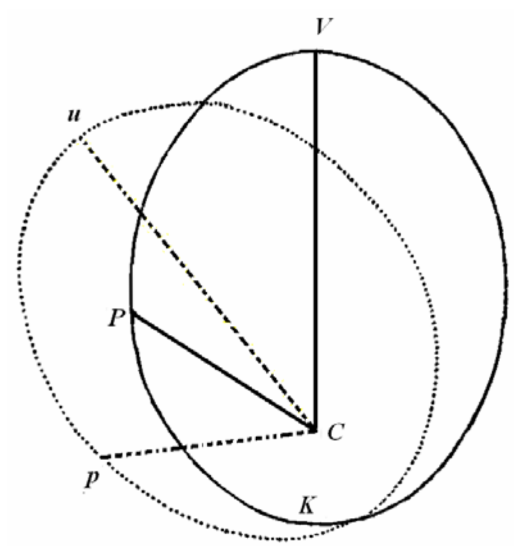
SECTION IX.

Concerning the motion of bodies in moving orbits, and from that the motion of the apses.

PROPOSITION XLIII. PROBLEM XXX.

It is required to bring about, that a body shall be able to move in a trajectory that likewise rotates about the centre of forces, and another body to remain [moving] *in the same stationary orbit.*



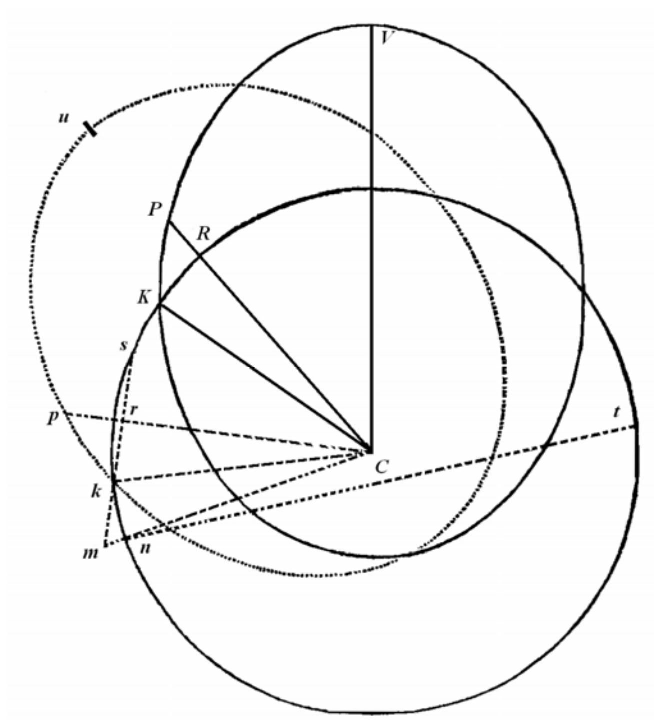
In the orbit *VPK* given stationary, the body *P* may be revolving by going from *V* towards *K*. From the centre *C* there may always be drawn *Cp*, which shall be equal to *CP* itself, and the angle *VCp* may be put in place proportional to the angle *VCP*; and the area, that the line *Cp* will describe, will be to the area *PCP*, that the line *CP* will describe likewise, as the velocity of the line describing *Cp* to the velocity of the line describing *CP;* that is, thus so that the angle *VCp* [will be] in a given ratio to the angle *VCP*, and therefore proportional to the time. Since the area that the line *Cp* will describe in the motionless plane shall be proportional to the time, it is evident that the body, urged by a centripetal force of the right size, together with the point *p* may be able to revolve in that curved line as the same point *p* now may be described in the fixed plane in the account established. The angle *VCu* is made equal to the angle *PCp,* and the *Cu* equal to the line *CV*, and the figure *uCp* equal to the figure *VCP,* and the body always present at *p* will be moving on the perimeter of the revolving figure *uCp,* and in the same time describes an arc of this *up* by which the body *P* can describe another arc similar and equal to *VP* in the stationary figure *VPK*. Therefore the centripetal force may be sought, by Corollary *V* of Proposition VI, by which a body may be able to revolve on that curved line that the point *p* will describe in the fixed plane, and the problem will be solved. *Q. E.F.*

[A number of ideas are to be presented here in succession, that perhaps need to be examined in more detail; Newton's understanding of a physical system is set out in the continued evolution of the positions, velocities, etc., of the bodies involved in time; indeed time is the customary free variable in Newton's differential calculus. Here he considers initially a stationary orbit, the curve traced out by a point attracted towards the focus by some force; any point on which line can be described using polar coordinates (r,θ), with the focus *C* as origin; such a body sweeps out equal areas in equal times. Another version of the same orbit is desired, that rotates about the first stationary orbit; since it is similar to the first orbit at any instant, the only difference is a rotation in the coordinates, and hence the corresponding point may be given by the coordinates $(r,\alpha\theta)$, for some value of α; in this way points in the stationary orbit are imaged onto points in the other rotating orbit : thus, the vertex *V* becomes *u*, a general point *P* becomes *p*, while the polar angle is increased in the ratio α, all at the same instant of time ; the identities involving the areas of the elliptic sectors correspond to the *'equal areas in equal times'* rule for Keplerian orbits (assumed to apply in this case, which we now know to be so from the conservation of angular momentum), are related by a constant $\alpha > 1$ for an anti-clockwise motion as defined in the diagrams, and clockwise for $\alpha < 1$; clearly in the moving case the body in the moving orbit sweeps out a larger area in the same time than the body in the stationary orbit.]

PROPOSITION XLIV. THEOREM XIV.

The difference of the forces, by which a body in a stationary orbit, and another body in a revolving orbit are able to be moving equally, is in the triplicate ratio [i.e. cube] of the common inverse altitude.

Take similar and equal parts *vp, pk* of the revolving orbit with the parts *VP* and *PK* of the stationary orbit; and the separation of the points *P* and *K* may be understood to be the smallest [*i.e.* increments].
[At this point we note in the diagram, that $CV = Cv$; $CP = Cp;$ while the angle between a stationary orbit line and its rotating image line are rotated through the same angle, so that $\angle VCv = \angle PCp$, etc.]



Send the perpendicular *kr* from the point *k* to the right line *pC*, and produce the same to *m*, so that *mr* shall be to *kr* as the angle *VCp* to the angle *VCP*.
[Note that the infinitesimal *rk* is the perpendicular from *k* to *pC*, and hence also is the perpendicular distance of *K* from *PC*, and by hyp. $kC = KC$, so that *K* and *k* lie on a circle with centre C; however, in the time Δt in which *P* has advanced to *K*, and *p* would have advanced to *k* in the other orbit – if stationary, the rotating orbit has moved forwards to a new position *m*, following the angle magnification α, according to which $rm = \alpha rk$. This becomes clear if we understand that at any instant, a point *p* on the moving orbit also at that instant is rotating in a circle with radius

Cp and centre *C*; thus, if we could 'freeze' the position of the body in the rotating orbit, we would be left with the pure circular motion of the body at *p,* which would present itself after the time increment at *n* ; thus the length *mn* corresponds to the incremental extra distance due to the effect of the motion of the body in the rotating ellipse, which Newton explains in detail.

Another point that emerges, concerning the angle amplification factor α , is the area increase or decrease ratio per unit time : for if *h* is the rate of change of the area in the stationary orbit, so that the increment in the area swept out by the body in the time increment Δt is $h\Delta t$, then likewise in the rotating orbit, the area swept out per unit time is *h'*, then the ratio of the areas swept out in the two orbits is $\frac{h'\Delta t}{h\Delta t} = \alpha$, which is the ratio of the areas of the elliptic sectors in the case of elliptic orbits.]

