SECTION VIII.

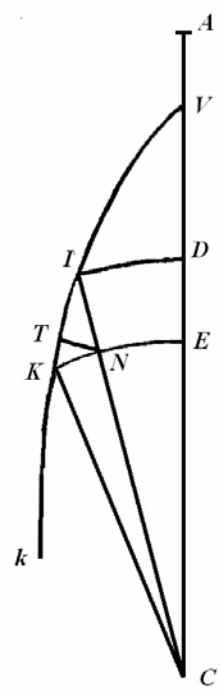
Concerning the finding of orbits in which rotating bodies are acted on by any centripetal forces.

PROPOSITION XL. THEOREM. XIII.

If a body may be moving in some manner, under the action of some centripetal force, and another body may ascend or descend, and the velocities of these in some case are equal at some altitude, then the velocities of these shall be equal at all equal altitudes.

[Although Newton of course does not use these exact words, this Proposition embodies the genesis of the idea of equipotential surfaces surrounding the source of a force field of some kind; in the diagram *DE* and *EK* are parts of such surfaces in the case of gravity, and a body falling or orbiting in some manner acquires an acceleration in moving from one surface to another, separated normally from it by an infinitesimal distance, and thus a change in velocity is produced; thus, the distances considered which Newton calls minimal are our infinitesimals, and likewise with the times involved. On the other hand, an orbiting body as well as a dropped body have to conserve the equal areas in equal times law, which amounts to the conservation of angular momentum, zero in the second case; all of these matters are attended to here. The same arguments are true of course for bodies projected upwards.]

Some body may descend from *A* through *D* and *E* to the centre *C*, and another body may be moving from *V* on a curve *VIKk*. With the centre *C*, for some radii the concentric circles *DI* and *EK* may be described meeting with the right line *AC* in *D* and *E*, and with the curve *VIK* in *I* and *K*. *IC* may be joined crossing *KE* at *N* itself*;* and onto *IK* there may be send the perpendicular *NT;* and the separation *DE* or *IN* of the circumferences of the circles shall be as a minimum, and the bodies at *D* and *I* may have equal velocities. Because the distances *CD* and *CI* are equal, the centripetal forces at *D* and *I* are equal. These forces may be expressed by the equal line elements *DE* and *IN;* and if the one force *IN* may be resolved into the two forces *NT* and *IT* (by Corol. 2 of the Laws.)*;* the force *NT,* by acting along the line *NT* perpendicular to the course *ITK* of the body, will not change the velocity of the body in that course, but will only draw the body away from moving in straight line, and it will always act to deflect the body from the tangent of the orbit itself, and the body to be progressing along the way of the curved line *ITKk*. That force will be completely used up in producing this effect: but the other force *IT,* by acting along the course of the body, the whole force will accelerate that body, and in the given time taken as the minimum possible will generate an acceleration proportional to that time itself. Hence the accelerations of the bodies at *D* and *I*



made in equal times (if the first ratios of the line elements arising are taken *DE, IN, IK, IT, NT*) become as the lines *DE* and *IT*: but with unequal times as these lengths and times jointly. But the times in which *DE* & *IK* are described, on account of the equality of the velocities are as the paths described *DE* and *IK*, and thus the accelerations, in the described paths along the lines *DE* & *IK*, are as *DE* and *IT*, *DE* and *IK* jointly, that is as DE^2 and the *rectangle* $IT \times IK$.

[If we let *v* be the common velocity at *D* and *I* initially, the first straight down, and the second along the curve, then after an increment in time $\Delta t_{DE} = \frac{DE}{v}$, the first body has acquired an extra velocity $g\Delta t_{DE} = g\frac{DE}{v}\ \alpha\ DE^2$; as the acceleration *g* (or force per unit mass) is expressed by the line *DE* :
(A lingering source of confusion in force diagrams at this time was that a line segment could represent a displacement, a velocity, a force, etc; only vector notation eventually removed some of this confusion.);
hence in this case, what we may call now the impulse or the change in velocity, $F\Delta t$, is proportional to DE^2.
For the other body on the curve, it has to travel a longer increment *IK*, but with the same initial velocity *v* as *D* (from the hypothesis) ; hence in this case $\Delta t_{IK} = \frac{IK}{v}$; in addition, the actual component of the force along the curve is diminished by the cosine factor $\frac{IT}{IN}$; or the impulse above, to which this must be equal, becomes

$$g\Delta t_{IK} \times \frac{IT}{IN} = IN \times \frac{IK}{v} \times \frac{IT}{IN}\ \alpha\ IK \times IT\ .$$

Chandrasekhar shows this on p. 167 ; he also talks about energy conservation, and it is quite wrong to do so, as there is no hint of conservation laws in the *Principia,* at least up to this point, apart from Kepler's 2nd Law, which Newton uses implicitly without naming it as such.]

