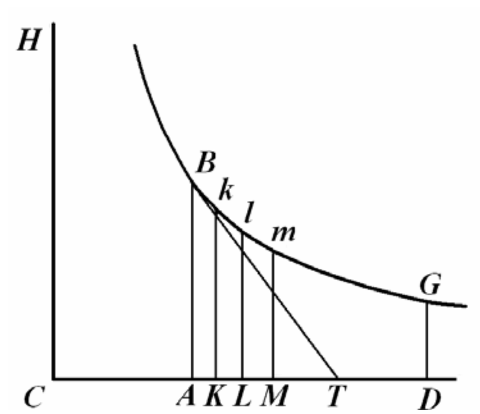
SECTlON II.

Concerning the motion of bodies resisted in the square ratio of the velocities.

PROPOSITION V. THEOREM III.

If there is resistance to a body in the square ratio of the velocity, and likewise it may be moving only by the inertial force through the same medium, truly the times in going from the smaller to the greater bounds may be taken in a geometric progression : I say that the velocities from the beginning of the individual times are inversely in the same geometric progression, and that the distances, which are described in the individual times, are equal.

For because the resistance of the medium is proportional to the square of the velocity, and the decrease of the velocity is proportional to the resistance ; if the time may be divided into innumerable equal parts, the differences of the same velocities will be proportional to the squares of the velocities from the beginnings of the individual times. Let these elements of time *AK, KL, LM,* &c. be taken on the right line *CD*, and the perpendiculars may be erected *AB, kK, Ll, Mm,* &c., meeting the hyperbola *BklmG* described, (with centre *C* and rectilinear asymptotes *CD*, C*H*) at *B, k, l, m,* &c., [where these vertical lines are taken as the velocities at these times] and *AB* shall be to *Kk* as *CK* to *CA,* and on separating the ratio, $AB-Kk$ shall be to *Kk* as *AK* to *CA,* & in turn $AB-Kk$ shall be to *AK* as *Kk* to C*A*, and thus as $AB\times Kk$ to $AB\times CA$. From which, since *AK* and $AB\times CA$ may be given, $AB-Kk$ will be as $AB\times Kk$, and finally, when *AB* and *Kk* merge, as AB^2 . And by like arguments $Kk-Ll$, $Ll-Mm$,etc. will be as Kk^2, Ll^2.etc. Therefore the squares of the lines *AB*, *Kk*, *Ll*, *Mm* are as the differences of the same and therefore since the squares of the velocities also were as the differences of these, the progression of both shall be similar.



[Thus, in terms of Newton's hyperbola :

$CA\times AB=CK\times kK$ or $\frac{CK}{CA}=\frac{AB}{kK}$; $\frac{CK-CA}{CA}=\frac{AB-kK}{kK}$ or $\frac{AK}{CA}=\frac{AB-kK}{kK}$

and $\frac{kK}{CA}=\frac{AB-kK}{AK}=\frac{AB\times Kk}{AB\times CA}$; hence $AB-Kk\propto AB\times Kk$; leading to the incremental change in the ordinate being proportional to the square of the ordinate when $Kk\to AB$.

The ordinates of the curve correspond to the velocities, and the abscissae correspond to the times : thus, the difference in the velocity between increments is proportional to the product of the velocities at the start of the increments. Newton has shown that the difference of neighbouring velocities is proportional to the product of these velocities, as required, and hence the gradient is proportional to the resistive force or acceleration.

Now, in analytical terms, the general drift of the solution follows if we consider :

$\frac{dv}{dt} = -kv^2$ or $\frac{dv}{v^2} = -kdt$ giving $\frac{1}{v} - \frac{1}{v_0} = k(t - t_0)$, for the neighbouring velocities $AB = v_0, Kk = v(\Delta t), Ll = v(2\Delta t)$, etc.; then $v(\Delta t) - v_0 = \Delta v = -kv_0 v(\Delta t) \to -kv_0^2$, as above; and from $\frac{1}{v} - \frac{1}{v_0} = k(t - t_0)$, the velocity varies inversely with the time, on choosing $\frac{1}{v_0} = kt_0$, with the starting point $t = t_0, v = v_0$; where for equal small changes in the time, $\Delta v = -kv^2 \Delta t$.

Again, to pass on to the next integration, we may write :
$\frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = v\frac{dv}{dx} = -kv^2$ or $\frac{dv}{dx} = -kv$ giving $\frac{dv}{v} = -kdx$, and $ln\frac{v}{v_0} = -kx$ or $v(x) = v_0 e^{-kx}$; giving $x = \frac{1}{k} ln\frac{v_0}{v}$; thus, the velocity decays exponentially with distance.]

With which demonstrated, it is a consequence also that the areas described by these lines shall be in a similar progression with the distances which are described by these velocities. Therefore if the initial velocity of the first time *AK* interval may be represented by the line *AB*, and if the initial velocity of the second *KL* by the line *Kk*, and with the distance from the first time described by the area *AKkB;* all the subsequent velocities may be represented by the following lines *Ll, Mm,* &c., and the distances described by the areas *Kl, Lm,* &c. And by adding these together, if the whole time may be expressed by the sum *AM* of all their parts, the whole length described may be represented by the sum of the parts *AMmB.* Now consider the time *AM* divided thus into the parts *AK, KL, LM,* &c. so that *CA, CK, CL, CM,* etc. shall be in a geometric progression ; and those parts will be in the same progression, and the velocities *AB, Kk, Ll, Mm,* &c. shall be [, from the nature of the hyperbola,] in the same progression inverted, and equal distances described *Ak, Kl, Lm,* &c. *Q.E.D.*

[Thus *AK, KL, LM,* etc., which are the differences of the lines *CA, CK, CM,* etc., are in the same progression. For the differences of any geometric progression are in the same geometric progression. Thus if there shall be :
$\frac{CA}{CK} = \frac{CK}{CL} = \frac{CL}{CM}$, etc.; on taking the previous term from each: $\frac{CA}{CK} = \frac{AK}{KL} = \frac{KL}{LM}$, etc., then the differences in the times are also in a geometric progression ;]

Corol. 1. Therefore it may be apparent, if the time may be represented by some part of the asymptote *AD,* and the velocity at the beginning of the time by the applied ordinate *AB;* then the velocity will be represented at the end of the time by the ordinate *DG,* and the whole distance described by the adjacent hyperbolic area *ABGD;* and also the distance, that some body can describe in the same time *AD.* with the first velocity *AB,* in a non-resisting medium, is given by the rectangle $AB \times AD$.
[Thus the mathematics is far simpler at this point than in the first section; the hyperbola is the velocity vs time curve itself, the area within a sector is the distance gone, and the gradient is proportional to the acceleration.]

Corol. 2. From which the distance is given in a resisting medium, by taking that likewise to the distance that it may be able to describe in a non-resisting medium with a uniform velocity *AB*, as the hyperbolic area *ABGD* to the rectangle $AB \times AD$.

Corol. 3. Also the resistance of the medium is given, by putting in place that initial motion to be itself equal to a uniform centripetal force, which by a body falling in the time *AC*, may be able to generate the velocity *AB*, in a non resisting medium. For if *BT* may be drawn touching the hyperbola in *B*, and crossing the asymptote at *T*; the right line *AT* will be equal to *AC* [from a property of the hyperbola], and the time may be represented, in which the initial resistance uniformly continued may be able to remove the whole velocity *AB.*

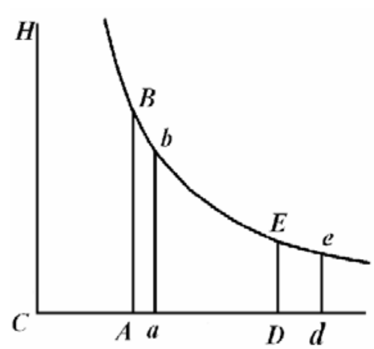
Corol. 4. And thence also the proportion of this resistance to the force of gravity is given, or some other centripetal force given.

Corol 5. And in turn, if the proportion of the resistance may be given to some centripetal force, then the time *AC* is given, in which a centripetal force equal to the resistance may be able to generate some velocity *AB :* and thence the point *B* may be given through which the hyperbola must be drawn, with the asymptotes *CH*, *CD* ; and so that a body by beginning its motion with that velocity *AB*, can describe that distance *ABGD* in some time *AD,* in a medium with a similar resistance.

PROPOSITION VI. THEOREM IV.

Equal homogeneous spherical bodies, impeded by resistances in the square ratio of the velocities, and moving by the inertial force alone, always describe equal distances in times which are inversely as the velocities at the start of the times, and lose proportional parts of their whole velocities.

With the rectangular asymptotes *CD, CH,* some hyperbola *BbEe* is described cut by the perpendiculars *AB, ab, DE, de,* in *B, b, E, e;* the initial velocities are represented by the perpendiculars *AB, DE,* and times by the lines *Aa*, *Dd.* Therefore as *Aa* is to *Dd* thus (by hypothesis) *DE* is to *AB*, and thus (from the nature of the hyperbola) as *CA* to *CD;* and on adding together, thus as *Ca* to *Cd.* Therefore the areas *ABba, DEed,* that is, the distances described are equal between themselves, and the initial velocities *AB, DE* are proportional both to the final ones *ab*, *de,* and therefore with the parts separated, also to the parts of these lost $AB - ab, DE - de$. *Q.E.D.*



[$\frac{Aa}{Dd} = \frac{DE}{AB} = \frac{CA}{CD} = \frac{CA+Aa}{CD+Dd} = \frac{Ca}{Cd}$. Thus, $\frac{CA}{CD}$ represents the ratio of the time intervals in which the velocity ratio equally changes according to $\frac{DE}{AB}$, and also the changes to the velocities ratio in the incremental times *Aa* and *Dd*: Now $v(x) = v_0 e^{-kx}$ or $x = \frac{1}{k} ln\left(\frac{v_0}{v(x)}\right)$ gives here in an obvious notation, $x_{Aa} = \frac{1}{k} ln\left(\frac{v_A}{v_a}\right)$ and $x_{Dd} = \frac{1}{k} ln\left(\frac{v_D}{v_d}\right)$; since the velocity ratios are equal, then so are the distances gone.]

PROPOSITION VII. THEOREM V.

Spherical bodies in which there is resistance in the square ratio of the velocities, in times, which are directly as the initial motions and inversely as the initial resistances, will lose proportional parts of the whole motion, and they will describe distances proportional jointly to these times and to these initial velocities.

For the parts of the motions lost are as the resistances and the times conjointly. Therefore as these parts shall be proportional to the whole, the resistance and the time conjointly must be as the motion. Hence the time will be as the motion directly and the resistance inversely. Whereby with the small differences [*i.e.* differentials] of the times taken in that ratio, the bodies will always lose small parts proportional to the whole motion, and thus they will retain velocities always proportional to their initial velocities. And on account of the given ratio of the velocities, they always describe distances which are as the initial velocities and times conjointly. *Q. E. D.*

[Here we follow the demonstration of Leseur & Janquier, in Note 89:

The whole demonstration of this proposition is set out by analysis in this manner. Let the mass of some globe be *m*, the initial velocity of the motion *c*, at the end of the time *t* it shall be *v*, with the initial resistance of the motion *r*, and because the resistances of the body at different places shall be as the square of the velocity, by hypothesis, there will be c^2 to v^2 as *r* to the resistance at the end of the time *t*, which will be $\frac{rv^2}{c^2}$. But the resistance $\frac{rv^2}{c^2}$ is as the decrement of the motion $-mdv$ directly, and as the time *dt* inversely, that is $\frac{rv^2}{c^2} = -\frac{mdv}{dt}$, and hence $dt = -\frac{mc^2dv}{rv^2}$, and with the fluents taken [*i.e.* integrals], $t = Q + \frac{mc^2}{rv}$. There is put $t = 0$, and $Q = -\frac{mc}{r}$, with which value substituted there becomes $t = \frac{mc^2 - mcv}{rv}$. The time may be taken directly as the first motion directly and as the first resistance *r* inversely, that is, $t \propto \frac{mc}{r}$, and there will be $\frac{mc}{r} \propto \frac{mc^2 - mcv}{rv}$; and thus $mcv \propto mc^2 - mcv$, and on dividing by *c*, $mv \propto mc - mv$; that is, the motion lost is as the initial motion; and hence on account of the given mass m, there will be also, c as $c - v$: that is the velocity lost as the initial velocity; thus also there will be c as $c - c + v$, or v, that is, the first velocity *c* is in a given ratio to the remaining velocity *v*. Now if the distance described is called *s* in the time *t*, there will be $ds = vdt$, and because *v* is as the given c, there will be *ds* as *cdt*, and with the fluents taken on account of *c* given, there becomes *s* as *ct*. Q.e.d.

Note 90. Because the distance $s \propto ct$, and $t \propto \frac{mc}{r}$, also there will be $s \propto \frac{mcc}{r}$; the mass m of a globe of which the diameter shall be *D*, and with the density for the globe given as the mass to the volume, that is, as the cube of diameter D^3; whereby there becomes $s \propto \frac{D^3cc}{r}$. If in addition with the velocity *c* given, the resistance *r* is as the diameter *D*, the index of the power of which is *n*, that is $r \propto D^n$, and hence with the velocity not given, the resistance *r*, as $D^n c^2$, will be as $s \propto \frac{D^3c^2}{D^nc^2}$, or as D^{3-n}. From which the Corollaries following are apparent.]

Corol. 1.Therefore if for bodies having the same velocity there is resistance in the square ratio of the diameters : homogeneous globes moving with some velocity or other, on describing distances proportional to their diameters, will lose parts of the motions proportional to the wholes. For the motion of any globe will be as the velocity and the mass conjointly, that is, as the velocity and the cube of the diameter, the resistance (by hypothesis) will be as the square of the diameter and the square of the velocity conjointly, and the time (by this proposition) is in the first ratio directly, and in the second inversely ; that is, as the diameter directly and the velocity inversely , and thus the distance, in proportion to the velocity and the time, is as the diameter.

Corol. 2, If there is resistance for bodies moving with equal velocities in the three on two ratio of the diameters : homogeneous globes moving with some velocity or other, by describing distances in the three on two ratio of the diameters, lose parts of the motions proportional to the wholes.

Carol. 3. And generally, if there is resistance for equally fast bodies in the ratio of some power of the diameters : the distances in which homogeneous globes, with some velocity of the motion, will lose parts of the motions proportional to the wholes, will be as the cubes of the diameters to that applicable power. Let the diameters be *D* and *E*; and if the resistances, when the velocities are put equal, shall be as D^n and E^n : the distances in which the globes, moving with some velocity or other, will lose parts of the motions proportional to the wholes, will be as D^{3-n} and E^{3-n}. And therefore homogeneous globes by describing proportional distances D^{3-n} and E^{3-n} themselves, will retain velocities in the same ratio in turn and from the beginning.

*Corol.*4. For if the globes may not be homogeneous, the distance described by the denser globe must be increased in the ratio of the density. For the motion, with equal velocity, is greater in the ratio of the density, and the time (by this proposition) may be increased in the ratio of the motion directly, and the distance described in the ratio of the time.

Corol. 5 And if globes may be moving in different mediums, the distance in the medium, that resists more than the other parts, will be diminished in the ratio of the greater resistance. For the time (by this proposition) may be diminished in the ratio of the increased resistance, and the distance in the ratio of the time.

LEMMA II.

The moment of a generating quantity [genitam] is equal to the moments arising from the generating quantities of the individual parts, with the indices of the powers of the same parts and coefficients continually multiplied out.

[*Genitam* is Newton's word describing a function of several variables, or more roughly as befits the times, a formula with several variables present ; the Latin word means : that brought forth, born, etc.; by *moment* Newton means the small momentary quantity that may be associated with a given generating quantity which itself is allowed to increase by its own moments ; Newton considered time as the free variable against which all other changes are to be measured. Please note that it is an anachronism to use the word *function* in the *Principia*; it was especially the arrival of Euler on the scene that started the change in people's ways of thinking about such things. Whereas for example, we think of a table of sines as representing special values of the sine function for discrete angles, at the time it was commonplace to consider such as a compilation of the ratios opp./hypot. for the collection of right-angled triangles with the corresponding angles, and nothing more ; likewise logarithms, etc. Some mathematicians such as Newton clearly thought further than this, of interpolations, and so forth, but there was the enormous task of setting up a framework to accommodate such activities by devising suitable notations, and so on. Thus Newton stumbles forward trying to find an appropriate name for something important, that we now call a function.

The interested observer may note that Newton's proofs of his propositions are little more than word descriptions of the analytical calculations he has performed : such proofs are very hard if not impossible at times to understand without the underlying calculus based methodology. So if you wonder : how on earth did he come upon this? The answer must be : he didn't do it this way initially, the proof presented is an afterthought, where he paints a pretty geometrical or graphical picture at the end of the analytical process. Hence, to understand Newton, read the enounciation of the proposition, work it out for yourself using calculus of the Leibniz kind, then read Newton's explanation, and you will, hopefully, understand everything. These thoughts are those of the translator: you may or may not agree with them; in any case, each person is entitled to their own opinion.]

I call a quantity a generating quantity [*i.e. gentitam*] , which arises from several parts or terms in arithmetic by multiplication, division, and extraction of roots, without addition or subtraction, or in geometry both by the discovery of contained quantities and sides [in the sense side of a square or cube, or the square , cube root, etc.], or of mean and extreme proportions, again without addition or subtraction. Products, quotients, roots, rectangles, squares, cubes, sides squared, sides cubed, and the like. These quantities I consider here, as indeterminate and variable, and as if continually increasing or decreasing in a state of motion or flux, and the momentary increments or decrements of these I understand by the name of moments : thus so that the increments may be obtained from added or positive quantities, and the decrements for subtracted or negative quantities. Yet beware that you have to understand small finite parts. The small finite parts are not the moments, but quantities arising themselves from the moments. Now the origins of finite magnitudes are required to be understood. For the magnitude of moments will not be seen in this Lemma, but the first proportion of these arising. It returns to the same thing if in place of moments there may be taken either the velocities of increments or decrements (which motion also is allowed to be called the changes and fluxions of quantities) or some finite quantity proportional to these velocities. But the coefficient is the quantity which arises from some part generated by applying the genitam to that part.

Therefore the sense of the lemma is, that if the moments of some quantities *A*, *B*, *C*, &c. shall always be increasing or decreasing by a motion , or from these the proportional velocities of the changes may be called *a*, *b*, *c*, &c.; the moment or change of the generating quantity of the rectangle *AB* were $aB + bA$, and the moment of the contained generating quantity ABC were $aBC + bAC + cAB$: and the moments of the powers of the generating quantities

$$A^2,\ A^3,\ A^4,\ A^{\frac{1}{2}}\ ,\ A^{\frac{3}{2}}, A^{\frac{1}{3}}, A^{\frac{2}{3}}, A^{-1},\ A^{-2},\ \&\ A^{-\frac{1}{2}}, \text{ were}$$
$$2.aA,\ 3aA^2, 4aA^3, \tfrac{1}{2}aA^{-\frac{1}{2}}, \tfrac{3}{2}aA^{\frac{1}{2}},\ \tfrac{2}{3}aA^{-\frac{2}{3}},\ \tfrac{2}{3}aA^{-\frac{1}{3}}, -aA^{-2},$$
$$-2A^{-3}, \&\ -\tfrac{1}{2}aA^{-\frac{3}{2}} \text{ respecive.}$$

And generally, so that the moment of any power $A^{\frac{n}{m}}$ were $\frac{n}{m}aA^{\frac{n-m}{m}}$. Likewise so that the moment of the generating quantity A^2B were $2aAB + bA^2$*;* and the moment of the generating quantity $A^3B^4C^2$ $3aA^2B^4C^2 + 4bA^3B^3C^2 + 2cA^3B^4C$; and the moment of the generating quantity $\frac{A^3}{B^2}$ or A^3B^{-2} becomes $3aA^2B^{-2} - 2bA^3B^{-3}$ and thus with the rest. Truly, the lemma may be demonstrated in this manner.

Case 1. Some rectangle *AB* always increased by a motion, where half of the moments $\frac{1}{2}a$ and $\frac{1}{2}b$ may be lacking from the sides *A* and *B*, which becomes $A - \frac{1}{2}a$ by $B - \frac{1}{2}b$, or $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$*;* and so that the initial sides *A* and *B* increased by the other moments added on, becomes $A + \frac{1}{2}a$ by $B + \frac{1}{2}b$ or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$.The first rectangle may be subtracted from this rectangle, and there will remain $aB + bA$. Therefore the increment $aB + bA$ of the rectangle is generated from the whole increments of the sides *a* and *b*. *Q.E.D.*

Case 2. *AB* may always be put equal to *G*, and the moment of the volume *ABC* or *GC* (by case 1, will be $gC + cG$, that is (if for *G* and *g* there may be written AB and $aB + bA$) $aBC + bAC + cAB$. And the account is the same for a generating quantity containing any number of sides [variables]. *Q.E.D.*

Case 3. The sides *A, B, C* may always themselves be put mutually equal ; and of A^2, that is of the rectangle *AB*, the moment $aB + bA$ will be 2*aA*, but of A^3, that is the content of

ABC, the moment $aBC + bAC + cAB$ will be $3aA^2$. And by the same argument the moment of any power A^n is naA^{n-1}. *Q.E.D.*

Case 4. From which since $\frac{1}{A}$ by *A* shall be 1, the moment of $\frac{1}{A}$ taken with *A*, together with $\frac{1}{A}$ multiplied by *a* will be the moment of 1, that is, zero.

Therefore the moment of $\frac{1}{A}$ or of A^{-1} itself is $\frac{-a}{A^2}$. And generally since $\frac{1}{A^n}$ by A^n shall be 1, the moment of $\frac{1}{A^n}$ itself taken by A^n together with $\frac{1}{A^n}$ with naA^{n-1} will be zero. And therefore the moment of $\frac{1}{A^n}$ or of A^{-n} itself will be $-\frac{na}{A^{n+1}}$. *Q.E.D.*

Case 5. And since $A^{\frac{1}{2}}$ by $A^{\frac{1}{2}}$ shall be *A*, the moment of $A^{\frac{1}{2}}$ itself multiplied by $2A^{\frac{1}{2}}$ will be *a*, and thus by case 3: the moment of $A^{\frac{1}{2}}$ will be $\frac{a}{2A^{\frac{1}{2}}}$, or $\frac{1}{2}aA^{-\frac{1}{2}}$. And generally if $A^{\frac{m}{n}}$ may be put equal to *B*, A^m will be equal to B^n, and thus maA^{m-1} is equal to nbB^{n-1}, and maA^{-1} equals nbB^{-1} or $nbA^{-\frac{m}{n}}$, and thus $\frac{m}{n}aA^{\frac{m-n}{n}}$ equals *b*, that is, equal to the moment of $A^{\frac{m}{n}}$. *Q.E.D.*

Case 6. Therefore the moment of any generated quantity A^mB^n is the moment of A^m taken with B^n, together with the moment of B^n taken with A^m, that is $maA^{m-1}B^n + nbB^{n-1}A^m$; and thus the power of the indices *m* and *n* shall be either whole or fractional numbers, and either positive or negative.
And the account is the same starting from several contained powers. *Q.E.D.*

Corol. I. Hence in continued proportions, if one term is given ;
the moments of the remaining the terms will be as the same terms multiplied by the number of intervals between these and the given term. Let *A, B, C, D, E, F* be continued proportionals ; and if the term *C* is given, the moments of the remaining terms will be amongst themselves as $-2A, -B,\ D,\ 2E,\ 3F$.

[For, since *A, B, C, D, E, F* are continued proportionals $D : C = C : B = \frac{C^2}{D} = C^2D^{-1}$ and similarly it is found that $A = \frac{C^3}{D^2} = C^3D^{-2}$, $E = \frac{D^2}{C}$, $F = \frac{D^3}{C^2}$, etc. Whereby on account of *C* given, the moment of this is zero, the moments of the remaining terms will be (by cases 2 & 3), $-2dC^3D^{-3}, -dC^2D^{-2}, d, \frac{2dD}{C}, \frac{3dD^2}{C^2}$, and on multiplying the individual terms by $\frac{D}{d}$, the proportions of the terms will remain :
$-2C^3D^{-2}, -C^2D^{-1}, D, \frac{2D^2}{C}, \frac{3D^3}{C^2}$, that is $-2A, -B, D, 2E, 3F$. But the number of terms between the term *A* and the given term *C* is 2, and thus is the interval between *E* and C; 1 is the interval between *B* and *C*, and between *C* and *D*; and 3 is the number of intervals between *C* and *F*. Whereby the truth of the Corollaries is verified.]

