SECTION VI.

Concerning the finding of motions in given orbits. (s)

(t) Thus, according to the annotated edition of the Principia by Fathers Le Seur and Jacquier, the last edition of which was pub. in 1833 in Glasgow, the so-called 'Jesuit Edition' , we have this summary :-

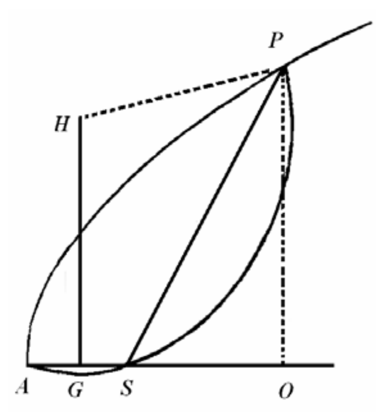
Note 338 (*s*), p. 202:

Newton in this whole section supposes the body thus to be moving in a given conic trajectory, so that with the radii drawn to the focus of the trajectory, areas or sectors will be described proportional to the times; for by that law in Book 3, all the planets have been shown by the phenomena to revolve in conic section orbits. In addition the time is required to be noted, in which the body will arrive at some given point of the trajectory of the orbit from some given point of the trajectory, e.g. from the principal vertex point of that to some other point on the same given trajectory ; and for the area to be given or the sector of the trajectory corresponding to this given time ; and from these given, the moving position on the trajectory may be sought at some other given time; or on the other hand the time may be sought in which the moving point will reach some given point in the trajectory; for since the areas shall be proportional to the times, in some given time, the area described in this time may be found, and in turn with the area described given, the time may be found in which the motion was described.

PROPOSITION XXX. PROBLEM XXII.

To find the position of a body in a given parabolic trajectory at a designated time.

Let *S* be the focus and *A* the principle vertex of the parabola, and let $4AS \times M$ be equal to the area of the parabola *APS* cut off, by which the radius *SP*, either after the height of the body had been described from the vertex, or before the approach of this to the vertex has been described. The magnitude of this area cut off is known to be proportional to the time itself. Bisect *AS* in *G,* and raise a perpendicular *GH* equal to 3*M*, and the circle described with centre *H*, radius *HS* will cut the parabola at the position sought *P*. For, with the perpendicular *PO* sent to the axis and with *PH* drawn, there is



$$AG^2 + GH^2 \ (= HP^2 = \overline{AO - AG}^{\,2} + \overline{PO - GH}^{\,2})$$
$$= AO^2 + PO^2 - 2AG \times AO - 2GH \times PO + AG^2 + GH^2.$$

From which

$$2GH \times PO = (AO^2 + PO^2 - 2AG \times AO) = AO^2 + \tfrac{3}{4}PO^2. \text{ [Since } \tfrac{OP^2}{4} = AS \times AO \text{,]}$$

For AO^2 write $AO \times \frac{PO^2}{4AS}$ [*i.e.* $2GH \times PO = AO \times \frac{PO^2}{4AS} + \frac{3}{4}PO^2$] ; and with all the applied terms divided by *3PO* and multiplied by 2*AS*, there becomes

$$\tfrac{4}{3}GH \times AS \ (= \tfrac{1}{6}AO \times PO + \tfrac{1}{2}AS \times PO$$
$$= \tfrac{AO+3AS}{6} \times PO = \tfrac{4AO-3SO}{6} \times PO = \text{area } \overline{APO - SPO}) = \text{area } APS.$$

[For the area under the parabola is $\frac{2}{3}AO \times OP$ (either by integration, or from Archimedes Prop. 17, quadr. Parab. sup. Theor. IV, *de Parabola*).]

But *GH* was equal to 3*M*, and thence $\frac{4}{3}GH \times AS$ is $4AS \times M$. Therefore the area cut *APS* is equal to the area $4AS \times M$, that was required to be cut.

Q. E. D.

*Corol.*1. Hence *GH* is to *AS*, as the time in which the body has described the arc *AP* to the time in which the body has described the arc between the vertex *A* and the perpendicular to the axis erected from the focus *S*. [Let *P'S* be the semi-latus rectum, for which $P'O = 2AS$, then the area *APS* : area *AP'S* = $\frac{4}{3}GH \times AS : \frac{2}{3}AS \times 2AS = GH{:}AS$.]

Corol. 2. And with the circle *ASP* always passing through the moving body at *P* , the velocity of the point *H* is to the velocity which the body had at the vertex *A* as 3 to 8; and thus also in that ratio is the line *GH* to the right line that the body in the time of its motion from *A* to *P*, that it may describe with the velocity it had at the vertex *A*. [See Note 339 (b)]

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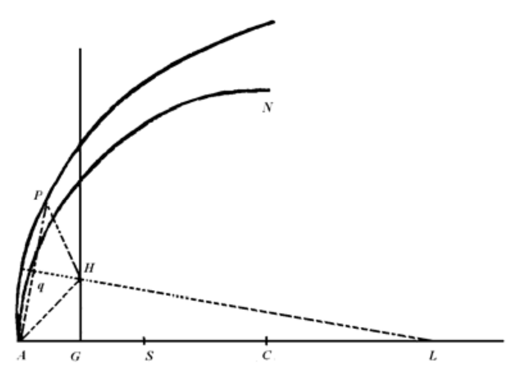
338 (*t*) : Let *S* be the focus, and *A*, the principal vertex of the parabola, and the time shall be given in which the body in moving on the parabola, as we have established (358.), from the vertex *A* to the point *P*, or arrives at the vertex *A* from the point *P* , or the time shall be given in which the sector *APS* is described.

Note 339 (b):

Join AP, and at the mid-point *q* of this, raise the perpendicular L, and since (from the demonstration) always $HP = HA$, and thus *AP* is the chord of this circle whose centre is H. And thus (by Book I, Sect. I Euclid's Elements) that perpendicular *qL* cuts the right line *GH* in *H* ; and on account of the similar triangles *LGH*, *LqA* there is $GH : qA$ or $\frac{1}{2}AP = LG : Lq$. There may be



taken $AC = 2AS$ clearly half of the latus rectum of the parabola, and with centre *C*, and with the radius *CA*, the circle *AN* may be described, this is a tangent to the parabola at *A* (note 241); truly with the points *P* and *A*, *H* and *G* coincident, and thus also the points *L* and *C* coincide. Let $Lq = LA = CA = 2AS = 4GS$ and $LG = CG = 3GS$, and with the arc *AP* equal to the chord *AP*, (Lem. VII) ; from which since in the above proportion there shall be $GH : \frac{1}{2}AP = LG : Lq$, in this case there shall be $GH : \frac{1}{2}AP = 3GS : 4GS$; that is, $GH : AP = 3 : 8$. Truly on account of the uniformity of the motion and equally by continuing in its state over the time through the points *A*, *P*, *G*, *H*, the velocity of the point *H* at *G*, is to the velocity of the body *P* at *A* as *GH* to *AP*, and because (from the Dem.) $\frac{4}{3}AS \times GH$ is always equal to the area *APS*, and $\frac{4}{3}AS$ is a constant quantity, *GH* will always be as the area *APS*, that is as the time in which the point *H*, has run through *GH*, and is hence whose motion is uniform and the same everywhere. Whereby the velocity of the point *H* is everywhere to the velocity that the body *P* has at *A*, as arising at *GH* , to that arising at *AP*, that is, as 3 to 8. Q.e.d.

