP* the right angle in the semicircle. Q.E.D.

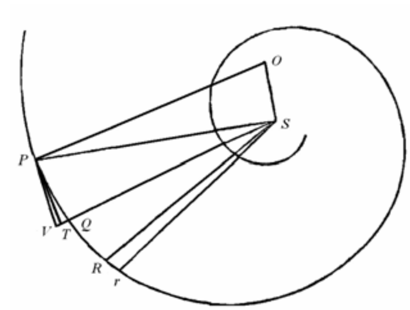
The perpendiculars *QD*, *SE* may be sent to *OP*, and the ultimate ratios of the lines will be of this kind: $\frac{TQ}{PD}=\frac{TS}{PE}=\frac{PS}{PE}$, or $\frac{2PO}{2PS}$; likewise $\frac{PD}{PQ}=\frac{PQ}{2PO}$; and from the disturbed equation, $\frac{TQ}{PQ}=\frac{PQ}{2PS}$. From which PQ^2 shall be equal to $TQ\times 2PS$. *Q.E.D.*

[Note from *L*. & *J*. : Because the lines *PT*, *DQ*, *ES* normal to *PO* are parallel, by Prop. X, Book VI, Euclid's *Elements* , there will be $\frac{TQ}{PD}=\frac{TS}{PE}=\frac{PS}{PE}$, and on account of the similar triangles *PSO*, *PES*, $\frac{PS}{PE}=\frac{PO}{PS}=\frac{2PO}{2PS}$ and thus $\frac{TQ}{PD}=\frac{2PO}{2PS}$. Now because the radii *OP* and *OQ* are perpendicular to the vanishing arc *PQ*, the point *O* is the centre, *PO* the radius, and 2*PO* the diameter of the osculating circle to the spiral at *P*. *PQ* the arc or chord of this circle, and thus the abscissa *PD* is to the chord *PQ* as *PQ* to the diameter 2*PO*. Whereby the result follows.]

PROPOSITION XV. THEOREM XII.

If the density of the medium at individual places shall be reciprocally as the distances of the places from the motionless centre, and the centripetal force shall be as the square of the density: I say that the body can rotate in a spiral, all the radii of which drawn from that centre may intersect at a given angle.

Everything may be put in place as in the above lemma, and *SQ* may be produced to *V*, so that *SV* may be equal to *SP*. At any time, in the resisting medium, the body may describe that minimum arc *PQ*, and in twice the time double that minimum arc *PR*; and the decrements of these arcs arising from the resistance, or the amounts taken from the arcs which that may be described in the same times without resistance, will in turn be as the squares of the times in which they are generated. [Thus, in modern terms, we may say that the decrease in the elemental arc length Δs due to the small resistive deceleration *a*, in the increment of the time Δt, is given by $\Delta s = \frac{1}{2}a \times \Delta t^2$ or $\Delta s \propto \Delta t^2$.] : And thus the decrement of the arc *PQ* is a quarter part of the decrement of the arc *PR*. From which also, if the area *PSQ* thus may be taken equal to an area *QSr*, the decrement of the arc *PQ* will be equal to half the decrement of the linule *Rr* [see note one below]; and thus the force of resistance and the centripetal force are in turn as the small lines $\frac{1}{2}Rr$ and TQ, which they produce at the same time. [If the incremental angle $\Delta\theta = T\hat{P}Q$ and *v* is the tangential velocity corresponding to *PT*, then in the time Δt, the tangent line or vector rotates through this angle, and the change in velocity is $v\Delta\theta$, corresponding to *TQ*, from which the centripetal acceleration is proportional to $v\frac{\Delta\theta}{\Delta t}$, towards the centre.] Because the centripetal force, by which the body is acted on at *P*, is reciprocally as SP^2 [from the Proposition] and (by Lem. X, Book I.) the line element *TQ*, which may be generated by that force, is in a ratio put together from the ratio of this force and the ratio of the time squared in which the arc *PQ* will be described (for I ignore the resistance in this case as infinitely less than the centripetal force), it follows that $TQ \times SP^2$, that is $\frac{1}{2}PQ^2 \times SP$ (by the most recent lemma), will be in the square ratio of the time, and thus the time $\Delta t \propto PQ \times \sqrt{SP}$; and the velocity of the body, by which the arc *PQ* may be described in that time, will be as $\frac{PQ}{PQ \times \sqrt{SP}}$ or $\frac{1}{\sqrt{SP}}$, that is, inversely in the square root ratio of *SP*.



[We may reiterate this as follows : $TQ \propto F_c$ and $TQ \propto \Delta t^2$ hence $TQ \propto F_c \Delta t^2$; but from the nature of the spiral as shown in the lemma above, $TQ = \frac{PQ^2}{2PS}$ and hence

$$\Delta t^2 \propto \frac{PQ^2}{F_c \times PS} \propto \frac{PQ^2 \times PS^2}{PS} = PQ^2 \times PS \text{ and } \Delta t \propto PQ \times \sqrt{PS}$$ as required.]

