CONCERNING THE MOTION OF BODIES

BOOK TWO.

SECTION I.

Concerning the motion of bodies being resisted in the ratio of the velocities.

PROPOSITION I. THEOREM I.

The motion [i.e. velocity] of a body removed by resistance, to which it is resisted in the ratio of the velocity, is as the distance completed in moving.

For since the loss of motion in equal small intervals of time shall be as the velocity, that is, as the small parts of the journey completed: the motion lost in the whole time shall be as the whole journey. *Q. E. D.*

[Thus, in each increment of time Δt for a given velocity *v*, we may write $\frac{\Delta v}{\Delta t} = -kv = -k\frac{\Delta x}{\Delta t}$; hence $\Delta v = -k\Delta x$ thus on adding the losses over the whole journey, the whole loss in velocity is proportional to the whole distance gone.]

Corol. Whereby if a body, freed from all weight, may be moving in free spaces by the action of its innate force only; and while the whole motion may be given at the start, as well as the remaining motion also, after the completion of some distances: the whole of the distance is given that the body is able to describe in an infinite time. For that distance will be to the distance now described, as the whole motion from the start, to that part of the motion lost.

[Expressing this idea in modern terms, if the rate of decrease of the velocity in the time *dt* is proportional to the velocity, then we have for a body of unit mass, $a(t) = \frac{dv(t)}{dt} = -kv = -k\frac{dx}{dt}$, giving $v(t) = v_0 e^{-kt}$; $x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)$. Thus, the whole journey is given by $x(\infty) = \frac{v_0}{k}$, while $\frac{x(\infty)}{x(t)} = \frac{1}{1-e^{-kt}}$; again, $\frac{v_0}{v_0\left(1-e^{-kt}\right)}$ is the whole motion at the start to the motion lost, which is in the same ratio, as asserted. Newton makes use of this simple idea in the following propositions. The luxury of the exponential function however, was not available to Newton; results had to be found without this convenience; use was made of the area under the rectangular hyperbola to measure the difference in times, as we shall see. Thus, we may write

$$a(t) = -kv_0e^{-kt} = -a_0e^{-kt}\ ;\ v(t) = v_0e^{-kt}\ ;\ kx(t) = v_0 - v(t)\ ;\ ln\left(\tfrac{v_o}{v_t}\right) = kt = k\int \tfrac{du}{u}\ ]$$

LEMMA I.

Proportional quantities formed from their differences are continued proportionals.

A shall be to $A-B$ as B to $B-C$ and as C to $C-D$, &c., and by being converted there becomes A to B as B to C and C to D, &c. *Q. E. D.*

[For if $\frac{A}{A-B} = \frac{B}{B-C} = \frac{C}{C-D} = \cdots,\text{etc.}$; then $\frac{A}{B-A} + 1 = \frac{B}{C-B} + 1 = \frac{C}{D-C} + 1 = \cdots,\text{etc.}$, giving $\frac{B}{B-A} = \frac{C}{C-B} = \frac{D}{D-C} = \cdots,\text{etc.}$ then $\frac{A}{B} = \frac{A-B}{B-C} = \frac{B}{C} = \frac{B-C}{C-D} = \frac{C}{D} = \frac{C-D}{D-E}$..... .; from which the result follows.]

PROPOSITION II. THEOREM II.

If, for a body resisted in the ratio of the velocity, and moving through a uniform medium by its innate force alone, and moreover equal time intervals may be taken, then the velocities at the beginnings of the individual time intervals are in a geometric progression, and the distances described in the individual time intervals are as the velocities.

Case. I. The time may be divided up into small equal parts ; and if the force of resistance may act at the beginning of the increments of time by single impulses, which shall be as the velocity: the decrease of the velocity in the individual increments of time will be as the same velocities. Therefore the velocities from these differences are proportionals, and on this account continued proportionals, (by Lem. I. Book. II.).

[For in $\frac{A}{A-B} = \frac{B}{B-C} = \frac{C}{C-D} = \cdots,\text{etc.}$, if $v_A,\ v_B,\ v_C$, etc. , are the velocities at the start of successive constant increments in time, then $v_A - v_B = kv_A,\ v_B - v_C = kv_B$, etc. ; and $\frac{v_A}{v_B} = \frac{v_A - v_B}{v_B - v_C} = \frac{v_B}{v_C} = \frac{v_B - v_C}{v_C - v_D} = \frac{v_C}{v_D} = \frac{v_C - v_D}{v_D - v_E}\ldots.. = \frac{1}{1-k} = \frac{1}{r}$, as the ratio of successive velocities is a constant r or $\frac{1}{1-k}$, by hypothesis . Thus we may write $v_B = rv_A,\ v_C = rv_B = r^2v_A,\ v_D = r^3v_A$, etc. in an inductive manner, as we have shown above, forming a geometric progression. In the limit, these ratios form a continuous logarithmic or exponential curve.]

Hence, if from a number of equal small time intervals, equal [larger] time intervals may be put in place, the velocities from the starts of these times will be as the terms in a continued [geometric] progression, which are taken in jumps, with an equal number of intermediate terms omitted everywhere. But the ratios have been composed from these terms, by ratios with the same repeated equality of the intermediate terms between themselves, and therefore on that account these too are ratios composed between themselves, and the velocities proportional to these terms are in a geometric progression also; [i.e. the larger steps in time are in arithmetic proportion, and the corresponding steps in the velocity are still in geometric proportion]. Now these equal small time intervals may be diminished, and the number of these increased indefinitely, so that from that the impulse of the resistance may be rendered continuous, and the velocities from the beginnings of the equal times, always continued proportionals, in this case will be continued proportionals. *Q. E. D.*

[Shades of Napier's algorithm for generating logarithms?]

Case 2. And the differences of the velocities, that is, the parts of these removed by the individual time intervals, are in proportion to the whole : but the distance described in the individual times, are as the parts of the velocity removed (by Prop. I, Book. II.), and on that account also are as the total distance. *Q.V.D.*

[As we have seen, in Eulerian or modern terms, $v(t) = v_0 e^{-kt}$; $x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)$; thus, when the time is advanced by dt , the (negative) change in the velocity is proportional to the velocity

$$dv(t) = -kv_0 e^{-k(t)} dt \propto v(t);$$

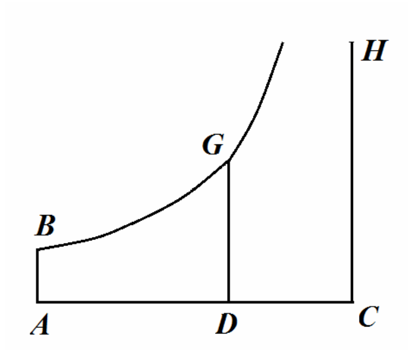
and the increment of distance

$$dx(t) = v_0 e^{-k(t)} dt \propto dv(t) \propto v(t);$$

i.e. proportional to the velocity lost or distance gained. These ideas have been expressed in terms of ratios of succeeding increments by Newton, either singly or in groups with the same number in each. In addition, we may note that as the resistance is proportional to the velocity, it also is exponential in nature, and we might denote it by $R(t) = R_0 e^{-kt}$. It is important to note here that the same exponential form of the decay occurs in the forced motion when gravity is present and the body is rising, although the multiplying constant is different.]

Corol. Hence a hyperbola *BG* may be described with the rectangular asymptotes *AC* and C*H*, and *AB*, *DG* shall be perpendiculars to the asymptote *AC*, and then the velocity of the body for a resisting medium, with some initial motion [velocity], may be shown by a given [*i.e.* constant] line *AC*, and with some elapse in time the velocity is now given by the indefinite [*i.e.* variable] line *DC:* it is possible to express the time by that area *ABGD,* and the distance described by the [velocity] line *AD* in that time. For if that area may be increased by the motion of the point *D* uniformly in the manner of the time, then the right line *D*C will decrease in a geometric ratio in the manner of the velocity, and the parts of the right line described *AC* will decrease in the same ratio in equal times.