Corol. 3. Hence also in turn it is possible to find the time in which the body has described some designated arc *AP*. Join *AP* and to the mid-point of this erect the perpendicular *GH* to the line crossing in *H*. [*i.e.* the same argument for the time can be applied to any point.]

LEMMA XXVIII.

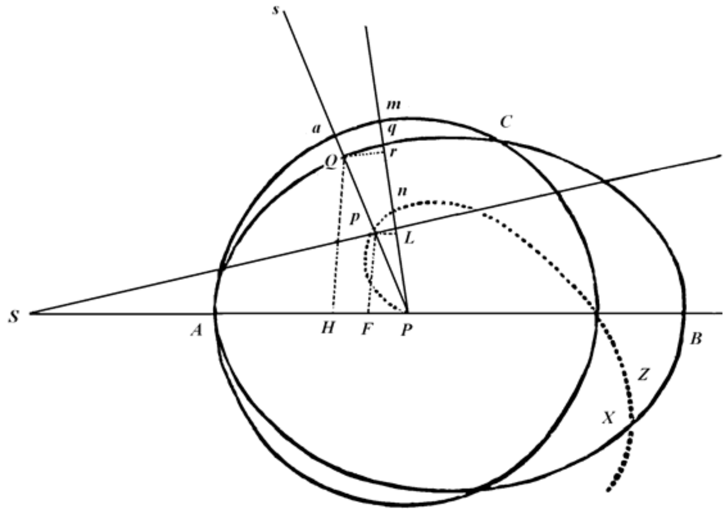
No figure is extant, cut by right lines as you please, of which the oval area may be able to be found generally by equations with a finite number of terms and dimensions.

Within an oval some point may be given, about which or pole a right line may rotate perpetually, with a uniform motion, and meanwhile on that right line a point may emerge moveable from the pole, and it may always go forwards with that velocity, which shall be as the square of that right line within the oval. [See note 359.] By this motion the point will describe a spiral with infinite rotations. Now if a part of the oval area can be found cut from that right line by a finite equation, also the distance may be found by the same equation of the point from the pole, which is proportional to this area, [See note 360 below.] and thus all the points of the spiral can be found by a finite equation: and therefore from any position of this right line the intersection with a given position with the spiral also can be found by a finite equation. And every right line produced infinitely cuts the spiral in an infinite number of points, and the equation, [See note 361 below.] by which some intersection of the two lines may be found, shows all the intersections of these with just as many roots, and thus it rises to just as many dimensions as there are intersections. Because two circles cut each other mutually in two points, a single intersection may not be found except by an equation of two dimensions, from which the other intersection also may be found. Because there can be four intersections of the two conic sections [See note 362 below.], it is not possible to find any of these generally except by an equation of four dimensions, from which all may be found at the same time.

For if these intersections themselves may be sought, since the law and the condition of all is the same, the calculation will be the same in each case, and therefore always the same conclusion, which therefore must include all like intersections and shown indifferently. From which also the intersections of conic sections and of curves of the third order, because from that there can be six, likewise they may be produced by equations of six dimensions, and six intersections of the two curves of the third power, because nine can be possible, likewise they may arise from equations of nine dimensions. Unless that by necessity may come about, all solid [*i.e.* volume or 3 dimensional] problems may be allowed to be reduced to the plane, and greater than solid [three dimensions] to three dimensions. I am concerned here with curves that are irreducible in power. For if the equation, by which a curve is defined, can be reduced to a lower power : the curve will not be single, but composed from two or more equations, whose intersections can be found separately by different calculations. In the same manner the intersections of two right lines and the sections of cones will always be produced by equations of two dimensions, of three right lines and of irreducible curves of the third power by equations of the third dimension, of four right lines and of irreducible curves of the fourth power by equations of the fourth dimension, and thus indefinitely. Therefore the intersections of right lines and spirals will require equations with an infinite number of dimensions, since this curve shall be simple and irreducible into many curves, and an infinitude of roots, by which all the intersections can be likewise shown. For this is the same law and the same calculation of everything. For if from the pole a perpendicular may be sent through that intersecting right line, and that perpendicular may be rotated together with the intersecting line about the pole, the intersections of the spiral will cross mutually from one into another, whatever shall be first or nearest, after one revolution will be the second, after two the third, and thus henceforth: nor meanwhile will the equation be changed except for the change in the magnitude of the quantities by which the position of the cuts may be determined. From which since these quantities after individual rotations may return to the first magnitudes, the equation will be returned to the first form, and thus one and the same will show all the intersections and therefore an infinite number of roots will be had, from which everything is able to be shown. Therefore the intersection of a right line and a spiral will be unable to be shown generally by a finite equation, and therefore nothing may be shown by such an equation generally with the area of which oval, cut by designated right lines.

By the same argument, if the distance between the pole and the point, by which the spiral may be described, should be taken proportional to the perimeter of the oval cut, it cannot be proved because the length of the perimeter cannot be shown generally by a finite equation. But here I talk about ovals which are not touched by conjugate figures going off to infinity.

Further notes from the Le Seur and Jacquier edition:



Note 359: Within the oval *ACBA* some point *P* may be given, or about which pole the right line *PS* may rotate perpetually, with a uniform motion, thus so that the given point *A* of this line may describe equal arcs of circles *AamX* in equal intervals of time, and meanwhile on that right line *PS*, the moving point *p* may emerge from the pole *P*, and will always go on the same right line *PS* with the velocity which shall [note the error here!] be as the square of that right line within the oval, that is, when the line *PS* arrives at the place *Ps*, and the mobile point *P* at *p*, the velocity of the point *p* shall be as the square of the right line *PQ* contained between the pole P and the oval *AQCP*, in this motion that point *p*, describes the spiral *PpnZ*, with infinite rotations.

Note 360. With these in place the right line *Pp* will always be as the area *PAQP*; for the circle *AamX* may be understood to be divided into innumerable equal arcs as *am*, and with the radii of the spiral *PQ, Pq* drawn, the perpendiculars *Qr, pL* sent from the circle to the oval crossing at *p* and *n, a* and *m, Q* and *q*, to *Pq*, and in the same time in which the point *a*, will traverse the arc *am*, the point *p* may run through the arc *am*, the point *p* may run through the line *Ln* ; on which account with the arc *am* arising, *Ln* will be as the velocity of the point *p* on the right line *Ps*, that is, (by hyp.) as the square of the line *PQ*; again on account of the similar triangles *Pam, PQr* $Pa : PQ = am : Qr = \frac{PQ\times am}{Pa}$, and hence the area of the sector arising *PQq*, $\frac{1}{2}Qr\times PQ = \frac{PQ^2\times am}{2Pa}$. Therefore since *am* and $2Pa$ shall be constant quantities (by hyp.), the area *PQq* arising or the flux of the area *PAQ* will be as PQ^2, and thus as the emerging *Ln*, or as the flux of the right line *Pp*, and hence the total area of the fluent *PAQ*, will be as the total right line of the flux *Pp* , (Cor. Lem. IV) Q.e.d.