But the *rectangle* $IT \times IK$ is equal to IN^2, that is, equal to DE^2 [from the similar triangles *ITN* and *IKN*] and therefore the accelerations in the transition of the bodies from *D* and *I* to *E* and *K* are produced equal. Therefore the velocities of the bodies are equal at *E* and *K* : and will always be found equal by the same argument in the subsequent equal distances. *Q.E.D.*

In addition by the same argument bodies both equidistant from the centre and with the same velocity, in ascending to equal distances, are equally retarded. *Q.E.D.*

Cor. 1. Hence if a body may be oscillating hanging from a thread, or restrained to be moving on some impediment perfectly lubricated and without friction, and another body may ascend or descent directly, the velocities of these at the same height shall be equal : the velocities of these will be equal to any others at all heights. For that same transverse force *NT* shall be presented either by the thread or the impediment of the completely slippery vessel. The body may not be retarded nor accelerated by that, but only constrained by the curve to cease being linear.

Cor. 2. Hence also if the quantity *P* shall be the maxima distance from the centre to which the body either oscillating or rotating in some trajectory may ascend, from some lower point of the trajectory, as here it may be able to rise projected upwards with a velocity ; and let the quantity *A* be the distance of the body from the centre at some other point of the orbit, and the centripetal force always shall be as some power A^{n-1} of *A*, the index $n-1$ of which is any number *n* diminished by one ; the velocity of the body at any height *A* will be as $\sqrt{P^n - A^n}$, and therefore is given. For the velocity of the body ascending and descending on a straight line (by Prop. XXXIX) is in this ratio itself.

PROPOSITION XLI. PROBLEM XXVIII.

With any kind of centripetal force in place, and the quadratures of the figures granted, then both the trajectories in which bodies are moving are required, as well as the times of the motions found in the trajectories.

Any force may be drawing [a body] towards the centre *C* and the trajectory *VIKk* shall be required to be found. The circle *VR* with centre *C* may be given described with some radius *CV*, and from the same centre some other circle [arcs] *ID* and *KE* are described cutting the trajectory in *I* and *K* and the right line C*V* in *D* and *E.* Then draw the right line *CNIX* cutting the circles *KE* and *VR* in *N* and *X,* and also the right line *CKY* meeting with the circle *VR* in *Y*. Moreover let the points *I* and *K* themselves in turn be the closest possible together, and the body may go from *V* by *I* and *K* to *k;* and the point *A* shall be that place from which the other body must fall, so that at the place *D* it will acquire a velocity equal to the velocity of the first body at *I.* And with the matters in place from Proposition XXXIX, the line element *IK,* as given described in the shortest time, will be as the velocity
[In modern terms we may write this velocity using polar coordinates (r,φ) corresponding to *CN* and the angle *NCK*, as $ds(r,\varphi) = \left(dr^2 + r^2 d\varphi^2\right)^{\frac{1}{2}}$ and $v(r,\varphi) = \left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{\frac{1}{2}}$],

and thus as the right line which can become [on integration] the area *ABFD,*

[Thus, the area $ABFD = \int vds$; and which now we call the energy integral, corresponding to the area is : $\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} = -\int F(r)dr$, where *F*(*r*) is the attracting force on the body. This equation arises from the force equation : $\ddot{r} - r\dot{\varphi}^2 = -F(r) + \frac{h^2}{r^3}$, since $r^2\dot{\varphi} = h$, the angular momentum equation, which Newton understood to be the 'equal area in equal times' law (of Kepler.) Thus, the radial velocity below becomes

$$\dot{r}=\left[-2\int F(r)dr-\tfrac{h^2}{r^2}\right]^{\frac{1}{2}}=\sqrt{\left(ABFD-ZZ\right)}\ .]$$

and the triangle *ICK* will be given proportional to the time, and thus *KN* will be inversely as the altitude *IC*, that is, if some [constant] quantity *Q* may be given, and the altitude *IC* may be called *A*, [not to be confused with the vertex *A*, then the length *KN* varies] as $\frac{Q}{A}$. Hence we may use the name *Z* for the quantity $\frac{Q}{A}$
[*i.e*. a length proportional to *KN*, normal to the radius at that point, so that $Z\times IC$ is proportional to the rate of change of area ; this quantity Z, which is just $\frac{h}{r}$, was introduced by Halley in editing the work, to ease the typesetting, according to Whiteside p.347 of Vol. VI],

and we may put the magnitude of *Q* to be that such that in some case there shall be $\sqrt{ABFD}$ to Z, as *IK* is to *KN*, and in every case there will be $\sqrt{ABFD}$ to *Z* as *IK* to *KN*, and *ABFD* to ZZ as IK^2 to KN^2, and separately *ABFD* –*ZZ* to *ZZ* as IN^2 to KN^2 [*i.e.* $\frac{ABFD-ZZ}{ZZ}=\frac{IK^2-KN^2}{KN^2}=\frac{IN^2}{KN^2}$.]