[The hyperbola is a device used in the calculation in conjunction with the logarithmic scale on the ordinate axis to ascertain the like times involved when the velocity and distance traversed change from one value to another via a series of continued proportionals, arising from the nature of the exponential or logarithmic curve. The correspondence between the natural logarithm and the area of a section of the rectangular hyperbola has been demonstrated above, in the final theorem of Leseur & Janquier, originating from the work of Gregorius, as presented by de Sarasa : (see the paper by R.P. Burn in *Historia Mathematica* Vol.28, (2001), pp.1-17). Thus, the difference of the starting and finishing times are associated with the logarithm of the ratio of the



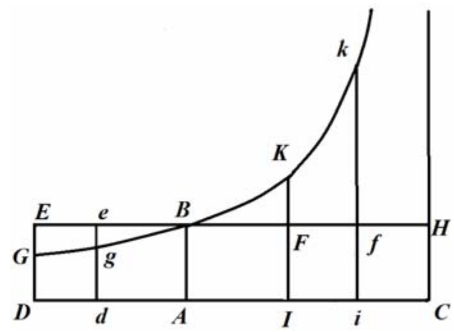
corresponding velocities (or of the abscissas of the logarithmic curve) at these times, and this logarithm is in turn equal to the area under the hyperbola; for if the speed of the point at *A* is v_0, and at some later time *t* along the line *CD* the speed is $v(t) = v_0 e^{-kt}$; the change in distance gone and acceleration or force are given by similar formulas involving logarithm ratios, then e.g. the area under the corresponding hyperbolic section *BADG* $= \int_{u=v_0}^{(u=v_0 e^{-kt})} \frac{du}{u} \propto$ time. See letter no. 297, *Halley to Wallis*, and subsequent notes in *The Correspondence of Isaac Newton*, 1676-1687, edited by H.W.Turnbull; Vol. II, p.456-p.462, CUP 1960; see also :*The Correspondence and Papers of Edmond Halley*, E.F.MacPike, Taylor & Francis, 1930.]

PROPOSITION III. PROBLEM I.

To define the motion of a body, for which, while it may ascend or descend in right lines through a similar medium, it is resisted in the ratio of the velocity, and that body is acted on by uniform gravity also.

With the body ascending, gravity may be put in place by some rectangle *BACH*, and the resistance of the medium at the beginning of the ascent by the rectangle *BADE*, taken on the opposite sides of the line *AB*. With rectilinear asymptotes *AC* and *CH*, a hyperbola may be described through the point *B*, cutting the perpendiculars *DE* and *de* in *G* and *g;* and by ascending in the time *DGgd* , the body will describe the distance *EGge*; and in the whole time of the ascent *DGBA*, the whole distance of the ascent *EGB* ;

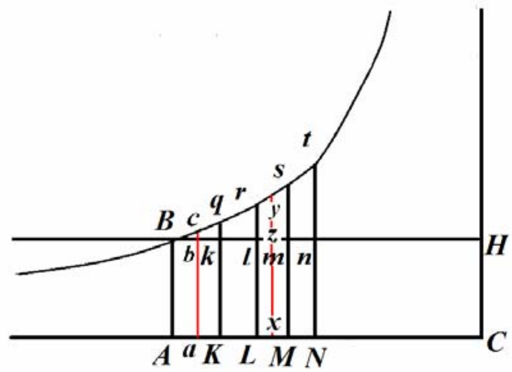
[Essentially, the whole motion of the ascent with an initial velocity v_0, and continued descent afterwards, is evaluated by making use of this diagram, after divinding the line under the hyperbola geometrically, the areas corrspond to equal times, and the forces, velocity changes, and distances gone are related by successively multipling these equal areas; for example, the force of gravity at the start of the latter downwards motion is represented by the line *AC*, which force subsequently diminishes geometrically or exponentially due to the resistance increasing ; the total force at the start of the motion upwards is shown by the line *DA*, which increases with the resistance decreasing with the velocity to the maximum value at *A*; the actual trajectory is not shown on this diagram (the problems discussed here by Newton are set out below in the L & J treatment, the trajectory of a projectile under this law of resistance is shown below in Note 56, and thereafter). During the ascent, the velocities are set out to decrease in a geometrical progression along the line *DA*, giving equal areas under the hyperbola in arithmetic progression, which we have seen correspond to equal time intervals; during each equal increment of time, the velocity lost, in a constant ratio to the previous increment, and thus as a fraction of the starting value *AD*, is



transformed into an equal increment in distance gone, so that the section *BEG* above the curve represents the whole distance gone upwards and the remaining area *ADGB* represents the loss in velocity. Newton's picture is hence a moving one, and we are given here only part of a graphical account of the solution of the associated differential equation.

If we consider the hyperbola to be simply $xy = g$, with origin *C*, and the positive ordinate axis *CD* acting to the right in a diagram (not shown) reflected in the vertical axis *CH*, so that we are dealing with a more conventional situation, where *g* is the acceleration of gravity for unit mass, then *B* is the point $(g,1)$. Furthermore conventionally, the acceleration at some point *D* is given by $a = \frac{dv}{dt} = -g - kv$ or $dt = \frac{-dv}{g+kv} = \frac{-1}{k}\frac{kdv}{g+kv}$, where $GD = \frac{g}{g+kv_0} = \frac{1}{1+\frac{kv_0}{g}}$, and where at once we have the time to reach *d*,

$t_{Dd} = \frac{1}{k} ln \frac{g+kv_0}{g+kv} = \frac{1}{k} ln \frac{1+\frac{kv_0}{g}}{1+\frac{kv}{g}}$ and $t_{DA} = \frac{1}{k} ln\left(1+\frac{kv_0}{g}\right)$. In additon, the time to travel the distance *AI* is given by $t_{AI} = \frac{1}{k} ln \frac{g}{g-kv}$, and the terminal velocity is given by $v_\infty = \frac{g}{k}$. Now, the time is specified to be increasing in constant increments Δt, and we need to find the velocities at these times; clearly the decrements $g\Delta t$ give the successive decreases from the initial velocity v_0 due to the retardation of gravity $v_0 - gt$, in addition, there is the retarding resistive force initially, kv_0. Now, $GD = \frac{g}{g+kv_0}$, $v_d = \frac{1}{k}\left(g + kv_0\right)e^{-kt_{Dd}} - \frac{g}{k}$, the change in distance in the next small time increment is
$\Delta x = v_d \Delta t = \frac{\Delta t}{k}\left(g + kv_0\right)e^{-kt_{Dd}} - \frac{g\Delta t}{k}$; with the signs reversed, we see that the second term in this formula corresponds to a constant change is distance, while the first term corresponds to a constant multiple of the speed decay resulting from the resistive force by Δt, and thus the areas are as indicated, and as set out below in a laborious manner below by *L* & *J*. The argument of Newton below applies to the other branch of the problem. One notes of course that this is a graphical method, and the actual curve is not plotted.]



In the time of the descent *ABKI* the body falls the distance *BFK*, and in the time *IKki* the distance fallen is *KFfk;* and the velocities of the body (proportional to the resistance of the medium) in the periods of time of these will be *ABED*, *ABed*, nothing, *ABFI*, *ABfi* respectively; and the maximum velocity, that the body can acquire on falling will be *BACH*.

