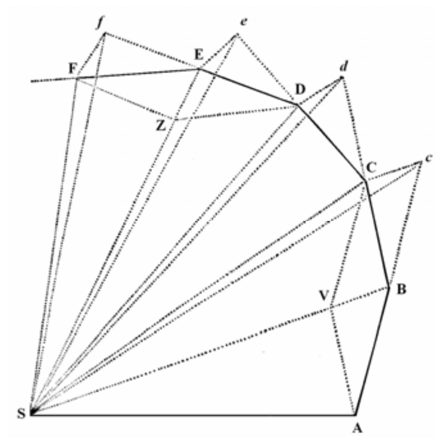
SECTION II.

On the finding of centripetal forces.

PROPOSITION I. THEOREM I.

The areas which bodies will describe driven in circles, with the radii drawn towards a motionless centre of the forces, remain together in immoveable planes, and to be proportional to the times.

The time may be divided into equal parts, and in the first part of the time the body may describe the right line *AB* with the force applied [at *B*]. Likewise in the second part of the time, if nothing may hinder, the body may go on along the right line to *c*, (by Law I.) describing the line *Bc* itself equal to *AB*; thus so that from the radii *AS*, *B*S, *cS* acting towards the centre, the equal areas *ASB*, *BSc* may be completed. Truly, when the body comes to *B*, by a single but large impulse the centripetal force acts, and brings about that the body deflects from the line *Bc* and goes along in the line *BC* ; *cC* is acting parallel to *BS* itself, crossing *B*C in *C;* and with the second part of the time completed, the body (by the corollary to Law I.) may be found at *C*, in the same plane with the triangle *ASB.* Join *SC;* and the triangle *SBC*, on account of the parallels *SB*, *Cc*, will be equal to the triangle *SBc,* and thus also equal to the triangle *SAB.* By a similar argument if the centripetal force acts successively at *C, D, E,* &c. making it so that the body in particular small times will describe the individual lines *CV*, *DE*, *EF*, &c. all these will lie in the same plane; and the triangle *SCD* to the triangle *SBC*, and *SDE* to *SCD* itself, and *SEF* will be equal to *SDE* itself. Therefore in equal times equal areas are described in the motionless plane : and by adding together, the sums of any areas *SADS*, *SAFS* are between themselves, as the times of describing. Now the number may be increased and the width of the triangles diminished indefinitely; and finally the perimeter of these *ADF,* (by the fourth corollary of the third lemma) will be a curved line: and thus the centripetal force, by which the body is drawn perpetually from the tangent of this curve, may act incessantly ; truly any areas described *SADS*, *SAFS* always proportional to the times of description, in this case will be proportional to the same times. *Q. E. D.*



Corol. I. The velocity of the body attracted towards the motionless centre, in intervals without resistance, is reciprocally as the perpendicular sent from that centre to the rectilinear tangent of the orbit. For the velocities at these locations *A, B, C, D, E,* are as the bases of equal [area] triangles *AB, BC, CD, DE, EF,* and these bases are reciprocally as the perpendiculars sent to themselves.

Corol. 2. If the chords *AB, BC* of two arcs described successively by the same body in equal times in non-resisting distances [or spaces] may be completed in the parallelogram *ABCV*; and of this diagonal, as it finally has that position when these arcs are diminished indefinitely, *BV* may be produced both ways, and will pass through the same centre of the forces.

Corol. 3. If the chords of arcs described in equal times in distances without resistance *AB, BC* and *DE, EF* may be completed into the parallelograms *ABCV, DEFZ;* the forces at *B* and *E* are in turn in the ratio of the final diagonals *BV, EZ,* when the arcs themselves are diminished indefinitely. For the motions of the body *BC* and *EF* are composed (by the Law I corollary) from the motions *Bc, BV* and *Ef, EZ*: but yet *BV* and *EZ* are equal to *Cc* and *Ff* themselves, in the demonstration of this proposition they were being generated by the impulses of the centripetal force at *B* and *E*, and thus they are proportional to these impulses.

Corol. 4. The forces by which any bodies in non-resisting intervals are drawn back from rectilinear motion and may be turned into curved orbits are between themselves as these versed sines of the arcs described in equal times which converge to the centre of forces, and they bisect the chords when these arcs are diminished indefinitely. For these versed sines are as the half diagonals, about which we have acted in the third corollary.

Corol.5. And thus the same forces are as the force of gravity, as these versed sines are to the perpendicular versed sines to the horizontal of the arcs of parabolas, which projectiles describe in the same time.

Corol. 6. All remains the same by Corollary V of the laws, when the planes, in which the bodies may be moving, together with the centres of forces, which in themselves have been placed, are not at rest, but may be moving uniformly in a direction.

[Some important features to note in these corollaries are :

1. The distances travelled by the body *AB, BC, CD*, etc., in successive constant time intervals δt are related to the constant velocities in the intervals from the start of each interval by the simple relations :

$$AB = v_A \delta t \,;\, BC = v_B \delta t \,;\, \text{etc.};$$

since the areas of the triangles $\Delta SAB, \Delta SCB,$ etc. described in the equal time intervals are equal, (and note how this has been achieved in diagram by means of the triangles of equal areas ΔScC and $\Delta CcB,$ etc.), then on calling the perpendicular distances from *S* to *AB, BC*, etc., $h_A, h_B, etc.$,

$$\Delta SAB = \Delta SBC = \Delta SCD, \text{etc.}; \text{ or, } h_A AB = h_B BC = h_C CD, \text{ etc.}, \text{ and hence}$$

$$h_A v_A = h_B v_B = h_C v_C, \text{ etc.},$$

and thus the perpendicular distance from the centre of force varies inversely as the velocity of the body in that section of the motion. We would now interpret this result in terms of conservation of angular momentum of the orbiting body.

2.When the parallelgrams *VBcC*, etc. collapse as δt tends to zero, the impelling force acts along the diagonal towards *S*.

3. The distances through which the centripetal force act, when applied, are given by the displacements *BV* or *cC*, *EZ* or *fF*, etc. By Lemma XI of Section I, these displacements tend towards the squares of the versed sines of the arcs *AC* and *DF* as δt tends to zero.

4. These deliberations apply to parabolic arcs with uniform gravity acting uniformly, and also under the circumstances that the whole system is moving with a uniform motion in some direction.

It should be noted that Newton, in the following propositions, has essentially let the number of sides of the polygon go to to infinity, and a geometrical analysis is performed on one of the small segments of the original polygon envelope of the resulting curve.]

PROPOSITION II. THEOREM II.

Any body, that is moving in some curved line described in a plane, and with the radius drawn to some point, either motionless or progressing uniformly in a rectilinear motion, will describe areas about that point proportional to the times, and is urged on by a centripetal force tending towards the same point.

Case I. For any body, because it is moving in a curved line, is turned from rectilinear motion by some force acting on itself (by Law I.). And that force, by which the body is turned from a rectilinear course, and it is known that the minima triangles *SAB, SBC, SCD,* &c. are described equally in equal intervals of time about the fixed point *S*, acts in the location *B* along a line parallel to *cC* itself (by Prop.XL, Book. I. *Elem*. and Law II.) that is, along the line *BS;* and in the place *C* along a line parallel to *dD* itself, that is, along the line *SC,* &c. Therefore the force always acts along lines tending towards that fixed point *S.*

Q. E. D.

Case:2. And, by the fifth corollary of the laws, it is likewise, either the surface is at rest in which the body will describe a curvilinear figure, or it may be moving together with the same body, and with its own point *S* moving uniformly in a direction, with the figure described.

