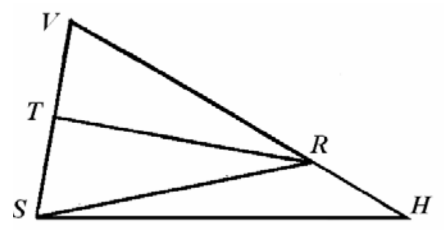
SECTION IV.

Concerning the finding of elliptical, parabolic and hyperbolic orbits from a given focus.

LEMMA XV.



If the two right lines SV and HV are changed in direction at some third point V, with an ellipse or hyperbola for which the two foci are S, H, of which the one line HV shall be equal* [*in length*] *to the principle axis of the figure, that is, to the axis on which the foci are placed, and the other line SV is bisected by a perpendicular TR sent from* T ; *that perpendicular TR will be a tangent at some point* [*R*] *on the conic section: and vice versa, if it touches, then HV will be equal* [*in length*] *to the principle axis of the figure.

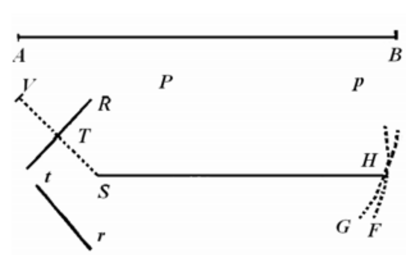
For the perpendicular *TR* cuts the right line *HV* at *R*, produced if there were a need ; and *SR* may be joined. On account of the equal lines *TS, TV,* the angles *TRS* and *TRV* and the lines *SR* and *VR* will be equal. From which the point *R* will be on the conic section, and the perpendicular *TR* will touch the same : and vice-versa.

Q.E.V.

[This section is concerned with the construction of conic sections satisfying various conditions of the focii and tangents, and so is not involved with mechanics directly; due to symmetry there will often be more than one point on the curve where a tangent with the same or a known gradient acts. The ellipse and hyperbola written in standard form are not of course functions as such, and are made up from the positive and negative square root functions which are symmetric.]

PROPOSITION XVIII. PROBLEM X.

With a focus and the principle axis given, to describe elliptic and hyperbolic trajectories, which will pass through a given point, and will be tangents to given lines in place.

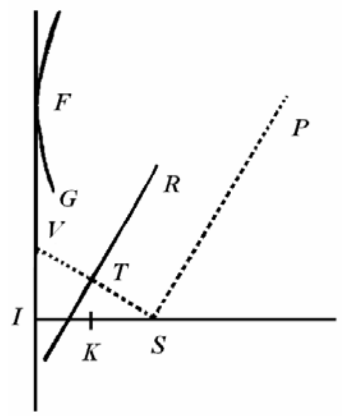


S shall be the common focus of the figures ; *AB* the length [i.e. 2*a* in modern notation] of the principle axis of any trajectory ; *P* the point through which the trajectory must pass ; and *TR* the right line that it must touch. With centre *P,* and with the interval [*i.e.* radius] $AB-SP$, if the orbit shall be an ellipse. or $AB+SP$, if that shall be a hyperbola, a circle *HG* may be described. The perpendicular *ST* may be sent to the tangent *TR*, and the same may be produced to *V* , so that *TV* shall be equal to *ST*; and the circle *FH* is described with centre

V, and the interval [or radius] *AB*. By this method from the two circles either two points *P*, *p* may be given, or two tangents *TR*, *tr*, or the point *P* and a tangent, are required to be described. *H* shall be the common intersection of these, and from the foci *S* and *H*, with that axis given, the trajectory may be described. I say that this has been accomplished. For the trajectory described (because therefore $PH + SP$ is equal to the axis in the case of an ellipse, and $PH - SP$ in the case of a hyperbola) will pass through the point *P*, and (by the above lemma) it touches the right line *TR*. And by the same argument the same will pass through two [given] points *P* and *p*, or touch two [given] right lines *TR* and *tr*. *Q.E.F*.

PROPOSITION XIX. PROBLEM XI.

To describe a parabolic trajectory about a given focus, which will pass through a given point, and touch a given right line in position.

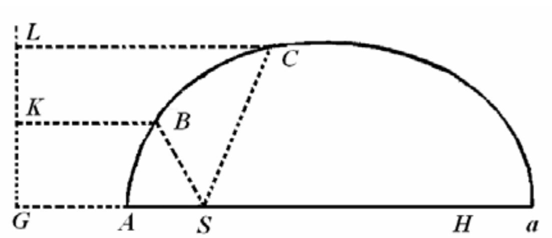


S shall be the focus, *P* the point and *TR* the tangent of the trajectory to be described. With centre *P*, with the interval [*i.e.* radius] *PS*, describe the circle *FG*. Send the perpendicular *ST* from the focus to the tangent, and produce the same to *V*, in order that *TV* shall be equal to *ST*. In the same manner another circle *fg* is required to be described, if another point *p* is given; or finding another point *v*, if another tangent *tr* is given ; then the right line *IF* must be drawn which touches the two circles *FG*, *fg* if the two points *P* and *p* are given, or it may pass through the two points *V* and *v*, if the two tangents *TR* and *tr* are given, or touch the circle *FG* and pass through the point *V*, if the point *P* and the tangent *TR* are given. Send the perpendicular *SI* to *FI*, and bisect the same in *K;* and the parabola may be described with the principle axis *SK* and vertex *K*. I say the proposition has been accomplished. For the parabola, on account of the equal lines *SK* and *IK*, *SP* and *FP*, will pass through the point *P;* and (by Lem. XIV, Corol. 3.) on account of the equal lines *ST* and *TV* and the right angle *STR*, touches the right line *TR*.