[In modern notation this becomes $\frac{ABFD-\frac{h^2}{r^2}}{\frac{h^2}{r^2}}=\left(\frac{dr}{rd\varphi}\right)^2=\left(\frac{v_r}{v_\varphi}\right)^2=\left(\frac{IN}{KN}\right)^2$.]

and thus $\sqrt{ABFD-ZZ}$ is to *Z* (or $\frac{Q}{A}$) is as *IN* to *KN*, and therefore $A\times KN$ equals $\frac{Q\times IN}{\sqrt{ABFD-ZZ}}$. From which since $YX\times XC$ shall be to $A\times KN$ as CX^2 to *AA*, $XY\times XC$ the rectangle will equal $\frac{Q\times IN\times CX^2}{AA\sqrt{ABFD-ZZ}}$.

$$[i.e.\ \ r^2d\varphi=\frac{Q\times IN\times CX^2}{AA\sqrt{ABFD-ZZ}}=\frac{h\times dr\times CX^2}{r^2\frac{dr}{dt}}=\frac{h\times dt\times CX^2}{r^2}=hdt.],$$

Therefore if *Db* and *Dc* may be taken on the perpendicular *DF* always equal respectively to $\frac{Q}{2\sqrt{ABFD-ZZ}}$ and $\frac{Q\times CX^2}{2AA\sqrt{ABFD-ZZ}}$,
[These are in turn $\frac{Q}{2\sqrt{ABFD-ZZ}}=\frac{h}{2\dot{r}}=\frac{hdt}{2dr}$ and $\frac{Q\times CX^2}{2AA\sqrt{ABFD-ZZ}}=\frac{h\times CX^2}{2r^2v_r}=\frac{r^2\dot{\varphi}\times CX^2}{2r^2\dot{r}}=\frac{CX^2}{2}\frac{d\varphi}{dr}$.
The first quadrature gives an area proportional to the time of descent, while the second gives an area proportional to the sector angle, and thus locates the position of the body on the curve.]

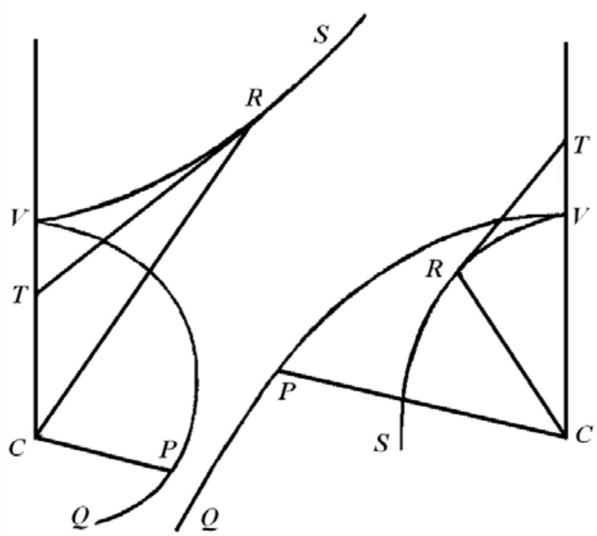
and the curved lines may be described *ab* and *ac* which always touch the points *b* and *c*; and from the point *V* to the line *AC* the perpendicular *Va* may be drawn cutting the curved areas *VDba* and *VDca*, and also the ordinates may be erected *Ez*, *Ex:* because the rectangle $Db\times IN$ or *DbzE* is equal to half of the rectangle $A\times KN$, or to the triangle *ICK;* and the rectangle $Dc\times IN$ or *DcxE* is equal to half of the rectangle $TX\times XC$ or to the triangle *XCT;* that is, since the small parts *DbzE* and *ICK* always equal the small parts arising of the areas *VDba* and *VIC*, and *DcxE* and *XCY* are always equal to the small parts arising of the areas *Dca* and *VCX*, the area generated *VDba* will always be an area equal to the area *VIC*, and thus proportional to the time, and the area produced *VDca* equal to the area generated *VCX*. Therefore with some time given from which the body has departed from the place *V*, the area will be given itself proportional to *VDba*, and thence the height of the body will be given *CD* or *CI;* and the area *BDca*, and for that the equal *VCX* together with the angle of this *VCI*. But with the angle *VCI* and the height *CI* given, the place *I* may be given, in which the body will be found in that time completed. *Q. E. I.*

[This would have been a very difficult proposition for those of Newton's contemporaries to follow, who were not conversant with Newton's calculus, and even now it is a little difficult in the Latin until one knows what is going on; Chandrasekhar sets this out in more detail on p. 170, where it becomes quite straight forwards as the double integration derived from the original differential equation, while Whiteside provides a similar historical enlightenment on p.347 of Vol. VI . Chandrasekhar admits to solving the problems himself, those he tackles, and then relating his solution to that of Newton, which has raised questions of anachronism, but these can be taken in one's stride; while Whiteside digs deep into Newton's methods from a historical standpoint; clearly the latter is more satisfactory, though the former has a lot to recommend it from the immediate nature of the solutions provided for the common reader; in note (209) Whiteside sets out the integrals for the angle and time of the orbiting body under a general force law; unfortunately for Newton, his solution was poorly received or even understood by his contemporaries, who went to great lengths subsequently to prove the same results using Leibniz's notation, which later caused Newton much unhappiness : see Whiteside's notes for further details on this.]