Corol. I. In intervals or in non-resisting mediums, if the areas are not proportional to the times, the forces do not tend to the meeting point of the radii, but thence are deviated as a consequence, or towards the direction in which the motion is made, but only if the description of the areas is accelerated: if it may be retarded ; they are deviated in the opposite way.

Corol. II. Also in mediums with resistance, if the description of the areas is accelerated, the directions of the forces deviate from the running together of the radii towards the direction in which there shall be motion.

Scholium.

A body can be urged by a centripetal force composed from several forces. In this case the understanding of the proposition is, because that force which is composed from all the forces, tend towards the point *S.* Again if some force may act always following a line described perpendicular to a surface ; this will act so that the body is deflected from the plane of its motion: but the magnitude of the surface described neither will be increased or diminished, and therefore in the composition of the forces is required to be ignored.

PROPOSITION III. THEOREM III.

Every body, which with a radius drawn to the centre of some other moving body will describe areas about that centre proportional to the times, is acted on by a force composed from the centripetal force tending towards that other body, and by the force arising from all the acceleration, by which that other body is acted on.

Let *L* be the first body, and *T* the other body : and (by corollary VI of the laws) if by a new force, which shall be equal and contrary to that, by which the other body *T* is driven, may urge each body along parallel lines ; the first body *L* goes on to describe the same areas as before about the other body *T*: but the force *T*, by which the other body was being, now is destroyed by the force equal and contrary to itself ; and therefore (by Law I) that other body *T* itself now left to itself, itself either will be at rest or it will be moving uniformly forwards : and the first body *L* by being pressed on by the difference of the forces, that is, by being urged by the remaining force, goes on to describe areas proportional to the times about the other body *T*. Therefore it tends (by theorem II.) by the difference of the forces to the other body *T* as centre.

Q. E. D.

Corol. 1. Hence if the one body *L,* by a radius drawn to the other *T,* will describe areas proportional to the times ; and concerning the total force (either simple, or from several forces joined together composed according to the second corollary of the laws,) by which the first body *L* is urged, the total accelerative force is subtracted (by the same corollary of the laws), by which the other body is impressed : all the remaining force, by which the first body is impressed, tends towards the other body *T* as centre.

Corol. 2. And, if these areas are approximately proportional to the times, the remaining force will tend approximately to the other body *T.*

Corol. 3. And in turn, if the remaining force tends approximately towards the other body *T,* these areas will be approximately proportional to the times.

Corol. 4. If the body *L* with the radius drawn to the other body *T* will describe areas, which with the times are deduced to be very unequal ; and that other body *T* either is at rest, or is moving uniformly in a direction : the action of the centripetal force tending towards the body *T* either is zero, or it is a mixture and composed from the actions of other very strong forces : and the total force from all, if there are several forces added together, is directed towards another centre (either fixed or moving). The same will be obtained, when the other body is moved by some other motion ; but only if the centripetal force is assumed, which remains after subtracting the total force acting on that other body *T* .

Scholium.

Because the description of the equal areas is an indication of the centre, which that force considers, by which the body is influenced especially, and by which it is drawn back from rectilinear motion, and retained in its orbit ; why may we not take in the following the equal delineation of the areas as the sign of a centre, about which the motion of all circles in free intervals is carried out ?

PROPOSITION IV. THEOREM IV.

The centripetal forces of bodies, which describe different circles with equal motion, tend towards the centres of these circles ; and to be between themselves, so that they are in the same time as the squares of the arcs described and as the radii of the circles.

These forces tend towards the centres of the circles by Prop. II. and Corol.2, Prop. I. and are between themselves as with the smallest arcs in equal times, as without doubt of the versed sines described by Corol.4, Prop. I, that is , as the squares of the arcs themselves applied to the diameters of the circles by Lem. VII. and therefore, since these arcs shall be as the arc described in any equal times, and the diameters shall be as the radii of these ; the forces will be as the squares of any arcs described in the same time applied to the radii of the circles.

Q. E. D.

Corol. I. Since these arcs shall be as the velocities of the bodies, [and inversely as the radii] the centripetal forces shall be composed from the square ratio of the velocities directly, and in the simple inverse ratio of the radii.

Corol. 2. And, since in the periodic times they shall be in the ratio composed from the ratio of the radii directly, and in the ratio of the velocities inversely ; the centripetal forces are in the ratio composed from the ratio of the radii directly, and in the square ratio of the periodic times inversely.

Corol. 3. From which if the periodic times may be equal, and therefore the velocities shall be as the radii; also the centripetal forces shall be as the radii; and vice versa.

Corol. 4. And if both the periodic times and the velocities shall be as the square roots of the radii ; the centripetal forces shall be equal among themselves ; and vice versa.

Corol.5 . If the periodic times shall be as the radii, and therefore the velocities are equal; the centripetal forces shall be inversely as the radii : and vice versa.

Corol. 6. If the periodic times shall be in the three on two power ratio of the radii, and therefore the velocities reciprocally in the square root ratio of the radii ; the centripetal forces shall be reciprocally as the square of the radii, and vice versa.

Corol. 7. And universally, if the time of the period shall be as any power R^n of *R*, and therefore the velocity inversely as the power R^{n-1} of the radius ; the centripetal force will be reciprocally as the power R^{2n-1} of the radius: and vice versa.

Corol. 8. All the same things described concerning times, velocities, and forces, by which similar bodies of any similar shape, and with centres having been put in place in these figures similarly, follow from the preceding demonstration and applied to these cases. But it is required for the description of the equality of areas to be substituted for the equality of motion, and the distances of the bodies from the centres to be taken for the radii.
[This had been shown experimentally by Kepler in his second law of planetary motion.]

Corol. 9. It also follows from the same demonstration ; that the arc, which a body will describe in a given circle by the centripetal force on rotating uniformly in some given time, is the mean proportional between the diameter of the circle, and the descent of the body on falling in the same time caused by the same given force.

[These results are set out in the analytical manner by *Routh* and *Brougham* in their *Analytical View of Sir Isaac Newton's Pricipia*, p. 36 – 37; and also are dealt with conclusively by modern authors.]

Scholium.

The case of corollary six prevails with celestial bodies, (as also our *Wren, Hook*, and *Halley* have deduced separately) and therefore who consider the centripetal force decreasing in the square ratio of the distances from the centre, I have decided to explain further in the following.

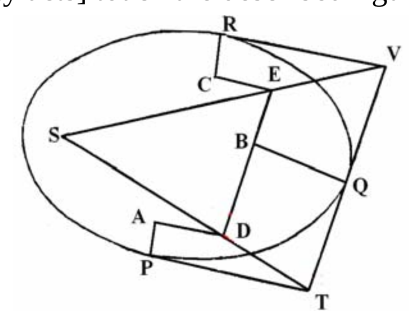
Again from the preceding propositions and from the benefit of the corollaries of this, also the proportion of the centripetal force to any known force is deduced, such as that of gravity. For if a body may be revolving in a circle concentric with the earth by its gravity, this weight is the centripetal force of that [motion]. But by Corol. IX of this proposition, from the descent of the weight and the time of one revolution, and the arc described in some given time is given. And from propositions of this kind *Huygens* in his exemplary treatise *de Horologio Oscillatorio* [Concerning the Oscillatory (*i.e.* Pendulum) Clock] brought together the force of gravity with the centrifugal forces of revolution.

