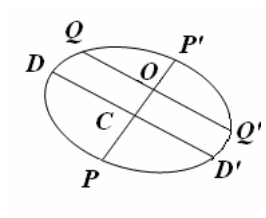
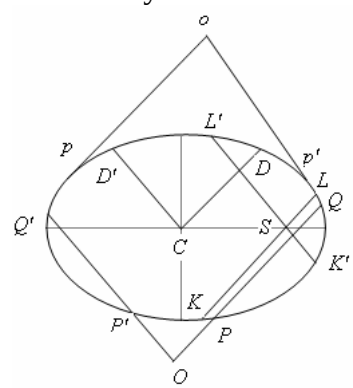
SECTION III.

Concerning the motion of bodies in eccentric conic sections.

[The ***Rectangle Theorem*** is a general theorem for Conic Sections with many applications: If two variable secants of a conic whose directions are fixed cut the conic in *P*, *Q* and *P'*, *Q'*, and intersect in *O*, then $\frac{OP.OQ}{OP'.OQ'}$ is constant for all positions of *O*. (For a coordinate geometry proof, see e.g. *Elements of Analytical Conics*, Gibson & Pinkerton, p.434. (1911)) We will demonstrate this theorem for an elliptical section, following this reference. Thus, if the chords *PQ and P'Q'* intersect at *O* outside the ellipse shown here, then if the tangents parallel to these chords in turn, *op* and *op'* , intersect at *o*, then $\frac{OP.OQ}{OP'.OQ'} = \frac{op^2}{op'^2}$.





Similarly, if the semi-diameters *CD* and *CD'* parallel to the chords are considered, then $\frac{OP.OQ}{OP'.OQ'} = \frac{op^2}{op'^2} = \frac{CD^2}{CD'^2}$. A useful case occurs when *O* lies on the focus *S* of the ellipse, in which case we have the added relations $\frac{OP.OQ}{OP'.OQ'} = \frac{op^2}{op'^2} = \frac{CD^2}{CD'^2} = \frac{SL.SK}{SL'.SK'}$. In this latter case, there is added property that the harmonic mean of the two segments of a focal chord is equal to the semi-latus rectum l , where $l = \frac{b^2}{a^2}$ for the standard ellipse: *i.e.* $\frac{1}{SL} + \frac{1}{SK} = \frac{2}{l}$, or $\frac{KL}{SL.SK} = \frac{2}{l}$; hence we have all these relations : $\frac{OP.OQ}{OP'.OQ'} = \frac{op^2}{op'^2} = \frac{CD^2}{CD'^2} = \frac{SL.SK}{SL'.SK'} = \frac{KL}{K'L'}$.

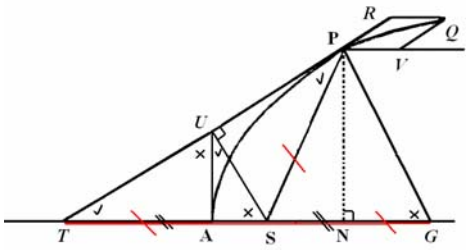
Thus, we have Newton's Theorem :

If VQ is an ordinate of the diameter PCP' of a conic, CP and CD are conjugate semi-diameters, then

$$\frac{P'V.VP}{VQ^2} = \frac{CP^2}{CD^2} .$$

From the given figure, which satisfies the requirements of the Rectangle Theorem, we have now in addition :
$\frac{P'C.CP}{DC.CD'} = \frac{P'V.VP}{QV.VQ'}$; giving $\frac{P'V.VP}{VQ^2} = \frac{CP^2}{CD^2}$ for conjugate axes, where C is the centre of the ellipse or hyperbola . A similar result holds for the hyperbola, though care must be taken with signs.

To prove a similar theorem for the parabola, used below by Newton, *i.e.* that $QV^2 = 4SP.PV$, we proceed as follows : . From the Rectangle Theorem applied to the infinite parallel chords $R\infty$ and $T\infty$, and with the conjugate semi-diameter formed by the tangent *TPR*:



$\frac{RP^2}{RQ.R\infty} = \frac{TP^2}{TA.T\infty}$ or $\frac{QV^2}{PV} = \frac{TP^2}{TA}$, since $\frac{R\infty}{T\infty}=1$ as *RQ* and *TA* are parallel. Therefore $QV^2 = \frac{PV.TP^2}{TA}$ Also, *PG* is normal to the tangent, and *PN* is the ordinate of the point P. Now, from the geometry of the figure, $TA = AN$ and $TS = SP = SG$; $\frac{TP}{TG} = \frac{TN}{TP}$ or $TP^2 = TN.TG = 2.TA.2SP$. Therefore $QV^2 = \frac{PV.TP^2}{TA} = 4SP.PV$, as required.

In addition, $2.UA = PN \therefore 4.UA^2 = PN^2 = 4TA.AS = 4AN.AS$, giving the familiar formula $y^2 = 4ax$.]

PROPOSITION Xl. PROBLEM VI.

A body may be revolving in an ellipse: the law of the centripetal force is required attracting towards the focus of the ellipse.

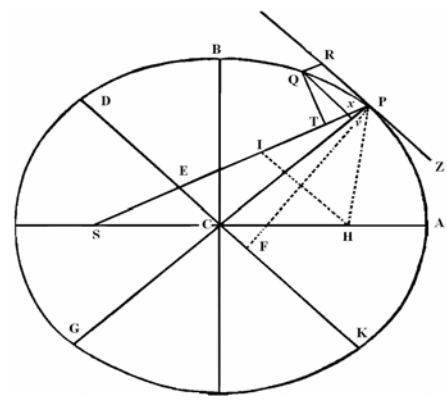
[Note : This is one of the main results of the *Principia*, that was of great interset at the time, and was the proposition that Newton had mislaid when first visited by Halley enquiring about such; surely part of the folklore that now extends around Newton, and indicative of his disregard for the intellectual pursuits of others in the years following his rebuttals by Hooke.

Thus, as always in these diagrams, we are looking for an expression for the area transcribed about the centre of force, here the focus *F*, in the element of time the body travels from *P* to *Q*, and this quantity squared is divided by a length corresponding to the versed sine in the limit, a distance proportional to and in the direction of the force within the arc of the motion (see Prop.I, Th.I, Sect. 2); thus the inverse of the acceleration is obtained as the limit is approached. Newton is careful enough to state in Sect. I that he does not take a limit, just approaches as near as you wish with smaller and greater magnitudes, effected by the longer sides of the parallelogram that encompass the arc that tends towards the long diagonal of the parallelogram.

This incremental parallelogram is itself of some interest, as its 'long' and 'short' sides are determined by different causes : the long side *PR* corresponds to a small increment of the tangent line – following the First Law of Motion, and the corresponding side *Qx* is an increment of a diameter of the section, the conjugate diameter of which passes through the centre *C* of the conic; the short side *QR* – following the Second Law of Motion corresponds to the versed sine of the force, and acts towards the focal point *F*. Thus, from

the point *P*, one line is sent to *S*, and another to *C*; while from *Q* one line is *Qv*, while the other drawn is *QT*, perpendicular to *SP*. Hence the area traced out is proportional to $PS \times QT$ which in turn is proportional to the time, while QR is the distance moved under the action of the central force. Hence the centripetal acceleration can be found proportional to $\frac{QR}{(PS \times QT)^2}$ .This ratio must be identified with known properties of the curve via proportionals, etc. derived from theorems of the conic section involved, one of which is : $\frac{Gv.vP}{Qv^2} = \frac{PC^2}{CD^2}$, shown above from elementary considerations . Thus Newton used what might be called geometrical dynamics in his calculations.]

Let *S* be a focus of the ellipse. *SP* may be drawn cutting both the diameter of the ellipse *DK* in *E*, and the applied ordinate Q*v* [i.e, a semi-chord of the axis *PCG*] in *x*, and the parallelogram Q*xPR* may be completed. It is apparent that *EP* is equal to the major semi-axis *AC* : because there, with the line *HI* drawn from the other focus of the ellipse parallel to *EC* itself, on account of the equal lines *CS*, *CH*, the lines *ES*, and *EI* are made equal, thus so that *EP* shall be half the sum of *PS*, *PI*, that is (on account of the parallel lines *HI*, *PR*, and the equal angles *IPR*, *HPZ*) half the sum of *PS*, *PH*, which themselves jointly are equal to the total axis length 2*AC*.