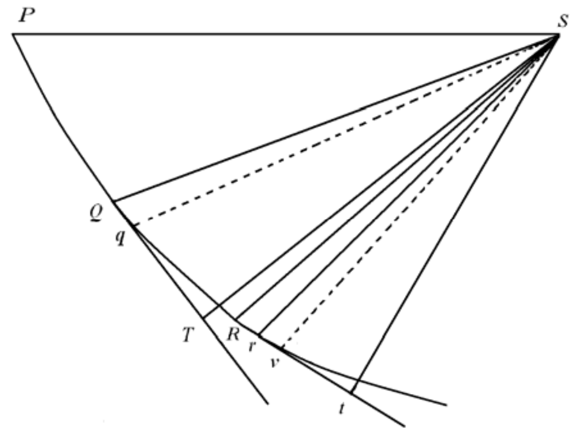
And by a like argument, the velocity by which the arc *QR* will be described is inversely as the square root ratio of *SQ* or $\frac{1}{\sqrt{SQ}}$, [for we already have the arc *PQ* described by a velocity as $\frac{1}{\sqrt{SP}}$; see note two below]. But these arcs *PQ* and *QR* are as the describing velocities in turn, that is, in the ratio $\sqrt{\frac{SQ}{SP}}$, or as $\frac{SQ}{\sqrt{SP\times SQ}}$; and on account of the equal angles SPQ and SQr [from the nature of the spiral], and the equal areas given *PSQ* and *QSr*, $\frac{\text{arc } PQ}{\text{arc } Qr}=\frac{SQ}{SP}=\frac{SQ}{\sqrt{SP\times SQ}}$. The differences of the following proportions may be taken, and there arises: $\frac{\text{arc } PQ}{\text{arc } Rr}=\frac{SQ}{SP-\sqrt{SP\times SQ}}=\frac{SQ}{\frac{1}{2}VQ}$

[Recall that $SP=SV$ by def., and $SQ=SP-VQ$ and hence $SP\times SQ=SP^2-SP\times VQ$; hence on extracting the square root, there becomes $\sqrt{SP\times SQ}=SP-\frac{1}{2}VQ-\frac{VQ^2}{8SP}-\text{etc.}$ The other terms past the second can be ignored, because with *P* and *Q* coinciding, they vanish with respect to VQ, and thus there will be $\sqrt{SP\times SQ}=SP-\frac{1}{2}VQ$, and hence $\frac{1}{2}VQ=SP-\sqrt{SP\times SQ}$, giving the ultimate ratio when *P* and *Q* coincide : $\frac{SQ}{SP-\sqrt{SP\times SQ}}\rightarrow\frac{SQ}{\frac{1}{2}VQ}$].

For with the points *P* and *Q* coinciding, the ultimate ratio is equal to $\frac{SP-\sqrt{SP\times SQ}}{\frac{1}{2}VQ}$.

Because the decrement of the arc *PQ*, arising from the resistance or from $2\times Rr$, is as the resistance and the square of the time jointly, the resistance will be as $\frac{Rr}{PQ^2\times SP}$. But there was [in the limit] $\frac{\text{arc } PQ}{\text{arc } Rr}=\frac{SQ}{\frac{1}{2}VQ}$, and thence $\frac{Rr}{PQ^2\times SP}$ shall be as $\frac{\frac{1}{2}VQ}{PQ\times SP\times SQ}$, or as $\frac{\frac{1}{2}OS}{OP\times SP^2}$. For with the points *P* and *Q* coinciding, *SP* and *SQ* coincide, and the angle PVQ shall be right ; and on account of the similar triangles *PVQ* and *PSO*, $\frac{PQ}{\frac{1}{2}VQ}=\frac{OP}{\frac{1}{2}OS}$. Therefore $\frac{OS}{OP\times SP^2}$ is to the resistance, in the ratio of the density at *P* and in the square ratio of the velocity jointly. The square ratio of the velocity may be taken, clearly the ratio $\frac{1}{SP}$, and the density of the medium at *P* will remain as $\frac{OS}{OP\times SP}$. The spiral may be given, and on account of the given ratio $\frac{OS}{OP}$, the density of the medium at *P* will be as $\frac{1}{SP}$. Therefore in a medium the density of which is inversely as the distance from the centre *SP*, a body is able to revolve on this spiral. *Q.E.D.*

[*L. & J.* note one: The body with that velocity which it may have at the place *P*, in equal times may describe as minimal the arcs *Pq*, *qv* in a non– resisting medium (the hatched lines), and the arcs *PQ*, *QR* in a medium with resistance, and from the demonstration above there will be had $4Qq=Rv$, but these areas *PSq* and *qSv* are equal, by Prop. I, Book I, and thus on account of the given equal areas *PSQ* and *QSr*, by hypothesis also $PSq-PSQ$ or the area *QSq* equals $qSv-QSr$ or $rSv-QSq$, and hence the area *rSv* is equal to 2*QSq* : but with the perpendiculars *ST* and *St* send from the centre *S* to the tangents *QT* and *rt* drawn through the points *Q* and *r*, the vanishing area *QSq* is $\frac{1}{2}ST\times Qq$, and the area *rSv* is $\frac{1}{2}St\times rv$. Whereby $ST\times Qq$ is equal to $\frac{1}{2}St\times rv$, and with the points *P* and *v* merging together, there becomes $St=ST$ and thus $Qq=\frac{1}{2}rv$, and $2Qq=rv$. Therefore since above we have found $4Qq=Rv$ there becomes $4Qq-2Qq$, or $2Qq=Rv-rv=Rr$, and thus $Qq=\frac{1}{2}Rr$. And thus in the same time in which the resistance generates the decrement *Qq*, or $\frac{1}{2}Rr$, the centripetal force, by which the body is drawn back from the tangent *PT* – see previous diagram ,as there is a difference in the labeling from the extra diagram here – to the point *Q* of the arc *PQ*, generates the decrement *TQ*, and thus the force of resistance is to the centripetal force as $\frac{1}{2}Rr$ to *TQ*, (by Cor. 4, Lem. X), and thus all these may be generally obtained, whatever were the curve *PQR*, whose properties we have not yet used, nor the centripetal force, the resistance, nor the velocity of the body.]