Corol. I. Hence the maximum and minimum heights of bodies, that is, the apses of trajectories may be found conveniently. For the apses are the points in that trajectory in which the right line *IC* drawn through the centre falls perpendicularly on the trajectory *VIK* : because that will be where the right lines *1K* and *NK* are equal [i.e. $\dot{r} = 0$], and thus the area *ABFD* is equal to *ZZ.*

Corol. 2. But also the angle *KIN*, in which the trajectory cuts that line *IC* somewhere, may be found conveniently from the height *IC* of the body; without doubt by taking the sine of this to the radius as *KN* to *IK*, that is, as *Z* to the square root of the area *ABFD.*

Corol. 3. If from the centre *C* and from the principal vertex *V* some conic section *VRS* may be described, and from some point *R* of this the tangent *RT* may be drawn meeting the axis *CV* produced indefinitely at the point *T;* then with *CR* joined there may be drawn the right line *CP*, which shall be equal to the abscissa *CT*, and the angle *VCP* is put in place proportional to the sector *VCR* may be put in place ; moreover a centripetal force may draw towards the centre *C* inversely proportional to the cube of the distance of the place [of the body], and the body will emerge from the place *V* with the just velocity along the perpendicular right line *CV* : that body may be progressing in the trajectory *VPQ* which the point *P* always touches ; and thus if the conic section *VRS* shall be a hyperbola, the same may fall to the centre. If that conic were an ellipse, that body will always ascend and depart to infinity. And conversely, if some body may emerge from the place *V* with a velocity, and likewise so that it has began either to descend obliquely to the centre, or to ascend obliquely from that, the figure *VRS* shall be either a hyperbola or an ellipse, the trajectory can be found either by increasing or diminishing the angle *VCP* in some given ratio. But also, with the centripetal force changed into a centrifugal force, the body will ascend obliquely in the trajectory *VRC* which is found by taking the angle *VCP* proportional to the elliptic sector *VRC*, and the length *CP* equal to the length *CT* as above. All these follow from the preceding proposition, by the quadrature of some curve, the discovery of which, as that may be done readily enough, I omit for the sake of brevity.

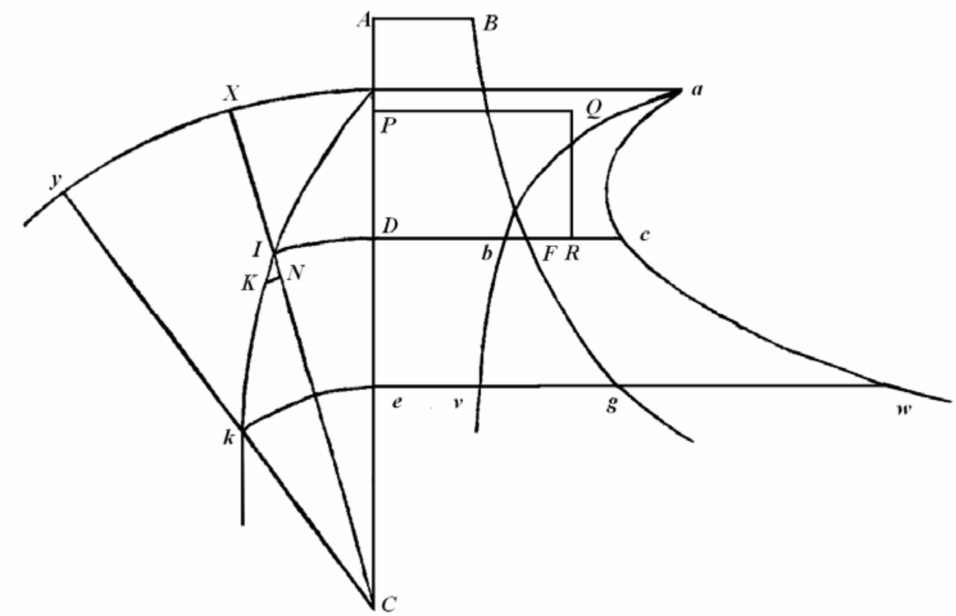


[Chandrasekhar, circa p.180, is not satisfied with this corollary , as it appears to contain uncorrected misprints, which do not entirely agree with his own derivations, and the diagrams have been altered in the third edition from the first two editions. Whiteside goes to some length to try to remove the confusion. Clearly this is a point in *Principia* still in need of complete final elucidation.]

PROPOSITION XLLI. PROBLEM XXIX.

With the law of the centripetal force given, the motion of the body from a given place is required with a given velocity, arising along a given right line.