Corol. 2. And if in four proportionalities the two middle moments may be given, the moments of the extremes will be as the same extremes. The same is understood concerning the sides of any given rectangle.

[Let $A : B = C : D$ or $B \times C = A \times D$, and $B \times C$, the given rectangle, will be by Case 1. $aD + dA = 0$, and hence $aD = -dA$, and thus $a : -d = A : D$.]

Corol. 3. And if the sum or difference of two squares may be given, the moments of the sides will be inversely as the sides.

[Let the sum of two squares equal the given square, or $A^2 + B^2 = C^2$ then, by Case 3, $2aA + 2bB = 0$, and thus $aA = -bB$ and hence $a : -b = B : A$. In these two corollaries it is necessary that with one of the variables increasing, the other is decreasing, and thus while the moment of one is positive, the other moment is negative.]

Scholium.

In a certain letter to our countryman *D.J. Collins*, dated10th December, 1672, since I may have been establishing a method of tangents that I suspected to be the same as that of Slusius, that had not yet been made public; I adjoined : ....*This is one particular instance or rather corollary of a general method, which can be brought to bear on any troublesome calculation, not only to drawing tangents to some curves, either geometrical or mechanical, or respecting right lines or any other curves, truly also to the resolution of other more obstruse general problems concerning curvatures, areas, lengths, centres of gravity of curves, etc. nor (as Hudden's method of maximas and minimas) is it restricted only to those equations in which irrational quantities are absent. I have carefully combined this method with that other so that I may reduce the equations arising to infinite series.* The letter up to this point. And these final words are with regard to that treatise I had written about these things in the year 1671. Truly the fundamentals of this general method are contained in the preceding lemma.

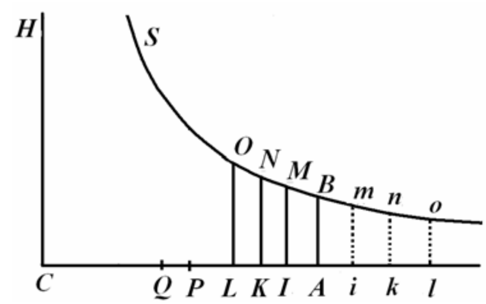
[Thus Newton puts his stamp on Calculus as being his invention, in the third edition of the *Principia*; earlier editions had contained references to Leibniz. Certainly, we must grant to Newton the wonderful ideas expressed in the above Lemma, which should be read by any serious student of mathematics : How many have or ever will ? However, we must also grant to Leibniz the invention of the notation which has stood the test of time, and which has been a cornerstone of the theory ever since. Both men made major contributions to the fledgling art, which had its beginnings in the work of others [see for example, Boyer's *The History of the Calculus*....Dover.]; in a world devoid of adverse human passions and frailties, one might hope that they could have shook hands, and said: *We did this together*. Such an event never happened; neither conceded honour to the other; Leibniz died a lonely old man in 1717, perhaps broken by the dispute: only his valet came to the funeral, while Newton reaped every award available to him, and now lies finally at rest in Westminster Abbey, the tomb jealously guarded by clerics who will stop you taking a photo, if they can ; one wonders who is treated with the greater respect regarding this controversy today.

A useful commentary on the dispute can be found by anyone, thanks to the generosity of JSTOR: *The manuscripts of Leibniz on his discovery of the differential calculus.* J.M. Child. *The Monist*, Vol. 26, No. 4 (Oct., 1916).]

PROPOSITION VIII. THEOREM VI.

If a body in a uniform medium, with gravity acting uniformly, may rise or fall in a straight line, and the whole distance may be separated into equal parts, and in which at the beginning of the individual parts (with the resistance of the medium added to the force of gravity, when the body rises, or by subtracting the same when the body falls) the absolute forces may be found; I say that these absolute forces are in geometric progression.

For the force of gravity may be shown by the given line *AC* [in the descent of the body from *A* at rest]; the resistance by the variable line *AK;* the absolute force in the descent of the body by the difference *KC;* the velocity of the body by the line *AP,* which shall be the mean proportional between *AK* and *AC*



[thus, Newton selects $AP^2 = AK \times AC$ to obtain a geometric progression, and *AC* is fixed, so that the resistance is proportional to the velocity squared, and *AC* is taken as the constant of proportionality],

and thus in the square root ratio of the resistance; the increment of the resistance made in a small time interval by the line *KL*, and the increment of the velocity at the same time by the short line *PQ*; and some hyperbola *BNS* will be described with the centre *C* and with the rectangular asymptotes *CA*, C*H*, with the perpendiculars *AB, KN, LO* erected meeting at *B, N, O*. Because *AK* is as AP^2, the moment of the one *KL* will be as the moment of the other 2*AP*.*PQ*. that is, as *AP* to *KC;* [Thus, $2AP \times PQ = AC \times KL$, or $2vdv = gdR$]; for the increment of the velocity *PQ* (by Law II of motion) is proportional to the force arising *KC*. The ratio of *KL* may be compounded with the ratio of *KN*, and the rectangle $KL \times KN$ [i.e. the increment in the area] becomes as $AP \times KC \times KN$ *;* that is, on account of the given rectangle $KC \times KN$, [from the nature of the hyperbola], $KL \times KN$ becomes as *AP*. And the vanishing ratio of the area of the hyperbola *KNOL* to the rectangle $KL \times KN$ is one of equality, when the points *K* and *L* coalesce. Therefore that vanishing hyperbolic area is as *AP*. Therefore the whole hyperbolic area *ABOL*, compounded from the individual areas *KNOL,* is always proportional to the velocity *AP*, and therefore proportional to the distance described with that velocity [in each incremental area]. Now that area may be divided in to the equal parts *ABMI; IMNK; KNOL,* &c. and the absolute forces *AC*, *IC*, *KC*, *LC,* &c. will be in a geometric progression. *Q.E.D.*

And by a similar argument [see following note], in the ascent of the body, by taking, at the contrary part of the point *A*, equal areas *ABmi, imnk, knol,* &c. and it will be agreed

that the absolute forces *AC, iC, kC, IC,* &c. are continued proportionals. And thus if all the distances in the ascent and in the descent may be taken equal; all the absolute forces *lC, k*C, *iC, AC, IC, KC, LC,* &c. will be continued proportionals, *Q.E.D.*
[In modern terms, we have the force equation : $\frac{dv}{dt} = v\frac{dv}{dx} = \pm g - kv^2$; initially with gravity absent as above, the equation can be integrated at once to give : $\frac{1}{v} - \frac{1}{v_0} = kt$. Again, from $v\frac{dv}{dx} = -kv^2$ we find : $v(x) = v_0 e^{-kx}$, and from which equations we can find at once: $x(t) = \frac{1}{k}ln(1+kv_0 t)$.

With the body projected downwards , the equation becomes with gravity acting :
$\frac{dv}{dt} = v\frac{dv}{dx} = g\left(1-\frac{k}{g}v^2\right)$, or $\frac{vdv}{1-\frac{k}{g}v^2} = gdx$, giving $ln\left(\frac{1-\frac{k}{g}v^2}{1-\frac{k}{g}v_0^2}\right) = -2kx$; or $g - kv^2 = \left(g - kv_0^2\right)e^{-2kx}$
this equation is also true if the body is projected upwards with the sign of *g* changed, and shows that the velocities are in a geometric progression.]

[L & J note : *And by a similar argument*.... For the force of gravity may be set out by the given line *AC*, the resistance by the indefinite line *Al*, the absolute force in the ascent of the body by the sum *Cl*, the velocity of the body by the line *Ap* which shall be the mean proportional between *Al* and *AC*, and thus in the square root ratio of the resistance ; the decrement of the resistance in the given small part of the time by the small line *pq*; and the hyperbola *SBo* will be described as above; because *Al* is as Ap^2 the moment of this *kl* will be as the moment of that 2*Apq*, that is, as *Ap* into *lC*; for the velocity of the decrement *pq* (by the Second Law of Motion) is proportional to the force generated *lC*, the ratio of this *kl* may be put together with the ratio of that *lo*, and the rectangle $kl \times lo$ as $Ap \times lC \times lo$, that is, on account of the rectangle $lC \times lo$, as *Ap*. Therefore, with the points *k*, *l* joined together, the hyperbolic area $knol = kl \times lo$ is as *Ap*. Therefore with the whole hyperbolic area 2*ABol* taken together from the individual *knol* always proportional to the velocity *Ap*, and therefore is proportional to the distance described with that velocity. Now the area may be divided up into equal parts *ABmi; imnk, knol,* etc., and the absolute forces *AC, iC, kC, lC,* etc. are in a geometric progression. *Q.e.d.*]

Corol. I. Hence if the distance described may be displayed by the hyperbolic area *ABNK;* the force of gravity, the velocity of the body and the resistance of the medium are able to be shown by *AC, AP*, and *AK* respectively , & vice versa.

[L & J note : *And vice versa*. In a similar manner if in the ascension of the body, the distance described until the motion is exhausted may be shown by the hyperbolic area *ABnk* ; the force of gravity, the velocity of the body, and the resistance of the medium can be put in place by the lines *AC, Ap*, and *Ak*.]

Corol. 2. And the line *AC* shows the maximum velocity, that the body by descending indefinitely, can acquire at some time.
[L & J note : For indeed $AP = AC$, and because, by construction, $AP^2 = AK \times AC$, there will also be $AK = AC$, and thus with the ordinate *KN* meeting the asymptote *CH*, the

hyperbolic area *ABNK* becomes infinite , and the distance described by descending in this proportion also will be infinite, truly the weight, the resistance, and the velocity of the body may be shown by the line *AC*, and then the resistance equals the weight, and therefore the velocity *AC* is a maximum.]

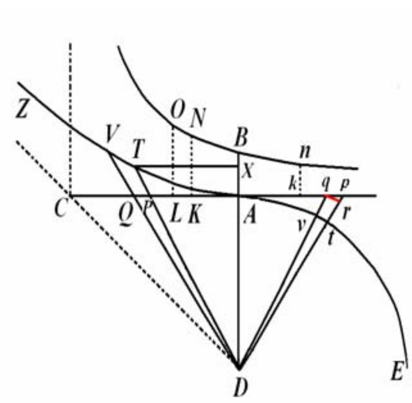
Corol. 3. Therefore if the resistance of the medium may be known for some given velocity, the maximum velocity may be found, by taking that to that known given velocity in the square root ratio, that the force of gravity has to that known resistance of the medium.

[Thus, when the velocity is a max., we have
$kv_{max}^2 = mg$; and $kv_{given}^2 = R$; then $\frac{v_{max}}{v_{given}} = \sqrt{\frac{mg}{R}}$.]

PROPOSITION IX. THEOREM VII.

With all now demonstrated in place, I say that, if the tangents of the angles of a circular section and of a hyperbolic section may be taken, with proportional velocities, with a radius of the correct magnitude present: the time for any ascent to the highest point will be as the sector of the circle, and the time for any descent from the highest place will be as the sector of the hyperbola.

To the right line *AC,* by which the force of gravity is shown, the perpendicular and equal right line *AD* may be drawn. With centre *D* and with semi diameter *AD* both the quadrant of the circle *AtE* may be drawn, as well as the rectangular hyperbola *AVZ* having the axis *AX*, the principle vertex *A,* and the asymptote *DC.* Now *Dp* and *DP* may be drawn and the sector of the circle *AtD* will be to the whole time ascending to the highest place; and the hyperbolic sector *ATD* will be as the whole time in descending from the highest place : But only if the tangents *Ap*, *AP* of the sectors shall be as the velocities.



Case 1. For *Dvq* may be drawn separating the moments of the sector *ADt* and of the triangle *ADp,* or the increments that likewise describe the incremental parts *tDv* and *qDp.* Since these increments, on account of the common angle *D*, are in the square ratio of the sides, the increment *tDv* will be as $\frac{qDp \times tD^2}{pD^2}$ (see first note below), that is, on account of the given *tD,* as $\frac{qDp}{pD^2}$. But pD^2 is as $AD^2 + Ap^2$, that is,

[Recall from the last proposition that *Ap* represents the variable velocity of the body, and *Ak* the associated resistance, proportional to the velocity squared.]

$AD^2 + AD \times Ak$, or $AD \times Ck$; and qDp is $\frac{1}{2} AD \times pq$.

[For $AC \times Ak$, or $AD \times Ak = Ap^2$, by Prop. VIII, and
$AD^2 + AD \times Ak = AD \times (AC + Ak) = AD \times Ck$.]

Therefore the increment of the sector *tDv* is as $\frac{pq}{Ck}$; that is, as the decrement of the velocity *pq* directly, and that force *Ck* which may diminish the velocity ; and thus inversely as the element of time corresponding to the decrement of the velocity. And on adding these together, the sum of all the increments *tDv* in the sector *ADt* shall be as the sum of all the individual increments of time with the velocity decreasing *Ap*, with the corresponding elements *pq* removed, until that velocity may have diminished to zero ; that is, the whole sector *ADt* is as the total time of ascending to the highest place.
Q. E. D.

[In modern terms, $\frac{dv}{dt} = g\left(1 - \frac{k}{g}v^2\right)$ gives $\frac{dv}{\left(1 - v\sqrt{\frac{k}{g}}\right)\left(1 + v\sqrt{\frac{k}{g}}\right)} = gdt$, and hence

$\frac{dv}{\left(1 - v\sqrt{\frac{k}{g}}\right)\left(1 + v\sqrt{\frac{k}{g}}\right)} = gdt$ and $\frac{dv}{1 - v\sqrt{\frac{k}{g}}} + \frac{dv}{1 + v\sqrt{\frac{k}{g}}} = \frac{\sqrt{kg}}{2}dt$, giving on integration

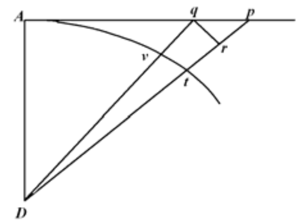
$ln\left(\frac{1 - v_0\sqrt{\frac{k}{g}}}{1 - v\sqrt{\frac{k}{g}}}\right) + ln\left(\frac{1 + v\sqrt{\frac{k}{g}}}{1 + v_0\sqrt{\frac{k}{g}}}\right) = \frac{\sqrt{kg}}{2}t$, provided $g > 0$, or if $g < 0$:

$\frac{dv}{dt} = g\left(1 + \frac{k}{g}v^2\right)$ giving $\frac{dv}{1 + \frac{k}{g}v^2} = -gdt$, and on integration,

$tan^{-1}\sqrt{\frac{k}{g}}v - tan^{-1}\sqrt{\frac{k}{g}}v_0 = -\sqrt{kg}t$. In the first case, the body is moving down, and in the second case, it is moving up.]

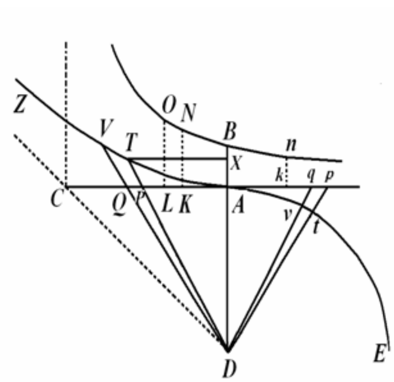
Case 2. *DQV* may be drawn cutting off the small parts *TDV* of the sector *DAV*, as well as the minimal part *PDQ* of the triangle *DAQ* ; and these small parts will be in turn as DT^2 to DP^2, that is (if *TX* and *AP* may be parallel) as DX^2 to DA^2 or TX^2 to AP^2, [on account of the similar triangles *DTX, DPA*], and on separating as $DX^2 - TX^2$ to $DA^2 - AP^2$. But from the nature of the hyperbola, [see second note below], $DX^2 - TX^2$ is AD^2, and by hypothesis AP^2 is $AD \times AK$ .Therefore the small parts are in turn as AD^2 to $AD^2 - AD \times AK$; that is, as *AD* to $AD - AK$ or *AC* to *CK:* and thus the small part of the sector *TDV* is $\frac{PDQ \times AC}{CK}$; [on account of *AC* to *AD*. For indeed $PDQ = \frac{1}{2} AD \times PQ$, and thus $TDV = \frac{\frac{1}{2} AD \times PQ \times AC}{CK}$.] ; and thus on account of *AC* and *AD* given, as $\frac{PQ}{CD}$, that is, directly as the increment of the velocity, and inversely as the force generating the increment ; and thus as the increment of the corresponding particle of time. And on adding together the sum of the elements of time, in which all the elements *PQ* of the

velocity *AP* may be generated, as the sum of the sectors *ATD*, that is, the total time as the whole sector.
Q. E. D.



[Here we follow the demonstration of Leseur & Janquier note one: *In the square ratio of the sides* ..... For if from the point *q* there is drawn to *Dp* the small line *qr* itself parallel to *vt*, the two vanishing triangles *Dqr, Dvt* are similar and [their areas] in the ratio of the squares of the sides *Dq*, *Dv* (from Euclid's *Elements*, Prop. XIX, Book VI) and triangle *Dqp* is equal to triangle *Dqr* with *pr* vanishing with respect to *Dq* ; therefore pD^2 is to tD^2, or AD^2, as triangle *qDp* to triangle *tDv*, and thus $tDv = \frac{AD^2 \times qDp}{pD^2}$, from which on account of the given radius of the circle *AD*, the element *tDv* is as $\frac{qDp}{pD^2}$.

Note 2 : *but from the nature of the hyperbola*, ... Since (by Theorem II, *Concerning the Hyperbola: Apollonius*), the rectangle $(2AD + AX) \times AX$, is to the square of the ordinate *TX*, as the transverse side is to the latus rectum, truly this hyperbola is equilateral, there will be (by Theorem II *Concerning the Hyperbola: Apollonius*)
$TX^2 = (2AD + AX) \times AX$. But $(2AD + AX) \times AX = DX^2 - DA^2$, (by Prop. VI, Book II, *Elements*),
$TX^2 = DX^2 - DA^2$ and hence $DX^2 - TX^2 = DA^2$.]



Corol. I. Hence if *AB* may be equal to the fourth part of *AC*, the distance that the body will describe in some time by falling, will be to the distance, that the body with the maximum velocity *AC*, can describe in the same time by progressing uniformly, as the area *ABNK* it can be shown to describe by falling which distance, to the area *ATD*, so that the time may be put in place. For since there shall be *A*C to *AP* as AP to *AK*, (per corol. I, Lem. II. of this section) *LK* will be to *PQ* as *2AK* to *AP*, that is, as 2*AP* to *AC*, and thence *LK* to $\frac{1}{2}PQ$ as *AP* to $\frac{1}{4}AC$ or *AB;* and *KN* is to *AC* or *AD*, as *AB* to *CK* ; [property of the hyperbola : Th. IV, *de Hyperb*.]*;* and thus from equation *LKNO* to *DPQ* as *AP* to C*K*. But there was *DPQ* to *DTV* as *CK* to *AC.* Again therefore from the equality *LKNO* is to *DTV* as *AP* to *AC*; that is, as the velocity of the falling body to the maximum velocity that the body by falling is able to acquire. Therefore since the moments of the areas *ABNK* and *ATD* , *LKNO & DTV* are as the velocities, all the parts of these areas taken together, likewise generated, as the distances likewise described, and thus the whole area generated from the beginning *ABNK* and *ATD* to the whole distance described from the beginning of the descent. *Q. E. D.*

Corol. 2. Likewise it follows also from the distance that is described in the ascent. Without doubt that whole distance shall be to the distance described in the same time with a uniform velocity *AC*, as the area *ABnk* is to the sector *ADt.*

Corol. 3. The velocity of the body falling in the time *ATD* is to the velocity, that it may acquire in the same time in a distance without resistance, as the triangle *APD* to the hyperbolic sector *ATD.* For the velocity in the non-resisting medium may be as the time *ATD,* and in the resisting medium it is as *AP*, that is, as the triangle *APD.* And these velocities at the starts of the descents are equal to each other, thus as these areas *ATD, APD.*

[S & J note 91, and following note: The velocity *Ap* of the body ascending in the medium with resistance to the maximum height *ABnk*, is to the velocity *AP* of the body in the same medium descending from rest through an equal distance *ABNK*, as the secant of the angle *ADp* to the radius, or what amounts to the same, as the tangent *Ap* of the angle *ADp*, to the sine of the same. For because, by hypothesis, the area *ABNK*, is equal to *ABnk*, there will be

$\frac{CK}{AC}=\frac{AC}{CK}$ and on separating $\frac{Ak}{AC}=\frac{AK}{CK}$, and on interchanging, $\frac{Ak}{AK}=\frac{AC}{CK}=\frac{Ck}{AC}=\frac{AC+Ak}{AC}$;

and thus $\frac{Ak\times AC}{AK\times AC}=\frac{AC^2+Ak\times AC}{AC^2}$; but by Prop. VIII, $AC\times Ak=Ap^2$, and $AC\times AK=AP^2$.

Whereby $\frac{Ap^2}{AP^2}=\frac{AC^2+Ap^2}{AC^2}=\frac{Dp^2}{AC^2}$, and hence $\frac{Ap}{AP}=\frac{Dp}{AC}=\frac{Dp}{AD}$. Q.e.d.

L & J Note : *And these velocities at the starts of the descents are equal to each other* on account of no resistance with respect to gravity, when a velocity arises. Therefore since the velocities in a non resisting medium shall always be among themselves as the areas *ATD*, and in a resisting medium they shall be as the triangle *APD*, the velocity acquired in a finite time in a medium with resistance *ATD* will be to the initial velocity in that medium with resistance will be as the finite triangle *APD*, to the triangle nascent triangle *APD*, and the initial velocity of descent in the non-resisting medium to the velocity acquired in the same medium in the finite time *ATD* will be as the nascent area *ATD* (equal to the nascent area *APD*) to the finite area *ATD* ; whereby, from the equality, the velocity of the body falling in the finite time *ATD* in the medium with resistance is to that velocity it may acquire falling in the same time in the non-resisting medium as the triangle *APD* to the sector of hyperbola *ATD.*]

Corol. 4. By the same argument the velocity in the ascent is to the velocity, by which the in the same time in a non-resisting distance may be able to lose all of its motion by ascending, as the triangle *ApD* to the circular sector *AtD;* or as the rectangle *Ap* to the arc *At.*

[*L & J*: *By the same argument*.... For the velocity in the non-resisting medium becomes as the time *AtD,* and in the resisting medium it is as Ap, that is, as the triangle ApD on account of the given *AD,* and these velocities at the end of the ascent when the vanish are equal to each other, hence as the vanishing areas *AtD*, *ApD*; but the triangle $ApD=\frac{1}{2}AD\times Ap$, and the sector of the circle $AtD=\frac{1}{2}AD\times At$. Whereby *ApD* is to *AtD* as *Ap* to *At*.]

Corol. 5. Therefore the time, in which a body may acquire a velocity *AP* by falling in a resisting medium, is to the time, in which it may acquire a maximum velocity *A*C by falling in a non-resisting distance, as the sector *ADT* to the triangle *ADC* [see following note one]*:* and the time, in which the velocity *Ap* may be lost in ascending in a resisting medium, is to the time in which the same velocity may be lost in a non-resisting distance, as the arc *At* to the tangent of this *Ap* [see following note two].

[Extended *S.* & *J.* Notes 92 and two notes, 93, 94 & 95 :

Hence if the velocity of ascent *Ap* in the resisting medium were equal to the maximum velocity *AC*, the velocity will be *Ap* or *AC*, to the velocity in which the body in the same time in a non-resisting space may be able to lose all its upwards motion, as the triangle ACD, to the eighth part of the circle, or as the radius to the eighth part of the periphery, or what amounts to the same thing, as the square of the circumscribed to the circle to the area of the circle. For while there becomes $Ap = AC$, the triangle *ApD* is equal to the triangle *ACD*, and the sector *AtD* , the eighth part of the circle, and thus the arc *At* is the eighth part of the periphery, and the triangle *ACD* is to the sector *AtD*, as AC to the arc At, and therefore the triangle ACD, on account of $AC = AD$, is the eighth part of the circumscribed square to the circle.

