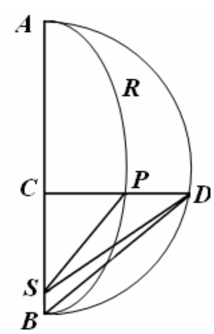
SECTION VII.

Concerning the rectilinear ascent and descent of bodies.

[Newton treats this problem as a limiting case of orbital motion, and there are three cases to consider: elliptic, parabolic, and hyperbolic orbits. It is easily shown in modern terms that in the elliptic case, the total energy of the body is given by $\frac{1}{2}v^2 - \frac{\mu}{r} < 0$, while in the second and third cases the total energy is zero, and > 0 respectively, where μ relates to the gravitational constant. Newton of course does not follow this approach. The task is to find the time to fall a given distance in a straight line towards the focus, or to be projected away likewise, with some given initial velocity and position. The method depends on finding the equivalent circular motion relating the areas, which are in proportion to the times as previously. The problem is then essentially a special case of Kepler's problem; arcs are related to areas.]

PROPOSITION XXXII. PROBLEM XXIV.

Because the centripetal force shall be inversely proportional to the square of the distance with the position from the centre, to define the intervals which a body by falling in a straight line will describe in given times.



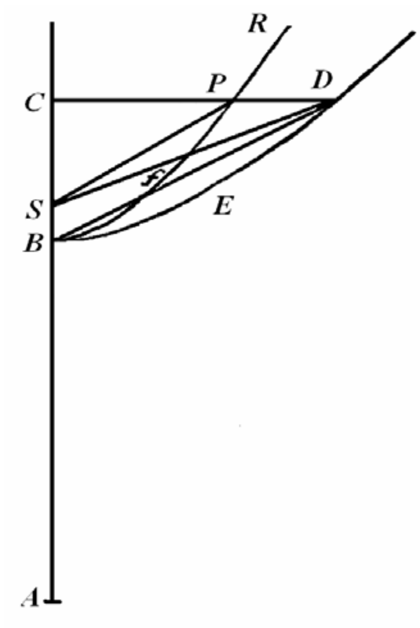
Case. I. If the body does not fall perpendicularly, it will describe some conic section (by Corol. I, Prop. XIII.) the focus of which agrees with the centre of forces. Let that conic section be *ARPB*, and *S* the focus of this . And initially, if the figure is an ellipse, upon the major axis *AB* of this the semicircle *ADB* may be described, and by falling the body may cross the right line *DPC* perpendicular to the axes; and with *DS* and *PS* drawn, the area *ASD* will be [proportional] to the area *ASP*, and thus also proportional to the time. With the axes *AB* remaining, the width of the ellipse may be continually become less, and always the area *ASD* will remain proportional to the time. That width may be decreased indefinitely : and with the orbit *APB* now coincident with the axes *AB* and the focus *S* with the end of the axis *B*, the body will descend on the right line *AC*, and the area *ABD* becomes proportional to the time. And thus there will be given the interval *AC*, which the body will describe by falling perpendicularly from the position *A* in the given time, but only if the area *ABD* may be taken proportional to the time, and the perpendicular *DC* may be sent from the point *D* to the right line *AB*

Q.E.I.

[See Chandrasekhar p.143 and beyond for a modern treatment, Routh & Brougham, p. 77 ; Whiteside Vol. VI, p. 325 onwards. In this case, the length of the latus rectum, given by $2a\sqrt{1-e^2}$, can approach zero as the eccentricity *e* approaches 1, making the ellipse more narrow and keeping the transverse length fixed, while the focus tends towards *B*, and the time is proportional to the area intercepted by a radius *SP* on the circle with diameter *DB*, where *S* coincides with *B* in the limiting case.]

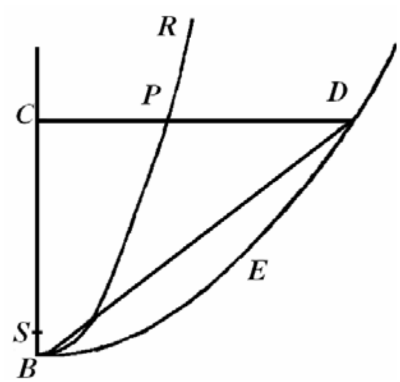
Case 2. If that figure *RPB* is a hyperbola [second diagram], the rectangular hyperbola *BED* may be described according to the same principal diameter *AB* : and because the areas *CSP*, *CBfP*, *SPfB* are in proportion to the areas *CSD*, *CBED*, *SDEB*, one to one, in the given ratio of the heights *CP*, *CD;* and the area *SPfB* is proportional to the time in which the body *P* will be moved through the arc *PfB;* the area *SDEB* will also be proportional to the same time. [Note that the here the rectangular hyperbola is the regular figure equivalent to the auxiliary circle for the ellipse, used in finding the area corresponding to the time.]The latus rectum of the hyperbola *RPB* may be diminished indefinitely with the transverse width remaining fixed, and the arc *PB* will coincide with the right line *CB* and the focus *S* with the vertex *B* and the right line *SD* with the right line *BD.* Therefore the area *BDEB* will be proportional to the time in which the body *C* by falling in a straight line will describe the line *CB.*

Q.E.I.



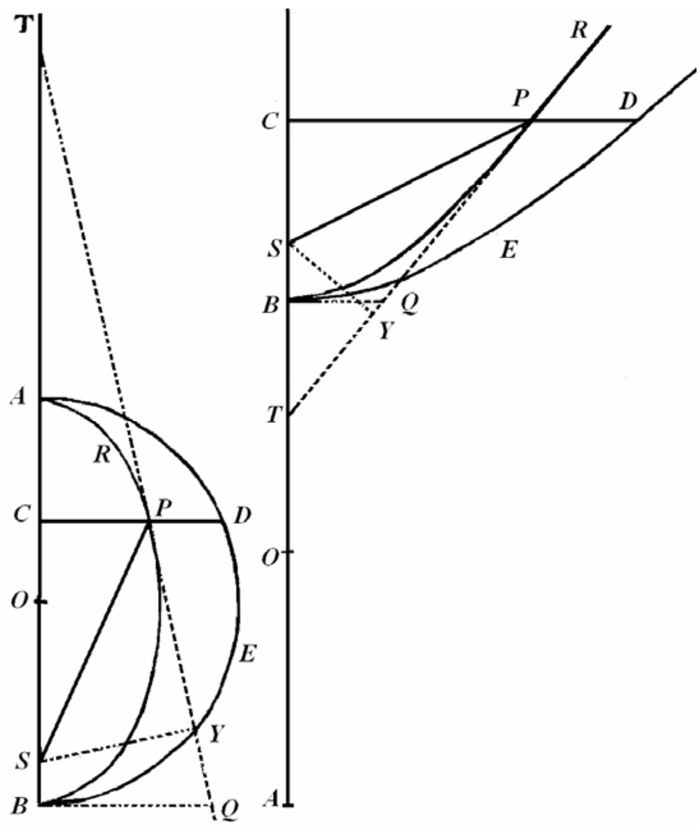
[In this case, the latus rectum or the focal chord, is given by $2a\sqrt{e^2-1}$, and as *e* approaches 1, the orbit becomes narrower, maintaining the same separation of the foci and centre.]

Case 3. And by a similar argument if the figure *RPB* is a parabola, and with the same principal vertex *B* another parabola *BED* may be described, which always may be given, then meanwhile the first parabola, on the perimeter of which the body *P* may be moving, and with the latus rectum of this reduced to zero, it may agree with the right line *CB;* the segment of the parabola *BDEB* will be proportional to the time in which both *P* or C will descend to the centre.



PROPOSITION XXXIII. THEOREM IX.

With the positions now found, I say that the velocity of the falling body at some place C is to the velocity of a body describing a circle with the centre B and radius BC, in the square root ratio that AC, the distance of the body from the circle or from the more distant vertex of the rectangular hyperbola A, has to the principal semi-diameter of the figure $\frac{1}{2}AB$.