With the matters remaining in the three preceding propositions : the body may arise from the place *I* along the line element *IK*, with that velocity which another body may acquire at *D* by falling from the place *P*, acted on by some uniform centripetal force : and this uniform force shall be to the first force by which the first body is acted on at *I*, as *DR* to *DF*. But the body may go towards *k* ; and with centre *C* and with the radius *Ck* it may describe the circle *ke,* meeting the right line *PD* in *e,* and there may be erected the ordinates *eg, ev, ew* of the curves *BFg, abv, acw.* From the given rectangle *PDRQ,* and with the given law of the centripetal force by which the first body is disturbed, the curved line *BFg* is given, by the construction of problem XXVII, and Corol. I of this. Then from the given angle *CIK* there is given



the arising proportion *IK, KN* is given, and thence, by the given construction of Prob. XXVIII. the quantity *Q* is given, together with the curved lines *abv* and *acw:* and thus, in some completed time *Dbve,* both the height of the body *Ce* or *Ck* is given, together with the the area *Dcwe,* and the sector *XCy* is equal to that, and the angle *ICk,* and the place *k* at which the body now will be moving. *Q. E.I.*

Moreover we may suppose the centripetal force in these propositions in receding from the centre to be varied in some manner according to some law, as with which able to be imagined at equal distances from the centre to be the same on all sides [*i.e.* symmetrical]. And up to this stage we have considered the motion of bodies in immoveable orbits. So that there remains a few things we may add concerning the motion of bodies in orbits, which are rotating around a centre of force.

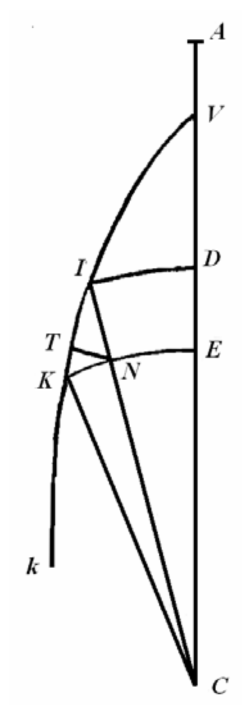
[This is the complete problem, given with the starting conditions of the body. The above integrations, which were indefinite, are now made definite.]

SECTIO VIII.

De inventione orbium in quibus corpora viribus quibuscunque centripetis agitata revolvuntur.

PROPOSITIO XL. THEOREMA. XIII.

Si corpus, cogente vi quacunque centripeta, moveatur utcunque, & corpus aliud recta ascendat vel descendat, sintque eorum velocitates in aliquo aequalium altitudinum casu aequales, velocitates eorum in omnibus aequalibus altitudinibus erunt aequales.



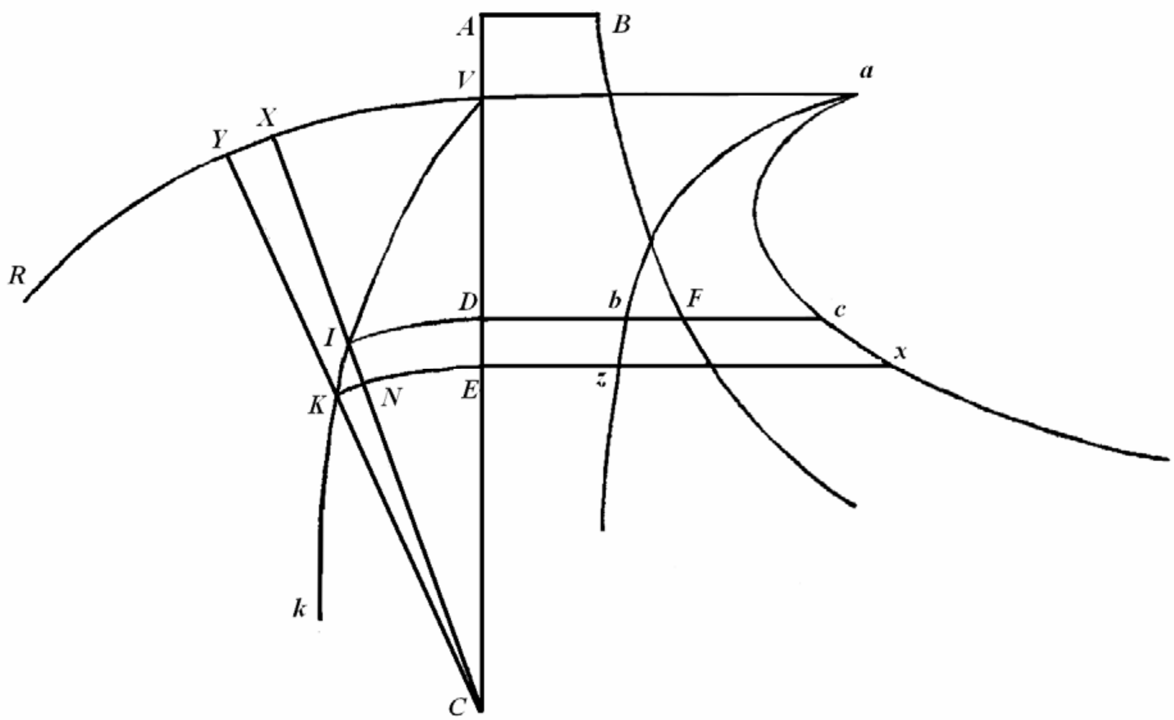
Descendat corpus aliquod ab *A* per *D*, *E,* ad centrum *C, &* moveatur corpus aliud a *V* in linea curva *VIKk.* Centro *C* intervallis quibusvis describantur circuli concentrici *DI, EK* rectae *AC* in *D* & *E,* curvaeque *VIK* in *I & K* occurrentes. Jungatur *IC* occurrens ipsi *KE* in *N;* & in *IK* demittatur perpendiculum *NT;* sitque circumferentiarum circulorum intervallum *DE* vel *IN* quam minimum, & habeant corpora in *D* & *I* velocitates aequales. Quoniam distantiae *CD, CI* aequantur, erunt vires centripetae in *D* & *I* aequales. Exponantur hae vires per aequales lineolas *DE, IN;* & si vis una *IN* (per Legum Corol. 2.) resolvatur in duas N*T & IT;* vis *NT,* agendo secundum lineam *NT* corporis cursui *ITK* perpendicularem, nil mutabit velocitatem corporis in cursu illo, sed retrahet solummodo corpus a cursu rectilineo, facietque ipsum de orbis tangente perpetuo deflectere, inque via curvilinea *ITKk* progredi. In hoc effectu producendo vis ilia tota consumetur: vis autem altera *IT,* secundum corporis cursum agendo, tota accelerabit illud, ae dato tempore quam minima accelerationem generabit sibi ipsi proportionalem. Proinde corporum in *D* & *I* accelerationes aequalibus tcmporibus factae (si sumantur linearum nascentium *DE, IN, IK*, *IT*, *NT* rationes primae) fiunt ut linae *DE*, *IT*: temporibus autem inaequalibus ut lineae illae & tempora conjunctim. Tempora autem quibus *DE* & *IK* describuntur, ob aequalitatem velocitatum sunt ut viae descriptae *DE* & *I K*, ideoque accelerationes, in cursu corporum per lineas *DE* & *IK*, sunt ut *DE* & *IT*, *DE* & *IK* conjunctim, id est ut *DE quad.* & $IT \times IK$ rectangulum. Sed rectangulum $IT \times IK$ aequale est *IN* quadrato, hoc est, aequale *DE quad.* & propterea accelerationes in transitu corporum a *D* & *I* ad *E* & *K* aequales generantur. Aequales igitur sunt corporum velocitates in *E* & *K* :& eodem argumento semper reperientur aequales in subsequentibus aequalibus distantiis. *Q.E.D.*