Note one : *as the sector ADT to the triangle ADC .....* For since AP may show the velocity acquired in the time *ATD* in the resisting medium, *AY* is taken such as to show the velocity produced in the same time in the non-resisting medium, and there will be by Corol. 2 $\frac{AP}{AY} = \frac{APD}{ATD}$, and since also *AC* may set out the maximum velocity, $\frac{AY}{AC}$ will be as the time in which the first swiftness can be acquired in the non-resisting medium to the time in which the maximum velocity can also be acquired in the non-resisting medium; and since the time in which the swiftness *AY* is acquired, may be expressed by the area *ATD*, $\frac{AY}{AC}$ as ATD to the area which may set out the time in which the maximum velocity is acquired in the non-resisting medium, and thus since there shall be $\frac{AP}{AY} = \frac{APD}{ATD}$ and $\frac{AY}{AC} = \frac{ATD}{\text{that area}}$, there will be from the equality $\frac{AP}{AC} = \frac{APD}{\text{that area}}$; but with the common altitude *DH* taken $\frac{AP}{AC} = \frac{APD}{ADC}$, therefore the area which expresses the time in which the maximum velocity is acquired in the medium without resistance, is the area *ADC*. From which it follows that the body in the resisting medium, cannot acquire the maximum velocity *AC* except by falling for an infinite time. For since there is made $AP = AC$, *DT* coincides with the asymptote *DC* of the hyperbola *ATV*, and the sector *ADT* becomes infinite.

Note two : *As the sector ADT to the triangle ADC....* For since *AP* may show the velocity acquired at the time *ATD* in the medium with resistance, *AY* may be taken such as to show the velocity produced at the same time in a medium without resistance, and there will be, by Corollary 2, $\frac{AP}{AY} = \frac{APD}{ATD}$, and also since *AC* may show the maximum velocity, there will be AY to AC as the time in which the first speed *AY* may be acquired in the non-resisting medium can be acquired, to the time in which the maximum speed *AC* also in the non-resisting medium may be acquired, and since the time in which the speed AY is acquired, may be expressed by the area *ATD*, *AY* to *AC* will be as *ATD* to the area which may show the time in which the maximum velocity in the non-resisting medium may be acquired, and thus since there shall be $\frac{AP}{AY}=\frac{APD}{ATD}$, and $\frac{AY}{AC}=\frac{APD}{\text{that area}}$, from the equality $\frac{AP}{AC}=\frac{APD}{\text{that area}}$, but with the common altitude *DH* taken, there is $\frac{AP}{AC}=\frac{APD}{ADC}$; therefore the area which will express the time in which the maximum velocity may be acquired, is the area *ADC*. From which it follows that the body in the resisting medium, to acquire the maximum velocity by falling is not possible unless in an infinite time. For since there is made $AP=AC$, *DT* will coincide with the asymptote *DC* of the hyperbola *ATV*, and the sector *ADT* shall be infinite.

Note three : *As the arc At, to the tangent of this Ap*..... If indeed, by Corollary 4, the velocity *Ap* required to be made zero in the resisting medium in the time *AtD*, is to the velocity requiring to be made zero in the same time in the non-resisting space, as the triangle *ApD* is to the sector *AtD*; and also as the time in which the velocity *Ap* in the non-resisting medium may be reduced to zero to the time *AtD* in which the other velocity in the non-resisting space is reduced to zero, that likewise is with that in which the velocity *Ap* in the resisting space is reduced to zero. Whereby the time in which the velocity *Ap*, may vanish in the non-resisting medium is to the time *AtD* in the resisting medium in which the velocity may vanish, as triangle *ApD* to the sector *AtD*, or the tangent *Ap* to the arc *At* of this. Therefore the proposition is apparent.

Extra Note 93. Hence the time in which the velocity of the body *Ap* can be lost by ascending in the resisting medium, is to the time in which the maximum velocity *AC* in the non-resisting medium may lose by ascending or acquired by descending as the sector of the circle *AtD*, to the triangle *ADC*, or as the arc *At* to the radius *AD*. For in the non-resisting medium the velocity *Ap* is to the velocity *AC*, as the time *ApD*, in which the velocity *AC* is generated or extinguished, that hence will be $\frac{AC\times ApD}{Ap}$ or $\frac{1}{2}AD\times AC$, that is, the triangle ADC.

Therefore since the time in which the velocity *Ap* is extinguished in the resisting medium, may be shown by the sector *AtD*, the proposition is apparent.

Extra Note 94. The time in which the body in the resisting medium may acquire the velocity *AP* by descending, of by ascending may lose the velocity *Ap*, is to the time in which it may acquire or lose the same velocity in the non-resisting medium, as the sector *ADT*, or *ADt*, to the triangle *ADP*, or *ADp*, respectively. And indeed, by Corollary 5 and note 93, the time in which the velocity *AP* may be generated in the non-resisting medium, or the velocity *Ap* removed, is to the time in which in which the maximum velocity *AC* may be generated or extinguished in the non-resisting medium, as *ADT* or *ADt*, to *ADC*; and the time in which the velocity *AC* may be generated or extinguished in the non-resisting space, is to the time in which the velocity *AP* or *Ap* may be generated or extinguished in the same non-resisting space, as *AC* to *AP* or *Ap*, and with the common altitude *DA* taken, as *ADC* to *APD* or *ApD*. Whereby, from the equality, the time in which the velocity *AP* may be generated in the medium with resistance, or the velocity Ap made zero, is to the time in which the same velocity may be produced or removed in the same distance in the non-resisting medium, as *ADT* to *ADP* or *ADp*.

Extra Note 95. If the speed *Ap* of the body ascending in the resisting medium were equal to the maximum *AC*, there will be $ADp = ADC$, and the sector *ADt* the eighth of the circle. Whereby the time in which the body ascending in the resisting medium can lose the maximum velocity AC is to the time in which it may lose the same in a non-resisting space, as the eighth of the circle to the triangle ADC, that is, as the area of the circle to the circumscribed square, or also as the 8th part of the periphery to the radius. We return to the *Principia*.]

Corol. 6. Hence from the given time the description of the distances of ascent or descent is given. For the maximum velocity is given in an infinite descent (by Corol. 2. & 3, Theorem VI. Book. II.) and thus the time is given in which that velocity may be acquired by falling in a non-resisting medium. And on taking the sector *ADT* or *ADt* to the triangle *ADC* in the ratio of the time given to the time just found ; then there will be given the velocity *AP* or *Ap*, then the area *ABNK* or *ABnk* [note one below], which is to the sector *ADT* or *ADt* as the distance sought to the distance, that in the given time, with that maximum velocity now found before, it can describe uniformly.

[*L.* & *J*. Note 96 : *And thus the time is given ...* For since the uniform accelerating forces, shall be as the velocities which they generate directly and the times in which these may be generated given inversely (13. Book I) to the uniform accelerating force by which the body may be acted on in some medium, or with the ratio of this force to some known other known force, for example, to the force of terrestrial gravity, and likewise with that given velocity that the accelerating force produced, will be giving the time in which that given velocity has arisen. For let the given accelerative force be to the known force of gravity, as *a* to *b*, the velocity *c* generated by that given accelerative force in the time *x*, and the velocity *C* that the force of gravity may generate in some given time *t*, there will be $\frac{a}{b} = \frac{c}{x} / \frac{C}{t}$. From which the time is found $x = \frac{bct}{aC}$.
Note one : *.....then the area ABNK or ABnk* : For there is (from the dem. of Prop. VIII) $\frac{AC}{AP} = \frac{AP}{AK}$, and $\frac{AC}{Ap} = \frac{Ap}{Ak}$, and thus with *AC* and *AP* or *Ap* given, *Ak* or *Ak* are given, and the corresponding areas *ABNK*, *ABnk*, which can be found by tables of logarithms.]

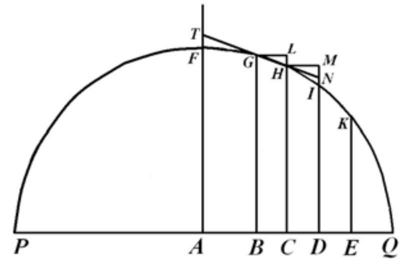
Corol. 7. And by going backwards, from the given ascent or descent distance *ABnk* or *ABNK*, the time *ADt* or *ADT* will be given.

[Note 97 : *And by going backwards....*Without doubt the area *ABnk* or *ABNK* is required to be taken to the triangle *ADC* in the give ratio of the distance ascended or descended be to the square of the distance, that the body in the non-resisting medium will describe in falling to the maximum velocity it may acquire, and thus *Ak* or *AK* will be given. And hence *Ap* or *AP* or the velocity will be given; moreover from these the sector *ADt* or *ADT* may be given, or the time (by Corollary 5). For the distance that the body in the non-resisting medium will describe in descending so that the maximum velocity *AC* may be acquired is called *A*, the time in which that distance is described *T*, the distance that it will describe in the resisting medium so that it may acquire the velocity *AP*, or the velocity lost *Ap*, is called *a*, the time *t*, and the distance that the body will describe in that time *t* and the maximum velocity *AC* by progressing uniformly shall be *S*, and because the body with the maximum velocity AC by progressing uniformly, it time T, will describe the distance 2A, there will be $\frac{S}{2A}=\frac{t}{T}$. But by Cor. 5 and note 93, $\frac{t}{T}=\frac{ADT \text{ or } ADt}{ADC}$, and thus $\frac{S}{2A}=\frac{ADT \text{ or } ADt}{ADC}$, and by Cor. 1 & 2, $\frac{s}{S}=\frac{ABNK}{ADT}$ or $\frac{ABnk}{ADt}$. Whereby from the equality, $\frac{s}{2A}=\frac{ABNK \text{ or } ABnk}{ADC}$.]

PROPOSITION X. PROBLEM III.

A uniform force of gravity may extend directly to the horizontal plane, and let the resistance [to motion] be conjointly as the density of the medium and as the square of the velocity: then both the density of the medium is required at individual places, which may be done so that the body may move along some given curved line, as well as the velocity of the body and the resistance of the medium at individual places.

Let *PQ* lie in a perpendicular plane to the horizontal scheme ; *PFHQ* a curved line crossing this plane at the points *P* and *Q; G, H, I, K* four locations of the body on this curve going from *F* to *Q* ; and *GB, HC, ID, KE* four parallel ordinates sent to the horizontal plane from these points, and the lines meeting the horizontal line *PQ* standing at the points *B, C, D, E* ; and *BC, CD, DE* shall be equal distances between the ordinates. From the points *G* and *H*, the right lines *GL, HN* may be drawn touching the curve in *G* & *H* , & with the ordinates *CH, DI* projected up meeting at *L* & *N* & the parallelogram *HCDM* may be completed. And the times [see note one below], in which the body will describe the arcs *GH, HI,* will be in the square root ratio of the altitudes *LH, NI,* that the body may be able to describe in these times, by falling from the tangents; and the velocities will be directly as the lengths described *GH, HI* and inversely as the times. The times may be shown by *T* and *t,* and the velocities by $\frac{GH}{T}$ and $\frac{HI}{t}$; and the decrement of the velocity made in the time *t* may be put in place by $\frac{GH}{T}-\frac{HI}{t}$ [see note two below]. This decrement arises from the resistance retarding the body, and with gravity accelerating the body. Gravity generates a velocity in the falling body and by describing a distance *NI* in falling, so that it may be able to describe twice that distance in the same time, as *Galileo* demonstrated, that is, a velocity $\frac{2NI}{t}$



: but with the body describing the arc *HI* [see note three below], that arc increases only by the length $HI-HN$ or $\frac{MI\times NI}{HI}$; and thus it generates only the velocity $\frac{2MI\times NI}{t\times HI}$ This velocity may be added to the aforesaid decrement, and the decrement of the velocity will be had arising from the resistance only, clearly $\frac{GH}{T}-\frac{HI}{t}+\frac{2MI\times NI}{t\times HI}$. And hence since gravity in the same time generates the velocity $\frac{2NI}{t}$ from the body falling ; the resistance will be to gravity as $\frac{GH}{T}-\frac{HI}{t}+\frac{2MI\times NI}{t\times HI}$ to $\frac{2NI}{t}$ or as $\frac{t\times GH}{T}-HI+\frac{2MI\times NI}{HI}$ to $2NI$.

Now for the abscissas *CB, CD, CE* there may be written $-o,\ o,\ 2o$. For the ordinate *CH* there may be written *P*, and for *MI* there may be written some series $Qo+Roo+So^3+$etc. And all the terms of the series after the first, clearly $Roo+So^3+$ etc. shall be *NI*, and the ordinates *DI*, *EK*, & *BG* will be $P-Qo-Roo-So^3-$ &c.$P-2Qo-4Roo\ -8S0^3$etc.& $P+Qo-Roo+So^3-$etc., respectively. And by squaring the difference of the orders $BG-CH$ & $CH-DI$, and to the square produced by adding the squares of *B*C, *CD*, the squares of the arcs *GH, HI* $oo+QQoo-2QRo^3+$etc. & $oo+QQoo+\ 2QRo^3+$etc. The roots of which $o\sqrt{1+QQ}-\frac{QRoo}{\sqrt{1+QQ}}$, and $o\sqrt{1+QQ}+\frac{QRoo}{\sqrt{1+QQ}}$ shall be the arcs *GH* and *HI*. Besides if from the ordinate C*H* there may be taken half the sum of the ordinates *BG* and *DI*, and from the ordinate *DI* there may be taken half the sum of the ordinates *CH* and *EK*, the sagitta of the arcs *GI* and *HK* will remain, Roo and $Roo+3So^3$. And these shall be proportional to the small lines *LH* and *NI*, and thus in the square ratio of the infinitely small times *T* and *t* and thence the ratio $\frac{t}{T}$ is $\sqrt{\frac{\mathrm{R}+3So}{R}}$ or $\frac{R+\frac{3}{2}So}{R}$; and $\frac{t\times GH}{T}-HI+\frac{2MI\times NI}{HI}$, by substituting the values now found of these $\frac{t}{T}$, G*H*, *HI*, *MI* & *NI*, there emerges $\frac{3Soo}{2R}\sqrt{1+QQ}$. And since $2NI$ becomes $2Roo$, the resistance now will be to gravity as $\frac{3Soo}{2R}\sqrt{1+QQ}$ to $2Roo$, that is, as $3S\sqrt{1+QQ}$ to $4RR$.

But this velocity is , whatever the body from any place *H*, going along the tangent *HN*, in a parabola by having a diameter *HC* and the latus rectum $\frac{HNq}{NI}$ or $\frac{1+QQ}{R}$, then can be moving in a vacuum.

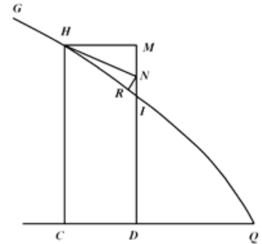
And the resistance is as the density of the medium and jointly with the square of the velocity, and therefore the density of the medium is as the resistance directly and inversely as the square of the velocity, that is, as $\frac{3S\sqrt{1+QQ}}{4RR}$ directly and $\frac{1+\mathrm{QQ}}{\mathrm{R}}$ inversely, that is, as $\frac{S}{R\sqrt{1+QQ}}$. *Q. E. I.*

[*L. & J.* Note one : *And the times*..... For in the same moment of time that a body in place at *G* may describe the tangent *GL* by the *in situ* force of the motion, a body may fall through such a height *LH* under the uniform force of gravity in a non-resisting medium in that time itself; for the effect of the resistance diminishes that height itself by an infinitely small amount , which thus is not to be considered here, and thus the body is considered to describe the complete arc *GH* by a force composed from the *in situ* force of the motion and the force of gravity. And in a similar manner, in the same time that it will describe the arc *HI*, by the force of gravity it may fall through the height *NI*. Whereby (by Lemma X of Book I), the times in which the body will describe the arcs *GH, HI*, or in which it falls through the heights *LH, NI*, are in square root ratio of these heights.

L. & J. Note two : *And the decrement of the velocity*..... For if the velocity through the arc *HI* were the same as the velocity through the arc *GH*, the one may be shown by $\frac{GH}{T}$, the other by $\frac{HI}{t}$. Whereby if the velocity may decrease, the decrement of that made in the time *t* may be expressed by $\frac{HI}{t}-\frac{GH}{T}$. But if indeed the velocity may increase, it may be expressed by $\frac{GH}{T}-\frac{HI}{t}$; this decrement or increment arises from the resistance of the body by retardation along the direction of the tangent HN or directly opposite to the arc HI, and from the motion of the body under gravity, for the force of gravity of the body descending may be seen to be divided into two forces normal and tangent; the motion of the body along the curve may be accelerated by the tangential force, and which normal force may neither accelerate nor retard. Whereby if the resistance is greater than the force of gravity, the motion is retarded, if less it will be accelerated, and if equal, neither accelerated nor retarded.

L. & J. Note three : *but with the body*....... For by the *in situ* force alone, the body may describe the tangent *HN* in the time *t*, and by the force of gravity alone the altitude *NI*, indeed with the forces taken together it will describe the arc *HI*. Whereby gravity may augment the distance along the direction *HN* or *HI* only by the amount $HI-HN$. But $HI-HN=\frac{MI\times NI}{HI}$. If indeed with centre *H* and radius *HN*, the arc of a circle *NR* may be considered described, cutting HI in R, the two triangles IRN, IMH will be similar, on account of the MIH common to the triangle on each side, and the angle IRN, IMH right, and thus equal; from which there is $HI:MI=NI:RI$ or $HI-HN$; and therefore $HI-HN=\frac{MI\times NI}{HI}$. Therefore since *RI* shall be the distance described in the time *t* by the tangential force of gravity, that velocity which that force may generate in the time *t*, may be expressed by $\frac{2RI}{t}=\frac{2MI\times NI}{t\times HI}$.

L. & J. extra notes : *And there will be had the decrement of the velocity arising from the resistance alone, evidently* $\frac{2RI}{t}=\frac{2MI\times NI}{t\times HI}$, not only in that case in which the resistance may be greater than the tangential force of gravity, but also in that case in which that may be superior to the other. For let the decrement of the velocity arising from the resistance alone be *V*, since the increment of the velocity arising from the tangential gravitational force shall be $\frac{2MI\times NI}{t\times HI}$, in the first case there will be $V-\frac{2MI\times NI}{t\times HI}=\frac{GH}{T}-\frac{HI}{t}$, and thus $V=\frac{GH}{T}-\frac{HI}{t}+\frac{2MI\times NI}{t\times HI}$; but in the second case there will be $\frac{2MI\times NI}{t\times HI}-V=\frac{HI}{t}-\frac{GH}{T}$, and therefore $V=\frac{2MI\times NI}{t\times HI}+\frac{GH}{T}-\frac{HI}{t}$, which is the same expression as the former.



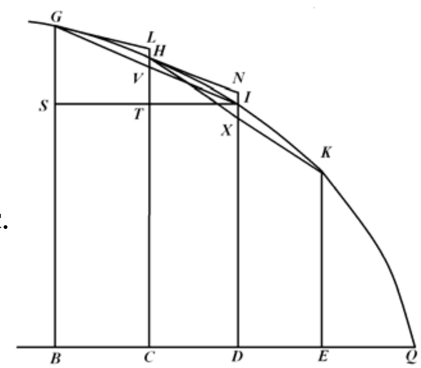
The resistance will be to gravity ...... For the accelerating and retarding forces are as the elements of the velocity which may be generated or removed in given moments of time.

There may be written –o, o, 2*o*.... For if the abscissas *CD, CE* may be taken positive, the abscissas in the opposite part *CB*, etc. must be assumed to be expressed negative.

And for MI some series may be written.... For the difference of the fluxions *MI* of the ordinates *CH, DN* can be expressed by an infinite series $Qo+Roo+So^3+\text{etc.}$, in which Q, R, S, etc. are finite quantities assumed here generally, and afterwards to be determined in the individual cases, and *o* is the nascent increment and a constant for the abscissa.

And the ordinates..... For indeed $DI=DM-MI=CH-MI=P-Qo-Roo-So^3-\text{etc.}$ (by Hypothesis) ; and because $CE=2o$, if in place of the value of the ordinate *DI* there may be written 2*o* , DI will become $EK=P-2Qo-4Roo-8So^3-\text{etc.}$; and in a similar manner because $CB=-o$, if in the value of the ordinate *DI* in place of + *o* there may be written – *o* , there becomes

$DI=BG=P+Qo-Roo+So^3-\text{etc.}$



....the squares of the arcs GH, HI, etc. may be had..... For indeed, on account of the right angle *HMI*, $HI^2=HM^2+MI^2$, and $HM=CD=o,$ and $MI=CH-DI=Qo+Roo+So^3+\text{etc.}$, and thus $HM^2=oo, MI^2=Q^2o^2+2QRo^3+R^2o^4+\text{etc.}$; thence $HI^2=o^2+QQR^2o^2+2QRo^3+\text{etc.}$ But the terms in which there o^4, o^5,etc. that vanish before the others may be ignored and contribute nothing to that expression. Whereby with the extraction of the square root made $HI=o\sqrt{1+QQ}+\frac{QRoo}{\sqrt{1+QQ}}$, with the remaining terms ignored; and in a similar manner there is found $GH=o\sqrt{1+QQ}-\frac{QRoo}{\sqrt{1+QQ}}$.

.....the sagitta of the arcs GI and HK will remain.... The chord *GI* may be joined cutting *CH* in *V*, and from the point *I* the perpendicular *IS* may be sent cutting *CH* in *T*. On account of the similar triangles *ITV, ISG*, there will be $\frac{IT}{IS}$ or $\frac{DC}{DB}$ *i.e.*$\frac{1}{2}$ as $\frac{TV}{GS}$, and thus $GS=2VT$, and $GB=2VT+SB=2VT+DI$ and $GB+DI=2VT+2DI$, whereby half the sum of the ordinates *GB* and *DI* is $VT+DI$, or *VC*, which if taken from the ordinate *CH*, there will remain the sagitta *VH* of the arc *GI*. And by similar reasoning it is apparent that the sagittam *IX* of the arc *HK* is equal to the difference between the ordinates *DI* and half the sum of the ordinates *CH* and *EK*.

...And these are the small proportional lines LH and NI.... For with the points *B, C, D, E* and *G, H, I, K* joined, the figures *NHIXH, LGHVG* become similar, and the homologous

sides *HV* and *IX*, *LH* and *NI* are proportional; but the small lines LH, NI are as the squares of the times T , t (from the demonstration), by which the arcs *GH*, *HI* are described.

...*And thence the ratio* $\frac{t}{T}$,etc. ..... For from the demonstration, $\frac{t^2}{T^2} = \frac{IX}{HV} = \frac{Roo+3So^3}{Roo} = \frac{R+3So}{R}$ and thus $\frac{t}{T} = \sqrt{\frac{R+3So}{R}} = \sqrt{\frac{RR+3SRo}{RR}} = \frac{\sqrt{RR+3SRo}}{R}$; but $\sqrt{RR+3SRo} = R + \frac{3SRo}{2}$, with negligible terms ignored : whereby there will be $\frac{t}{T} = \frac{R+\frac{3SRo}{2}}{R} = 1 + \frac{3So}{2R}$.

...... *there emerges* $\frac{3Soo}{2R}\sqrt{1+QQ}$ ..... For there is $\frac{t\times GH}{T} = o\sqrt{1+QQ} - \frac{QRoo}{\sqrt{1+QQ}} + \frac{\frac{3}{2}Soo\sqrt{1+QQ}}{R}$, which with the term in which there is found o^3 ignored, which vanishes before the rest. From which there becomes $\frac{t\times GH}{T} - HI = \frac{\frac{3}{2}Soo\sqrt{1+QQ}}{R} - \frac{2QRoo}{\sqrt{1+QQ}}$; but with $2MI = 2Qo$ and $NI = Roo$ ignored with the remainder of the terms of the series vanishing, and hence $\frac{2MI\times NI}{HI} = \frac{2QRo^2}{\sqrt{1+QQ}}$, with the term $\frac{QRoo}{\sqrt{1+QQ}}$ ignored in the value of the arc *HI*. Whereby there will be $\frac{t\times GH}{T} - HI + \frac{2MI\times NI}{HI} = \frac{3Soo\sqrt{1+QQ}}{2R}$.

....*Then it can move in a vacuum*....... For since the velocity along the arc *HI*, or along the emerging tangent *HN*, may be able to be considered equal, and the body in the same moment of time in which it may describe *HN* by the vi insitu, by the uniform force of gravity, with the resistance ignored which here may be considered as zero, falls through the height *NI*; the nascent arc *HI*, that the body with the forces taken together will describe, can be taken as the arc of a parabola, the diameter of which is *HC*, the tangent *HN* with the ordinates parallel, and *NI* parallel and equal to the abscissa to which the equal ordinate *HN* may correspond. Whereby the latus rectum of this parabola will be $\frac{HN^2}{NI}$, by the geometry of the parabola, or (by Lemma VII of Book I), $\frac{HI^2}{NI} = \frac{oo+QQoo}{Roo} = \frac{1+QQ}{R}$, with the negligible terms to be ignored. And if that body hence may be moving in a vacuum, it will describe this parabola.

