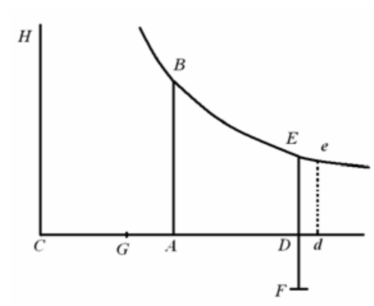
SECTION III.

Concerning the motion of bodies in which it is resisted partially in the ratio of the velocity and partially as the square ratio of the same.

PROPOSITION XI. THEOREM VIII.

If for a body resisted partially in the ratio of the velocity, partially in the ratio of the square of the velocity, and likewise that may be moved only by the force of inertia in a similar medium : and moreover the times may be taken in an arithmetic progression ; magnitudes inversely proportional to the velocities increased by a certain amount, shall be in a geometric progression.



With centre *C*, with the rectangular asymptotes *CADd* and *CH*, the hyperbola *BEe* may be described, and *AB*, *DE*, *dc* shall be parallel to the asymptote *CH.* The points *A, G* may be given on the asymptote *CD:* And if the time may be put in place by the hyperbolic area *ABED* increasing uniformly ; I say that the velocity can be shown by the length *DF*, the reciprocal of which *GD* together with the reciprocal of the given *CG* shall produce the length *CD* increasing in a geometric progression.

For the small area *DEed* given shall be as the smallest increment of the time, and *Dd* will be inversely as *DE*, and thus directly as *CD.* [For if $DE \times Dd = \Delta t$ then $Dd = \frac{\Delta t}{DE} \propto CD$.] But the decrement of $\frac{1}{GD}$, which (by Lemma III of this) is $\frac{Dd}{GD^2}$, will be as $\frac{CD}{GD^2}$ or $\frac{CG+GD}{GD^2}$, that is, as $\frac{1}{GD}+\frac{CG}{GD^2}$. Therefore in the time *ABED* increased uniformly by the addition of the given small part *EDde*, $\frac{1}{GD}$ decreases in the same ratio with the velocity. For the decrease of the velocity is as the resistance, that is (by hypothesis) as the sum of two quantities, one of which is as the velocity, the other as the square of the velocity, and the decrement of $\frac{1}{GD}$ is as the sum of the quantities $\frac{1}{GD}$ and $\frac{CG}{GD^2}$, the former of which is $\frac{1}{GD}$ and the latter $\frac{CG}{GD^2}$ is as $\frac{1}{GD^2}$ *:* hence $\frac{1}{GD}$ is as the velocity, on account of the analogous decrement. And if the quantity *GD*, inversely proportional to $\frac{1}{GD}$ itself, may be increased by the qiven quantity *CG* ; the sum *CD*, with the time *ABED* increasing uniformly, increases in a geometric progression. *Q. E. D.*

Corol. 1. Therefore if, with the points *A,* and *G* given, the time may be shown by the hyperbolic area *ABED,* the velocity is able to be shown by the reciprocal of *GD* itself.

Corol. 2. But with *GA* to *GD* taken as the reciprocal of the velocity from the start, to the reciprocal of the velocity at the end of some time *ABED*, the point *G* may be found. Moreover with that found, the velocity from some other given time is able to be found.

[We follow Brougham and Routh, p. 205., changing their notation a little:
The resistance is represented by $R = kv + \frac{k}{\alpha}v^2$ per unit mass, and the equation of motion, in the presence of gravity g, becomes $\frac{dv}{dt} = v\frac{dv}{dx} = g - kv - \frac{k}{\alpha}v^2$. Thus, in the absence of gravity, we have $\frac{dv}{dt} = -kv - \frac{k}{\alpha}v^2$, which we can write in the form :

$$\frac{dv}{v} - \frac{dv}{\alpha+v} = -\frac{k}{\alpha}dt; \text{ integrate : } ln\frac{v}{\alpha+v} - ln\frac{v_0}{\alpha+v_0} = ln\frac{1}{\frac{\alpha}{v}+1} - ln\frac{1}{\frac{\alpha}{v_0}+1} = -\frac{kt}{\alpha}.$$

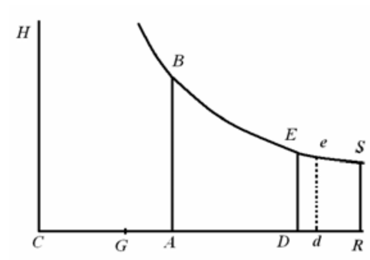
$$\text{or } \frac{1+\frac{\alpha}{v}}{1+\frac{\alpha}{v_0}} = e^{\frac{kt}{\alpha}}, \text{ and } \frac{1}{\alpha} + \frac{1}{v} = \left(\frac{1}{\alpha} + \frac{1}{v_0}\right)e^{\frac{kt}{\alpha}}$$

Thus when the times are in an arithmetical progression, quantities associated with the reciprocals of the velocity with an added given quantity are in a geometric progression. We can proceed to a demonstration of the next proposition here, again with zero gravity: $\frac{dv}{dx} = -\frac{k}{\alpha}(v+\alpha)$ hence $\frac{dv}{v+\alpha} = -\frac{k}{\alpha}dx$, giving $v+\alpha = (v_0+\alpha)e^{-\frac{kx}{\alpha}}$. In this case, if the distances are augmented by regular intervals, the velocities with an added constant are increased in a geometric progression. Proposition XIV finds the velocities that correspond to uniform increases in the distance gone, in the case of gravity being present.]

PROPOSITION XII. THEOREM IX.

With the same in place, I say that if the distances described may be taken in an arithmetic progression, the velocities increased by a certain given amount will be in a geometric progression.

The point *R* may be given on the asymptote *CD*, and with the perpendicular *RS* erected, which meets the hyperbola at *S*, a description of the distances may be shown by the hyperbolic area *RSED ;* and the velocity will be as the length *GD*, which with the given *CG* puts together the length *CD* decreasing in a geometric progression, while meanwhile the distance *RSED* may be increased in an arithmetic progression. And indeed on account of the given increment of the distance *EDde*, the linelet *Dd*, which is the decrement of *GD* itself, will be inversely as *ED*, and thus directly as *CD*, that is, as the sum of the same *GD* and of the given length *CG*. But the decrement of the velocity in a time inversely proportional to itself, in which a small amount of distance *DdeE* is described, is jointly as the resistance and the time, that is, directly as the sum of the two quantities, of which the one is as the velocity, the other is as the square of the velocity, and inversely as the velocity, and thus directly as the sum of the two quantities, with one of which given, the other is as the velocity. Therefore the decrement of the velocity as well as of the line *GD*, is as the given quantity, and the decreasing quantity jointly; and will always be analogous decreasing quantities ; without doubt the velocity and the line *GD*. *Q.E.D.*



Corol. 1. If the velocity may be shown by the length *GD*, the distance described will be as the area of the hyperbola *DESR*.