The rectangle *BACH* may be resolved into innumerable [equal] rectangles *Ak, Kl, Lm, Mn*, etc. which shall be as equal increments in the velocity made in the same equal increments of time; and so that zero, *Ak*, *Al, Am*, *An*, etc. shall be the whole speeds [accomplished by multiplying these constant time increments by the constant acceleration of gravity], and thus (by hypothesis [on multiplying these speeds by *k*,]) as the resistances of the medium from the beginning of the individual equal time intervals. Thus, *AC* becomes to *AK* or *ABHC* to *ABkK* as the force of gravity to the resistance at the beginning of the second time, and thence the resistances are taken away from the force of gravity, and *ABHC*, *KkHC*, *LlHC*, *MmHC*, etc. will remain as the absolute forces by which the body is acted on at the beginning of the individual time intervals, and thus (by the second law of motion) as the increments of the velocities, that is, as the rectangles *Ak*, *Kl*, *Lm*, *Mn*, &c., and on that account (by Lem. I. Book II.) in a geometric progression.
Whereby if the right lines *Kk*, *Ll*, *Mm*, *Nn*, etc. produced meet the hyperbola in *q,r,s,t*, etc. the areas *ABqK*, *KqrL*, *LrsM*, *MstN*, etc. will be equal, and thus then always in the same ratio both with the times as well as the force of gravity. But the area *ABqK* (by Corol. 3. Lem. VII. & Lem. VIII. Book1.) is to the area *Bkq* as *Kq* to $\frac{1}{2}kq$ or *AC* to $\frac{1}{2}AK$ that is, in proportion to the force of gravity to the resistance in the middle of the first time increment. And by a similar argument the areas *qKLr*, *rLMs*, *sMNt*, etc. are to the areas *qklr*, *rlms; smnt*, etc. as the forces of gravity to the resistances in the middle of the second, third, fourth, time increments, etc. Hence since the equal areas *BAKq*, *qKLr*, *rLMs*, *sMNt*, &c. shall be in the same ratio to the forces of gravity, the areas *Bkq*, *qktr*, *rlms*, *smnt*, etc. shall be to the resistances in the middle of the individual times, that is (per hypothesis) with the velocities, and thus analogous to the distances described.
The sum of the analogous quantities may be taken, and the areas will be in the same ratio as the total distances described *Bkq*, *Blr*, *Bms*, *Bnt*, etc.; and also the areas *ABqK*, *ABrL*, *ABsM*, *ABtN*, etc. to the times. Therefore the body, while descending, in some time *ABrL*, will describe the distance *Blr*, and in the time *LrtN* the distance *rlnt*.
Q. E. D.
And the demonstration of the motion in the ascent is set out in a similar manner. *Q. E. D.*

Corol. 1. Therefore the maximum velocity, that the falling body is able to acquire, is to the given velocity acquired by falling in some time, as the given force of gravity, by which that body is continually urged on, to the force of resistance, from which it may be impeded up to the end of that time.

[From $v = \frac{g}{k}\left(1 - e^{-kt}\right)$, the maximum velocity becomes $v_{max} = \frac{g}{k}$, and hence $\frac{v_{max}}{v(t)} = \frac{1}{\left(1-e^{-kt}\right)} = \frac{g}{kv(t)}$.]

Corol. 2. But with the time increasing in an arithmetic progression, the sum of that maximum velocity and of the velocity in ascending, and also of the difference of these in the descent, is described by a geometric progression.

[For $v = \frac{g}{k}\left(\left(1+\frac{kv_0}{g}\right)e^{-kt} - 1\right) + \frac{g}{k} = \frac{g}{k}\left(1+\frac{kv_0}{g}\right)e^{-kt}$ etc.]

Corol. 3. But also the difference of the distances, which are described in equal difference of the time, decrease in the same geometric progression.

Corol. 4. Truly the distance described by the body is the difference of two distances, of which the one is as the time taken from the beginning of the descent, and the other as the velocity, which distances also at the beginning of the descent itself are equal to each other.

[For $x(t) = \frac{g}{k}\left(t + \frac{e^{-kt}}{k} - \frac{1}{k}\right) = \frac{g}{k}\left(t - \frac{v}{g}\right) = \frac{gt-v}{k}$.]

[Notes by the translator: This is as in the previous corollary with the motion starting with some value v_0, except that a constant downwards force is added, in addition to the resistance decelerating the body to rest at *AB*, so that the velocity increments in the region *EDAB* become narrower geometrically, for constant time increments, and then afterwards accelerating the body, while the resistive force always opposes the motion in the downwards motion. A reference line *AB* is taken on the hyperbola representing the point where the body is momentarily at rest, the areas between the vertical lines mark out equal time intervals in the motion, while the starting point is *DE* on the axis *DC*; we may consider the asymptotes *CD* and *CH* as representing the axes of the dimensionless variables *x* and $y = \frac{1}{x}$. As previously, the ascent time is found by multiplying the area under the hyperbola by a scaling constant; Thus, the time to rise to the maximum height, given by the area under the hyperbola *BADE*, is less than the time to fall, given by the area *kiAB*; both times are different from the time of the journey without the constant force.

Again we may express this proposition in modern terms, if the rate of decrease of the velocity in the time *dt* is proportional to the velocity, then we have for a body of unit mass ascending, say under constant acceleration of gravity *g* acting downwards, with an initial velocity upwards v_0 : $\frac{dv(t)}{dt} = -g - kv = -g - k\frac{dx}{dt}$, thus, the first equation gives $\frac{dv(t)}{g+kv} = -dt$, $\rightarrow$ $\frac{1}{k} ln\left(\frac{g+kv}{g+kv_0}\right) = -t$; or $\frac{g+kv}{g+kv_0} = e^{-kt}$. In this case, the dimensionless variable

$z = \frac{g+kv}{g+kv_0}$ is used to find the time by integrating under the hyperbola. It follows from $\frac{dv(t)}{dt} = -g - k\frac{dx}{dt}$ that $dv(t) = -gdt - kdx$, giving $v = v_0 - gt - kx$, or $x = \frac{(v_0 - v)}{k} - \frac{g}{k}t$.
In the first part of the motion, the speed decreases as the body rises, where the time is given by $-\frac{1}{k} ln u = -\frac{1}{k}\int_1^u \frac{dz}{z} = t$, and the speed is zero when $\frac{1}{k} ln\left(1 + \frac{k}{g}v_0\right) = t_A$, where the dimensionless variable z or its upper limit u is less than one. The distance ascended in the time t_A is $x = \frac{v_0 - \frac{g}{k} ln\left(1+\frac{k}{g}v_0\right)}{k} = \frac{v_0}{k} - \frac{g}{k^2} ln\left(\frac{g+kv_0}{g}\right) = \frac{1}{k}\left(v_0 - gt_A\right) = BADE - BGDA = BEG$.
The first term *BADE* represents the distance gone with no forces acting, while the second term is the distance removed by 'falling' under gravity in the same time. Similar reasoning can be applied for the downwards motion.

For the second downward part of the motion, note that we have established above, for the rising motion, that

$\frac{1}{k} ln\left(\frac{g+kv_0}{g+kv}\right) = t$; while subsequently, $\frac{dv}{dt} = g - kv$, or $-\frac{d(g-kv)}{kdt} = g - kv$

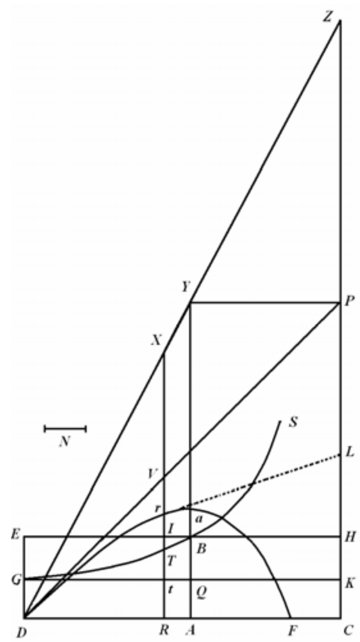
giving $ln(g - kv) = -kt$,

for the falling motion ; hence $g - kv = ge^{-kt}$ or $v = \frac{g}{k}\left(1 - e^{-kt}\right)$. The distance gone in time t is hence $x(t) = \frac{g}{k}\left(t - \frac{1}{k} + \frac{e^{-kt}}{k}\right) = \frac{g}{k}\left(t - \frac{v}{g}\right) = \frac{gt-v}{k}$. Newton's ratio, from $\frac{dv}{dt} = g - kv$ gives $\Delta v = g\Delta t - kv\Delta t$ then becomes $g\Delta t$ to $-k\Delta x$: or $\frac{g}{kv}$, or the ratio of the gravitational force to the resistive force, and the motions lost become equal to the distances gone.]