AB may be bisected in *O*, the diameter of each common figure *RPB*, *DEB*; and the right line *PT* may be drawn, which may touch the figure *RPB* in *P*, and also cuts that common diameter *AB* (if there is a need for extending) in *T;* and let *SY* be perpendicular to that right line, and let *BQ* be perpendicular to this diameter, and *L* may be put as the latus rectum of the figure *RPB* . It may be agreed by Corol. IX, Prop. XVI, that the velocity at some place *P* of the body moving on the line *RPB* about the centre *S,* shall be to the velocity of the body being described about the circle with the same centre and radius *SP* as the square root of the ratio of the rectangle $\frac{1}{2}L \times SP$ to SY^2.

[Recall that by Corol. IX, Prop. XVI, the velocity at *P* for the conic is as $\frac{\sqrt{\frac{1}{2}L}}{SY}$, and for the circle with radius *SP*, for which the velocity is as $\frac{1}{\sqrt{SP}}$ [i.e., in modern terms, from the force equation for motion in a circle, we have $\frac{v^2}{r}\,\alpha\,\frac{1}{r^2}$ or $v\,\alpha\,\frac{1}{\sqrt{r}}$], we have $\frac{v^2_{conicAPB}}{v^2_{cir.rad.=SP}} = \frac{L}{SY^2}\times SP = \frac{\frac{1}{2}L\times SP}{SY^2}$, where we note that the circle is an ellipse with equal semi-major and minor axes, and the latus rectum is the diameter, while *SY* becomes *SP*.]

But from the theory of conics there is $AC.CB$ to CP^2 as $2AO$ to L, and thus $\frac{2CP^2\times AO}{ACB}$ equals *L*.

[To show this analytically for an ellipse, note initially from $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ that $y^2 = b^2\left(1-\frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a+x)(a-x)$; hence $\frac{AC.CB}{CP^2} = \frac{a^2}{b^2} = \frac{2a}{L} = \frac{2AO}{L}$, hence $L = \frac{CP^2\times 2AO}{AC.CB}$, as $L = \frac{2b^2}{a}$.]

Therefore these velocities in turn are in the square root ratio $\frac{CP^2\times AO\times SP}{AC.CB}$ to SY^2

$$[\text{ or } \frac{v^2_{conic}}{v^2_{circle}} = \frac{CP^2\times AO\times SP}{AC.CB\times SY^2}].$$

Again from the theory of conics *CO* is to *BO* as *BO* to *TO*, and by adding or taking from each other as *CB* to *BT*. From which either by taking or adding there shall be *BO* – or + *CO* to *BO* as *CT* to *BT*, that is, *AC* to *AO* as *CP* to *BQ*

[*i.e.* $\frac{CO}{BO} = \frac{BO}{TO} = \frac{BO+CO}{TO+BO} = \frac{BC}{BT}$; then $1-\frac{BC}{BT} = \frac{TC}{BT}$, while $1-\frac{OC}{OB} = 1-\frac{OC}{OA} = \frac{AC}{OA}$ because of the similar triangles *CPT* and *BQT*, $CP = \frac{BQ.AC}{OA}$];

and thence $\frac{CP^2\times AO\times SP}{AC.CB}$ is equal to $\frac{BQ^2\times AC\times SP}{AO\times BC}$. Now with the width *CP* of the figure *RPB* diminished indefinitely, thus so that the point *P* coincides with the point *C*; and the point *S* with the point *B*, and the line *SP* with the line *BC*, and the line *ST* with the line *BQ* ; and the velocity of the body now descending on the right line *CB* becomes to the velocity of the body describing the circle *BC* with the radius *B*, in the square root ratio of $\frac{BQ^2\times AC\times SP}{AO\times BC}$ to SY^2, that is (with the equal ratios *SP* to *BC* and BQ^2 to SY^2 ignored), in the square root ratio *AC* to *AO* or $\frac{1}{2}AB$, [*i.e.* $\frac{v_{C\ conic}}{v_{circle}} = \sqrt{\frac{AC}{AO}}$.]

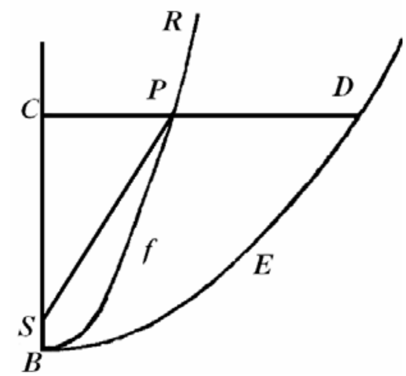
Q. E.D.

Corol. I. With the points *B* and *S* coinciding, *TC* shall be to *TS* as *AC* to *AO*.

Corol. 2. With the body rotating in some circle at a given distance from the centre of the circle, it may rise up by its own motion to twice is distance from the centre.

PROPOSITION XXXIV. THEOREM X.

If the figure BED is a parabola, I say that the velocity of the falling body at some place C is equal to the velocity by which the body can describe a circle uniformly with centre B and with radius half of its interval BC.



For the velocity of the body describing a parabola *RPB* about a centre *S* at some place *P* (by Corol. Prop. XVI.) is equal to the velocity of the body describing a circle uniformly about the same centre *S*, with a radius half of the interval *SP*. There the width of the parabola *CP* may be diminished indefinitely, so that the arc of the parabola *PfB* may coincide with the right line *CB*, the centre *S* with the interval *B*, and the radius *SP* with the interval *BC*, and the proposition will be agreed upon.

[Recall $\frac{v_{conic}^2}{v_{circle}^2} = \frac{CP^2 \times AO \times SP}{AC.CB \times SY^2}$; in this case $\frac{AC.CB}{CP^2} = \frac{a^2}{b^2} = \frac{2a}{L} = \frac{1}{2}$, and hence $\frac{v_{conic}^2}{v_{circle}^2} = \frac{2 \times AO \times SP}{SY^2} = 1$ when $AO = \frac{1}{2}BC$ and $SP = SY = BC$.]

PROPOSITION XXXV. THEOREM XI.

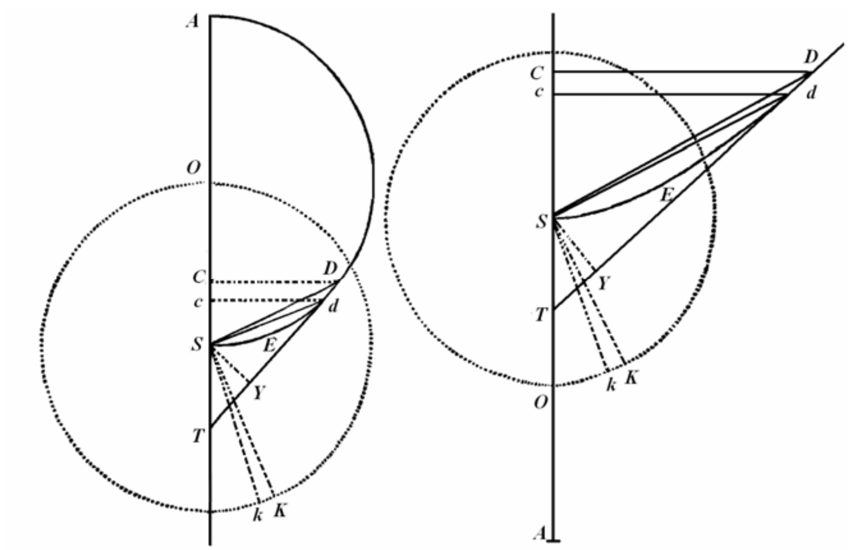
With the same in place, I say that the area of the figure, described by the indefinite radius SD, shall be equal to the area that the body can describe in the same time, with a radius equal to half of the latus rectum of the rectilinear figure DES, by rotating uniformly about the centre S.