Because the altitudes of the bodies *PC* and *pC, KC* and *kC* always shall be equal, it is evident that the increments or decrements of the lines *PC* and *pC* always shall be equal, and thus if at the present places of the bodies *P* and *p*, the individual motions may be separated (by Corol. 2 of the Laws.) into two [motions] : these towards the centre, or along the lines *PC* and *pC* may be determined, and the others which shall be transverse to the former, and may have a direction perpendicular to the lines *PC* and *pC* themselves may also be determined. The motions towards the centres will be equal [as $CP = Cp$], and the transverse motions of the body *p* will be to the transverse motion of the body *P,* as the angular motion of the line *p*C to the angular motion of the line *PC,* that is, as the angle *VCp* to the angle *VCP.* Therefore, in the same time in which the body *P* arrives at the point *K* by its own motion in both directions, the body *p* will be moved by an equal amount towards the centre from *p* towards *C* equally, and thus in that completed time it will be found somewhere on the line *mkr,* which is perpendicular to the line *pC* through the point *k* . From the transverse motion, it will acquire a distance from the line *pC,* which shall be to the distance that the other body *P* acquires from the line *PC*, as the transverse motion of the body *p* is to the transverse motion of the other body *P.* Whereby since *kr* shall be equal to the distance that the body *P* acquires from the line *PC,* and *mr* shall be to *kr* as the angle *VCp* to the angle *VCP,* that is, as the transverse motion of the body *p* to the transverse motion of the body *P,* it is evident that the body *p* will be found in that completed time at the place *m.*

Thus these [motions] themselves will be in place when the bodies *p* and *P* are moving equably along the lines *pC* and *PC*, and thus the bodies are acted on by equal forces along these lines, [*i.e*. if the force on the body *p* in the mobile ellipse is equal to the force on the other body *P* in the stationary ellipse, due to the equal distances.] But if the angle *pCn* may be taken to the angle *pCk* as the angle *VCp* is to the angle *VCP*, and thus *nC* equals *kC,* and the body *p* in that time completed may actually be found at *n* . Thus [since this is not the case] it [i.e. *p*] shall be acted on by a greater force than the body *P,* but only if the angle *pCn* is greater than the angle *pCk,* that is if the orbit *upk* either is advancing, or is moving with a greater speed that shall be double of that by which the line *CP* is carried in regression ; and with a smaller force if it is moving slower in the orbit. And the difference of the forces, so that the interval of the places *mn,* through which that body *p* from the action of this [extra force], in that given interval of time, must be

transferred. With the centre *C* and radius *Cn* or *Ck*, a circle may be described cutting the lines *mr* and *ms* product to *s* and *t*, and the rectangle $mn \times mt$ will be described equal to the rectangle $mk \times ms$, and thus *mn* equals $\frac{mk \times ms}{mt}$. But since the triangles *pCk* and *pCn* may be given in magnitude at a given time, *kr* and *mr* and so their sum and difference *mk* and *ms* are inversely as the altitude *pC*
[these triangles correspond to the areas swept out by the body in the mobile trajectory relative to that frame, and the corresponding area swept out by the body in the mobile trajectory relative to the stationary frame, corresponding to the their respective lengths by the common altitude *Cp*],
and thus the rectangle $mk \times ms$ is inversely as the square of the altitude *pC.* And *mt* is directly as $\frac{1}{2}mt$, that is, as the altitude *pC*. These are the first ratios of the nascent lengths
[Note that Newton calls differentials arising in pending integrations *nascent*, or the smallest of magnitudes coming into being; while the differentials associated with differentiation are called *evanescent*, or vanishing ones.];
and hence $\frac{mk \times ms}{mt}$ shall be, that is, the total increment arising *mn*, proportional to the inverse difference of the forces, as the cube of the altitude *pC. Q. E. D.*

[Thus, an extra force in addition to the central force responsible for the stationary trajectory, and varying as the inverse cube of the distance from the focus, is introduced in these circumstances, to produce the rotating orbit. For an analytical solution see, e.g. Whittaker's *Analytical Dynamics*, on the solvable problems of particle dynamics, p.83 ; and also of course Chandrasekhar, circa p.187; Brougham & Routh also treat this problem, p. 88, known as Newton's theorem of revolving orbits.

In the following Corollaries we find the ratio of the difference of the forces to a circular force derived from the versed sine is given by $\frac{mk \times ms}{mt}$ to $\frac{rk^2}{2kC}$. Note that ms or $mk = rm \pm rs = rm \pm rk = (\alpha \pm 1)rk$; Again, *rk* is the altitude of the elemental sector *pkC*, and the area of this can be related to $h' \Delta t$ by being equal to $\frac{1}{2} pC \times rk$; hence $ms = rm + rs = (\alpha + 1)rk = \frac{h(\alpha+1)}{PC}\Delta t$ and $mk = rm - rs = (\alpha - 1)rk = \frac{h(\alpha-1)}{PC}\Delta t$;
and therefore $\frac{mk \times ms}{mt} = (\alpha^2 - 1)\frac{rk^2}{mt} = \frac{h^2(\alpha^2-1)}{2PC^3}\Delta t^2$, and the result $mn = \frac{h^2(\alpha^2-1)}{2PC^3}\Delta t^2$ follows on taking *mt* as almost a diameter equal to 2*kC*. These results imply that the orbits are almost circular, for the centre of curvature is taken as *C*.
This is the total difference in the centripetal forces acting on the bodies in the two orbits, at the same distance from *C*, and is established in this manner by Chandrasekhar.]

Corol. I. Hence the difference of the forces at the places *P* and *p*, or *K* and *k*, to the force by which the body shall be able to revolve in a circular motion from *R* to *K* in the same time in which the body *P* will describe the arc *PK* in the stationary arc, is as the line element arising *mn*, to the versed sine of the arc *RK* arising, that is as $\frac{mk \times ms}{mt}$ to $\frac{rk^2}{2kC}$, or as $mk \times ms$ to *rk* squared [as the latter trajectory is circular.]; this is, if the given quantities may be taken *F, G* in that ratio in turn as the angle *VCP* may have to the angle *VCp,* as *GG–FF* to *FF.*

[These results follow as a simplification above, on taking the ratio *G* to *F* as *h'* to *h* and in turn equal to α . Again, Chandrasekhar elaborates, and comes to the same conclusion.]

And therefore, if from the centre *C* with some radius *CP* or *Cp* a sector of the circle may be described equal to the total area *VPC,* that the body *P* has described revolving at some time in the stationary orbit with the radius drawn to the centre of the circle: the difference of the forces, by which the body *P* in the stationary orbit and the body *p* in the mobile orbit are revolving, will be to the centripetal force, by which some body, with the radius drawn to the centre, may be able to describe that sector, by which the area *VPC* will be described uniformly, as *GG–FF* to *FF.* And if that sector and area *pCk* are in turn as the times in which they are described.

[If we start from Kepler's first and second laws, then we can assume a stationary ellipse of the form : $\frac{l}{r} = 1 + e\cos\varphi$. Now for an ellipse, the semi-latus rectum is given by $l = \frac{b^2}{a}$ the radial acceleration is given by $\ddot{r} - r\dot{\varphi}^2$, while the angular momentum or area is changing at a constant rate : $\frac{1}{2}r^2\dot{\varphi} = h = \frac{dA}{dt}$. The radial acceleration is readily shown to be $\ddot{r} = -\frac{h^2}{lr^2}$; this result is needed for the next Corollary.]

Corol.2. If the orbit *VPK* shall be an ellipse having the focus *C* and the higher apse *V*; and similar and equal to that there may be put the ellipse *upk,* thus so that *pC* equals *PC* always, and the angle *VCp* shall be in the given ratio *G* to *F* to the angle *VCP* ; but for the height *PC* or *pC* there may be written *A*, and for the ellipse 2*R* may be put in place for the latus rectum: the force will be, by which the body can revolve in the rotating ellipse, as $\frac{FF}{AA} + \frac{RGG-RFF}{A^3}$ and conversely.

[Note that *R* has been absorbed into the proportionality, as the force should be : $\frac{1}{R}\left[\frac{F^2}{A^2} + \frac{RG^2-RF^2}{A^3}\right]$.]

For the force by which the body may revolve in the stationary ellipse may be set out by the amount $\frac{FF}{AA}$ [from Prop. XI, Section III], and the force at *V* will be $\frac{FF}{CA^2}$. But the force by which the body may be able to revolve in a circle at the distance *CV* with that velocity that the body may have in the ellipse at *V*, is to the force by which the body revolving in the ellipse may be acted on from the apse *V*, as half of the latus rectum of the ellipse to the semi diameter of the circle *CV,* and thus there arises $\frac{RFF}{CV^3}$: and the force which shall be to that as *GG–FF* as *FF*, becomes $\frac{RGG-RFF}{CV^3}$ *:* and this force (by Corol. I. of this section) is the difference of the forces at *V* by which the body *P* in the stationary ellipse *VPK,* and the body *p* in the moving ellipse *upk* are revolving. From which since (by this proposition) that difference at some other height *A* shall be to the that written height *CV* as $\frac{1}{A^3}$ to $\frac{1}{CV^3}$, the same difference $\frac{RGG-RFF}{CV^3}$ will prevail at any height *A*. Therefore to the force $\frac{FF}{AA}$, by which the body can revolve in the stationary ellipse *VPK*, the extra force must be added $\frac{RGG-RFF}{A^3}$; and the total force may be put together $\frac{FF}{AA}+\frac{RGG-RFF}{A^3}$, by which the body shall be able to rotate in the moving ellipse *upk*.