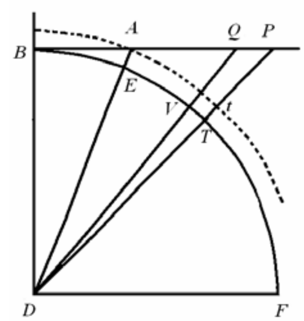
Corol. 2. And if some point *R* may be assumed, the point *G* may be found by taking *GR* to *GD*, as the velocity is from the start, to the velocity after some described distance *RSED*. Moreover with the point *G* found, the distance is given from the velocity, and conversely.

Corol. 3. From which since (by Prop. XI.) the velocity may be given from the given time, and by that proposition the distance may be given from the given velocity, the distance may be given from the given time, and conversely.

PROPOSITION XIII. THEOREM X.

Because on putting in place a body attracted uniformly downwards by gravity, it may ascend or descent along a right line; and because it may be resisted in part in the ratio of the velocity and in part in the same ratio squared : I say that, if parallel right lines may be drawn parallel to the diameters through the ends of the conjugate diameters of a circle and hyperbola, and the velocities shall be as the segments of certain parallels drawn from a given point ; the times will be as the sectors of the areas, with right lines drawn to the ends of the segments cut off : and conversely.

Case I. In the first place we may consider the body rising, and with centre *D* and with some radius *DB*, the quadrant of a circle *BETF* is described, and through the end of the radius *DB* the indefinite line *BAP* may be drawn, parallel to the radius *DF*. On that the point *A* may be given, and the segment *AP* may be taken proportional to the velocity. And since the one part of the resistance shall be as the velocity and the other part as the velocity squared ; the whole resistance shall be as $AP^2 + 2BA.AP$; *DA* and *DP* may be joined cutting the circle at *E* and *T*, and gravity may be shown by DA^2 thus so that gravity shall be to the resistance as DA^2 to $AP^2 + 2BA.AP$: and the time of the whole ascent shall be as the sector of the circle *EDT*.



[Note that the physical problem in this case has been replaced by purely geometrical one. In this case the velocity of a point along the line *BP* corresponds to the rate at which the area of the triangle *PAD* is swept out in the quadrant of the circle, the radius of which is $\sqrt{g}$]

Indeed *DVQ* may be drawn, and marking off the moment *PQ* of the velocity *AP*, and the moment *DTV* of the sector *DET* corresponding to the given moment of the time , and that decrement *PQ* of the velocity will be as the sum of the forces of gravity DA^2 and of the resistance $AP^2 + 2BA.AP$, that is (by Prop. 12. Book 2. *Euclid Elem*.) as DP^2.

[From elementary geometry, we have
$DP^2 = DB^2 + (BA + AP)^2 = \left(DB^2 + BA^2 =\right) DA^2 + AP^2 + 2BA.AP \propto g + kv + \frac{k}{\alpha}v^2$ below.]
Therefore the area *DPQ*, itself proportional to *PQ*, is as DP^2; [essentially
$\frac{dv}{dt} = -g - kv - \frac{k}{\alpha}v^2$ as below.]

and the area *DTV*, which is to the area *DPQ* as DT^2 to DP^2, is as DT^2. Therefore the area *EDT* decreases uniformly in the manner of the future time, by the subtraction of the given small elements *DTV*, and therefore is proportional to the time of the whole ascent. *Q.E.D.*

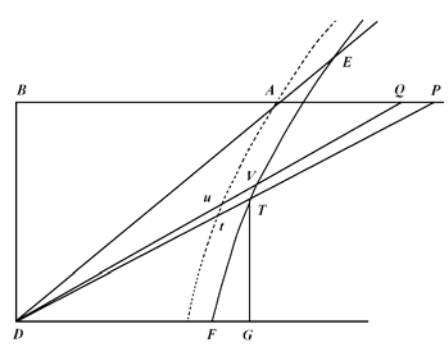
[Following Brougham & Routh, in the general case with gravity present for the falling body, as discussed below by Newton, we have, $\frac{dv}{dt} = v\frac{dv}{dx} = g - kv - \frac{k}{\alpha}v^2$ giving $\frac{dv}{\frac{1}{\alpha}v^2 + v - \frac{g}{k}} = -kdt$ or $\frac{dv}{v^2 + \alpha v - \frac{\alpha g}{k}} = -\frac{k}{\alpha}dt$, and thence $\frac{dv}{\left(v + \frac{\alpha}{2}\right)^2 - \frac{\alpha g}{k} - \frac{\alpha^2}{4}} = -\frac{k}{\alpha}dt$, which we may write as $\frac{dv}{\left(v + \frac{\alpha}{2}\right)^2 - c^2} = -\frac{k}{\alpha}dt$, which may be written as : $\frac{dv}{\left(v + \frac{\alpha}{2}\right) - c} - \frac{dv}{\left(v + \frac{\alpha}{2}\right) + c} = -\frac{k}{2c\alpha}dt$, and which hence may be integrated to give : $ln\frac{\left(\left(v + \frac{\alpha}{2}\right) - c\right)}{\left(\left(v_0 + \frac{\alpha}{2}\right) - c\right)} - ln\frac{\left(\left(v + \frac{\alpha}{2}\right) + c\right)}{\left(\left(v_0 + \frac{\alpha}{2}\right) + c\right)} = -\frac{kt}{2c\alpha}$ and

$ln\frac{\left(v + \frac{\alpha}{2} + c\right)\left(v_0 + \frac{\alpha}{2} - c\right)}{\left(v + \frac{\alpha}{2} - c\right)\left(v_0 + \frac{\alpha}{2} + c\right)} = \frac{kt}{2c\alpha}$ or $\frac{\left(v + \frac{\alpha}{2} - c\right)\left(v_0 + \frac{\alpha}{2} + c\right)}{\left(v + \frac{\alpha}{2} + c\right)\left(v_0 + \frac{\alpha}{2} - c\right)} = e^{\frac{-kt}{2c\alpha}}$,

thus enabling *t* to be found in terms of *v*, and by solving the equation, vice versa.
On the other hand, if the body is rising, as Newton considers above, we have to solve $\frac{dv}{dt} = v\frac{dv}{dx} = -g - kv - \frac{k}{\alpha}v^2$, giving : $\frac{dv}{\frac{1}{\alpha}v^2 + v + \frac{g}{k}} = -kdt$ or $\frac{dv}{v^2 + \alpha v + \frac{\alpha g}{k}} = -\frac{k}{\alpha}dt$, or $\frac{dv}{\left(v + \frac{\alpha}{2}\right)^2 + \frac{\alpha g}{k} - \frac{\alpha^2}{4}} = -\frac{k}{\alpha}dt$; we now have the problem whether $\frac{g}{k} > \frac{\alpha}{4}$ or $\frac{g}{k} < \frac{\alpha}{4}$; if $\frac{g}{k} > \frac{\alpha}{4}$ then we have the equation, $\frac{dv}{\left(v + \frac{\alpha}{2}\right)^2 + b^2} = -\frac{k}{\alpha}dt$, where $b^2 = \frac{\alpha g}{k} - \frac{\alpha^2}{4}$. Now the solution becomes :