[Thus, in the customary equation for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and from the construction : $SP + PH = 2a$; from the reflection theorem for a ray travelling from one focus to the other, the angles *IPR*, *HPZ* are equal; then $PH = PI$; $SE = EI$; therefore, $SP + PI = 2a$ and $\frac{1}{2}(SP + PI) = a$.]

The perpendicular *QT* may be sent to *SP*, and I call *L* the principal latus rectum of the ellipse [a useful constant, the length of the vertical focal chord, equal to $\frac{2b^2}{a}$] (or $\frac{2BC^2}{AC}$), there will be $L \times QR$ to $L \times Pv$ as *QR* to *Pv*, that is, as *PE* or *AC* to *PC* [from the similar triangles *Pvx* and *PFC*]; and $L \times Pv$ to *Gv.vP* as *L* to *Gv;* and *Gv.vP* to Qv^2 as PC^2 to CD^2,

[thus we have the ratios : $\frac{L \times QR}{L \times Pv} = \frac{QR}{Pv} = \frac{PE}{PC} = \frac{AC}{PC}$; $\frac{L \times Pv}{Gv.vP} = \frac{L}{Gv}$; $\frac{Gv.vP}{Qv^2} = \frac{PC^2}{CD^2}$],

and (per Corol.2, Lem. VII.) Qv^2 to Qx^2 with the points *Q* and *P* merged together is the ratio of equality; Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2 or (per Lem. XII.) as CD^2 to CB^2. [Note 261 (*i*) Leseur & Janquier: From the nature of conics,

the diameters of the parts made in terms of squares of the ordinates : as the square of the transverse diameter to the square of the conjugate of that : $PF \times CD = AC \times BC$, and thus $PF^2 \times CD^2 = AC^2 \times BC^2$, and thus $\frac{AC^2}{PF^2} = \frac{CD^2}{BC^2}$. Note also that the lines *DK, IH, Qv,* and *RP* are all parallel; hence the angle *FEP* in the right angle triangle *PEF* is equal to the angle *QxT* in the small right angled triangle *QxT*, which triangles are hence similar.]

$$[i.e.\ \frac{Qx^2}{QT^2} = \frac{Qv^2}{QT^2} = \frac{EP^2}{PF^2} = \frac{CA^2}{PF^2}.]$$

And with all these ratios taken together, $L \times QR$ becomes to QT^2 as $AC \times L \times PC^2 \times CD^2$, or $2.CB^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$, or as 2.*PC* to *Gv*.

$$[i.e.\ \frac{L \times QR}{QT^2} = \frac{AC \times L \times PC^2 \times CD^2}{PC \times Gv \times CD^2 \times CB^2} = \frac{2.CB^2 \times PC^2 \times CD^2}{PC \times Gv \times CD^2 \times CB^2} = \frac{2.PC}{Gv}.]$$

But with the points *Q* and *P* coalescing, 2.*PC* and *Gv* are equal. Therefore from these proportionalities $L \times QR$ and QT^2 are equal. These equalities are multiplied by $\frac{SP^2}{QR}$, and there becomes $L \times SP^2$ equals $\frac{SP^2 \times QT^2}{QR}$. Therefore (by Corol.1 & 5, Prop.VI.) the centripetal force is reciprocal as $L \times SP^2$ that is, reciprocally in the squared ratio of the distance *SP*.

Q E.I.

[Note the desire to be rid of vanishing quantities, and to replace these with finite lengths ; the triangles *QTx* and *PCE* fulfil this role; the first has vanishing sides, and the latter does not; hence a ratio of vanishing sides in the first gives rise to a finite ratio in the second. This can be accomplished by setting one vanishing ratio equal to a finite ratio times by a vanishing ratio common to both the desired final quantities : considerable skill must be exercised to do this. Thus, no quantities are allowed to vanish, merely to cancel to a finite number. A crucial deduction in this matter is finding the length *EP* equal to *AC*. There is also the ratios of the sections of chords: $\frac{Gv.vP}{Qv^2} = \frac{PC^2}{CD^2}$; thus it is necessary also to get rid of *Pv* somehow, as well as Qx^2 – the latter from the similar triangles mentioned; the former from the vanishing triangle *Pxv* which is similar to the triangle *PFE*, which gives the ratio $\frac{Pv}{Px} = \frac{PC}{PE} = \frac{PC}{AC}$. Thus, regarding the finite value of the vanishing quantities $\frac{QR}{QT^2}$, we can set $QR = Px = PE \times \frac{Pv}{PC} = AC \times \frac{Pv}{PC}$; while $Pv = \frac{PC^2}{CD^2} \times \frac{Qv^2}{Gv}$; thus, in place of *QR*, we may write $\frac{AC}{PC} \times \frac{PC^2}{CD^2} \times \frac{Qv^2}{Gv} = AC \times \frac{PC}{CD^2} \times \frac{Qv^2}{Gv}$. Again, QT^2 can be replaced by $Qx^2 \times \frac{PF^2}{CA^2}$. It then follows that $\frac{QR}{QT^2} = AC \times \frac{PC}{CD^2} \times \frac{1}{Gv} \times \left(\frac{Qv^2}{Qx^2}\right) \times \frac{AC^2}{PF^2}$; but from above,

$\frac{AC^2}{PF^2} = \frac{CD^2}{BC^2}$, and hence $\frac{QR}{QT^2} = AC \times \frac{PC}{CD^2} \times \frac{1}{Gv} \times \left(\frac{Qv^2}{Qx^2}\right) \times \frac{CD^2}{BC^2} = AC \times \frac{PC}{BC^2} \times \frac{1}{Gv} = \frac{AC}{2BC^2}$, since Gv tends to $2PC$ in the limit; hence $\frac{QR}{QT^2} \times \frac{2BC^2}{AC} = 1$, or $L \times \frac{QR}{QT^2} = 1$, as required.]

The Same Otherwise.

Since the force attracting the body towards the centre of the ellipse, by which the body is able to rotate about that, shall be (by Corol.I, Prop. X.) as the distance *CP* from the centre of the ellipse *C;* C*E* is taken parallel to the tangent of the ellipse *PR* ; and the force, by which the same body *P* can revolve about some other point of the ellipse *S*, if *CE* and *PS* are concurrent in *E*, will be as $\frac{PE^3}{SP^2}$ (by Corol.3, Prop.VII.) that is, if the point *S* shall be the focus of the ellipse, and thus *PE* may be given, as reciprocally as SP^2. *Q.E.1.*

By the same brevity, with which we have extended the fifth problem to the parabola and the hyperbola, here it is allowed to be used likewise; truly on account of the worth of the problem, and the use of this in the following, it will be a pleasure to confirm the other propositions by demonstration.

PROPOSITION XII. PROBLEM VII.

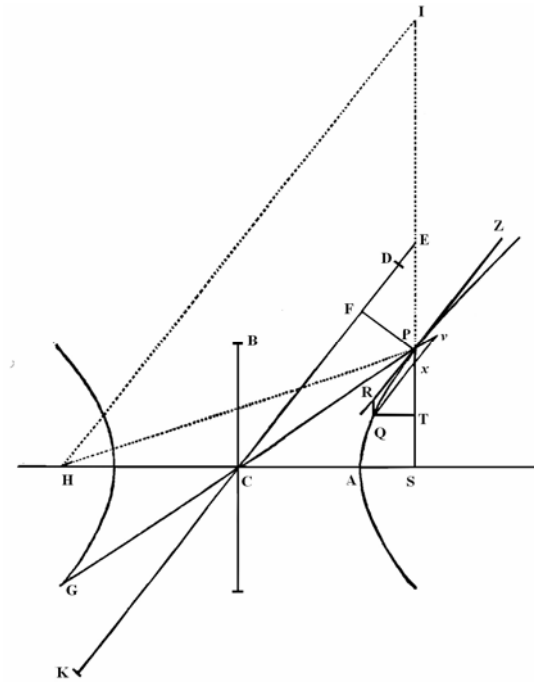
A body may be moving in a hyperbola : the law of the centripetal force is required tending towards the focus of the figure.