[*L. & J.* note two: For since from the demonstration $\frac{PQ}{SQ}=\frac{QR}{\sqrt{SP\times SQ}}$ and $\frac{PQ}{SQ}=\frac{Qr}{SP}$, there will also be $\frac{Qr}{SP}=\frac{QR}{\sqrt{SP\times SQ}}$,from which there will be $\frac{PQ}{SQ}=\frac{Qr-QR(=Rr)}{SP-\sqrt{SP\times SQ}}$ and hence $\frac{PQ}{Rr}=\frac{SQ}{SP-\sqrt{SP\times SQ}}$.]

[Brougham & Routh note using analytical methods, p. 213:

The equiangular spiral, by definition, has the property that the tangent at any point makes contact with the radius vector at the same angle always; call this angle α. Let *r* be the radius, *v* the velocity along the spiral1, the central force *P*, and kv^2 the resistance at any point on the spiral; thus, Newton's unusual density and force descriptions are not adhered to in this description to follow – we note 'unusual' in the sense 'non physical', and of course it was adopted by Newton to render his methods of analysis – being a combination of intuition, geometry, and his early form of calculus – more tractable. The equation satisfied at any point is clearly $\frac{dv}{dt}=-kv^2-P\cos\alpha$; but on all continuous plane curves, $\frac{dr}{dt}=v\cos\alpha$ and hence $\frac{dv}{dt}=\frac{dv}{dr}\frac{dr}{dt}=\frac{dv}{dr}v\cos\alpha=\frac{1}{2}\frac{dv^2}{dr}\cos\alpha$; hence the above equation becomes $\frac{dv^2}{dr}+\frac{2k}{\cos\alpha}v^2=-2P$, and this is a general equation for all plane curves. The equation giving the motion perpendicular to the arc under these circumstances is well-known in terms of the local radius of curvature *R* : $\frac{v^2}{R}=P\sin\alpha$; but for the equiangular spiral, the radius of curvature is $\frac{r}{\sin\alpha}$ (consult triangle *POS* in the first diagram above); hence we have $v^2 = Pr$; if we substitute this in above equation $\frac{dv^2}{dr} + \frac{2k}{\cos\alpha}v^2 = -2P$, we obtain $\frac{d(Pr)}{dr} + \frac{2k}{\cos\alpha}Pr = -2P$, giving $\frac{d(P)}{P} = -3\frac{dr}{r} - \frac{2kd\,r}{\cos\alpha}$ and $\ln P = C - \int\left(3 + \frac{2kr}{\cos\alpha}\right)\frac{dr}{r}$. Now, if the force varies inversely as the distance, we have $P = \frac{\mu}{r^2}$ then we have $k = -\frac{\cos\alpha}{2}\frac{1}{r} = \frac{D}{r}$; that is, the density varies inversely as the distance, and the negative sign indicates that the body is approaching the centre of force, or that the angle α is greater than a right angle.]

Corol. 1. That velocity at any place *P* is always the same, with which a body in a non-resisting medium can rotate in a circle, at the same distance from the centre *SP*.
[This also follows from $v^2 = Pr$, and when $P = \frac{\mu}{r^2}$ we have $v = \sqrt{\frac{\mu}{r}}$.]
Corol. 2. The density of the medium, if the distance *SP* may be given, is as $\frac{OS}{OP}$, if that distance is not given, as $\frac{OS}{OP\times SP}$: From thence the spiral can be adapted to any density of the medium.
[This also follows from $k = -\frac{\cos\alpha}{2}\frac{1}{r} = \frac{D}{r}$, where $D = -\frac{\cos\alpha}{2}$. For, if *k* and *r* are given, then α can be found.]

Corol. 3. The force of resistance at any place *P*, is to the centripetal force at the same place as $\frac{1}{2}OS$ to *OP*. For these forces are in turn as $\frac{1}{2}Rr$ and *TQ* or as $\frac{\frac{1}{4}VQ\times PQ}{SQ}$ and $\frac{\frac{1}{2}PQ^2}{SP}$, that is, as $\frac{1}{2}VQ$ and *PQ*, or $\frac{1}{2}OS$ and *OP*. Therefore for a given spiral the proportion of the resistance to the centripetal force is given, and in turn from that given proportion the spiral is given.

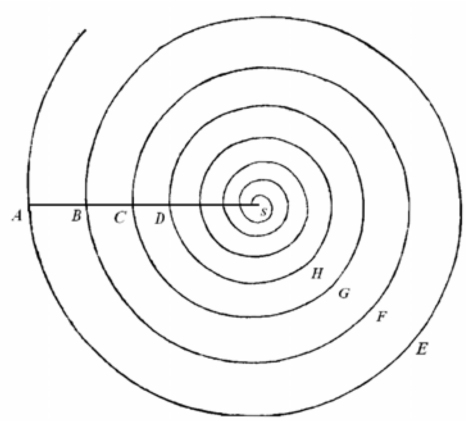
Corol. 4. And thus the body is unable to rotate in a spiral except when the force of resistance is less than half of the centripetal force. Make the resistance equal to half of the centripetal force, and the spiral agrees with the right line *PS*, and in accordance with this the body may fall along the right line to the centre with that velocity, which shall be to the velocity, as we have proven in the above in the parabolic case (Theorem X, Book I) able to fall in a non resisting medium, in the square root ratio of one to two. And here the descent times will be inversely as the velocities, and thus may be given.