$tan^{-1}\left(\frac{v + \frac{\alpha}{2}}{b}\right) - tan^{-1}\left(\frac{v_0 + \frac{\alpha}{2}}{b}\right) = -\frac{b}{\alpha}kt$.]

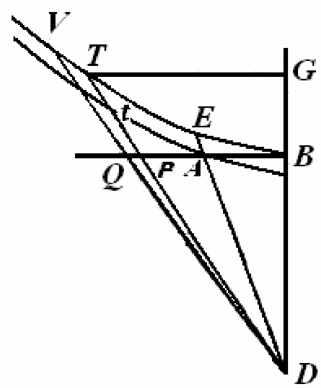
Case 2. If the velocity of the body in the ascent may be expressed by the length *AP* as before, and the resistance may be put to be as $AP^2 + 2BA \times AP$, and if the force of gravity shall be less than may be possible to express by DA^2; *BD* is taken of this length so that $AB^2 - BD^2$ shall be proportional to gravity, and let *DF* be perpendicular and equal to *DB* itself, and through



the vertex *F* the hyperbola *FTVE* may be described, of which *DB* & *DF* shall be the conjugate semi diameters, which will cut *DA* in *E*, and *DP*, *DQ* in *T* and *V*; the total time of the ascent will be as the sector *TDE* of the hyperbola.

For the decrement made in the velocity *PQ* in a given small time interval, is as the sum of the resistance $AP^2+2BA\times AP$ and of gravity AB^2-BD^2 that is, as BP^2-BD^2. Moreover the area *DTV* to the area *DPQ*, is as DT^2 to DP^2; and thus, if *DF* may be sent to the perpendicular *GT*, as GT^2 or GD^2-DF^2 to BD^2, and as GD^2 to BP^2, and separately, as DF^2 to BP^2-BD^2.

Whereby since the area *DPQ* shall be as *PQ* that is, as BP^2-BD^2; the area *DTV* will be as DF^2 given. Therefore the area *EDT* is described uniformly in individual equal elements of time, by subtraction of small parts from just as many given parts of the given *DTV*, and therefore is proportional to the time. *Q. E. D.*



Case 3. Let *AP* be the velocity of the body in the descent of the body, and $AP^2+2BA\times AP$ the resistance, and BD^2-AB^2 the force of gravity, with a right angle *DBA* present. And if from the centre *D*, with the principle vertex *B*, the rectangular hyperbola *BETV* may be described cutting *DA*, *DP* and *DQ* produced at *E*, *T* and *V*; *DET* will be to a sector of this hyperbola as the total time of descent.

For the increment of the velocity *PQ*, and proportional to that the area *DPQ*, is as the excess of gravity over the resistance, that is, as $BD^2-AB^2-2BA\times AP-AP^2$ or BD^2-BP^2. [As above, $dv\propto\left(g-kv-\frac{k}{\alpha}v^2\right)dt$.]

And the ratio of the areas of the triangles $\frac{\Delta DTV}{\Delta DPQ}=\frac{DT^2}{DP^2}$,

[As $\Delta DTV\propto PD\times QD\times \sin P\widehat{D}Q$ and $\Delta TDV\propto TD\times VD\times \sin P\widehat{D}Q$; since the triangles are evanescent, the relevant sides are equal, and thus in the same ratio.],

and thus $\frac{\Delta DTV}{\Delta DPQ}=\frac{DT^2}{DP^2}=\frac{GT^2}{BP^2}=\frac{GD^2-BD^2}{BP^2}$, [For if $y^2-x^2=b^2$ then $GD^2-GT^2=BD^2$]

and thus as $\frac{GD^2}{BD^2}$, and separately as $\frac{BD^2}{BD^2-BP^2}$ [For we can write $\frac{-GD^2+BD^2}{-BP^2}$ and to this may be added the corresponding terms of another equal ratio: in this case $\frac{-GD^2+BD^2+GD^2}{+BD^2-BP^2}$].

Whereby since the area *DPQ* shall be as BD^2-BP^2, the area *DTV* will be as the given BD^2. Therefore the area *EDT* increases uniformly with the individual increments of the time, by adding just as many of the given increments *DTV*, and therefore is proportional to the time of the descent. *Q.E.D.*

Corol. If with the centre *D* the arc *At* may be drawn similar to the arc *ET* with the radius *DA* drawn through the vertex *A* subtending the angle *ADT*: the velocity *AP* will be to the velocity, that the body in the time *EDT*, in a distance without resistance on ascending may lose, or in descending may acquire, as the area of the triangle *DAP* to the

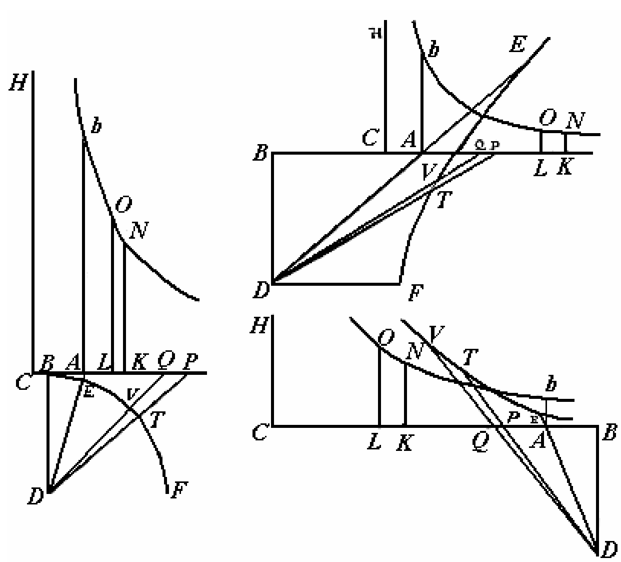
area of the sector *DAt*; and thus may be given from a given time. For the velocity, in a non-resisting medium, is proportional to the time and thus to this sector; in a medium with resistance it is as the triangle, and in each medium, when that is a minimum, it approaches the ratio of equality, for the customary sectors and triangles.
[The elemental triangles coincide at the start if there is no resistance, and do so finally when the resistance vanishes.]