Note 361. The points *p* and *Q* may be referred to the right line *AB*, in the given position with the perpendiculars *Q, H, pF* sent to *AB* and the area *PAQ* shall be equal to a finite quantity *E* from the variable lines *PH, QH* and composed from other constants as desired, and since the line *Pp* is proportional to the area *PAQ* or to the finite quantity *E* (360), that line will be able to be expressed by a factor from the quantity *E* into a constant quantity *B*, and $Pp = E\times B$ will be a finite equation. Truly from the similar triangles *PFp, PHQ* and the right angle to *H*, $Pp : pF = PQ$, or $\sqrt{PH^2 + QH^2} : QH$, and $Pp : PF = PQ$, or $\sqrt{PH^2 + QH^2} : PH$, and besides from the nature of the oval *AQCB*, another equation may be given between *PH and QH*, four finite equations therefore may be found, which likewise contain five variables, to wit *Pp, PF, pF, PH, QH*, and which hence will be able to be reduced to a single finite equation in which only two variables *PF, pF* may be found, and thus by this finite equation all the points of the spiral will be able to be found, and therefore with the position given of any right line *Sp*, the intersection *p* of any given right line *Sp* with the spiral also can be found from a finite equation; for since two right lines *Sp, SB* shall be given in position, the magnitude of the line *SP* and the nature of the triangle *SpF* may be given, and hence the ratio of the lines *SF* or $SP \mp PF$ to *Fp*, and a new equation may be found between *PF* and *Fp*; therefore by this equation and by the other which is to the spiral, *PF* may be determined, and *Fp*, and the point of intersection *p* may be found by a finite equation.

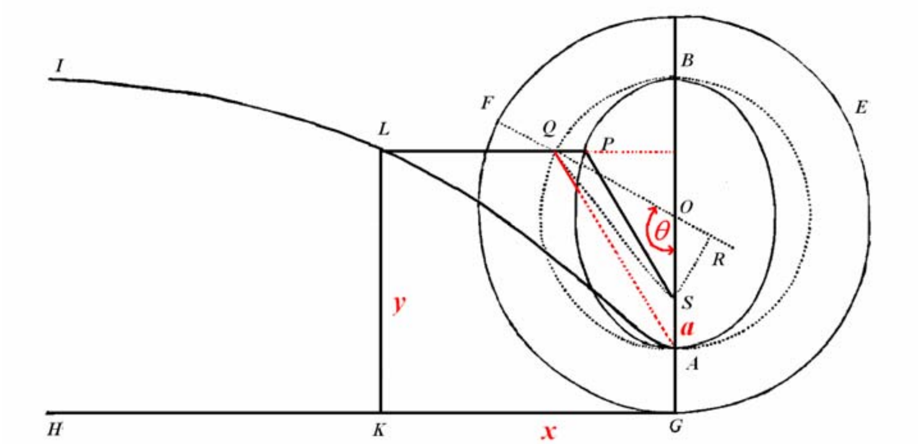
Note 362. The two lines *AMS, Sms* may be referred to the same right line *AQ* given in place, mutually intersecting in the points *S, s* , and let AQ, AP be the common abscissae, and *QS, PM, Pm* the ordinates accorded to these; because with the common intersections of the lines *SMs, Sms*, the ordinates *PM, Pm* are equal, if in the two equations for the lines *SMs, Sms*, with the abscissae remaining common, in place of the ordinates *PM, Pm*, the same letters may be written, such as *y*, and then from these equations the letter which expressed the common abscissa may be eliminated, an equation will be obtained composed from *y* and constants only. Again this final equation no more will determine the first common ordinate *SQ*, or the first intersection *S*, than the third of the fourth, etc, since there shall be the same law for everything and likewise the same condition in the calculation; therefore this equation must show completed and indifferently all the common coordinates *QS*, and likewise all the intersections *S*, and thus just as many roots or the values of *y* to be returned as there are common ordinates or intersections, but the equation has just as many dimensions as the number of roots; and thus if the intersections *S, s* of the lines *SMs, Sms*, shall be finite in number, also the equation which may determine those is finite ; but if the intersections were infinite in number, the equation will be of infinite dimensions and with an infinitude of roots.
See also Chandrasekbar, p. 133. Here the point is made that 'smooth' curves are geometrically rational or algebraic, and thus their area can be found; curves with points of discontinuity, called 'geometrically irrational' by Newton, such as a sector of the ellipse, are not smooth, and do not satisfy an algebraic equation of a finite number of dimensions, and therefore cannot be integrated exactly. However, we find in what follows that such sectors can be related to the area of the corresponding circular circle, and approximate solutions can be found for the angle, if we know the area : which is of course the aim of this section – the area is known and the angle or position in the orbit is required. Open curves such as the parabola and hyperbola do not suffer from this condition.

Corollary.

Hence the area of [the sector] of an ellipse, which will be described by a radius drawn from the focus to a moving body, will not be produced by a finite equation for the time given; and therefore cannot be determined geometrically from the drawing of rational curves . I call curves geometrically rational [*i.e.* algebraic], all of the points of which are defined by lengths, that is, they are able to be determined by complicated ratios of lengths ; and the others (as spirals, quadratrixes, trochoides) I call geometrically irrational. For the lengths which are or are not as number to number (just as in the tenth Book of the Elements) are arithmetically rationals or irrationals. [Whiteside notes that it was Barrow who made this connection on innumerability, and not Euclid, which Newton corrected.] Therefore I cut off an area of an ellipse proportional to the time as follows, by a geometrically irrational curve [*i.e*. a cycloid].

PROPOSITION XXXI. PROBLEM XXIII.

To find the position of a body in a given elliptical trajectory at a given designated time.



Let *A* be the principle vertex of an ellipse *APB*; *S* the focus, and *O* the centre, and let *P* be the position of the body required to be found. Produce *OA* to *G*, so that *OG* shall be to *OA* as *OA* to *OS* [In modern terms, if we let the eccentricity be *e*, than $OS = ae$ and $OG = a / e$, where $OA = a$.] Erect the perpendicular *GH*, and with centre *O* and radius *OG* describe the circle *GEF*, and upon the ruler or established right line *GH*, the wheel *GEF* may progress by rotating about its axis, meanwhile with its own point *A* tracing out the trochoid *ALI.* With which done, take *GK* in the ratio to the perimeter of the wheel *GEG*, so that the time, in which the body by progressing from *A* will describe the arc *AP*, is to the time of one revolution of the ellipse. The perpendicular *KL* can be erected crossing the trochoid [or prolate cycloid] at *L*, and with *LP* itself drawn parallel to *KG* will meet the ellipse in the position of the body sought *P*.