Also the preceding can be demonstrated in this manner. In any circle a polygon is understood to be described of some number of sides. And if the body be required to move with a given velocity along the sides of the polygon, and is to be reflected by the circle, according to the individual angles of this; the force, with which the individual reflections impinge on the circle, will be as the velocity of this: and thus the sum of the forces in a given time will be as that velocity, and the number of reflections jointly : that is (if the kind of polygon may be given) as the distance described in that given time, and increased or decreased in the ratio of the length of the same to the aforementioned radius ; *i.e.* , as the square of this length applied [*i.e.* multiplied by] to the radius: and thus, if a polygon with an infinite number of sides may coincide with the circle, as the square of the arc described in the given time applicable to the radius. This is the centrifugal force, by which the body acts on the circle ; and to this the contrary force is equal, by which the circle continually repels the body towards the centre**.**

PROPOSITION V. PROBLEM I.

With the velocity given in some places, by which a body will describe some given figure with forces commonly tending towards some given centre, to find that centre.

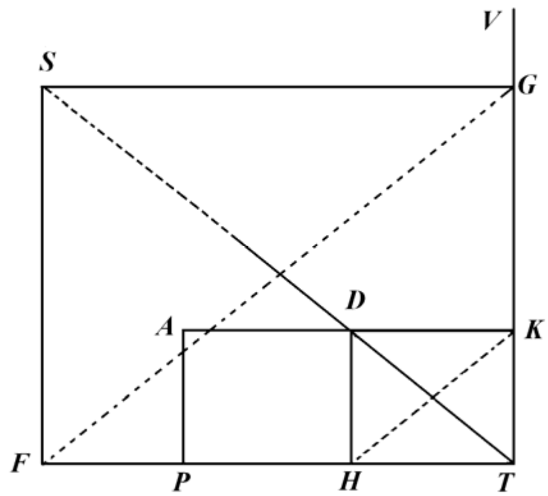
The three lines *PT*, *TQV*, *VR* [along which the velocity acts] touch the described figure in just as many points *P*, Q, *R*, concurring in *T* and *V*. To the tangents the perpendiculars *PA*, *QB*, *RC* are erected reciprocally proportional to the velocities of the body at these points *P, Q, R*, from which they have been raised ; that is, thus so that *PA* to *QB* shall be as the velocity at *Q* to the velocity at *P*, and *QB* to *RC* shall be as the velocity at *R* to the velocity at *Q* . Through the ends of the perpendiculars *A*, *B*, *C* are drawn *AD*, *DBE*, *EC* at right angles concurring in *D* and *E* [parallel to the respective tangents]*:* And *TD* and *VE* are drawn concurrent at the centre sought *S*. For the perpendiculars sent from the centre *S* to the tangents *PT, QT* (by Corol. I, Prop. I.) are inversely as the velocities of the body at the points *P* and *Q*; and thus by the construction [these perpendiculars from *S* to the tangents from *T* and *V* will be] directly as the perpendiculars *AP* and *BQ* , that is: as the perpendiculars are sent from the point *D* to the tangents. From which it is easily deduced that the points *S, D, T* are on a single right line. And by a similar argument the points *S*, *E*, *V* also are on a single right line; and therefore the body is turning about the centre *S* at the concurrence of the lines *TD* and *VE*.



Q. E. D.

[We give here the Le Seur and Janquier proof of Prop. V ; note 206, p. 78 :

The points *S, D*, and *T* do lie on a single straight line. For from the centre *S*, with the perpendiculars *SG, SF* sent to the tangents *TV*, *TF*; and from the point *D*, with the perpendiculars *DK*, *DH*, it is apparent the angles *FSG, HDK* to be equal and contained betwee parallel lines, and on account of the sides *SF, SG, DH*, *DK*, in an analogous manner, the triangles *FGS*, *HKD* are similar ; and thus the angle *SFG, DHK* are equal; and hence there will be :



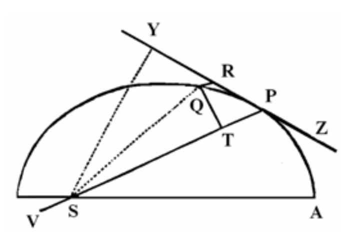
$$TH : TF = HK : FG = DH : SF,$$
&
$$TK : TG = HK : FG = DK : SG.$$

On account of which *TD*, produced, will pass through the centre *S*.]

PROPOSITION VI. THEOREM V.

If a body is revolving in some orbit in a non resisting space about an immobile centre, and some arc just arisen may be described in the minimum time, and the versed sine of the arc may be understood to be drawn, which bisects the chord, and which produced passes through the centre of forces : the centripetal force at the middle of the arc will be directly as the versed sine and inversely with the time squared.

For the versed sine at the given time is as the force (Per Cor.4. Prop. I.), and with an increase in the time in some ratio, on account of the increase of the arc in the same ratio, the versed sine may be increased in that ratio squared (By Cor.2. & 3, Lem. XI.) and thus the versed sine is as the force to the first power and the time to the second power. The squared ratio of the time is rearranged on both sides of the proportionality, and the force becomes directly as the versed sine and inversely as the time squared.



Q. E. D.

The same is easily shown also by Cor.4., Lem. X.

[*i.e.* in an obvious notation, $vSin \sim F_c$; $\Delta t \to \alpha\Delta t$ and the arc $\Delta\vartheta \to \alpha\Delta\vartheta$; the versed sine increases as α^2, and thus $vSin \to F_c\left(\Delta t\right)^2$. See note to Cor.1 below.]

Corol. I. If the body *P* may describe the curved line *APQ* by revolving about the centre *S*; truly the right line *ZPR* may touch the curve at some point *P*, and to the tangent from some other point of the curve *Q* , *QR* is acting parallel to the length *SP*, and *QT* may be sent perpendicular to that distance *SP:* the centripetal force will be reciprocally as the volume $\frac{SP^2 \times QT^2}{QR}$; but only if the magnitude of this volume is always taken as that, which it shall be finally, when the points *P* and *Q* coalesce. For *QR* is equal to the versed sine of twice the arc *QP*, in the middle of which is *P*, and twice the triangle *SQP* or $SP \times QT$ is proportional to the time, in which the double of this arc will be described; and thus can be written for the time to be expressed.

Q. E. D.

[Recall that the versed sine, or the turned sine, is the line segment *DB* ; see the added

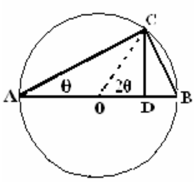


diagram here; given by $vSin = r - r\cos 2\theta = 2r\sin^2\theta$, or for very small arcs, by $vSin = 2r\theta^2$; thus $vSin = \frac{s^2}{2r}$ in this case, and the elemental area of the corresponding sector *OCB* is $r^2\theta = \frac{1}{2}rs$. Now, in these motions equal areas are described in equal times, and thus the elemental [i.e. instantaneous] area is proportional to the elemental time and vice versa : $\Delta t \; \alpha \; r^2\theta = \frac{1}{2}rs$ and thus the centripetal force or acceleration for unit mass, for areas such as found above becomes

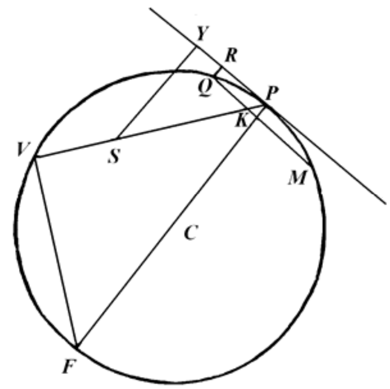
$$F_c \; \alpha \; \frac{vSin}{(\Delta t)^2} \; \alpha \; r\left(\frac{\Delta\theta}{\Delta t}\right)^2 = \omega^2 r\,,$$

the modern form, which is inversely as $\frac{SP^2 \times QT^2}{QR}$ as required. We may also note that the idea of angular velocity was not apparent at the time, it seem that Euler was responsible for this, so that Newton's derivation of the centripetal acceleration goes as far as the deviation from the tangent towards *S*, *QR*; this distance is divided by the area of the associated triangle $SP \times QT$ squared, proportional to the time squared, as one expects for an acceleration.]