With the semi-axes *CA, CB* of the hyperbola taken; *PG, KD* are other conjugate diameters; *PF* is the perpendicular to the diameter *KD;* and *Qv* the applied ordinate to the diameter *GP*. Construct *SP* cutting with the diameter *DK* in *E*, and then with the applied ordinate Q*v* in *x*, and the parallelogram *QRPx* may be completed. It is apparent that *EP* is equal to the transverse semi-axis *AC*, because there, from the other focus of the hyperbola *H* the line *HI* parallel to *EC* is drawn, on account of which the equal quantities *CS, CH* may be equal to *ES, E1* ; thus so that *EP* shall be the semi-difference of *PS, PI*, that is (on account of the parallel lines *IH, PR* and the equal angles *IPR, HPZ*) of *PS, PH*, which amounts to the total difference of the axis *2.AC* .

[This is a special case, where the chord *SP* is perpendicular to the axis, perhaps chosen to ease the diagram : from symmetry and from the construction, $CH = CS$ and $EI = ES$; for the hyperbola, $PH - PS = 2.AC$; now $PI = PH$, i.e. the triangle *PHI* is isosceles : for if we assume the physical result that a ray from H striking at *P* appears to come from *S* on reflection, then the angles *RPH* and *IPZ* are equal, hence the normal line *FP* bisects the angle *HPI*, and so the triangle is isosceles, and $PI = PH$. Hence $PI - PS = 2.AC$ or $PE = AC$ on dividing by 2.]

To *SP* there may be sent the perpendicular *QT*. And I call *L* the principal latus rectum of the hyperbola (or $\frac{2BC^2}{AC}$), there will be $L \times QR$ to $L \times Pv$ as *QR* to *Pv*, or *Px* to *Pv*, that is : (on account of the similar triangles *Pxv*, *PEC*) as *PE* to *PC*, or *AC* to *PC*. Also there will be $L \times Pv$ to $Gv \times Pv$ as *L* to *Gv;* and (from the nature of conics) the rectangle *Gv.vP* to Qv^2 as PC^2 to CD^2 *;* and (by Corol.2, Lem.VII.) Qv^2 to Qx^2 with the points *Q* and *P* coalescing shall become the ratio of equality ; and Qx^2 or Qv^2 is to QT^2 as EP^2 to PF^2, that is, as CA^2 to PF^2, or (by Lem.XII.) as CD^2 to CB^2 : and with all these ratios taken together $L \times QR$ shall be to QT^2 as $AC \times L \times PC^2 \times CD^2$, or $2CB^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$, or as 2*PC* to *Gv*. [See below for modern notation.] But with the points *P* and *Q* coalescing 2*PC* and *Gv* are equal. And therefore from these proportions $L \times QR$ and QT^2 are equal. These equalities are multiplied by $\frac{SP^2}{QR}$, and there becomes $L \times SP^2$ equal to $\frac{SP^2 \times QT^2}{QR}$. Therefore (by Corol.I & V, Prop.VI.) the centripetal force is reciprocally as $L \times SP^2$, that is, reciprocally in the ration of the square of the distance *SP*. *Q. E. I.*

[The derivation and notation follow the ellipse above exactly.]



$$[\frac{L\times QR}{L\times Pv}=\frac{QR}{Pv}=\frac{Px}{Pv}=\frac{PE}{PC}=\frac{AC}{PC};\ \frac{L\times Pv}{Gv\times Pv}=\frac{L}{Gv};\ \frac{Gv.vP}{Qv^2}=\frac{PC^2}{CD^2};$$

as $Q \rightarrow P, \frac{Qx^2}{QT^2}=\frac{EP^2}{PF^2}=\frac{CA^2}{PF^2}=\frac{CD^2}{CB^2}$.

Finally, $\frac{L\times QR}{QT^2}=\frac{AC\times L\times PC^2\times CD^2}{PC\times Gv\times CD^2\times CB^2}=\frac{2CB^2\times PC^2\times CD^2}{PC\times Gv\times CD^2\times CB^2}=\frac{2PC}{Gv}$.]

The same otherwise.

The force may be found, by which the body tends from the centre of the hyperbola *C*. This will be produced proportional to the distance *CP*. Truly thence (by Carol.III, Prop.VII.) the force tending towards the focus *S* will be as $\frac{PE^3}{SP^2}$ that is, on account of *PE* given reciprocally as SP^2.

Q.E.1.

In the same manner it may be shown, because the body by this force turned from centripetal to centrifugal will be moving in the above hyperbola.

[The idea of free positive and bound negative energy orbits was not of course available at the time, whereby a body with total energy > 0 describes a hyperbola about an attracting source of force such as the sun, a body with total energy < 0 describes an ellipse, while the parabola corresponds to zero energy, such as a comet starting from rest at essentially an infinite distance from the sun. Thus, an attractive gravitational force can still give rise to this hyperbolic motion. Such comets with zero or positive energy have been observed occasionally. In the diagram, the attractive force may act from the one

focus, while a repulsive force may act from the other, and in this respect they have the same effect.]

LEMMA XIII.

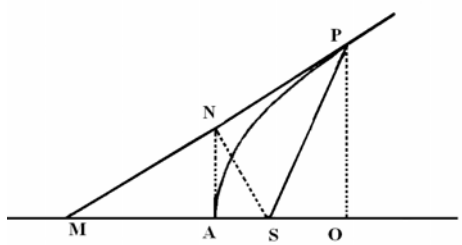
The latus rectum of a parabola pertaining to some vertex is four times the distance of that vertex from the focus of the figure.

This is apparent from conic sections.

[For the modern reader, the delightful *Book of Curves* by E. H. Lockwood (CUP) can be consulted for the more common properties referring to conic sections, as well as for details about many other well-known curves.]

LEMMA XIV.

The perpendicular, which is sent from the focus of the parabola to the tangent of this curve, is the mean proportional between the distance of the focus from the point of contact and from the principal vertex of the figure.



For let *AP* be the parabola, *S* the focus of this, *A* the principle vertex, *P* the point of contact of the tangent, *PO* the applied ordinate to the principle diameter, *PM* the tangent crossing the principal axis *M*, and *SN* the perpendicular from the focus to the tangent. *AN* is joined and on account of the equal lines *MS* and *SP*, *MN* and *NP*, *MA* and *AO* , the right lines *AN* and *OP* are parallel *;* and thence the triangle *SAN* will be right-angled at *A*, and similar to the equal triangles *SNM* and *SNP:* therefore *PS* is to *SN* as *SN* is to *SA. Q.E.D.*

$$[\,i.e.\ \frac{PS}{SN} = \frac{SN}{SA}\,.]$$

[For the envelope of the parabola can be constructed from the tangents by rotating the right angle *SAN* about *S* while *N* moves progressively along the coordinate line *AN* from *N* to some *N'* (not shown here) whereby *SNN'P* is a cyclic quadrilateral and the angle *ANS* is the angle between the chord and the tangent, equal to the angle in the alternate segment *NPS*, on letting *N'* tend towards *N*. See Lockwood p.4 for the details.]

Corol. 1. PS^2 is to SN^2 as *PS* to *SA*. [For $SN^2 = PS.SA$ and $\frac{PS^2}{PS.SA} = \frac{PS}{SA}$.]

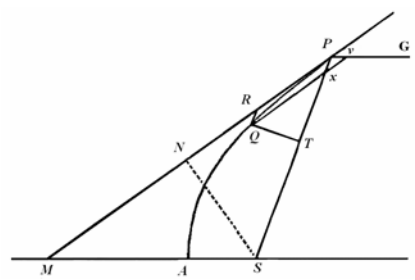
Corol. 2. And on account of *SA* given there is SN^2 as *PS*.

Corol. 3. And the running together of any tangent *PM* with the right line *SN*, which is from the focus into the perpendicular itself, falls on the right line *AN*, which is a tangent to the parabola at the principle vertex.

PROPOSITION XIII. PROBLEM VIII.

A body may be moving in the perimeter of a parabola :the law of the centripetal force tending towards the focus of this figure is to be acquired.