Scholium.

Also it may be possible to show the case in the ascent of the body, when the force of gravity is less than what may be able to show by DA^2 or $AB^2 + BD^2$, and greater than what can be shown by $AB^2 - BD^2$, and must be shown by AB^2. But I will hurry on to other matters.

PROPOSITION XIV. THEOREM XI.

With everything the same in place, I say that the distance described in ascending or descending, is as the difference of the area put in place in that time, and of a certain other area which may be augmented or diminished in an arithmetic progression; if the forces from the resistance and from gravity added together may be taken in a geometric progression.



AC may be taken (in the three final figures) proportional to gravity, and *AK* to the resistance. Moreover they may be taken in the same part of the point *A* if the body descends, otherwise in the contrary part, *Ab* may be erected which shall be to *DB* as DB^2 to $4BA \times AC$: and the hyperbola *bN* described to the rectangular asymptotes *CK* and *CH*, and with *KN* to be perpendicular to *CK*, the area *AbNK* will be increased or diminished in an arithmetical progression, while the forces *CK* are taken in a geometric progression. Therefore I say that the distance of the body from the maximum height of this shall be as the excess area *AbNK* above the area *DET*.

For since *AK* shall be as the resistance, that is, as $AP^2 + 2BA \times AP$; some given quantity *Z* may be assumed, and *AK* may be put equal to $\frac{AP^2+2BA\times AP}{Z}$; and (by Lemma II of this section) the moment *KL* of this *AK* will be equal to $\frac{2AP\times PQ+2BA\times PQ}{Z}$ or $\frac{2BP\times PQ}{Z}$ and the moment *KLON* of the area *AbNK* equals $\frac{2BP\times PQ\times LO}{Z}$ or $\frac{BP\times PQ\times BD^3}{2Z\times CK\times AB}$.

[For, by hypothesis from Theorem IV, there is

$\frac{LO}{Ab} = \frac{CA}{CK}$, and by construction, $\frac{Ab}{DB} = \frac{DB^2}{4BA\times AC}$, and thus

$\frac{LO}{DB} = \frac{DB^2}{4BA\times CK}$ and hence $LO = \frac{DB^3}{4BA\times CK}$, $\frac{2BP\times PQ\times LO}{Z} = \frac{BP\times PQ\times DB^3}{2Z\times CK\times AB}$.]

Case 1. Now if the body ascends, and gravity shall be as $AB^2 + BD^2$, with the circle *BET* present (in the first figure), the line *AC* which is proportional to gravity, will be $\frac{AB^2+BD^2}{Z}$, or $AP^2 + 2BA \times AP + AB^2 + BD^2$ will be $AK \times Z + AC \times Z$ or $CK \times Z$; and thus the area DTV will be to the area DPQ as DT^2 or VB^2 to $CK \times Z$.

Case 2. But if the body ascends, and gravity shall be as $AB^2 - BD^2$, the line *AC* (in the second figure) will be $\frac{AB^2-BD^2}{Z}$ and DT^2 will be to DP^2 as DF^2 or DB^2 to $AB^2 - BD^2$ or $AP^2 + 2BA \times AP + AB^2 - BD^2$, that is, as $AK \times Z + AC \times Z$ or $CK \times Z$. And thus the area DTV will be to the area *VPQ* as VB^2 to $CK \times Z$.

Case 3. And by the same argument, if the body descends, and therefore gravity shall be as $BD^2 - AB^2$, and the line *AC* (in the third figure) may be equal to $\frac{BD^2-AB^2}{Z}$, the area *DTV* will be to the area *DPQ* as DB^2 to $CK \times Z$: as above.

Therefore since these areas shall always be in this ratio ; if for the area *DTV*, by which the moment of the time may always be shown equal to itself, there may be written some rectangle requiring to be determined, for example $BD \times m$, the area will be *DPQ*, that is, $\frac{1}{2}BD \times PQ$ to $BD \times m$ as $CK \times Z$ to BD^2. And thus there may be $PQ \times BD^3$ equal to $2BD \times m \times CK \times Z$, and the moment *KLON* of the area *AbNK* found above shall be $\frac{BP\times BD\times m}{AB}$. The moment, *DTV* or $BD \times m$, of the area *DET* may be taken away, and $\frac{AP\times BD\times m}{AB}$ will remain. Therefore the difference of the moments, that is, the moment of the difference of the areas, equals $\frac{AP\times BD\times m}{AB}$; and therefore on account of the given $\frac{BD\times m}{AB}$, as the velocity *AP*, that is, as the moment of the distance that the body will describe either by ascending or descending. And thus the difference of the areas and that distance shall be proportional to the moments, either increasing or decreasing, and likewise either arising or vanishing. *Q. E. D.*

[Following Brougham & Routh as above, in the general case with gravity present for the falling body, as discussed above by Newton, we have, $\frac{dv}{dt} = v\frac{dv}{dx} = g - kv - \frac{k}{\alpha}v^2$ giving

$ln\frac{(v+\frac{\alpha}{2}+c)(v_0+\frac{\alpha}{2}-c)}{(v+\frac{\alpha}{2}-c)(v_0+\frac{\alpha}{2}+c)}=\frac{kt}{2c\alpha}$ or $\frac{(v+\frac{\alpha}{2}-c)(v_0+\frac{\alpha}{2}+c)}{(v+\frac{\alpha}{2}+c)(v_0+\frac{\alpha}{2}-c)}=e^{\frac{-kt}{2c\alpha}}$, thus enabling *t* to be found in terms of *v*, and by solving the equation, vice versa. The equation $v\frac{dv}{dx}=g-kv-\frac{k}{\alpha}v^2$ for the falling body [case 3] can now be solved : $\frac{vdv}{(v+\frac{\alpha}{2})^2-(\frac{\alpha^2}{4}+\frac{\alpha g}{k})}=-\frac{kdx}{\alpha}$; setting $c^2=\frac{\alpha^2}{4}+\frac{\alpha g}{k}$, we find :

$\frac{vdv}{(v+\frac{\alpha}{2})^2-c^2}=-\frac{kdx}{\alpha}$ and $\frac{\frac{1}{2}d\left((v+\frac{\alpha}{2})^2-c^2\right)}{(v+\frac{\alpha}{2})^2-c^2}-\frac{\frac{1}{2}\alpha dv}{(v+\frac{\alpha}{2})^2-c^2}=-\frac{kdx}{\alpha}$; leading to

$ln\left(\left(v+\frac{\alpha}{2}\right)^2-c^2\right)+\frac{\alpha}{2c}ln\left(\frac{v+\frac{\alpha}{2}+c}{v+\frac{\alpha}{2}-c}\right)=C-\frac{2kx}{\alpha}$, where *C* corresponds to the l.h.s. when $x=0$ and $v=v_0$.