...*That is, so that, etc*. For because the resistance is to constant gravity as $3S\sqrt{1+QQ}$ to $4RR$, the resistance will be as $\frac{3S\sqrt{1+QQ}}{4RR}$. But the velocity is as $\frac{HI}{t}$, and the square of that as $\frac{HI^2}{t^2}$; and HI^2 is $oo + QQoo$, neglecting smaller quantities, t^2 is indeed as NI, or as Roo (from the demonstration) ; and thus the square of the velocity is as $\frac{1+QQ}{R}$. Whereby the density of the medium will be as $\frac{3S\sqrt{1+QQ}}{4R(1+QQ)}$, and from the given number $\frac{3}{4}$, as $\frac{S\sqrt{1+QQ}}{R(1+QQ)} = \frac{S}{R\sqrt{1+QQ}}$. Back to Newton.]

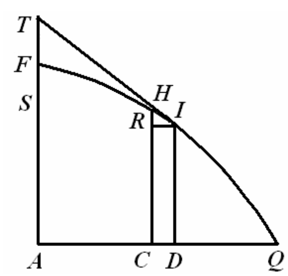
Corol. 1. If the tangent *HN* may be produced to the side then it may cross the ordinate *AF* at some point *T:* $\frac{HT}{AC}$ will be equal to $\sqrt{1+QQ}$, and thus can be written for $\sqrt{1+QQ}$ in the above. n which ratio the resistance will be to gravity as $3S\times HT$ to $4RR\times AC$, the velocity will be as $\frac{HT}{AC\sqrt{R}}$ and the density of the medium will be as $\frac{S\times AC}{R\times HT}$.

[$\frac{HT}{AC}$ *will be equal....* From the points *H* and *I* the perpendiculars *HS* and *IR* may be sent to *AF* and *CH*, and on account of the similar triangles *IRH*, *HST*, there will be *HT* to *HS* as *AC* to *HI* or *CD*, and thus $\frac{HT}{AC}=\frac{HI}{CD}=\frac{o\sqrt{1+QQ}}{o}=\sqrt{1+QQ}$.]

Corol. 2. And hence, if the curved line *PFHQ* may be defined by a relation between the base or the abscissa *A*C and the applied ordinate *CH*, as is customary; and the value of the applied ordinate may be resolved into a converging series : the problem may be solved expeditely by the first terms of the series, as in the following examples.

*Example*1. Let the semicircular line *PFH* be described on the diameter *PQ*, and the density of the medium may be required which may act so that the projectile may be moving along this line.

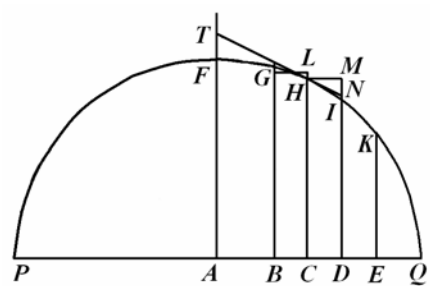
The diameter *PQ* may be bisected at *A ;* call *AQ, n; AC, a; CH, e;* and *CD, o* : and there will be DI^2 or $AQ^2-AD^2=nn-aa-2ao-oo$, or $ee-2ao-oo$, and with the root extracted by our method, there becomes



$DI=e-\frac{ao}{e}-\frac{oo}{2e}-\frac{aaoo}{2e^3}-\frac{ao^3}{2e^3}-\frac{a^3o^3}{2e^5}-$ etc. Here there may be written *nn* for $ee+aa$, and there emerges

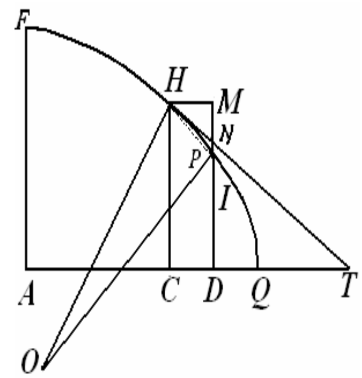
$DI=e-\frac{ao}{e}-\frac{nnoo}{2e^3}-\frac{anno^3}{2e^5}-$ etc.

A series of this kind may be separated into successive terms in this manner. I call the term the first, in which the infinitely small quantity *o* does not appear , the second, in which that quantity is of one dimension ; the third, in which it arises of dimension two ; the fourth, in which it is of the third, and thus indefinitely. And the first term, which here is *e*, always will denote the length of the ordinate *CH* standing at the start of the variable quantity *o*. The second term, which here is $\frac{ao}{e}$, will denote the difference *e* between *CH* and *DN*, that is, the small line *MN*, which is cut off by completing the parallelogram *HCDM*, and thus may always determine the position of the tangent *HN* [note one below]; as in this case by taking $\frac{MN}{HM}=\frac{ao}{eo}=\frac{a}{e}$. The third term $\frac{nnoo}{2e^3}$, which here will the small line *IN*, which lies between the tangent and the curve, and thus determines the angle of contact *IHN* or the curvature that the curved line has at *H* [note two below] . If that small line *IN* is of finite magnitude, it will be designated by the third term together with the following terms



indefinitely. But if that small line may be diminished indefinitely, the following terms arise infinitely smaller than the third, and thus may be able to be ignored. The fourth term determines the variation of the curvature, the fifth the variation of the variation, and thus henceforth. From which along the way the use of these series in the solution of problems is apparent, and not to be dismissed, which depend on tangents and the curvature of curves.

The series $e-\frac{ao}{e}-\frac{nnoo}{2e^3}-\frac{anno^3}{2e^5}-\text{etc.}$ now may be established, with the series $P-Qo-Roo-So^3-\text{etc.}$ and from which by writing $e, \frac{a}{e}, \frac{nn}{2e^3}$, and $\frac{ann}{2e^5}$, for *P*, *Q*, *R* and *S* and by writing $\sqrt{1+\frac{aa}{ee}}$ for $\sqrt{1+QQ}$ or $\frac{n}{e}$, the density of the medium will be produced as $\frac{a}{ne}$, that is (on account of *n* given), as $\frac{a}{e}$ or $\frac{AC}{CH}$, that is, as that length of the tangent *HT*, which may be terminated on the radius *AF* standing normally on *PQ* itself: and the resistance will be to gravity as 3*a* to 2*n*, that is, as 3*AC* to the diameter *PQ* of the circle: moreover the velocity will be as $\sqrt{CH}$. Whereby if a body may leave from the position *F* with the correct velocity along a line parallel to *PQ* itself, and the density of the medium at the individual places *H* shall be as the length of the tangent *HT*, and the resistance also at some place *H* shall be to the force of gravity as *3A*C to *PQ* that body will describe the quadrant of a circle *FHQ*. *Q.E.I.*



But if the same body may be proceeding from the same place *P*, along a line perpendicular to *PQ*, and it may begin to move in the arc of a circle *PFQ*, it shall be necessary to take *AC* or *a* on the opposite side of the centre *A*, and therefore the sign of this must change and it is required to write – *a* for + *a*. With which done the density of the medium may appear as $-\frac{a}{e}$. But a negative density, that is, which accelerates the motion of bodies, is not allowed by nature : and therefore cannot happen naturally, so that the body by ascending from *P* may describe the quadrant of a circle *PF*. Towards this effect the body must be impelled to accelerate by the medium, and not to be impeded by resistance, [thus only one quadrant of the circle can be considered; either the left-hand quadrant for the initially vertically rising body, or the right-hand part for the initially horizontally projected descending body.].

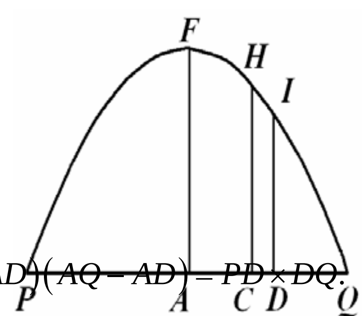
[*L. & J*. Note one : The tangent *HN* may be produced so that it crosses the diameter *AQ* at *T*; and because of the similarity of the triangles *HMN*, *TCH*, there will be $\frac{CT}{HC}=\frac{HM}{MN}$. Now truly in general there is $HM=o$ and $MN=Qo$, and *Q* is the coefficient of the second term of the general series for any curve (from the demonstration of Prop. X) ; whereby if there is taken $\frac{CT}{HC}=\frac{1}{Q}$, the subtangent CT will be found.

S & J note two : Let *O* be the centre of curvature of the curve *FHQ* osculating at *H*; *OH*, *OI* the radii, *HPI* the chord of the arc *HI*, *NP* the radii, *HPI* the chord of the arc *HI*, *NP* the small arc of the circle with centre *H* and radius *HN* described. The two triangles *IPN*, *IMH* will be similar, on account of the right angle at *P* and *M*, and the common angle at *I*; and therefore $\frac{HI}{HM}=\frac{NI}{NP}$ and hence $NP=\frac{HM\times NI}{HI}$. The measure of the angle *NHI*, that the tangent *HN* makes with the subtended arc *HPI*, is half the arc *HI*, and to the angle at the centre *HOI* the measure is the whole arc *HI* (from the nature of the circle) ; from which NP or $\frac{HM\times NI}{HI}:HN\left(\text{or } HI\right)=\frac{1}{2}HI:HO$, and thus the radius of osculation $HO=\frac{HI^3}{2HM\times NI}$. And because (from the demonstration of Prop. X),

$HI=o\sqrt{1+QQ}$; $HM=o$; and $NI=Roo$; there will be $HO=\frac{(1+QQ)^{\frac{3}{2}}}{2R}$. But the contact angle and the curvature of the curved line *FHQ* in *H* is as the radius of osculation HO inversely, that is, as $\frac{2R}{(1+QQ)^{\frac{3}{2}}}$. Whereby that angle, or the curvature at *H* will be determined, by the given second and third terms of the series into which the value of the applied ordinate is resolved.]

Example 2. The line *PFQ* shall be a parabola, having the axis *AF* perpendicular to the horizontal *PQ*, and the density of the medium is required, that enables the projectile to be moving along this curve.

From the nature of the parabola [see note], the rectangle *PDQ* is equal to the rectangle under the ordinate *DI* and some given right line ; that is, if that right line may be called *b*; *PC, a; PQ, c; CH, e;* & *CD, o* ; the rectangle $a+o$ by $c-a-o$ or $ac-aa-2ao+co-oo$ is equal to the rectangle *b* by *DI*, and thus *DI* equals $\frac{ac-aa}{b}+\frac{c-2a}{b}o-\frac{oo}{b}$. Now the second term of this series is required to be written, $\frac{c-2a}{b}o$ for *Qo*, likewise the third term $\frac{oo}{b}$ for Roo. Since indeed there are no more terms, the coefficient *S* of the fourth term must vanish, and therefore the quantity $\frac{S}{R\sqrt{1+QQ}}$, to which the density of the medium is proportional, will be zero. Therefore the projectile does not move in a parabola for any density of the medium, as *Galileo* showed at one time. *Q. E. I.*

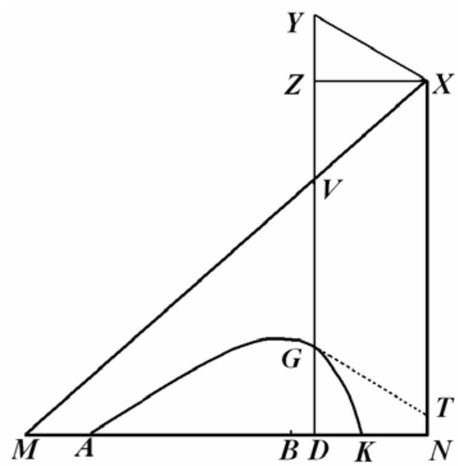


[*L. & J.* Note : From the point *I* to the axis of the parabola *FA* a perpendicular *IR* may be sent, and let the latus rectum axis be equal to *b*; from the nature of the parabola, there will be : $b\times FR=RI^2=AD^2$ and $b\times FA=AQ^2$. Whereby

$b\times FR-b\times FA$ or $b\times RA$ or $b\times DI=AQ^2-AD^2=\left(AQ+AD\right)\left(AQ-AD\right)=PD\times DQ$.
]

Example. 3. The line *AGK* shall be a hyperbola, having the asymptote *NX* perpendicular to the horizontal plane *AK*, and the density of the medium is sought, which may enable the projectile to move along this line.

MX shall be the other asymptote, crossing the applied ordinate produced *DG* at *V;* and from the nature of the hyperbola, the rectangle *XV* by *VG* may be given [from the nature of the hyperbola : see *de Hyper*. Th. 4; *Apoll*.].
Moreover the ratio $\frac{DN}{VX}$, and therefore also the rectangle *DN* by *VG* may be given [from similar triangles.] That shall be *bb*, and on completing the parallelogram *DNXZ;* there may be called : *BN, a; BD, o; NX,* c; and the given ratio *VZ* to Z*X* or *DN* may be put equal to $\frac{m}{n}$. And there will be *DN* equal to $a-o$, *VG* equal to $\frac{bb}{a-o}$, *VZ* equal to $\frac{m}{n}\overline{a-o}$, and *GD* or $NX-VZ-VG$ equal to the same term $c-\frac{m}{n}a+\frac{m}{n}o-\frac{bb}{a-o}$. The term $\frac{bb}{a-o}$ may be resolved into the converging series $\frac{bb}{a}+\frac{bb}{aa}o+\frac{bb}{a^3}oo+\frac{bb}{a^4}o^3$ etc., and there becomes *GD* equal to $c-\frac{m}{n}a-\frac{bb}{a}+\frac{m}{n}o-\frac{bb}{aa}o-\frac{bb}{a^3}o^2-\frac{bb}{a^4}o^3$ etc. The second term of this series $\frac{m}{n}o-\frac{bb}{aa}o$ is required to be taken for *Qo* [as this series is equal to the series $P-Qo-Roo-So^3-$etc., and so the terms may be equated]*;* the third with the sign changed $\frac{bb}{a^3}o^2$ for Ro^2 and the fourth also with the sign changed $\frac{bb}{a^4}o^3$ for So^3, the coefficients of these $\frac{m}{n}-\frac{bb}{aa}$, $\frac{bb}{a^3}$ and $\frac{bb}{a^4}$ are required to be written in the above rules for *Q*, *R* & *S*. With which done the density of the medium will be as [the numerator and denominator in $\frac{bb}{a^4}$:]



$$\frac{\frac{bb}{a^4}}{\frac{bb}{a^3}\sqrt{1+\frac{mm}{nn}-\frac{2mbb}{naa}+\frac{b^4}{a^4}}} \text{ or } \frac{1}{\sqrt{aa+\frac{mm}{nn}aa-\frac{2mbb}{n}+\frac{b^4}{aa}}}$$

that is, [note one] if in *VZ* there may be taken *VY* equal to *VG*, as $\frac{1}{XY}$. For if aa and $\frac{mm}{nn}aa-\frac{2mbb}{n}+\frac{b^4}{aa}$ are squares of *XZ* and *ZY* themselves. Moreover the resistance is found in the ratio to gravity that $3XY$ has to $2YG$ [note two]*;* and the velocity is that, [note three], by which a body may proceed in the parabola, with vertex *G*, the diameter *DG*, and by having the latus rectum $\frac{XY^2}{VG}$. And thus put in place that the densities of the medium at the individual places *G* shall be inversely as the distances *XY,* and that the resistance at some place *G* shall be to gravity as 3*XY* to 2*YG;* and the body sent from that place *A*, with the correct velocity, will describe that hyperbola *AGK. Q. E. I.*

[*L. & J.* Note one : For indeed $VG = \frac{bb}{a-o} = \frac{bb}{a}$, and $VG = \frac{m}{n}\overline{a-o} = \frac{m}{n}a$, where BD or o vanishes. Whereby $VY - VZ = ZY = \frac{bb}{a} - \frac{m}{n}a$; and because ZX or $DN = a$, and $YX^2 = YZ^2 + ZX^2$, there becomes $YX^2 = aa + \frac{mm}{nn}aa - \frac{2mbb}{n} + \frac{b^4}{aa}$; and thus the density of the medium as $\frac{1}{XY}$.

Note two : The resistance is to gravity in the ratio $\frac{3S\sqrt{1+QQ}}{4RR}$, that is, as $\frac{3bb}{a^4} \times \sqrt{1 + \frac{mm}{nn} - \frac{2mb^2}{naa} + \frac{b^4}{a^4}}$ to $\frac{4b^4}{a^6}$, or on dividing by $\frac{bb}{a^5}$, as $3\sqrt{aa + \frac{mm}{nn}aa - \frac{2mb^2}{n} + \frac{b^4}{a^2}}$ to $\frac{4b^2}{a}$, or as $3XY$ to $4VG = 2YG$.

Note three : The latus rectum of this parabola is

$$\frac{1+QQ}{R} = \frac{1 + \frac{mm}{nn} - \frac{2mbb}{naa} + \frac{b^4}{a^4}}{\frac{bb}{a^3}} = \frac{aa + \frac{mm}{nn}aa - \frac{2mbb}{n} + \frac{b^4}{aa}}{\frac{bb}{a}} = \frac{YX^2}{VG}$$

. But the velocity is as $\sqrt{\frac{1+QQ}{R}}$ and thus as $\frac{YX}{\sqrt{VG}}$.]

Example 4. It may be considered generally, that the line *AGK* shall be a hyperbola, with centre *X*, asymptotes *MX* and *NX* described by that rule, that with the rectangle constructed *XZDN* the side of which *ZD* may cut the hyperbola in *G* and its asymptote in *V*, *VG* were inversely as some power DN^n of *ZX* or of *DN*, the index of which is the number *n* [note one]: and the density of the medium is sought, by which the projectile may progress along this curve.

For *BN*, *BD*, *NX* there may be written *A*, *O*, *C* respectively, and let $\frac{VZ}{XZ \text{ or } DN} = \frac{d}{e}$, and *VG* is equal to $\frac{bb}{DN^n}$, and *DN* equals $A - O$, $VG = \frac{bb}{\overline{A-O}^n}$, $VZ = \frac{d}{e}\overline{A-O}$, and *GD* or $NX - VZ - VG$ equals $C - \frac{d}{e}A + \frac{d}{e}O - \frac{bb}{\overline{A-O}^n}$. This term $\frac{bb}{\overline{A-O}^n}$ may be resolved into an infinite series [*i.e.* by the binomial expansion of $(A-O)^{-n}$:]

$$\frac{bb}{A^n} + \frac{nbb}{A^{n+1}}O + \frac{nn+n}{2A^{n+2}}bbO^2 + \frac{n^3+3nn+2n}{6A^{n+3}}bbO^3 \text{ etc.}$$

and G*D* equals

$$C - \frac{d}{e}A - \frac{bb}{A^n} + \frac{d}{e}O - \frac{nbb}{A^{n+1}}O - \frac{+nn+n}{2A^{n+2}}bbO^2 - \frac{+n^3+3nn+2n}{6A^{n+3}}bbO^3 \text{ etc.}$$

The second term of this series $\frac{d}{e}O - \frac{nbb}{A^{n+1}}O$ is required to be taken for *Qo*, the third $\frac{+nn+n}{2A^{n+2}}bbO^2$ for Ro^2, the fourth $\frac{+n^3+3nn+2n}{6A^{n+3}}bbO^3$ for So^3. And thence the density of the medium $\frac{S}{R\sqrt{1+QQ}}$, in place of some *G*, shall be

$$\frac{n+2}{3\sqrt{A^2 + \frac{dd}{ee}A^2 - \frac{2dnbb}{eA^n}A + \frac{nnb^4}{A^{2n}}}}$$

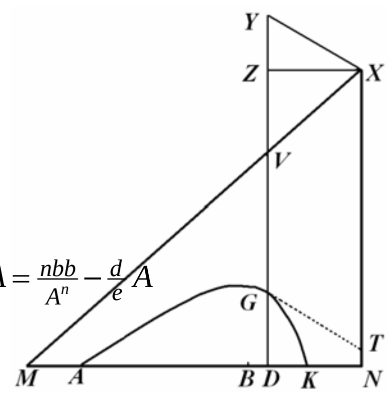
[note two] , and thus if in *VZ* , *VY* may be taken equal to $n \times VG$, that density is inversely as *XY*. Indeed A^2 and $\frac{dd}{ee}A^2 - \frac{2dnbb}{eA^n}A + \frac{nnb^4}{A^{2n}}$ are the squares of *XZ* and *ZY* [note three]. But the resistance in the same place G [note four] shall be to gravity as $3S\times\frac{XY}{A}$ to $4RR$, that is, as XY to $\frac{2nn+2n}{n+2}\times VG$. And the velocity in that place itself is, by which the projected body may progress in a parabola, vertex G, diameter GD [note five] and with giving the latus rectum $\frac{1+QQ}{R}$ or $\frac{2XY^2}{\overline{nn+n}\times VG}$. *Q.E.I.*

[*L. & J.* Note one: But hence the hyperbola, while it is produced, the lines XM, XN also produced approach continually, and it is evident that these can only touch at an infinite distance.

Note two : For it is found that $\frac{S}{R}=\frac{n+2}{SA}$, and

$\sqrt{1+QQ}=\sqrt{1+\frac{dd}{ee}-\frac{2dnbb}{eA^{n+1}}+\frac{nnb^4}{A^{2n+2}}}$; and thus, on account of the given number

$\frac{n+2}{3}$, $\frac{S}{R\sqrt{1+QQ}}$ is as $\frac{1}{\sqrt{AA+\frac{dd}{ee}AA-\frac{2dnbbA}{eA^n}+\frac{n^2b^4}{A^{2n}}}}$.



Note three : For

$XZ=DN=A$ by Hypoth. and $ZY=VY-VZ=n\times VG-\frac{d}{e}A=\frac{nbb}{A^n}-\frac{d}{e}A$;

or $ZY=VZ-VY=\frac{d}{e}A-\frac{nbb}{A^n}$, because YV is greater or less than VZ. Whereby since there shall be $XY^2=XZ^2+ZY^2$, the density will be as $\frac{1}{XY}$.

Note four : Because from the demonstration,

$\frac{XY}{A}=\sqrt{1+QQ}$ there will be $3S\sqrt{1+QQ}=\frac{3S\times XY}{A}$, and thence the resistance to gravity will be as $\frac{3S\times XY}{A}$ to $4RR$, or as XY to $\frac{4RR\times A}{3S}$, but $4RR\times A=\frac{(nn+n)^2\times b^4}{A^{2n+3}}$ and

$3S=\frac{(nn+n)\times(n+2)b^2}{2A^{n+3}}$, and hence $\frac{4RRA}{3S}=\frac{(2nn+2n)\times b^2}{(n+2)\times A^n}=\frac{2nn+2n}{n+2}\times VG$, on account of $VG=\frac{bb}{A^n}$.

Whereby the resistance is to gravity as XY to $\frac{2nn+2n}{n+2}\times VG$.

Note five : Indeed there is $\frac{XY^2}{A^2}=1+QQ$, and hence $\frac{1+QQ}{R}=\frac{2XY^2\times A^n}{(nn+n)\times bb}=\frac{2XY^2}{(nn+n)\times VG}$, on account of $VG=\frac{bb}{A^n}$. From which the velocity which is as $\sqrt{\frac{1+QQ}{R}}$, will be as $\frac{XY}{\sqrt{VG}}$, on account of the given number $\frac{2}{nn+n}$. End of notes.]

Scholium.

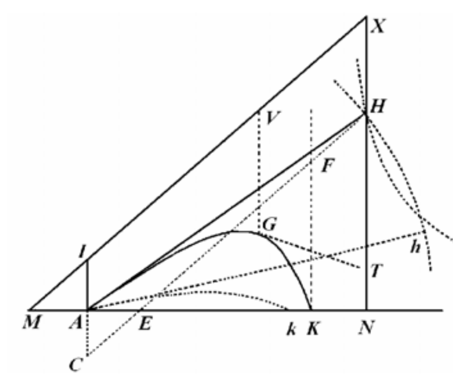
By the same reasoning that gave rise to the density of the medium as $\frac{S\times AC}{R\times HT}$ in the first corollary, if the resistance may be put as some power V^n of the velocity V the density of the medium will be produced as $\frac{S}{R^{\frac{4-n}{2}}}\times\left(\frac{AC}{HT}\right)^{n-1}$. [note one] And therefore if the curve can be found by that law, so that the ratio may be given $\frac{S}{R^{\frac{4-n}{2}}}$ to $\left(\frac{HT}{AC}\right)^{n-1}$, or $\frac{S^2}{R^{4-n}}$ to $\left(1+QQ\right)^{n-1}$: the body will be moving on this curve in a uniform medium with resistance that shall be as the power of the velocity V^n. But we may return to simpler curves.