For consider the body *C* as falling in the shortest interval of time to describe the element of length *Cc*, and meanwhile another body *K*, by rotating uniformly in a circle *OKk* about the centre *S*, to describe the arc *Kk*. The perpendiculars C*D* and *cd* may be erected meeting the figure *DES* in *D* and *d*. *SD, Sd*, *SK*, *Sk* may be joined and *Dd* may be drawn meeting the axis *AS* in *T*, and to that the perpendicular *SY* may be sent.

Case. 1. Now if the figure *DES* is a circle or a rectangular hyperbola, the diameter *AS* may be the transverse bisector of this at *O*, and *SO* will be half of the latus rectum. And because *TC* is to *TD* as *Cc* to *Dd*, and *TD* to *TS* as *CD* to *SY*, so that from the equation there will be *TC* to *TS* as $CD \times Cc$ to $SY \times Dd$.

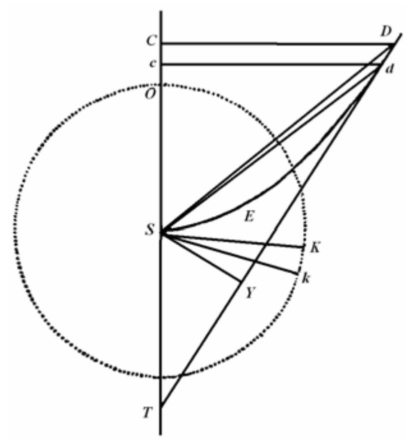
[*i.e.* $\frac{TC}{TD} = \frac{Cc}{Dd}$,and $\frac{TD}{TS} = \frac{CD}{ST}$; $\frac{TC}{TS} = \frac{CD \times Cc}{SY \times Dd}$]

But (by Coroll. Prop. XXXIII.) *TC* is to *TS* as *AC* to *AO*, for example if the final ratios of the lines may be taken on placing the points *D* and *d* together. Therefore *AC* is to *AO* or *SK* as $CD \times Cc$ to $SY \times Dd$. Again the velocity of the descending body at *C* is to the velocity of the body described around the circle with radius *SC* and centre *S* in the square root *AC* to *AD* or *SK* (by Prop. XXXIII.) And this velocity to the velocity of the body describing the circle *OKk* is in the square root ratio *SK* to *SC* (by Corol. VI. Prop. IV.) and from that equation the first velocity to the final, that is the line element *Cc* to the arc *Kk* is in the square root ratio *AC* ad *SC*, that is in the ratio *AC* to *CD*. Whereby $CD \times Cc$ is equal to $AC \times Kk$, and therefore *AC* to *SK* as $AC \times Kk$ to $SY \times Dd$, and thus $SK \times Kk$ equals $SY \times Dd$, and $\frac{1}{2} SK \times Kk$ equals $\frac{1}{2} SY \times Dd$, that is the area *KSk* is equal to the area *SDd.* Therefore in the individual small increments of time



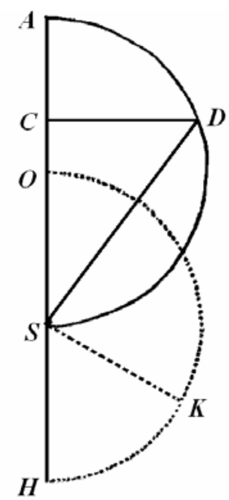
the small increments of the two areas *KSk* and *SDd* may be generated, which, if the magnitude of these may be diminished and the number increased indefinitely, maintain a ratio of equality, and therefore (by the Corollaries of Lemma IV.) the whole areas generated likewise are always equal. *Q.E.D.*

Case. 2. But if the figure *DES* shall be a parabola, there may be found to be as above $CD \times Cc$ is to $SY \times Dd$ as *TC* to *TS*, that is as 2 to 1, and thus $\frac{1}{4} CD \times Cc$ is equal as above $\frac{1}{2} SY \times Dd$. But the velocity of the falling body at *C* is equal to the velocity by which the circle with radius $\frac{1}{2} SC$ may be able to be described uniformly (by Prop. XXXIV.) And this velocity to the velocity by which the circle with radius *SK* may be able to be described, that is, the element *Cc* to the arc *Kk* (by Corol. VI., Prop. IV.) is in the square root ratio *SK* to $\frac{1}{2} SC$, that is, in



the ratio *SK* to $\frac{1}{2}CD$. Whereby $\frac{1}{2}SK \times Kk$ is equal to $\frac{1}{4}CD \times Cc$ and thus equal to $\frac{1}{2}SY \times Dd$, that is, the area *KSk* is equal to the area *SDd* as above. *Q.E.D.*

PROPOSITION XXXVI. PROBLEM XXV.



With the place of the falling body A given to determine the descent times.

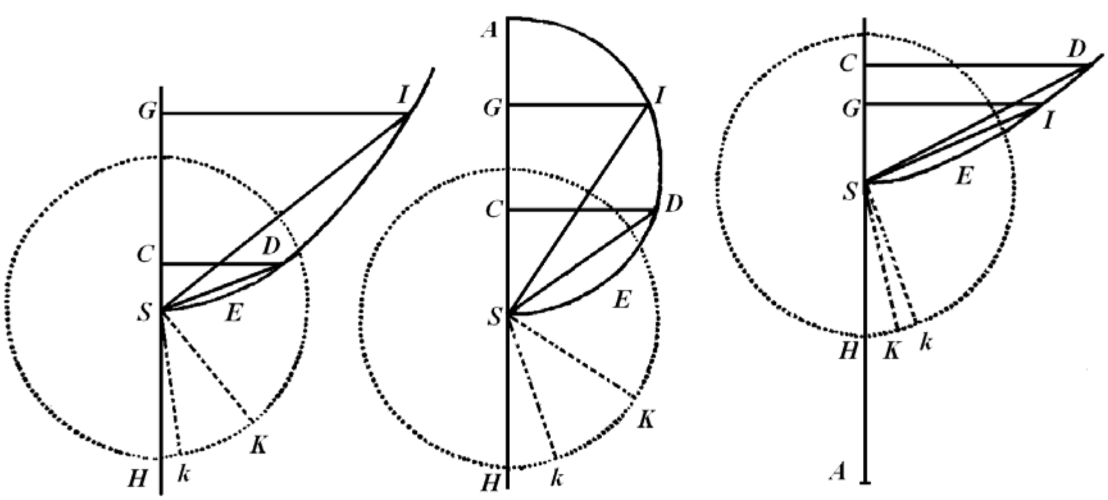
Upon the diameter *AS*, describe the semi-circle *ADS*, the distance of the body at the start, and so that the semicircle *OKH* about the centre *S* is equal to this. From some position of the body *C* erect the applied ordinate *CD*. Join *SD*, and put in place the sector *OSK* equal to the area *ASD*. It is apparent by Prop. XXXV that the body on falling describes the distance *AC* in the same time that the other body, by rotating uniformly about the centre *S*, can describe the arc *OK* in the same time. *Q. E. F.*

PROPOSITION XXXVII. PROBLEM XXVI.

To find the times of ascent or descent of a body projected from some given place.

[There is as need to classify the motion of a body, projected either up or down, into one of the three types, corresponding to degenerate motion on and ellipse, hyperbola, or a parabola.]