Corol. 2. By the same argument the centripetal force is reciprocally as the volume $\frac{SY^2 \times QP^2}{QR}$, but only if the perpendicular *ST* shall be sent from the centre of forces to the tangent *PR* of the orbit. For the rectangles $SY \times QP$ and $SP \times QT$ are equal.

Corol. 3. If the orbit either is a circle, or either touches or cuts a circle concentrically : that is, the angle of contact, or which contains the smallest section with a circle, and having the same radius of curvature at the point *P* [as the circle]*;* and if *PV* shall be a chord of this circle for a body acted on by a centre of forces: the centripetal force [which was previously inversely as $\frac{SY^2 \times QP^2}{QR}$,] shall be inversely as the volume $SY^2 \times PV$.

For PV is as $\frac{QP^2}{QR}$.



[We give here the Le Seur and Janquier proof of Prop. VI, Cor. III ; note 211, p. 81 :

PV shall be as $\frac{QP^2}{QR}$. For let $PQVF$ be the osculating circle, and with the chord QM drawn, as well as the other chord PV drawn through the centre of forces S, the former may be bisected in K, and there will be (by Prop. 35, Book III, *Euclid Elements*),

$QK^2 = VK \times PK$; but with PK evanescent, $QR = PK$,

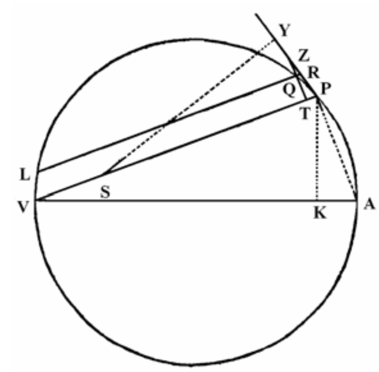
and (by Cor. I, Lemma VII) $QK = QP$, hence $QP^2 = PV \times QR$, and $VP = \frac{QP^2}{QR}$.]

Corol. 4. With the same in place, the centripetal force is directly as the velocity squared , and inversely as that chord [PV]. For the velocity is reciprocally as the perpendicular SY by Corol. I. Prop. I.

Corol. 5. Hence if some curvilinear figure APQ is given, and in that there may be given also a point S, towards which the centripetal force is always directed, the law of the centripetal force can be found, from which some body P drawn back from a straight line always will be retained in the perimeter of that figure, and by revolving describes it. Without doubt either the volume $\frac{SP^2 \times QT^2}{QR}$ or the volume $SY^2 \times PV$ is required to be computed, for this is reciprocally proportional to the force. We will give examples of this matter in the following problems.

PROPOSITION VII. PROBLEM II.

A body may rotate in the circumference of a circle, the law of the attracting centripetal force towards some given point is required.



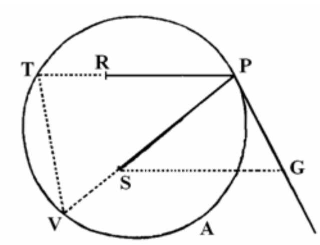
Let $VQPA$ be the circumference of the circle; S the point given, to which the force as it were tends towards its centre ; P the body brought to the circumference ; Q the nearest point, into which it will be moved; and PRZ the tangent to the circle at the previous point. The chord PV

may be drawn through the point *S;* and with the diameter of the circle *VA* drawn, *AP* is joined*;* and to *SP* there may be sent the perpendicular *QT*, which produced may run to meet tangent *PR* in *Z;* and then *LR* may act through the point *Q*, which shall be parallel to *SP* itself, and then crosses to the circle at *L*, and then with the tangent *PZ* at *R.* And on account of the similar triangles *ZQR, ZTP, VPA;* there will be RP^2 (that is $RL \times RQ$) to QT^2 as AV^2 to PV^2. And thus $\frac{RL \times RQ \times PV^2}{AV^2}$ is equal to QT^2. [Note the degenerate cyclic quadrilateral *LQPP*.] This equality may be multiplied by $\frac{SP^2}{QR}$, and with the points *P* and *Q* coalescing there may be written *PV* for *RL*. Thus $\frac{SP^2 \times PV^3}{AV^2}$ becomes equal to $\frac{SP^2 \times QT^2.}{QR}$. Therefore (by Corol.1. & 5. Prop. VI.) the centripetal force is inversely as $\frac{SP^2 \times PV^3}{AV^2}$ as the square of the distance or of the height *SP*, and jointly as the cube of the chord *PV*. *Q. E. D.*

The Same Otherwise.

To the tangent *PR* produced there may be sent the perpendicular *SY:* and on account of the similar triangles *SYP*, *VPA;* there will be *AV* to *PV* as *SP* to *SY* : and thus $\frac{SP \times PV}{AV}$ equals *SY*, and $\frac{SP^2 \times PV^3}{AV^2}$ equals $SY^2 \times PV$. And therefore (by Coral.3. & 5. Prop. VI.) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$, that is, on account of *AV* given, reciprocally as $SP^2 \times PV^3$. *Q. E. D.*

Corol. I. Hence if the given point *S*, towards which the centripetal force always tends, may be located on the circumference of this circle, for example at *V*, the centripetal force will be reciprocally as the fifth power of the height *SP*.

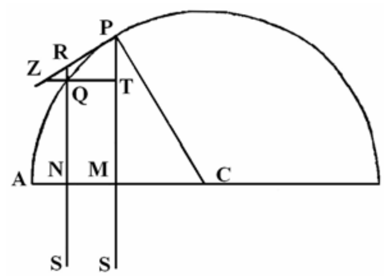


Corol. 2. The force, [proportional to $SP^2 \times PV^3$] by which the body *P* is revolving in a circle *APTV* about the centre of forces *S*, is to the force, by which the same body *P* in the same circle and in the same periodic time can be revolving about some other centre of forces *R*, as $RP^2 \times SP$ to the cube of the right line *SG*, which is a right line drawn from the centre of the first force *S* to the tangent of the orbit *PG*, and which is parallel to the distance of the body from the second centre of forces. For from the construction of this proposition, the first force is to the second force as $RP^2 \times PT^3$ to $SP^2 \times PV^3$ that is, as $SP \times RP^2$ to $\frac{SP^3 \times PV^3}{PT^3}$, or (on account of the similar triangles *PSG*, *TPV* [for : $\frac{SG}{PV} = \frac{SP}{TP} = \frac{PG}{TV}$]) to SG^3.

Corol.3. The force, by which a body *P* is revolving in some orbit about a centre of forces *S* is that force, by which the same body *P* in the same orbit and in the same periodic time can revolve about some other centre of forces *R*, as $SP \times RP^2$; and certainly contained under the distance of the body from the first centre of forces *S* and with the square of the distance of this from the second centre of force *R*, to the cube of the right line *SG*, which is drawn from the centre of the first forces *S* to the tangent of the orbit *PG*, and is parallel to the distance *RP* of the body from the second centre of forces. For the forces in this orbit at some point *P* of this are in a circle of the same curvature.