Because the motion shall not be parabolic unless in a non-resisting medium, with the motion here described in hyperbolas indeed it shall be through a constant resistance ; it is evident that the line, that a projectile will describe in a uniformly resisting medium, approaches closer to these hyperbolas than to a parabola [note two]. Certainly that line is of the hyperbolic kind, but which is described more about the vertex. Truly there is not much difference between this and that, why may not they be put to use with no inconvenience in place of those in practical matters. And perhaps these will soon be more useful, more accurate than the hyperbola and likewise better arranged. Truly in use thus they may follow.

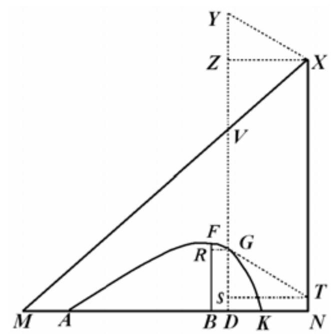
The parallelogram *XYGT* may be completed [note 3], and the right line *GT* touches the hyperbola in *G*, and thus the density of the medium at *G* is inversely as the tangent *GT*, and the velocity likewise as $\sqrt{\frac{GTq}{GV}}$, moreover the resistance is to the force of gravity as *GT* to $\frac{2nn+2n}{n+2}\times GV$.

Therefore if a body projected from the place *A* along the right line *AH* may describe the hyperbola *AGK*, and *AH* produced may meet the asymptote *SNX* in *H*, and with *AI* drawn parallel to the same may meet the other asymptote *MX* in *I* [note 4] : the density of the medium at *A* will be inversely as *AH*, and the velocity of the body as $\sqrt{\frac{AH^2}{AI}}$, and the resistance at the same place as *AH* to $\frac{2nn+2n}{n+2}\times AI$. From which the following rules may be produced.

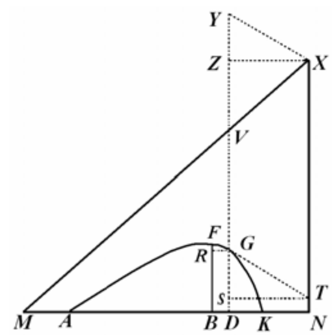
[*L. & J*. Note one : If indeed there were $\frac{S}{R^{\frac{4-n}{2}}}$ to $\left(\frac{HT}{AC}\right)^{n-1}$ in the ratio *a* to *b*, there will be $\frac{S}{R^{\frac{4-n}{2}}}=\frac{a}{b}\times\left(\frac{HT}{AC}\right)^{n-1}$ and $\frac{S\times AC^{n-1}}{R^{\frac{4-n}{2}}\times HT^{n-1}}=\frac{a}{b}$, and therefore uniform. But by Corollary 1. Prop. X, $\frac{HT}{AC}=\sqrt{1+QQ}$: whereby if there were given the ratio $\frac{S}{R^{\frac{4-n}{2}}}$ to $\left(\frac{HT}{AC}\right)^{n-1}$, also the ratio of the squares will be given : $\frac{S^2}{R^{4-n}}$ to $\left(1+QQ\right)^{n-1}$, and conversely.



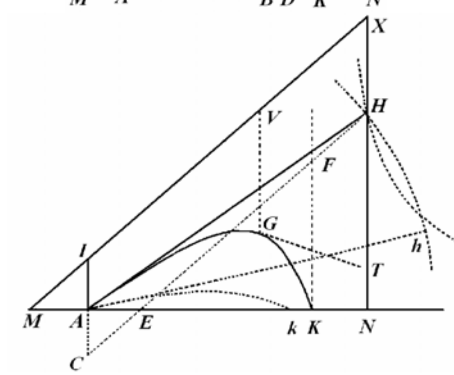
Note two : Since yet from these it cannot be agreed completely, that from these hyperbolas the density of the medium shall be inversely proportional to the variable right line *XY*, and besides it may not be shown satisfactorily that the curve, which the projectile will describe in the uniform medium in the hypothesis of the resistance proportional to the square of the velocity, to have a vertical asymptote as *XN* : since especially in this hypothesis of the resistance the horizontal distance described by the motion present, separated from gravity, may emerge infinite (by Cor.1, Prop.V.). Yet indeed hyperbolas can be found in which for that small part of the curve *AGK*, which in the course of practical matters is necessary, the right line *XY* shall be as approximately constant, and therefore the density of the medium as approximately uniform; from which it comes about that these curves in practice shall not be inconvenient to use.



Note three : From the point *G* draw the ordinate *BF* through *B*, and from the point *T* to the ordinate *DG* there shall be sent the perpendiculars *GR* and *TS*, and let *GT* be a tangent at *G*. There will be $\frac{FR}{RG} = \frac{FR}{BD} = \frac{GS}{ST}$, on account of the similar triangles *FRG, GST*. But *FR* is *Qo* or $\frac{nbbO}{A^{n+1}} - \frac{d}{e}O$, BD is O, and $ST = ZX = A$. Whereby there will be $\frac{nbbO}{A^{n+1}} - \frac{d}{e}$ to 1, or $\frac{nbb}{A^{n+1}} - \frac{d}{e}A$ to *A*, as $\frac{GS}{ZX}$ or *A*. Therefore $ZY = GS$; and therefore the tangent *GT* is equal and parallel to the right line *YX*. But from the demonstration the density of the medium at *G* is inversely as the tangent *GT*, the velocity at that place as $\sqrt{\frac{GT^2}{GV}}$, and the resistance to gravity as GT to $\frac{2nn+2n}{n+2} \times GV$.

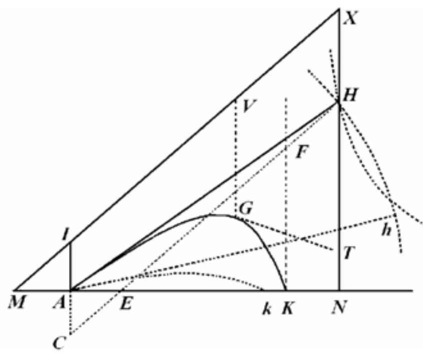


Note four : With the point *G* coinciding with the point *A*. the tangent *GT* agrees with the tangent *AH*, and the right line *VG* with *AI*, and hence the density of the medium at *A* is inversely as the *AH*. End of notes.]

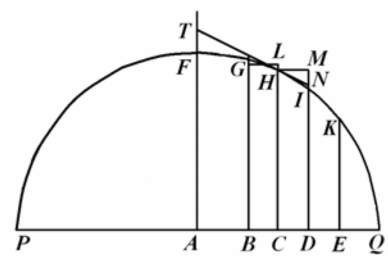


Rule 1. If both the density of the medium as well as the velocity remain the same by which some body may be projected from *A*, and the angle *NAH* may be changed; the lengths *AH, AI, HX* will remain. And thus if these lengths may be found in some case, thereupon the hyperbola from some given angle *NAH* can be conveniently found.

[*L*. & *J*. note on Rule 1 : With the index *n* of the hyperbola remaining the same and with the density of the medium at *A*, the length of the tangent remains *AH* which is inversely proportional to the density. With which remaining velocity the body is projected from the place A, the line $\sqrt{\frac{AH^2}{AI}}$ remains which is as the velocity; and thus since given there shall be *AH* and *AI* given. On account of the parallels *GT*, *YX*, there is $TX = GY = GV + VY = GV + n\times GV$ (Example 4), and because with the point *G* coinciding with the point *A*, there becomes $GV = AI$, and $TX = HX$; there will be $HX = AI + n\times$



Whereby on account of the given quantities *AI* and *n*, also *HX* is given. From which if these lengths *AH*, *AI*, and *HX* may be found in some case, the hyperbola henceforth may be conveniently found from some given angle *NAH*. Indeed with these given, the points *A*, *H*, and *I* are given. Through *H* the line *XHN* is drawn vertical to the horizontal *AN*, and the point *N* will be given; and because *HX* is given, the point *X* is also given; indeed with the two points *X* and *I* given, the right line *XIM* is given with the point *M* by which the horizontal line *MN* may be cut. From which with some right line drawn *VD* normal to the horizontal *AN*, and if on that there may be taken $\frac{VG}{AI} = \frac{AN^n}{DN^n}$, or as $\frac{XI^n}{XV^n}$, the point G will be given on the trajectory AGK. For indeed, by example 4, some ordinate *VG* to another ordinate *IA*, *i.e.* $\frac{VG}{IA}$ is as $\frac{AN^n}{DN^n}$, or as $\frac{XI^n}{XV^n}$.]



Rule 2. If both the angle *NAB*, as well as the density of the medium may remain the same at *A*, and the velocity may be changed by which some body is projected; the length *AH* will remain, and *AI* will be changed inversely in the square ratio of the velocity.

[*L.* & *J.* note on Rule 2 : With the density of the medium serving its purpose at *A*, the length of the tangent *AH* will be maintained, which is inversely as the density. And because the velocity at *A* is as $\sqrt{\frac{AH^2}{AI}}$, and the square of the velocity as $\frac{AH^2}{AI}$, that is, as $\frac{1}{AI}$, on account of *AH* given; *AI* will be inversely proportional to the square of the velocity.]

Rule 3. If both the angle *NAH* , as well as the velocity of the body at *A*, and the accelerating gravity may be maintained, and the proportion of the resistance at *A* may be increased a little to gravity by some ratio ; the proportion *AH* to *AI* will be increased in the same ratio, with the latus rectum of the aforementioned parabola remaining, and proportional to that length $\frac{AH^2}{AI}$ *:* and therefore *AH* may be diminished in the same ratio, and *AI* may be diminished in that ratio squared. Truly the ratio of the resistance to the weight may be increased, when either the specific gravity shall be smaller under an equal magnitude [in its ratio to the resistance] or the density of the medium increased, or the resistance from the diminished magnitude, is diminished in a smaller ratio than the weight [is diminished].

[*L*. & *J*. note on Rule 3 : With the velocity of the body and with the acceleration of gravity given at the point A, the length $\frac{AH^2}{AI}$ is giving both by the square of the velocity as well as by the latus rectum of the proportional parabola (Example 4). But the motive resistance, if it may be called that, to the motive gravity, is as AH to $\frac{2nn+2n}{n+2}\times AI$ (Example 4). Whereby if the proportion of the motive resistance at A to the motive gravity may be increased in some ratio, the proportion of AH to $\frac{2nn+2n}{n+2}\times AI$, or, on account of the given number $\frac{2nn+2n}{n+2}$, will increase $\frac{AH}{AI}$ in the same proportion, and because the length $\frac{AH^2}{AI}$ is constant, and hence $\frac{AH}{AI}$ is as $\frac{1}{AH}$, and AI is as AH^2, it is necessary that *AH* may be decreased in the ratio by which $\frac{AH}{AI}$ may be increased, and so that *AI* may be reduced in that ratio squared.]

Rule 4. Because the density of the medium near the vertex of the hyperbola is greater than at the place *A;* so that it may have a lesser density, the ratio of the smallest tangent *GT* to the tangent *AH* must be found, and the density at *A* to be augmented in a slightly greater ratio than half the sum of these tangents to the minimum tangent *GT*.

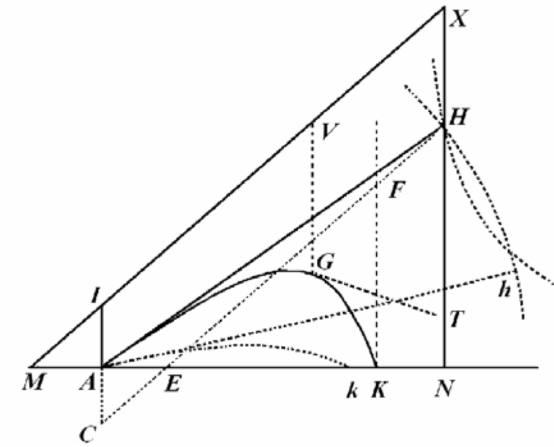
[*L*. & *J*. note on Rule 4 : Because the density at some place *G* is inversely as the tangent *GT*, which is less near the vertex of the hyperbola than at the place *A*; it is evident that the density of the medium near the vertex of the hyperbola is greater than at the position A. The density at the place *A* may be called *K*, at the place *G* through which the minimal tangent *GT* may be drawn, may be called *B* ; and there will be
$\frac{K}{B}=\frac{GT}{AH}$ and hence $\frac{K+B}{K}=\frac{GT+AH}{GT}$, and $\frac{\frac{K+B}{2}}{K}=\frac{\frac{GT+AH}{2}}{GT}$. But $\frac{K+B}{2}$ must be the mean density, if the tangent *AH* becomes the maximum of all, and thus so that *GT* is the minimum of all; and thus, so that the density of the medium may be had as nearly uniform, it may be required to increase the density at *A* in the ratio of half the sum of the tangents $\frac{GT+AH}{2}$ to the minimum tangent *GT*. Truly because the tangent AH is not the maximum of all, but other tangents drawn to the parts of the curve towards K are greater; the density at *A* is to be increased in a ratio a little greater than half the sum $\frac{GT+AH}{2}$ to *GT*, so that the medium may be considered to be almost uniform. And from this agreement errors arising from that because the medium at the place *A* may be supposed denser, they may be almost corrected by other errors which arise from that because at *G* the medium may be placed rarer than for ratio of the curve *AGK*.]

Rule 5. If the lengths *AH*, *AI*, may be given and the figure *AGK* is required to be described : produce *HN* to *X*, so that there shall be *HX* to *AI* as $n+1$ to 1, and with centre *X* and with the asymptotes *MX*, *NX* a hyperbola may be described through the point *A*, by that rule, so that there shall be *AI* to some *VG* as XV^n to XI^n.

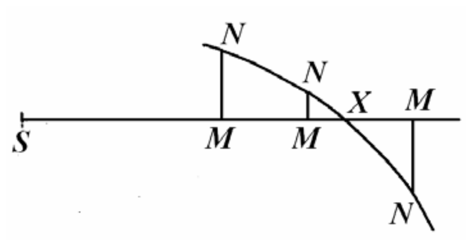
[*L*. & *J*. note on Rule 5 : If the lengths *AH*, *AI* with the angle *HAN* may be given, and the figure *AGK* may be described : from the point *H* to the horizontal *AN* send the perpendicular *HN* ; produce *HN* to *X*, so that $HX = (n+1) \times AI$, as we have shown before for Rule 1, and with centre *X* and asymptotes *MX*, *NX* the hyperbola is described through the point A, by that law, so that $\frac{AI}{VG} = \frac{XV^n}{XI^n}$, for some *VG* : for by example 4, $\frac{VG}{AI} = \frac{AN^n}{DN^n} = \frac{XI^n}{XV^n}$.]

Rule 6. From which the greater the number *n* is, from that the more accurate these hyperbolas shall be described in the ascent of the body from *A*, and the less accurate in the descent of this body to *K;* and conversely. The conic hyperbola maintains a mean ratio, and is simpler than the other curves. Therefore if the hyperbola shall be of this kind, and the point *K* is sought, where the projected body falls on some line *AN* passing through the point *A*: *AN* produced may cross the asymptotes *MX*, *NX* at *M* and *N*, and *NK* may be taken equal to *AM*.

[*L*. & *J*. Extended Note on Rule 6 : Since the larger the number *n* becomes, there the greater these hyperbolas in the ascent of the body from *A* approach to trajectories described in a uniform medium, and there they are the less accurate in the descent to *K*; and conversely. For since the greater the number n, there the smaller the tangent *GT*, which is inversely proportional to the density, in the ascent of the body from *A* it may be varied; and there the more it may be changed in the descent to *K*, certainly the density of the medium shall be given at the *A* with the angle of projection *HAN*, and the quantity $\frac{n+2}{AH}$ proportional to the density at *A* will be given, by Example 4, and thus there the tangent *AH* will be longer as the number *n* becomes greater; and because from the given angle *HAN*, the kind of right angled triangle may be given, and thus the ratio of the sides *AH*, *AN*, *HN* also are given, it is evident that with the increase in *AH* or in the number n, also the sides *AN* and *HN* may increase. From the demonstration in Example 4, with the body ascending, the square of the tangent *GT*, $GT^2 = DN^2 + (ZV - nVG)^2$, and with the body descending the square is $GT^2 = DN^2 + (nVG - ZV)^2$. From the nature of the hyperbola AGK, $\frac{DN^n}{AN^n} = \frac{AI}{VG}$ and thus $nVG = \frac{nAI \times AN^n}{DN^n}$. From the demonstration of the 1st rule, $HX = (n+1) \times AI$ and thus $NX = HN + (n+1) \times AI$, and $NX - AI = HN + nAI$. But on account of the similar triangles *XZV, MNX, MAI*, $\frac{ZX(\text{or } DN)}{ZV} = \frac{MN}{NX} = \frac{MA}{AI}$, and separating, $\frac{DN}{ZV} = \frac{AN}{NX - AI} = \frac{AN}{HN + nAI}$; from which there becomes $ZV = \frac{DN \times HN + nAI \times DN}{AN}$. Whereby in the ascent of the body, $GT^2 = DN^2 + \left(\frac{DN \times HN + nAI \times DN}{AN} - \frac{nAI \times AN^n}{DN^n}\right)^2$, and in the



descent
$GT^2 = DN^2 + \left(\frac{nAI\times AN^n}{DN^n} - \frac{DN\times HN + nAI\times DN}{AN}\right)^2$.
Now truly if the number *n* were large enough in the ascent, the lines AH, AN, HN both in the ascent and in the descent of the body are longer, and in the ascent from A, DN is almost equal to AN, in the descent indeed DN may be permitted to be a little less than AN. From which in the ascent from A there is nearly $\frac{nAI\times AN^n}{DN^n} = nAI$, and thus
$GT^2 = DN^2 + \left(\frac{DN\times HN}{AN}\right)^2 = DN^2 + HN^2$ nearly. But $AH^2 = AN^2 + HN^2$: whereby the ratio *GT* to *AH* in the ascent of the body from *A* is almost one of equality, while the number *n* may be supposed large enough, and hence the density does not vary very much ; in the descent indeed to *K*, *DN* becomes as some small amount with respect to the given *AN*, and thus the quantity $\frac{nAI\times AN^n}{DN^n}$ will increase markedly, and hence the tangent *GT* is changed greatly when the number *n* is large. The opposite happens, if that number shall be exceedingly small. Again since the number *n* can be any integer or fraction, and in the hyperbolic conic *n* shall be equal to 1, which just as the medium may maintain a place among all the whole and fractional numbers, it is shown well enough that the conical hyperbola maintains a mean ratio between all the greater and lesser hyperbolas, and because it is simpler than the others, the true trajectory of the projectile in the medium can be given. Therefore if the hyperbola *AGK* shall be of this kind, and the point *K* is sought where the projected body strikes some right line, horizontal or oblique to the horizontal passing through the point *A* : *AN* produced meets the asymptotes *MX*, *NX* at *M* and *N*, and *NK* may be taken equal to *AM*, and the point K will be found, by Theorem I, *Conics*, *Apoll*. End of note.]



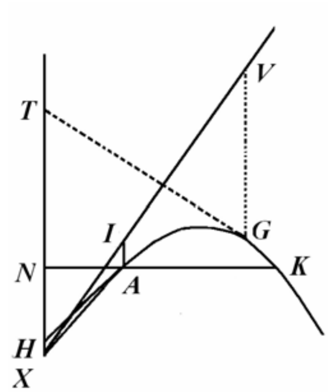
Rule 7. And hence the method set out may be clear for determining this hyperbola from phenomena. Two similar and equal bodies may be projected, with the same velocity, at different angles *HAK*, *hAk*, and incident in the horizontal plane at *K* & *k;* and the proportion *AK* to *Ak* may be noted. Let this be as *d* to *e*. Then with some perpendicular *AI* length erected, [note one] assume some length *AH* to *Ah*, and thence deduce graphically the lengths *AK*, *Ak*, by rule 6. If the ratio *AK* to *Ak* shall be the same as with the ratio *d* to *e*, with the length *AH* had been correctly assumed [note two]. But if less take on the infinite line *SM* a length *SM* equal to the assumed *AH*, and erect the perpendicular *MN* equal to the difference of the ratios $\frac{AK}{Ak} - \frac{d}{e}$ drawn on some given line. By a similar method from several assumed lengths *AB* several points *N* may be found, and through all by drawing a regular curved line *NNXN*, cutting the line *SMMM* in *X*. Finally *AH* may be assumed equal to the abscissa *SX*, and thence with the length *AK* found anew*;* and the lengths, which shall be to the assumed length, which shall be the assumed length *AI* and hence finally *AH*, as the length *AK* known by trial and error to the final length found *AK*, truly will be these lengths *AI* and *AH* [note three], that it was necessary to find. And also truly with these given the resistance of the medium at the place, clearly which shall be to the force of gravity as *AH* to *2AI*. But the density of the medium is required to be increased by rule 4 and the resistance found in this manner, if it may be augmented in the same manner, and it becomes more accurate.

[*L*. & *J*. Note one : For with the tangent *AH* given, both in magnitude and position, the vertical *HN* may be given together with the point *N*; and because also AI is assumed, also there is $HX = 2AI$ by the demonstraton of Rule 1, because $n = 1$; the centre *X* of the hyperbola is given, and thence on account of the given point I, the other asymptote *XIM* is given with the point *M* on the horizontal line *MN*; and by taking *NK* equal to the given *MA*, the point *K* is given, and hence the length *AK* will be obtained. And the other length *Ak* may be found in the same way.

Note two : With the density of the medium given at *A* with the velocity of the body projected at differing angles *HAK*, *hAk*, the perpendicular *AI* remains, and the tangent *AH* is equal to the tangent *Ah*, by the 1st Rule. With the tangent *AH* and the angle *HAK* given, the hyperbola *AGK* can be described by the 6th Rule and the preceding note, and thus it is given both in kind and magnitude. From which finally the angle *HAK* and the ratio of the tangents *HA* to *AI*, the kind of hyperbola will be given finally, that is, all the hyperbolas will be similar that may be described from these two given. Whereby if in hyperbola *AGK*, which may be supposed to be described in the diagram, the tangent assumed *AH* shall be to the perpendicular *AI*, as the tangent of the hyperbola (that the body projected with an angle equal to *HAK* will describe in the resisting medium), is to its own perpendicular *AI*; the hyperbola *AGK* described on the page will be similar to the hyperbola which is described in the medium. And by the same argument the other hyperbola, whose amplitude is *Ak*, and tangent *Ah*, with the perpendicular *AI* remaining, will be similar to that hyperbola which the body described projected at an angle equal to *hAk* , in the second experiment. From which therefore, on account of the similitude of the figures described on the page and in the resisting medium, the amplitudes *AK*, *Ak* will be between themselves as the homologous amplitudes of the hyperbolas which were described in the experiments, that is $\frac{AK}{Ak} = \frac{d}{e}$.

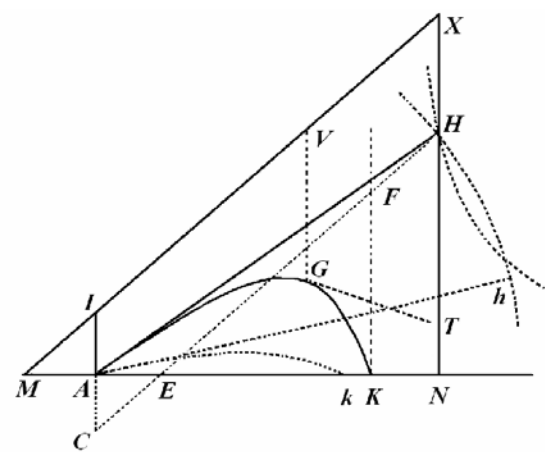
Note three : For since the abscissa *SM* shall be assumed to be equal to the length *AH*, and the difference of the ratios $\frac{AK}{Ak} - \frac{d}{e}$ may be shown be the ordinate *MN*; where there becomes $SM = SX$ and hence $MN = 0$, also $\frac{AK}{Ak} - \frac{d}{e} = 0$, and thus $\frac{AK}{Ak} = \frac{d}{e}$, and *SX* indeed is equal to *AH*, by the preceding note. Thus if from the given perpendicular *AI* and the true length found *AH* with the angle *HAN* required to be found, as above, with the length *AK* ; on account of the similar figures in the resisting medium and described on the page, the length *AK* found by experiment and the length *AK* finally found on the page, will be as the length *AH* in the resisting medium to the length *AH* drawn on the figure, and also as the perpendicular AI in the resisting medium assumed in the diagram. With which found, it will be possible to describe a hyperbola equal and similar to the hyperbola the body describes in the resisting medium. End of notes.]