[Note that the ellipse does not itself rotate, just the point *P* on its curve, along with the imaginary wheel , and the diagram is counter intuitive, as the circle must rotate clockwise while it moves to the left to produce the curve shown so that *P* falls on *L* at the position shown; this appears to be rather strange; the diagram by Wren who originally constructed this solution, had the cycloid going to the right ; thus we may consider the point *P* to coincide with *A* when $\theta = 0$, which is also a point on the cycloid, and the wheel to be rolling to the left, so that *P* later actually coincides with *L* in the position shown. In this case, in rotating through the positive angle θ , the point P moves to the left a distance *GK* or $\frac{a\theta}{e}$, while in the same time shortening this distance by the amount $a\sin\theta$; hence relative to the origin *O* fixed in space in the position shown, we have the point *L* situated at $x = \frac{a}{e}(\theta - e\sin\theta)$; the position of the *y* coordinate is given by $y = \frac{a}{e}(1 - e\cos\theta)$. Thus, this point is assumed given, from which we may deduce the area swept out by P in the orbit.]

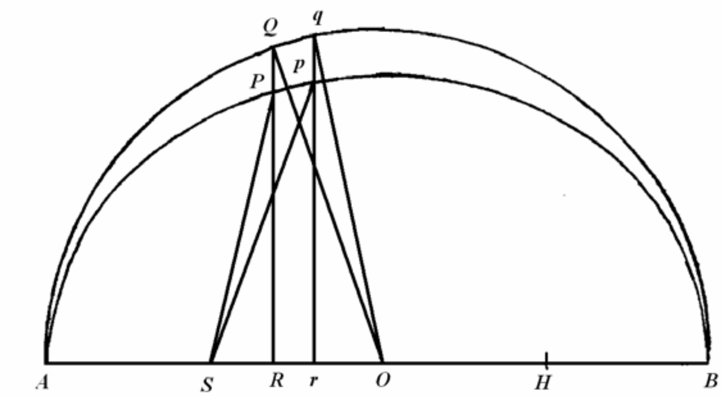
For with centre *O*, with the radius *OA* the semicircle *AQB* may be described, and for *LP* to meet the arc *AQ* if there is a need by producing *Q*, and *SQ*, *OQ* may be joined. For the arc *EFG* may cross *OQ* in *F*, and to the same *OQ* there may be sent the perpendicular *SR*. The area *APS* is as the area *AQS*, that is as the difference between the sector area *OQA* and the triangle area *OQS*, or, as the difference of the products $\frac{1}{2}OQ \times arcAQ$ and $\frac{1}{2}OQ \times SR$, [recall that for different ellipses or for an ellipse and a circle, on the same diameter and with the ordinates in a given ratio, then the corresponding areas of elliptic (and circular) sectors are in the same ratio *b/a*.] that is, on account of the given $\frac{1}{2}OQ$, as the difference between the arc *AQ* and the right line *SR*, [*i.e.* the circular sector $AQS = \frac{1}{2}OQ\left(arcAQ - SR\right)$, and the elliptic sector $APS = \frac{b}{a}AQS$], and thus (since the equal ratios shall be given, *SR* to the sine of the arc *AQ*, *OS* to *OA*, *OA* to *OG*, and the *arc AQ* to the *arc GF*, and on separating, $arcAQ - SR$ to $GF - \sin arcAQ$), as *GK* to the difference between the arc *GF* and the sine of arc *AQ*.
[*i.e.* $APS = \frac{a}{2}\left(arcAQ - SR\right)$ giving $SR = OS\sin arcAQ = ae\sin\theta$, and hence the area of the elliptic sector $APS = \frac{ab}{2}\left(\theta - e\sin\theta\right)$, which relates to the known *x* coordinate, which is moving along the *x* axis at a constant speed. There is still the problem of finding the angle from the coordinate *x* analytically, if we are not content with simply measuring it from a mechanical model.]

Q.E.D.

[Whiteside, in VI note134, points put that this mechanical solution of the problem of determining the time to travel along the arc of the ellipse from the apse *A* is essentially the same as that found by Wren in 1658, and published by Wallis in his tract *De Cycloide* the following year. Note that this mechanical solution uses arcs rather than areas. Further information can be found in the notes provided by Whiteside, especially note 138. Part of the confusion about the following method was due to a page being mislaid by Humphrey Newton, Newton's amanuensis, which resulted in several pages being deleted, as Newton was unable in the short time availabe due to the printing in process, to reproduce that page. The ellipse is now put upright, and a numerical method is used to solve Kepler's equation.]

Scholium.

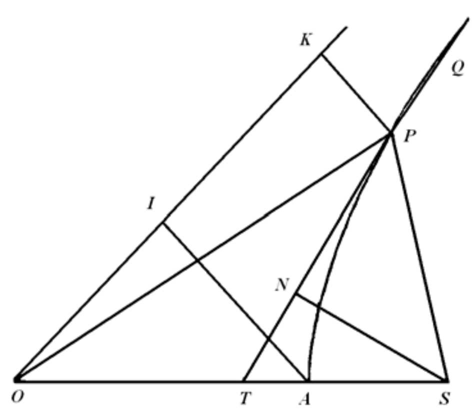
The remaining description of this curve shall be had with difficulty, it is better to give an approximate solution. Then a certain angle *B* may be found, which shall be to the angle of the degrees 57.29578, that an arc subtends equal to the radius, as it is the distance *SH* of the focus to the diameter of the ellipse *AB;* then also a certain length L, which shall be to the radius [*r*] in the same ratio inverted. [Thus, in modern terms, we set



the eccentricity equal to the size of the angle *B* in radians or $\frac{HS}{AB} = \frac{SO}{AO} = B = \frac{r}{L} = e$, the eccentricity.] With which once found, the problem then may be put together by the following analysis. By some construction, or in some manner by making a guess, the position of the body *P* may be known near to the true position *p*. And with the applied ordinate *PR* sent to the axis of the ellipse, from the proportion of the diameters of the ellipse, the ordinate *RQ* of the circumscribed circle *AQB* is given, which is the sine of the angle *AOQ* with the radius being *AO* [by def.], and which cuts the ellipse at *P*. That will suffice in a rough calculation to find the angle in approximate numbers. [That is, $\frac{PR}{QR} = \frac{SH}{AB} = e$.] Also the angle may be known to be proportional to the time, that is, which may be to four right angles, as the time is in which the body will describe the arc *AP*, to the time of one revolution in the ellipse. Let this angle be *N*. [Thus, $\frac{t}{T} = \frac{arcAp}{2\pi r} = \frac{N}{2\pi}$.] [Clearly, the solution is required to the problem: area of sector of ellipse *ApS* : area of sector of circle *AqS* = *b/a*. This may be developed as above, and we come upon an equation similar to that above : the elliptic sector $ApS = \frac{ab}{2}\left(\theta - e\sin\theta\right)$; or $N = \theta - e\sin\theta$, and the value of θ is required to satisfy this equaton. Following Whiteside, an approximate solution is set up $\theta_2 = \theta_1 + \varepsilon_1$, where ε_i is small; from which successive approximations follow. The initial value θ_1 it taken as the angle *AOQ*, while the values for $\varepsilon_1, \varepsilon_2$, etc., are successively *E*, *G*... We may, for clarification, invoke Newton's Method for finding a better approximation to the root of an equation : if x_0 is an approximate root of the equation $f(x) = 0$, then a better approximation is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, and so on; here $f(\theta) = N - \theta + e\sin\theta$, in which case