Q *E.F.*

PROPOSITION XX. PROBLEM XII.

To describe a trajectory of some given kind about a given focus, which will pass through given points and touch a given right line in place.



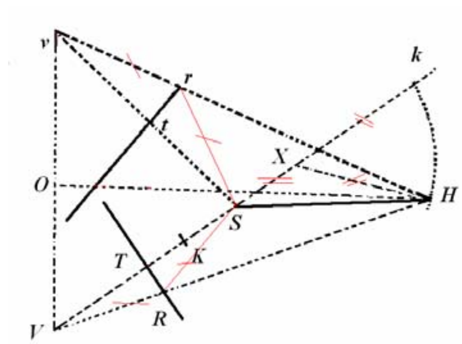
Case I. With *S* the given focus, the trajectory *ABC* shall be described through the two points *B* and *C*. Because with the kind of trajectory given, the ratio of the principle axis to the separation of the focal points will be given. On that account take *KB* to *BS*, and *LC* to *CS*. From the centres *B* and *C*, with the distances *BK*, *CL*, describe two circles, and to the right line *KL*, which may touch the same [circles] at *K* and *L*, send the perpendicular *SG*, and cut the same line at *A* and *a*, thus so that *GA* is to *AS* and *Ga* to *aS* as *KB* is to *BS*, and with the axis *Aa*, vertices *A*, *a*, the trajectory may be described. I say the construction is complete. For the other focus of the figure described shall be *H*, and since *GA* shall be to *AS* as *Ga* to *aS*, there will be separately $Ga - GA$ or *Aa* to $aS - AS$ or *SH* in the same ratio, and thus in the ratio that the principle axis of the figure described has to the separation of the foci of this figure; and therefore the figure described is of the same kind as that required to be described. And since *KB* to *BS* and *LC* to *CS* shall be in the same ratio, this figure will pass through the points *B*, *C*, as has been shown from the conics.

Q *E.F.*

[Here the proof indicated relies on the focus directrix property between *S* and *GL* for a point on the ellipse, for which the ratios $\frac{CS}{CL}$, etc., are constant. Thus,

$$\frac{AS}{AG} - 1 = \frac{aS}{aG} - 1;\ \frac{AG}{AS} = \frac{aG}{aS},\text{or}\ \frac{AG}{aG} = \frac{AS}{aS},\text{giving}\ \frac{Aa}{aG} = \frac{HS}{aS}\ \text{or}\ \frac{Aa}{HS} = \frac{aG}{aS}.]$$

Case 2. With *S* the given focus, the trajectory is to be described which may touch the two right lines *TR* and *tr* at some points. [Note : The original diagram in the 3rd edition has *r* at the wrong side of *t*, if we want to apply Lemma VI, assuming the trajectory is an ellipse.] Send the perpendiculars *ST* and *St* from the focus to the tangents and produce the same to *V* and *v*, so that *TV* and *tv* shall be equal to *TS* and *tS*. Bisect *Vv* in *O*, and erect the indefinite perpendicular *OH*, and cut the right line *VS* produced indefinitely in *K* and *k*, thus so that *VK* shall be to *KS* and *Vk* to *kS* as the principal axis of the described trajectory is to the separation of the foci. [Thus, *TR* is a tangent at the point *R* and $\frac{VK}{KS} = \frac{Vk}{kS} = \frac{VH}{SH}$.] With diameter *Kk*, a circle is described cutting *OH* in *H;* and with the foci *S, H*, with the principal axis itself made equal to *VH*, the trajectory is described. I say the construction is complete. For bisect *Kk* in *X*, and join *HX*, *HS*, *HV*, and *Hv*. Because *VK* to *KS* is as *Vk* to *kS* [*i.e.* $\frac{VK}{KS} = \frac{Vk}{kS}$]*;* and on adding together as $VK + Vk$ to $KS + kS$



$$[\textit{i.e.}\ \frac{VK+Vk}{Vk} = \frac{KS+kS}{kS}\ ;\ \text{or}\ \frac{VK+Vk}{KS+kS} = \frac{2VX}{2KX} = \frac{Vk}{kS}\ ];$$

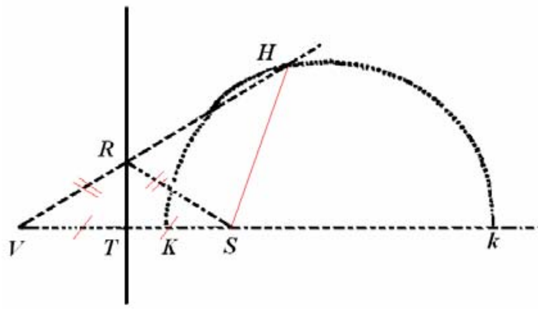
and separately as $Vk - VK$ to $kS - KS$, that is, as 2*VX* to 2*KX* and 2*KX* to 2*SX*

$$[\text{i.e.}\ \frac{Vk-VK}{Vk} = \frac{kS-KS}{kS}\ ;\ \text{or}\ \frac{Vk-VK}{kS-KS} = \frac{2KX}{2SX} = \frac{Vk}{kS}\ ];$$

and thus, as *VX* to *HX* and *HX* to *SX*,[*i.e.* as $\frac{VX}{HX}$ and $\frac{HX}{SX}$ which are hence equal to $\frac{Vk}{kS}$,] there will be the similar triangles *VHX* and *HXS*, and therefore *VH* will be to *SH* as *VX* to *XH*, and thus as *VK* to *KS* [*i.e.* $\frac{VH}{SH} = \frac{VX}{XH} = \frac{VK}{KS}$]. Therefore the principal axis of the described trajectory *VH* has that ratio to the separation of the foci *SH*, and therefore is of the same kind. Since in addition *VH* and *vH* may be equal in length to the principal axis, and *VS* and *vS* may be bisected by the perpendicular lines *TR* and *tr*, it is clear (from Lem. XV.) these right lines touch the described trajectory.