[It appears that someone misread the diagram at some stage, for Whiteside refers to the vertex of the mobile ellipse as *v* from Newton's original notes, and it appears as such in the first 1687 edition, but is taken as *u* in the two later editions of the *Principia*.]

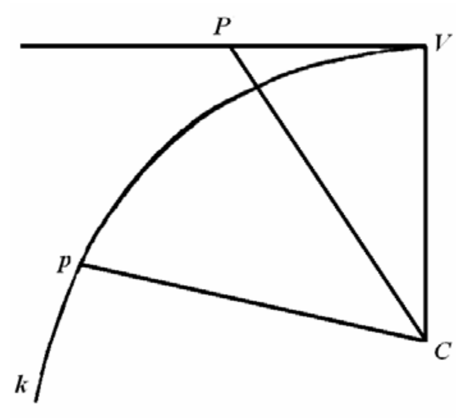
Corol.3. It may be deduced in the same manner, if the stationary orbit *VPK* shall be an ellipse having the centre of forces at *C;* and to this there may be put in place the similar, equal, and concentric mobile ellipse *upk;* and let 2*R* be the principle latus rectum of this ellipse, and 2*T* transverse side or the major, and the angle *VCp* always shall be to the angle *VCP* as *G* to *F*; the forces, by which the bodies in the stationary ellipse and in the mobile ellipse are able to revolve in equal times, will be as $\frac{FFA}{T^3}$ and $\frac{FFA}{T^3}+\frac{RGG-RFF}{A^3}$ respectively.

Corol. 4. And generally, if the maximum height *CV* of some body may be called *T*, and the radius of curvature that the orbit *VPK* has at *V*, that is the radius of the circle equal to the curvature, may be called *R*, and the centripetal force, by which the body can revolve in some stationary trajectory *VPK* at the place *V*, may be called $\frac{F^2}{RT^2}$,

[all the editions of the *Principia* have the nonsensical $\frac{F^2V}{T^2}$ here, which both Whiteside and Chandrasekhar discretely point out as being in error],
and for the other mobile ellipses may be called indefinitely *X* at the positions *P*, with the altitude *CP* called *A*, and *G* to *F* may be taken in the given ratio of the angle *VCp* to the angle *VCP:* the centripetal force will be as the sum of the forces $X+\frac{VRG^2-VRF^2}{A^3}$, by which the body can move circularly in the same motion in the same trajectory *upk* in the same times with the same motions.

Corol. 5. Therefore in some given motion of the body in the stationary orbit, the motion of this about the centre of forces can be increased or decreased in a given ratio, and thence new stationary orbits can be found in which bodies may be rotated by new centripetal forces.

Corol. 6. Therefore a perpendicular *VP* of indefinite length is erected to the given right line in place *CV*, and *Cp* may be drawn equal to *CP* itself, making the angle *VCp,* which shall be in a given ratio to the angle *VCP*; the force by which the body can revolve on that curve *Vpk* that the point *p* always touches, [or traces out] will be reciprocally as the cube of the altitude *Cp.* For the body *P* by the force of inertia, with no other force acting, can progress uniformly along the line *VP.* A force may be added at the centre *C,* inversely proportional to the cube of the altitude *CP* or *Cp*, and (by the present demonstration) that rectilinear motion will be turned away into the curved line *Vpk.* But this curve *Vpk* is that same curve as that curve *VPQ* found in Corol. 3. Prop. XLI , in which we have said there bodies attracted by forces of this kind ascend obliquely.



PROPOSITION XLV. PROBLEM XXXI.

The motions of the apses are required of orbits that are especially close to circles.

The problem is required to be solved by using arithmetic because the orbit, that the body revolving in the mobile ellipse (as in Corol. 2. or 3. of the above proposition) will describe in the stationary place, may approach to the form of this orbit of which the apses are required, and by seeking the apses of the orbit that the body will describe in that stationary plane. But the orbits may acquire the same form, if the centripetal forces by which they are described, gathered together, may return proportionals with equal altitudes. Let the point *V* be the upper apse, and there may be written, *T* for the maximum altitude *CV, A* for some other altitude *CP* or *Cp,* and *X* for the difference of the altitudes *CV*– *CP;* and the force, by which the body is moving in an ellipse about its focus *C* (as in Corol. 2.), and which in Corol. 2. was as $\frac{F^2}{A^2}+\frac{RG^2-RF^2}{A^3}$, that is as $\frac{F^2A+RG^2-RF^2}{A^3}$, by substituting $T-X$ for *A*, will be as $\frac{RG^2-RF^2+TF^2-F^2X}{A^3}$. Any other centripetal force may be reduced similarly to a fraction whose denominator shall be A^3 and the numerators, made from the gathering of the homogeneous terms, may be put in place in an analogous manner. The matter will be apparent from examples.

Example 1. We may put the centripetal force to be uniform [in the two cases, some rotating apse *u* is compared with the upper apse *V*], and thus the force to be as $\frac{A^3}{A^3}$, or (by writing $T-X$ for the height *A* in the numerator), as $\frac{T^3-3T^2X+3TX^2-X^3}{A^3}$;

[The use of $T-X$ is just a computational convenience for the distance *A*, which is to be expanded out in the numerator of $\frac{A^3}{A^3}$, while the denominator is taken as constant, so that small differences from the maximum value T can be found easily; here :
T denotes the max. height of the body in the stationary ellipse;
A denotes the altitude *CP* or *Cp* at some time; and
X denotes the difference $CP-CV=A-T$, which can be made infinitesimally small.]

and with the corresponding terms of the numerators gathered together [for this particular arrangement in comparison with the general above], without doubt 'the given' with 'the given' and the 'not given' with 'not given', the expression becomes

$$RG^2-RF^2+TF^2 \text{ to } T^3; \text{ as } -F^2X \text{ to } -3T^2X+3TX^2-X^3$$

$$\text{or, as } -F^2 \text{ to } -3T^2+3TX-X^2.$$

Now since the orbit may be put as especially close to being circular, the orbit may begin with a circle and on account of *R* and *T* made equal, and *X* diminished indefinitely, the final ratios become RG^2 to T^3 as $-F^2$ to $-3T^2$, or G^2 to T^2 as F^2 to $3T^2$, and in turn G^2 to F^2 as T^2 to $3T^2$, that is, as 1 to 3; and thus *G* to *F*, that is the angle *VCp* is to the angle *VCP*, as 1 to $\sqrt{3}$.
[*i.e.* $\frac{G^2}{T^2}=\frac{F^2}{3T^2}$ or $\frac{G^2}{F^2}=\frac{T^2}{3T^2}=\frac{1}{3}$. Also, the rate of change of the respective areas is $\alpha=\frac{h'}{h}=\frac{\angle CVp}{\angle CVP}$.]
Therefore since the body in the stationary ellipse, by descending from the highest apse to the lowest may make the angle *VCP* (thus as I may say) 180 degrees ; the other body in the mobile ellipse, and thus in the stationary orbit on which we have acted, from the upper apse to the lower apse on descending may make an angle *VCp* of $\frac{180}{\sqrt{3}}$ that therefore on account of the similitude of which orbit, that the body will describe with a uniform centripetal force acting, and will be described performed in the plane at rest by rotating in the revolving ellipse. Through the above collection of the similar terms these orbits are returned, not universally but on that occasion since they may approach especially to the circular form. Therefore a body revolving with a uniform centripetal force in an almost circular orbit, between the upper and lower apses may always make an angle of $\frac{180}{\sqrt{3}}$ degrees, or 103 *gr. 55 m.* 23 *sec.* to the centre ; arriving from the upper apse to the lower apse once this angle has been made, and thence returning to the upper apse when again it has made the same angle ; and thus henceforth indefinitely.