The construction of the lemma may remain, and let *P* be a body on the perimeter of the parabola, and from the closest place *Q*, into which the body may move, with the line *QR* drawn acting parallel to *Sp* itself and the perpendicular *QT*, and also *Qv* parallel to the tangent, and running to meet both with the diameter *PG* in *v*, as well as the interval *SP* in *x*. Now on account of the similar triangles *Pxv*, *SPM*, and the equality of the sides *SM*, *SP*, the other sides *Px* or *QR* and *Pv* are equal. But from the theory of conics [See note at start of this section.] the square of the ordinate *Qv* is equal to the rectangle under the latus rectum and the segment of the diameter *Pv*, that is (by Lem. XIII.) to the rectangle $4PS \times Pv$, or to $4PS \times QR$; and with the points *P* and *Q* merging together, the ratio *Qv* to *Qx* (by Corol 2, Lem.VII.) shall be one of equality. Hence Qx^2 in that case is equal to the rectangle $4PS \times QR$. But (on account of the similar triangles QxT, SPN) Qx^2 is to QT^2 as PS^2 to SN^2, that is (by Corol.I, Lem.XIV.) as *PS* to *SA*, that is, as $4PS \times QR$ to $4SA \times QR$, and thence (by Prop. IX, Lib. V. Elem.) QT^2 and $4SA \times QR$ are equal. This equality is multiplied by $\frac{SP^2}{QR}$, and there becomes $\frac{SP^2 \times QT^2}{QR}$ equal to $SP^2 \times 4SA$: and therefore (by Corol.I. and V, Prop. VI.) : the centripetal force is reciprocally as $SP^2 \times 4SA$, that is, on account of 4*SA* given, reciprocally in the square ration of the distance *SP*.



Q.E.I.

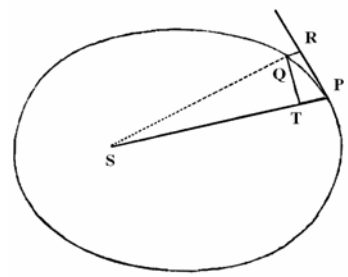
[$Qv^2 \to Qx^2 = 4PS \times QR$; $\frac{Qx^2}{QT^2} = \frac{PS^2}{SN^2} = \frac{PS}{SA} = \frac{4PS \times QR}{4SA \times QR}$; $QT^2 = 4SA \times QR$.]

Corol. I. From the three latest propositions it follows, that if some body *P* may emerge from the position *P* and follows some line *PR* with whatever velocity, and with the centripetal force, which shall be reciprocally proportional to the square of the distance of the position from the centre, likewise may be acting ; this body will be moved in some conic sections having the focus in the centre of forces ; and vice versa. For from given focus and point of contact, and from the position of the tangent, it is possible to describe a conic section, which will have a given curvature at that point. But the curvature is given from the given centripetal force, and from the velocity of the body : and it is not possible to describe two orbits mutually touching each other with the same centripetal force and with the same velocity.

Corol 2. If the velocity, with which the body leaves from its position *P*, shall be that, by which the short line *PR* may be able to be described in some smallest part of time ; and the centripetal force shall be able in the same time to move through the distance *QR:* this body will be moved in some conic section, the principal latus rectum is that quantity $\frac{QT^2}{QR}$ which finally is made when the small lines *PR* and *QR* are diminished indefinitely. The circle I refer to the ellipse in these corollaries ; and I rule out the case, where the body descends to the centre along a straight line.

PROPOSITION XIV. THEOREM VI.

If several bodies are rotating about a common centre, and the centripetal force shall be in the reciprocal square ratio of the distances of the places from the centre ; I say that the principal latera recta are in the square ratio of the areas, which the bodies describe by the radii to the centre in the same time.



For (by Corol 2, Prop. XIII, [and Prop. XI]) the latus rectum *L* is equal to the quantity $\frac{QT^2}{QR}$ which finally comes about, when the points *P* and *Q* coincide. But the minimum line *QR* in the given time is as the generating centripetal force, that is (by hypothesis) reciprocally as SP^2. Therefore $\frac{QT^2}{QR}$ is as $QT^2 \times SP^2$, that is, the latus rectum *L* is in the square ratio of the area $QT \times SP$.

Q.E.D.

Corol. Hence the total area of the ellipse, and to which the rectangle under the axis is proportional, is composed from the square root ratio of the latus rectum, and from the ratio of the periodic time. For the total area is as the area $QT \times SP$, which will be described in the given time, taken into the periodic time.

PROPOSITION XV. THEOREM VII.

With the same in position, I say that the periodic times for ellipses are in the ratio of the three on two power of the major axis.

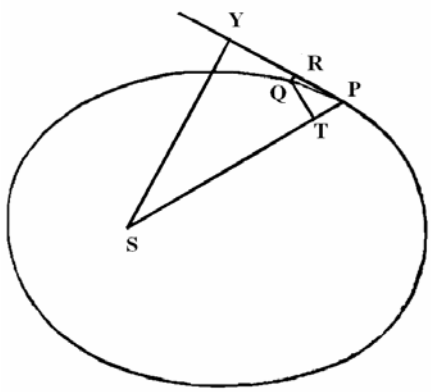
For the minor axis is the mean proportional between the major axis and the latus rectum, and thus the rectangle under the axes is in the ratio of the major axis. But this rectangle (by Corol. Prop. XIV.) is in the ratio composed from the square root of the latus rectum and in the ratio of the periodic time. The ratio of the square root of the lengths of the lines are subtracted from both sides, and there will remain the three on two ratio of the major axis with the ratio of the periodic time.

Q.E.D.

Corol. Therefore the periodic times are the same in ellipses and circles, of which the diameters are equal to the major axes of ellipses.

PROPOSITION XVI. THEOREM VIII.

With the same in place, and with the right lines drawn to the bodies, which at that instant are tangents to the orbits, and with perpendiculars sent from the common focus to these tangents: I say that the velocities of the bodies are in a ratio composed from the inverse ratio of the perpendiculars, and directly with the square root ratio of the principal latera recta.



Send the perpendicular *SY* from the focus *S* to the tangent *PR*, and the velocity of the body *P* will be reciprocally in the ratio of the square root of the quantity $\frac{SY^2}{L}$. For that velocity is as the minimum arc *PQ* described in the given moment of time, that is (by Lem.VII.) as the tangent *PR*, that is, on account of the proportionals *PR* to *QT* and *SP* to *SY*, as $\frac{SP \times QT}{SY}$ or as *SY* reciprocally and $SP \times QT$ directly; and $SP \times QT$ shall be as the area described in the given time, that is (by Prop.XIV.) in the square root ratio of the latus rectum.

Q *E.D.*

*Corol.*I. The principal latera recta are in a ratio composed from the square of the ratio of the perpendiculars and, and in the squared ratio of the velocities.

Corol. 2. The velocities of bodies, at the maximum and minimum distances from the common focus, are in a ratio composed from the inverse ratio of the distances, and directly as the square root of the principal latera recta. For the perpendiculars now are these distances themselves.

*Corol.*3. And thus the velocities in a conic section, at the maximum or minimum distances from the focus, is to the velocity in a circle at the same distance from the centre, in the square root ratio of the principal latus rectum to twice that distance.

Corol. 4. The velocities of bodies gyrating in ellipses at their mean distances from the common focus are the same as of bodies gyrating in circles at the same distances; that is (by Corol.6, Prop. IV.) reciprocally in the square root ratio of the distances. For the perpendiculars now are the minor semi-axes, and these are as the mean proportionals between the distances and the latera recta. This ratio may be taken inversely with the

square root ratio of the latus rectums directly, and it becomes the ratio of the inverse square of the distances.

Corol. 5. In the same figure, or even in different figures, of which the principal latera recta are equal, the velocity of the body is inversely as the perpendicular sent from the focus to the tangent.

Corol. 6. In a parabola the velocity is reciprocally in the square root ratio of the distance of the body from the focus of the figure; in an ellipse it is changed more, in a hyperbola less than in this ratio. For (by Corol. 2, Lem. XIV.) the perpendicular sent from the focus to the tangent of the parabola is in the inverse square ratio of the distance. In a hyperbola the perpendicular is varied less, in an ellipse more.

Corol. 7. In the parabola the velocity of a body at some distance from the focus is as the velocity of the body revolving in a circle at the same distance from the same centre in the square root ratio of two to one; in an ellipse it is less, in a hyperbola it is greater than in this ratio. For (by Corollary 2 of this Prop.)the velocity at the vertex of the parabola is in this ratio, and (by Corollary 6 of this Prop. and Proposition IV) the same proportion is maintained between all the distances. Hence also in the parabola the velocity everywhere is equal to the velocity of the body revolving in a circle at half the distance, in an ellipse it is less, greater in a hyperbola.