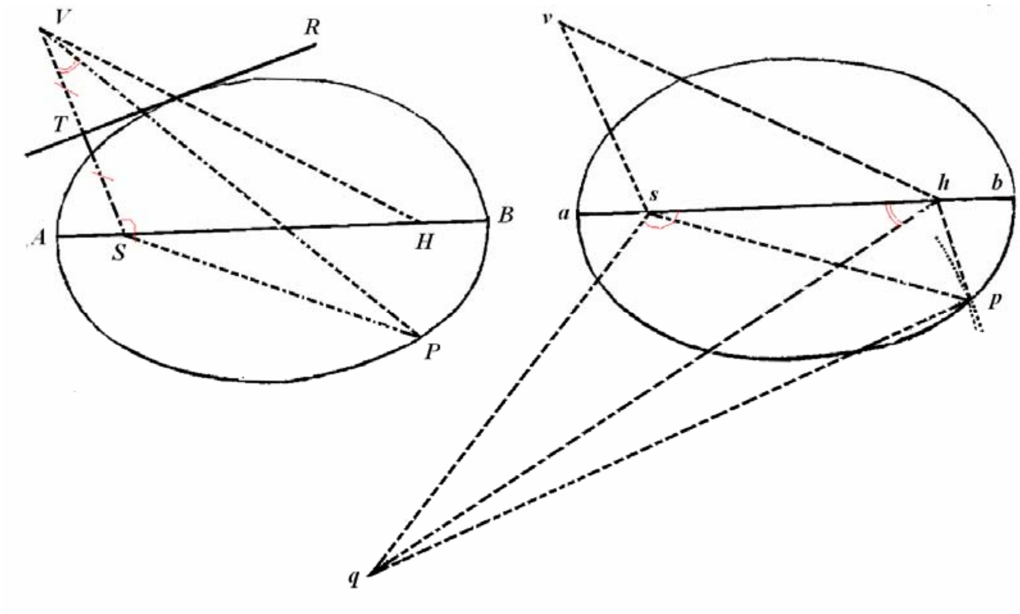
Q *E.F.*

Case. 3. With the focus *S* given, a trajectory shall be described which touches the right line *TR* at the given point *R*. Send the perpendicular *ST* to the right line *TR*, and produce the same to *V*, so that *TV* shall be equal to *ST*. Join *VR* and cut the right line *VS* produced indefinitely in *K* and *k*, thus so that *VK* to *SK* and *Vk* to *Sk* shall be as the principal axis of the ellipse required to be described to the separation of the foci [*i.e.* $\frac{VK}{SK} = \frac{Vk}{Sk} = \frac{VH}{SH}$] ; and with a circle described on the diameter *Kk*, with the right line *VR* produced to be cut in *H*, and with the foci *S*, *H*, with the principal axis made equal to the right line *VH*, the trajectory will be described. I say that the construction is complete. For *VH* is to *S*H as *VK* to *SK*, and thus as the principal axis of the trajectory described to the separation of the foci of this, as may be apparent from the demonstration in the second case, and therefore the trajectory described to be of the same kind with that to be described, truly the right line *TR* by which the angle *VRS* may be bisected, to touch the trajectory in the point *R*, is apparent from the theory of conics.



Q *E. F.*

Case 4. Now the trajectory *APB* shall be described about the focus *S*, which may touch the right line *TR*, and may pass through some point *P* beyond the given tangent, and which shall be similar to the figure *apb*, with the principal axis *ab*, and described with the foci *s*, *h*. Send the perpendicular *ST* to the tangent *TR*, & produce the same to *V*, so that *TV* shall equal *ST*. Moreover make the angles *VSP*, *SVP* equal to the angles *hsq*, *shq* ; and with the centre *q* and with an interval [*i.e.* radius] which shall be to *ab* as *SP* to *VS* describe a circle cutting the figure *apb* in *p*. Join *sp* and with *SH* acting which shall be to *sh* as *SP* is to *sp*, and which angle *PSH* may be put in place equal to the angle *psh* and the angle *VSH* equal to the angle *psq*. And then with the foci *S*, *H*, and with the principle axis distance *AB* equal to *VH*, the section of a cone may be described. I say that the construction is done. For if *sv* is acting which shall be to *sp* as s*h* is to *sq*, and which put in place the angle *vsp* equal to the angle *hsq* and the angle *vsh* equal to the angle *psq*, the triangles *svh* and *spq* will be similar, and therefore *vh* will be to *pq* as *sh* to *sq*, that is (on account of the similar triangles *VSP*, *hsq*) so that *VS* is to *SP* or *ab* to *pq*. Therefore *vh* and *ab* are equal.

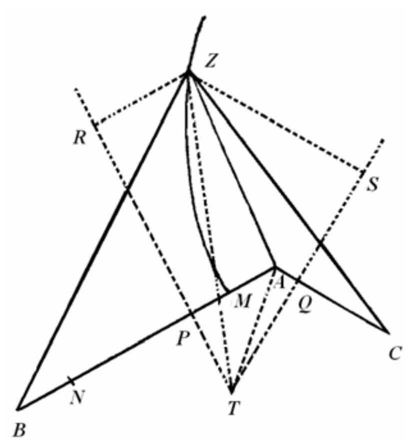


Again on account of the similar triangles *VSH, vsh, VH* is to *SH* as *vh* to *sh,* that is, the axis of the conic section now described is to the interval of separation of the foci, as the axis *ab* to the separation of the foci *sh;* and therefore the figure now described is similar to the figure *apb.* But this figure passes through the point *P,* on account of which the triangle *PSH* shall be similar to the triangle *psh;* and because *VH* is equal to the axis itself and *VS* may be bisected perpendicularly by the right line *TR,* it will touch the same right line *TR.*

Q.E.F.