PROPOSITION VIII. PROBLEM III.

A body may move in the semicircle* PQA : *towards effecting this, a law of the centripetal force is required thus tending towards a remote point* S, *so that all the lines* PS, RS *drawn towards that, will be had as parallel.



From the semicircle with centre *C*, the radius *CA* is drawn, cutting such parallels in *M* and *N*, and *CP* may be joined. On account of the similar triangles *CPM*, *PZT* and *RZQ* there is CP^2 to PM^2 as PR^2 to QT^2, and from the nature of the circle PR^2 is equal to the rectangle $QR \times \overline{RN + QN}$, or if with the points *P* and *Q* merged together to the rectangle $QR \times 2PM$. Therefore CP^2 to PM^2 is as $QR \times 2PM$ to QT^2 and thus $\frac{QT^2}{QR}$ equals $\frac{2PM^3}{CP^2}$, and [the ratio for the inverse of the centripetal force]
$\frac{QT^2 \times SP^2}{QR}$ equals $\frac{2PM^3 \times SP^2}{CP^2}$.
Therefore (by Corol.I.&5, Prop. VI.) the centripetal force is reciprocally as $\frac{2PM^3 \times SP^2}{CP^2}$, that is (with the ratio determined $\frac{2SP^2}{CP^2}$ ignored) reciprocally as PM^3.

Q. E. D.

The same may be deduced easily from the preceding proposition.

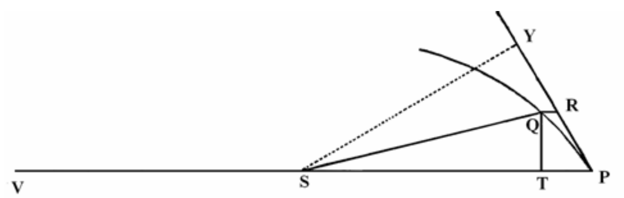
Scholium.

And from an argument not much dissimilar the body may be found to move in an ellipse, or also in a hyperbola or a parabola, by a central force which shall be reciprocally as the cube of the applied ordinate to a centre of forces acting at a great distance.

PROPOSITION IX. PROBLEM IV.

A body may be rotating on the* [equiangular] *spiral PQS with all the radii SP, SQ, &c. cut in a given angle : the law of the centripetal force is required tending towards the centre of the spiral.

A small indefinite angle *PSQ* may be given, and on that account all the given angles will be given by a figure with the appearance *SPRQT* .



[Recall that we need to evaluate $\frac{QT^2 \times SP^2}{QR}$ which varies inversely as the centripetal force.] Therefore the ratio $\frac{QT}{QR}$ is given [*i.e.* constant], and $\frac{QT^2}{QR}$ is as *QT*, that is (on account of the kind of that given figure) as *SP*. Now the angle *PSQ* may be changed in some manner, and the right line *QR* subtending the contact angle *QPR* will be changed (by lemma XI) in the square ratio of *PR* itself or in the square ratio of *QT*. Therefore $\frac{QT^2}{QR}$ will remain the same which it was first, that is as *SP*. Whereby $\frac{QT^2 \times SP^2}{QR}$ is as SP^3 and thus (by Corol.1 & 5, Prop. VI), the centripetal force is reciprocally as the cube of the distance *SP*.

Q. E. D.

The Same Otherwise.

The perpendicular *SY* sent to the tangent, and the chord *PV* of the circle concentrically cutting the spiral are in given ratios to the altitude *SP* [i.e. $\frac{SY}{SP}$ and $\frac{PV}{SP}$ are both constant]; and thus SP^3 is as $SY^2 \times PV$, that is (by Corol.3.& 5, Prop. VI.) reciprocally as the centripetal force.

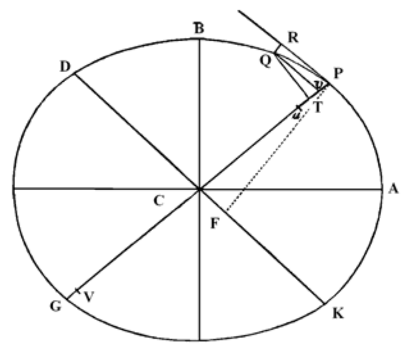
LEMMA XII.

All the parallelograms drawn about any conjugate diameters of a given ellipse or hyperbola are equal to each other.

Shown in works on conic sections.
[Note that conjugate diameters for the ellipse are defined from the chords parallel to parallel tangents on either side of the ellipse; the one parallel chord passing through the centre is the one diameter, while the conjugate diameter bisects all these chords, beginning and ending on the contact points of the parallel tangents.]

PROPOSITION X. PROBLEM V.

A body may be rotating in an ellipse : the law of the force is required tending towards the centre of the ellipse.



With the semi-axes *CA*, *CB* of the ellipse taken ; *GP*, *DK* the other conjugate diameters ; *PF*, *QT* the perpendiculars to the diameters ; *Qv* the applied ordinate to the diameter *G P;* and if the parallelogram *Qv*PR may be completed, there will be (from works on conics);
[Newton defers proving the following theorem, which is not quite trivial : but which follows from a like theorem for the circle, as the ellipse can be regarded as a projected circle, for which some theorems apply with a little modification. It is perhaps the case that theorems used by Newton for the circle were thus extended to the other conic sections where appropriate. Using the diagram given here, this theorem states that $Pv \times vG{:}Qv^2 = PC^2{:}CD^2$, where *PC* and *CD* are semi-conjugate diameters and thus *PR* is parallel to *CD* ; now, in the case of the circle, the corresponding theorem is $Pv \times vG{:}Qv^2 = 1{:}1$, where *QT* and *Cv* are perpendicular to *GP* and $PC^2{=}CD^2$, a well-known result ; but the ratios based on parallel lines are not altered on projection, and the result follows with the others accounted for. This extension to the ellipse in the form given is shown, for example, in Theorem 9, § 136, *Elements of Analytical Geometry*, Gibson & Pinkerton, Macmillan & Co., 1911. A branch of geometry not studied much these days. To return to the translation :]
the rectangle *PvG* [*i.e.* $Pv \times vG$] is to Qv^2 as PC^2 is to CD^2 and (on account of the similar triangles *QvT*, *PCF*) Qv^2 to QT^2 as PC^2 to PF^2 ; and with the ratios taken jointly, the rectangle *PvG* to QT^2 as PC^2 to CD^2 and PC^2 to PF^2 that is,

$$vG \text{ to } \frac{QT^2}{Pv} \text{ as } PC^2 \text{ to } \frac{CD^2 \times PF^2}{PC^2}.$$

$$[i.e. \frac{Pv \times vG}{Qv^2} = \frac{PC^2}{CD^2} \text{ and } \frac{Qv^2}{QT^2} = \frac{PC^2}{PF^2}; \text{ or}$$

$$\therefore \frac{Pv \times vG}{QT^2} = \frac{PC^2}{CD^2} \times \frac{PC^2}{PF^2}, \text{ or}$$

$$\frac{vG}{QT^2 / Pv} = \frac{PC^2}{CD^2 \times PF^2 / PC^2}.]$$

Write QR for Pv, and (by Lemma XII.) $BC \times CA$ for $CD \times PF$, also (with the points P and Q coinciding) $2PC$ for vG, and with the extremes and means multiplied in turn there becomes $\frac{QT^2 \times PC^2}{QR}$ equal to $\frac{2BC^2 \times CA^2}{PC}$. Therefore there is, (by Coral.5, Prop. VI.) the centripetal force varies reciprocally as $\frac{2BC^2 \times CA^2}{PC}$; that is (on account of the given $2BC^2 \times CA^2$) reciprocally as $\frac{1}{PC}$; that is, directly as the distance PC.