On the other hand, if the body is rising, as Newton considers above, we have to solve $\frac{dv}{dt}=v\frac{dv}{dx}=-g-kv-\frac{k}{\alpha}v^2$, giving : $tan^{-1}\left(\frac{v+\frac{\alpha}{2}}{b}\right)-tan^{-1}\left(\frac{v_0+\frac{\alpha}{2}}{b}\right)=-\frac{b}{\alpha}kt$, for the velocity at time *t*. In this case, in a similar manner, if *g* is made negative above, and $g>\frac{k\alpha}{4}$, then the quantity c^2 becomes imaginary [case 1]; however, on putting $b^2=\frac{\alpha g}{k}-\frac{\alpha^2}{4}$, the integral becomes $ln\left(\left(v+\frac{\alpha}{2}\right)^2+b^2\right)-\frac{\alpha}{b}tan^{-1}\left(\frac{v+\frac{\alpha}{2}}{b}\right)=C-\frac{2kx}{\alpha}$, with a result similar to the above if c^2 is still positive, and thus $g<\frac{k\alpha}{4}$ [case 2].]

Cor. If the length may be called *M*, which arises by applying the area *DET* to the line *BD* ; and some other length *V* may be taken in this same ratio to the length *M*, that the line *DA* has to the line *DE*: the distance that the body will describe in the whole ascent or descent in a medium with resistance, will be to the distance that the body in a medium without resistance can describe by falling from rest in the same time, as the difference of the aforesaid areas to $\frac{BD\times V^2}{AB}$: and thus is given from the time. For the distance in this non-resisting medium in the square ratio of the time, or as V^2; and on account of *BD* and *AB* given as $\frac{BD\times V^2}{AB}$. This area is equal to the area $\frac{DA^2\times BD\times M^2}{DE^2\times AB}$, and the moment of *M* itself is *m*; and therefore the moment of this area is $\frac{DA^2\times BD\times 2M\times m}{DE^2\times AB}$. But this moment is to the moment of the difference of the aforesaid areas *DET* and *AbNK*, viz. to $\frac{AP\times BD\times m}{AB}$ as $\frac{DA^2\times BD\times M}{DE^2}$ to $\frac{1}{2}BD\times AP$, or as $\frac{DA^2}{DE^2}$ by *DET* to *DAP*; and thus, when the areas *DET* and *DAP* shall be as small as possible, in the ratio of equality. Therefore the area $\frac{BD\times V^2}{AB}$, and the difference of the areas *DET* and *AbNK*, when all these areas shall be as small as possible, have equal moments; and thus are equal. From which since the velocities, and therefore also the distances in each medium descending from the start or ascending to the end described in the same time approach equality ; and thus they shall be in turn then as the area $\frac{BD\times V^2}{AB}$, and the difference of the areas *DET* and *AbNK*; and therefore since the distance in the non-resisting medium always shall be as $\frac{BD\times V^2}{AB}$, and the distance in the resisting medium always shall be as the difference of the areas *DET* and *AbNK* : it is necessary that the distances in each medium, described in some equal moments of time, shall be in turn as that area $\frac{BD\times V^2}{AB}$, and the difference of the areas *DET* and *AbNK*. *Q.E.D*.

Scholium.

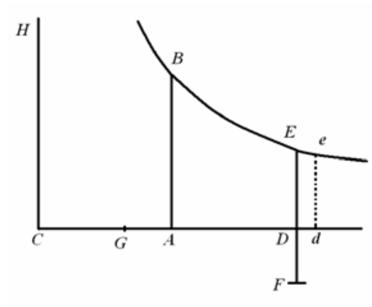
The resistance of spherical bodies in fluids arises in part from the tenacity, partially from the friction, and partially from the density of the medium. And that part of the resistance which arises from the density of the fluid we have said to be in the square ratio of the speed; the other part, which arises from the tenacity of the fluid, is uniform, or as the moment of the time: and thus now may be allowed to go to the motion of the body, by which it is resisted partially by a uniform force or in the ratio of the moments of time, and partially in the ratio of the square of the velocity. But it suffices to have revealed the approach regarding this observation in Propositions VIII. & IX. which preceded, and the corollaries of these. Certainly for the same ascent of the body with uniform resistance, which arises from the weight of this, it is possible to substitute a uniform resistance, which arises from the tenacity of the medium, when the body may be moving by the inertial force alone , and with the body ascending in a straight line it is allowed to add this uniform resistance to the force of gravity , and to subtract the same, when the body falls along as straight line. Also it may be allowed to go on to the motion of bodies, in which the resistance is partially uniform, and partially in the ratio of the velocity, and partially in the ratio of the square of the velocities. And I have shown the way in the preceding Propositions XIII and XIV, in which also uniform resistance, which arises from the tenacity of the medium can be substituted for the force of gravity, or since the same as before may be added to it. But I hurry on to other matters.

SECTIO III.
De motu corporum quibus resistitur partim in ratione velocitatis, partim in eiusdem ratione duplicata.

PROPOSITIO XI. THEOREMA VIII.
Si corpori resistitur partim in ratione velocitatis, partim in velocitatis ratione duplicata, & idem sola vi insita in medio similari movetur: sumantur autem tempora in progressione arithmetica quantitates velocitatis; reciproce proportionales, data quadam quantitate auctae, erunt in progressione geometrica.

Centro *C*, asymptotis rectangulis *CADd* & *C*H, describatur hyperbola *BEe,* & asymptoto *CH* parallelae sint *AB*, *DE*, *dc.* In asymptoto *CD* dentur puncta *A*, *G:* Et st tempus exponatur per aream hyperbolicam *ABED* uniformiter crescentem ; dico quod velocitas exponi potest per longitudinem *DF*, cuius reciproca *GD* una cum data *CG* componat longitudinem *CD* in progressione geometrica crescentem.