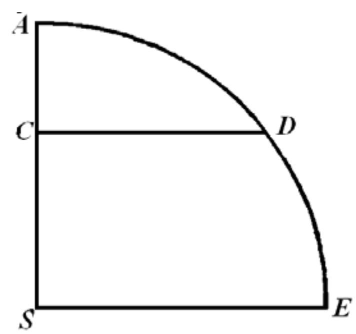
The body may emerge from a given place *G* along the line *GS* with some velocity. In the square ratio of this velocity to the uniform velocity in a circle, by which a body may be able to rotate about the centre *S* for a given radius *SG*,



take *GA* to $\frac{1}{2}AS$. If that ratio is of the number two to one, the point *A* infinitely far away, in which case a parabola is being described with vertex *S*, axes *SG*, with some latus rectum. This is apparent from Prop. XXXIV. But if that ratio were greater or smaller than 2 to 1, in the first case a circle, in the latter a rectangular hyperbola, must be described on the diameter *SA*. It is apparent by Prop. XXXIII. Then with centre *S*, with a radius equal to half the latus rectum, the circle *HkK* may be described, and at the position of the body *G* either descending or ascending, and at some other place C, the perpendiculars *GI* and C*D* meeting the conic section or the circle in *I* and *D.* Then with *SI* an*d SD* joined, the sectors *HSK* and *HSk* are made equal to the segments *SEIS*, *SEDS*, and by Prop. XXXV the body *G* describes the interval *GC* in the same time in which the body *K* can describe the arc *Kk*. *Q.E.F*.

PROPOSITION XXXVIII. THEOREM XII.

Because the centripetal force may be put proportional to the height or distance of the places from the centre, I say, that the times of falling, the velocities and the distances described, are proportional to the arcs, to the sines of the arcs and to the versed sines respectively.



[Recall that an inverse square law of force acting on a body in orbit from the focus of the ellipse may be replaced by one of proportionality acting from the centre (Prop. IV) ; thus there is proportional motion between uniform rotation on the auxiliary circle, on any ellipse with the same semi-major axis, and the S.H.M. on the vertical line *AS* in the limiting case. Here the forces are in proportion to the distances *SC*, the speeds to the chords *CD*, and the distances fallen to the versed sines *AC*.]

A body may fall from some position *A* along the right line *AS;* and from the centre of forces *S*, with a radius *AS*, the quadrant of a circle may be described *AE,* and let *CD* be the [right] sine of any arc *AD;* and the body *A*, in the time *AD*, by falling will describe the interval *AC,* and at the place *C* it will acquire the velocity *CD.*

In the same manner it may be shown from Proposition X, as by which Proposition XXXII was demonstrated from Proposition XI.

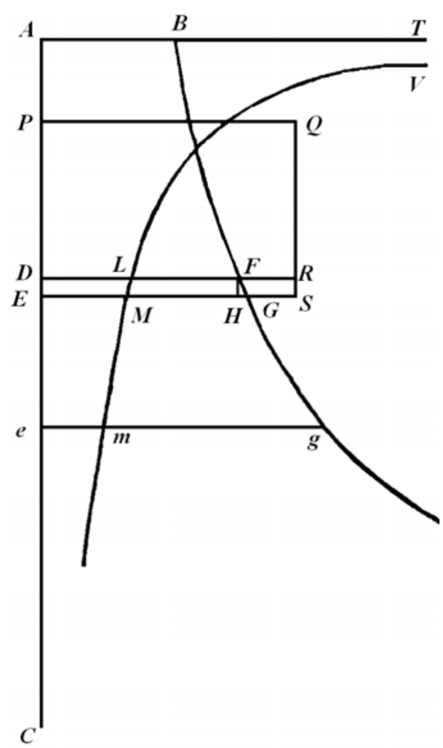
Corol. I. Hence the times are equal, in which a single body by falling from the place *A* arrives at the centre *S,* and another body by rotating will describe the fourth part of the arc *ADE.*

Corol. 2. Hence all the times are equal in which bodies fall from any places as far as the centre. For all the periodic times of revolution may be equal (by Corol. III. Prop. IV.).

PROPOSITION XXXIX. PROBLEM XXVII.

With centripetal forces of any kind put in place, and with the quadratures of the curvilinear figures agreed upon, the straight rise or fall of a body is required, as well as the velocity at individual places , and the time in which the body may arrive at some place: And conversely.

The body *E* may fall from some place *A* on the right line *ADEC*, and from its place *E* a perpendicular line *EG* may always be put in place, proportional to the centripetal force at that place tending towards the centre *C*: And let *BFG* be the curved line that point *G* always touches [*i.e.* the curve traced out by the centripetal force or acceleration]. Moreover, at the start the line *EG* may coincide with the perpendicular *AB* itself, and the velocity of the body at some place *E* will be as the right line, [the square of which] can be as the curvilinear area *ABGE. Q.E.I.*
[This is essentially an exercise in integration; the first curve *BFG* is integrated w.r.t. *z* from $z=0$ at *A* to some general point, giving the area under the force vs. distance, or the acceleration vs. distance curve, which we can interpret as the kinetic energy acquired, or as the work done by the force; we can show this readily starting from $\frac{ddz}{dt^2}=-F(z)$, which can be written in the form $\frac{dv}{dz}\cdot\frac{dz}{dt}=v\frac{dv}{dz}=-F(z)$, or as the indefinite integral $\frac{1}{2}v^2=-\int F(z)dz$. Clearly Newton was familiar with, and indeed was the originator of this wonderful new approach to solving problems, but chose not to divulge his method.]



On *EG, EM* may be taken for the right line, which can be inversely proportional to [the square root of] the area *ABGE*, and *VLM* shall be the curved line, that the point *N* always touches, and the asymptote of this is the right line *AB* produced; and the time will be, in which the body by falling describes the line *AE*, as the area under the curve *ABTVME.*

Q. E. I.

[Following on from the last note, we now have $\left[\frac{dt}{dz}\right]_{z=E}=\left[2\int F(z)dz\right]^{-\frac{1}{2}}\alpha\, 1:\sqrt{area\ ABGE}$ $=EM$.
If we call *EM* the function $T(z)$, then $t=\int T(z)dz$; the limits of integration are chosen to fit the circumstances, from *A* to *E*. There now

follows the verification of these integrations by the inverse process of differentiation.]

And indeed on the right line *AE* there may be taken that line of the shortest length *DE,* and *DLF* shall be the locus of the line *EMG,* when the body will be moving through *D;* and if this shall be the centripetal force, as the right line, which can be the area *ABGE*, it shall be as the velocity of the descent: that area will be in the square ratio of the speed, that is, if for the velocities at *D* and *E,* there may be written V and $V+I$, the area *ABFD* will be as *VV*, and the area *ABGE* as $VV+2.VI+II$, and separately the area *DFGE* as $2.VI+II$, and thus $\frac{DFGE}{DE}$ as $\frac{2VI+II}{DE}$, that is, if the ratios are taken of the first vanishing quantities, the length *DF* shall be as the quantity $\frac{2VI}{DE}$, and thus also as the quantity of half of this $\frac{I\times V}{DE}$. But the time, in which the body falling will describe the element of line *DE,* is as that element directly and as the velocity *V* inversely, and the force is as the velocity increment *I* directly and the time inversely, and thus if the first vanishing ratios are taken, as $\frac{I\times V}{DE}$, that is, as the length *DF.* Therefore *DF* or *EG* becomes proportional to the force itself so that the body may descend with that velocity, which shall be as the right line which can be as the [square root] of the area *ABGE.*

Q.E.D.

Again since the time, in which, in which any line element *DE* of a given length may be described, shall be inversely as the velocity, and thus inversely as the right line which can become the area *ABFV*; and let it be *DL,* and thus the area arising *DLME,* as the same right line inversely: the time will be as the area *DLME,* and the sum of all the times as the sum of all the areas, that is (by Corol. Lem. IV.) the total time in which the line *AE* is described will be as the total area *ATVME.*

Q.E.D.

Corol. 1. If *P* shall be the place, from which the body must fall, as urged by some uniform known centripetal force (such as gravity generally is supposed) it may acquire a velocity at the place *D* equal to the velocity, that another body falling by some other force has acquired at the same place *D*, and on the perpendicular *DF, DR* may be taken, which shall be to *DF* as that uniform force [*PQ*] to the other force at the place *D;* and the rectangle *PDRQ* may be completed, and an area *ABFD* equal to this may be cut off;

[Thus, $PQ\times PD=\int F(z)dz$ with suitable limits chosen.]