Q. E. I.

The same otherwise.

In the right line PG from the one side of the point T there may be taken u so that Tu shall be equal to Tv itself*;* then take uV, which shall be to vG as DC^2 is to PC^2. And since from the theory of conics there is Qv^2 to $Pv \times vG$ as DC^2 to PC^2 , Qv^2 is equal to $PV \times uV$. With the rectangle uPv added to both sides, and the square of the chord of the arc PQ will be produced equal to the rectangle VPv *;* and thus the circle, which touches the conic section at P and passed through the point Q, will also pass through the point V. The points P and Q run together, and the ratio uV to vG, which is the same as the ratio DC^2 to PC^2, becomes the ratio PV to PG or PV to $2PC$; and thus PV will be equal to $\frac{2DC^2}{PC}$. Therefore the force, by which the body P is revolving in an ellipse, will be reciprocally as $\frac{2DC^2}{PC}$ into PF^2 (by Carol.3, Prop. VI) that is (on account of $2DC^2$ into PF^2 given) directly as PC.

Q. E. I.

Scholium

Corol. I. Therefore the force is as the distance of the body from the centre of the ellipse : and in turn, if the force shall be as the distance, the body will move in an ellipse having the centre at the centre of forces, or perhaps in a circle, into which certainly the ellipse can be transported.

Corol.2. And the periodic times of the revolutions made will be equal in all ellipses around the same centre. For these times are equal in similar ellipses (by Corol. 3. and 8. Prop. IV.) but in ellipses having a common major axis they are accordingly in turn as the whole total elliptic areas, and inversely likewise of the particular areas described ; that is, directly as the minor axes, and the velocities of the bodies inversely at the principal vertices ; that is, as the minor axes directly, and inversely as the ordinates at the same point of the common axis ; and therefore (on account of the equality of the direct and inverse ratios) in the ratio of equality.

Scholium.

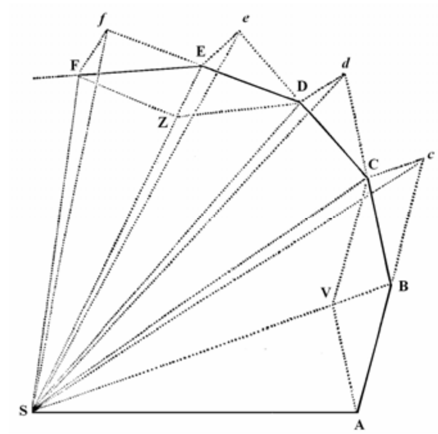
If with the centre of the ellipse departing to infinity it may change into a parabola, the body will be moving in this parabola ; and a constant force now emerges at an infinite distance from the centre. This is *Galileo's* theorem. And if with a parabolic section of a cone (with the inclination of the plane to the section of the cone changed) may be changed into a hyperbola, the body will be moved in the perimeter of this with the centripetal force turned into a centrifugal force. And just as in a circle or ellipse, if the forces tending towards the centre of the figure placed on the abscissa [*i.e.* the *x* or ordinate axis]; these forces by augmenting or diminishing the ordinates in some given ratio, either by changing the angle of inclination of the ordinates to the abscissae, always may be augmented or diminished in some ratio of the distances from the centre, but only if the periodic times remain constant; thus also in general figures, if the ordinates be augmented or diminished in some given ratio, or the angle of the ordinates may be changed in some manner, with the periodic times remaining ; the forces tending towards some centre placed on the abscissa tending to particular ordinates may be augmented or diminished in the ratio of the distances from the centre.

SECTIO II.

De inventione virum centripetarum.

PROPOSITIO I. THEOREMA I.

Areas, quas corpora in gyros acta radiis ad immobile centrum virium ductis describunt, & in planis immobilibus consistere,& esse temporibus proportionales.



Dividatur tempus in partes aequales, & prima temporis parte describat corpus vi insita rectam *AB.* Idem secunda temporis parte, si nil impediret, recta pergeret ad *c*, (per leg. I.) describens lineam *Bc* aequalem ipsi *AB;* adeo ut radiis *AS*, *B*S, *cS* ad centrum actis, confectae forent aequales areae *ASB, BSc.* Verum ubi corpus venit ad *B*, agat vis centripeta impulsu unico sed magno, efficiatque ut corpus de recta *Bc* declinet & pergat in recta *BC.* Ipsi *BS* parallela agatur *cC*, occurrens *B*C in *C;* & completa secunda temporis parte, corpus (per legum corol.I.) reperietur in *C*, in eodem plano cum triangulo *ASB.* Iunge *SC;* & triangulum *SBC,* ob parallelas *SB, Cc,* aequale erit triangulo *SBc,* atque ideo etiam triangulo *SAB.* Simili argumento si vis centripeta successive agat in *C, D, E,* &c. faciens ut corpus singulis temporis particulis singulas describat rectas *CV*, *DE, E F,* &c. iacebunt hae omnes in eodem plano; & triangulum *SCD* triangulo *SBC,* & *SDE* ipsi *SCD,* & *SEF* ipsi *SDE* aequale erit. Aequalibus igitur temporibus aequales areae in plano immoto describuntur: & componendo, sunt arearum summae quaevis *SADS*, *SAF S* inter se, ut sunt tempora descriptionum. Augeatur jam numerus & minuatur latitudo triangulorum in infinitum; & eorum ultima perimeter *ADF*, (per corollarium quartum lemmatis tertii) erit linea curva: ideoque vis centripeta, qua corpus a tangente huius curvae perpetuo retrahitur, aget indesinenter; areae vera quaevis descriptae *SADS*, *SAFS* temporibus descriptionum semper proportionales, erunt iisdem temporibus in hoc casu proportionales. *Q. E. D.*

Corol. I. Velocitas corporis in centrum immobile attracti est in spatiis non resistentibus reciproce ut perpendiculum a centro illo in orbis tangentem rectilineam demissum. Est enim velocitas in locis illis *A*, *B, C, D, E,* ut sunt bases aequalium triangulorum *AB*, *BC, CD, DE, EF;* & hae bases sunt reciproce ut perpendicula in ipsas demissa.

Corol. 2. Si arcuum duorum aequalibus temporibus in spatiis non resistentibus ab eodem corpore successive descriptorum chordae *AB*, *BC* compleantur in parallelogrammum *ABCV,* & huius diagonalis *BV* in ea positione quam ultimo habet ubi arcus illi in infinitum diminuuntur, producatur utrinque; transibit eadem per centrum virium.

Corol. 3. Si arcuum aequalibus temporibus in spatiis non resistentibus descriptorum chordae A *B, B C* ac *DE, EF* compleantur in parallelogramma *ABCV, DEFZ;* vires in *B* & *E* sunt ad invicem in ultima ratione diagonalium *BV, EZ,* ubi arcus isti in infinitum diminuuntur. Nam corporis motus *BC* & *EF* componuntur (per legum corol. I.) ex motibus *Bc, BV* & *Ef, EZ:* atqui *BV* & *E Z,* ipsis *Cc* & *Ff* aequales, in demonstratione propositionis huius generabantur ab impulsibus vis centripetae in *B* & *E,* ideoque
sunt his impulsibus proportionales.