LEMMA XVI.

From three given points, to put in place three right lines to a fourth point which is not given, of which the differences are given or are zero.



Case I. Let these points be given *A, B, C* and let *Z* be the fourth point, that it is required to find; on account of the given difference of the lines *AZ, BZ,* the point *Z* will be found on a hyperbola of which the foci are *A* and *B,* and that given difference the principal axis. Let *MN* be that axis. Take *PM* to *MA* so that it is as *MN* to *AB,* [*i.e.* $\frac{PM}{MA}=\frac{MN}{AB}$; note that *PR* is the directrix of this branch of the hyperbola.] and with *PR* erected perpendicular to *AB,* and with the perpendicular *ZR* sent to *PR;* there will be, from the nature of this hyperbola, *ZR* to *AZ* as *MN* is to *AB* [*i.e.* $\frac{ZR}{AZ}=\frac{MN}{AB}$; here Newton has used the inverse of the eccentricity to define a constant ratio]. By similar reasoning the point *Z* will be located on another hyperbola, of which the foci are *A* and *C* and the principal axis the difference between *AZ* and *CZ*, and *QS* itself can be drawn perpendicular to *AC*, to which if from some point *Z* of this hyperbola the normal *ZS* may be sent, this will be to *AZ* as the difference is between *AZ* and *CZ* is to *AC*. Therefore the ratios of *ZR* and *ZS* to *AZ* are given, and on that account the same ratio of *ZR* and *ZS* in turn is given ; therefore if the right lines *RP* and *SQ* meet in *T*, and *TZ* and *TA* may be drawn, a figure of the kind *TRZS* will be given, and with the right line *TZ* on which the point *Z* will be given in place somewhere. Also there will be given the right line *TA*, and also the angle *ATZ ;* and on account of the given ratios of *AZ* and *TZ* to *ZS* the ratio of these will be given in turn; and thence the triangle *ATZ* will be given, the vertex of which is the point *Z*.

Q.E.I.

Case 2. If two from the three lines such as *AZ* and *B* Z may be made equal, thus draw the right line *TZ*, so that it may bisect the angle *AB* ; then find the triangle *ATZ* as above.

Case 3. If all three are equal, the point *Z* may be located in the centre of the circle passing through the points *A*, *B*, *C*.

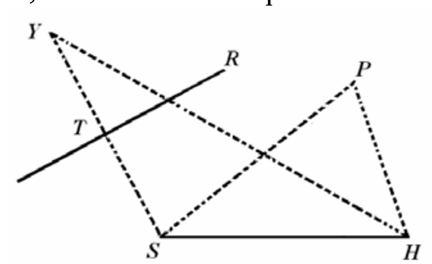
Q.E.1.

This lemma problem is solved also in the book of *Appolonius* on tangents restored by *Vieta*.

PROPOSITION XXI. PROBLEM XIII.

To describe a trajectory around a given focus, which will pass through given points and touch given right lines in place.

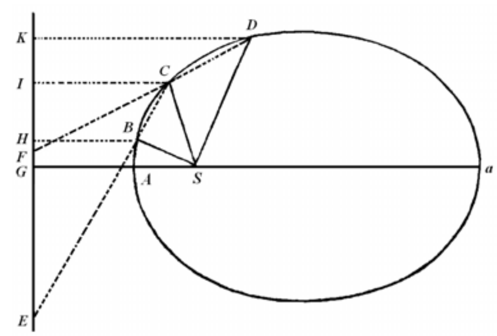
The focus *S* may be given, a point *P*, and touching *TR*, and it shall be required to find the other focus *H*. To the tangent send the perpendicular *ST* and produce the same to *Y*; so that *TY* shall be equal to *ST*, and *YH* will be equal to the principal axis. Join *SP* and *HP*, and *SP* will be the difference between *HP* and the principal axis. In this manner if several tangents *TR* may be given, or more points *P*, always just as many lines *TH*, or *PH*, drawn from the said points *Y* or *P* to the focus *H*,



which either shall be equal to the axis, or to some given lengths *SP* different from the same, and thus which either are equal among themselves in turn, or have some given differences; and thence, by the above lemma, that other focus *H* is given. But with the foci in place together with the length of the axis (which either is *YH;* or, if the trajectory be an ellipse, $PH+SP$ *;* or $PH-SP$ for a hyperbola,) the trajectory may be had.

Q *E.1*.

Scholium.



When the trajectory is a hyperbola, I cannot deal with the opposite hyperbola under the name of this trajectory. For a body by progressing in its motion cannot cross over into the opposite hyperbola.

The case where three points are given can be solved expediently thus. The points B, C, D may be given. Join BC, CD produced to E, F, so that there shall be EB to EC as SB to SC, and FC to FD as SC to SD. To EF drawn and produced send the normals SG, BH, and on GS produced indefinitely take GA to AS and Ga to aS as HB is to BS; and A will be the vertex, and Aa the principal axis of the trajectory: which, just as Gd shall be greater, equal, or less than AS, will be an ellipse, parabola or hyperbola; the point a falling in the first case on the same part of the line GF with the point A; in the second case departing to infinity; in the third case on the opposite side of the line GF. For if the perpendiculars CI, DK may be sent to GF ; IC will be to HB as EC to EB, that is, as SC to SB; and in turn IC to SC as HB to SB or as GA to SA. And by a like argument it is approved that KD to SD be in the same ratio. Therefore place the points B, C, D in a conic section around the focus S thus described, so that all the right lines, drawn from the focus to the individual points of the section, shall be in that given ratio to the perpendiculars sent from the same points to the line GF.