A will be the place from which the other body has fallen. For with the rectangle *DRSE* completed, since there shall be the area *ABFD* to the area *DFGE* as *VV* to 2.*VI*, and thus as $\frac{1}{2}V$ to *I*, that is, as of half of the whole velocity to the increment of the velocity of the body falling by the unequal force; and likewise the area *PQRD* to the area *DRSE* as of half of the whole velocity to the increment of the velocity of the body falling under the uniform force; and these increments shall be (on account of the equality of the increments of time arising) as the generating forces, that is, as the applied lines *DF, DR* in order, and thus as the areas arising *DFGE* and *DRSE;* from the equality of the total area, *ABFD* and *PQRD* are as the half of the total speeds, and therefore, on account of the equal speeds, equal in turn.

Corol. 2. From which if some body may be projected from some place *D* with some given velocity either up or down, and the law of the centripetal force may be given, the velocity of this will be found at any other place *e*, on erecting the ordinate *e.g.*, and by taking that velocity at *e* to the velocity at the place *D* as the right line, which can become [on squaring] the rectangular area *PQRD* either increase by the curved area *DFge*, if the place *e* is below the place *D*, or decreased by it, if this is above, to the right line which can become [on squaring] the area *PQRD* only.

$$[\text{Thus}, vel_1^2 = vel_2^2 \pm \int F(z)dz\,.]$$

Corol. 3. The time too will become known by erecting the ordinate *em* inversely proportional to the square root of the side from *PQRD* + or – *DFge*, and by taking the time in which the body has described the line *De* to the time in which the other body fell with a uniform force from *P* and on falling arrives at *D*, as the curvilinear area *DLme* to the rectangle $2.PD \times DL$. For the time in which the body falling under the uniform force has described the line *PD*, is to the time in which likewise the body has described the line *PE* in the square root ratio *PD* to *PE*, that is (with the element of the line now arising) in the ratio *PD* to $PD + \frac{1}{2}DE$ or 2.*PD* to $2PD + DE$, and separately, to the time in which the same body has described the line element *DE* as *2PD* to *DE*, and thus as the rectangle $2PD \times DL$ to the area *DLME;* and the time in which the one body has described the line element *DE* to the time in which the other body in the non uniform motion has described the line *De*, as the area *DLME* to the area *DLme*, and from the equality the first time to the final time as the rectangle $2PD \times DL$ to the area *DLme.*

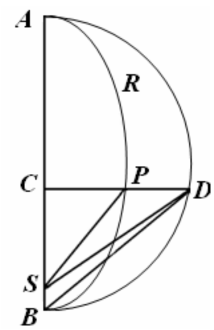
SECTIO VII.

De corporum ascensu & *descensu rectilineo.*

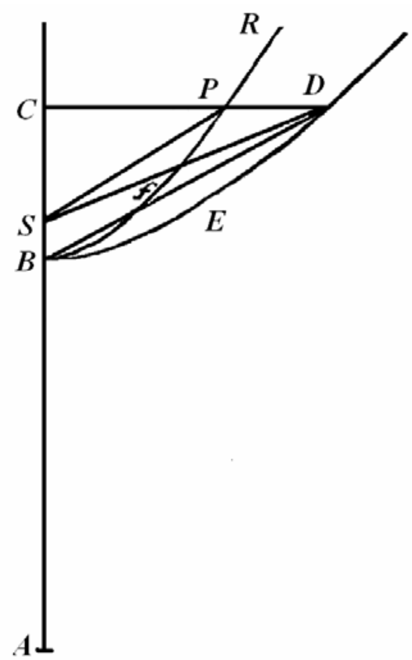
PROPOSITIO XXXII. PROBLEMA XXIV.

Posito quod vis centripeta sit reciproce proportionalis quadrato distantitae locorum a centro, spatia definire quae corpus recta cadendo datis temporibus describit.

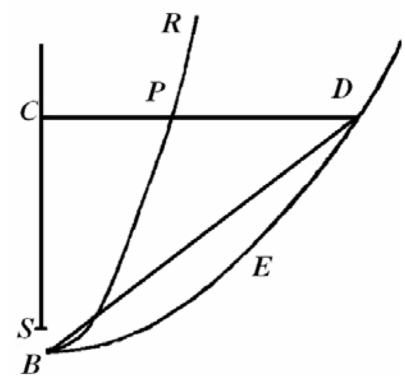
Cas. I. Si corpus non cadit perpendiculariter, describet id (per Corol. I. Prop. XIII.) sectionem aliquam conicam cuius umbilicus congruit cum centro virium. Sit sectio illa conica *ARPB* & umbilicus eius *S.* Et primo si figura ellipsis est; super huius axe majore *AB* describatur semicirculus *ADB,* & per corpus decidens transeat recta *DPC* perpendicularis ad axem; actisque *D S, PS* erit area *ASD* areae *ASP*, atque ideo etiam tempori proportionalis. Manente axe *AB* minuatur perpetuo latitudo ellipseos, & semper manebit area *ASD* tempori proportionalis. Minuatur latitudo illa in infinitum: & orbe *APB* iam coincidente cum axe *AB* & umbilico *S* cum axis termino *B,* descendet corpus in recta .*AC,* & area *ABD* evadet tempori proportionalis. Dabitur itaque spatium *AC*, quod corpus de loco *A* perpendiculariter cadendo tempore dato describit, si modo tempori proportionalis capiatur area *ABD*, & a puncto *D* ad rectam *AB* demittatur perpendicularis *DC Q.E.I.*



Cas 2. Si figura illa *RPB* hyperbola est, describatur ad eandem diametrum principalem *AB* hyperbola rectangula *BED:* & quoniam areae *CSP*, *CBfP*, *SPfB* sunt ad areas *CSD, CBED, SDEB,* singulae ad singulas, in data ratione altitudinum *CP, CD;* & area *SPfB* proportionalis est tempori quo corpus *P* movebitur per arcum *PfB;* erit etiam area *SDEB* eidem tempori proportionalis. Minuatur latus rectum hyperbolae *RPB* in infinitum manente latere transverso, & coibit arcus *PB* cum recta *CB* & umbilicus *S* cum vertice *B* & recta *SD* cum recta *BD.* Proinde area *BDEB* proportionalis erit tempori quo corpus *C* recto descensu describit lineam *CB. Q.E.I.*



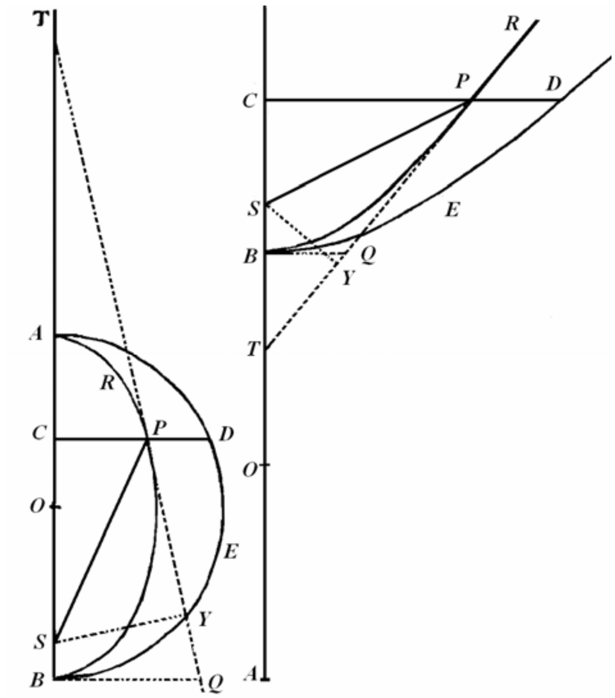
Cas.3. Et simili argumento si figura *RPB* parabola est, & eodem vertice principali *B* describatur alia parabola *BED*, quae semper maneat data, interea dum parabola prior, in cuius perimetro corpus *P* movetur, diminuto & in nihilurn redacto eius latere recto, conveniat cum linea *CB;* fiet segmentum parabolicum *BDEB* proportionale tempori quo corpus illud *P* vel C descendet ad centrum



PROPOSITIO XXXIII. THEOREMA IX.