PROPOSITION IV. PROBLEM II.

Because a uniform force of gravity may be put in some similar medium, and it may act perpendicularly to the horizontal ; to define the motion of a projectile in the same, with the resistance experienced proportional to the velocity.

The projectile may be sent off from some place *D* following some right line *DP*, and the velocity of the same may be set out by the length *DP* at the start of the motion. The perpendicular *PC* may be sent from the point *P* to the horizontal line *DC*, and *DC* may be cut at *A*, [for the greatest height reached in the trajectory] so that *DA* shall be to *AC* as the resistance of the medium, arising from the initial vertical motion, to the force of gravity; or (likewise) as the rectangle under *DA* and *DP* shall be to the rectangle under *AC* and *CP* as the whole resistance from the start of the motion to the force of gravity.



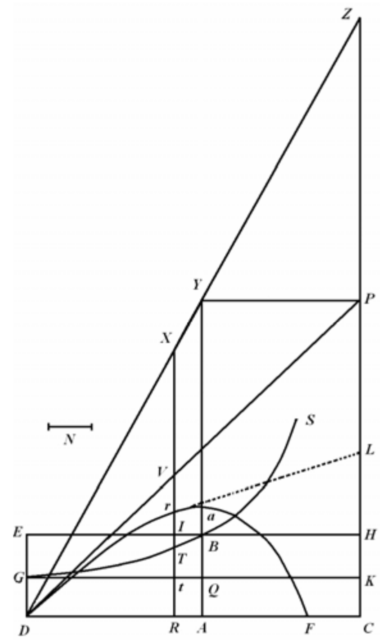
[*i.e.* $\frac{DA}{AC}=\frac{kv_0(up)}{g}$ or as $\frac{DA\times DP}{AC\times CP}=\frac{kv_0}{g}$, since $kv_0\left(up\right)=v_0\times\frac{PC}{DP}$; this is a small modification of Cor. I, Prop. III.]

Some hyperbola *GTBS* is described with asymptotes *DC* and *CP*, cutting the perpendiculars *DG* and *AB* in *G* and *B;* and the parallelogram *DGKC* may be completed, the side of which *GK* may cut *AB* in *Q*; The line of length *N* may be taken in the ratio to *QB* in which *DC* shall be to *CP*; and at some point *R* of the line *DC* erect the perpendicular *RT*, that may meet the hyperbola in *T*, and the right lines *EH*, *GK*, *DP* in *I, t* and *V* , and on that line take *Vr* equal to $\frac{tGT}{N}$, or what is the same thing, take *Rr* equal to $\frac{GTIE}{N}$;

[*i.e*. since we have an obliquity factor $\frac{N}{QB}=\frac{DC}{CP}=\frac{\cos\alpha}{\sin\alpha}$, where α is the initial angle of projection to the horizontal, and $\frac{DC}{CP}=\frac{DR}{RV}$, on account of the similar triangles *DRV* and *DCP*; there will be $\frac{N}{QB}=\frac{DR}{RV}$, and thus $RV=\frac{DR\times QB}{N}$. But the rectangle $GEIt=Gt\times GE=DR\times QB=GTIE+tGT$, and thus $\frac{GTIE}{N}=\frac{DR\times QB-tGT}{N}=RV-\frac{tGT}{N}$. Whereby if there is taken $Vr=\frac{tGT}{N}$, there will be $\frac{GTIE}{N}=RV-Vr=Rr$.],

and for the projectile to arrive at the point *r* in the time *DRTG*, describing the curved line *DraF*, that it may always be a tangent at the point *r*, moreover arriving at the maximum height *a* on the perpendicular *AB*, and afterwards always approaching towards the asymptote *PC.* And the velocity of this projectile at some point *r* is as the tangent of the curve *rL. Q. E. I.* [See the L. & J. notes in the other file for a complete explanation.]

For indeed *N* is to *QB* as *DC* to *CP* or *DR* to *RV*, and thus *RV* equals $\frac{DR\times QB}{N}$, [*i.e.* $\frac{N}{QB}=\frac{DC}{CP}=\frac{DR}{RV}:RV=\frac{DR\times QB}{N}$;] then *Rr* (that is $RV-Vr$ or $\frac{DR\times QB-tGT}{N}$) equals $\frac{DR\times AB-RDGT}{N}$. Now the time may be put in place by the area *RDGT*, and (by Corol. 2. of the laws) the motion may be separated into two parts, the one vertical, and the other horizontal. And since the resistance shall be as the motion, this also may be separated into two parts with parts proportional to and opposing the motion: and thus the length described by the motion horizontally, will be (by Prop. II. of this) as the line *DR*, truly with the height (by Prop. III. of this Book) as the area $DR\times AB-RDGT$, that is, as the line *Rr*. But the motion in the very beginning itself , the area *RDGT*, is equal to the rectangle $DR\times AQ$ and thus that line *Rr* (or $\frac{DR\times AB-DR\times AQ}{N}$ then will be to *DR* as $AB-AQ$ or *QB* to *N*, that is, as *CP* to *DC* ; and thus as the vertical motion to the longitudinal motion at the beginning. Therefore since *Rr* shall always be as the height, and *DR* always as the length, and *Rr* to *DR* at the start as the height to the length horizontally: it is necessary that *Rr* shall be always to *DR* as the height to the horizontal distance, and therefore so that the body may be moving on the line *DraF*, as it may always be a tangent at the point *r*. *Q. E. D.*



[Translator's note : See letter no. 297, *Halley to Wallis*, and subsequent notes in *The Correspondence of Isaac Newton*, 1676-1687, edited by H.W.Turnbull; Vol. II, p.456-p.462, CUP 1960. We give now the standard analytical solution. The initial velocity may be separated into its horizontal and vertical components $v_0 \cos\alpha$ and $v_0 \sin\alpha$, where v_0 is the initial velocity, and α is the angle to the horizontal. The motions are described by the two equations from the second law for the horizontal and vertical motions :

$\frac{dv_x}{dt}=-kv_x$ and $\frac{dv_y}{dt}=-kv_y-g$.

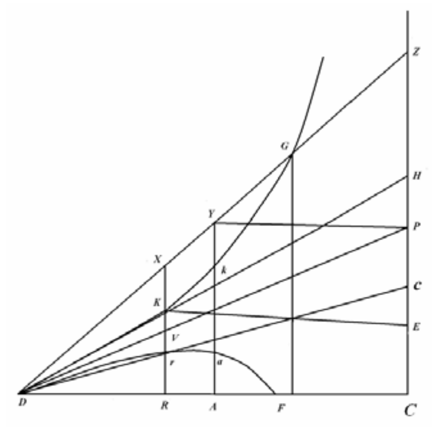
The solutions to both these have been found already ; for the horizontal motion throughout we have the equation for the speed :

$v_x=v_0\cos\alpha e^{-kt}$ and for the vertical up: $\frac{g+kv_y}{g+kv_0\sin\alpha}=e^{-kt}$; while for the vertical down we have $v_y=\frac{g}{k}\left(1-e^{-kt}\right)$. The maximum height is reached when $t=\frac{1}{k}ln\left(1+\frac{kv_0}{g}\sin\alpha\right)$, and the height risen is found from $\frac{dy}{dt}=\left(\frac{g}{k}+v_0\sin\alpha\right)e^{-kt}-\frac{g}{k}$, giving

$y=\frac{1}{k}\left(\frac{g}{k}+v_0\sin\alpha\right)\left(1-e^{-kt}\right)-\frac{gt}{k}$; in which case the maximum height is

$y=\frac{1}{k}v_0\sin\alpha-\frac{g}{k^2}ln\left(1+\frac{kv_0}{g}\sin\alpha\right)$, and the corresponding horizontal distances are given by $x=\frac{1}{k}v_0 \cos\alpha\left(1-e^{-kt}\right)$. Thus the time to reach a certain horizontal distance x is given by $t=\frac{1}{k}\ln\left(\frac{1}{v_0\cos\alpha-kx}\right)$ The time of flight occurs when the downwards distance $y=\frac{1}{k}\left(\frac{g}{k}+v_0\sin\alpha\right)\left(1-e^{-kt}\right)-\frac{gt}{k}=0$; this can only be solved approximately, as likewise the range.]