By a method not much different the solution of this problem has been treated most clearly in the geometry of *de la Hire;* Book VIII, Prop. XXV of his book on conic sections.

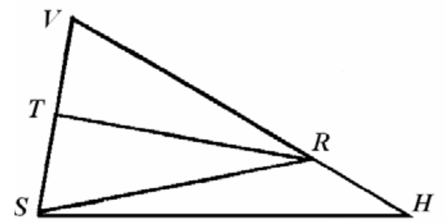
[*The New Elements of Conick Sections* by Philip de la Hire was translated from Latin and French editions into English in 1724; it is available on microfilm.]

SECTIO IV.

De inventione orbium ellipticorum, parabolicorum & *hyperbolicorum ex umbilico dato.*

LEMMA XV.

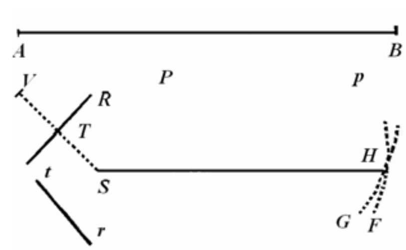
Si ab ellipseos vel hyperbolae cuiusvis umbilicis duobus S*, H, *ad punctum quodvis tertium* V *inflectantur rectae duae* SV, HV *quarum una* HV *aequalis sit axi principali figurae, id est, axi in quo umbilici iacent, altera* SV *a perpendiculo* TR *in se demisso bisecetur in* T ; *perpendiculum illud* TR *sectionem conicam alicubi tanget: & contra, si tangit, erit* HV *aequalis axi principali figurae.



Secet enim perpendiculum *TR* rectam *HV* productam, si opus fuerit, in *R;* & iungatur *SR.* Ob aequales *TS, TV,* aequales erunt & rectae *SR, VR* & anguli *TRS, TRV.* Unde punctum *R* erit ad sectionem conicam, & perpendiculum *T R* tanget eandem : & contra. *Q.E.V.*

PROPOSITIO XVIII. PROBLEMA X.

Datis umbilico & axibus principalibus describere traiectorias ellipticas & hyperbolicas, quae transibunt per puncta data, & rectas positione datas contingent.

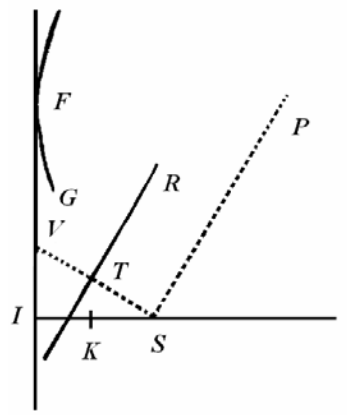


Sit *S* communis umbilicus figurarum; *AB* longitudo axis principalis traiectoria cuiusvis; *P* punctum per quod traiectoriae debet transire ; & *TR* recta quam debet tangere. Centro *P* intervallo $AB-SP$, si orbita sit ellipsis, vel $AB+SP$, si ea sit hyperbola, describatur circulus *HG*. Ad tangentem *TR* demittatur perpendiculum *ST*, & producatur idem ad *V* , ut sit *TV* aequalis *ST*; centroque *V* & intervallo *AB* describatur circulus *FH*. Hac methodo sive dentur duo puncta *P, p,* sive duae tangentes *TR, tr,* sive punctum *P* & tangens, describendi sunt circuli duo. Sit *H* eorum intersectio communis, & umbilicis *S*, *H,* axe illo dato describatur traiectoria. Dico factum. Nam trajectoria descripta (eo quod $PH+SP$ in ellipsi, & $PH-SP$ hyperbola aequatur axi) transibit per punctum *P*, & (per lemma superius) tanget rectam *TR*. Et eodem argumento vel transibit eadem per puncta duo *P, p* vel tanget rectas duas *TR*, *tr*.

Q.E.F.

PROPOSITIO XIX. PROBLEMA XI.

Circa datum umbilicum traiectoriam parabolicam describere, quae transibit per puncta data, & rectas positione datas continget.

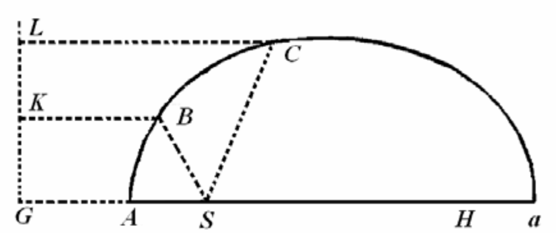


Sit *S* umbilicus, *P* punctum & *TR* tangens traiectoriae describendae. Centro *P*, intervallo *PS* describe circulum *FG.* Ab umbilico ad tangentem demitte perpendicularem *ST,* & produc eam ad *V*, ut sit *TV* aequalis *ST.* Eodem modo describendus est alter circulus *fg*, si datur alterum punctum *p;* vel inveniendum alterum punctum *v*, si datur altera tangens *tr;* dein ducenda recta *IF* quae tangat duos circuIos *FG*, *fg* si dantur duo puncta *P,p,* vel transeat per duo puncta *V, v,* si dantur dua tangentes *TR, tr,* vel tangat circulum *FG* & transeat per punctum *V,* si datur punctum *P* & tangens *TR.* Ad *FI* demitte perpendicularem *SI,* eamque biseca in *K;* & axe *SK*, vertice principali *K* describatur parabola. Dico factum. Nam parabola, ob aequaIes *SK* & *IK*, *SP* & *FP,* transibit per punctum *P;* & (per Lem. XIV, Corol. 3.) ob aequaIes *S T* & *TV* & angulum rectum *STR,* tanget rectam *TR.*

Q *E.F.*

PROPOSITIO XX. PROBLEMA XII.