Corol. 4. Vires quibus corpora quaelibet in spatiis non resistentibus a motibus rectilineis retrahuntur ac detorquentur in orbes curvos sunt inter se ut arcuum aequalibus temporibus descriptorum sagittae illae quae convergunt ad centrum virium, & chordas bisecant ubi arcus illi in infinitum diminuuntur. Nam hae sagittae sunt semisses diagonalium, de quibus egimus in corollario tertio.

Corol.5. Ideoque vires eaedem sunt ad vim gravitatis, ut hae sagittae ad sagittas horizonti perpendiculares arcuum parabolicorum, quos projectilia eadem tempore describunt.

Corol. 6. Eadem omnia obtinent per legum corol.v. ubi plana, in quibus corpora moventur, una cum centris virium, quae in ipsis sita sunt, non quiescunt , sed moventur uniformiter in directum.

PROPOSITIO II. THEOREMA II.

Corpus omne, quod movetur in linea aliqua curva in plano descripta, & radio ducto ad punctum vel immobile, vel motu rectilineo uniformiter progrediens, describit areas circa punctum illud temporibus proportionales, urgetur a vi centripeta tendente ad idem punctum.

Cas I. Nam corpus omne, quod movetur in linea curva, detorquetur de cursu rectilineo per vim aliquam in ipsum agentem (per leg. I.) Et vis illa, qua corpus de cursu rectilineo detorquetur, & cogitur triangula quam minima *SAB, SBC, SCD,* &c. circa punctum immobile *S* temporibus aequalibus aequalia describere, agit in loco *B* secundum lineam parallelam ipsi *cC* (per prop. XL.lib.I. elem. & leg.II.) hoc est, secundum lineam *BS;* & in loco *C* secundum lineam ipsi *dD* parallelam, hoc est, secundum lineam *SC,* &c. Agit ergo semper secundum lineas tendentes ad punctum illud immobile *S. Q. E. D.*

Cas.:2. Et, per legum corollarium quintum, perinde est, sive quiestar superficies, in qua corpus describit figuram curvilineam, sive moveatur eadem una cum corpore, figura descripta, & puncto suo *S* uniformiter in directum.

Corol. I. In spatiis vel mediis non resistentibus, si areae non sunt temporibus proportionales, vires non tendunt ad concursum radiorum; sed inde declinant in consequentia, seu versus plagam in quam fit motus, si modo arearum descriptio acceleratur: sin retardatur; declinant in antecedentia.

Corol. II. In mediis etiam resistentibus, si arearum descriptio acceleratur, virium directiones declinant a concurfsu radiorum versus plagam, in quam sit motus.

Scholium.

Urgeri potest corpus a vi centripeta composita ex pluribus viribus. In hoc casu sensus propositionis est, quod vis illa quae ex omnibus componitur, tendit ad punctum *S.* Porro si vis aliqua agat perpetuo secundum lineam superficiei descriptae perpendicularem; haec faciet ut corpus deflectatur a plano sui motus: sed quantitatem superficiei descriptae nec augebit nec minuet, & propterea in compositione virium negligenda est.

PROPOSITIO III. THEOREMA III.

Corpus omne, quod radio ad centrum corporis alterius utcunque moti ducto describit areas circa centrum illud temporibus proportionales, urgetur vi composita ex vi centripeta tendente ad corpus illud alterum, & ex vi omni acceleratrice qua corpus illud alterum urgetur.

Sit corpus primum *L*, & corpus alterum *T:* & (per legum corol. vi.) si vi nova, quae aequalis & contraria sit illi, qua corpus alterum *T* urgetur, urgeatur corpus utrumque secundum lineas parallelas ; perget corpus primum *L* describere circa corpus alterum *T* areas easdem ac prius: vis autem, qua corpus alterum *T* urgebatur, iam destructur per vim sibi aequalem & contrariam ; & propterea (per leg. I.) corpus illud alterum *T* sibimet ipsi iam relictum vel quiescet, vel movebitur uniformiter in directum: & corpus primum *L* urgente differentia virium. id est, urgente vi reliqua perget areas temporibus proportionales circa corpus alterum *T* describere. Tendit igitur (per theor. II.) differentia virium ad corpus illud alterum *T* ut centrum. *Q. E. D.*

Corol. 1. Hinc si corpus unum *L* radio ad alterum *T* ducto describit areas temporibus proportionales; atque de vi tota (sive simplici, sive ex viribus pluribus iuxta legum corollarium secundum composita,) qua corpus prius *L* urgetur, subducatur (per idem legum corollarium) vis tota acceleratrix, qua corpus alterum urgetur: vis omnis reliqua, qua corpus prius urgetur, tendet ad corpus alterum *T* ut centrum.

Corol. 2. Et, si areae illae sunt temporibus quamproxime proportionales, vis reliqua tendet ad corpus alterum *T* quamproxime.

Corol. 3. Et vice versa, si vis reliqua tendit quamproxime ad corpus alterum *T*, erunt areae illae temporibus quamproxime proportionales.

Corol. 4. Si corpus *L* radio ad alterum corpus *T* ducto describit areas, quae cum temporibus collatae sunt valde inaequales; & corpus ilIud alterum *T* vel quiescit, vel movetur uniformiter in directum: actio vis centripetae ad corpus illud alterum *T* tendentis vel nulla est, vel miscetur & componitur cum actionibus admodum potentibus aliarum virium: visque tota ex omnibus, si plures sunt vires, composita ad aliud (sive immobile sive mobile) centrum dirigitur. Idem obtinet, ubi corpus alterum motu quocunque movetur; si modo vis centripeta sumatur, quae restat post subductionem vis totius in corpus iliud alterum *T* agentis.

Scholium.

Quoniam aequabilis arearum descriptio index est centri, quod vis illa respicit, qua corpus maxime afficitur, quaque retrahitur a motu rectilineo, & in orbita sua retinetur; quidni usurpemus in sequentibus aequabilem arearum descriptionem ut indicem centri, circum quod motus omnis circularis in spatiis liberis peragitur?

PROPOSITIO IV. THEOREMA IV.

Corporum, quae diversos circulos aequabili motu describunt, vires centripetas ad centra eorundem circulorum tendere; & esse inter se, ut sunt arcuum simul descriptorum quadrata applicata ad circulorum radios.

Tendunt hae vires ad centra circulorum per prop. II. & corol. 2. prop. I. & sunt inter se ut arcuum aequalibus temporibus quam minimis descriptorum sinus verfi per corol. 4. prop. I. hoc est, ut quadrata arcuum eorundem ad diametros circulorum applicata per lem.vii. & propterea, cum hi arcus sint ut arcus temporibus quibusvis aequalibus descripti, & diametri sint ut eorum radii; vires erunt ut arcuum quorumvis simul descriptorum quadrata applicata ad radios circulorum. *Q. E. D.*

Corol. I. Cum arcus illi sint ut velocitates corporum, vires centripetae erunt in ratione composita ex duplicata ratione velocitatum directe, & ratione simplici radiorum inverse.

Corol. 2. Et, cum tempora periodica sint in ratione composita ex ratione radiorum directe, & ratione velocitatum inverse; vires centripetae sunt in ratione composita ex ratione radiorum directae, & ratione duplicata temporum periodicorum inverse.

Corol. 3. Unde si tempora periodica aequentur, & propterea velocitates sint ut radii; erunt etiam vires centripetae ut radii: & contra.