Example 2. We may put the centripetal force A^{n-3} to be as some power of the altitude *A* or as $\frac{A^n}{A^3}$ where $n-3$ and *n* indicate some whole numbers or fractional indices of the powers, either rational or irrational, positive or negative. That number A^n or $|T-X|^n$ may be reduced into a converging infinite series, and the series

$T^n - nXT^{n-1} + \frac{nn-n}{2}XXT^{n-2}$ etc. emerges. And with the terms of this collected together with the terms of the other numerator $RGG - RFF + TFF - FFX$, there shall be $RGG - RFF + TFF$ to T^n as $-FF$ to $-nXT^{n-1} + \frac{nn-n}{2}XXT^{n-2}$ etc. And with the final ratios taken when the orbits approach to a circular form, there shall be *RGG* to T^n as – *FF* to $-nXT^{n-1}$, or *GG* to T^{n-1} as *FF* to nT^{n-1}, and in turn *GG* to *FF* as T^{n-1} to nT^{n-1}; that is as I to *n;* and thus *G* to *F*, that is the angle *VCp* to the angle *VCP*, as 1 to $\sqrt{n}$. Whereby the angle *VCP*, in the falling of the body performed in the ellipse from the upper apse to the lower apse, shall be 180 degrees; the angle *VCp* will be made, in the descent of the body from the upper apse to the lower apse, that the body will describe in an almost circular orbit proportional to some centripetal force of the power A^{n-3}, equal to an angle of $\frac{180}{\sqrt{n}}$; and with this angle repeated the body will return from the lower apse to the higher apse, and thus henceforth indefinitely. So that if the centripetal force shall be as the distance of the body from the centre, that is, as *A* or $\frac{A^4}{A^3}$, *n* will be equal to 4 and $\sqrt{n}$ equals 2; and thus the angle between the upper apse and the lower apse equals $\frac{180}{2}$ or 90 degrees. Therefore with a fourth part of one revolution completed the body will arrive at the lower apse, and with another quarter part at the upper apse, and thus henceforth in turn indefinitely. That which also had been shown from proposition X : for the body may be revolving in the stationary ellipse, acted on by this centripetal force, whose centre is at the centre of the forces. But if the centripetal force shall be reciprocally as the distance, that is directly as $\frac{1}{A}$ or $\frac{A^2}{A^3}$, *n* will be equal to 2, and thus between the upper apse and the lower apse the angle will be $\frac{180}{\sqrt{2}}$ degrees, or 127 *degrees,* 16*m.*, 45 *sec.* and therefore will be revolving by such a force, always by the repetition of this angle, in turns with the others, from the upper apse to the lower apse, and eternally may arrive at the lower apse from. Again if the centripetal force shall be reciprocally as the square root of the square root of the eleventh power of the altitude, that is inversely as $A^{\frac{11}{4}}$, and thus directly as $\frac{1}{A^{\frac{11}{4}}}$, or as $\frac{A^{\frac{1}{4}}}{A^3}$ *n* will be equal to $\frac{1}{4}$ and $\frac{180}{\sqrt{n}}$ degrees equals 360 degrees and therefore the body descending from the upper apse and thus perpetually descending, it will arrive at the lower apse when it will have completed a whole revolution, then by completing another revolution by ascending always, it will return to the upper apse : and thus eternally in turn.

Example 3. We may take *m* and *n* for any indices of the powers of the altitude, and taking *b* and *c* for any given numbers, we may put the centripetal force to be as $\frac{bA^m + cA^n}{A^3}$, that is as $\frac{b \times \overline{T-X}^m + c \times \overline{T-X}^n}{A^3}$, or (by the same method of our converging series) as

$$\frac{bT^m + cT^n - mbXT^{m-1} - ncXT^{n-1} + \frac{mm-m}{2}bXXT^{m-2} + \frac{nn-n}{2}cXXT^{n-2} \text{ etc.}}{A^3}$$

and with the terms of the numerator gathered together, it becomes $RGG - RFF + TFF$ to $bT^m + cT^n$, as $-FF$ to $-mbT^{m-1} - ncT^{n-1} + \frac{mm-m}{2}bXT^{m-2} + \frac{nn-n}{2}cXT^{n-2}$ etc. And on taking the final ratios which arise, here the orbits may approach to a circular form , let G^2 be to $bT^{m-1} + cT^{n-1}$, as F^2 to $mbT^{m-1} + ncT^{n-1}$, and in turn G^2 to F^2 as $bT^{m-1} + cT^{n-1}$ to $mbT^{m-1} + ncT^{n-1}$. Which proportion, by setting the maximum altitude *CP* or *T* arithmetically to one, shall be G^2 to F^2 as $b + c$ to $mb+nc$, and thus as 1 to $\frac{mb+nc}{b+c}$. From which *G* shall be to *F*, which shall be the angle *VCp* to the angle *VCP,* as 1 to $\sqrt{\frac{mb+nc}{b+c}}$. And therefore since the angle *VCP* between the upper apse and the lower apse of the stationary ellipse shall be 180 degrees, the angle *VCp* between the same apses, in the orbit that the body described by a centripetal force proportional to the quantity $\frac{bA^m+cA^n}{A\,cub.}$, is equal to an angle of $180\sqrt{\frac{b+c}{mb+nc}}$ degrees. And by the same argument if the centripetal force shall be as $\frac{bA^m-cA^n}{A^3}$, an angle of $180\sqrt{\frac{b-c}{mb-nc}}$ may be found between the apses. Nor will problems be resolved otherwise in difficult cases. A quantity, to which the centripetal force is proportional, must always be resolved into a converging series having the denominator A^3. Then the given part of the numerator which arises from that operation to the other part of this which is not given, and the part given of the numerator of this $RG^2 - RF^2 + TF^2 - F^2X$ to the other part of this not given, are to be put in the same ratio: And by deleting the superfluous quantities, and on writing one for *T*, the proportion *G* to *F* will be obtained.

Corol. I. Hence if the centripetal force shall be as some power of the altitude, it is possible to find that power from the motion of the apses; and conversely. Without doubt if the whole of the angular motion, by which a body returns to the same apse, shall be to the angular motion of one revolution, or of 360 degrees, as some number *m* to another number *n,* and the altitude may be called *A* : the force will be as that power of the altitude $A^{\frac{n^2}{m^2}-3}$, the index of which is $\frac{n^2}{m^2} - 3$. That which has been shown by the second example. From which it is clear that that force, in receding from the centre, cannot decrease in a ratio greater than the cube of the altitude. A body acted on by such a force revolving and descending from that apse, if it begins to descent, at no time will arrive at the lower apse or the minimum altitude, but will descend as far as the centre, describing that curved line which we have discussed in Corol. 3. Prop. XII. But if that body should have started from the descending [*i.e.* lower] apse, or to ascend minimally, it will ascend indefinitely, nor at any time will it arrive at the upper apse. For it describes that curved line which have been discussed in the same Corol. and in Corol. VI. of Prop. XLIV. And thus where the force, in receding from the centre, decreases in a ratio greater than the cube of the altitude, the body descending from the lower apse, likewise as it may have started either to descend or ascend, descends to the centre or ascend to infinity. But if the force, in receding from the centre either may decrease to be less than the cube of the altitude, or increase in some ratio of the altitude ; the body at no time may descend as far as the centre, but will arrive at some lower apse: and conversely, if a body is ascending and descending from one apse to the other in turn, then at no time will it be called to the centre; the force in receding from the centre either will be greater, or may decrease in a ratio less than the cube of the altitude: and from which the body will have returned quicker from one apse to the other, there the further the ratio of the forces recedes from that cubic ratio. So that if the body may be returned in revolving either 8 , 4 , 2 or $1\frac{1}{2}$ times from the upper apse to the upper apse by ascending and descending ; that is, if *m* to *n* were as 8, 4, 2 or $1\frac{1}{2}$ to 1, and thus $\frac{nn}{mm}-3$ may give rise to $\frac{1}{64}-3$, $\frac{1}{16}-3$, $\frac{1}{4}-3$, or $\frac{4}{9}-3$: the force will be as $A^{\frac{1}{64}-3}$, $A^{\frac{1}{16}-3}$, $A^{\frac{1}{4}-3}$, or $A^{\frac{4}{9}-3}$, that is, reciprocally as $A^{\frac{1}{64}-3}$, $A^{3-\frac{1}{16}}$, $A^{3-\frac{1}{4}}$ or $A^{3-\frac{4}{9}}$. If the body in individual rotations may have returned to the same stationary apse; there will be *m* to *n* as 1 to 1, and thus $A^{\frac{nn}{mm}-3}$ equals A^{-2} or $\frac{1}{A^2}$; and therefore the decrease of the forces are in the square ratio of the altitudes, as has been shown in the preceding. If the body may return to the same apse in the parts of a revolution, either in three quarters, or two thirds, or one third, or in one quarter ; *m* to *n* will be as $\frac{1}{4}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{1}{4}$ to 1, and thus $A^{\frac{nn}{mm}-3}$ equals either $A^{\frac{16}{9}-3}$, $A^{\frac{9}{4}-3}$, A^{9-3}, or A^{16-3} ; and therefore the force varies reciprocally either as $A^{\frac{11}{9}}$, $A^{\frac{3}{4}}$, or directly as A^6, A^{13}. And then if the body on progressing from the upper apse to the same upper apse has completed a whole number of revolutions, and three degrees beyond, and therefore that apse by a single revolution of the body will have been completed as a consequence in three degrees; *m* to *n* will be as 363 degrees to 360 degrees or as 121 to I20, and thus the force $A^{\frac{nn}{mm}-3}$ will be equal to $A^{-\frac{29523}{14641}}$; and therefore the centripetal force varies reciprocally as $A^{\frac{29523}{14641}}$ or reciprocally as $A^{2\frac{4}{243}}$ approximately. Therefore the centripetal force decreases in a ratio a little greater than the square, but which in turn approaches $59\frac{3}{4}$ closer to the square than to the cube.