Corol. 8. The velocity of gyration in some conic section is to the velocity of gyration in a circle at a distance of half the principle latus rectum of the section, as that distance to the perpendicular from the focus sent to the tangent of the section. This is apparent from corollary five.

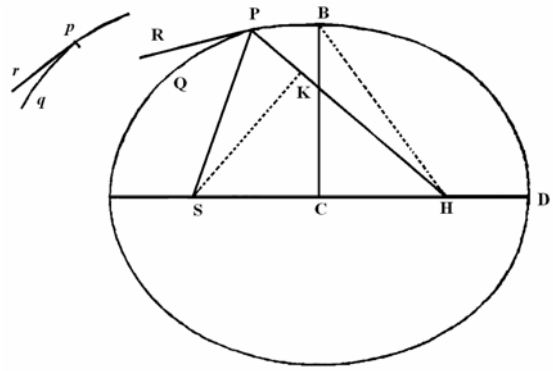
Corol. 9. From which when (by Corol 6. Prop. IV.) the velocity of gyration in this circle shall be to the velocity of gyration in some other circle reciprocally in the square root ratio of the distances ; there becomes from the equality the velocity of gyration in the conic section to the velocity of gyration in a circle at the same distance, as the mean proportional between that common distance and half the principle latus rectum of the section, to the perpendicular sent from the common focus to the tangent of the section.

PROPOSITION XVII. PROBLEM IX.

Because the centripetal force shall be reciprocally proportional to the square of the distance of the places put in place from the centre, and because the magnitude of that force shall be known absolutely; the line is required, that the body will describe from that place progressing with a given velocity along a given right line.

The centripetal force tending towards the point *S* shall be that, by which the body *p* may gyrate in some given orbit *pq*, and the velocity of this may be known at the position *p*. From the place *P* the body *P* may emerge following the line *PR* with a given velocity, and thence soon, with the centripetal force acting, that may be deflected in the section of

a cone *PQ*. Therefore this right line *PR* is a tangent at *P*. Likewise *pq* may touch the other orbit in *p*, and if from *S* these perpendiculars are understood to be sent, (by Corol. I, Prop. XVI.) the principle latus rectum of the conic section will be to the principle latus rectum of the orbit in a ratio composed from the squared ratio of the perpendiculars and from the squared ratio of the velocities, from which thus it is given. Let *L* be the latus rectum of the conic section. Therefore the focus *S* of this conic section is given.



The complement of the angle *RPS* to two right angles makes the angle *RPH;* and from the position given of the line *PH*, on which the other focus *H* may be located. With the perpendicular *SK* sent from *PH*, the conjugate semi-axis *BC* is understood to be erected, and there will be

$$SP^2 - 2KPH + PH^2 = SH^2 = 4CH^2 = 4BH^2 - 4BC^2 =$$
$$\overline{SP+PH}^2 - L \times \overline{SP+PH} = SP^2 + 2SPH + PH^2 - L \times \overline{SP+PH}.$$

On both sides there may be added $2.KPH - SP^2 - PH^2 + L \times \overline{SP+PH}$, and there becomes $L \times SP + PH = 2SPH + 2KPH$, or $SP + PH$ to PH as $2.SP + 2.KP$ to L. From which *PH* is given both in length as well as in position. Without doubt if that shall be the velocity of the body at *P*, so that the latus rectum *L* were less than $2.SP + 2.KP$, *PH* will be added to the same direction of the tangent *PR* with the line *PS;* and thus the figure will be an ellipse, and it will be given from the given foci *S, H,* and the principle axis $SP + AH$. But if the velocity of the body shall be so great, so that the latus rectum *L* were equal to $2.SP + 2.KP$, *PH* will be infinitely long; and therefore the figure will be a parabola having the axis *SH* parallel to the line *PK*, and thence it will be given. But if the body at this point emerges with a greater velocity from its place *P* , a length may be required to be taken *PH* at another direction of the tangent; and thus with the tangent arising between the two foci, the figure will be a hyperbola having the principle axis equal to the difference of the lines *SP* and *PH*, and thence it will be given. For if the body in these cases may revolve in a conic section thus found, it has been shown in Prop. XI, XII, and XIII, that the centripetal force will be inversely as the square of the distance from the centre of the forces *S*; and thus the line *PQ* is shown to be correct, as the body may describe by such a force, from some given place *P*, with a given velocity, emerging along the given right line *PR* in place. *Q.E.F.*

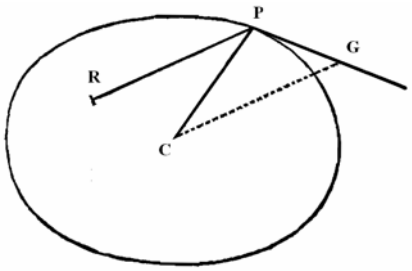
Corol. I. Hence in every conic section from a given vertex *D* , with the latus rectum *L*, and with the focus *S*, another focus *H* is given on taking *DH* to *DS* as the latus rectum to the difference between the latus rectum and 4*DS*. For the proportion $SP + PH$ to *PH* is as $2SP + 2KP$ to L in the case of this corollary, shall be $DS + DH$ to *DH* as 4*DS* to *L*, and separately *DS* to *DH* as $4DS - L$ to *L*.

Corol. 2. From which if the velocity of the body is given at the principal vertex *D,* the orbit may be found readily, clearly on taking the latus rectum of this to twice the distance *DS,* in the square ratio of the velocity of this given to the velocity of the body in a circle at a distance of rotation *DS* (by Corol.3, Prop. XVI. ;) then *DH* to *DS* shall be as the latus rectum to the difference between the latus rectum and 4*DS.*

Corol. 3. Hence also if the body may be moving in some conic section, and it may be disturbed from it orbit by some external impulse ; it is possible to know, what course it will pursue afterwards. For on compounding the proper motion of the body with that motion, that the impulse alone may generate, the motion of the body will be had with which it will emerge along a given line in place with the given impulse in place,.

Corol. 4. And if that body may be disturbed continually by some extrinsic impressed force, as the course may become know approximately, by requiring the changes to be deduced which the force induces on that body at some points, and from analogous series the continuous changes can be judged at the intermediate places.

Scholium.



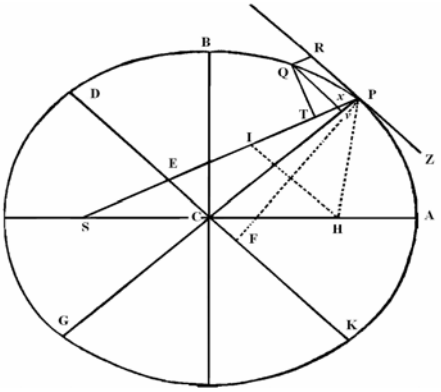
If the body *P* tending towards some given point *R* by the centripetal force, may be moving on the perimeter of some given conic section, the centre of which shall be *C;* and the centripetal law of the force may be required : *CG* is drawn parallel to the radius *RP* , and crossing to the tangent of the orbit *PG* at *G*; and that force (by Corol.I. and Schol. Prop. X. & Corol.3, Prop.VII.) will be as $\frac{CG^3}{RP^2}$.

SECTION III.

De motu corporum in conicis sectionibus excentricis.

PROPOSITIO Xl. PROBLEMA VI.

Revolvatur corpus in ellipsi: requiritur lex vis centripetae tendentis ad umbilicum ellipseos.