Sed &. eodem argumento corpora aequivelocia & aequallter a centro distantia, in ascensu ad aequales distantias aequaliter retardabuntur. *Q.E.D.*
Cor. 1. Hinc si corpus vel oscilletur pendens a fila, vel impedimento quovis politissimo & perfecte lubrico cogatur in linea curva moveri, & corpus aliud recta ascendat vel descendat, sintque velocitates eorum in eadem quacunque altitudine aequales: erunt velocitates eorum in allis quibuscunque aequalibus altitudinibus aequales. Namque corporis penduli filo vel impedimento vasis absolute lubrici idem praestatur quod vi transversa *NT.* Corpus eo non retardatur, non acceleratur, sed tantum cogitur de cursu rectilineo discedere.
Cor. 2. Hinc etiam si quantitas *P* sit maxima a centro distantia ad quam corpus vel oscillans vel in trajectoria quacunque revolvens, deque quovis trajectoriae puncto, ea quam ibi habet velocitate sursum projectum ascendere possit; sitque quantitas *A* distantia corporis a centro in allo quovis orbitae puncto, & vis centripeta semper sit ut ipsius *A* dignitas quaelibet A^{n-1}, cujus index $n-1$est numerus quilibet *n* unitate diminutus; velocitas corporis in omni altitudine *A* erit ut $\sqrt{P^n - A^n}$, atque ideo datur. Namque velocitas recta ascendentis ac descendentis (per prop. XXXIX) est in hac ipsa ratione.

PROPOSITIO XLI. PROBLEMA XXVIII.

Posita cuiuscunque generis vi centripeta & concessis figurarum curvilinearum quadraturis, requiruntur tum trajectoriae in quibus corpora movebuntur, tum tempora motuum in trajectoriis inventis.