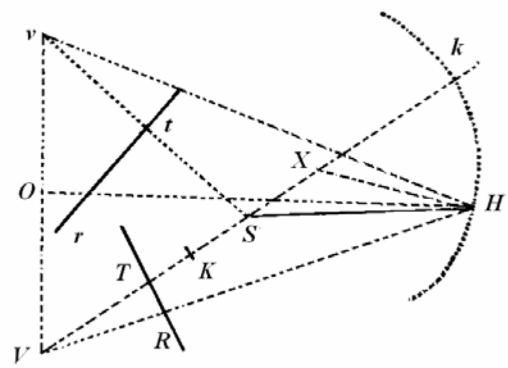
Circa datum umbilicum traiectoriam quamvis specie datam describere, quae per data puncta transibit & rectas tanget positione datas.



Cas: I. Dato umbilico *S*, describenda sit traiectoria *ABC* per puncta duo *B, C.* Quoniam traiectoria datur specie, dabitur ratio axis principalis ad distantiam umbilicorum. In ea ratione cape *KB* ad *BS,* & *LC* ad *CS.* Centris *B,C,* intervallis *BK*, *CL,* describe circulos duos, & ad rectam *KL,* quae tangat eosdem in *K* & *L,* demitte perpendiculum *SG*, idemque seca in *A* & *a*, ita ut sit *GA* ad *AS* & *Ga* ad *aS* ut est *KB* ad *BS* & axe *Aa*, verticibus *A, a* describatur traiectoria. Dico factum. Sit enim *H* umbilicus alter figurae descriptae, & cum sit *GA* ad *AS* ut *Ga* ad *aS*, erit divisim $Ga - GA$ seu *Aa* ad $aS - AS$ seu *SH* in eadem ratione, ideoque in ratione quam habet axis principalis figurae describendae ad distantiam umbilicorum eius; & propterea figura dcscripta est eiusdem speciei cum describenda. Cumque sint *KB* ad *BS* & *LC* ad *CS* in eadem ratione, transibit haec figura per puncta *B, C,* ut ex conicis manifestum est.

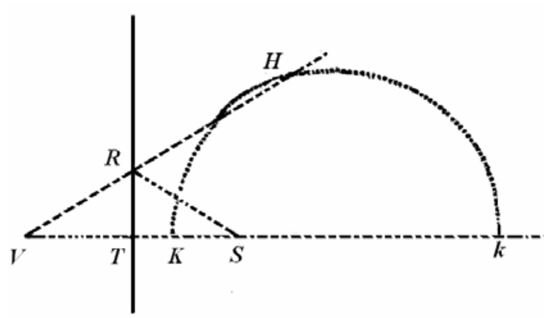
Q *E.F.*

Cas. 2. Dato umbilico *S*, describenda sit traiectria quae rectas duas *TR*, *tr* alicubi contingat. Ab umbilico in tangentes demitte perpendicula *ST*, *St* & produc eadem ad *V*, *v*, ut sint *TV*, *tv* aequales *TS*, *tS*. Biseca *Vv* in *O*, & erige perpendiculum infinitum *OH*, rectamque *VS* infinite productam seca in *K* & *k*, ita ut sit *VK* ad *KS* & *Vk* ad *kS* ut est traiectoriae describendae axis principalis ad umbilcorum distantiam. Super diametro *Kk* describatur circulus secans *OH* in *H;* & umbilicis *S*, *H*, axe principali ipsam *VH* aequante, describatur traiectoria. Dico factum. Nam biseca *Kk* in *X*, & iunge *HX*, *HS*, *HV*, *Hv*. Quoniam est *VK* ad *KS* ut *Vk* ad *kS;* & composite ut $VK+Vk$ ad $KS+kS$ *;* divisimque ut $Vk-VK$ ad $kS-KS$, id est, ut 2*VX* ad 2*KX* & 2*KX* ad 2*SX*, ideoque ut *VX* ad *HX* & *HX* ad *SX*, similia erunt triangula *VXH*, *HXS*, & propterea *VH* erit ad *SH* ut *VX* ad *XH*, ideoque ut *VK* ad *KS*. Habet igitur traiectoriae descriptae axis principalis *VH* eam rationem ad ipsius umbilicorum distantiam *SH*, quam habet traiectoriae describendae axis principalis ad ipsius umbilicorum distantiam, & propterea eiusdem est speciei. Insuper cum V *H*, *vH* aequentur axi principali, & V*S*, *vS* a rectis *TR*, *tr* perpendiculariter bisecentur, liquet (ex Lem. xv.) rectas illas traiectoriam descriptam tangere.