[This also follows from $D = -\frac{\cos\alpha}{2}$, where we observe that the magnitude $D \leq \frac{1}{2}$; when the angle α is zero, the spiral degenerates into the right line *SP* .

Corol. 5. And because at equal distances from the centre the velocity is the same in the spiral *PQR* and along the right line *SP*, and the length of the spiral to the length of the right line *PS* is in a given ratio, evidently in the ratio *OP* to *OS* ; the descent time in the spiral will be in the same given ratio to the descent time along the line *SP*, and hence is given.

Corol. 6. If from the centre S with any two given radii, two circles may be described ; and with these two circles remaining, the angle that the spiral maintains with the radius *PS* may be changed in some manner : the number of revolutions that the body can complete within the circumferences of the circles, by going along the spiral from circumference to circumference, is as $\frac{PS}{OS}$ or as the tangent of that angle that the spiral maintains with the radius *PS*; truly the time of the same revolutions is as $\frac{OP}{OS}$, that is, as the secant of the same angle, or also inversely as the density of the medium.

[For if $\frac{dr}{dt} = v\cos\alpha = \sqrt{\frac{\mu}{r}}.\cos\alpha$, then $dt = \frac{dr}{v\cos\alpha} = \frac{dr\sqrt{r}}{\sqrt{\mu}\cos\alpha}$, and thus the time to go from a distance r_1 to a distance r_2 is given by : $T = \frac{2}{3\sqrt{\mu}\cos\alpha}\left(r_2^{\frac{3}{2}} - r_1^{\frac{3}{2}}\right)$; if the angle α is almost right, then the time is very long, and if $\alpha = 0$, then the same formula T' gives the time of descent to any part of the radius vector ; hence we may write $T = \frac{T'}{\cos\alpha}$; and the number of revolutions also may be found, for if θ be the angle the radius r makes with some fixed line, then $d\theta = \frac{dr}{r}\tan\alpha$, which can be found from the first diagram above in triangle *PQV*, where $dr = TV$, etc., hence $\theta_2 - \theta_1 = ln\frac{r_2}{r_1}\tan\alpha$, and the number of revolutions will be $ln\frac{r_2}{r_1}\frac{\tan\alpha}{2\pi}$; this includes corollaries 5 & 6.]



Corol. 7. If a body in a medium, the density of which is inversely as the distance of the locations from the centre, has made a revolution along some curve *AEB* about the centre, and may have cut the first radius *AS* at *A* and with the same angle at *B* as the first, and with a velocity which was inversely in the square root ratio of its distance from the centre to its first velocity at *A* (that is, so that *AS* is to the mean proportional between *AS* and *BS*) that body goes on to make innumerable similar revolutions *BFC*, *CGD*, &c., and from the intersections it may separate the radius *AS* into the parts *AS*, *BS*, *CS*, *DS*, &c. in continued proportions. Truly the times of the revolutions will be directly as the perimeters of the orbits *AEB*, *BFC*, *CGD*, &c., and inversely with the starting velocities at *A*, *B*, *C* ; that is, as $AS^{\frac{3}{2}}, BS^{\frac{3}{2}}, CS^{\frac{3}{2}}$. And the total time, in which the body may arrive at the centre, will be to the time of the first revolution, as the sum of all the continued proportionals $AS^{\frac{3}{2}}, BS^{\frac{3}{2}}, CS^{\frac{3}{2}}$, going off to infinite, to the first term $AS^{\frac{3}{2}}$; that is, as that first term $AS^{\frac{3}{2}}$ to the difference of the two first terms

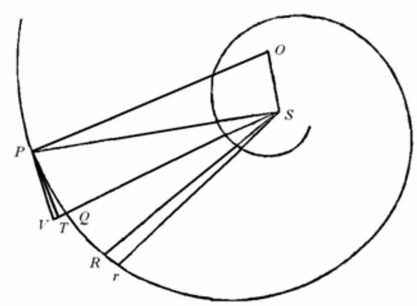
$AS^{\frac{3}{2}} - BS^{\frac{3}{2}}$, or as $\frac{2}{3}AS$ to AB as an approximation. From which that whole time is found expeditiously.

Corol. 8. In addition from these it is possible also to deduce the motion of bodies in mediums, the density of which is not uniform, or which may observe some other designated law. With centre *S*, with the radii in continued proportions *SA*, *S B*, *SC*, &c. describe as many circles, and put in place the time of the revolutions between the perimeters of any two of these circles, in the medium which we have treated, to be to the time of revolution between the same in the proposed medium, as the mean density of the proposed medium between these circles, to the density of the medium which we have considered, taken approximately as the mean density between the same circles: But also the secant of the angle by which the determined spiral, in the medium we have treated, may cut the radius *AS*, to be in the same ratio to the secant of the angle by which the new spiral may cut the radius in the proposed medium; also so that the tangents of the same angles thus to be approximately the number of all the revolutions between the same two circles. If these are made everywhere between two circles, the motion will be continued for all the circles. And with this agreed upon we can imagine without difficulty in what manners and times bodies in some medium ought to rotate regularly.

Corol. 9. And although eccentric motions may be completed in spirals approaching the form of ovals; yet be considering the individual revolutions of these to stand apart in turn with the same radius, and to approach the centre by the same steps with the above spiral described, we will understand also how the motions of bodies may be completed in spirals of this kind.