Corol. 2. Hence also if the body, by a centripetal force which shall be reciprocally as the square of the altitude, may be rotating in an ellipse having a focus at the centre of forces, and to this centripetal force there may be added or taken away some other external force ; the motion of the apses can be known (by example three) which may arise from that external force: and conversely. So that if the force by which the body shall be revolving in an ellipse shall be as $\frac{1}{A^2}$ and the external force taken away shall be as *c*A, and thus the force remaining shall be as $\frac{A-cA^4}{A\,cub.}$; there will be (in the third example) *b* equals 1, *m* equals 1, and *n* equals 4, and thus the angle of rotation between the apses equals an angle of $180\sqrt{\frac{1-c}{1-4c}}$. We may put that external force to be 357.45 less in parts than the other

force by which the body is rotating in the ellipse, that is c becomes $\frac{100}{35745}$, with A or T present equaling 1, and $180\sqrt{\frac{1-c}{1-4c}}$ may become $180\sqrt{\frac{35645}{35345}}$, or 180.7623, that is, 180 degrees, 45m. 44sec. Therefore the body descending from the upper apse, by moving through an angle of 180 degrees, 45m. 44sec, will reach the lower apse, and with this motion doubled it will return to the upper apse : and thus the upper apse by progressing will make 1 degree. 31 m. 28 sec. in individual rotations
The apses of the moon progress around twice as fast.

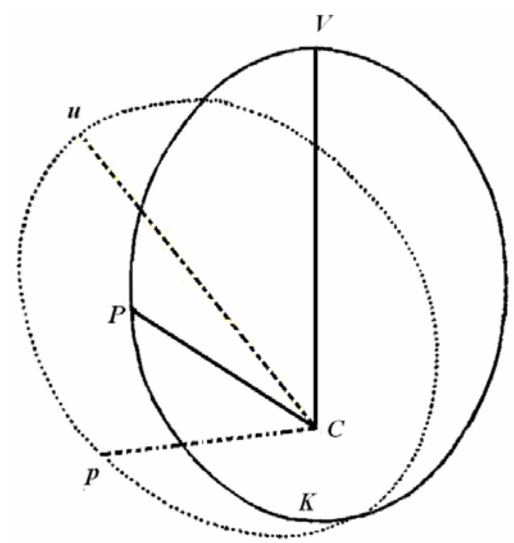
Up until now we have been concerned with the motion of bodies in orbits of which the planes pass through the centre of forces. It remains that the motion of bodies also may be determined in eccentric planes. For the writers who treat the motion of weights, are accustomed to consider the oblique ascent and descent of weights in any planes given, as well as straight up and down: and equally to judge the motions of bodies with any forces for whatever centres desired, and of the dependence on eccentric planes this comes to be considered. Moreover we may suppose the planes to be the most polished and completely slippery lest they may retard bodies. Certainly, in these demonstrations, on whatever of these planes on which in turn the bodies touch by pressing on, we may take the planes parallel to these, in which the centres of the bodies are moving and the orbits they may describe by moving. And at once we may determine by the same law the motions of bodies carried out on curves surfaces.

SECTIO IX.

De motu corporum in orbibus mobilibus, deque motu apsidum.

PROPOSITIO XLIII. PROBLEMA XXX.

Efficiendum est ut corpus in trajectoria quacunque circa centrum virium revolvente perinde moveri possit, atque corpus aliud in eadem trajectoria quiescente.

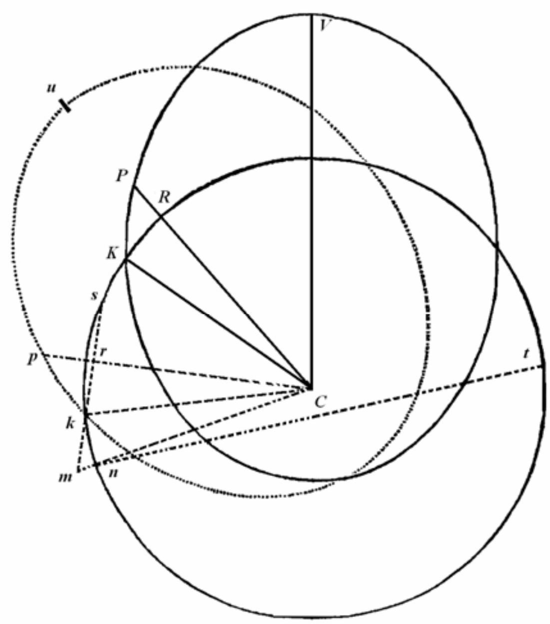


In orbe *VPK* positione dato revolvatur corpus *P* pergendo a *V* versus *K.* A centro *C* agatur semper *Cp,* quae sit ipsi *CP* aequalis, angulumque *VCp* angulo *VCP* proportionalem constituat; & area, quam linea *Cp* describit, erit ad aream *PCP,* quam linea *CP* simul describit, ut velocitas lineae describentis *Cp* ad velocitatem lineae describentis *CP;* hoc est, ut angulus *VCp* ad angulum *VCP,* ideoque in data ratione, & propterea tempori proportionalis. Cum area tempori proportionalis sit quam linea *Cp* in plano immobili describit, manifestum est quod corpus, cogente justae quantitatis vi centripeta, revolvi possit una cum puncto *p* in curva illa linea quam punctum idem *p* ratione jam exposita describit in plano immobili. Fiat angulus *VCu* angulo *PCp,* & linea *Cu* lineae *CV,* atque figura *uCp* figurae *VCP* aequalis, & corpus in *p* semper existens movebitur in perimetro figurae revolventis *uCp,* eodemque tempore describet arcum ejus *up* quo corpus aliud *P* arcum ipsi similem & aequalem *VP* in figura quiescente *VPK* describere potest. Quaeratur igitur, per Corollarium quintum Propositionis VI, vis centripeta qua corpus revolvi possit in curva illa linea quam punctum *p* describit in plano immobili, & solvetur problema. *Q. E.F.*

PROPOSITIO XLIV. THEOREMA XIV.

Differentia virium, quibus corpus in orbe quiescente, & corpus aliud in eodem orbe revolvente aequaliter moveri; possunt, est in triplicata ratione communis altitudinis inverse.

Partibus orbis quiescentis *VP, PK* sunto similes & aequales orbis revolventis partes *up, pk;* & punctorum *P, K* distantia intelligatur esse quam minima. A puncto *k* in rectam *pC* demitte perpendiculum *kr*, idemque produc ad *m*, ut sit *mr* ad *kr* ut angulus *VCp* ad angulum *VCP*. Quoniam corporum altitudines *PC* & *pC, KC, & kC* semper aequantur, manifestum est quod linearum *PC* & *pC* incrementa vel decrementa semper sint aequalia, ideoque si corporum in locis *P* & *p* existentium distinguantur motus singuli (per Legum Corol. 2.) in binos, quorum hi versus centrum, sive secundum lineas *PC, pC* determinentur, & alteri prioribus transversi sint, & secundum lineas ipsis *PC, pC* perpendiculares directionem habeant; motus ,versus centrum erunt aequales, & motus transversus corporis *p* erit ad motum transversum corporis *P*, ut motus angularis lineae *p*C ad motum angularem lineae *PC*, id est, ut angulus *VCp* ad angulum *VCP*. Igitur eodem tempore quo corpus *P* motu suo utroque pervenit ad punctum *K*, corpus *p* aequali in centrum motu aequaliter movebitur a *p* versus *C*, ideoque completo illo tempore reperietur alicubi in linea *mkr*, quae per punctum *k* in lineam *pC* perpendicularis est; & motu transverso acquiret distantiam a linea *pC*. quae sit ad distantiam quam corpus alterum *P* acquirit a linea *PC*, ut est motus transversus corporis *p* ad motum transversum corporis alterius *P*. Quare