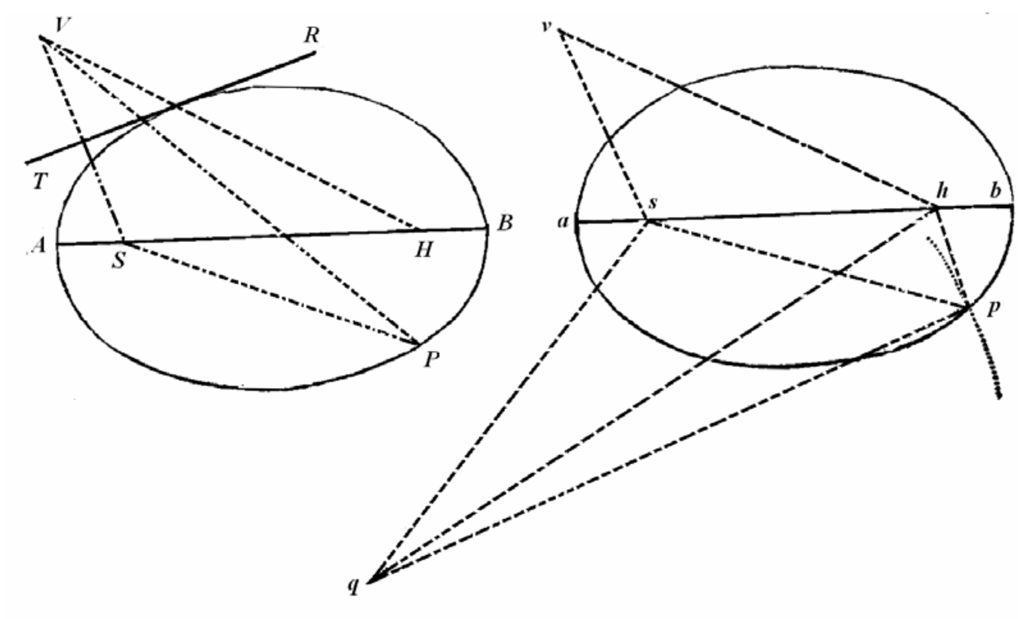
Q *E.F.*

Cas. 3. Dato umbilico *S* describenda sit traiectoria quae rectum *TR* tanget in puncto dato *R*. In rectam *TR* demitte perpendicularem *ST*, & produc eandem ad *V*, ut sit *TV* aequalis *ST*. Iunge *VR* & rectam *VS* infinite productam seca in *K* & *k*, ita ut sit *VK* ad *SK* & *Vk* ad *Sk* ut ellipseos describendae axis principalis ad distantiam umbilicorum; circuloque super diametro *Kk* descripto secetur producta recta *VR* in *H*, & umbilicis *S*, *H*, axe principali rectam *VH* aequante, describatur traiectoria. Dico factum. Namque *VH* esse ad *S*H ut *VK* ad *S K*, atque ideo ut axis principalis traiectoriae describendae ad distantiam umbilicorum eius, patet ex demonstratis in casu secundo, & propterea traiectoriam descriptam eiusdem esse speciei cum describenda, rectum vero *TR* qua angulus *VRS* bisecatur, tangere traiectoriam in puncto *R*, patet ex conicis.



Q *E. F.*

Cas. 4. Circa umbilicum *S* describenda iam sit traiectoria *APB* quae tangat rectum *TR,* transeatque per punctum quodvis *P* extra tangentem datum, quaeque similis sit figurae *aph,* axe principali *a*s, & umbilicis *s, h* descriptae. In tangentem *TR* demitte perpendiculum *ST,* & produc idem ad *V,* ut sit *TV* aequalis *ST.* Angulis autem *VSP, SVP* fac angulos *hsq, shq* aequales; centroque *q* & intervallo quod sit ad *ab* ut *SP* ad *VS* describe circulum secantem figuram *apb* in *p.* Iunge *sp* & age *SH* quae sit ad *sh* ut est *SP* ad *sp,* quaeque angulum *PSH* angulo *psh* & angulum *VSH* angulo *psq* aequales constituat. Denique umbilicis *S*, *H,* & axe principali A*B* distantiam *VH* aequante, describatur sectio conica. Dico factum. Nam si agatur *sv* quae sit ad *sp* ut est s*h* ad *sq,* quaeque constituat angulum *vsp* angulo *hsq* & angulum *vsh* angulo *psq* aequates, triangula *sv h, spq* erunt similia, & propterea *vh* erit ad *pq* ut *sh* ad *sq*, id est (ob similia triangula *VSP, hsq*) ut est *VS* ad *SP* seu *ab* ad *pq.* Aequantur ergo *vh* & *ab.*



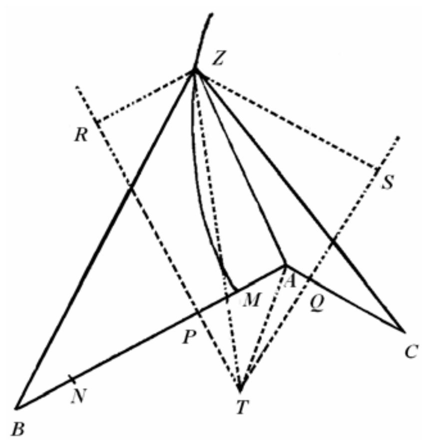
Porro ob similia triangula *VSH, v*s*h,* est *VH* ad *SH* ut *vh* ad *sh,* id est, axis conicae sectionis iam descriptae ad illius umbilicorum intervallum, ut axis *ab* ad umbilicorum intervallum *sh; &* propterea figura iam descripta similis est figurae *apb.* Transit autem haec figura per punctum *P,* eo quod triangulum *PSH* simile sit triangulo *psh;* & quia *VH* aequatur ipsius axi & *VS* bisecatur perpendiculariter a recta *TR,* tangit eadem rectam *TR.*

Q.E.F.

LEMMA XVI.