$f'(\theta) = -1 + e\cos\theta$, and $\theta_2 = AOQ - \frac{N - AOQ + B\sin AOQ}{B\cos AOQ} = AOQ - \frac{(N-AOQ)L + r\sin AOQ}{L\cos AOQ}$;

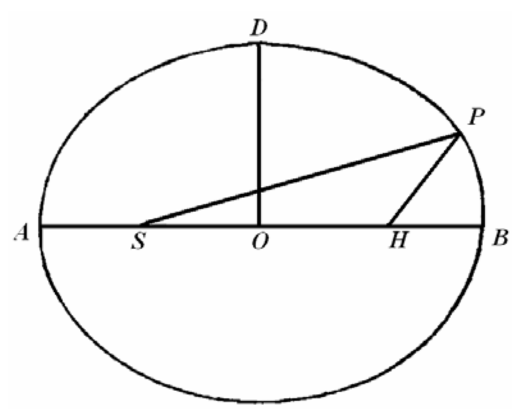
thus, the correction is the angle $-\frac{(N-AOQ)L + r\sin AOQ}{L\cos AOQ} = \frac{(N-AOQ)L}{L\cos AOQ} + \frac{r\sin AOQ}{L\cos AOQ}$; Newton sets the correction into two parts, $D = \frac{r\sin AOQ}{L\cos AOQ}$; and applies the correction *D* to the other part, giving : $E = \frac{L(N - AOQ + D)}{L\cos AOQ}$.] Then an angle *D* may be taken to the angle *B*, as the sine of the angle *AOQ* to the radius itself, and an angle *E* to the angle $N - AOQ + D$, as the length *L* to the same length *L* diminished by the cosine of the angle *AOQ* [this statement applies to both parts above], where that angle is less than a right angle, but increased when it becomes greater. Afterwards then the angle *F* may be taken to the angle *B*, so that the sine of the angle $AOQ + E$ is to the radius, and the angle *G* to the angle $N - AOQ - E + F$ as the length *L* to the same length diminished by the cosine of the angle $AOQ + E$, when that is less than a right angle, increased when greater. In the third place in turn, the angle *H* may be taken to the angle *B*, as the sine of the angle $AOQ + E + G$ is to the radius; and the angle *I* to the angle $N - AOQ - E - G + H$, and the length L to the same length diminished by the cosine of the angle $AOQ + E + G$, when this is less than a right angle, increased when greater. And thus it is permitted to go on indefinitely. Finally the angle *AOQ* may be taken equal to the angle $AOQ + E + G + I + \&c$. And from the cosine of this *Or* and with the ordinate *pr*, which is to the sine of this *qr* as the minor axis of the ellipse to the major axis, the correct position *p* of the body is obtained. If when the angle $N - AOQ + D$ is negative, the + sign of *E* everywhere is changed into –, and the – sign into +. Likewise it is to be understood concerning the signs of *G* and I, where the angles $N - AOQ - E + F$, and $N - AOQ - E - G + H$ may be produced negative.
But as the infinite series $AOQ + E + G + I + \text{etc}$. converges rapidly, thus so that scarcely will there be any need to progress further than to the second term *E*. And the calculation may be based on this theorem, that the area *APS* shall be as the difference between the arc *AQ* and the right line from the focus *S* sent perpendicularly to the radius *OQ*.



A similar calculation is put in place with the hyperbola problem. Let *O* be the centre of this, the vertex *A*, the focus *S* and the asymptote *OK*. The magnitude of the area cut off is requiring to be proportional to the time. Let that be *A*, and a guess is made concerning the position of the right line *SP*, which shall be cut approximate to the true area *APS*. *OP* may be joined, and from *A* and *P* to the asymptote draw *AI* and *PK* parallel to the other asymptote, and the area *AIKP* will be given by a table of logarithms, and from the area equal to *OPA*, which taken from the triangle *OPS*, the area cut *APS* will be left. From the difference of the area requiring to be cut off *A* and of the area cut off *AP*S the double $2APS - 2A$ or

$2A-2APS$ to the line *SN*, which is perpendicular from the focus *S* into the tangent *TP*, the length of the chord *PQ* may arise. But that chord *PQ* may be inscribed between *A* and *P*, if the area cut *APS* shall be greater than the area required to be cut *A*, otherwise cut at the opposite side of the point *P*: and the point *Q* will be a more accurate position of the body. And with the computation repeated the same may be found more accurately indefinitely.

And from these calculations the problem generally can be put together analytically. Truly the calculation which follows is more fitted to astronomical uses. With the semi-axis of the ellipse *AO*, *OB*, *OD* present, and *L* its latus rectum, and *D* the difference between the semi-minor axis *OD* and the semi-latus rectum $\frac{1}{2}L$; whereby then the angle *Y*, the sine of which shall be to the radius as the rectangle formed from that difference *D*, and the half sum of the axes $AO+OD$ to the square of the major axis *AB;* then the angle *Z*, the sine of which shall be to the radius as twice the rectangle formed by the distance between the foci *SH* and that difference *D* to the triple of the square of the major semi-axis *AD.* With these angles thus found ; the position of the body thus can be determined. Take the angle *T* proportional to the time in which the arc *BP* has been described, or equal to the mean motion (as they say) ; and the angle *V*, as it is the first equation of the mean motion, to the angle *Y*, the first maximum equation, as the sine of double the angle *T* to the radius ; and the angle *X*, the second equation, to the angle *Z*, the second maximum equation, as the cube of the sine of the angle *T* is to the cube of the radius. [See Whiteside note 159 Vol. VI etc.] Take the sum of the angles *T, V, X* or the sum $T+X+V$, if the angle *T* is less than a right angle , or the difference $T+X-V$, if this greater than a right angle and less than two right angles, equal to the angle *BHP*, the mean motion equation ; and if *HP* should cut the ellipse at *P*, with *SP* drawn it will cut the area *BSP* approximately in proportion to the time. This practice may be seen to be work well enough, because of the very small angles *V* and *X* therefore arising, in seconds of arc, if it pleases to be sufficient to find two or three figures of the places. And also this is accurate enough for the theory of the planets. For in the orbit of Mars itself, the equation of the centre of which is ten degrees, the error scarcely will exceed a second of arc. But when the equated mean motion has been found (the angle *BHP*), then the true motion (the angle *BSP*) and the distance (*SP*)are had readily by the well-known method [of Seth-Ward; this useful reference is missing in the 3rd edition, in place in the 1st and 2nd edition].