Corol. 1. Therefore *Rr* equals $\frac{DR\times AB}{N}-\frac{RDGT}{N}$: and thus if *RT* may be produced to *X* so that *RX* shall be equal to *DR* to $\frac{DR\times AB}{N}$; that is, if the parallelogram *ACPY* may be completed and *DY* joined cutting C*P* at *Z,* and *RT* may be produced then it may meet *DT* in *X; Xr* will be equal to $\frac{RDGT}{N}$ and therefore proportional to the time.

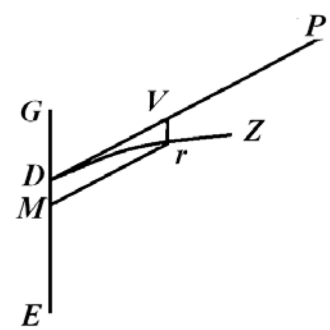


[L& J: Note 57. For with $RX=\frac{DR\times AB}{N}$, *RX* will be to *DR* as *AB* given to *N* given, and thus the positions of the points *X* for the right line which passes through the point *D*, with *DR* vanishing everywhere, *RX* also vanishes. With the point *R* coinciding with *A*, there becomes *RX* or $\frac{AY}{DA}=\frac{AB}{N}$, and by the properties of the hyperbola as previously, $\frac{DC}{AC}=\frac{AB}{GD \text{ or } AQ}$; and separating, $\frac{DC}{DA}=\frac{AB}{BQ}$, truly by construction, $\frac{CP}{DC}=\frac{BQ}{N}$, and thus by equality, $\frac{CP}{DA}=\frac{AB}{N}=\frac{AY}{DA}$ and hence $AY=CP$. Thence if the parallelogram *ACPY* may be completed, and *DY* may be joined cutting *CP* in *Z*, *DZ* will be a right line that the point *X* always touches. Therefore since $RX=\frac{DR\times AB}{N}$, and $Xr=RX-Rr=\frac{DR\times AB}{N}-\frac{DR\times AB+RDGT}{N}$; $Xr=\frac{RDGT}{N}$, and therefore, on account of *N* given, *Xr* is as the area *RDGT*, and thus as the time in which the body arrives at the place *r* from the place *D*.]

Corol. 2. From which if some innumerable lines C*R* may be taken, or, which is likewise, innumerable lines *ZX* may be taken in a geometric progression ; there will be just as many *Xr* in an arithmetic progression. And hence the curve *DraF* may be easily delineated by a table of logarithms.

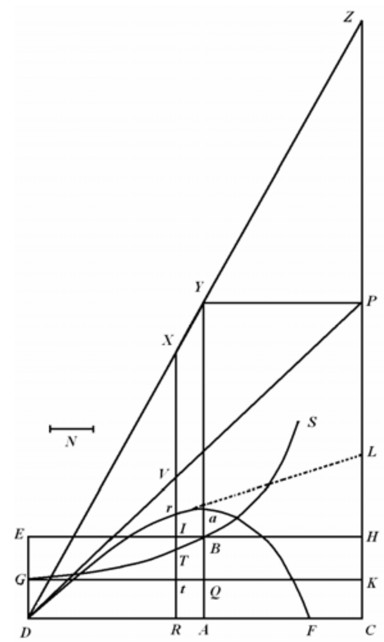
Corol. 3. If from the vertex D, with the diameter DG produced downwards, and with the latus rectum which shall be to $2DP$ as the whole resistance from the start of the motion, to the force of gravity, a parabola may be constructed : the velocity with which the body may depart from the position D along the right line DP, as in a medium with uniform resistance may describe the curve $DraF$, that itself will be with some body to depart from the same place D, along the same right line DP, as in a space with no resistance may describe a parabola. For the latus rectum of this parabola, with the same initial motion, is $\frac{DV^2}{Vr}$; and Vr is $\frac{tGT}{N}$ or $\frac{DR\times Tt}{2N}$. But the right line which, if it be drawn, may touch the hyperbola GTS at G, is parallel to DK itself, and thus Tt is $\frac{CK\times DR}{DC}$, & N is $\frac{QB\times DC}{CP}$. And therefore Vr is $\frac{DRq\times CK\times CP}{2DCq\times QR}$, that is (on account of the proportionals DR and DC, DV and DP) $\frac{DVq\times CK\times CP}{2DPq\times QB}$ and the latus rectum $\frac{DVquad.}{Vr}$ produced $\frac{2DPq\times QB}{CK\times CP}$, that is (on account of the proportionals QB and CK, DA and AC), $\frac{2DPq\times DA}{AC\times CP}$, and thus to $2DP$, as $DP\times DA$ to $CP\times AC$; that is, as the resistance to gravity. *Q. E. D.*

Corol. 4. From which if the body may be projected from some place D, with a given velocity, along some given right line DP in place , and the resistance of the medium from the start of the motion may be given: it is possible to find the curve $DraF$, that the same body describes. For from the given velocity the latus rectum of the parabola is given, as has been noted. And on taking $2DP$ to that latus rectum, as the force of gravity is to the force of resistance, DP may be given. Then by cutting DC in A, so that $CP\times AC$ shall be to $DP\times VA$ in that same ratio of gravity to resistance, the point A will be given. And thence the curve $DraF$ may be given.

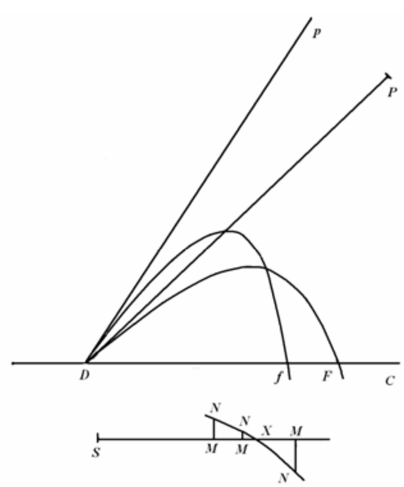


Corol. 5 . And conversely, if the curve $DraF$ may be given, both the velocity of the body and the resistance of the medium will be given at the individual points r. For from the given ratio $CP\times AC$ to $DP\times VA$, then the resistance of the medium from the start of the motion, as well as the latus rectum of the parabola is given: and thence also the velocity from the beginning of the motion is given. Then from the length of the tangent rL, and the velocity proportional to this is given, and the resistance proportional to the velocity at some place r.

Corol. 6. But since the length 2*DP* shall be to the latus rectum of the parabola as gravity to the resistance at *D;* and from the increase in the velocity the resistance may be augmented in the same ratio, but the latus rectum if the parabola may be augmented in that square ratio: it is apparent the length to be increased in that simple ratio, and thus always to be proportional to the velocity, nor to be increased or decreased by a change in the angle *CDP*, unless also the velocity may be changed.