Rule 8. With the lengths *AH, HX* found*;* if now it may be wished to put in place the position of the line *AH,* along which the projectile sent with some given velocity, it will fall on some point *K:* at the points *A* and *K* the right lines *AC*, *KF* may be erected to the horizontal, of which *AC* tends downwards, and may be equated to *AI* or $\frac{1}{2}HX$, [by Rule 5, as $n = 1$.] .With asymptotes *AK*, *KF* a hyperbola may be described, the conjugate of which may pass through the point *C,* and with centre *A* and with and radius *AH* a circle may be described cutting that hyperbola at the point *H;* and the projectile sent along the right line *AH* will fall at the point *K. Q. E. I.*

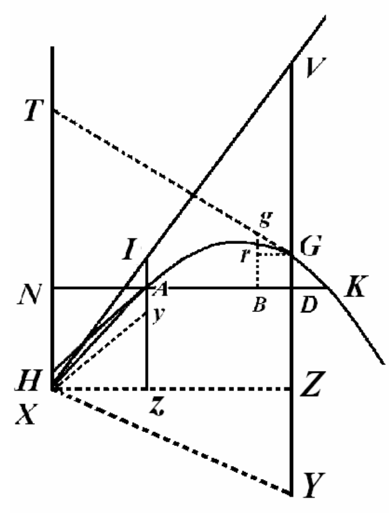


For the point *H,* on account of the given length *AH,* is located somewhere on the circle described. *CH* may be drawn passing through *AK* and *KF,* the one in *E,* the other in *F;* and on account of the parallel lines *CH*, *MX* and the equal lines *AC, AI, AE* equals *AM,* and therefore also equals *KN* [note one]. But $\frac{CE}{AE} = \frac{FH}{KN}$, and therefore *CE* and *FH* are equal. Therefore the point *H* falls on the hyperbola described with the asymptotes *AK, KF*, the conjugate of which passes through the point *C,* and thus is found at the common intersection of this hyperbola and of the described circle. *Q. E. D.*

Moreover it is to be noted that this operation thus may be had itself, whether with the right line *AKN* shall be parallel to the horizontal, or inclined at some angle to the horizontal : each produce two angles *NAH, NAH* from the two intersections *H, h ;* and because it suffices to describe the circle mechanically once in practice, then apply the endless ruler *CH* to the point *C* thus, so that the part *FH* of this, intersected by the circle and the right line *FK*, shall be equal to the part of this C*E* situated between the point *C* and the right line *AK*. [note 2]



What has been said about hyperbolas may also be applied to parabolas. For if *XAGK* may designate a parabola that the right line *XV* may touch at the vertex *X*, and let the applied ordinates be *IA*, *VG* so that any powers of the abscissas *XI, XV* : XI^n , XV^n may be drawn *XT*, *GT*, *AH,* of which *XT* shall be parallel to *VG,* and *GT*, *AH* may touch the parabola at *G* and *A:* and the body projected with the correct velocity from some place *A*, along the right line *AH* produced, describes this parabola, only if the density of the medium, at individual places *G*, shall be inversely as the tangent *GT*. But the velocity at *G* will be that by which a projectile may go on, in a space without resistance, in a conical parabola having the vertex *G*, diameter *VG* produced upwards, and the latus rectum $\frac{2GT^2}{nn-n\times VG}$. And the resistance at *G* will be to the force of gravity as *GT* to $\frac{2nn-2n}{n-2}\times VG$. From which if *NAK* may designate a horizontal straight line, and with the density of the medium then remaining at *A*, while the velocity by which the body may be projected at some angle *NAH* may be changed; the lengths *AH*, *AI*, *HX*, will remain, and thence the vertex *X* of the parabola, and the position of the right line *XI*, and on taking *VG* to *AI* as XV^n to XI^n, all the points of the parabola *G* will be given, through which the projectile will pass. [Extended final note.]



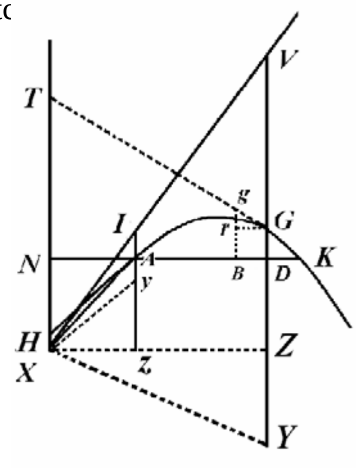
[*L*. & *J*. Note one : For if we suppose *H* to be the point sought, through which it is required to draw the line *AH*, by the construction there will be *HX* equal and parallel to *IC*, and thus *CH* parallel to *IX* or *MX*, and hence the triangles *CAE*, *IAM* similar and thus since there shall be $CA = AI$, by construction, also there will be $AE = MA = KN$, by Th. I *Conics*, *Apoll*. But on account of the similar triangles *CAE, HNE*, and on account of the parallel lines *KF*, *NH*, there is $\frac{CE}{AE} = \frac{EH}{EN} = \frac{FH}{KN}$; and hence the point *H* falls on the hyperbola, again by Th.1.

Note two : Since the point *H* may be determined by the intersection of a circle with the hyperbola, from the demonstration, and the circle can cut the hyperbola in two points, from the two points of intersection *H, h*, two angles are produced, or there are two positions of the tangent *AH*, along which the projectile sent with a given velocity falls on the point *K*.

Extended final note : The line *VG* may be produced so that it cuts the horizontal line *NK* in *D*, and the line XZ parallel to the horizontal in *Z*. For *BN, BD, NX* there may be written *A, O, c*. respectively ; and *M* shall be the intersection of the lines *XV, NK*; and *XN* to *NM*, or on account of the similar triangles *XNM, VZX* , $\frac{VZ}{ZX \text{ or } DN} = \frac{d}{e}$; and thus $DN = A + O$, and $VZ = \frac{d}{e}\times(A+O)$. Indeed because $\frac{VG}{XV^n} = \frac{VX}{XZ \text{ or } DN}$, in the given ratio $\frac{XN}{NM}$; also there will be *VG* as DN^n. Therefore there may be put $VG = \frac{DN^n}{bb} = \frac{(A+O)^n}{bb} = \frac{A^n}{bb} + \frac{nA^{n-1}O}{bb} + \frac{n(n-1)A^{n-2}O^2}{1.2bb} + \frac{n(n-1)(n-2)A^{n-3}O^3}{1.2.3bb} + \text{etc.}$, and there will be

$$GD = VZ - NZ - VG = \tfrac{d}{e} \times A + O - c - \tfrac{(A+O)^n}{bb}$$

$$= \tfrac{d}{e}A - c - \tfrac{A^n}{bb} + \tfrac{d}{e}O - \tfrac{nA^{n-1}O}{bb} - \tfrac{n(n-1)A^{n-2}O^2}{1.2bb} - \tfrac{n(n-1)(n-2)A^{n-3}O^3}{1.2.3bb} - \text{etc}$$



Whereby there will be

$Q = \frac{nA^{n-1}O}{bb} - \frac{d}{e}; R = \frac{n(n-1)A^{n-2}}{2bb}$, and $S = \frac{n(n-1)(n-2)A^{n-3}}{6bb}$.

The ordinate *Bg* may be drawn through the point *B*, to which there may be sent the perpendicular *Gr* from *G*, and *XY* shall be equal and parallel to the tangent *GT*; and on account of the similar triangles *Grg* and *XZY*, $\frac{Gr^2}{Gg^2} = \frac{XZ^2}{XY^2} = \frac{DN^2}{GT^2}$; but there is $Gr^2 = O^2, rg^2 = QQOO$, and thus $Gg^2 = OO \times (1+QQ)$; whereby since also there shall be *BN* or $DN = A$, there will be $GT^2 = AA \times (1+QQ), GT = A\sqrt{1+QQ}$. Per Corollary 1, Prop. X, the density of the medium at the place *G* is as $\frac{S \times A}{R \times GT}$ and from the demonstration, $\frac{S}{R} = \frac{n-2}{3A}$, and thus $\frac{S \times A}{R \times GT}$ is as $\frac{n-2}{3GT}$; whereby, on account of the given number $\frac{n-2}{3GT}$, the density varies inversely as the tangent *GT*. The velocity at *G*, by Prop. X, is that, since by which the projectile may go forwards, in a space without resistance, in the parabolic conic having the vertex *G*, the diameter *GD*, and the latus rectum $\frac{1+QQ}{R}$; and thus since there shall be $\frac{1+QQ}{R} = \frac{GT^2}{A^2R} = \frac{2GT^2}{(nn-n)\times\frac{A^n}{bb}} = \frac{2GT^2}{(nn-n)\times VG} \to$, from the demonstration, the latus rectum of the parabola will be $\frac{2GT^2}{(nn-n)\times VG}$. The resistance at *G*, by Cor. I, Prop. X, is to the force of gravity $3S \times GT$ to $4RR \times A$ that is, as GT to $\frac{4RR\times A}{3S}$; but $4RR \times A = \frac{(nn-n)^2 \times A^{2n-3}}{b^4}$, and $3S = \frac{(nn-n)\times(n-2)}{2} \times \frac{A^{2n-3}}{bb}$, and thus $\frac{4RR\times A}{3S} = \frac{nn-2n}{n-2} \times \frac{A^n}{bb} = \frac{2nn-2n}{n-2} \times VG$. Therefore the resistance will be to gravity, as *GT* to $\frac{2nn-2n}{n-2} \times VG$. The velocity at the place *G*, by Prop. X, is as $\sqrt{\frac{1+QQ}{R}} = \sqrt{\frac{2GT^2}{(nn-n)\times VG}}$, and thus on account of the given number $\frac{2}{nn-n}$, as $\frac{GT}{\sqrt{VG}}$.

Therefore when the body is at *A*, the resistance of the medium is as $\frac{1}{AH}$, and the velocity is as $\frac{AH}{\sqrt{AI}}$; from which with both the density of the medium at A, as well as the velocity with which the body is projected, and with some change in the angle *NAH*, *AH* will remain, and $\frac{AH}{\sqrt{AI}}$, and hence AI. Again because

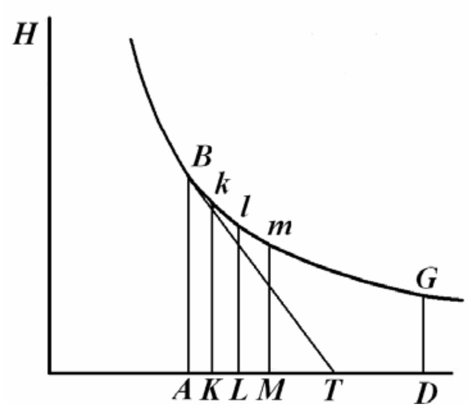
$ZY^2 = XY^2 - XZ^2 = GT^2 - DN^2 = AA \times (1+QQ) - AA = AAQQ$, and thus $ZY = Q \times A = \frac{nA^n}{bb} - \frac{d}{e}A = nVG - VZ$, and $ZY + VZ = VY = nVG$, there will be in place of A, $Iy = n \times AI$, and hence $Ay = XH = nAI - AI$. Whereby with *AI* remaining, also *HX* will remain, on account of the given number $n-1$. The lengths *AH, AI* and hence *HX* may be found using the 7th Rule for the hyperbola made; and thence the point *H* will be given, by which if there is drawn *THX* perpendicular to the horizontal, with *XH* given, the position of the line *XI* will be given, and on taking $\frac{VG}{IA} = \frac{XV^n}{XI^n}$, all the points *G* of the parabola will be given, through which the projectile will pass.

The most elegant of problems concerning the finding of trajectories that a body will describe in a medium with the resistance close to the square of the velocity was passed over by Newton in his *Principia*. The matter was completely resolved later by the most distinguished of mathematicians, Johan Bernoulli, Hermann, and Euler, who found analytically the trajectory described in some medium that had a resistance as some power of the velocity. End of extended note.]

SECTIO II.
De motu corporum quibus resistitur in duplicata ratione velocitatum.

PROPOSITIO V. THEOREMA III.
Si corpori resistitur in velocitatis ratione duplicata, & idem sola vi insita per medium similare movetur , tempora vero sumantur in progressione geometrica a minoribus terminis ad maiores pergente: dico quod velocitates initio singulorum temporum sunt in eadem progressione geometrica inverse, & quod spatia sunt aequalia, quae singulis temporibus describuntur.

Nam quoniam quadrato velocitatis proportionalis est resistentiae medii, & resistentia proportionale est decrementum velocitatis; si tempus in particulas innumeras aequales dividatur, quadrata velocitatum singulis temporum initiis erunt velocitatum earundem differentiis proportionalia. Sunto temporis particulae illa *AK*, *KL*, *LM*, &c. in recta *CD* sumptae, & erigantur perpendicula *AB*, *kK*, *Ll*, *Mm*, &c. hyperbolae *BklmG*, centro *C* asymptotis rectangulis *CD*, C*H* descriptae, occurentia in *B*, *k*, *l*, *m*, &c. & erit *AB* ad *Kk* ut *CK* ad *CA*, & divisim $AB-Kk$ ad *Kk* ut *AK* ad *CA*, & vicissim $AB-Kk$ ad *AK* ut *Kk* ad C*A*, ideoque ut $AB\times Kk$ ad $AB\times CA$. Unde, cum *A K* & $AB\times CA$ dentur, erit $AB-Kk$ ut $AB\times Kk$, & ultimo, ubi coeunt *AB* & *Kk*, ut *ABq*. Et simili argumento erunt $Kk-Ll$, $Ll-Mm$,&c. ut $Kk\ quad$,$Ll\ quad$.&c. Linearum igitur *AB*, *Kk*, *Ll*, *Mm* quadrata sunt ut earundem differentiae & idcirco cum quadrata velocitatum fuerint etiam ut ipsarum differentiae, similis erit ambarum progressio. Quo demonstrato, consequens est etiam ut areae his lineis descriptae sint in progressione consimili cum spatiis quae velocitatibus describuntur. Ergo si velocitas initio primi temporis *AK* exponatur per lineam *AB*, & velocitas initio secundi *KL* per lineam *Kk*, & longitudo primo tempore descripta per aream *AKkB;* velocitates omnes subsequentes exponentur per lineas subsequentes *Ll*, *Mm*, &c. & longitudines descriptae per areas *Kl*, *Lm*, &c. Et composite, si tempus totum exponatur per summam partium suarum *AM*, longitudo tota descripta exponetur per summam partium suarum *AMmB.* Concipe iam tempus *AM* ita dividi in partes *AK*, *KL*, *LM*, &c. ut sint *CA*, *CK*, *CL*, *CM*, etc. in progressione geometrica; & erunt partes illae in eadem progressione, & velocitates *AB*, *Kk*, *Ll*, *Mm*, &c. in progressione eadem inversa, atque spatia descripta *Ak*, *Kl*, *Lm*, &c. aequalia. *Q.E.D.*



Corol. 1. Patet ergo quod, si tempus exponatur per asymptoti partem quamvis *AD*, & velocitas in principio temporis per ordinatim applicatam *AB;* velocitas in fine temporis exponetur per ordinatam *DG*, & spatium totum descriptum per aream hyperbolicam adiacentem *ABGD;* necnon spatium, quod corpus aliquod eodem tempore *AD*. velocitate prima *AB*, in medio non resistente describere posset, per rectangulum $AB\times AD$.

Corol. 2, Unde datur spatium in media resistente descriptum, capiendo illud ad spatium quod velocitate uniformi *A B* in medio non resistente simul describi posset, ut est area hyperbolica *ABGD* ad rectangulum $AB \times AD$.

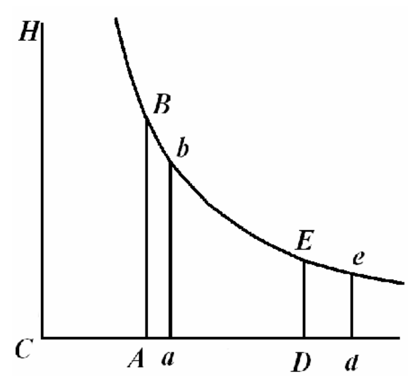
Corol. 3. Datur etiam resistentia medii, Statuendo eam ipso motus initio aequalem esse vi uniformi centripetae , quae in cadente corpore, tempore *AC*, in medio non resistente, generare posset velocitatem *AB.* Nam si ducatur *BT* quae tangat hyperbolam in *B,* & occurrat asymptoto in *T;* recta *AT aequalis* erit ipsi *AC*, & tempus exponet, quo resistentia prima uniformiter continuata tollere posset velocitatem totam *AB.*

Corol. 4. Et inde datur etiam proportio huius resistentiae ad vim gravitatis, aliamve quamvis datam vim centripetam.

Corol 5. Et vice versa, si datur proportio resistentiae ad datam quamvis vim centripetam , datur tempus *AC,* quo vis centripeta resistentiae aequalis generare possit velocitatem quamvis *AB: &* inde datur punctum *B* per quod hyperbola, asymptotis *CH, CD* describi debet; ut & spatium *ABGD*, quod corpus incipiendo motum suum cum velocitate illa *AB,* tempore quovis *AV,* in medio similari resistente describere potest,

PROPOSITIO VI. THEOREMA IV.

Corpora sphaerica homogenea & *aequalia, resistentiis in duplicata ratione velocitatum impedita,* & *solis viribus insitis incitata, temporibus, quae sunt reciproce ut velocitates sub initio, describunt semper aequalia spatia, & amittunt partes velocitatum proportionales totis.*



Asymptotis rectangulis *CD, CH* descripta hyperbola quavis *BbEe* secante perpendicula *AB, ab, DE, de*, in *B, b, E, e,* exponantur velocitates initiales per perpendicula *AB, DE, &* tempora per lineas *A*a, *Dd.* Est ergo ut *Aa* ad *Dd* ita (per hypothesin) *DE* ad *AB,* & ita (ex natura hyperbolae) *CA* ad *CD;* & componendo, ita *CA* ad *Cd*. Ergo areae *ABba, DEed*, hoc est, spatia descripta aequantur inter se, & velocitates primae *AB, DE* sunt ultimis *ab, de, &* propterea dividendo partibus etiam suis amissis $AB - ab, DE - de$ proportionales. *Q.E.D.*

PROPOSITIO VII. THEOREMA V.

Corpora sphaerica quibus resistitur in duplicata ratione velocitatum,temporibus, quae sunt ut motus primi directe & *resistentiae primae inverse, amittent partes motuum proportionales totis,* & *spatia describent temporibus istis* & *velocitatibus primis coniunctim proportionalia.*

Namque motuum partes amissae sunt ut resistentiae & tempora coniunctim. Igitur ut partes illae sint totis proportionales, debebit resistentia & tempus coniunctim esse ut motus. Proinde tempus erit ut motus directe & resistentia inverse. Quare temporum particulis in ea ratione sumptis, corpora amittent semper particulas motuum proportionales totis, ideoque retinebunt velocitates velocitatibus suis primis semper proportionales. Et ob datam velocitatum rationem, describent semper spatia, qua sunt ut velocitates primae & tempora coniunctim. *Q. E. D.*

Corol. 1. Igitur si aequivelocibus corporibus resistitur in duplicata ratione diametrorum : globi homogenei quibuscunque cum velocitatibus moti, describendo spatia diametris suis proportionalia, amittent partes motuum proportionales totis. Motus enim globi cuiusque erit ut eius velocitas & massa, coniundim, id est, ut velocitas & cubus diametri , resistentia (per hypothesin) erit ut quadratum diametri & quadratum velocitatis coniunctim , & tempus (per hanc propositionem) est in ratione priore directe & ratione posteriore inverse ; id est, ut diameter directe & velocitas inverse , ideoque spatium, tempori & velocitati proportionale, est ut diameter.
Corol. 2, Si aequivelocibus corporibus resistitur in ratione sesquiplicata diametrorum : globi homogenei quibuscunque cum velocitatibus moti, describendo spatia in sesquiplicata ratione diametrorum, amittent partes motuum proportionales totis.
Carol. 3. Et universaliter, si aequivelocibus corporibus resistitur in ratione dignitatis cuiuscunque diametrorum : spatia quibus globi homogenei, quibuscunque cum vclocitatibus moti, amittent partes motuum proportionales totis, erunt ut cubi diametrorum ad dignitatem illam applicati. Sunto diametri *D* & *E*; & si resistentiae, ubi velocitates aequales ponuntur, sint ut D^n & E^n : spatia quibus globi, quibuscunque cum velocitatibus moti, amittent partes motuum proportionales totis, erunt ut D^{3-n} & E^{3-n}. Et propterea globi homogenei describendo spatia ipsis D^{3-n} & E^{3-n} proportionalia, retinebunt velocitates in eadem ratione ad invicem ac sub initio.
*Corol.*4. Quod si globi non sint homogenei, spatium a globo densiore descriptum augeri debet in ratione densitatis. Motus enim, sub pari velocitate, maior est in ratione densitatis, & tempus (per hanc propositionem) augetur in ratione motus directe, ac spatium descriptum in ratione temporis.
Corol. 5 Et si globi moveantur in mediis diversis; spatium in medio, quod cateris paribus magis resistit, diminuendum erit in ratione maioris resistentiae. Tempus enim (per hanc propositionem, diminuetur in ratione resistentiae auctae, & spatium in ratione temporis.

L E M M A II.

Momentum genitae aequatur momentis laterum singulorum generantium in eorundem laterum indices dignitatum & *coefficientia continue ductis.*

Genitam voco quantitatem omnem, quae ex lateribus vel terminis quibuscunque in arithmetica per multiplicationem, divisionem, & extractionem radicum , in geometria per inventionem vel contentorum & laterum, vel extremarum & mediarum proportionalium, sine additione & subductione generatur, Eiusmodi quantitates sunt facti, quoti, radices, rectangula, quadrata, cubi, latera quadrata , latera cubica, & similes. Has quantitates, ut indeterminates & instabiles, & quasi motu fluxuve perpetuo crescentes vel decrescentes, hic considero , & earum incrementa vel decrementa momentanea sub nomine momentorum intelligo : ita ut incrementa pro momentis addititiis seu affirmativis, ac decrementa pro subductitiis seu negativis habeantur. Cave tamen intellexeris particulas finitas. Particulae finitae non sunt momenta, sed quantitates ipsae ex momentis genitae. Intelligenda sunt principia iamiam nascentia finitarum magnitudinum. Neque enim spectatur in hoc lemmate magnitudo momentorum, sed prima nascentium proportio. Eodem recidit si loco momentorum usurpentur vel velocitates incrementorum ac decrementorum (quas etiam motus, mutationes & fluxiones quantitatum nominare Iicet) vel finite quaevis quantitates velocitatibus hisce proportionales. Lateris autem cuiusque generantis coefficiens est quantitas, quae oritur applicando genitam ad hoc latus.

Igitur sensus lemmatis est, ut,si quantitatum quarumcunque perpetuo motu crescentium vel decrescentium *A, B, C,* &c. momenta, vel his proportionales mutationum velocitates dicantur *a, b, c,* &c. momentum vel mutatio geniti rectanguli *AB* fuerit $aB+bA$, & geniti contenti *ABC* momentum fuerit $aBC+bAC+cAB$: & genitarum digititatum $A^2, A^3, A^4, A^{\frac{1}{2}}, A^{\frac{3}{2}}, A^{\frac{1}{3}}, A^{\frac{2}{3}}, A^{-1}, A^{-2}$, & $A^{-\frac{1}{2}}$, momenta $2.aA$, $3aA^2, 4aA^3, \frac{1}{2}aA^{-\frac{1}{2}}, \frac{3}{2}aA^{\frac{1}{2}}, \frac{2}{3}aA^{-\frac{2}{3}}, \frac{2}{3}aA^{-\frac{1}{3}}, -aA^{-2}$, $-2A^{-3}$,& $-\frac{1}{2}aA^{-\frac{3}{2}}$ respecive.

Et generaliter, ut dignitatis cuiuscunque $A^{\frac{n}{m}}$ momentum fuerit $\frac{n}{m}aA^{\frac{n-m}{m}}$. Item ut gentae A^2B momentum fuerit $2aAB+bA^2$; & genitae $A^3B^4C^2$ momentum $3aA^2B^4C^2+4bA^3B^3C^2+2cA^3B^4C$; & genitae $\frac{A^3}{B^2}$ sive A^3B^{-2} momentum $3aA^2B^{-2}-2bA^3B^{-3}$ & sic in caeteris. Demonstratur vero lemma in hunc modum.