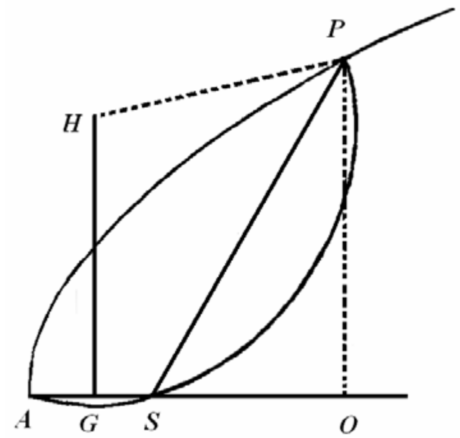
Up to the present we have been concerned with the motion of bodies on curved lines. But it may also happen that the body ascends or descend by a right line, and I now go on to to set out matters relating to motions of this kind.

SECTIO VI.

De inventione motuum in orbibus datis.

PROPOSITIO XXX. PROBLEMA XXII.

Corporis in data traiectoria parabolica moti invenire locum ad tempus assignatum.



Sit *S* umbilicus & *A* vertex principalis parabolae, sitque $4AS \times M$ aequale areae parabolicae abscindendae *APS*, quae radio *SP*, vel post excelsum corporis de vertice descripta fuit, vel ante appulsum eius ad ad verticem describenda est. Innotescit quantitas areae illius abscindendae ex tempore ipsi proportionali. Biseca *A S* in *G*,
erigeque perpendiculum *GH* aequale 3*M*, & circulus centro *H*, intervallo *HS* descriptus secabit parabolam in loco quaesito *P*. Nam, demissa ad axem perpendiculari *PO* & ducta *PH*, est

$$AGq + GHq\ (= HPq = \overline{AO - AG} : quad. + \overline{PO - GH} : quad.)$$
$$= AOq + POq - 2GAO - 2GH \times PO + AGq + GHq.$$

Unde

$$2GH \times PO \left(= AOq + POq - 2GAO\right) = AOq + \tfrac{3}{4} POq.$$

Pro
AOQ scribe $AO \times \frac{POq}{4AS}$; & applicatis terminis omnibus ad *3PO* ductisque in 2*A S*, fiet

$$\tfrac{4}{3} GH \times \mathrm{A}S(= \tfrac{2}{6} AO \times PO + \tfrac{1}{2} AS \times PO$$
$$- \tfrac{AO+3AS}{6} \times PO = \tfrac{4AO-3SO}{6} \times PO = \text{areae } \overline{APO - SPO})$$

=areae *APS*. Sed *GH* erat 3*M*, & inde $\frac{4}{3} GH \times AS$ est $4AS \times M$. Ergo area abscissa *APS* aequalis est abscindenae $4AS \times M$

Q. E. D.

*Corol.*1 Hinc *GH* est ad *AS*, ut tempus quo corpus descripsit arcum *AP* ad tempus quo corpus descripsit arcum inter verticem *A* & perpendiculum ad axem ab umbilico *S* erectum.
Corol. 2. Et circulo *ASP* per corpus motum *P* perpetuo transceunte, velocitas puncti *H* est ad velocitatem quam corpus habuit in vertice *A* ut 3 ad 8; ideoque in ea etiam ratione est linea *GH* ad lineam rectam quam corpus tempore motus sui ab *A* ad *P*, ea cum velocitate quam habuit in vertice *A*, describere posset.

Corol. 3. Hinc etiam vice versa inveniri potest tempus quo corpus descripsit arcum quemvis assignatum *AP*. Iunge *AP* & ad medium eius punctum erige perpendiculum rectae *GH* occurrens in *H*.

LEMMA XXVIII.

Nulla extat figura ovalis cuius area, rectis pro lubitu abscissa, possit per aequationes numero terminorum ac dimensionum finitas generaliter inveniri.

Intra ovalem detur punctum quodvis, circa quod ceu polum revolvatur perpetuo linea recta, uniformi cum motu, & interea in recta illa exeat punctum mobile de polo, pergatque semper ea cum velocitate, quae sit ut rectae illius intra ovalem quadratum. Hoc motu punctum illud describet spiralem gyris infinitis. Jam si areae ovalis a recta illa abscissae portio per finitam aequationem inveniri potest, invenietur etiam per eandem aequationem distantia puncti a polo, quae huic areae proportionalis est, ideoque omnia spiralis puncta per aequationem finitam inveniri possunt: & propterea rectae cuiusvis positione datae intersectio cum spirali inveniri etiam potest per aequationem finitam. Atqui recta omnis infinite producta spiralem secat in punctis numero infinitis, & aequatio, qua intersectio aliqua duarum linearum invenitur, exhibet earum intersectiones omnes radicibus totidem, ideoque ascendit ad tot dimensiones quot sunt intersectiones. Quoniam circuli duo se mutuo secant in punctis duobus, intersectio una non invenietur nisi per aequationem duarum dimensionum, qua intersectio altera etiam inveniatur. Quoniam duarum sectionum conicarum quatuor esse possunt intersectiones, non potest aliqua earum generaliter inveniri nisi per aequationem quatuor dimensionum, qua omnes simul inveniantur. Nam si intersectiones illae seorsim quaerantur quoniam eadem est omnium lex & conditio, idem erit calculus in casu unoquoque, & propterea eadem semper conclusio, quae igitur debet omnes intersectiones simul complecti & indifferenter exhibere. Unde etiam intersectiones sectionum conicarum & curvarum tertiae potestatis, eo quod sex esse possunt, simul prodeunt per aequationes sex dimensionum, sex & intersectiones duarum curvarum tertiae potestatis, quia novem esse possunt, simul prodeunt per aequationes dimensionum novem. Id nisi necessario fieret, reducere liceret problemata omnia solida ad plana, & plusquam solida ad solida. Loquor hic de curvis potestate irreducibilibus. Nam si aequatio, per quam curva definitur, ad inferiorem potestatem reduci possit: curva non erit unica, sed ex duabus vel pluribus composita, quarum intersectiones per calculos diversos seorsim inveniri possunt. Ad eundem modum intersectiones binae rectarum & sectionum conicarum prodeunt semper per aequationes duarum dimensionum, ternae rectarum & curvarum irreducibilium quartae potestatis per aequationes trium, quaternae rectarum & curvarum irreducibilium quartae potestatis per aequationes dimensionum quatuor, & sic in infinitum. Ergo rectae & spiralis intersectiones numero infinitae, cum curva haec sit simplex & in curvas plures irreducibilis, requirunt aequationes numero dimensionum & radicum infinitas, quibus intersectiones omnes possunt simul exhiberi. Est enim eadem omnium lex & idem calculus. Nam si a polo in rectam illam secantem demittatur perpendiculum, & perpendiculum illud una cum secante revolvatur circa polum, intersectiones spiralis transibunt in se mutuo, quaeque prima erat seu proxima, post unam revolutionem secunda erit, post duas tertia, & sic deinceps: nec interea mutabitur aequatio nisi pro mutata magnitudine quantitatum per quas positio secantis determinatur. Unde cum quantitates illae post singulas revolutiones redeunt ad magnitudines primas, aequatio redibit ad formam primam, ideoque una eademque exhibebit intersectiones omnes & propterea radices habebit numero infinitas, quibus omnes exhiberi possunt. Nequit ergo intersectio rectae & spiralis per aequationern finitam generaliter inveniri, & idcirco nulla extat ovalis cuius area, rectis imperatis abscissa, possit per talem aequationem generaliter exhiberi.