Sit enim areola *DEed* datum temporis incrementum quam minimum, & erit *Dd* reciproce ut *DE*, ideoque directe ut *CD.* Ipsius autem $\frac{1}{GD}$ decrementum, quod (per huius lem. II.) est $\frac{Dd}{GDq}$, erit ut $\frac{CD}{GDq}$ seu $\frac{CG+GD}{GDq}$, id est, ut $\frac{1}{GD}+\frac{CG}{GDq}$. Igitur tempore *ABED* per additionem datarum particularum *EDde* uniformiter crescente, decrescit $\frac{1}{GD}$ in eadem ratione cum velocitate. Nam decrementum velocitatis est ut resistentia, hoc est (per hypothesin) ut summa duarum quantitatum, quarum una est ut velocitas, altera ut quadratum velocitatis , & ipsius $\frac{1}{GD}$ decrementum est ut summa quantitatum $\frac{1}{GD}$ & $\frac{CG}{GDq}$ quarum prior est $\frac{1}{GD}$ & posterior $\frac{CG}{GDq}$ est ut $\frac{1}{GDq}$ *:* proinde $\frac{1}{GD}$, ob analogum decrementum, est ut velocitas. Et si quantitas *GD*, ipsi $\frac{1}{GD}$ reciproce proportionalis, quantitate data *CG* augeatur; summa *CD*, tempore*ABED* uniformiter crescente, crescet in progressione geometrica. *Q. E. D.*



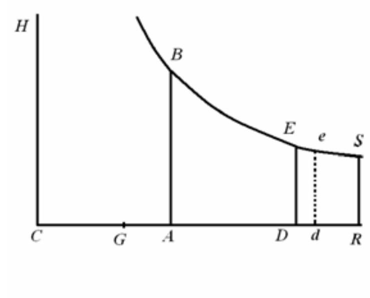
Corol. 1. Igitur si, datis punctis *A*, *G,* exponatur tempus per aream hyperbolicam *ABED,* exponi potest velocitas per ipsius *GD* reciprocam.

Corol. 2. Sumendo autem *GA* ad *GD* ut velocitatis reciproca sub initio, ad velocitatis reciprocam in fine temporis cuiusvis *ABED,* invenietur punctum *G.* Eo autem invento, velocitas ex dato quovis alia tempore inveniri potest.

PROPOSITIO XII. THEOREMA IX.

Iisdem positis, dico quod si spatia descripta sumantur in progressione arithmetica, velocitates data quadam quantitate auctae erunt in progressione geometrica.

In asymptoto *CD* detur punctum *R*, & erecto perpendiculo *R S*, quod occurrat hyperbolae in *S*, exponatur descriptum spatium per aream hyperbolicam *RSED ;* & velocitas erit ut longitudo *GD*, quae cum data *CG* componit longitudinem *CD* in progressione geometrica decrescentem, interea dum spatium *RSED* augetur in arithmetica. Etenim ob datum spatii incrementum *EDde*, lineola *Dd*, quae decrementum est ipsius *GD*, erit reciproce ut *ED*, ideoque directe ut *CD*, hoc est, ut summa eiusdem *GD* & longitudinis datae *CG*. Sed velocitatis decrementum, tempore sibi reciproce proportionali, quo data spatii particula *DdeE* describitur, est ut resistentia & tempus coniunctim, id est, directe ut summa duarum quantitatum, quarum una est ut velocitas, altera ut velocitatis quadratum, & inverse ut velocitas , ideoque directe ut summa duarum quantitatum, quarum una datur, altera est ut velocitas. Decrementum igitur tam velocitatis quam lineae *GD*, est ut quantitas data & quantitas decrescens coniunctim, & propter analoga decrementa, analogae semper erunt quantitates decrescentes ; nimirum velocitas & linea *GD*. *Q.E.D.*



Corol. 1. Si velocitas exponatur per longitudinem *GD*, spatium descriptum erit ut area hyperbolica *DESR.*

Corol. 2. Et si utcunque assumatur punctum *R*, invenietur punctum *G* capiendo *GR* ad *GD*, ut est velocitas sub initio ad velocitatem post spatium quodvis *RSED* descriptum. Invento autem puncto *G*, datur spatium ex data velocitate, & contra.

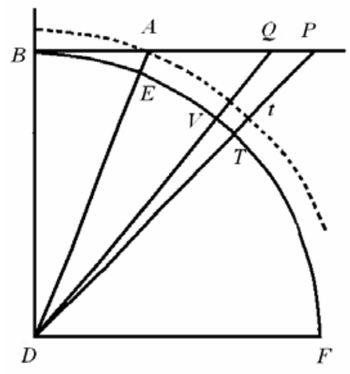
Corol. 3. Unde cum (per prop. XI.) detur velocitas ex dato tempore, & per hanc propositionem detur spatium ex data velocitate; dabitur spatium ex dato tempore: & contra.

PROPOSITIO XIII. THEOREMA X.

Posito quod corpus ab uniformi gravitate deorsum attractum recta ascendit vel descendit; & quod eidem resistitur partim in ratione velocitatis, partim in eiusdem ratione duplicata : dico quod, si circuli & hyperbolae diametris parallelae rectae per coniugatarum diametrorum terminos ducantur, & velocitates sint ut segmenta quaedam parallelarum a dato puncto ducta; tempora erunt ut arearum sectores, rectis a centro ad segmentorum terminos ductis abscissi: & contra.

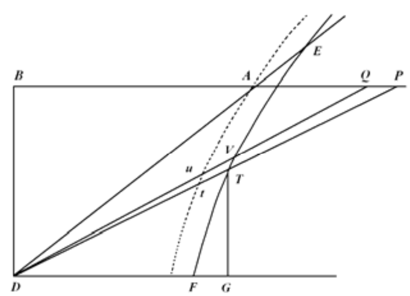
Cas. I. Ponamus primo quod corpus ascendit, centroque *D* & semidiametro quovis *DB* describatur circuli quadrans *BETF*, & per semidiametri *DB* terminum *B* agatur infinita *BAP*, semidiametro *DF* parallela. In ea detur punctum *A*, & capiatur segmentum

AP velocitati proportionale. Et cum resistentiae pars altera sit ut velocitas & pars altera ut velocitatis quadratum ; sit resistentia tota ut $APquad.+2BAP$. Iungantur *DA, DP* circulum secantes in *E* ac *T*, & exponatur gravitas per *DAquad*. ita ut sit gravitas ad resistentiam ut *DAq* ad $APq+2BAP$: & tempus ascensus totius erit ut circuli sector *EDT*.