Esta ellipseos umbilicus *S.* Agatur *SP* secans ellipseos tum diametrum *DK* in *E*, tum ordinatim applicatam Q*v* in *x*, & compleatur parallelogrammum Q*xPR.* Patet *EP* aequalem esse semiaxi majori *AC*, eo quod, acta ab altero ellipseos umbilico *H* linea *HI* ipsi *EC* parallela, ob aequales *CS, CH* requentur *ES, E1*, adeo ut *EP* semisumma sit ipsarum *PS, PI*, id est (ob parallelas *HI, PR*, & angulos aequales *IPR,HPZ*) ipsarum *PS, PH*, quae coniunctim axem totum 2.*AC* adaequant. Ad *SP* demittatur perpendicularis *QT*, & ellipseos latere recto principali (seu $\frac{2.BCquad.}{AC}$) dicto *L*, erit $L \times QR$ ad $L \times Pv$ ut *QR* ad *Pv*, id est, ut *PE* seu *AC* ad *PC;* & $L \times Pv$ ad *GvP* ut L ad *Gv;* & *GvP* ad *Qvquad*.
ut *PC quad*. ad *CD quad*., &. (per Corol.2., Lem. VII.) *Qv quad*. ad *Qx quad*. punctis *Q* &.*P* coeuntibus est ratio aequalitatis; & *Qx quad*. seu *Qv quad*. est ad *QT quad*. ut *EP quad*. ad *PF quad*., id est, ut *CA quad*. ad *PF quad*. sive (per Lem. XII.) ut *CD quad*. ad *CBquad*. Et conjunctis his omnibus rationibus, $L \times QR$ fit ad *QTquad*. *ut* $AC \times L \times PCq \times CDq$, seu $2.CBq \times PCq \times CDq$ ad $PC \times Gv \times CDq \times CBq$, sive ut 2.*PC* ad *Gv.* Sed punctis *Q* & *P* coeuntibus aequantur 2.*PC* & *Gv.* Ergo & his proportionalia $L \times QR$ & *QTquad.* aequantur. Ducantur haec aequalia in $\frac{SPq}{QR}$, & fiet $L \times SPq$ aequale $\frac{SPq \times QTq}{QR}$. Ergo (per corol.I & , Prop.VI.) vis centripeta reciproce est ut $L \times SPq$ id est, reciproce in ratione duplicata distantiae *SP*. *Q E.I.*

Idem aliter.

Cum vis ad centrum ellipseos tendens, qua corpus *P* ellipsi illa revolvi potest, sit (per Corol.I, Prop.X.) ut *CP* distantia corporis ab ellipseos centro *C;* ducatur C*E* parallela ellipseos tangenti *PR* ; & vis, qua corpus idem *P* circum aliud quodvis ellipseos punctum *S* revolvi potest, si *CE* & *P S* concurrant in *E*, erit ut $\frac{PEcub.}{SPq}$ (per Corol.3, Prop.VII.) hoc est, si punctum *S* fit umbilicus ellipseos, ideoque *PE* detur, ut *SPq* reciproce. *Q.E.1.*

Eadem brevitate, qua traduximus problema quintum ad parabolam, & hyperbolam, liceret idem hic facere; verum ob dignitatem problematis, & usum ejus in sequentibus non pigebit carus ceteros demonstratione confirmare.

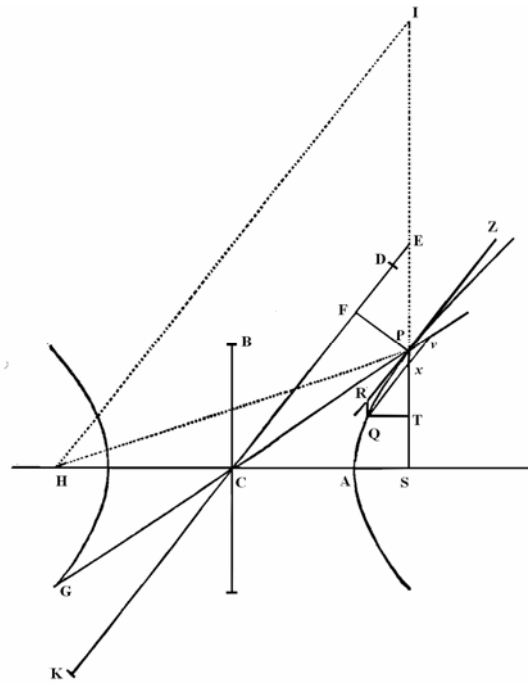
PROPOSITIO XII. PROBLEMA VII.

Moveatur corpus in hyperbola: requiritur lex vis centripetae tendentis ad umbilicum figurae.

Sumto *CA, CB* semiaxes hyperbolae; *PG, KD* diametri aliae conjugatae; *PF* perpendiculum ad diametrum *KD;* & *Qv* ordinatim applicata ad diametrum *GP.* Agatur *SP* secans cum diametrum *DK* in *E*, tum ordinatim applicatam Q*v* in *x*, & compleatur parallelogrammum *QRPx.* Patet *EP* aequalem esse semiaxi transverso *AC,* eo quod, acta ab altero hyperbolae umbilico *H* linea *HI* ipsi *EC* parallela, ob aequales *CS, CH* aequentur *ES, E1* ; adeo ut *EP* semidifferentia sit ipsarum *PS, PI,* id est (ob parallelas *IH, PR* & angulos aequales *IPR, HPZ)* ipsarum *PS, PH,* quarum differentia axem totum *2.AC* adaequat. Ad *S P* demittatur perpendicularis *QT.* Et hyperbolae latere recto principali (seu $\frac{2BCq}{AC}$) dicto *L*, erit $L \times QR$ ad $L \times Pv$ ut *QR* ad *Pv*, seu *x* ad *Pv*, id est : (ob similia triangula *Pxv, PEC*) ut *PE* ad *PC,* seu *AC* ad *PC.* Erit etiam $L \times Pv$ ad $Gv \times Pv$ ut *L* ad *Gv;* & (ex natura conicorum) rectangulum *GvP* ad *Qv quad.* ut *PCq* ad *CDq; &* (per Corol.2, Lem.VII.) *Qv quad.* ad *Qx quad.* punctis *Q* & *P* coeuntibus fit ratio aequalitatis ; & *Qx quad.* seu *Qv quad.* est ad *QTq* ut *EPq* ad *PFq,* id est, ut *CAq* ad *PFq,* sive (per Lem.XII.) ut *CDq* ad *CB q*: & conjunctis his omnibus rationibus $L \times QR$ sit ad *QTq* ut $AC \times L \times PCq \times CDq$, seu 2. $2CBq \times PCq \times CDq$ ad $PC \times Gv \times CDq \times CBq$ sive ut 2*PC* ad *Gv.* Sed punctis *P* & *Q* coeuntibus aequantur 2*PC* & *Gv*. Ergo & his proportionalia $L \times QR$ & *QTq* aequantur. Ducantur haec aequalia in $\frac{SPq}{QR}$, & fiet $L \times SPq$ aequalia $\frac{SPq \times QTq}{QR}$. Ergo (per Corol.I & V, Prop.VI.) vis centripeta reciproce est ut $L \times SPq$, id est, reciproce in ratione duplicata distantiae *SP. Q. E. I.*

Idem aliter.

Inveniatur vis, que tendit ab hyperbole centro *C.* Prodibit haec distantiae *CP*



proportionalis. Inde vera (per Carol.III, Prop.VII.) vis ad umbilicum *S* tendens erit ut $\frac{PEcub.}{SPq}$ hoc est, ob datam *PE* reciproce ut *SPq*. *Q.E.1.*

Eadem modo demonstratur, quod corpus hac vi centripeta in c:entrifugam versa movebitur in hyperbola opposita.

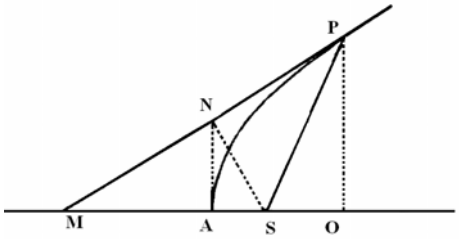
LEMMA XIII.

Latus rectum parabolae ad verticem quemvis pertinens est quodruplum distantiae verticis illius ab umbilio figurae.

Patet ex conicis.

LEMMA XIV.

Perpendiculum, quod ab umbilico parabolae ad tangentem eius demittitur, medium est proportionale inter distantias umbilici a puncto contactus & a vertice principali figurae.