Eodem argumento, si intervallum poli & puncti, quo spiralis describitur, capiatur Ovalis perimetro abscissae proportionale, probari potest quod longitudo perimetri nequit per finitam aequationem generaliter exhiberi. De ovalibus autem hic loquor quae non tanguntur a figuris conjugatis in infinitum pergentibus.

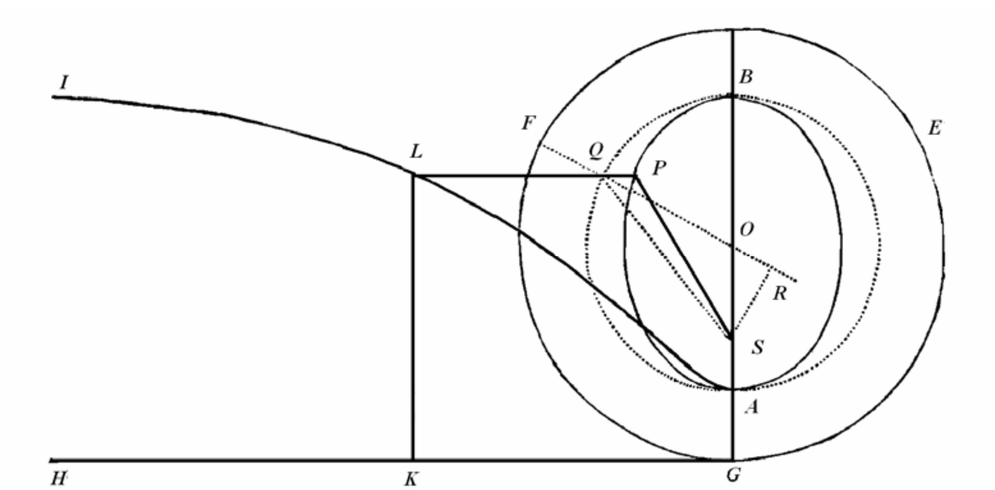
Corollarium.

Hinc area ellipseos, quae radio ab umbilico ad corpus mobile ducto describitur, non prodit ex dato tempore, per aequationem finitam; & propterea per descriptionem curvarum geometrice rationalium determinari nequit. Curvas geometrice rationales appello.quarum puncta~omnia per longitudines aequationibus definitas, id est, per longitudimum rationes complicatas, determinari possunt ; caeterasque (ut spirales, quadratrices, trochoides) geometrice irrationales. Nam longitudines quae sunt vel non sunt ut numerus ad numerum (quemadmodum in decimo elementorum) sunt arithmetice rationales vel irrationales. Aream igitur ellipseos tempori proportionalem abscindo per curvam geometrice irrationalem ut sequitur.

PROPOSITIO XXXI. PROBLEMA XXIII.

Corporis in data traiectoria elliptica moti invenire locum ad tempus assignatum.

Ellipseos *APB* sit *A* vertex principalis; *S* umbilicus, & *O* centrum, sitque *P* corporis locus inveniendus. Produc *OA* ad *G*, ut sit *OG* ad *OA* ut *OA* ad *OS.* Erige perpendiculum *GH*, centroque *O* & intervallo *O*G describe circulum *GEF*, & super regula *GH*, ceu fundo, progrediatur rota *GEF* revolvendo circa axem suum, & interea puncto suo *A* describendo trochoidem *ALI*. Quo facto, cape *GK* in ratione ad rotae perimetrum *GEFG*, ut est tempus, quo corpus progrediendo ab *A* descripsit arcum *AP*, ad tempus revolutionis unius in ellipsi. Erigatur perpendiculum *K L*



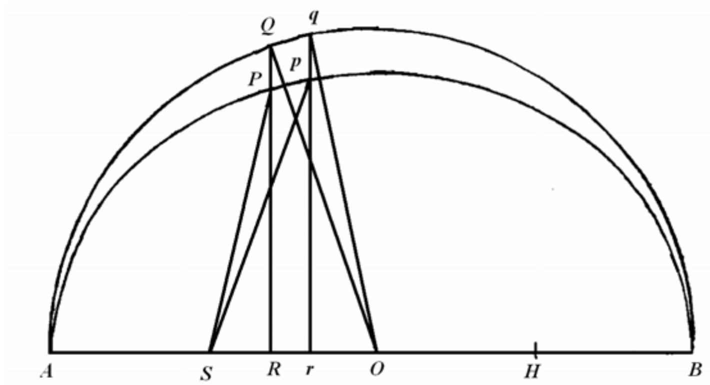
occurrens trochoidi in *L*, & acta *LP* ipsi *KG* parallela occurret ellipsi in corporis loco quaesito *P*.

Nam centro *O*, intervallo *OA* describatur semicirculus *AQB*, & arcui *AQ* occurrat *LP* si opus est producta in *Q*, iunganturque *SQ*, *OQ*. Arcui *EFG* occurrat *OQ* in *F*, & in eandem *OQ* demittatur perpendiculum *SR*. Area *APS* est ut area *AQS*, id est, ut differentia inter sectorem *OQA* & triangulum *OQS*, sive ut differentia rectangulorum $\frac{1}{2}OQ \times AQ$ & $\frac{1}{2}OQ \times SR$, hoc est, ob datam $\frac{1}{2}OQ$, ut differentia inter arcum *AQ* & rectam *SR*, ideoque (cum eaedem sint datae rationes *SR* ad finum arcus *AQ*, *OS* ad *O*A, *OA* ad *OG*, *AQ* ad *GF*, & divisim $AQ - SR$ ad $GF -$ sinu arcus *AQ*) ut *GK* differentia inter arcum *GF* & sinum arcus *A*Q.

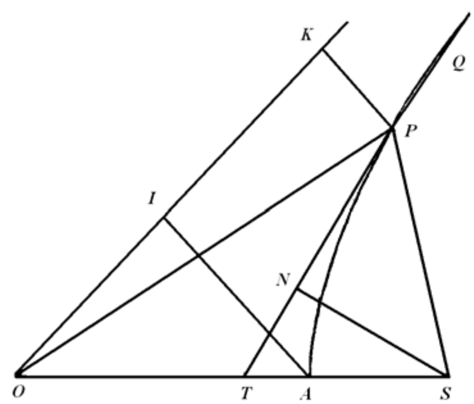
Q.E.D.

Scholium.