Positis iam inventis, dico quod corporis cadentis velocitas in loco quovis C est ad velocitatem corporis centro B intervallo BC circulum describentis, in subduplicata ratione quam AC, distantia corporis a circuli vel hyperbolae rectangulae venice ulteriore A, habet ad figurae semidiametrum principalem $\frac{1}{2}AB$.

Bisecetur *AB*, communis utriusque figurae *RPB*, *DEB* diameter, in *O* ; & agatur recta *PT,* quae tangat figuram *RPB* in *p,* atque etiam secet communem illam diametrum *A B* (si opus eft pro ductam) in *T;* sitque *SY* ad hanc rectam, & *BQ* ad hanc diametrum perpendicularis, atque figurae *RPB* latus rectum ponatur *L*. Constat per Corol. IX. Prop. XVI. quod corporis in linea *RPB* circa centrum *S* moventis velocitas in loco quovis *P* sit ad velocitatem corporis intervallo *SP* circa idem centrum circulum describentis in



subduplicata ratione retltanguli $\frac{1}{2}L \times SP$ ad *SY* quadratum. Est autem ex conicis *ACB* ad *CPq* ut 2*AO* ad *L*, ideoque $\frac{2CPq\times AO}{ACB}$ equale *L*. Ergo velocitates illae sunt ad invicem in subduplicata ratione $\frac{2CPq\times AO\times SP}{ACB}$. Porro ex conicis est *CO* ad *BO* ut *BO* ad *TO*, & composite vel divisim ut *CB* ad *BT.* Unde vel dividendo vel componendo sit *BO* – vel + *CO* ad *BO* ut *CT* ad *BT*, id est, *AC* ad *AO* ut *CP* ad *BQ;* indeque $\frac{2CPq\times AO\times SP}{ACB}$ aequale est $\frac{BQq\times AC\times SP}{AO\times BC}$. Minuatur iam in infinitum figurae *RPB* latitudo *CP*, sic ut punctum *P* coeat cum puncto *C*; punctumque *S* cum puncto *B*, & linea *SP* cum linea *BC*, lineaque *ST* cum linea *BQ* ; & corporis iam recta descendentis in linea *CB* velocitas fiet ad velocitatem corporis centro *B* intervallo *BC* circulum describentis, in subduplicata ratione ipsius $\frac{BQq\times AC\times SP}{AO\times BC}$ ad *SYq*, hoc est (neglectis aequalitatis rationibus *SP* ad *BC* & *BQq* ad *STq*) in subduplicata ratione *AC* ad *AO* sive $\frac{1}{2}AB$. *Q. E.D.*

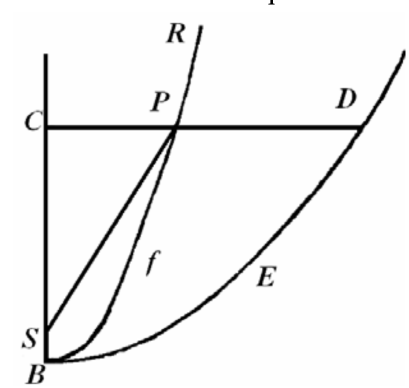
Corol. I. Punctis *B* & *S* coeuntibus, sit *TC* ad *TS* ut *AC* ad *AO.*

Corol. 2. Corpus ad datam a centro distantiam in circulo quovis revolvens, motu suo sursum verso ascendet ad duplam suam a centro distantiam.

PROPOSITIO XXXIV. THEOREMA X.

Si figura BED parabola est, dico quod corporis cadentis velocitas in loco quovis C aequalis est velocitati qua corpus centro B dimidio intervalli sui BC circulum uniformiter describere potest.

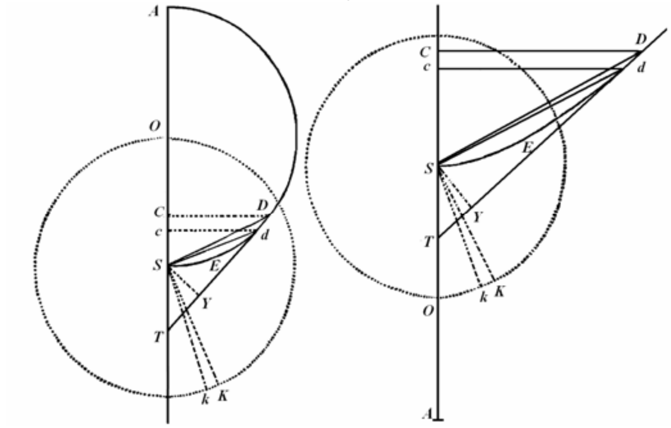
Nam corporis parabolam *RPB* circa centrum *S* describentis velocitas in loco quovis *P* (per Corol. Prop. XVI.) aequalis est velocitati corporis dimidio intervalli *SP* circulum circa idem centrum *S* uniformiter describentis. Minuatur parabolae latitudo *CP* in infinitum eo, ut arcus parabolicus *PfB* cum recta *eB*, centrum *S* cum vertice *B*, & intervallum *SP* cum intervallo *BC* coincidat, & constabit propositio.



PROPOSITIO XXXV. THEOREMA XI.

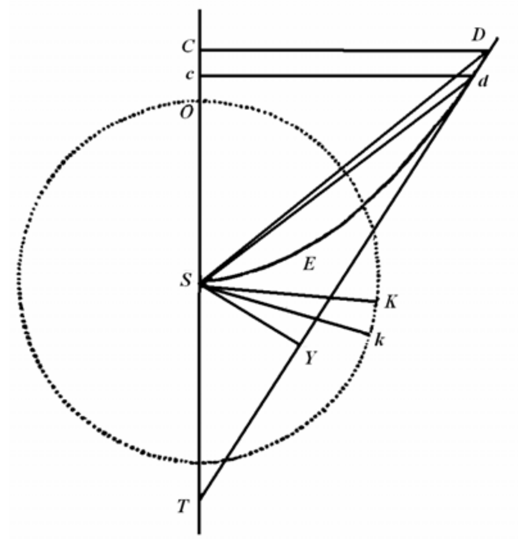
Iisdem positis, dico quod area figurae DES, radio indefinitio SD descripta, aequalis sit areae quam corpus, radio dimidium lateris recti figurae DES aequante, circa centrum S uniformiter gyrando, eodem tempore destribere potest.

Nam concipe corpus *C* quam minima temporis particula lineolam *Cc* cadendo describere, & interea corpus aliud *K*, uniformiter in circulo *OKk* circa centrum *S* gyrando, arcum *Kk* describere. Erigantur perpendicula C*D*, *cd* occurrentia figurae *DES* in *D, d.* Iungantur *SD, Sd, SK, Sk* & ducatur *Dd* axi *AS* occurrens in *T*, & ad eam demittatur perpendiculum *SY*.
Cas. 1. Iam si figura *DES* circulus est vel hyperbola rectangula, bisecetur eius transversa diameter *AS* in *O*, & erit *SO* dimidium lateris recti. Et quoniam est *TC* ad *TD* ut *Cc* ad *Dd*, & *TD* ad *TS* ut *CD* ad *SY*, erit ex aequo *TC* ad *TS* ut $CD\times Cc$ ad $SY\times Dd$.
Sed (per Coroll. Prop. XXXIII.) est *TC* ad *TS* ut *AC* ad *AO*, puta si in coitu punctorum *D, d* capiantur linearum rationes ultimae. Ergo *AC* est ad *AO* seu *SK* ut $CD\times Cc$ ad $SY\times Dd$. Porro corporis descendentis velocitas in *C* est ad velocitatem corporis circulum intervallo *SC* circa centrum *S* describentis in subduplicata ratione *AC* ad *AD* vel *SK* (per Prop. XXXIII.) Et haec velocitas ad velocitatem corporis describentis circulum *OKk* in subduplicata ratione *SK* ad *SC* (per Corol. VI. Prop. IV.) & ex aequo velocitas prima ad ultimam, hoc est lineola *Cc* ad arcum *Kk* in subduplicata ratione ratione *AC* ad *SC*, id est in ratione *AC*ad *CD*. Quare est $CD\times Cc$ equate $AC\times Kk$, & propterea *AC* ad *SK* ut $AC\times Kk$ ad $SY\times Dd$,indeque $SK\times Kk$ aeqale $SY\times Dd$, & $\frac{1}{2}SK\times Kk$ aequale $\frac{1}{2}SY\times Dd$, id est area *KSk* aequalis areae *SDd.* Singulis igitur temporis