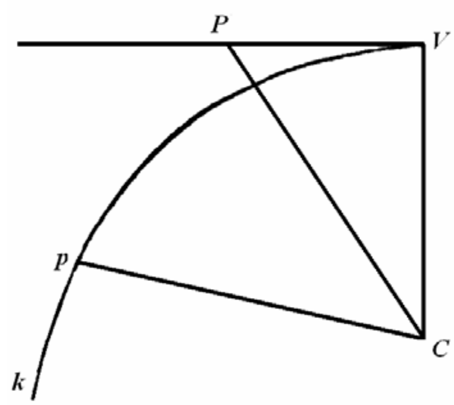
cum *kr* aequalis sit distantiae quam corpus *P* acquirit a linea *PC*, sitque *mr* ad *kr* ut angulus *VCp* ad angulum *VCP*, hoc est, ut motus transversus corporis *p* ad motum transversum corporis *P*, manifestum est quod corpus *p* completo illo tempore reperietur in loco *m*. Haec ita se habebunt ubi corpora *p* &*P* aequaliter secundum lineas *pC* & *PC* moventur, ideoque aequalibus viribus secundum lineas illas urgentur. Capiatur autem angulus *pCn* ad angulum *pCk* ut est angulus *VCp* ad angulum *VCP*, sitque *nC* aequalis *kC*, & corpus *p* completo illo tempore revera reperietur in *n* ; ideoque vi majore urgetur quam corpus *P*, si modo angulus *nCp* angulo *kCp* major est, id est si orbis *upk* vel movetur in consequentia, vel movetur in antecedentia majore celeritate quam sit dupla ejus qua linea *CP* in consequentia fertur; & vi minore si orbis tardius movetur in antecedentia. Estque virium differentia ut locorum intervallum *mn*, per quod corpus illud *p* ipsius actione, dato illo temporis spatio, transferri debet. Centro *C* intervallo *Cn* vel *Ck* describi intelligatur circulus secans lineas *mr*, *mn* productas in *s* & *t*, & erit rectangulum $mn \times mt$ aequale rectangulo $mk \times ms$, ideoque *mn* aequale $\frac{mk \times ms}{mt}$. Cum autem triangula *pCk*, *pCn* dato tempore dentur magnitudine, sunt *kr* & *mr*, earumque differentia *mit* & summa *ms* reciproce ut altitudo *pC*, ideoque rectangulum $mk \times ms$ est reciproce ut quadratum altitudinis *pC*. Est & *mt* directe ut $\frac{1}{2}mt$, id est, ut altitudo *pC*. Hae sunt primae rationes linearum nascentium; & hinc sit $\frac{mk \times ms}{mt}$, id est lineola nascens *mn*, eique proportionalis virium differentia reciproce ut cubus altitudinis *pC*. *Q. E. D.*

Corol. I. Hinc differentia virium in locis *P* & *p*, vel *K* & *k*, est ad vim qua corpus motu circulari revolvi possit ab *R* ad *K* eodem tempore quo corpus *P* in orbe immobili describit .arcum *PK*, ut lineola nascens *mn*, ad sinum versum arcus nascentis *RK*, id est ut $\frac{mk \times ms}{mt}$ ad $\frac{rkq}{2kC}$, vel ut $mk \times ms$ ad *rk* quadratum; hoc est, si capiantur datae quantitates *F*, *G* in ea ratione ad invicem quam habet angulus *VCP* ad angulum *VCp*, ut *GG–FF* ad *FF*. Et propterea, si centro *C* intervallo quovis *CP* vel *Cp* describatur fector circularis aequalis areae toti *VPC*, quam corpus *P* tempore quovis in orbe immobili revolvens radio ad centrum ducto descripsit: differentia virium, quibus corpus *P* in orbe immobili & corpus *p* in orbe mobili revolvuntur, erit ad vim centripetam, qua corpus aliquod, radio ad centrum ducto, sectorem illum eodem tempore, quo descripta sit area *VPC*, uniformiter describere potuisset, Ut *GG–FF* ad *FF*. Namque sector ille & area *pCk* sunt ad invicem ut tempora quibus describuntur.

Corol.2. Si orbis *VPK* ellipsis sit umbilicum habens *C* & apsidem summam *V;* eique similis & aequalis ponatur ellipsis *upk*, ita ut sit semper *p C* aequalis *PC*, & angulus *VCp* sit ad angulum *VCP* in data ratione *G* ad F ; pro altitudine autem *PC* vel *pC* scribatur *A*, & pro ellipseos latere recto ponatur 2*R*: erit vis, qua corpus in ellipsi mobili revolvi potest, ut $\frac{FF}{AA} + \frac{RGG-RFF}{Acub.}$ & contra. Exponatur enim vis qua corpus revolvatur in immota ellipsi per quantitatem $\frac{FF}{AA}$, & vis in *V* erit $\frac{FF}{CAquad.}$. Vis autem qua corpus in circulo ad distantiam *CV* ea cum velocitate revolvi posset quam corpus in ellipsi revolvens habet in *V*, est ad vim qua corpus in ellipsi revolvens urgetur in apside *V*, ut dimidium lateris recti ellipseos ad circuli semidiametrum *CV*, ideoque valet $\frac{RFF}{CVcub.}$: & vis quae sit ad hanc ut *GG–FF* ad *FF*, valet $\frac{RGG-RFF}{CVcub.}$ *:* estque haec vis (per hujus Corol. I.) differentia virium *V* quibus corpus *P* in ellipti immota *VPK*, & corpus *p* in ellipsi mobili *upk* revolvuntur Unde cum (per hanc prop.) differentia illa in alia quavis altitudine *A* sit ad scripsam in altitudine *CV* ut $\frac{1}{Acub.}$ ad $\frac{1}{CVcub.}$, eadem differentia in omni altudine *A* valebit $\frac{RGG-RFF}{CVcub.}$. Igitur ad vim $\frac{FF}{AA}$, qua corpus revolvi potest in ellipsi immobili *VPK,* addatur excessus $\frac{RGG-RFF}{Acub.}$; & componetur vis tota $\frac{FF}{AA} + \frac{RGG-RFF}{Acub.}$ qua corpus in elIipsi mobili *upk* iisdem temporibus revolvi possit.

Corol.3. Ad eundem modum colligetur quod, si orbis immobilis *VPK* ellipsis sit centrum habens in virium centro *C;* eique similis, aequalis & concentrica ponatur ellipsis mobilis *upk;* sitque 2*R* ellipseos hujus latus rectum principale, & 2*T* latus transversum sive axis major, atque angulus *VCp* semper sit ad angulum *VCP* ut *G* ad *F*; vires, quibus corpora in ellipsi immobili & mobili temporibus aequalibus revolvi possunt, erunt ut $\frac{FFA}{Tcub.}$ & $\frac{FFA}{Tcub.}+\frac{RGG-RFF}{Acub.}$ respective.

Corol. 4. Et universaliter, si corporis altitudo maxima *CV* nominetur *T*, & radius curvaturae quam orbis *VPK* habet in *V*, id est radius circuli aequaliter curvi, nominetur *R*, & vis centripeta, qua corpus in trajectoria quascunque immobili *VPK* revolvi potest in loco *V*, dicatur $\frac{VFF}{TT}$, atque aliis in locis *P* indefinite dicatur *X*, altitudine *CP* nominata *A*, & capiatur *G* ad *F* in data ratione anguli *YCp* ad angulum *VCP:* erit vis centripeta, qua corpus idem eosdem motus in eadem trajectoria *upk* circulariter mota temporibus iisdem peragere poteii, ut summa virium $X+\frac{VRGG-VRFF}{Acub.}$.



Corol. 5. Dato igitur motu corporis in orbe quocunque immobili, augeri vel minui potest ejus motus angularis circa centrum virium in ratione data, & inde inveniri novi orbes immobiles in quibus eorpora novis viribus centripetis gyrentur.

Corol. 6. Igitur si ad rectam *CV* positione datam erigatur perpendiculum longitudinis indeterminatae, jungaturque *CP*, & ipsi aequalis agatur *Cp*, constituens angulum *vCp*, qui sit ad angulum *VCP* in data ratione; vis qua corpus gyrari potest in curva illa *Vpk* quam punctum *p* perpetuo tangit, erit reciproce ut cubus altitudinis *Cp.* Nam corpus *P* per vim inertiae, nulla alia vi urgente, uniformiter progredi potest in recta *VP.* Addatur vis in centrum *C*, cubo altitudinis *CP* vel *Cp* reciproce proportionalis, & (per jam demonstrata) detorquebitur motus ille rectilineus in lineam curvam *Vpk.* Est autem haec curva *Vpk* eadem cum curva illa *VPQ* in Corol. 3. Prop. XLI inventa, in qua ibi diximus corpora hujusmodi viribus attracta oblique ascendere.

PROPOSITIO XLV. PROBLEMA XXXI.

Orbium qui sunt circulis maxime finitimi requiruntur motus apsidum.