Sit enim *AP* parabola, *S* umbilicus eius, *A* vertex principalis, *P* punctum contactus, *PO* ordinatim applicata ad diametrum principalem, *PM* tangens diametro principali occurrens in *M*, & *SN* linea perpendicularis ab umbilico in tangentem. Iungatur *AN* & ob aequales *MS* & *SP*, *MN*, & *NP*, *MA* & *AO* parallelae erunt rectae *AN* & *OP;* & inde triangulum *SAN*

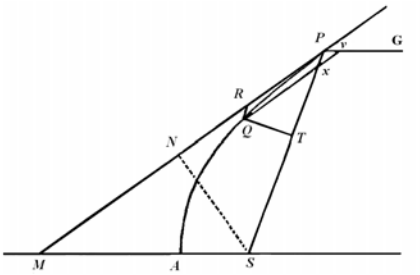
rectangulum erit ad *A*, & simile triangulis aequalibus *SNM, SNP:* ergo *PS* est ad *SN* ut *SN* ad *SA. Q.E.D.*

Corol. 1. *PSq* est ad *SNq* ut *PS* ad *SA.*
Corol. 2. Et ob datam *SA* est *SNq* ut *PS.*
Corol. 3. Et concursus tangentis cuiusvis *PM* cum recta *SN*, que ab umbilico in ipsam perpendicularis ea, incidit in rectam *AN* quae parabolam tangit in vertice principali.

PROPOSITIO XIII. PROBLEMA VIII.

Moveatur corpus in perimetro parabolae: acquiritur lex vis centripetae tendentis ad umbilicum hujus figurae.

Maneat constructio lemmatis, sitque *P* corpus in perimetro parabolae, & a loco *Q*, in quem corpus proxime movetur, age ipsi *Sp* parallelam *QR* & perpendicularem *QT*, necnon *Qv* tangenti parallelam, & occurrentem tum diametro *PG* in *v*, tum distantiae *SP* in *x*. Iam ob similia triangula *Pxv*, *SPM*, & aequalia unius latera *SM*, *SP*, aequalia sunt alterius latera *Px* seu *QR* & *Pv*. Sed ex conicis quadratum ordinatae *Qv* aequate est rectangulo sub latere recto & segmento diametri *Pv*, id est (per Lem. XIII.) rectangulo $4PS \times Pv$, seu $4PS \times QR$; & punctis *P* & *Q* coeuntibus, ratio *Qv* ad *Qx* (per Corol 2, Lem.VII.) sit ratio aequalitatis. Ergo *Qx quad.* eo in casu aequale est rectangulo $4PS \times QR$. Est autem (ob similia triangula *QxT* , *SPN*) *Qxq* ad *QTq* ut *PSq* ad *SNq*, hoc est (per Corol.I, Lem.XIV.) ut *PS* ad *SA*, id est, ut $4PS \times QR$ ad $4SA \times QR$, & inde (per Prop. IX, Lib. V. elem.) *QTq* & $4SA \times QR$ aequantur. Ducantur haec equalia in $\frac{SPq}{QR}$, & fiet $\frac{SPq \times QTq}{QR}$ aequale $SPq \times 4SA$: & propterea (per corol.I. & V, Prop. VI.) : vis centripeta est reciproce ut $SPq \times 4SA$, id est, ob datam 4 *SA* reciproce in duplicata ratione distantiae *SP*. *Q.E.I.*



Corol. I. Ex tribus novissimis propositionibus consequens est, quod si corpus quodvis *P* secundum lineam quamvis rectam *PR* quacunque cum velocitate exeat de loco *P*, & vi centripeta, quae sit reciproce proportionalis quadrato distantiae locorum a centro, simul agitetur; movebitur hoc corpus in aliqua sectionum conicarum umbilicum habente in centro virium ; & contra. Nam datis umbilico, & puncto contactus, & positione tangentis, describi potest sectio conica, quae curvaturam datam ad punctum illud habebit. Datur autem curvatura ex data vi centripeta, & velocita corporis: & orbes duo se mutua tangentes eadem vi centripeta eademque velocitate describi non possunt.

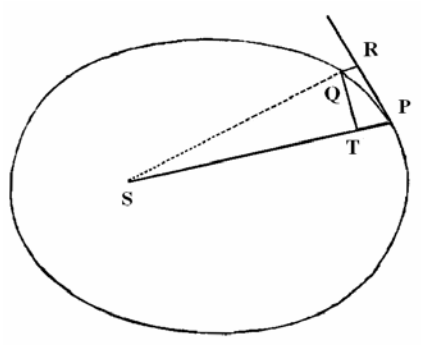
Corol 2. Si velocitas, quacum corpus exit de loco suo *P*, ea sit, qua lineola *PR* in minima aliqua temporis particula describi possit ; & vis centripeta potis sit eadem tempore corpus idem movere per spatium *QR:* movebitur hoc corpus in conica aliqua sectione,

cuius latus rectum principale est quantitas illa $\frac{QTq}{QR}$ quae ultimo fit ubi lineolae *PR, QR* in infinitum diminuuntur. Circulum in his corollariis refero ad ellipsin; & casum excipio, ubi corpus recta descendit ad centrum.

PROPOSITIO XIV. THEOREMA VI.

Si corpora plura revolvantur circa centrum commune, & vis centripeta sit reciproce in duplicata ratione distantiae locorum a centro; dico quod orbium latera recta principalia sunt in duplicata ratione arearum, quas corpora radiis ad centrum ductis eadem tempore describunt.

Nam (per Corol 2, Prop. XIII.) latus rectum *L* aequalia est quantitati $\frac{QTq}{QR}$ quae ultimo fit, ubi coeunt puncta *P* & *Q*. Sed linea minima *QR* dato tempore est ut vis centripeta generans, hoc est (per hypothesin) reciproce ut *SPq*. Ergo $\frac{QTq}{QR}$ est ut $QTq \times SPq$, hoc est, latus rectum *L* in duplicata ratione areae $QT \times SP$. *Q.E.D.*



Corol. Hinc ellipseos area tota, eique proportionale rectangulum sub axibus est in ratione composita ex subduplicata ratione lateris recti, & ratione temporis periodici. Namque area tota est ut area $QT \times SP$, quae dato tempore describitur, ducta in tempus periodicum.

PROPOSITIO XV. THEOREMA VII.

Iisdem positis, dico quod tempora periodica in ellipsibus sunt in ratione sesquiplicata majorum axium.

Namque axis minor est medius proportionalis inter axem majorem & latus rectum, atque ideo rectangulum sub axibus est in ratione axis maioris. Sed hoc rectangulum (per Corol. Prop. XIV.) est in ratione composita ex subduplicata ratione lateris recti & ratione periodici temporis. Dematur utrobique subduplicata ratio lateris recti, & manebit sesquiplicata ratio maioris axis eadem cum ratione periodici temporis. *Q.E.D.*

Corol. Sunt igitur tempora periodica in ellipsibus eadem ac in circulis, quorum diametri aequantur maioribus axibus ellipseon.

PROPOSITIO XVI. THEOREMA VIII.

Iisdem positis, & *actis ad corpora lineis rectis, quae ibidem tangant orbitas, demissisque ab umbilico communi ab has tangentes perpendicularibus: dico quod velocitates corporum sunt in ratione composita ex ratione perpendiculorum inverse, & subduplicata ratione laterum rectorum principalium directe.*

Ab umbilico *S* ad tangentem *PR* demitte perpendiculum *SY*, & velocitas

corporis *P* erit reciproce in subduplicata ratione quantitatis $\frac{SYq}{L}$. Nam velocitas illa est ut arcus quam minimus *PQ* in data temporis particula descriptus, hoc est (per Lem.VII.) ut tangens *PR*, id est, ob proportionales *PR* ad *QT* & *SP* ad *SY*, ut $\frac{SP \times QT}{SY}$ sive ut *SY* reciproce & $SP \times QT$ directe; estque $SP \times QT$ ut area dato tempore descripta, id est (per Prop.XIV.) in subduplicata ratione lateris recti. Q *E.D.*

*Corol.*I. Latera recta principalia sunt in ratione composita ex duplicata ratione perpendiculorum, & duplicata ratione velocitatum.

Corol. 2. Velocitates corporum, in maximis & minimis ab umbilico communi distantiis, sunt in ratione composita ex ratione distantiarum inverse, & subduplicata ratione laterum rectorum principalium directe. Nam perpendicula iam sunt ipsae distantiae.

*Corol.*3. Ideoque velocitas in conica sectione, in maxima vel minima ab umbilico distantia, est ad velocitatem in circulo in eadem a centro distantia in subduplicata ratione lateris recti principalis ad duplam illam distantiam.

Corol. 4. Corporum in ellipsibus gyrantium velocitates in mediocribus distantiis ab umbilico communi sunt eaedem, quae corporum gyrantium in circulis ad easdem distantias; hoc est (per Corol.6, Prop. IV.) reciproce in subduplicata ratione distantiarum. Nam perpendicula iam sunt semi-axes minores, & hi sunt ut mediae proportionales inter distantias & latera casum. Componatur haec ratio inverse cum subduplicata ratione laterum rectorum directe, & fiet ratio subduplicata distantiarum inverse.