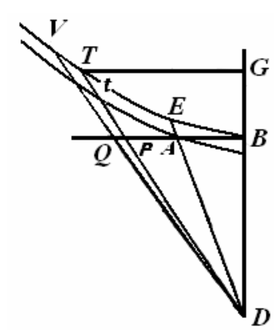
Agatur enim *DVQ*, abscindens & velocitatis *AP* momentum *PQ*, & sectoris *DET* momentum *DTV* dato temporis momento respondens , & velocitatis decrementum illud *PQ* erit ut summa virium gravitatis *DAq* & resistentiae $APq+2BAP$, id est (per Prop. 12. Lib. 2. Elem.) ut *DPquad*. Proinde area *DPQ*, ipsi *PQ* proportionalis, est ut *DPquad*. & area *DTV*, quae est ad aream *DPQ* ut *DTq* ad *DPq*, est ut datum *DTq*. Decrescit igitur area *EDT* uniformiter ad modum temporis futuri, per subductionem datarum particularum *DTV*, & propterea tempori ascensus totius proportionalis est. *Q.E.D*.

Cas. 2. Si velocitas in ascensu corporis exponatur per longitudinem *AP* ut prius, & resistentia ponatur esse ut $APq+2BAP$, & si vis gravitatis minor sit quam quae per *DAq* exponi possit; capiatur *BD* eius longitudinis, ut sit $ABq-BDq$ gravitati proportionale, sitque *DF* ipsi *DB* perpendicularis & aequalis, & per verticem *F* describatur hyperbola *FTVE*, cuius semidiametri coniugate sint *DB* & *DF*, quaeque secet *DA* in *E*, & *DP*, *DQ* in *T* & *V*; erit tempus ascensus totius ut hyperbolae sector *TDE*.



Nam velocitatis decrementum *PQ* in data temporis particula factum, est ut summa resistentiae $APq+2BAP$ & gravitatis $ABq-BDq$ id est, ut $BPq-BDq$. Est autem area *DTV* ad aream *DPQ*, ut *DTq* ad *DPq*; ideoque, si ad *DF* demittatur perpendiculum *GT*, ut *GTq* seu $GDq-DFq$ ad *BDq*, utque *GDq* ad *BPq*, & divisim ut *DFq* ad $BPq-BDq$. Quare cum area *DPQ* sit ut *PQ* id est, ut $BPq-BDq$; erit area *DTV* ut datum *DFq*. Decrescit igitur area *EDT* uniformiter singulis temporis particulis aequalibus, per subductionem particularum totidem datarum *DTV*, & propterea tempori proportionalis est. *Q. E. D.*

Cas. 3. Sit *AP* velocitas in descensu corporis, & $APq+2BAP$ resistentia, & $BDq-ABq$ vis gravitatis, existente angulo *DBA* recto. Et si centro *D*, vertice principali *B*, describatur hyperbola rectangula *BETV* secans productas *DA*, *DP* & *DQ* in *E*, *T* & *V*; erit hyperbolas huius sector *DET* ut tempus totum descensus.



Nam velocitatis incrementum *PQ*, eique proportionalis area *DPQ*, est ut excessus gravitatis supra resistentiam, id est, ut

$BDq - ABq - 2BAP - APq$ seu $BDq - BPq$. Et area *DTV* est ad aream *DPQ* ut *DTq* ad *DPq*, ideoque ut *GTq* seu $GDq - BDq$ ad *BPq*, utque *GDq* ad *BDq*, & divisim ut *BDq* ad $BDq - BPq$. Quare cum area *DPQ* sit ut $BDq - BPq$, erit area *DTV* ut datum *BDq*. Crescit igitur area *EDT* uniformiter singulis temporis particulis aequalibus, per additionem totidem datarum particularum *DTV*, & propterea tempori descensus proportionalis est. *Q.E.D.*

Corol. Si centro *D* semidiametro *DA* per verticem *A* ducatur arcus *At* similis arcui *ET*, & similiter subtendens angulum *ADT*: velocitas *AP* erit ad velocitatem, quam corpus tempore *EDT*, in spatio non resistente, ascendendo amittere vel descendendo acquirere posset, ut area trianguli *DAP* ad aream sectoris *DAt*; ideoque ex dato tempore datur. Nam velocitas, in medio non resistente, tempori, atque ideo sectori huic proportionalis est; in media resistente est ut triangulum; & in medio utroque, ubi quam minima est, accedit ad rationem aequalitatis, pro more sectoris & trianguli.

Scholium.

Demonstrari etiam posset casus in ascensu corporis, ubi vis gravitatis minor est quam quae exponi possit per *DAq* seu $ABq + BDq$, & maior quam quae exponi possit per $ABq - BDq$, & exponi debet per *ABq*. Sed propero ad alia.

PROPOSITIO XIV. THEOREMA XI.

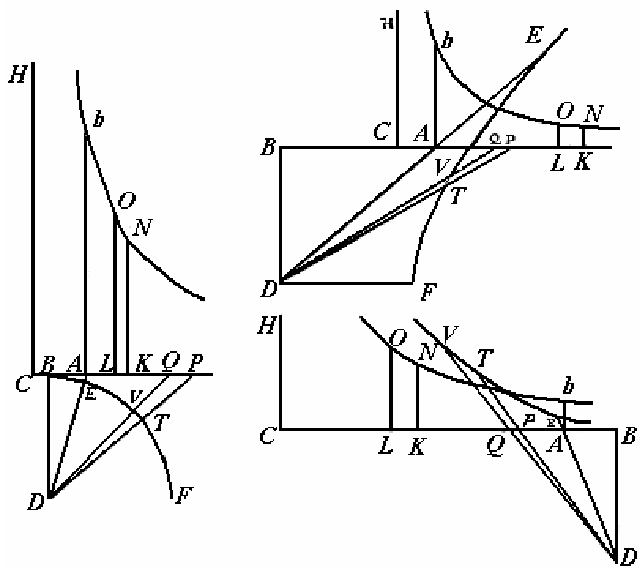
Iisdem positis, dico quod spatium ascensu vel descensu descriptum, est ut differentia areae per quam tempus exponitur, & areae cuiusdam alterius quae augetur vel diminuitur in progressione arithmetica; si vires ex restistentia & gravitate compositae sumantur in progressione geometrica.

Capiatur *AC* (in fig. tribus ultimis) gravitati, & *AK* resistentiae proportionalis. Capiantur autem ad easdem partes puncti *A* si corpus descendit, aliter ad contrarias. Erigatur *Ab*, quae sit ad *DB* ut *DBq* ad $4BAC$: & descripta ad asymptotos rectangulas *CK*, *CH* hyperbola *bN*, erectaque *KN* ad *CK* perpendiculari, area *AbNK* augebitur vel diminuetur in progressione arithmetica, dum vires CK in progressione geometrica sumuntur. Dico igitur quod distantia corporis ab eius altitudine maxima sit ut excessus area: *AbNK* supra aream *DET*.