Tendat vis quaelibet ad centrum *C* & invenienda sit trajectoria *VIKk.* Detur circulus *VR* centro *C* intervallo quovis *CV* descriptus, centroque eodem describantur alii quivis circuli *ID, KE* trajectoriam secantes in *I* & *K* rectamque C*V* in *D* & *E.* Age tum rectam *CNIX* secantem circulos *KE, VR* in *N* & *X,* tum rectam *CKY* occurrentem circulo *VR* in *Y*. Sint autem puncta *I* & *K* sibi invicem vicinissima, & pergat corpus ab *V* per *I* & *K* bet, ut in loco D velocitatem acquirat aequalem velocitati corporis prioris in *I.* Et stantibus quae in Propositione XXXIX, lineola *IK*, dato tempore quam minimo descripta, erit ut velocitas, atque ideo ut recta quae potest aream *ABFD,* & triangulum *ICK* tempori proportionale dabitur, ideoque *KN* erit reciproce ut altitudo *IC,* id est, si detur quantitas aliqua *Q*, & altitudo *IC* nominetur *A*, ut $\frac{Q}{A}$. Hanc quantitatem *A* nominemus Z, & ponamus eam esse magnitudinem ipsius *Q* ut sit in aliquo casu $\sqrt{ABFD}$ ad Z ut eft *IK* ad *KN*, & erit in omni casu $\sqrt{ABFD}$ ad *Z* ut *IK* ad *KN, &*

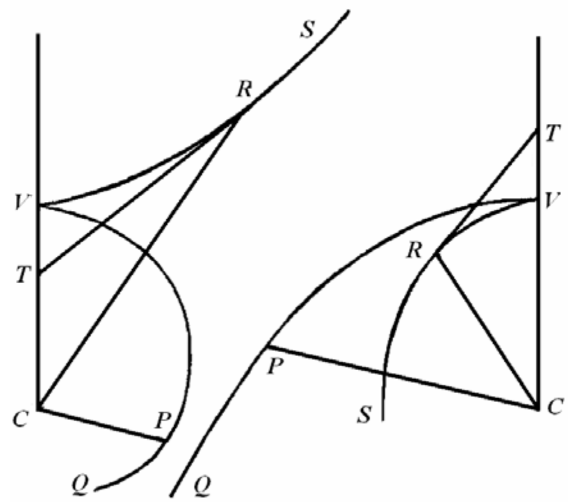


ad *ZZ* ut *IKq* ad *KNq*, & divisim *ABFD-ZZ* ad ut *IN quad.* ad *KN quad.* ideoque $\sqrt{ABFD-ZZ}$ ad *Z* seu $\frac{Q}{A}$ ut *IN* ad *KN*, & propterea $A\times KN$ aequale $\frac{Q\times IN}{\sqrt{ABFD-ZZ}}$. Unde cum $YX\times XC$ sit ad $A\times KN$ ut *CX q* ad *AA*, erit rectanguum $XY\times XC$ aequale $\frac{Q\times IN\times CX\ quad.}{AA\sqrt{ABFD-ZZ}}$. Igitur si in perpendiculo *DF* capiantur semper *Db*, *Dc* ipsis $\frac{Q}{2\sqrt{ABFD-ZZ}}$, $\frac{Q\times CX\ quad.}{2AA\sqrt{ABFD-ZZ}}$ aequales respective, & describantur curvae lineae *ab, ac* quas puncta *b, c* perpetuo tangunt; deque puncto *V* ad lineam *AC* erigatur perpendiculum *Va* abscindens areas curvilineas *VDba, VDca,* & erigantur etiam ordinatae *Ez, Ex:* quoniam rectangulum $Db\times IN$ seu *DbzE* aequale en dimidio rectanguli $A\times KN$ seu triangulo *ICK;* & rectangulum $Dc\times IN$ seu *DcxE* aequale est dimidio rectanguli $TX\times XC$ seu triangulo *XCT;* hoc est, quoniam arearum *VDba, VI*C aequales semper sunt nascentes particulae *DbzE, ICK,* & arearum *VDca, VCX* aequales semper sunt nascentes particulae *DcxE, XCY.* erit area genita *VDba* aequalis areae genitae *VIC,* ideoque tempori proportionalis, & area genita *VDca* aequalis sectori genito *VCX.* Dato igitur tempore quovis ex quo corpus discessit de loco *V,* dabitur area ipsi proportionalis *VDba,* & inde dabitur corporis altitudo *CD* vel *CI;* & area *BDca,* eique aequalis sector *VCX* una cum ejus angulo *VCI.* Datis autem angulo *VCI &* altitudine *CI* datur locus *I,* in quo corpus completo illo tempore reperietur. *Q. E. I.*

Corol. I. Hinc maximae minimaeque corporum altitudines, id est, apsides trajectoriarum expedite inveniri possunt. Sunt enim apsides puncta illa in quibus recta *IC* per centrum ducta incidit perpendiculariter in trajectoriam *VIK* : id quod sit ubi rectae *1K* & *NK* aequantur, ideoque ubi area *ABFD* aequalis est *ZZ*.

Corol. 2. Sed & angulus *KIN*, in quo trajectoria alicubi secat lineam illam *IC*, ex data corporis altitudine *IC* expedite invenitur; nimirum capiendo sinum eius ad radium ut *KN* ad *IK*, id est, ut *Z* ad latus quadratum areae *ABFD*.