Corol. 5. In eadem figura, vel etiam in figuris diversis, quarum latera casum principalia sunt aequalia, velocitas corporis est reciproce at perpendiculum demitrum ab umbilico ad tangentem.

Corol. 6. In parabola velocitas est reciproce in subduplicata ratione distantiae corporis ab umbilico figurae; in ellipsi magis variatur, in hyperbola minus quam in hac ratione. Nam (per Corol. 2, Lem. XIV.) perpendiculum demissum ab umbilico ad tangentem parabole est in subduplicata ratione distantiae. In hyperbola perpendiculum minus variatur, in ellipse magis.

Corol. 7. In parabola velocitas corporis ad quamvis ab umbilico distantiam est ad velocitatem corporis revolventis in circulo ad eandem a centro distantiam in subduplicata ratione numeri binarii ad unitatem; in ellipsi minor est, in hyperbola major quam in hac ratione. Nam per hujus corollarium secundum velocitas in vertice parabolae est in hac ratione, & per corollaria sexta hujus & propositionis quartae servatur eadem proportio in omnibus distantiis. Hinc etiam in parabola velocitas ubique aequalis est velocitati corporis revolventis in circulo ad dimidiam distantiam, in ellipsi minor est, in hyperbola major.

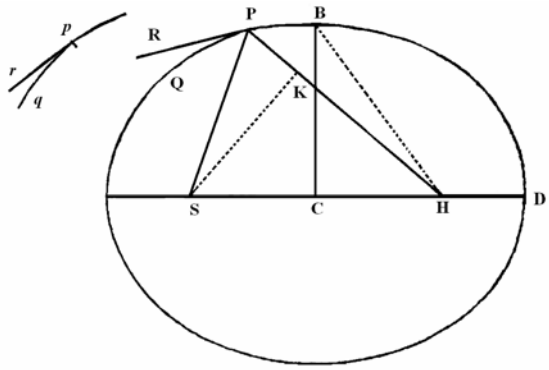
Corol. 8. Velocitas gyrantis in sectione quavis conica est ad velocitatem gyrantis in circulo in distantia dimidii lateris recti principalis sectionis, ut distantia illa ad perpendiculum ab umbilico in tangentem sectonis demissum. Patet per corollarium quintum.

Corol. 9. Unde cum (per Corol 6. Prop. IV.) velocitas gyrantis in hoc circulo sit ad velocitatem gyrantis in circulo quovis alio reciproce in subduplicata ratione distantiarum ; fiet ex aequo velocitas gyrantis in conica sectione ad velocitatem gyrantis in circulo in eadem distantia, ut media proportionalis inter distantiam illam communem & semissem principalis lateris recti sectionis, ad perpendiculum ab umbilico communi in tangentem sectionis demissum.

PROPOSITIO XVII. PROBLEMA IX.

Posito quod vis centripeta sit reciproce proportionalis quadrato distantiae locorum a centro, & quod vis illius quantitas absoluta sit cognita; requiritur linea, quam corpus describit de loco dato cum data velocitate secundum datam rectam egrediens.

Vis centripeta tendens ad punctum *S* ea sit, qua corpus *p* in orbita quavis data *pq* gyretur, & cognoscatur hujus velocitas in loco *p*. De loco *P* secundum lineam *PR* exeat corpus *P* cum data velocitate, & mox inde, cogente vi centripeta, deflectat illud in coni sectionem *PQ*. Hanc igitur recta *PR* tanget in *P*. Tangat itidem casum aliqua *pr* orbitam *pq* in *p*, & si ab *S* ad eas tangentes demitti intelligantur perpendicula, erit (per Corol. I, Prop. XVI.) latus rectum principale coni fectionis ad latus rectum principale orbit in ratione composita ex duplicata ratione perpendiculorum & duplicata ratione velocitatum, a que ideo datur. Sit *L* coni sectionis latus rectum. Datur praeterea eiusdem coni sectionis umbilicus *S*. Anguli *RPS* complementum ad duos rectos fiat angulus *RPH;* & dabitur positione linea *PH*, in qua umbilicus alter *H* locatur. Demisso ad *PH* perpendiculo *SK*, erigi intelligatur semiaxis conjugatus *BC*, & erit



$$SPq - 2KPH + PHq = SHq = 4CHq = 4BHq - 4BCq =$$
$$\overline{SP+PH}:quad. - L \times \overline{SP+PH} = SPq + 2SPH + PHq - L \times \overline{SP+PH}.$$

Addantur utrobique $2.KPH - SPq - PHq + L \times \overline{SP+PH}$, & fiet $L \times SP + PH = 2SPH + 2KPH$,seu $SP + PH$ ad PH ut $2.SP + 2.KP$ ad L. Unde datur *PH* tam longitudine quam positione. Nimirum si ea sit corporis in *P* velocitas, ut latus rectum *L* minus fuerit quam $2.SP + 2.KP$, iacebit *PH* ad eandem partem tangentis *PR* cum linea *PS;* ideoque figura erit ellipsis, & ex datis umbilicis *S*, *H*, &, axe principali

$SP + AH$, dabitur. Sin tanta sit corporis velocitas, ut latus rectum *L* aequale fuerit $2.SP + 2.KP$, longitudo *PH* infinita erit; & propterea figura erit parabola axem habens *SH* parallelum lineae *P K*, & inde dabitur. Quod si corpus maiori adhuc cum velocitate de loco suo *P* exeat, capienda erit longitudo *PH* ad alteram partem tangentis; ideoque tangente inter umbilicos pergente, figura erit hyperbola axem habens principalem aequalem differentiae linearum *SP& PH*, & inde dabitur. Nam si corpus in his casibus revolvatur in conica sectione sic inventa, demonstratum est in Prop. XI, XII, & XIII, quod vis centripeta erit ut quadratum distantiae corporis a centro virium *S* reciproce; ideoque linea *PQ* recte exhibetur, quam corpus tali vi describet, de loco dato *P*, cum data velocitate, secundum rectam positione datam *PR* egrediens. *Q.E.F.*

Corol. I. Hinc in omni coni sectione ex dato vertice *D* principali, latere recto *L*, & umbilico *S*, datur umbilicus alter *H* capiendo *DH* ad *DS* ut est latus rectum ad differentiam inter latus rectum & 4*DS*. Nam proportio $SP + PH$ ad *PH* ut $2SP + 2KP$ ad L in casu hujus corollarii, sit $DS + DH$ ad *DH* ut 4*DS* ad *L*, & divisim *DS* ad *DH* ut $4DS - L$ ad *L*.

Corol. 2. Unde si datur corporis velocitas in vertice principali *D*, invenietur orbita expedite, capiendo scilicet latus rectum eius ad duplam distantiam *DS*, in duplicata ratione velocitatis huius datae ad velocitatem corporis in circulo ad distantiam *DS* gyrantis (per Corol.3, Prop. XVI. ;) dein *DH* ad *DS* ut latus rectum ad differentiam inter latus rectum & 4*DS*.

Corol. 3. Hinc etiam si corpus moveatur in sectione quacunque conica, & ex orbe suo impulsu quocunque exturbetur; cognosci potest orbis, in quo postea cursum suum peraget. Nam componendo proprium corporis motum cum motu illo, quem impulsus solus generaret, habebitur motus quocum corpus de dato impulsus loco, secundum rectum positione datam, exibit.

Corot. 4. Et si corpus illud vi aliqua extrinsecus impressa continuo perturbetur, innotescet cursus quam proxime, colligendo mutationes quas vis illa in punctis quibusdam inducit, & ex seriei analogia mutationes continuas in locis intermediis aestimando.

Scholium.

Si corpus *P* vi centripeta ad punctum quodcunque datum *R* tendente moveatur in perimetro datae cuiuscunque sectionis conicae, cujus centrum sit *C;* & requiratur lex vis centripetae: ducatur *CG* radio *RP* parallela, & orbis tangenti *PG* occurrens in *G*; & vis illa (per Corol.I. & Schol., Prop. X. & Corol.3, Prop. VII.) erit ut $\frac{CGcub.}{RPquad.}$.