PROPOSITION XVI. THEOREM XIII.

If the density of the medium at individual places shall be inversely as the distances from a fixed centre, and the centripetal force shall be as some power of the same distance : I say that the body can rotate in a spiral that cuts all the radii drawn from the centre at a given angle.



This may be demonstrated by the same method as the above proposition. For if the centripetal force at *P* shall be inversely as any power SP^{n+1} of the distance *SP*, of which the index is $n+1$: it may be deduced as above that the time, in which the body may describe some arc *PQ* will be as $PQ \times PS^{\frac{1}{2}n}$; and the resistance at *P* will be as $\frac{Rr}{PQ^2 \times SP^n}$, [these result follow at once by regarding the curve locally as circular], or as $\frac{\overline{1-\frac{1}{2}n} \times VQ}{PQ \times SP^n \times SQ}$, and thus as $\frac{\overline{1-\frac{1}{2}n} \times OS}{OP \times SP^{n+1}}$, that is, from $\frac{\overline{1-\frac{1}{2}n} \times OS}{OP}$ given, reciprocally as SP^{n+1}. And therefore, since the velocity shall be inversely as $SP^{\frac{1}{2}n}$, the density at *P* will be inversely as *SP*.

Corol. 1. The resistance is to the centripetal force as $\overline{1-\frac{1}{2}n}\times OS$ to *OP*.

Corol.2. If the centripetal force shall be inversely as SP^3, there will be $1-\frac{1}{2}n=0$; and thus the resistance, and the density of the medium will be zero, as in proposition nine of the first book.

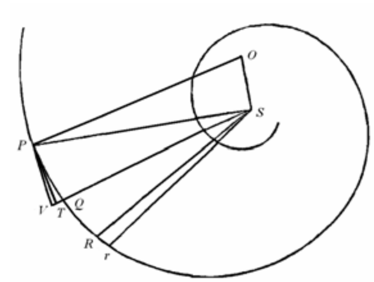
Corol. 3. If the centripetal force shall be inversely as some power of the radius *SP* of which the index is greater than the number 3, the positive resistance will be changed into a negative resistance.

Scholium.

Besides this proposition and the above ones, which consider unequally dense mediums, they are required to be understood concerning the motion of very small bodies, so that the greater density from one side of the body to the other may not be required to be considered. I suppose also that the resistance to be proportional to the density, with all other things being equal. From which in media, in which the force of resistance is not as the density, the density in that must be increased or diminished to such an extent that the excess resistance may be removed or the deficiency may be supplied.

PROPOSITION XVII. PROBLEM IV.

To find both the centripetal force and the resistance of the medium, by which the body in a given spiral can rotate with a given law of the velocity.



Let *PQR* be that given spiral. From the velocity, by which the body runs through that minimum arc *PQ*, the time will be given, and the force will be given from the altitude *TQ*, which is as the centripetal force and the square of the time. Then from the difference *RSr* of the areas *PSQ* and *QSR* completed in equal small intervals of time, the retardation of the body is given, and from the retardation the resistance may be found and the density of the medium.

PROPOSITION XVIII. PROBLEM V.

With a given law of the centripetal force, to find the density of the medium at individual places, that the body will describe in the given spiral.

From the centripetal force the velocity is required to be found at individual places, then from the retardation of the velocity the density of the medium may be found, as in the above proposition.

Truly I have uncovered the method required to treat this problem in proposition ten of this section and the second lemma ; and I do not wish to detain the reader with lengthy inquires into complexities of this kind. Now other matters are required to be added towards the progression of bodies by forces, and from the density and resistance of the mediums, in which the motion up to this point has been treated, and may be completed from these relations.

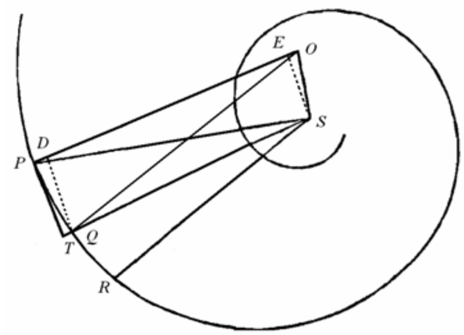
SECTIO IV.

De corporum circulari motu in mediis resistentibus.

LEMMA III.

Sit PQR spiralis quae secet radios omnes SP, SQ, SR, &c. in aequalibus angulis . Agatur recta PT quae tangat eandem in puncto quovis P, secetque radium SQ in T; & ad spiralem erectis perpendiculis PO, QO concurrentibus in O, iungatur SO. Dico quod si puncta P & Q accedant ad invicem ac coeant, angulus PSO evadet rectus, & ultima ratio rectanguli $TQ \times 2PS$ *ad PQquad. erit ratio aequalitatis.*

Etenim de angulis rectis *OPQ*, *OQR* subducantur anguli aequales *SPQ*, *SQR*, & manebunt anguli aequales *OPS*, *OQS*. Ergo circulus qui transit per puncta *O*, *S*, *P* transibit etiam per punctum *Q*. Coeant puncta *P* & *Q* & hic circulus in loco coitus *PQ* tanget spiralem, ideoque perpendiculiter secabit rectam *OP*. Fiet igitur *OP* diameter circuli huius, & angulus *OSP* in semicirculo rectus. Q.E.D.