Nam cum *AK* sit ut resistentia, id est, ut $APq + 2BAP$; assumatur data quaevis quantitas *Z*, & ponatur *AK* aequalis $\frac{APq+2BAP}{Z}$; & (per huius lemma II.) erit ipsius *AK* momentum *KL* aequale $\frac{2APQ+2BA\times PQ}{Z}$ seu $\frac{2BPQ}{Z}$ & areae *AbNK* momentum *KLON* aequale $\frac{2PBQ\times LO}{Z}$ seu $\frac{PBQ\times BDcub.}{2Z\times CK\times AB}$.

Cas. 1. Iam si corpus ascendit, sitque gravitas ut $ABq + BDq$, existente *BET* circulo (in figura prima) linea *AC* quae gravitati proportionalis est, erit $\frac{ABq+BDq}{Z}$, seu $APq + 2BAP + ABq + BDq$ erit $AK \times Z + AC \times Z$ seu $CK \times Z$; ideoque area *DTV* erit ad aream *DPQ* ut *DTq* vel *VBq* ad $CK \times Z$.

Cas. 2. Sin corpus ascendit, & gravitas sit ut $ABq - BDq$, linea *AC* (in figura secunda)



erit $\frac{ABq-BDq}{Z}$ & *DTq* erit ad *DPq* ut *DFq* seu *DBq* ad $ABq - BDq$ seu $APq + 2BAP + ABq - BDq$, id est, ad $AK \times Z + AC \times Z$ seu $CK \times Z$. Ideoque area *DTV* erit ad aream *VPQ* ut *VBq* ad $CK \times Z$.

Cas. 3. Et eodem argumento, si corpus descendit, & propterea gravitas sit ut $BDq - ABq$, & linea *AC* (in figura tertia) aequetur $\frac{BDq-ABq}{Z}$ erit area *DTV* ad aream *DPQ* ut *DBq* ad $CK \times Z$: ut supra.

Cum igitur areae illae semper sint in hac ratione; si pro area *DTV*, qua momentum temporis sibimet ipsi semper aequale exponitur, scribatur determinatum quodvis rectangulum, puta $BD \times m$, erit area *DPQ*, id est, $\frac{1}{2} BD \times PQ$; ad $BD \times m$ ut $CK \times Z$ ad *BDq*. Atque inde sit $PQ \times BDcub.$ aequale $2BD \times m \times CK \times Z$, & areae *AbNK* momentum *KLON* superius inventum sit $\frac{BP \times BD \times m}{AB}$. Auferatur area *DET* momentum *DTV* seu $BD \times m$, & restabit $\frac{AP \times BD \times m}{AB}$. Est igitur differentia momentorum, id est, momentum differentiae arearum, aequalis $\frac{AP \times BD \times m}{AB}$; & propterea ob datum $\frac{BD \times m}{AB}$, ut velocitas *AP*, id est, ut momentum spatii quod corpus ascendendo vel descendendo describit. Ideoque differentia arearum & spatium illud, proportionalibus momentis crescentia vel decrescentia & simul incipientia vel simul evanescentia, sunt proportionalia. *Q. E. D.*

Corol. Si longitudo, quae oritur applicando aream *DET* ad linea *BD*, dicatur *M*; & longitude alia *V* sumatur in ea ratione ad Iongitudinem *M*, quam habet linea *DA* ad lineam *DE*: spatium, quod corpus ascensu vel descensu toto in medio resistente describit, erit ad spatium, quod corpus in medio non resistente e quiete cadendo eodem tempore describere potest, ut arearum predictarum differentia ad $\frac{BD\times V^2}{AB}$: ideoque ex dato tempore datur. Nam spatium in medio non resistente ea in duplicata ratione temporis, sive ut V^2; & ob datas *BD* & *AB* ut $\frac{BD\times V^2}{AB}$. Haec area aequalis est areae $\frac{DAq\times BD\times M^2}{DEq\times AB}$, & ipsius *M* momentum est *m*; & propterea huius areae momentum est $\frac{DAq\times BD\times 2M\times m}{DEq\times AB}$. Hoc autem momentum est ad momentum differeniae arearum praedictarum *DET* & *AbNK*, viz. ad $\frac{AP\times BD\times m}{AB}$ ut $\frac{DAq\times BD\times M}{DEq}$ ad $\frac{1}{2}BD\times AP$, sive ut $\frac{DAq}{DEq}$ in *DET* ad *DAP*; ideoque, ubi areae *DET* & *DAP* quam minimae sunt, in ratione aequalitatis. Area igitur $\frac{BD\times V^2}{AB}$, & differentia arearum *DET* & *AbNK*, quando omnes hae areae quam minime sunt, aequalia habent momenta; ideoque sunt aequales. Unde cum velocitates, & propterea etiam spatia in medio utroque in principio dcscensus vel fine ascensus simul descripta accedant ad aequalitatem; ideoque tunc sint ad invicem ut area $\frac{BD\times V^2}{AB}$, & arearum *DET* & *AbNK* differentia; & praeterea cum spatium in medio non resistente sit perpetuo ut $\frac{BD\times V^2}{AB}$, & spatium in media resistente sit perpetuo ut arearum *DET* & *AbNK* differentia: necesse est, ut spatia in medio utroque, in aequalibus quibuscunque temporibus descripta, sint ad invicem ut area illa $\frac{BD\times V^2}{AB}$, & arearum *DET* & *AbNK* differentia. *Q.E.D.*

Scholium.

Resistentia corporum sphericorum in fluidis oritur partim ex tenacitate, partim ex friciione, & partim ex densitate medii. Et resistentiae partem illam, quae oritur ex densitate fluidi diximus esse in duplicata ratione velocitatis; pars altera, quae oritur ex tenacitate fluidi, est uniformis, sive ut momentum temporis: ideoque iam pergere liceret ad motum corporum, quibus resistitur partim vi uniformi seu in ratione momentorum temporis, & partim in ratione duplicata velocitatis. Sed sufficit aditum patefecisse ad hanc speculationem in propositionibus VIII. & IX. quae praecedunt, & eorum corollariis. In iisdem utique pro corporis ascendentis resistentia uniformi, quae ex eius gravitate oritur, substitui potest resistentia uniformis, quae oritur ex tenacitate medii, quando corpus sola vi insita movetur , & corpore recta ascendente addere licet hanc uniformem resistentiam vi gravitatis , eandemque subducere, quando corpus recta descendit. Pergere etiam liceret ad motum corporum, quibus resistitur partim uniformiter, partim in ratione velocitatis, & partim in ratione duplicata velocitatis. Et viam aperui in propositionibus praecedentibus XIII & XIV in quibus etiam resistentia uniformis, quae oritur ex tenacitate medii pro vi gravitatis substitui potest, vel cum eadem, ut prius, componi. Sed propero ad alia.