A datis tribus punctis ad quartum non datum inflectere tres rectas quarum differentiae vel dantur ***vel nulIae sunt.***

Cas. I. Sunto puncta illa data *A, B, C* & punctum quartum *Z,* quod invenire oportet; ob datam differentiam linearum *AZ, BZ,* locabitur punctum *Z* in hyperbola cuius umbilici sunt *A* & *B,* & principalis axis differentia illa data. Sit axis ille *MN*. Cape *PM* ad *M A* ut est *MN* ad *AB,* & erecta *PR* perpendiculari ad *AB,* demissaque *ZR* perpendiculari ad *PR;* erit, ex natura huius hyperbolae, *ZR* ad *AZ* ut est *MN* ad *AB.* Simili discursu punctum *Z* locabitur in alia hyperbola, cujus umbilici sunt *A, C* & principalis axis differentia inter *AZ* & *CZ,* ducique potest; *QS* ipsi *AC* perpendicularis, ad quam si ab hyperbolae huius puncto quovis *Z* demittatur normalis *ZS,* haec fuerit ad *AZ* ut est differentia inter *AZ & CZ* ad *AC.* Dantur ergo rationes ipsarum *ZR* & *ZS* ad *AZ,* & idcirco datur earundem *ZR* & *ZS* ratio ad invicem ; ideoque si rectae *RP, SQ* concurrant in *T,* & agantur *TZ* & *TA,* figura *TRZS* dabitur specie, & recta *TZ* in qua punctum *Z* alicubi locatur, dabitur positione. Dabitur etiam recta *TA,* ut & angulus *ATZ ;* & ob datas rationes ipsarum *AZ* ac *TZ* ad *ZS* dabitur earundem ratio ad invicem; & inde dabitur triangulum *ATZ,* cujus vertex est punctum *Z.*



Q.E.I.

Cas. 2. Si duae ex tribus lineis, puta *AZ* & *B* Z, aequantur, ita age rectam *TZ,* ut bisecet rectam *AB* ; dein quaere triangulum *ATZ* ut supra.

Cas. 3. Si omnes tres aequantur, locabitur punctum *Z* in centro circuli per puncta *A, B, C* transeuntis.

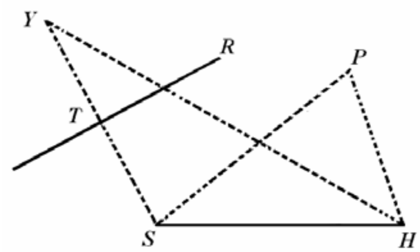
Q.E.1.

Solvitur etiam hoc lemma problematicum per librum tactionum *Apollonii* a *Vieta* restitutum.

PROPOSITIO XXI. PROBLEMA XIII.

Traiectoriam circa datum umbilicum describere, quae transibit per puncta data & rectas positione datas continget.

Detur umbilicus *S*, punctum *P*, & tangens *TR*, & inveniendus sit umbilicus alter *H*. Ad tangentem demitte perpendiculum *ST* & produc idem ad *Y*; ut sit *TY* aequalis *ST*, & erit *YH* aequatis axi principali. Iunge *SP*, *HP*, & erit *SP* differentia inter *HP &* axem principalem. Hoc modo si dentur plures tangentes *TR*, vel plura puncta *P*, devenietur semper ad lineas totidem *TH*, vel *PH*, a dictis punctis *Y*vel *P* ad umbilicum *H* ductas,



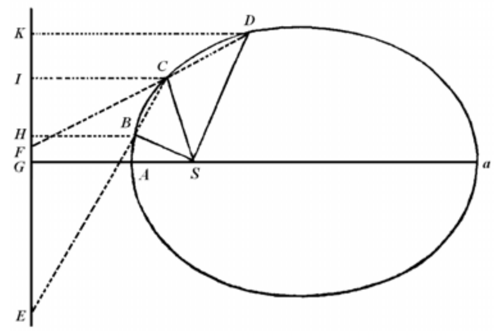
quae vel aequantur axibus, vel datis longitudinibus *SP* differunt ab iisdem, atque ideo quae vel aequantur sibl invicem, vel datas habent differentias; & inde, per lemma superius, datur umbilicus ille alter *H*. Habitis autem umbilicis una cum axis longitudine (quae vel est *YH;* vel, si traiectoria ellipsis est, $PH + SP$*;* sin hyperbola, $PH - S\ P$) habetur traiectoria.

Q *E.1*.

Scholium.

Ubi traiectoria est hyperbola, sub nomine huius traiectoriae oppositam hyperbolam non comprehendo. Corpus enim pergendo in motu suo in oppositam hyperbolam transire non potest.

Casus ubi dantur tria puncta sic solvitur expeditius. Dentur puncta *B*, *C*, *D*. Iunctas *BC*, *CD* produc ad *E*, *F*, ut sit *EB* ad *EC* ut *SB* ad *SC*, & *FC* ad *FD* ut *SC* ad *SD*. Ad *EF* ductam & produttam demitte normales *SG*, *BH*, inque *GS* infinite producta cape *GA* ad *AS* & *Ga* ad *aS* ut est *HB* ad *BS;* & erit *A* vertex, & *Aa* axis principalis traiectoriae: quae, perinde ut *Gd* major, aequalis, vel minor fuerit quam *AS*, erit



ellipsis, parabola vel hyperbola; puncta *a* in primo casu cadente ad eandem partem lineae *GF* cum puncto *A;* in secundo casu abeunte in infinitum; in tertio cadente ad contrariam partem lineae *GF*. Nam si demittantur ad *GF* perpendicula *CI*, *DK;* erit *IC* ad*H B* ut *EC* ad *EB*, hoc est, ut *SC* ad *SB;* & vicissim *IC* ad *SC* ut *HB* ad *SB* sive ut *GA* ad *SA*. Et simili argumento probabitur esse *KD* ad *SD* in eadem ratione. Iacent ergo puncta *B*, *C*, *D* in coni sectione circa umbilicum *S* ita descripta, ut rectae omnes, ab umbilico *S* ad singula sectionis puncta ductae, sint ad perpendicula a punctis iisdem ad rectam *GF* demissa in data illa ratione.

Methodo haud multum dissimili huius problematis solutionem tradit clarissimus geometra *de la Hire;* conicorum suorum Lib.VIIl, Prop. XXV.