Corol. 7. From which a method of determining the curve *DraF* approximately from phenomena is apparent, and thence by deducing the resistance and velocity by which a body is projected. Two similar and equal bodies may be projected with the same velocity from the same place *D*, following different angles *CDP*, *CDp* and the locations may be known *F*, *f*, where they are incident on the horizontal plane *DC*. Then, with some length assumed for *DP* or *Dp*, which resistance in some ratio to gravity may be put in place at *D*, and that ratio may be established by some length *SM*. Then by calculation, from that assumed length *DP*, the lengths *DF*, *Df*, from the ratio $\frac{Ff}{DF}$ are found by calculation: the same ratio found by experiment may be taken, and the difference may be put in place by the perpendicular *MN*. Do the same over again and a third time, always by assuming a new ratio *SM* of the resistance to gravity, and by putting in place a new difference *MN*. But the differences may be drawn positive on one side of the line *SM* and negative on the other; and through the points *N*, *N*, *N* there may be drawn a curve of the form *NNN* cutting the right line *SMMM* in *X*, and *SX* will be the true ratio of the resistance to gravity, as it was required to find. From that ratio the length *DF* is required to be deduced by calculation ; and the length, which shall be to the assumed length *DP*, as the length *DF* known through experiment to the length *DF* found in this manner, will be the true length *DP*. With which found, both the curved line *DraF* will then be had that the body will describe, as well as the velocity and resistance of the body at individual places.



Scholium.

Besides, the resistance of bodies to be in the ratio of the velocities, is a hypothesis more mathematical than natural. In mediums, which are free from all stiffness, the resistance of the bodies are in the square ratio of the velocities. And indeed by the action of faster bodies there is communicated to the same quantity of the medium, in a smaller time, a greater motion on account of the greater velocities ; and thus in equal times, on account of the greater quantity of the medium disturbed, a greater motion is communicated in the square ratio; and the resistance (by laws of motion II & III) is as the motion communicated. Therefore we may see how a motion arises from this law of resistance.

DE MOTU CORPORUM

LIBER SECUNDUS.

SECTIO I.

De motu corporum quibus resistitur in ratione velocitatis.

PROPOSITIO I. THEOREMA I.

Corporis, cui resistitur in ratione velocitatis, motus ex resistentia amissus est ut spatium movendo confectum.

Nam cum motus singulis temporis particulis aequalibus amissus sit ut velocitas, hoc est, ut itineris confecti particula: erit, componendo, motus toto tempore amissus ut iter totum. *Q. E. D.*

Corol. Quare si corpus, gravitate omni destitutum, in spatiis liberis sola vi insita moveatur; ac detur tum motus totus sub initio, tum etiam motus reliquus post spatium aliquod confectum: dabitur spatium totum quod corpus infinito tempore describere potest. Erit enim spatium illud ad spatium jam descriptum, ut motus totus sub initio ad motus illius partem amissam.

LEMMA I.

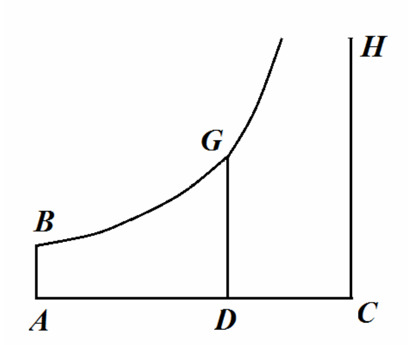
Quantitates differentiis suis proportionales sunt continue proportionales.

Sit A ad $A-B$ ut B ad $B-C$ & C ad $C-D$, &c. & convertendo fiet A ad B ut B ad C & C ad D, &c. *Q. E. D.*

PROPOSITIO II. THEOREMA II.

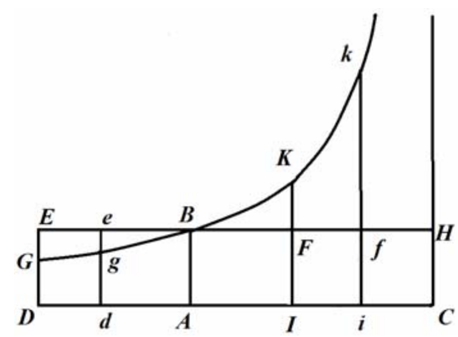
Si corpori resistitur in ratione velocitatis, & *idem sola vi insita per medium similare moveatur, sumantur autem tempora aequalia: velocitates in principiis singulorum temporum sunt in progressio geometrica,* & *spatia singulis temporibus descripta sunt ut velocitates.*

Cas. I. Dividatur tempus in particulas aequales; & si ipsis particularum initiis agat vis resistentiae impulsu unico, quae sit ut velocitas : erit decrementum velocitatis singulis temporis particulis ut eadem velocitas. Sunt ergo velocitates differentiis suis proportionales, & propterea (per Lem. I. Lib. II.) continue proportionales. Proinde si ex aequali particularum numero componantur tempera quaelibet aequalia, erunt velocitates ipsis temporum initiis, ur termini in progressione continua, qui per saltum capiuntur, omisso passim aequali terminorum intermediorum numero. Componuntur autem horum terminorum rationes ex rationibus inter se iisdem terminorum intermediorum aequaliter repetitis & propterea eae quoque rationes compositae inter se eaedem sunt igitur velocitates, his terminis proportionales, sunt in progressione geometrica. Minuantur jam aequales illae temporum particulae, & augeatur earum numerus in infinitum, eo ut resistentiae impulsus reddatur continuus , & velocitates in principiis aequalium temporum, semper continue proportionales, erunt in hoc etiam casu continue proportionales. *Q. E. D.*



Cas. 2. Et divisim velocitatum differentie, hoc est, earum partes singulis temporibus amissae, sunt ut totae : spatia autem singulis temporibus descripta sunt ut velocitatum partes amissae (per Prop. I, Lib. II.) & propterea etiam ut totae. *Q.V.D.*

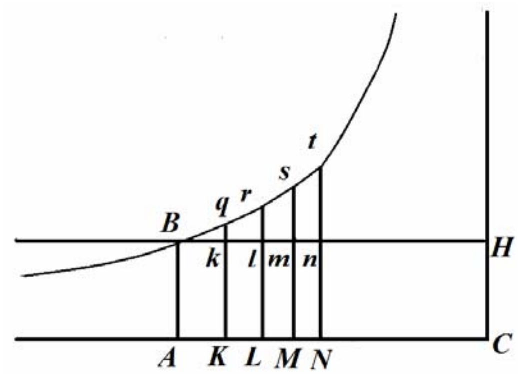
Corol. Hinc si asymptotis rectangulis *AC* ,C*H* describatur hyperbola *BG*, sintque *AB, DG* ad asymptoton *AC* perpendiculares, & exponatur tum corporis velocitas tum resistientia medii, ipso motus initio, per lineam quamvis datam *AC*, elapso autem tempore aliquo per lineam indefinitam *DC:* exponi potest tempus per aream *ABGD,* & spatium eo tempore descriptum per lineam *AD.* Nam si area illa per motum puncti *D* augeatur uniformiter ad modum temporis, decrescet recta *D*C in ratione geometrica ad modum velocitatis, & partes rectae *AC* aequalibus temporibus descriptae decrescent in eadem ratione.



PROPOSITIO III. PROBLEMA I.