Problema solvitur arithmetice faciendo ut orbis, quem corpus in ellipsi mobili (ut in propositionis superioris Corol. 2.vel 3.) revolvolvens describit in plano immobili, accedat ad formam orbis cujus apsides requiruntur, & quaerendo apsides orbis quem corpus illud in plano immobili describit. Orbes autem eandem acquirent formam, si vires centripetae quibus describuntur, inter se collatae, in aequalibus altitudinibus reddantur proportionales. Sit punctum *V* apsis summa, & scribantur *T* pro altitudine maxima *CV*, *A* pro altitudine quavis alia *CP* vel *Cp*, & *X* pro altitudinum differentia *CV*– *CP*; & vis, qua corpus in ellipsi circa umbilicum suum *C* (ut in Corol. 2.) revolvente movetur, quaeque in Corol. 2. erat ut $\frac{RGG-RFF}{Acub.}$, id est ut $\frac{FFA+RGG-RFF}{Acub.}$, substituendo $T-X$ pro *A*, erit ut $\frac{RGG-RFF+TFF-FFX}{Acub.}$. Reducenda similiter est vis alia quaevis centripeta ad fractionem cujus denominator sit *Acub.* & numeratores, facta homologorum terminorum collatione, statuendi sunt analogi. Res exemplis patebit.

Exempl. I. Ponamus vim centripetam uniformem esse, ideoque ut $\frac{Acub.}{Acub.}$, sive (scribendo $T-X$ pro *A* in numeratore) ut $\frac{Tcub.-3TTX+3TXX-Xcub.}{Acub.}$; & collatis numeratorum terminis correspondentibus, nimirum datis cum datis & non datiscum non datis, fiet $RGG-RFF+TFF$ ad $Tcub.$ ut $-FFX$ ad $-3TTX+3TXX-Xcub.$ sive ut $-FF$ ad $-3TT+3TX-XX$. Jam cum orbis ponatur circulo quam maxime finitimus, coeat orbis cum circulo & ob factas *R*, *T* aequales, atque *X* in infinitum diminutam, rationes ultimae erunt *RGG* ad *T cub.* ut $-FF$ ad $-3TT$, seu *GG* ad *TT* ut *FF* ad 3*TT*, & vicissim *GG* ad *FF* ut *TT* ad 3*TT*, id est, ut 1 ad 3; ideoque *G* ad *F*, hoc est angulus *VCp* ad angulum *VCP*, ut I ad $\sqrt{3}$. Ergo cum corpus in ellipsi immobili, ab apside summa ad apsidem imam descendendo conficiat angulum *VCP* (ut ita dicam) graduum 180; corpus aliud in ellipsi mobili, atque ideo in orbe immobili de quo agimus, ab apside summa ad apsidem imam descendendo conficiet angulum *VCp* graduum $\frac{180}{\sqrt{3}}$ id ideo ob similitudinem orbis hujus, quem corpus agente uniformi vi centripeta describit, & orbis illius quem corpus in ellipsi revolvente gyros peragens describit in plano quiescente. Per superiorem terminorum collationem similes redduntur hi orbes, non universaliter sed tunc cum ad formam circularem quam maxime appropinquant. Corpus igitur uniformi cum vi centripeta in orbe propemodum circulari revolvens, inter apsidem summam & apsidem imam conficiet semper angulum $\frac{180}{\sqrt{3}}$ graduum, seu 103 *gr. 55 m.* 23 *sec.* ad centrum; perveniens ab apside summa ad apsidem imam ubi semel confecit hunc angulum, & inde ad apsidem summam rediens ubi iterum confecit eundem angulum; & sic deinceps in infinitum.

Exempl. 2. Ponamus vim centripetam esse ut altitudinis *A* dignitas quaelibet A^{n-3} seu $\frac{A^n}{A^3}$ ubi $n-3$ & *n* fignificant dignitatum indices quoscunque integros vel fractos, rationales vel irrationales, affirmativos vel negativos. Numerator ille A^n seu $|T-X|^n$ in seriem indeterminatam per methodum nostram serierum convergentium reducta, evadit $T^n-nXT^{n-1}+\frac{nn-n}{2}XXT^{n-2}$ etc. Et collatis hujus terminis cum terminis numeratoris alterius $RGG-RFF+TFF-FFX$, sit $RGG-RFF+TFF$ ad T^n ut $-FF$ ad $-nXT^{n-1}+\frac{nn-n}{2}XXT^{n-2}$ etc.. Et sumendo rationes ultimas ubi orbes ad formam circularem accedunt, sit *RGG* ad T^n ut $-FF$ ad $-nXT^{n-1}$, seu *GG* ad T^{n-1} ut *FF* ad nT^{n-1}, & vicissim *GG* ad *FF* ut T^{n-1} ad nT^{n-1}; id est ut 1 ad *n;* ideoque *G* ad *F*, id est angulus *VCp* ad angulum *VCP*, ut 1 ad $\sqrt{n}$. Quare cum angulus *VCP*, in descensu corporis ab apside summa ad apsidem imam in ellipsi confectus, sit graduum 180; conficietur angulus *VCp*, in descensu corporis ab apside summa ad apsidem imam, in orbe propemodum circulari quem corpus quodvis vi centripeta dignitati A^{n-3} proportionali describit, aequalis angulo graduum $\frac{180}{\sqrt{n}}$; & hoc angulo repetito corpus redibit ab apside ima ad apsidem summam, & sic deinceps in infinitum. Ut si vis centripeta sit ut distantia corporis a centro, id est, ut *A* seu $\frac{A^4}{A^3}$, erit *n* aequalis 4 & $\sqrt{n}$ aequalis 2; ideoque angulus inter apsidem summam & apsidem imam aequalis $\frac{180}{2}$ seu 90 *gr*. Completa igitur quarta parte revolutionis unius corpus perveniet ad apsidem imam, & completa alia quarta parte ad apsidem summam, & sic deinceps per vices in infinitum. Id quod etiam ex propositione X, manifestum est. Nam corpus urgente hac vi centripeta revolvetur in ellipsi immobili, cujus centrum est in centro virium. Quod si vis centripeta sit reciproce ut distantia, id est directe ut $\frac{1}{A}$ seu $\frac{A^2}{A^3}$, erit *n* aequalis 2, ideoque inter apsidem summam & imam angulus erit graduum $\frac{180}{\sqrt{2}}$ seu I27 *gr*.16*m*. 45 *sec*. & propterea corpus tali vi revolvens, perpetua anguli hujus repetitione, vicibus alternis ab apside summa ad imam & ab ima ad summam perveniet in aeternum. Porro si vis centripeta sit reciproce ut latus quadrato–quadratum undecimae dignitatis altitudinis, id est reciproce ut $A^{\frac{11}{4}}$, ideoque directe ut $\frac{1}{A^{\frac{11}{4}}}$ seu ut $\frac{A^{\frac{1}{4}}}{A^3}$ erit *n* aequalis $\frac{1}{4}$ & $\frac{180}{\sqrt{n}}$ *gr*. aequalis 360 *gr*. & propterea corpus de apside summa discedens & subinde perpetuo descendens, peveniet ad apsidem imam ubi complevit revolutionem integram, dein perpetuo ascensu complendo aliam revolutionem integram, redibit ad apsidem summam: & sic per vices in aeternum.

Exempt. 3. Assumentes *m* & *n* pro quibusvis indicibus dignitatum altitudinis, & *b, c* pro numeris quibusvis datis, ponamus vim centripetam esse ut

$\frac{bA^m+cA^n}{A\,cub.}$, id est ut $\frac{b \text{ in } \overline{T-X}^m+c \text{ in } \overline{T-X}^n}{A\,cub.}$ seu (per eandem methodum nostram serierum eonvergentium) ut

$$\frac{bT^m+cT^n-mbXT^{m-1}-ncXT^{n-1}+\frac{mm-m}{2}bXXT^{m-2}+\frac{nn-n}{2}cXXT^{n-2} \text{ etc.}}{A\,cub.}$$