Cas: 1. Rectangulum quodvis motu perpetuo auctum *AB*, ubi de lateribus *A* & *B* deerant momentorum dimidia $\frac{1}{2}a$ & $\frac{1}{2}b$, fuit $A-\frac{1}{2}a$ in $B-\frac{1}{2}b$, seu $AB-\frac{1}{2}aB-\frac{1}{2}bA+\frac{1}{4}ab$; & quam primum latera *A* & *B* alteris momentorum dimidiis aucta sunt, evadit $A+\frac{1}{2}a$ in $B+\frac{1}{2}b$ seu $AB+\frac{1}{2}aB+\frac{1}{2}bA+\frac{1}{4}ab$. De hoc rectangulo subducatur rectangulum prius, & manebit excessus $aB+bA$. Igitur laterum incrementis totis *a* & *b* generatur rectanguli incrementum $aB+bA$. *Q.E.D.*

Cas. 2. Ponatur *AB* semper aequale *G*, & contenti *ABC* seu *GC* momentum (per cas. 1, erit $gC+cG$, id est (si pro *G* & *g* scribantur AB & $aB+bA$) $aBC+bAC+cAB$. Et par est ratio contenti sub lateribus quotcunque, *Q.E.D.*

Cas. 3. Ponantur latera *A, B, C* sibi mutuo semper aequalia; & ipsius A^2, id est rectanguli *AB*, momentum $aB+bA$ erit $2aA$, ipsius autem A^3, id est contenti *ABC*, momentum $aBC+bAC+cAB$ erit $3aA^2$. Et eodem argumento momentum dignitatis cuiuscunque A^n est naA^{n-1}. *Q.E.D.*

Cas. 4. Unde cum $\frac{1}{A}$ in *A* sit 1, momentum ipsius $\frac{1}{A}$ ductum in *A*, una cum $\frac{1}{A}$ ducto in *a* erit momentum ipsius 1, id est, nihil.

Proinde momentum ipsius $\frac{1}{A}$ seu ipsius A^{-1} est $\frac{-a}{A^2}$. Et generaliter cum $\frac{1}{A^n}$ in A^n sit 1, momentum ipsius $\frac{1}{A^n}$ ductum in A^n una cum $\frac{1}{A^n}$

in naA^{n-1} erit nihil. Et propterea momentum ipsius $\frac{1}{A^n}$ seu A^{-n} erit $-\frac{na}{A^{n+1}}$. *Q.E.D.*

Cas. 5. Et cum $A^{\frac{1}{2}}$ in $A^{\frac{1}{2}}$ sit *A*, momentum ipsius $A^{\frac{1}{2}}$ ductum in $2A^{\frac{1}{2}}$ erit *a*, per cas. 3: ideoque momentum ipsius $A^{\frac{1}{2}}$ erit $\frac{a}{2A^{\frac{1}{2}}}$, sive $\frac{1}{2}aA^{-\frac{1}{2}}$. Et generaliter si ponatur $A^{\frac{m}{n}}$ aequale *B*, erit A^m aequale B^n, ideoque maA^{m-1} aequale nbB^{n-1}, & maA^{-1} aequale nbB^{-1} seu $nbA^{-\frac{m}{n}}$, ideoque $\frac{m}{n}aA^{\frac{m-n}{n}}$ aequale *b*, id est, aequale momento ipsius $A^{\frac{m}{n}}$. *Q.E.D.*

Cas. 6. Igitur genitae cuiuscunque $A^m B^n$ momentum est momentum ipsius A^m ductum in B^n, una cum momento ipsius B^n ducto in A^m, id est $maA^{m-1}B^n + nbB^{n-1}A^m$; idque sive dignitatum indices *m* & *n* sint integri numeri vel fracti, sive affirmativi vel negativi. Et par est ratio contenti sub pluribus dignitatibus. *Q.E.D.*

Corol. I. Hinc in continue proportionalibus, si terminus unus datur; momenta terminorum reliquorum erunt ut iidem termini multiplicati per numerum intervallorum inter ipsos & terminum datum. Sunto *A, B, C, D, E, F* continue proportionales ; & si detur terminus *C*, momenta reliquorum terminorum erunt inter se ut $-2A - B$, D, $2E$, $3F$.

Corol. 2. Et si in quatuor proportionalibus duae mediae dentur momenta extremarum erunt ut eaedem extremae, Idem intelligendum est de lateribus rectanguli cuiuscunque dati.

Corol. 3. Et si summa vel differentia duorum quadratorum detur, momenta laterum erunt reciproce ut latera.

Scholium.

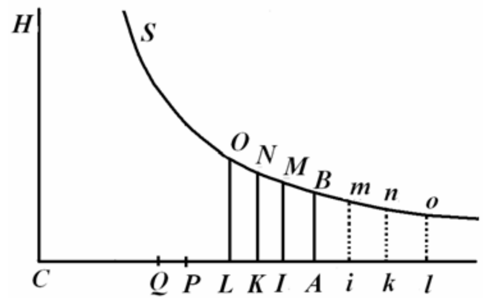
In epistola quadam ad *D.J. Collinium* nostratem 10 Decem. 1672 data, cum descripsissem methodum tangentium quam suspicabar eandem esse cum methodo *Slusii* tum nondum communicata; subiunxi :

Hoc est unum particulare vel corollarium potius methodi generalis, quae extendit se citra molestum ullum calculum, non modo ad ducendum tangentes ad quasvis curvas sive geometricus sive mechanicas vel quomodocunque rectas lineas aliasve curvas respicientes,verum etiam ad resolvendum alia abstrusiora problematum genera de curvitatibus, areis, longitudinibus, centris gravitatis curvarum etc. neque (quemadmodum Huddenii *methodus de maximis* & *minimis)ad solas restringitur aequationes illas quae quantitatibus surdis sunt immunes. Hanc methodum intertexui alteri isti qua aequationum exegesin instituo reducendo eas ad series infinitas.* Hactenus epistola. Et haec ultima verba spectant ad tractatum quem anno 1671 de his rebus scripseram. Methodi vero huius generalis fundamentum continetur in lemmate precedente.

PROPOSITIO VIII. THEOREMA VI.

Si corpus in medio uniformi, gravitate uniformiter agente, recta ascendat vel descendat, & spatium totum descriptum distinguatur in partes aequales, inque principiis singularum partium (addendo resistentiam medii ad vim gravitatis, quando corpus ascendit, vel subducendo ipsam quando corpus descendit) investigentur vires absolutae; dico quod vires illae absolutae sunt in progressione geometrica.

Exponatur enim vis gravitatis per datam lineam *AC;* resistentia per lineam indefinitam *AK;* vis absoluta in descensu corporis per differentiam *KC;* velocitas corporis per lineam *AP,* quae sit media proportionalis inter *AK* & *AC*, ideoque in subduplicata ratione resistentiae ; incrementum resistentiae data temporis particula factum per lineolam *KL,* & contemporaneum velocitatis incrementum per lineolam *PQ* ; & centro *C* asymptotis rectangulis *CA*, C*H* describatur hyperbola quaevis *BNS*, erectis perpendiculis A*B*, *KN, LO* occurrens in *B, N, O.* Quoniam *AK* est ut *APq,* erit huius momentum *KL* ut illius momentum 2*AP*Q. id est, ut *AP* in *KC;* nam velocitatis incrementum *PQ* (per motus leg. II) proportionale est vi generanti *KC*. Componatur ratio ipsius *KL* cum ratione ipsius *KN,* & fiet rectangulum $KL\times KN$ ut $AP\times KC\times KN$; hoc est, ob datum rectangulum $KC\times KN$, ut *AP*. Atqui areae hyperbolicae *KNOL* ad rectangulum $KL\times KN$ ratio ultima, ubi coeunt puncta *K* & *L,* est aequalitatis. Ergo area illa hyperbolica evanescens est ut *AP.* Componitur igitur area tota hyperbolica *ABOL* ex particulis *KNOL* velocitati *AP* semper proportionalibus, & propterea spatio velocitate ista descripto proportionalis est. Dividatur iam area illa in partes aequales *ABMI; IMNK; KNOL*, &c. & vires absolutae *AC*, *IC*, *KC*, *LC,* &c. erunt in progressione geometrica. *Q.E.D.*



Et simili argumento, in ascensu corporis, sumendo , ad contrariam partem puncti *A*, aequales areas *ABmi, imnk*, *knol,* &c. constabit quod vires absolutae *AC*, *iC*, *kC, IC,* &c. sunt continue proportionales. Ideoque si spatia omnia in ascensu & descensu capiantur aequalia; omnes vires absolutae *lC, k*C, *iC*, *AC*, *IC, KC*, *LC,* &c. erunt continue proportionales, *Q.E.D.*

Corol. I. Hinc si spatium descriptum exponatur per aream hypelbolicam *ABNK;* exponi possunt vis gravitatis, velocitas corporis & resistentia medii per lineas *AC, AP, & AK* respective , & vice versa.

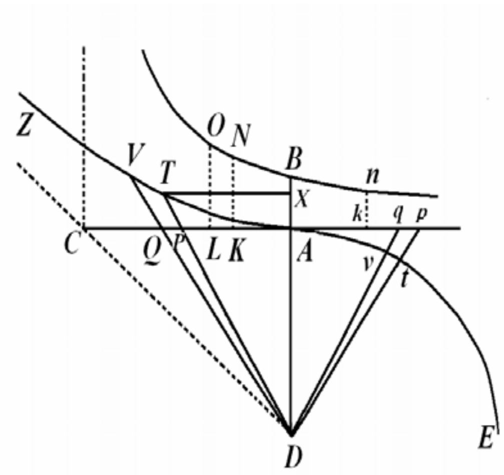
Carol. 2. Et velocitatis maximae, quam corpus in infinitum descendendo potest unquam acquirere, exponens est linea *AC.*

Corol. 3. Igitur si in data aliqua velocitate cognoscatur resistentia medii, invenietur velocitas maxima, sumendo ipsam ad velocitatem illam datam in subduplicata ratione, quam habet vis gravitatis ad medii resistentiam illam cognitam.

PROPOSITIO IX. THEOREMA VII.

Positis iam demonstratis, dico quod, si tangentes angulorum sectoris circularis & *sectoris hyperbolici sumantur velocitatibus proportionales, existente radio iustae magnitudinis: erit tempus omne ascendendi ad locum summum ut sector circuli,*
& *tempus omne descendendi a loco summo ut sector hyberbolae.*

Rectae *AC,* qua vis gravitatis exponitur, perpendicularis & aequalis ducatur *AD.* Centro *D* semidiametro *AD* describatur tum circuli quadrans *AtE;* tum hyperbola rectangula *AVZ* axem habens *AX,* verticem principalem *A*, & asymptoton *DC.* Ducantur *Dp, DP,* & erit sector circularis *AtD* ut tempus omne ascendendi ad locum summum; & sector hyperbolicus *ATD* ut tempus omne descendendi a loco summo: Si modo sectorum tangentes *Ap, AP* sint ut velocitates.



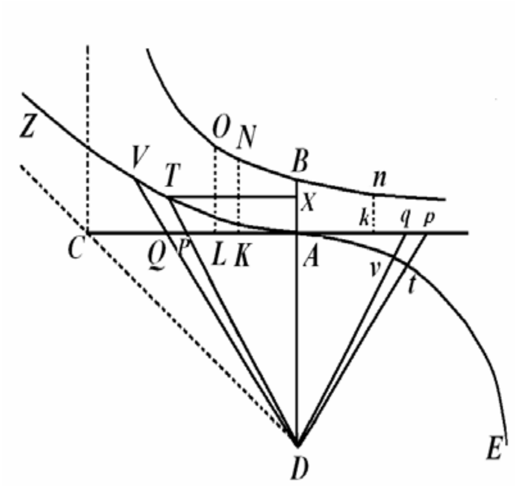
Cas. 1. Agatur enim *Dvq* abscindens sectoris *ADt* & trianguli *ADp* momenta, seu particulas quam minimas simul descriptas *tDv & qDp.* Cum particulae illae, ob angulum communem *D*, sunt in duplicata ratione laterum, erit particula *tDv* ut $\frac{qDp \times tD\ quad}{pD\ quad}$.

id est, ob datam *tD*, ut $\frac{qDp}{pD\ quad}$. Sed *pD quad.* est $AD\ quad. + Ap\ quad.$, id est, $AD\ quad. + AD \times Ak$, seu $AD \times Ck$; & qDp est $\frac{1}{2} AD \times pq$. Ergo sectoris particula *tDv* est ut $\frac{pq}{Ck}$; id est, ut velocitatis decrementum quam minimum *pq* directe, & vis illa *Ck* quae velocitatem diminuit inverse; atque ideo ut particula temporis decremento velocitatis respondens. Et componendo sit summa particularum omnium *tDv* in sectore *ADt*, ut summa particularum temporis singulis velocitatis decrescentis *Ap* particulis amissis *pq* respondentium, usque dum velocitas illa in nihilum diminuta evanuerit;
hoc est, sector totus *ADt* est ut tempus totum ascendendi ad locum summum. *Q. E. D.*

Cas. 2. Agatur *DQV* abscindens tum sectoris *DAV*, tum trianguli *DAQ* particulas quam minimas *TDq* & *PDQ*; & erunt hae particulae ad invicem ut *DTq* ad *DPq*, id est (si *TX* & *AP* parallelae sint) ut *DXq* ad *DAq* vel *TXq* ad *APq,* & divisim ut $DXq - TXq\ ad\ DAq - APq$. Sed ex natura hyperbolae $DXq - TXq$ est ADq, & per hypothesin *APq* est $AD \times AK$. Ergo particulae sunt ad invicem ut *ADq* ad $ADq - AD \times AK$; id est, ut *AD* ad $AD - AK$ seu *AC* ad *CK:* ideoque sectoris particula *TDV* est $\frac{PDQ \times AC}{CK}$; atque ideo ob datas *AC* & *AD*, ut $\frac{PQ}{CD}$, id est, ut incrementum ve1ocitatis directe, utque vis generans incrementum inverse; atque ideo ut particula temporis incremento respondens. Et componendo sit summa particularum temporis, quibus omnes velocitatis *AP* particulae *PQ* generantur, ut summa particularum sectoris *ATD,* id est, tempus totum ut sector totus.

Q. E. D.

Corol. 1. Hinc si *AB* aequetur quartae parti ipsius *AC*, spatium quod corpus tempore quovis cadendo describit, erit ad spatium, quod corpus velocitate maxima *AC*, eadem tempore uniformiter progrediende describere potest, ut area *ABNK*, qua spatium cadendo descriptum exponitur, ad aream *ATD*, qua tempus exponitur. Nam cum sit *A*C ad *AP* ut AP ad *A K*, erit (per corol. I, Lem. II. huius) *L K* ad *PQ* ut *2AK* ad *AP*, hoc est, ut 2*AP* ad *AC*, & inde *LK* ad $\frac{1}{2}PQ$ ut *AP* ad $\frac{1}{4}AC$ vel *AB;* est & *KN* ad *AC* vel *AD*, ut *AB* ad *CK;* itaque ex aequo *LKNO* ad *DPQ* ut *AP* ad C*K*. Sed erat *DPQ* ad *DTV* ut *CK* ad *AC.* Ergo rursus ex aequo *LKNO* est ad *DTV* u*t* *AP* ad *AC*; hoc est, ut velocitas corporis cadentis ad velocitatem maximam quam corpus cadendo potest acquirere. Cum igitur arearum *ABNK* & *ATD* momenta *LKNO &* *DTV*sunt ut velocitates, erunt arearum illarum partes omnes simul genitae ut spatia simul descripta, ideoque arem totae ab initio genitae *ABNK* & *ATD* ut spatia tota ab initio descensus descripta. *Q. E. D.*



Corol. 2. Idem consequitur etiam de spatio quod in ascensu describitur. Nimirum quod spatium illud omne sit ad spatium, uniformi cum velocitate *AC* eodem tempore descriptum, ut est area *ABnk* ad sectorem *ADt.*

Corol. 3. Velocitas corporis tempore *ATD* cadentis est ad velocitatem, quam eodem tempore in spatio non resistente acquireret, ut triangulum *APD* ad sectorem hyperbolicum *ATD.* Nam velocitas in medio non resistente foret ut tempus *ATD,* & in medio resistente est ut *AP,* id est, ut triangulum *APD.* Et velocitates illae initio descensus aequantur inter se, perinde ut areae illae *ATD, APD.*

Corol. 4. Eodem argumento velocitas in ascensu est ad velocitatem, qua corpus eodem tempore in spatio non resistente omnem suum ascendendi motum amittere posset, ut triangulum *ApD* ad sectorem circularem *AtD;* sive ut recta *Ap* ad arcum *At.*

Corol. 5. Est igitur tempus, quo corpus in medio resistente cadendo velocitatem *AP* acquirit, ad tempus, quo velocitatem maximam *A*C in spatio non resistente cadendo acquirere posset, ut sector *ADT* ad triangulum *ADC:* & tempus, quo velocitatem *Ap* in medio resistente ascendendo possit amittere, ad tempus quo velocitatem eandem in spatio non resistente ascendendo posset amittere, ut arcus *At* ad eius tangentem *Ap.*

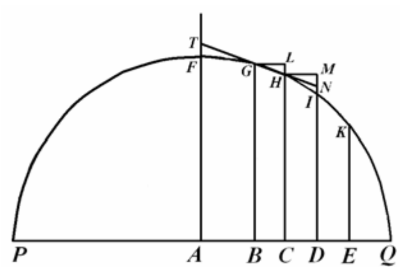
Corol. 6. Hinc ex dato tempore datur spatium ascensu vel descensu descriptum. Nam corporis in infinitum descendentis datur velocitas maxima (per Corol. 2. & 3. theor. VI. Lib. II.) indeque datur tempus quo corpus velocitatem illam in spatio non resistente cadendo posset acquirere. Et sumendo sectorem *ADT* vel *ADt* ad triangulum *ADC* in ratione temporis dati ad tempus modo inventum ; dabitur tum velocitas *AP* vel *Ap*, tum area *ABNK* vel *ABnk*, quae est ad sectorem *ADT* vel *ADt* ut Spatium quesitum ad spatium, quod tempore dato, cum velocitate illa maxima jam ante inventa, uniformiter describi potest.

Corol. 7. Et regrediendo, ex dato ascensus vel descensus spatio *ABnk* vel *ABNK*, dabitur tempus *ADt* vel *ADT*.

PROPOSITIO X. PROBLEMA III.

Tendat, uniformis vis gravitatis directe ad planum horizontis, sitque resistentia ut medii densitas & quadratum velocitatis coniunctim: requiritur tum medii densitas in locis singulis, quae faciat ut corpus in data quavis linea curva moveatur ; tum corporis velocitas & medii resistentia in locis singulis.

Sit *PQ* planum illud plano schematis perpendiculare ; *PFHQ* linea curva plano huic occurrens in punctis *P* & *Q; G, H, I, K* loca quatuor corporis in hac curva ab *F* ad *Q* pergentis; & *GB,HC, ID, KE* ordinatae quatuor parallela ab his punctis ad horizontem demisse, & lineae horizontali *PQ* ad puncta *B, C, D, E* insistentes ; & sint *BC, CD, DE* distantiae ordinatarum inter se aequales. A punctis *G* & *H* ducantur rectae *GL, HN* curvam tangentes in G & *H*, & ordinatis *CH, DI* sursum productis occurrentes in *L* & *N*



& compleatur parallelogrammum *HCDM*. Et tempore, quibus corpus describit arcus *GH, HI*, erunt in subduplicata ratione altitudinum *LH, NI*, quas corpus temporibus illis describere posset, a tangentibus cadendo; & velocitates erunt ut longitudines descripae *GH, HI* directe & tempora inverse. Exponantur tempora per *T & t*, & velocitates per $\frac{GH}{T}$ & $\frac{HI}{t}$ & decrementum velocitatis tempore *t* factum exponetur per $\frac{GH}{T}-\frac{HI}{t}$. Hoc decrementum oritur a resistentia corpus retardante, & gravitate corpus accelerante. Gravitas, in corpore cadente & spatium *NI* cadendo describente, generat velocitatem, qua duplum illud spatium eodem tempore describi potuisset, ut *Galilaeus* demonstravit , id est, velocitatem $\frac{2NI}{t}$: at in corpore arcum *HI* describente, auget arcum illum sola longitudine $HI-HN$ seu $\frac{MI\times NI}{HI}$; ideoque generat tantum velocitatem $\frac{2MI\times NI}{t\times HI}$ Addatur haec velocitas ad decrementum praedictum, & habebitur decrementum velocitatis ex resistentia sola oriundum, nempe $\frac{GH}{T}-\frac{HI}{t}+\frac{2MI\times NI}{t\times HI}$. Proindeque cum gravitas eodem tempore in corpore cadente generet velocitatem $\frac{2NI}{t}$; resistentia erit ad gravitatem ut $\frac{GH}{T}-\frac{HI}{t}+\frac{2MI\times NI}{t\times HI}$ ad $\frac{2NI}{t}$ sive ut $\frac{t\times GH}{T}-HI+\frac{2MI\times NI}{HI}$ ad $2NI$.

Iam pro abscissis *CB, CD, CE* scribantur $-o,\ o,\ 2o$. Pro ordinata *CH* scribatur *P*, & pro *MI* scribatur series qualibet $Qo+Roo+So^3+\text{etc.}$ Et seriei termini omnes post primum, nempe $Roo+So^3+$ etc.. erunt *NI, &* ordinatae *DI, EK*, & *BG* erunt

$P-Qo-Roo-So^3-$ &c.$P-2Qo-4Roo-8S0^3$ &c. & $P+Qo-Roo+So^3-$&c.

respective. Et quadrando differentias ordinatarum $BG-CH$ & $CH-DI$, & ad quadrata prodeuntia addendo quadrata ipsarum *B*C, *CD,* habebuntur arcuum *GH, HI* quadrata

$oo+QQoo-2QRo^3+$&c. & $oo+QQoo+2QRo^3+$&c. Quorum radices

$o\sqrt{1+QQ}-\frac{QRoo}{\sqrt{1+QQ}}$, & $o\sqrt{1+QQ}+\frac{QRoo}{\sqrt{1+QQ}}$ sunt arcus *GH* & *HI*. Preaterea si ab ordinata C*H* subducatur semisumma ordinatarum *BG* ac *DI*, & ab ordinata *DI* subducatur semisumma ordinatarum *CH* & *EK*, manebunt arcuum *GI* & *HK* sagittae

Roo & $Roo+3So^3$. Et hae sunt lineolis *LH* & *NI* proportionales, ideoque in duplicata ratione temporum infinite parvorum *T* & *t* . & inde ratio

$\frac{t}{T}$ est $\sqrt{\frac{\text{R}+3S0}{R}}$ seu $\frac{R+\frac{3}{2}So}{R}$; & $\frac{t\times GH}{T}-HI+\frac{2MI\times NI}{HI}$, substituendo ipsorum $\frac{t}{T}$, G*H, HI, MI* & *NI* valores iam inventos,evadit $\frac{3Soo}{2R}\sqrt{1+QQ}$. Et cum $2NI$ fit $2Roo$, resistentia iam erit ad gravitatem ut $\frac{3Soo}{2R}\sqrt{1+QQ}$ ad $2Roo$, id est, ut $3S\sqrt{1+QQ}$ ad $4RR$.

Velocitas autem ea est, quacum corpus de loco quovis *H*, secundum tangentem *HN* egrediens, in parabola diametrum *HC* & latus rectum $\frac{HNq}{NI}$ seu $\frac{1+QQ}{R}$ habente, deinceps in vacuo moveri potest,

Et resistentia est ut medii densitas & quadratum velocitatis coniunctim, & propterea medii densitas est ut resistentia directe & quadratum velocitatis inverse, id est, ut $\frac{3S\sqrt{1+QQ}}{4RR}$ directe & $\frac{1+\text{QQ}}{\text{R}}$ inverse, hoc est, ut $\frac{S}{R\sqrt{1+QQ}}$. *Q. E. I.*

Corol. I. Si tangens *HN* producatur utrinque donec occurrat ordinatae cuilibet *AF* in *T:* erit $\frac{HT}{AC}$ aequalis $\sqrt{1+QQ}$, ideoque in superioribus pro $\sqrt{1+QQ}$ scribi potest. Qua ratione resistentia erit ad gravitatem ut $3S\times HT$ ad $4RR\times AC$, velocitas erit ut $\frac{HT}{AC\sqrt{R}}$ & medii densitas erit ut $\frac{S\times AC}{R\times HT}$.