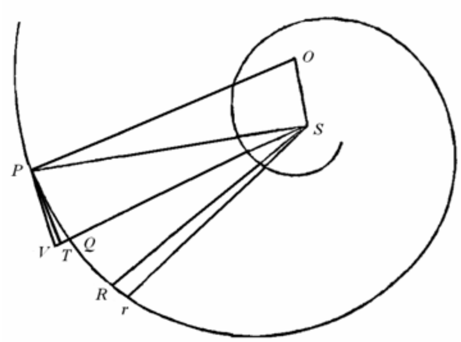


Ad *OP* demittantur perpendicula *QD*, *SE*, & linearum rationes ultime erunt huiusmodi: *TQ* ad *PD* ut *TS* vel *PS* ad *PE*, seu 2*PO* ad 2*PS*; item *PD* ad *PQ* ut *PQ* ad 2*PO*; & ex aequo perturbate *TQ* ad *PQ* ut *PQ* ad *2PS*. Unde sit *PQq* equale $TQ \times 2PS$. *Q.E.D.*

PROPOSITIO XV. THEOREMA XII.

Si medii densitas in locis singulis sit reciproce ut distantia locorum a centro immobili, sitque vis centripeta in duplicata ratione densitatis: dico quod corpus gyrari potest in spirali, quae radios omnes a centro illo ductos intersecat in angulo data.

Ponantur quae in superiore lemmate, & producatur *SQ* ad *P*, ut sit *SV* aequalis *SP*. Tempore quovis, in medio resistente, describat corpus arcum quam minimum *PQ*, & tempore duplo arcum quam minimum *PR*; & decrementa horum arcuum ex resistentia oriunda, sive defectus ab arcubus, qui in medio non resistente iisdem temporibus describerentur, erunt ad invicem ut quadrata temporum in quibus generantur: Est itaque decrementum arcus *PQ* pars quarta decrementi arcus *PR*. Unde etiam, si areae *PSQ* ita aequalis capiatur area *QSr*, erit decrementum arcus *PQ* aequale dimidio lineole *Rr*



; ideoque vis resistentiae & vis centripeta sunt ad invicem ut lineolae $\frac{1}{2}Rr$ & TQ quas simul generant. Quoniam vis centripeta, qua corpus urgetur in *P*, est reciproce ut *SPq* & (per Lem. X., Lib. I.) lineola *TQ*, quae vi illa generatur, est in ratione composita ex ratione huius vis & ratione duplicata temporis quo arcus *PQ* describitur (nam resistentiam in hoc casu, ut infinite minorem quam vis centripeta, negligo), erit $TQ \times SPq$, id est (per lemma novisiimum) $\frac{1}{2}PQq \times SP$, in ratione duplicata temporis, ideoque tempus est ut $PQ \times \sqrt{SP}$; & corporis velocitas, qua arcus *PQ* illo tempore describitur, ut $\frac{PQ}{PQ \times \sqrt{SP}}$ seu $\frac{1}{\sqrt{SP}}$, hoc est, in subduplicata ratione ipsius *SP* reciproce, Et simili argumento, velocitas qua arcus *QR* describitur, est in subduplicata ratione ipsius *SQ* reciproce. Sunt autem arcus illi *PQ* & *QR* ut velocitates descriptrices ad invicem, id est, in subduplicata ratione *SQ* ad *SP*, sive ut *SQ* ad $\sqrt{SP \times SQ}$; & ob aequales angulos *SPQ, QSr*, & aequales areas *PSQ, QSr*, est arcus *PQ* ad arcum *Qr* ut *SQ* ad *SP*. Sumantur proportionalium consequentium differentiae, & fiet arcus *PQ* ad arcum *Rr* ut *SQ* ad $SP - \sqrt{SP \times SQ}$; seu $\frac{1}{2}VQ$. Nam punctis *P* & *Q* coeuntibus, ratio ultima $SP - \sqrt{SP \times SQ}$ ad $\frac{1}{2}VQ$ est aequalitatis. Quoniam decrementum arcus *PQ*, ex resistentia oriundum, sive huius duplum *Rr*, est ut resistentia & quadratum temporis coniunctim; erit resistentia ut $\frac{Rr}{PQq \times SP}$. Erat autem *PQ* ad *Rr*, ut *SQ* ad $\frac{1}{2}VQ$, & inde $\frac{Rr}{PQq \times SP}$ sit ut $\frac{\frac{1}{2}VQ}{PQ \times SP \times SQ}$, sive ut $\frac{\frac{1}{2}OS}{OP \times SPq}$. Namque punctis *P* & *Q* coeuntibus, *SP* & *SQ* coincidunt, & angulus *PVQ* sit rectus ; & ob similia triangula *PVQ, PSO*, sit *PQ* ad $\frac{1}{2}VQ$ ut *OP* ad $\frac{1}{2}OS$. Est igitur $\frac{OS}{OP \times SPq}$ ut resistentia, id est, in ratione densitatis medii in P &. ratione duplicata velocitatis coniunctim. Auferatur duplicata ratio velocitatis, nempe ratio $\frac{1}{SP}$, & manebit medii densitas in P ut $\frac{OS}{OP \times SP}$. Detur spiralis, & ob datam rationem *OS* ad *OP*, densitas medii in *P* erit ut $\frac{1}{SP}$. In medio igitur cuius densitas est reciproce ut distantia a centro *SP*, corpus gyrari potest in hac spirali. *Q.E.D.*

Corol. 1. Velocitas in loco quovis *P* ea semper eft, quacum corpus in medio non resistente eadem vi centripeta gyrari potest in circulo, ad eandem a centro distantiam *SP*.