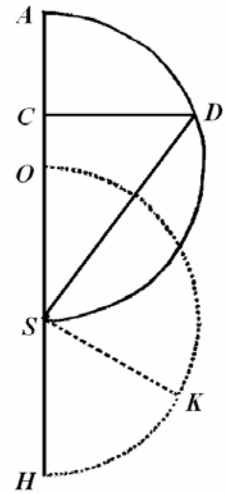


generantur arearum duarum particulae *KSk,* & *SDd*, quae, si magnitudo earum minuatur & numerus augeatur in infinitum rationem obtinent aequalitatis, & propterea (per corollarium lemmatis IV.) areae totae simul genitae sunt semper aequales. *Q.E.D.*

Cas. 2. Quod si figura *DES* parabola sit, invenietur esse ut supra $CD \times Cc$ ad $SY \times Dd$ ut *TC* ad *TS*, hoc est ut 2 ad 1, ideoque $\frac{1}{4}CD \times Cc$ aequale esse ut supra $\frac{1}{2}SY \times Dd$. Sed corporis cadentis velocitas in *C* aequalis est velocitati qua circulus intervallo $\frac{1}{2}SC$ uniformiter describi possit (per Prop. XXXIV.) Et haec velocitas ad velocitatem qua circulus radio *SK* describi possit, hoc est, lineola *Cc* ad arcum *Kk* (per corol. VI. Prop. IV.) est in subduplicata ratione *SK* ad $\frac{1}{2}SC$, id est, in ratione *S K* ad $\frac{1}{2}CD$. Quare est $\frac{1}{2}SK \times Kk$ aequale $\frac{1}{4}CD \times Cc$ ideoque aequale $\frac{1}{2}SY \times Dd$, hoc est, area *KSk* *a*equalis areae *SDd* ut supra. *Q.E.D.*



PROPOSITIO XXXVI. PROBLEMA XXV.



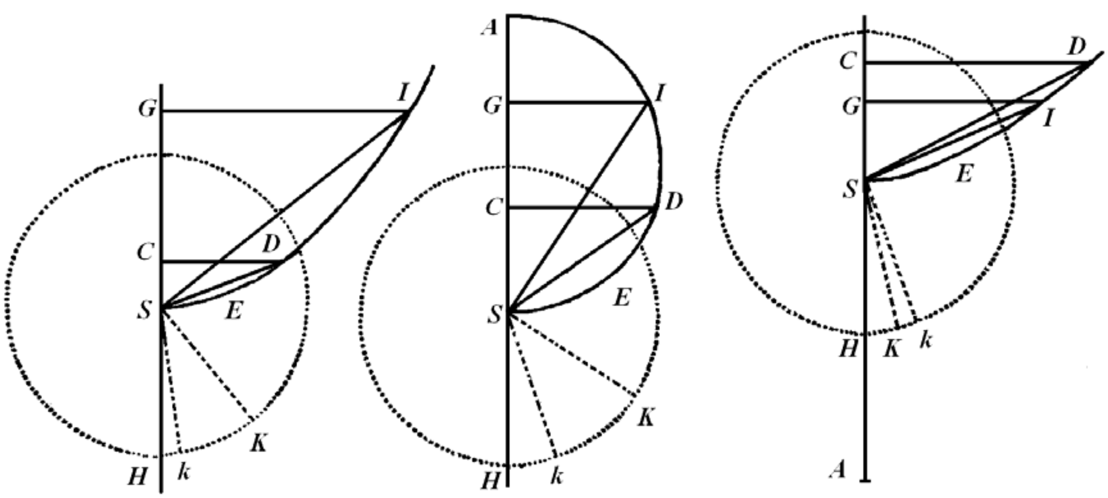
Corporis de loco dato A cadentis determinare tempora descensuso

Super diametro *AS*, distantia corporis a centro sub initio, describe semicirculum *ADS*, ut & huic aequalem semicirculum *OKH* circa centrum *S*. De corporis loco quovis *C* erige ordinatim applicatam *CD*. Iunge *SD*, & areae *ASD* aequalem constitue sectorem *OSK*. Patet per Prop. XXXV quod corpus cadendo describet spatium *AC* eodem tempore quo corpus aliud, uniformiter circa centrum *S* gyrando, describere potest arcum *OK*. *Q. E. F.*

PROPOSITIO XXXVII. PROBLEMA XXVI.

Corporis de loco dato sursum vel deorsum projecti definire tempora ascensus vel descensus.

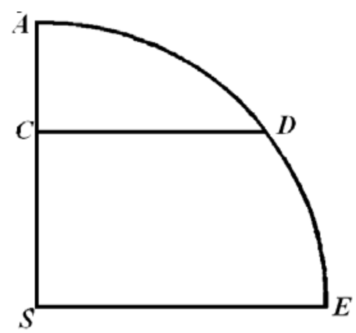
Exeat corpus de loco dato *G* secundum lineam *GS* cum velocitate quacunque. In duplicata ratione huius velocitatis ad uniformem in circulo velocitatem, qua corpus ad intervallum datum *SG* circa centrum



S revolvi posset, cape *GA* ad $\frac{1}{2}AS$. Si ratio illa est numeri binarii ad unitatem, punctum *A* infinite distat, quo casu parabola vertice *S*, axe *SG*, latere quovis recto describenda est. Patet hoc per Prop. XXXIV. Sin ratio illa minor vel major est quam 2 ad 1, priore casu circulus, posteriore hyperbola rectangula super diametro *SA* describi debet. Patet per prop. XXXIII. Tum centro *S*, intervallo aequante dimidium lateris recti, describatur circulus *HkK*, *&* ad corporis descendentis vel ascendentis locum *G*, & locum alium quemvis C, erigantur perpendicula *GI*, C*D* occurrentia conicae sectioni vel circulo in *I* ac *D.* Dein iunctis *SI*, *SD*, fiant segmentis *SEIS*, *SEDS* sectores *HSK*, *HSk* aequales, & per Prop. XXXV corpus *G* describet spatium *GC* eadem tempore quo corpus *K* describere potest arcum *K*k. *Q.E.F.*

PROPOSITIO XXXVIII. THEOREMA XII.

Posito quod vis centripeta proportionalis sit altitudini seu distantiae locorum a centro, dico quod cadentium tempora, velocitates & spatia descripta sunt arcubus, arcuumque sinibus rectis & sinibus versis respective proportionalia.



Cadat corpus de loco quovis *A* secundum rectam *AS;* & centro virium *S*, intervallo *AS*, describatur circuli quadrans *AE*, sitque *CD* sinus rectus arcus cuiusvis *AD;* & corpus *A*, tempore *AD*, cadendo describit spatium *AC*, inque loco *C* acquiret velocitatem *CD.*

Demonstratur eodem modo ex propositione x, quo Propositio XXXII, ex propositione XI demonstrata fuit.