Corol. 2. Et hinc, si curva linea *PFHQ* definiatur per relationem inter basem seu abscissam *A*C & ordinatim applicatam *CH,* ut moris est; & valor ordinatim applicatae resolvatur in seriem convergentem: Problema per primos seriei terminos expedite solvetur, ut in exemplis sequentibus.

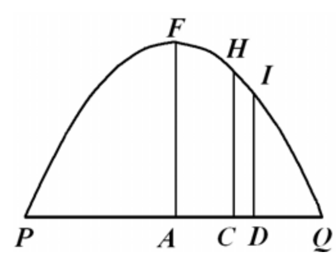
Exempl. 1. Sit linea *PFH* semicirculus super diametro *PQ* descriptus, & requiratur medii densitas quae faciat ut proiectile in hac linea moveatur.

Bisecetur diameter *PQ* in *A;* dic *AQ,n; A C, a; CH, e; & CD, o* : & erit *DIq* seu $AQq-ADq=nn-aa-2ao-oo$,seu $ee-2ao-oo$, & radice per methodum nostram extracta, fiet $DI=e-\frac{ao}{e}-\frac{oo}{2e}-\frac{aaoo}{2e^3}-\frac{ao^3}{2e^3}-\frac{a^3o^3}{2e^5}-\text{etc.}$ Hic scribatur *nn* pro $ee+aa$, & evadet $DI=e-\frac{ao}{e}-\frac{nnoo}{2e^3}-\frac{anno^3}{2e^5}-$ etc.

Huiusmodi series distinguo in terminos successivos in hunc modum. Terminum primum appello, in quo quantitas infinite parva *o* non extat , secundum, in quo quantitas illa est unius dimensionis ; tertium, in quo extat duarum; quartum, in quo trium est ; & sic in infinitum. Et primus terminus, qui hic est *e*, denotabit semper longitudinem ordinatae *CH* insistentis ad initium indefinitae quantitatis *o*. Secundus terminus, qui hic est $\frac{ao}{e}$, denotabit differentiam *e* inter *CH* & *DN*, id est, lineolam *MN*, quae abscinditur complendo parallelogrammum *HCDM*, atque ideo positionem tangentis *HN* semper determinat; ut in hoc casu capiendo *MN* ad *HM* ut est $\frac{ao}{e}$ ad *o*, seu *a* ad *e*. Terminus tertius, qui hic est $\frac{nnoo}{2e^3}$ designabit lineolam *IN*, quae iacet inter tangentem & curvam, ideoque determinat angulun contactus *IHN* seu curvaturam quam curva linea habet in *H*. Si lineola illa *IN* finite est magnitudinis, designabitur per terminum tertium una cum sequentibus in infinitum. At si lineola illa minuatur in infinitum, termini subsequentes evadent infinite minores tertio, ideoque negligi possint. Terminus quartus determinat variationem curvaturae, quintus variationem variationis, & sic deinceps. Unde obiter patet usus non contemnendus harum serierum in solutione problematum, quae pendent a tangentibus & curvature curvarum.

Conferatur iam series $e-\frac{ao}{e}-\frac{nnoo}{2e^3}-\frac{anno^3}{2e^5}-$ etc. cum serie $P-Qo-Roo-So^3-$ etc. perinde pro *P, Q, R & S* scribatur $e, \frac{a}{e}, \frac{nn}{2e^3}, \& \frac{ann}{2e^5}$, & pro $\sqrt{1+QQ}$ scribatur $\sqrt{1+\frac{aa}{ee}}$ seu $\frac{n}{e}$, prodibit medii densitas ut $\frac{a}{ne}$, hoc est (ob datam *n*), ut $\frac{a}{e}$ seu $\frac{AC}{CH}$, id est, ut tangentis longitudo illa *HT*, quae ad semidiametrum *AF* ipsi *PQ* normaliter insistentem terminatur : & resistentia erit ad gravitatem ut 3*a* ad 2*n*, id est, ut 3*AC* ad circuli diametrum *PQ* : velocitas autem erit ut , $\sqrt{CH}$. Quare si corpus iusta cum velocitate secundum lineam ipsi *PQ* parallelam exeat de loco *F*, & medii densitas in singulis locis *H* sit ut longitudo tangentis *HT*, & resistentia etiam in loco aliquo *H* sit ad vim gravitatis ut *3A*C ad *PQ* corpus illud describet circuli quadrantem *FHQ*. *Q.E.I.*

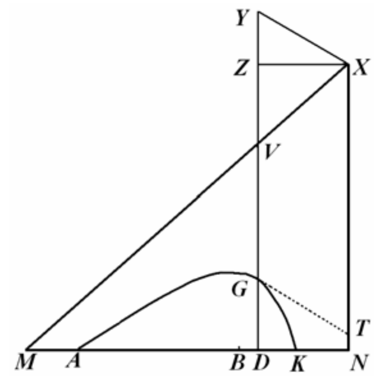
At si corpus idem de loco *P*, secundum lineam ipsi *PQ* perpendicularem egrederetur, & in arcu semicirculi *PFQ* moveri inciperet, sumenda esset *AC* seu *a* ad contrarias partes centri *A*, & propterea signum eius mutandum esset & scribendum – *a* pro + *a*. Quo pacto prodiret medii densitas ut $-\frac{a}{e}$. Negativam autem densitatem, hoc est, quae motus corporum accelerat, natura non admittit: & propterea naturaliter fieri non potest, ut corpus ascendendo a *P* describat circuli quadrantem *PF*. Ad hunc effectum deberet corpus a medio impellente accelerari, non a resistente impediri,



Exempl. 2. Sit linea *PFQ* parabola, axem habens *AF* horizonti *PQ* perpendicularem, & acquiratur medii densitas, quae faciat ut proiectile in ipsa moveatur.

Ex natura parabolae, rectangulum *PDQ* aequale est rectangulo sub ordinata *DI* &

recta aliqua data: hoc est, si dicantur recta illa *b* ; *PC, a; PQ, c; CH, e;* & *CD, o* ; rectangulum $a+o$ in $c \quad -a-o$ seu $ac-aa-2ao+co-oo$ seu aequale est rectangulo *b* in *DI,* ideoque *DI* aequale $\frac{ac-aa}{b}+\frac{c-2a}{b}o-\frac{oo}{b}$. Iam scribendus esset huius seriei secundus terminus $\frac{c-2a}{b}o$, pro *Qo*, tertius item terminus $\frac{oo}{b}$ pro Roo. Cum vero plures non sint termini, debebit quarti coefficiens S evanscere, and propterea quantitas $\frac{S}{R\sqrt{1+QQ}}$, cui medii densitas proportionalis est, nihil erit, Nulla igitur medii densitate movebitur proiectile in parabola, uti olim demonstravit *Galiilaeus. Q. E. I.*



Exempl. 3. Sit linea *AGK* hyperbola, asymptoton habens *NX* plano horizontali *AK* perpendicularem , & quaeratur medii densitas, quae faciat ut proiectile moveatur in hac linea.

Sit *MX* asymptotos altera, ordinatim applicatae *DG* productae occurrens in *V;* & ex natura hyperbolae, rectangulum *XV* in *VG* dabitur. Datur autem ratio *DN* ad *VX,* & propterea datur etiam rectangulum *DN* in *VG.* Sit illud *bb*: & completo parallelogrammo *DNXZ;* dicatur *BN, a; BD, o; NX,* c; & ratio data *VZ* ad Z*X* vel *DN* ponatur esse $\frac{m}{n}$. Et erit *DN* aequalis $a-o$, *VG* aequalis $\frac{bb}{a-o}$, *VZ* aequalis $\frac{m}{n}\overline{a-o}$, & *GD* seu $NX-VZ-VG$ aequalis item terminus $c-\frac{m}{n}a+\frac{m}{n}o-\frac{bb}{a-o}$. Resolvatur terminus $\frac{bb}{a-o}$ in seriem convergentum $\frac{bb}{a}+\frac{bb}{aa}o+\frac{bb}{a^3}oo+\frac{bb}{a^4}o^3$ etc., & fiet *GD* aequalis $c-\frac{m}{n}a-\frac{bb}{a}+\frac{m}{n}o-\frac{bb}{aa}o-\frac{bb}{a^3}o^2-\frac{bb}{a^4}o^3$ etc. *;* Huius seriei terminus secundus $:\frac{m}{n}o-\frac{bb}{aa}o$ usurpandus est pro *Qo.* tertius cum signo mutato $\frac{bb}{a^3}o^2$ pro Ro^2 & quartus cum signa etiam mutato $\frac{bb}{a^4}o^3$ pro So^3, eorumque coefficientes $\frac{m}{n}-\frac{bb}{aa}, \frac{bb}{a^3}$ & $\frac{bb}{a^4}$ scribendae sunt in regula superiore pro *Q*, *R* & *S*. Quo facto prodit medii densitas ut

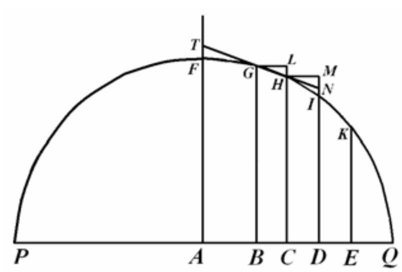
$$\frac{\frac{bb}{a^4}}{\frac{bb}{a^3}\sqrt{1+\frac{mm}{nn}-\frac{2mbb}{naa}+\frac{b^4}{a^4}}} \text{ seu } \frac{1}{\sqrt{aa+\frac{mm}{nn}aa-\frac{2mbb}{n}+\frac{b^4}{aa}}}$$

est, si in *VZ* sumatur *VY* aequalis *VG,* ut $\frac{1}{XY}$. Namque aa & $\frac{mm}{nn}aa-\frac{2mbb}{n}+\frac{b^4}{aa}$ sunt ipsarum *XZ* & *ZY* quadrata. Restistentia autem invenitur in ratione ad gravitatem quam habet $3XY$ ad $2YG$ *;* & velocitas ea est, quacum corpus in parabola pergeret verticem *G,* diametrum *DG,* & latus rectum $\frac{XY\ quad.}{VG}$ habente. Ponatur itaque quod medii densitates in locis singulis *G* sint reciproce ut distantiae *XY,* quodque resistentia in loco aliquo *G* sit ad gravitatem ut 3*XY* ad 2*YG;* & corpus de loco *A,* iusta cum velocitate emissum, describet hyperbolam illam *AGK. Q. E. I.*

Exempl. 4. Ponatur indefinitae, quod linea *AGK* hyperbola sit, centro *X*, asymptotis *MX*, *NX* ea lege descripta, ut constructo rectangulo *XZDN* cuius latus *ZD* secet hyperbolam in *G* & asymptoton eius in *V*, fuerit *VG* reciproce ut ipsius *ZX* vel *DN* dignitas aliqua DN^n, cuius index est numerus *n* : & quaeratur medii densitas, qua proiectile progrediatur in hac curva.

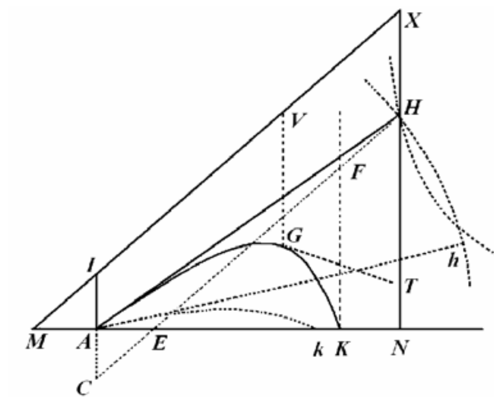
Pro *BN*, *BD*, *NX* scribantur *A*, *O*, *C* respective, sitque *VZ* ad *XZ* vel *DN* ut *d* ad *e*, & *VG* aqualis $\frac{bb}{DN^n}$, & erit *DN* aequalis $A-O, VG=\frac{bb}{\overline{A-O}}VZ=\frac{d}{e}\overline{A-O}$, & *GD* seu $NX-VZ-VG$ aequalis $C-\frac{d}{e}A+\frac{d}{e}O-\frac{bb}{\overline{A-O}^n}$. Resolvatur terminus illae $\frac{bb}{\overline{A-O}^n}$ in seriem infinitam $\frac{bb}{A^n}+\frac{nbb}{A^{n+1}}O+\frac{nn+n}{2A^{n+2}}bbO^2+\frac{n^3+3nn+2n}{6A^{n+3}}bbO^3$ etc. &c. ac fiet G*D* aequalis $C-\frac{d}{e}A-\frac{bb}{A^n}+\frac{d}{e}O-\frac{nbb}{A^{n+1}}O-\frac{+nn+n}{2A^{n+2}}bbO^2-\frac{+n^3+3nn+2n}{6A^{n+3}}bbO^3$ etc. Huius seriei terminus secundus $\frac{d}{e}O-\frac{nbb}{A^{n+1}}O$ usurpandus est pro *Qo*, tertius $\frac{+nn+n}{2A^{n+2}}bbO^2$ pro Ro^2, quartus $\frac{+n^3+3nn+2n}{6A^{n+3}}bbO^3$ pro So^3. Et inde medii densitas $\frac{S}{R\sqrt{1+QQ}}$, in loco quovis *G*, sit $\frac{n+2}{3\sqrt{A^2+\frac{dd}{ee}A^2-\frac{2dnbb}{eA^n}A+\frac{nnb^4}{A^{2n}}}}$, ideoque si in *VZ* capiatur *VY* aequalis $n\times VG$, densitas illa est reciproce ut *XY*. Sunt enim A^2 & $\frac{dd}{ee}A^2-\frac{2dnbb}{eA^n}A+\frac{nnb^4}{A^{2n}}$ ipsarum *XZ* & *ZY* quadrata. Resistentia autem in eodem loco *G* sit ad gravitatem ut 3*S* in $\frac{XY}{A}$ ad $4RR$, id est, ut *XY* ad $\frac{2nn+2n}{n+2}VG$. Et velocitas ibidem ea ipsa est, quacum corpus proiectum in parabola pergeret, verticem *G*, diametrum *GD* & latus rectum $\frac{1+QQ}{R}$ seu $\frac{2XY\ quad.}{\overline{nn+n}\ \text{in}VG}$ habente. *Q.E.l.*

Scholium.



Eadem ratione qua prodiit densitas medii ut $\frac{S\times AC}{R\times HT}$ in corollorio primo, si resistentia ponatur ut velocitatis V dignitas quaelibet V^n prodibit densitas medii ut $\frac{S}{R^{\frac{4-n}{2}}}\times\left(\frac{AC}{HT}\right)^{n-1}$. Et propteria si curva invenire potest ea lege, ut data fuerit ratio $\frac{S}{R^{\frac{4-n}{2}}}$ ad $\left(\frac{HT}{AC}\right)^{n-1}$, vel $\frac{S^2}{R^{4-n}}$ ad $\left(1+QQ\right)^{n-1}$: corpus movebitur in hac curva in uniformi media cum resistentia qua sit ut velocitatis dignitas V^n. Sed redeamus ad curvas simpliciores.

Quoniam motus non sit in parabola nisi in media non resistente, in hyperbolis vera hic descriptis sit per resistentiam perpetuam ; perspicuum est quod linea, quam proiectile in media uniformiter resistente describit, propius accedit ad hyperbolas hasce quam ad parabolam. Est utique linea illa hyperbolici generis, sed quae circa verticem magis descripsi. Tanta vero non est inter has & illam differentia, quin illius loco possint hae in rebus practicis non incommode adhiberi. Et utiliores forsan future sunt hae, quam hyperbola magis accurata & simul magis composita. Ipse vero in usum sic deducentur.



Compleatur parallelogrammum *XYGT*, & recta *GT* tanget hyperbolam in *G*, ideoque densitas medii in *G* est reciproce ut tangens *GT*, & velocitas ibidem ut $\sqrt{\frac{GTq}{GV}}$, resistentia autem ad vim gravitatis ut *GT* ad $\frac{2nn+2n}{n+2}$ in *GV*.

Proinde si corpus de loco *A* secundum rectam *AH* proiectum describat hyperbolam *AGK*, & *AH* producta occurrat asymptoto *SNX* in *H*, actaque *AI* eidem parallela occurrat alteri asymptoto *MX* in *I:* erit medii densitas in *A* reciproce ut *AH*, & corporis velocitas ut $\sqrt{\frac{AHq}{AI}}$, ac resistentia ibidem ad gravitatem ut *AH* ad $\frac{2nn+2n}{n+2}$ in *AI*. Unde prodeunt sequentes regula.

Reg. 1. Si servetur tum medii densitas in *A*, tum velocitas quacum corpus proiicitur, & mutetur angulus *NAH;* manebunt longitudines *AH*, *AI*, *HX*. Ideoque si longitudines illae in aliquo casu inveniantur, hyperbola deinceps ex dato quovis angulo *NAH* expedite determinari potest.

Reg. 2. Si servetur tum angulus *NAB*, tum medii densitas in *A*, & mutetur velocitas quacum corpus proiicitur; servabitur longitudo *AH*, & mutabitur *AI* in duplicata ratione velocitatis reciproce.

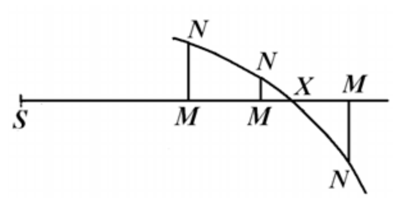
*Reg.*3. Si tam angulus *NAH* , quam corporis velocitas in *A*, gravitasque acceleratrix servetur, & proportio resistentiae in *A* ad gravitatem motricem augeatur in ratione quacunque; augebitur proportio *AH* ad *AI* in eadem ratione, manente parabola praedictae latere recto, eique proportionali longitudine $\frac{AHq}{AI}$ *: &* propterea minuetur *AH* in eadem ratione, *& AI* minuetur in ratione illa duplicate. Augetur vero proportio resistentiae ad pondus, ubi vel gravitas specifica sub aequali magnitudine sit minor, vel medii densitas maior, vel resistentia, ex magnitudine diminuta, diminuitur in minore ratione quam pondus.

Reg. 4. Quoniam densitas medii prope verticem hyperbolae maior estquam in loco *A;* ut habeatur densitas mediocris, debet ratio minimae tangentium *GT* ad tangentem *AH* inveniri, & densitas in *A* augeri in ratione paulo maiore quam semisummae harum tangentium ad minimam tangentium *GT*.

*Reg:*5. Si dantur Longitudines *AH*, *AI*, & describenda sit figura *AGK* : produc *HN* ad *X*, ut sit *HX* ad *AI* ut $n+1$ad 1, centroque *X* & asymptotis *MX*, *NX* per punctum *A* describatur hyperbola, ea lege, ut sit *AI* ad quamvis *VG* ut XV^n ad XI^n .

Reg. 6. Quo maior est numerus *n*, eo magis accuratae sunt hae hyperbolae in ascensu corporis ab *A*, & minus accuratae in eius descensu ad *K;* & contra. Hyperbola conica mediocrem rationem tenet, estque caeteris simplicior. Igitur si hyperbola sit huius generis, & punctum *K*, ubi corpus proiectum incidet in rectam quamvis *AN* per punctum *A* transeuntem, quaeratur : occurrat producta *AN* asymptotis *MX*, *NX* in *M* & *N*, & sumatur *NK* ipsi *AM* aequalis.

Reg. 7. Et hinc liquet methodus expedita determinandi hanc hyperbolam ex phaenomenis. Proiiciantur corpora duo similia & aequalia, eadem velocitate, in angulis diversis *HAK*, *hAk*, incidantque in planum horizontis in *K* & *k;* & notetur proportio *AK* ad *Ak*. Sit ea *d* ad *e*. Tum erecto cuiusvis longitudinis perpendiculo *AI*, assume utcunque longitndinem *AH* vel *Ah*, & inde collige graphice longitudines *AK*, *Ak*, per reg. 6. Si ratio *AK* ad *Ak* sit eadem cum ratione *d* ad *e*, longitudo *AH* recte assumpta fuit. Sin minus cape in recta infinita *SM* longitudinem *SM* aequalem assumptae *AH*, & erige perpendiculum *MN* aequale rationum differentae $\frac{AK}{Ak}-\frac{d}{e}$ ductae in rectam quamvis datam. Simili methodo ex assumptis pluribus longitudinibus *AB* invenienda sunt plura puncta *N*, & per omnia agenda curva linea regularis *NNXN*, secans rectam *SMMM* in *X*. Assumatur demum *AH* aequalis abscissae *SX*, & inde denuo inveniatur longitudo *AK;* & longitudines, que sint ad assumptam longitudinem *AI* & hanc ultimam *AH*, ut longitudo *AK* per experimentum cognita ad ultimo inventam longitudinem *AK*, erunt verae illae longitudines *AI* & *AH*, quas invenire oportuit, Hisce vero datis dabitur & resistentia medii in loco *A*, quippe quae sit ad vim gravitatis ut *AH* ad *2AI*. Augenda est autem densitas medii per reg. 4. & resistentia modo inventa, si in eadem ratione augeatur, fiet accuratior.



Reg. 8. Inventis longitudinibus *AH*, *HX;* si iam desideretur positio rectae *AH*, secundum quam proiectile, data illa cum velocitate emissum, incidit in punctum quodvis *K:* ad puncta *A* & *K* erigantur rectae *AC*, *KF* horizonti perpendiculares, quarum *AC* deorsum tendat, & requetur ipsi *AI* seu $\frac{1}{2}HX$. Asymptotis *AK*, *KF* describatur hyperbola, cuius coniugata transeat per punctum *C*, centroque *A* & intervallo *AH* describatur circulus secans hyperbolam illam in puncto *H;* & proiectile secundum rectam *AH*emissum incidet in punctum *K*. *Q. E. I.* Nam punctum *H*, ob datam longitudinem *AH*, locatur alicubi in circulo descripto. Agatur *CH* occurrens ipsis *AK* & *KF*, illi in *E*, huic in *F;* & ob parallelas *CH*, *MX* & aequales *AC*, *AI*, erit *AE* aqualis *AM*, & propterea etiam aequaelis *KN*. Sed *CE* est ad *AE* ut *FH* ad *KN*, & propterea *CE* & *FH* aequantur. Incidit ergo punctum *H* in hyperbolam asymptotis *AK*, *KF* descriptam, cuius coniugata transit per punctum *C*, atque ideo reperitur in communi intersectione hyperbola huius & circuli descripti. *Q. E. D.*

Notandum est autem quod haec operatio perinde se habet, sive recta *AKN* horizonti parallels sit, sive ad horizontem in angulo quovis inclinata: quodque ex duabus intersectionibus *H*, *h* duo prodeunt anguli *NAH*, *NAH;* & quod in praxi mechanica sufficit circulum semel describere, deinde regulam interminatam *CH* ita applicare ad punctum *C*, ut eius pars *FH*, circulo & rectae *FK* interiecta, aequalis sit eius parti C*E* inter punctum *C* & rectam *AK* sitae.

Quae de hyperbolis dicta sunt facile applicantur ad parabolas. Nam si *XAGK* parabolam designet quam recta *XV* tangat in vertice *X*, sintque ordinatim applicatae *IA*, *VG* ut quaelibet abscissarum *XI*, *XV* dignitates XI^n, XV^n; agantur *XT*, *GT*, *AH*, quarum *XT* parallela sit *VG*, & *GT*, *AH* parabolam tangant in *G* & *A*: & corpus de loco quovis *A*, secundum rectam *AH* productam, iusta cum velocitate proiectum, describet hanc parabolam, si modo densitas medii, in locis singulis *G*, sit reciproce ut tangens *GT*. Velocitas autem in *G* ea erit quacum proiectile pergeret, in spatio non resistente, in parabola conica verticem *G*, diametrum *VG* deorsum productam, & latus rectum $\frac{2GTq}{\overline{nn-n}\times VG}$ habente. Et resistentia in *G* erit ad vim gravitatis ut *GT* ad $\frac{2nn-2n}{n-2}VG$. Unde si *NAK* lineam horizontalem designet, & manente tum densitate medii in *A*, tum velocitate quacum corpus proiicitur, mutetur utcunque angulus *NAH*; manebunt longitudines *AH*, *AI*, *HX*, & inde datur parabolas vertex *X*, & positio rectae *XI*, & sumendo *VG* ad *AI* ut XV^n ad XI^n, dantur omnia parabolae puncta *G*, per quae proiectile transibit.