Corol. 2. Medii densitas, si datur distantia *SP*, est ut $\frac{OS}{OP}$, sin distantia illa non datur, ut $\frac{OS}{OP \times SP}$: Et inde spiralis ad quamlibet medii densitatem aptari potest.

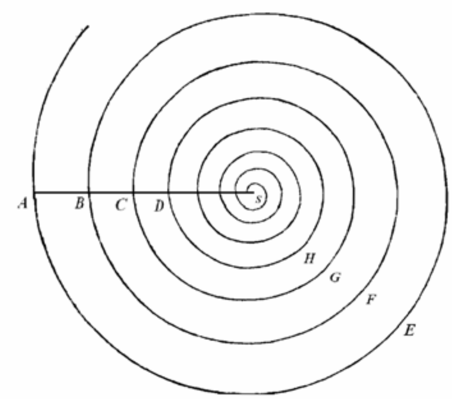
Corol. 3. Vis resistentiae in loco quovis *P*, est ad vim centripetam in eodem loco ut $\frac{1}{2}OS$ ad *OP*. Nam vires illae sunt ad invicem ut $\frac{1}{2}Rr$ & TQ sive ut $\frac{\frac{1}{4}VQ \times PQ}{SQ}$ & $\frac{\frac{1}{2}PQq}{SP}$, hoc est, ut $\frac{1}{2}VQ$ & *PQ*, seu $\frac{1}{2}OS$ & *OP*. Data igitur spirali datur proportio resistentiae ad vim centripetam, & vice versa ex data illa proportione datur spiralis.

Corol. 4. Corpus itaque gyrari nequit in hac spirali, nisi ubi vis restistentiae minor est quam dimidium vis centripetae. Fiat resistentia aequalis dimidio vis centripetae, & spiralis conveniet cum linea recta *PS*, inque hac recta corpus descendet ad centrum ea cum velocitate, quae sit ad velocitatem, qua probavimus in superioribus in casu parabolas (Theor. X, Lib. I) descensum in medio non resistente fieri, in subduplicata ratione unitatis ad numerum binarium. Et tempora descensus hic erunt reciproce ut velocitates, atque ideo dantur.

Corol. 5. Et quoniam in aequalibus a centro distantiis velocitas eadem est in spirali *PQR* atque in recta *SP*, & longitudino spiralis ad longitudinem rectae *PS* est in data ratione, nempe in ratione *OP* ad *OS* ; tempus descensus in spirali erit ad tempus descensus in recta *SP* in eadem illa data ratione, proindeque datur.

Corol.6. Si centro S intervallis duobus quibuscunque datis describantur duo circuli; & manentibus hisce circulis, mutetur utcunque angulus quem spiralis continet cum radio *PS*: numerus revolutionum quas corpus intra circulorum circumferentias, pergendo in spirali a circumferentia ad circumferentiam, complere potest, est ut $\frac{PS}{OS}$ sive ut tangens anguli illius quem spiralis contin et cum radio *PS*; tempus vero revolutionum earundem ut $\frac{OP}{OS}$, id est, ut secans anguli eiusdem, vel etiam reciproce ut medii densitas.

Corol.7. Si corpus in medio, cuius densitas est reciproce ut distantia locorum a centro, revolutionem in curva quacunque *AEB* circa centrum illud fecerit, & radium primum *AS* in eodem angulo secuerit in *B* quo prius in *A*, idque cum velocitate que fuerit ad velocitatem suam primam in *A* reciproce in subduplicata ratione distantiarum a centro (id est, ut *AS* ad mediam proportionalem inter *AS* & *BS*) corpus illud perget innumeras consimiles revolutiones *BFC*, *CGD*, &c. facere, & intersectionibus distinguet radium *AS* in partes *AS*, *BS*, *CS*, *DS*, &c. continue proportionales. Revolutionum vero tempora erunt ut perimetri orbitarum *AEB*, *BFC*, *CGD*, &c. directe, & velocitates in principiis *A*, *B*, *C*, inverse; id est, ut $AS^{\frac{3}{2}}, BS^{\frac{3}{2}}, CS^{\frac{3}{2}}$. Atque tempus totum, quo corpus perveniet ad centrum, erit ad tempus revolutionis primae, ut summa omnium continue proportionalium $AS^{\frac{3}{2}}, BS^{\frac{3}{2}}, CS^{\frac{3}{2}}$, pergentium in infinitum, ad terminum primum $AS^{\frac{3}{2}}$; id est, ut terminus ille primus $AS^{\frac{3}{2}}$ ad differentiam duorum primorum $AS^{\frac{3}{2}} - BS^{\frac{3}{2}}$, sive ut $\frac{2}{3}AS$ ad AB quam proxime. Unde tempus illud totum expedite invenitur.