Corporis, cui, dum in medio similari recta ascendit vel descendit, resistitur in ratione velocitatis, quodque ab uniformi gravitate urgetur, definire motum,

Corpore ascendente, exponatur gravitas per datum quodvis rectangulum *BACH,* & resistentia medii initio ascensus per rectangulum *BADE* sumptum ad contrarias partes rectae *AB.* Asymptotis rectangulis *AC, CH,* per punctum *B* describatur hyperbola secans perpendicula *DE, de in G, g;* & corpus ascendendo tempore *VGgd* describet spatium *EGge,* tempore *DGBA* spatium ascensus totius *EGB* ; tempore *ABKI* spatium descensus *BFK,* atque tempore *IKki* spatium descensus *KFfk;* & velocitates corporis (resistentiae medii proportionales) in horum temporum periodis erunt *ABED, ABed,* nulla, *ABFI, ABfi* respective; atque maxima velocitas, quam corpus descendendo potest acquirere, erit *BACH.*



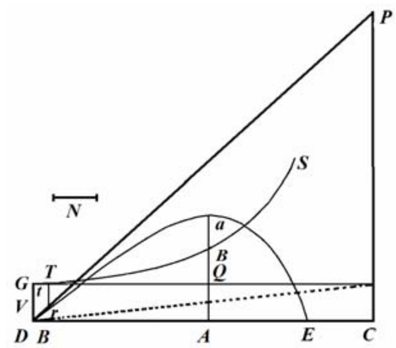
Resolvatur rectangulum *BACH* in rectangula innumera *Ak,Kl, Lm, Mn, etc.* quae sint ut incrementa velocitatum aequalibus totidem temporibus facta; & erunt nihil, *Ak, Al, Am, An,* &c. ut velocitates totae, atque ideo (per hypothesin) ut resistentiae medii principio singulorum temporum aequalium. Fiat *AC* ad *AK* vel *ABHC* ad *ABkK* ut vis gravitatis ad resistentiam in principia temporis secundi, deque vi gravitatis subducantur resisteniae, & manebunt *ABHC, KkHC, LlHC, MmHC,* &c. ut vires absolutai quibus corpus in principio singulorum temporum urgetur, atque ideo (per motus legem II) ut incrementa velocitatum, id est, ut rectagula *Ak, Kl, Lm, Mm,* &c., & propterea (per Lem. I. Lib. II.) in progressione geometrica. Quare si rectae *Kk, Ll, Mm, Nn,* &c. productae occurrant hyperbole in *q,r,s,t,* &c. erunt area *ABqK, KqrL, LrsM, MstN,* &c. aequales, ideoque tum temporibus tum viribus gravitatis simper aequalibus analogae. Est autem area *slB qK* (per Corol. 3. Lem. VII. & Lem. VIII. Lib. 1.) ad aream *Bkq* ut *Kq* ad $\frac{1}{2}kq$ seu *AC* ad $\frac{1}{2}AK$, hoc est, ut vis gravitatis ad resistentiam in medio temporis primi.
Et simili argumento areae *qKLr, rLMs, sMNt,* &c. sunt ad areas *qklr, rlms; smnt,* &c. ut vires gravitatis ad resistentias in medio temporis secundi, tertii, quarti, &c. Proinde cum area: aequales *BAKq, qKLr, rLMs, sMNt,* &c. sint viribus gravitatis analogae, erunt areae *Bkq, qktr, rlms, smnt,* &c. resistentiis in mediis singulorum temporum, hoc est (per hypothesin) velocitatibus, atque ideo descriptis spariis analogae. Sumantur analogarum summae, & erunt areae *Bkq, Blr, Bms, Bet,* &c. spatiis totis descriptis analogae; necnon areae *ABqK, ABrL, ABsM,.ABtN,* &c. temporibus. Corpus igitur inter descendendum, tempore quovis *ABrL,* describit spatium *Blr,* & tempore *LrtN* spatium *rlnt. Q. E. D.* Et similis est demonstratio motus expositi in ascensu. *Q. E. D.*

Corol. 1. Igitur velocitas maxima, quam corpus cadendo potest acquirere, est ad velocitatem dato quovis tempore acquisitam, ut vis data gravitatis, qua corpus illud perpetuo urgetur, ad vim resisteniae, qua in sine temporis illius impeditur,
Corol. 2. Tempore autem aucto in progressione arithmetica, summa velocitatis illius maximse ac velocitatis in ascensu, atque etiam earundem differentia in descensu decrescit in progressione geometrica.
Corol. 3. Sed & differentiae spatiorum, quae in aequalibus temporum differentiis describuntur, decrescunt in eadem progressione geometrica.
Corol. 4. Spatium vero a corpore descriptum differentia est duorum spatiorum, quorum alterum en ut tempus sumptum ab initio descensus, & alterum ut velocitas, que etiam ipso descensus initio aequantur inter se.

PROPOSITIO IV. PROBLEMA II.

Posito quod vis gravitatis in medio aliquo similari uniformis sit, ac tendat perpendiculariter ad planum horizontis; definire motum projectilis in eadem, resistentiam velocitati proportionalem patientis.

E loco quovis *D* egrediatur projectile secundum lineam quamvis rectam *DP,* & per longitudinem *DP* exponatur eiusdem velocitas sub initio motus. A puncto *P* ad lineam horizontalem *DC* demittatur perpendiculum *PC,* & secetur *DC* in *A,* ut sit *DA* ad *AC* ut resistentia medii, ex motu in altitudinem sub initio orta, ad vim gravitatis; vel (quod perinde est) ut sit rectangulum sub *DA* & *DP* ad rectagulum sub *AC* & *CP* ut resistentia tota sub initio motus ad vim gravitatis. Asymptotis *DC, CP* describatur hyperbola quaevis *GTBS* secans perpendicula *DG, AB* in *G* & *B;* & compleatur parallelogrammum *DGKC,* cujus latus *GK* secet *AB* in *Q*; Capiatur linea *N* in ratione ad *QB* qua *D C* sit ad *CP;* & ad rectae *DC* punctum quodvis *R* erecto perpendiculo *RT,* quod hyperbolae in *T,* & rectis *EH, GK, DP* in *I, t* & *V* occurrat , in eo cape *Vr* equalem $\frac{tGT}{N}$, vel quod perinde est, cape *Rr* aequalem $\frac{GTIE}{N}$; & projectile tempore *DRTG* perveniet ad punctum *r,* describens curvam lineam *DraF,* quam punctum *r* semper tangit, perveniens autem ad maximam altitudinem *a* in perpendiculo *AB,* & postea semper appropinquans ad asymptoton *PC.* Estque velocitas ejus in puncto quovis *r* ut curvae tangens *rL. Q. E. I.*



Est enim *N* ad *QB* ut *DC* ad *CP* seu *DR* ad *RV,* ideoque *RV* aequalis $\frac{DR\times QB}{N}$, et *Rr* (id est $RV - Vr$ seu $\frac{DR\times QB - tGT}{N}$) aequalis $\frac{DR\times AB - RDGT}{N}$. Exponatur jam tempus per aream *RDGT*, & (per legum Corol. 2.) distinguatur motus corporis in duos, unum ascensus, alterum ad latus. Et cum resistentia sit ut motus, distinguetur etiam haec in partes duas partibus motus proportionales & contrarias : ideoque longitudo, a motu ad latus descripta, erit (per Prop. II. huius) ut linea *DR,* altitudo vero (per Prop. III. huius) ut area $DR\times AB - RDGT$, hoc est, ut linea *Rr.* Ipso autem motus initio area *RDGT* aequalis est rectangulo $DR\times AQ$ ideoque linea illa *Rr* (seu $\frac{DR\times AB - DR\times AQ}{N}$ tunc est ad *DR* ut $AB - AQ$ seu *QB* ad *N*, id est, ut *CP* ad *DC* ; atque ideo ut motus in altitudinem ad motum in longitudinem sub initio. Cum igitur *Rr* semper sit ut altitudo, ac *DR* semper ut longitudo,
atque *Rr* ad *DR* sub initio ut altitudo ad longitudinem: necesse est ut *Rr* semper sit ad *DR* ut altitudo ad longitudinem,& propterea ut corpus moveatur in linea *DraF,* quam punctum *r* perpetuo tangit. *Q. E. D.*