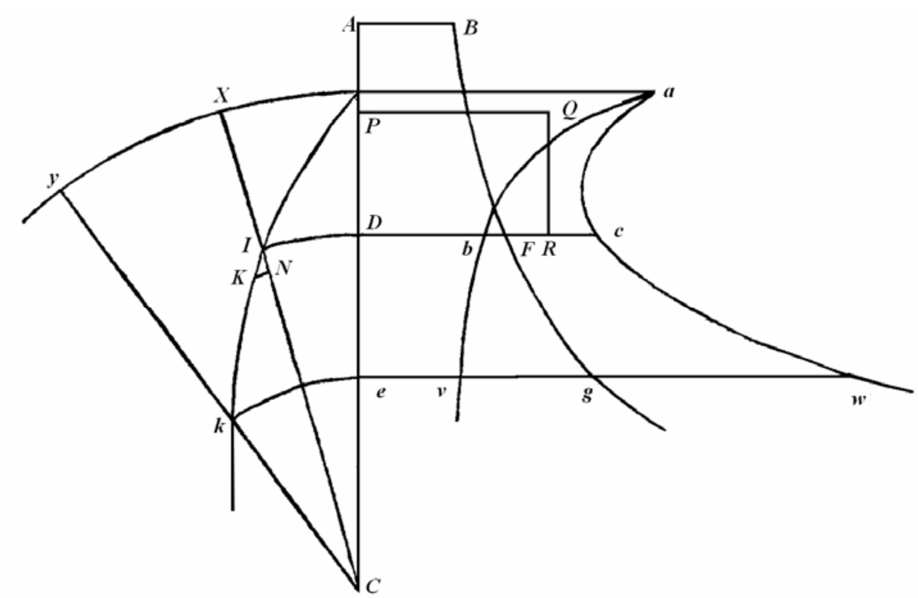
Corol. 3. Si centro *C* & vertice principali *V* describatur sectio quaelibet conica *VRS*, & a quovis ejus puncto *R* agatur tangens *RT* occurrens axi infinitae producto *CV* in puncto *T;* dein juncta *CR* ducatur recta *CP*, quae aequalis sit abscissae *CT*, angulumque *VCP* sectori *VCR* proportionalem constituat; tendat autem ad centrum *C* vis centripeta cubo disiantiae locorum a centro reciproce proportionalis, & exeat corpus de loco *V* justa cum velocitate secundum lineam rectae *CV* perpendicularem : progredietur corpus illud in trajectoria *VPQ* quam punctum *P* perpetuo tangit; ideoque si conica sectio *VRS* hyperbola sit, descendet idem ad centrum :. Sin ea elllpsis sit, ascendet illud perpetuo & abibit in infinitum. Et contra, si corpus quacunque cum velocitate exeat de loco *V*, & perinde ut incoeperit vel oblique descendere ad centrum, vel ab eo oblique ascendere, figura *VRS* vel hyperbola sit vel elllpsis, inveniri potest trajectoria augendo vel minuendo angulum *VCP* in data aliqua ratione. Sed &, vi centripeta in centrifugam versa, ascendet corpus oblique in trajectoria *VRC* quae invenitur capiendo angulum *VCP* sectori elliptico *VRC* proportionalem, & longitudinem *CP* longitudini *CT* aequalem ut supra. Consequuntur haec omnia ex propositione praecedente, per curvae cujusdam quadraturam, cujus inventionem, ut satis facilem, brevitatis gratia missam facio.



PROPOSITIO XLLI. PROBLEMA XXIX.

Data lege vis centripetae, requiritur motus corporis de loco dato, data cum velocitate, secundum datam rectam egressi.

Stantibus quae in tribus propositionibus praecedentibus: exeat corpus de loco *I* secundum lineolam *IK,* ea cum velocitate quam corpus aliud, vi aliqua uniformi centripeta, de loco *P* cadendo acquirere porret in *D:* sitque haec vis uniformis ad vim, qua corpus primum urgetur in *I,* ut *DR* ad *DF.* Pergat autem corpus versus *k* ; centroque *C* & intervallo *Ck* describatur circulus *ke* occurrens rectae *PD* in *e,* & erigantur curvarum *B Fg, abv, acw* ordinatim applicatae *eg, ev, ew.* Ex dato rectangulo *PDRQ* dataque



lege vis centripetae qua corpus primum agitatur, datur curva linea *BFg,* per constructionem problematis XXVII, & ejus Carol. I. Deinde ex dato angulo *CIK* datur proportio nascentium *IK, KN,* & inde, per constructionem Prob. XXVIII. datur quantitas *Q*, una cum curvis lineis *abv, acw:* ideoque, completo tempore quovis *Dbve,* datur tum corporis altitudo *Ce* vel *Ck*, tum area *Dcwe,* eique aequalis sector *XCy* angulusque *ICk,* & locus *k* in quo corpus tunc versabitur. *Q. E.I.*

Supponimus autem in his propositionibus vim centripetam in recessu quidem a centro variari secundum legem quamcunque, quam quis imaginari potest, in aequalibus autem a centro distantiis esse undique eandem. Atque hactenus motum corporum in orbibus immobilibus consideravimus. Superest ut de motu eorum in orbibus, qui circa centrum virium revolvuntur, adjiciamus pauca.