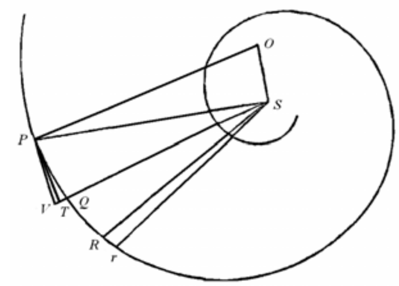


Corol. 8. Ex his etiam praeter propter colligere licet motus corporum in mediis, quorum densitas aut uniformis est, aut aliam quaemcunque legem assignatam observat. Centro S, intervallis continue proportionalibus *SA*, *S B*, *SC*, &c.describe circulos quotcunque, & statue tempus revolutionum inter perimetros duorum quorumvis ex his circulis, in medio de quo egimus, esse ad tempus revolutionum inter eosdem in medio proposito, ut medii propositi densitas mediocris inter hos circulos ad medii, de quo egimus, densitatem mediocrem inter eosdem quam proxime: Sed & in eadem quoque ratione esse secantem anguli quo spiralis praefinita, in medio de quo egimus, secat radium *AS*, ad secantem anguliquo spiralis nova secat radium eundem inmedio proposito: Atque etiam ut sunt eorundem angulorum tangentes ita esse numeros revolutionum omnium inter circulos eosdem duos quam proxime. Si haec fiant passim inter circulos binos, continuabitur motus per circulos omnes. Atque hoc pacto haud difficulter imaginari possimus quibus modis ac temporibus corpora in media quocunque regulari gyrari debebunt.

Corol. 9. Et quamvis motus excentrici in spiralibus ad formam ovalium accedentibus peragantur; tamen concipiendo spiralium illarum singulas revolutiones iisdem ab invicem intervallis distare, iisdemque gradibus ad centrum accedere cum spirali superius descripta, intelligemus etiam quomodo motus corporum in huiusmodi spiralibus peragantur.

PROPOSITIO XVI. THEOREMA XIII.

Si medii densitas in locis singulis sit reciproce ut distantia locorum a centro immobili, sitque vis centripeta reciproce ut dignitas quaelibet eiusdem distantiae: dico quod corpus gyrari potest in spirali quae radios omnes a centro illo ductos intersecat in angulo dato.



Demonstratur eadem methodo cum propositione superiore. Nam si vis centripeta in *P* sit reciproce ut distantiae *SP*, dignitas quaelibet SP^{n+1} cuius index est $n+1$: colligetur ut supra, quod tempus, quo corpus describit arcum quemvis *PQ* erit ut $PQ\times PS^{\frac{1}{2}n}$; & resistentia in *P* ut $\frac{Rr}{PQq\times SP^{n}}$, sive ut $\frac{\overline{1-\frac{1}{2}n}\times VQ}{PQ\times SP^{n}\times SQ}$ ideoque ut $\frac{\overline{1-\frac{1}{2}n}\times OS}{OP\times SP^{n+1}}$, ideoque ut $\frac{\overline{1-\frac{1}{2}n}\times OS}{OP\times SP^{n+1}}$ hoc est, ab datum $\frac{\overline{1-\frac{1}{2}n}\times OS}{OP}$, reciproce ut SP^{n+1}. Et propterea, cum velocitas sit reciproce ut $SP^{\frac{1}{2}n}$, densitas in *P* erit reciproce ut *SP*.

Corol. I. Resistentia est ad vim centripetam ut $\overline{1-\frac{1}{2}n}\times OS$ ad OP.

Corol.2. Si viscentripeta sit reciproce ut *SPcub*., erit $1-\frac{1}{2}n=0$; ideoque resistenia, & densitas medii nulla erit, ut in propositione nona libri primi.

Corol. 3. Si vis centripeta sit reciproce ut dignitas aliqua radii *SP* cuius index est maior numero 3, resistentia affirmativa in negativam mutabitur.

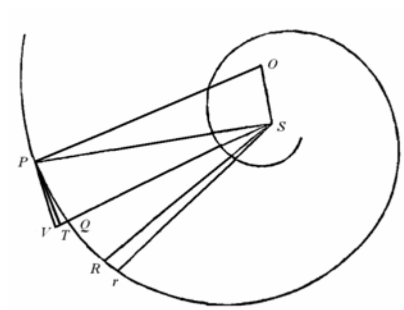
Scholium.

Caeterum haec propositio & superiores, quae ad media inequaliter densa spectant, intelligendae sunt de motu corporum adeo parvorum, ut medii ex uno corporis latere maior densitas quam ex altere non consideranda veniat. Resistentiam quoque caeteris paribus densitati proportionalem esse suppono. Unde in mediis, quorum vis resistendi non est ut densitas, debet densitas eo usque augeri vel diminui, ut resistentiae vel tollatur excessus vel defectus suppleatur.

PROPOSITIO XVII. PROBLEMA IV.

Invenire & vim centripetam & medii resistentiam, qua corpus in data spirali, data velocitatis lege, revolvi potest.

Sit spiralis illa *PQR*. Ex velocitate, qua corpus percurrit arcum quam minimum *PQ* dabitur tempus, & ex altitudine *TQ*, quae est ut vis centripeta & quadratum temporis, dabitur vis. Deinde ex arearum, aequalibus temporum particulis confectarum *PSQ* & *QSR*, differentia *RSr*, dabitur corporis retardatio, & ex retardatione invenietur resistentia ac densitas medii.



PROPOSITIO XVIII. PROBLEMA V.

Data lege vis centripetae, invenire medii densitatem in locis singulis, qua corpus datam spiralem describet.

Ex vi centripeta invenienda est velocitas in locis singulis, deinde ex velocitatis retardatione quarenda medii densitas , ut in propositione superiore.

Methodum vero tractandi haec problemata aperui in huius propositione decima, & lemmate secundo ; & lectorem in huiusmodi perplexis disquisitionibus diutius detinere nolo. Addenda iam sunt aliqua de viribus corporum ad progrediendum, deque densitate & restistentia mediorum, in quibus motus hactenus expositi & his affines peraguntur,