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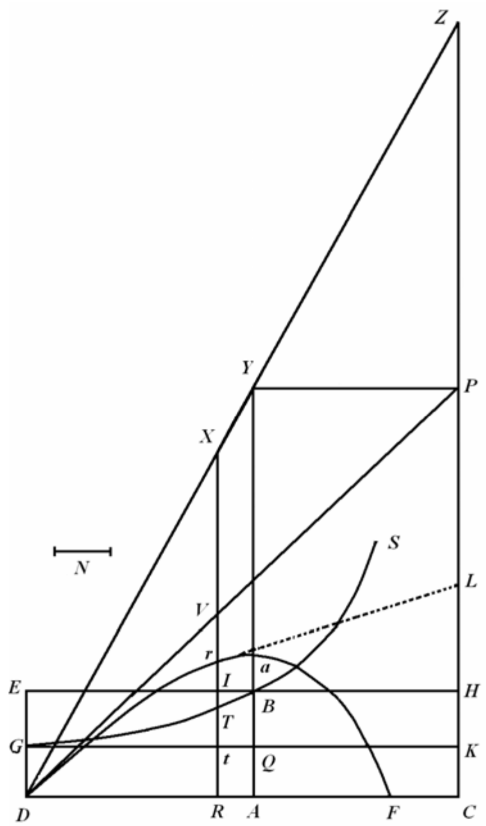
Corol. I. Est igitur *Rr* aequalis $\frac{DR\times AB}{N} - \frac{RDGT}{N}$: ideoque si producatur *RT* ad *X* ut sit *RX* aequalis *DR* ad $\frac{DR\times AB}{N}$; id est, si compleatur parallelogrammum *ACPY* ac iungatur *DY* secans C*P* in *Z*, & producatur *RT* donec occurrat *DT* in *X;* erit *Xr* aequalis $\frac{RDGT}{N}$ & propterea tempori proportionalis.

Corol. 2. Unde si capiantur innumerae C*R*, vel, quod perinde est, innumerae *ZX* in progressione geometrica; erunt totidem *Xr* in progressione arithmetica. Et hinc curva *DraF* per tabulam logarithmorum facile delineatur.

Corol. 3. Si vertice *D*, diametro *DG* deorsum producta, & latere recto quod sit ad 2*DP* ut resistentia tota ipso motus initio ad vim gravitatis, parabola construatur: velocitas quacum corpus exire debet de loco *D* secundum rectam *DP*, ut in media uniformi resistente describat curvam *DraF*, ea ipsa erit quacum exire debet de eodem loco *D* est, secundum eandem rectam *DP*, ut in spatio non resistentae describat parabolam. Nam latus rectum parabole huius, ipso motus initio, est $\frac{DVquad.}{Vr}$; & *Vr* est $\frac{tGT}{N}$ seu $\frac{DR\times Tt}{2N}$. Recta autem quae, si duceretur, hyperbolam G*TS* tangeret in G, parallela est ipsi *DK*, ideoque *Tt* est $\frac{CK\times DR}{DC}$, & *N* erat $\frac{QB\times DC}{CP}$. Et propterea *Vr* est $\frac{DRq\times CK\times CP}{2DCq\times QR}$, id est (ob proportionales *DR & DC*, *DV& DP*) $\frac{DVq\times CK\times CP}{2DPq\times QB}$ & latus rectum $\frac{DVquad.}{Vr}$ prodit $\frac{2DPq\times QB}{CK\times CP}$, id est (ob proportionales *QB* & *CK*, *DA* & *AC*), $\frac{2DPq\times DA}{AC\times CP}$, ideoque ad 2*DP*, ut $DP\times DA$ ad $CP\times AC$; hoc est, ut resistentia ad gravitatem. *Q. E. D.*

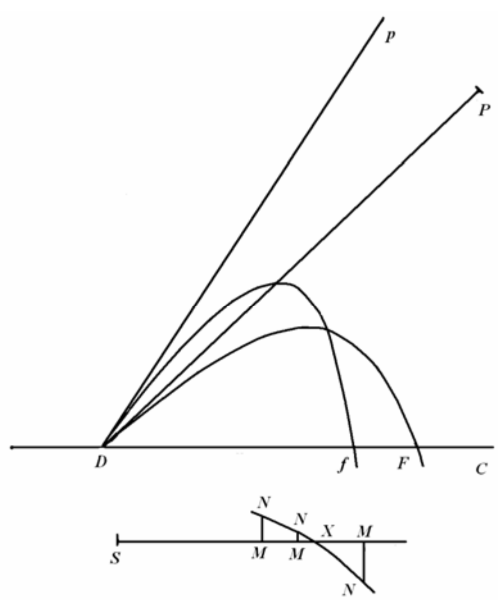
Corol. 4. Unde si corpus de loco quovis *D*, data cum velocitate, secundum rectam quamvis positione datam *DP* proiiciatur , & resistentia medii ipso motus initio detur: inveniri potest curva *DraF*, quam corpus idem describet. Nam ex data velocitate datur latus rectum parabolae, ut notum est. Et sumendo 2*DP* ad latus illud rectum, ut est vis gravitatis ad vim resistentiae, datur *DP*. Dein secando *DC* in *A*, ut sit $CP\times AC$ ad $DP\times VA$ in eadem illa ratione gravitatis ad resistentiam, dabitur punctum *A*. Et inde datur curva *DraF*.

Corol. 5 . Et contra, si datur curva *DraF*, dabitur & velocitas corporis & resistentia medii in locis singulis *r*. Nam ex data ratione $CP\times AC$ ad $DP\times VA$, datur tum resistentia medii sub initio motus, tum latus rectum parabolae: & inde datur etiam velocitas sub initio motus. Deinde ex longitudine tangentis *rL*, datur & huic proportionalis velocitas, & velocitati proportionalis resistentia in loco quovis *r*.



Corol. 6. Cum autem longitudo 2*DP* sit ad latus rectum parabolae ut gravitas ad resistentiam in *D;* & ex aucta velocitate augeatur resistentia in eadem ratione, at latus rectum parabolae augeatur in ratione illa duplicata: pater longitudinem augeri in ratione illa simplici, ideoque velocitati semper proportionalem esse, neque ex angulo *CDP* mutate augeri vel minui, nisi mutetur quoque velocitas.

Corol. 7. Unde liquet methodus determinandi curvam *DraF* ex phaenomenis quamproxime, & inde colligendi resistentiam & velocitatem quacum corpus proiicitur. Proiiciantur corpora duo similia & aequalia eadem cum velocitate, de loco *D*, secundum angulos diversos *CDP*, *CDp* & cognoscantur loca *F*, *f*, ubi incidunt in horizontale planum *D C.* Tum, assumpta quacunqne longitudine pro *DP* vel *Dp,* fingatur quod resistentia in *D* sit ad gravitatem in ratione qualitet, & exponatur ratio illa per longitudinem quamvis *S M.*



Deinde per computationem, ex longitudine illa assumpta *DP*, inveniantur longitudines *DF*, *Df*, ac de ratione: ac per calculum inventa, auferatur ratio eadem per experimentum inventa, & exponatur differentia per perpendiculum *MN.* Idem fac iterum ac tertio, assumendo semper novam resistentiae ad gravitatem rationem *SM*, & colligendo novam differentiam *MN.* Ducantur autem differentiae affirmative ad unam partem rectae *SM*, & negativae ad alteram; & per puncta *N*, *N*, *N* agatur curva regularis *NNN* secans rectam *SMMM* in *X*, & erit *SX* vera ratio resistentiae ad gravitatem, quam invenire oportuit, Ex hac ratione colligenda est Iongitudo *DF* per calculum ; & longitude, quae sit ad assumptam longitudinem *DP*, ut longitudo *DF* per experimentum cognita ad longitudinem *DF* modo inventam, erit vera longitudo *DP*. Qua inventa, habetur tum curva linea *DraF* quam corpus describit, tum corporis velocitas & resistentia in locis singulis.

Scholium.

Caeterum, resistentiam corporum esse in ratione velocitatis, hypothesis est magis mathematica quam naturalis. In mediis, quae rigore omni vacant, resisteniae corporum sunt in duplicata ratione velocitatum, Etenim actione corporis velocioris communicatur eidem medii quantitati, tempore minore, motus maior in ratione maioris velocitatis; ideoque tempore aequali, ob maiorem medii quantitatem perturbatam, communicatur motus in duplicata ratione maior; estque resistentia (per motus leg. II. & III.) ut motus communicartus. Videamus igitur quales oriantur motus ex hac lege resistentiae.