& collatis numeratorum terminis, fiet $RGG-RFF+TFF$ ad bT^m+cT^n, ut $-FF$ ad $-mbT^{m-1}-ncT^{n-1}+\frac{mm-m}{2}bXT^{m-2}+\frac{nn-n}{2}cXT^{n-2}$ etc. Et sumendo rationes ultimas quae prodeunt ubi orbes ad formam circularem accedunt, sit *GG* ad $bT^{m-1}+cT^{n-1}$,ut *FF* ad $mbT^{m-1}+ncT^{n-1}$, & vicissim *GG* ad *FF* ut $bT^{m-1}+cT^{n-1}$ ad $mbT^{m-1}+ncT^{n-1}$. Quae proportio, exponendo altitudinem maximam *CP* seu *T* arithmetice per unitatem, sit *GG* ad *FF* ut $b+c$ ad $mb+nc$, ideoque ut 1 ad $\frac{mb+nc}{b+c}$. Unde est *G* ad *F*, id est angulus *VCp* ad angulum *VCP*, ut I ad $\sqrt{\frac{mb+nc}{b+c}}$. Et propterea cum angulus *VCP* inter apsidem summam & apsidem imam in ellipsi immobili sit 180 *gr.* erit angulus *VCp* inter easdem apsides, in orbe quem corpus vi centripeta quantitati $\frac{bA^m+cA^n}{A\,cub.}$ proportionali describit, aequalis angulo graduum $180\sqrt{\frac{b+c}{mb+nc}}$. Et eodem argumento si vis centripeta sit ut $\frac{bA^m-cA^n}{A\,cub.}$ angulus inter apsides invenietur graduum $180\sqrt{\frac{b-c}{mb-nc}}$. Nec secus resolvetur problema in casibus difficilioribus. Quantitas, cui vis centripeta proportionalis est, resolvi semper debet in series convergentes denominatorem habentes A *cub.* Dein pars data numeratoris qui ex illa operatione provenit ad ipsius partem alteram non datam, & pars data numeratoris hujus $RGG-RFF+TFF-FFX$ ad ipsius partern alteram non datam in eadem ratione ponendae sunt: Et quantitates superfluas delendo, scribendoque unitatem pro *T*, obtinebitur proportio *G* ad *F*.

Corol. I. Hinc si vis centripeta sit ut aliqua altitudinis dignitas, inveniri potest dignitas illa ex motu apsidum; & contra. Nimirum si motus totus angularis, quo corpus redit ad apsidem eandem, sit ad motum angularem revolutionis unius, seu graduum 360, ut numerus aliquis *m* ad numerum alium *n*, & altitudo nominetur *A* : erit vis ut altitudinis dignitas illa $A^{\frac{n^2}{m^2}-3}$, cujus index est $\frac{n^2}{m^2}-3$. Id quod per exempla secunda manifestum est. Unde liquet vim illam in majore quam triplicata altitudinis ratione, in recessu a centro, decrescere non posse: Corpus tali vi revolvens deque apside discedens, si coeperit descendere nunquam perveniet ad apsidem imam seu altitudinem minimam, sed descendet usque ad centrum, describens curvam illam lineam de qua egimus in Corol. 3. Prop. XlI. Sin coeperit illud, de apside discedens, vel minimum ascendere; ascendet in infinitum, neque unquam perveniet ad apsidem summam. Describet enim curvam illam lineam de qua actum est in eadem Corol. & in Corol. VI. Prop. XLIV. Sic & ubi vis, in recessu a centro, decrescit in majore quam triplicata ratione altitudinis, corpus de apside discedens. perinde ut coeperit descendere vel ascendere, vel descendet ad centrum usque vel ascendet in infinitum. At si vis, in recessu a centro, vel decrescat in minore quam triplicata ratione altitudinis, vel crescat in altitudinis ratione quacunque ; corpus nunquam descendet ad centrum usque, sed ad apsidem imam aliquando perveniet: & contra, si corpus de apside ad apsidem alternis vicibus descendens & ascendens nunquam appellat ad centrum; vis in recessu a centro aut augebitur, aut in minore quam triplicata altitudinis ratione decrescet: & quo citius corpus de apside ad apsidem redierit, eo longius ratio virium recedet a ratione illa triplicata. Ut si corpus revolutionibus 8 vel 4 vel 2 vel $1\frac{1}{2}$ de apside summa ad apsidem summam alterno descensu & ascensu redierit; hoc est, si fuerit *m* ad *n* ut 8 vel 4 vel 2 vel $1\frac{1}{2}$ ad 1, ideoque $\frac{nn}{mm}-3$ valeat $\frac{1}{64}-3$; vel $\frac{1}{16}-3$ vel $\frac{1}{4}-3$ vel $\frac{4}{9}-3$: erit vis ut $A^{\frac{1}{64}-3}$ vel $A^{\frac{1}{16}-3}$ vel $A^{\frac{1}{4}-3}$ vel $A^{\frac{4}{9}-3}$, id est, reciproce ut

$A^{\frac{1}{64}-3}$ vel $A^{3-\frac{1}{16}}$ vel $A^{3-\frac{1}{4}}$ vel $A^{3-\frac{4}{9}}$,. Si corpus singulis revolutionibus redierit ad apsidem eandem immotam; erit *m* ad *n* ut 1 ad 1, ideoque $A^{\frac{nn}{mm}-3}$ aequalis A^{-2} seu $\frac{1}{AA}$; & propterea decrementum virium in ratione duplicata altitudinis, ut in praecedentibus demonstratum est. Si corpus partibus revolutionis unius vel tribus quartis, vel duabus tertiis, vel una tertia, vel una quarta, ad apsidem eandem redierit; erit *m* ad *n* ut $\frac{1}{4}$ vel $\frac{2}{3}$ vel $\frac{1}{3}$ vel $\frac{1}{4}$ ad 1, ideoque $A^{\frac{nn}{mm}-3}$ aequalis $A^{\frac{16}{9}-3}$ vel $A^{\frac{9}{4}-3}$ vel A^{9-3} vel A^{16-3} ; & propterea vis aut reciproce ut $A^{\frac{11}{9}}$ vel $A^{\frac{3}{4}}$ aut directe ut A^{6} vel A^{13}. Denique si corpus pergendo ab apside summa ad apsidem summam confecerit revolutionem integram, & praeterea gradus tres, ideoque apsis illa singulis corporis revolutionibus confecerit in consequentia gradus tres; erit *m* ad *n* ut 363 *gr.* ad 360 *gr.* sive ut 121 ad I20, ideoque $A^{\frac{nn}{mm}-3}$ erit aequale $A^{-\frac{29523}{14641}}$; & propterea vis centripeta reciproce ut $A^{\frac{29523}{14641}}$ seu reciproce ut $A^{2\frac{4}{243}}$ proxime. Decrescit igitur vis centripeta in ratione paulo majore quam duplicata, sed que vicibus $59\frac{3}{4}$ propius ad duplicatam quam ad triplicatam accedit.

Corol. 2. Hinc etiam si corpus, vi centripeta quae sit reciproce ut quadratum altitudinis, revolvatur in ellipsi umbilicum habente in centro virium, & huic vi centripetae addatur vel auferatur vis alia quaevis extranea; cognosci potest (per exempIa tertia) motus apsidum qui ex vi illa extranea orietur: & contra. Ut si vis qua corpus revolvitur in ellipli sit ut $\frac{1}{AA}$ & vis extranea ablata ut *c*A, ideoque vis reliqua ut $\frac{A-cA^{4}}{A\,cub.}$; erit (in exempla tertia) *b* aequalis 1, *m* aequalis 1, & *n* aequalis 4, ideoque angulus revolutionis inter apsides aequalis angulo graduum $180\sqrt{\frac{1-c}{1-4c}}$. Ponamus vim illam extraneam esse 357.45 partibus minorem quam vis altera qua corpus revolvitur in ellipsi, id est *c* esse $\frac{100}{35745}$, existente *A* vel *T* equali 1, &$180\sqrt{\frac{1-c}{1-4c}}$ evadet $180\sqrt{\frac{35645}{35345}}$, seu 180.7623, id est, *180gr.* 45*m.* 44*sec.* Igitur corpus de apside summa discedens, motu angulari 180 *gr.* 45*m.* 44*s* perveniet ad apsidem imam, & hoc motu duplicato ad apsidem summam redibit : ideoque apsis summa singulis revolutionibus progrediendo conficiet I *gr.* 31 *m.* 28 *sec.* Apsis lunae est duplo velocior circiter.

Hactenus de motu corporum in orbibus quorum plana per centrum virium transeunt. Superest ut motus etiam determinemus in planis excentricis. Nam scriptores qui motum gravium tractant, considerare solent ascensus & descensus ponderum, tam obliquos in planis quibuscunque datis, quam perpendiculares: & pari jure motus corporum viribus quibuscunque centra petentium, & planis excentricis innitentium hic considerandus venit. Plana autem supponimus esse politissima & absolute lubrica ne corpora retardent. Quinimo, in his demonsrationibus, vice planorum quibus corpora incumbunt quaeque tangunt incumbendo, usurpamus plana his parallela, in quibus centra corporum moventur & orbitas movendo describunt. Et eadem lege motus corporum in superficiebus curvis peractos subinde determinamus.