Caeterum, cum difficilis sit hujus curvae descriptio, praestat folutionem vero proximam adhibere. Inveniatur tum angulus quidam *B*, qui sit ad angulum graduum 57.29578, quem arcus radio aequalis subtendit, ut est umbilicorum distantia *SH* ad ellipseos diametrum *AB;* tum etiam longitudo quaedam L, quae sit ad radium in eadem



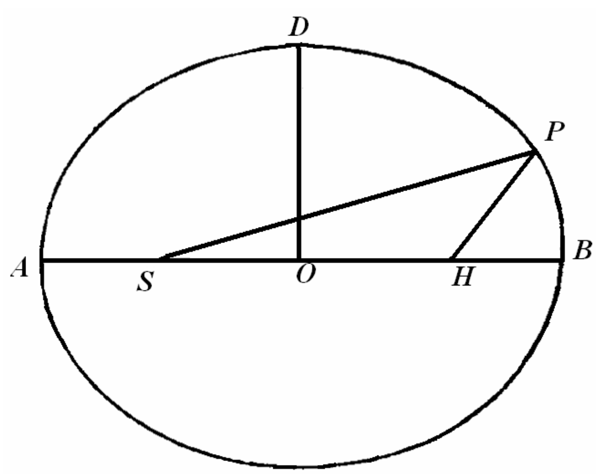
ratione inverse. Quibus semel inventis, problema deinceps consit per sequentem analysin. Per constructionem quamvis, vel utcunque coniecturam faciendo, cognoscatur corporis locus *P* proximus vero eius loco *p.* Demissaque ad axem ellipseos ordinatim applicata *PR,* ex proportione diametrorum ellipseos, dabitur circuli circumscripti *AQB* ordinatim applicata *RQ*, quae sinus est anguli *AOQ* existente *AO* radio, quaeque ellipsin secat in *P*. Sufficit angulum illum rudi calculo in numeris proximis invenire. Cognoscatur etiam angulus tempori proportionalis, id est, qui sit ad quatuor rectos, ut est tempus.
quo corpus descripsit arcum *AP*, ad tempus revolutionis unius in ellipsi. Sit angulus iste *N*. Tum captatur & angulus *D* ad angulum *B*, ut est sinus iste anguli *AOQ* ad radium, & angulos *E* ad angulum $N-AOQ+D$, ut est longitudo *L* ad longitudinem eandem *L* cosinu anguli *AOQ* diminutam, ubi angulus iste recto minor est, auctam ubi major. Postea capiatur tum angulus *F* ad angulum *B*, ut est sinus anguli $AOQ+E$ ad radium, tum angulus *G* ad angulum $N-AOQ-E+F$ ut esi longitudo *L* ad longitudinem eandem cosinu anguli *AOQ* + *E* diminutam ubi angulus iste recto minor est, auctam ubi major. Tertia vice capiatur angulus *H* ad angulum *B*, ut est sinus anguli $AOQ+E+G$ ad radium; & angulus *I* ad angulum $N-AOQ-E-G+H$, ut est longitudo *L* ad eandem longitudinem cosinu anguli $AOQ+E+G$ diminutam, ubi angulus iste recto minor est, auctam ubi maior. Et sic pergere licet in infinitum. Denique capiatur angulus *AOQ* aequalis angulo $AOQ+E+G+I+\&c$. Et ex cosinu ejus *Or* & ordinata *pr*, quae est ad sinum eius *qr* ut ellipseos axis minor ad axem maiorem, habebitur corporis locus correctus *p.* Si quando angulus $N-AOQ+D$ negativus est, debet signum + ipsius E ubique mutari in – , & signum – in +. Idem intelligendum est de signis ipsorum *G* & I, ubi anguli $N-AOQ-E+F$, & $N-AOQ-E-G+H$ negativi prodeunt.



Convergit autem series infinita $AOQ+E+G+I+\&c$. quam celerrime, adeo ut vix unquam opus fuerit ultra progredi quam ad terminum secundum *E*. Et fundatur calculus in hoc theoremate, quod area *APS* sit ut differentia inter arcum *AQ* & rectam ab umbilico *S* in radium *OQ* perpendiculariter demissam.

Non dissimili calculo conficitur problema in hyperbola. Sit eius centrum *O*, vertex *A*, umbilicus *S* & asymptotos *OK*. Cognoscatur quantitas areae abscindendae tempori proportionalis. Sit ea *A*, & fiat coniectura de positione rectae *SP*, quae aream *APS* abscindat verae proximam. Iungatur *OP*, & ab *A* & *P* ad asymptoton agantur *AI*, *PK* asymptoto alteri parallelae, & per tabulam logarithmorum dabitur area *AIKP*, eique aequalis area *OPA*, quae subducta de triangulo *OPS* relinquet aream abscissam *APS*. Applicando areae abscindendae *A* & abscissae *APS* differentiam duplam $2APS-2A$ vel $2A-2APS$ ad lineam *SN*, quae ab umbilico *S* in tangentem *TP* perpendicularis est, orietur longitudo chordae *PQ*. Inscribatur autem chorda illa *PQ* inter *A* & *P*, si area abscissa *APS* maior sit area abscindenda *A*, secus ad puncti *P* contrarias partes: & punctum *Q* erit locus corporis accuratior. Et computatione repetita invenietur idem accuratior in perpetuum.

Atque his calculis problema generaliter consit analytice. Verum usibus astronomicis accommodatior est calculus particularis, qui sequitur. Existentibus *AO*, *OB*, *OD* semiaxibus ellipseos, & *L* ipsius latere recto, ac *D* differentia inter semiaxem minorem *OD* & lateris recti semissem $\frac{1}{2}L$;



quaere tum angulum *Y*, cuius sinus sit ad radium ut est rectangulum sub differentia illa *D*, & semisumma axium $AO+OD$ ad quadratum axis maioris *AB;* tum angulum *Z*, cuius sinus sit ad radium ut est duplum rectangulum sub umbilicorum distantia *SH* & differentia illa *D* ad triplum quadratum semiaxis majoris *AD*. His angulis semel inventis; locus corporis sic deinceps determinabur. Sume angulum *T* proportionalem tempori quo arcus *BP* descriptus est, seu motui medio (ut loquuntur) aequalem; & angulum *V*, primam ut est medii motus aequationem, ad angulum *Y*, aequationem maximam primam, ut est sinus dupli anguli *T* ad radium; atque angulum X, aequationem secundam, ad angulum *Z*, aequationem maximam secundam, ut est cubus sinus anguli *T* ad cubum radii. Angulorum *T*, *V*, *X* vel summae $T+X+V$, si angulus *T* recto minor est, vel differentiae $T+X-V$, si is recto maior est rectisque duobus minor, aequalem cape angulum *BHP*, motum medium aequatum; & si *HP* occurrat ellipsi in *P*, acta *SP* abscindet aream *BSP* in tempori proportionalem quamproxime.Haec praxis satis expedita videtur, propterea quod angulorum perexiguorum *V* & *X*, in minutis secundis, si placet, positorum, figuras duas tresve primas invenire sufficit. Sed & satis accurata est ad theoriam planetarum. Nam in orbe vel Martis ipsius, cuius aequatio centri maxima est graduum decem, error vix superabit minutum unum secundum. Invento autem angulo motus medi aequati *BHP*, angulus veri motus *BSP* & distantia *SP* in promptu sunt per methodum notissimam.

Hactenus de motu corporum in lineis curvis. Fieri autem potest ut mobile recta descendat vel recta ascendat, & quae ad istiusmodi motus spectant, pergo jam exponere.