Corol. I. Hinc aequalia sunt tempora, quibus corpus unum de loco *A* cadendo pervenit ad centrum *S*, & corpus aliud revolvendo describit arcum quadrantalem *ADE.*

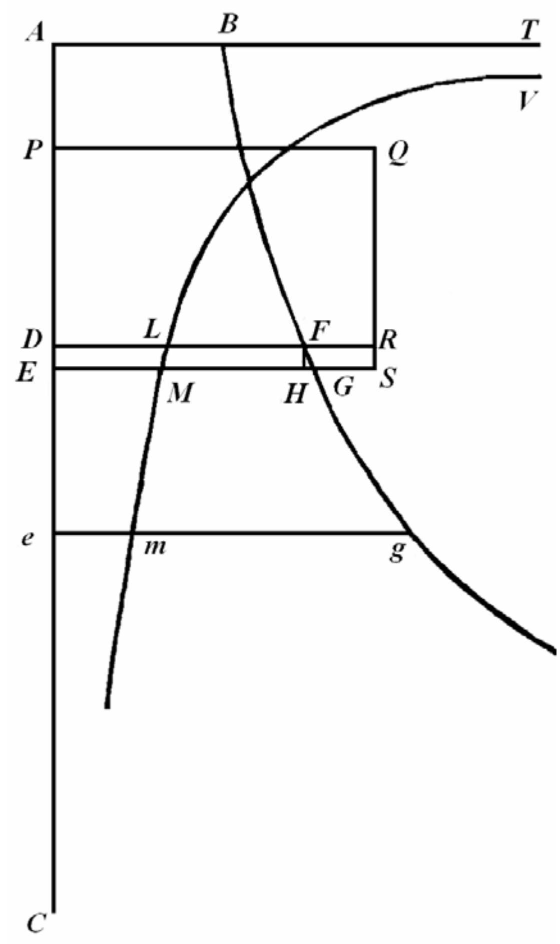
Corol. 2. Proinde aequalia sunt tempora omnia quibus corpora de locis quibusvis ad usque centrum cadunt. Nam revolventium tempora omnia periodica (per Corol. III. Prop. IV.) aequantur.

PROPOSITIO XXXIX. PROBLEMA XXVII.

Posita cuiuscunque generis vi centripeta, & *concessis figurarum curvilinearum quadraturis, requiritur corporis recta ascendentis vel descendentis tum velocitas in locis singulis, tum tempus quo corpus ad locum quemvis perveniet: Et contra.*

De loco quovis *A* in recta *ADEC* cadat corpus *E,* deque loco eius *E* erigatur semper perpendicularis *EG,* vi centripetae in loco illo ad centrum *C* tendenti proportionalis: Sitque *BFG* linea curva quam punctum *G* perpetuo tangit. Coincidat autem *EG* ipso motus initio cum perpendiculari *AB,* & erit corporis velocitas in loco quovis *E* ut recta, que potest aream curvilineam *ABGE.*

Q.E.I.



In *EG* capiatur *EM* rectae, que potest aream *ABGE,* reciproce proportionalis, & sit *VLM* linea curva, quam punctum *N* perpetuo tangit, & cuius asymptotos est recta *AB* producta; & erit tempus, quo corpus cadendo describit lineam *AE,* ut area curvilinea *ABTVME.*

Q. E. I.

Etenim in recta *AE* capiatur linea quam minima *DE* datae longitudinis, sitque *DLF* locus lineae *EMG,* ubi corpus versabatur in *D;* & si ea sit vis centripeta, ut recta, quae potest: aream *ABGE,* sit ut descendentis velocitas: erit area ipsa in duplicata ratione velocitatis, id est, si pro velocitatibus in *D* & *E,* scribantur V & $V+I$, erit area *ABFD* ut *VV,* & area *ABGE* ut $VV+2.VI+II$, & divisim area *DFGE* ut $2.VI+II$, ideoque $\frac{DFGE}{DE}$ ut $\frac{2VI+II}{DE}$, id est, si primae quantitatum nascentium rationes sumantur, 1ongitudo *DF* ut quantitas $\frac{2VI}{DE}$, ideoque etiam ut quantitis huius dimidium $\frac{I\times V}{DE}$. Est autem tempus, quo corpus cadendo describit lineolam *DE,* ut lineola illa directe & velocitas *V* inverse, estque vis ut velocitatis incrementum *I* directe & tempus inverse, ideoque si primae nascentium rationes sumantur, ut $\frac{I\times V}{DE}$, hoc est, ut longitudo *DF.* Ergo vis ipsi *DF* vel *EG* proportionalis facit ut corpus ea cum velocitate descendat, quae sit ut recta quae potest aream *ABGE. Q.E.D.*

Porro cum tempus, quo qurelibet longitudinis datae lineola *DE* describatur, sit ut velocitas inverse, ideoque inverse ut linea recta quae potest aream *ABFV*; sitque *DL*, atque ideo area nascens *DLME*, ut eadem linea recta inverse: erit tempus ut area *DL ME*, & summa omnium temporum ut summa omnium arearum, hoc est (per Corol. Lem. IV.) tempus totum quo linea *AE* describitur ut area tota *ATVME. Q.E.D.*

Corol. 1. Si *P* sit locus, de quo corpus cadere debet, ut urgente aliqua uniformi vi centripeta nota (qualis vulgo supponitur gravitas) velocitatem acquirat in loco *D* aequalem velocitati, quam corpus aliud vi quacunque cadens acquisivit eodem loco *D*, & in perpendiculari *DF* capiatur *DR*, quae sit ad *DF* ut vis illa uniformis ad vim alteram in loco *D*, & compleatur rectangulum *PDRQ*, eique aequalis abscindatur area *ABFD;* erit *A* locus de quo corpus alterum cecidit. Namque completo rectangulo *DRSE*, cum sit area *ABFD* ad aream *DFGE* ut *VV* ad 2.*VI*, ideoque ut $\frac{1}{2}V$ ad *I*, id est, ut semissis velocitatis totius ad incrementum velocitatis corporis vi inaequabili cadentis; & similiter area *PQRD* ad aream *DRSE* ut semissis velocitatis totius ad incrementum velocitatis corporis uniformi vi cadentis; sintque incrementa illa (ob aequalitatem temporum nascentium) ut vires generatrices, id est, ut ordinatim applicatae *DF*, *DR*, ideoque ut areae nascentes *DFGE, DRSE;* erunt ex aequo areae totae *ABFD, PQRD* ad invicem ut semisses totarum velocitatum, & propterea, ob aequalitatem velocitatum, aequantur.

Corol. 2. Unde si corpus quodlibet de loco quocunque *D* data cum velocitate vel sursum vel deorsum proiiciatur, & detur lex vis centripetae, invenietur velocitas eius in alio quovis loco *e*, erigendo ordinatam *eg*, & capiendo velocitatem illam ad velocitatem in loco *D* ut est recta, quae potest rectangulum *PQRD* area curvilinea *DFge* vel auctum, si locus *e* est loco *D* inferior, vel diminutum, si is superior est, ad rectam quae potest rectangulum solum *PQRD.*

Corol. 3. Tempus quoque innotescet erigendo ordinatam *em* reciproce proportionalem lateri quadrato ex *PQRD + vel - DFge, &* capiendo tempus quo corpus descripsit lineam *De* ad tempus quo corpus alterum vi uniformi cecidit a *P* & cadendo pervenit ad *D*, ut area curvilinea *DLme* ad rectangulum $2.PD\times DL$. Namque tempus quo corpus vi uniformi descendens descripsit lineam *PD*, est ad tempus quo corpus idem descripsit lineam *PE* in subduplicata ratione *PD* ad *PE*, id est (lineola iamiam nascente) in ratione *PD* ad $PD+\frac{1}{2}DE$ seu 2.*PD* ad $2PD+DE$, & divisim, ad tempus quo corpus idem descripsit lineolam *DE* ut *2PD* ad *DE*, ideoque ut rectangulum $2PD\times DL$ ad aream *DLME;* estque tempus quo corpus utrumque descripsit lineolam *DE* ad tempus quo corpus alterum inaequabili motu descripsit lineam *De*, ut area *DLME* ad aream *DLme*, & ex aequo tempus primum ad tempus ultimum ut rectangulum $2PD\times DL$ ad aream *DLme*.