Corol. 4. Si & tempora periodica, & velocitates sint in ratione subduplicata radiorum; aequales erunt vires centripetae inter se: & contra.

Corol.5 . Si tempora periodica sint ut radii, & propterea velocitates aequates; vires centripetae erunt reciproce ut radii: & contra.

Corol. 6. Si tempora periodica sint in ratione sesquiplicata radiorum, & propterea velocitates reciproce in radiorum ratione subduplicata; vires centripetae erunt reciproce ut quadrata radiorum: & contra.

Corol. 7. Et universaliter, si tempus periodicum sit ut radii R potestas quaelibet R^n , & propterea velocitas reciproce ut radii potestas R^{n-1}*;* erit vis centripeta reciproce ut radii potestas R^{2n-1}: & contra.

Corol. 8. Eadem omnia de temporibus, velocitatibus, & viribus, quibus corpora similes figurarum quarumcunque similium, centraque in figuris illis similiter posita habentium, partes describunt, consequuntur ex demonstratione praecedentium ad hosce casus applicata. Applicatur autem substituendo aequabilem arearum descriptionem pro aequabili motu, & distantias corporum a centris pro radiis usurpando.

Corol. 9. Ex eadem demonstratione consequitur etiam; quod arcus, quem corpus in circulo data vi centripeta uniformiter revolvendo tempore quovis describit, medius est proportionalis inter diametrum circuli, & descensum corporis eadem data vi eodemque tempore cadendo confectum.

Scholium.

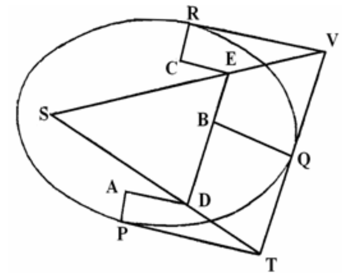
Casus corollarii sexti obtinet in corporibus coelestibus, (ut seorsum collegerunt etiam nostrates *Wrennus, Hookius* & *Hallaeus*) & propterea quae spectant ad vim centripetam decrescentem in duplicata ratione distantiarum a centris, decrevi fusius in sequentibus exponere.

Porro praecedentis propositionis & corollariorum eius beneficio, colligitur etiam proportio vis centripetae ad vim quamlibet notam, qualis est ea gravitatis. Nam si corpus in circulo terrre concentrico vi gravitatis suae revolvatur, haec gravitas est ipsius vis centripeta. Datur autem ex descensu gravium & tempus revolutionis unius, & arcus dato quovis tempore descriptus, per huius corol. ix. Et huiusmodi propositionibus *Hugenius* in eximio suo tractatu *de Horologio Oscillatorio* vim gravitatis cum revolventium viribus centrifugis contulit.

Demonstrari etiam possunt praecedentia in hunc modum. In circulo quovis describi intelligatur polygonum laterum quotcunque. Et si corpus in polygoni lateribus data cum velocitate movendo ad eius angulos singulos a circulo reflectatur; vis, qua singulis reflexionibus impingit in circulum, erit ut eius velocitas: ideoque summa virium in dato tempore erit ut velocitas illa, & numerus reflexionum conjunctim: hoc est (si polygonum detur specie) ut longitudo dato illo tempore descripta, & aucta vel diminuta in ratione longitudinis eiusdem ad circuli praedicti radium; id est, ut quadratum longitudinis illius applicatum ad radium: ideoque, si polygonum lateribus infinite diminutis coincidat cum circulo, ut quadratum arcus dato tempore descripti applicatum ad radium. Haec est vis centrifuga, qua corpus urget circulum; & huic aequalis est vis contraria, qua circulus continuo repellit corpus centrum versus.

PROPOSITIO V. PROBLEMA I.

Data quibuscunque in locis velocitate, qua corpus figuram datam viribus ad commune aliquod centrum tendentibus describit, centrum illud invenire.



Figuram descriptam tangant rectae tres *PT, TQV, VR* in punctis totidem *P*, Q, *R*, concurrentes in *T* & *V*. Ad tangentes erigantur perpendicula *PA, QB, RC* velocitatibus corporis in punctis illis *P, Q, R,* a quibus eriguntur, reciproce proportionalia; id est, ita ut sit *PA* ad *QB* ut velocitas in *Q* ad velocitatem in *P*, & *QB* ad *RC* ut velocitas in *R* ad velocitatem in *Q* . Per perpendiculorum terminos *A, B, C* ad angulos rectos ducantur *AD, DBE, E C* concurrentes in *D* & *E:* Et actae *TD, VE* concurrent in centro quaesito *S.*

Nam perpendicula a centro *S* in tangentes *PT; QT* demissa (per corol.I. prop.I.) sunt reciproce ut velocitates corporis in punctis *P* & *Q*; ideoque per constructionem ut perpendicula *AP*, *BQ* directe, id est: ut perpendicula a puncto *D* in tangentes demissa. Unde facile colligitur quod puncta *S, D, T* sunt in una recta. Et simili argumento puncta *S, E, V* sunt etiam in una recta; & propterea centrum *S* in concursu rectarum *TD, VE* versatur. *Q. E. D.*

PROPOSITIO VI. THEOREMA V.

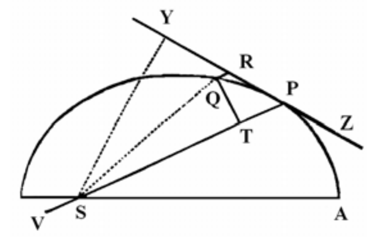
Si corpus in spatio non resistente circa centrum immobile in orbe quocunque revolvatur, & arcum quemvis iamiam nascentem tempore quam minimo describat, & sagitta arcus duci intelligatur, quae chordam bisecet, & producta transeat per centrum virium: erit vis centripeta in medio arcus, ut sagitta directe & tempus his inverse.

Nam sagitta dato tempore est ut vis (per corol. 4. prop. I.) & augendo tempus in ratione quavis, ob auctum arcum in eadem ratione sagitta augetur in ratione illa duplicata (per corol. 2. & 3, lem. XI.) ideoque est ut vis semel & tempus bis. Subducatur duplicata ratio temporis utrinque, & fiet vis ut sagitta directe & tempus bis inverse. *Q. E. D.*

Idem facile demonstratur etiam per Corol.4.Lem.X.

Corol. I. Si corpus *P* revolvendo circa centrum *S* describat lineam curvam *APQ* ; tangat vera recta *ZPR* curvam illam in puncto quovis *P,* & ad tangentem ab alio quovis curvae puncto *Q* agatur *QR* distantiae *SP* parallela, ac demittatur *QT* perpendicularis ad distantiam illam *SP:* vis centripeta erit reciproce ut solidam $\frac{SPquad.\times QTquad.}{QR}$; si modo solidi illus ea semper sumatur quantitas, quae ultimo sit, ubi coeunt puncta *P* & *Q* Nam *QR* aequalis est sagittae dupli arcus *QP*, in cuius medio est *P*, & duplum trianguli *SQP* sive $SP\times QT$, tempori, quo arcus iste duplus describitur, proportionate est; ideoque pro temporis exponente scribi potest.

Corol. 2. Eodem argumento vis centripeta est reciproce ut solidam $\frac{SYq \times QPq}{QR}$ si modo *ST* perpendiculum sit a centro virium in orbis tangentem *PR* demissum. Nam rectangula $SY \times QP$ & $SP \times QT$ aequantur.



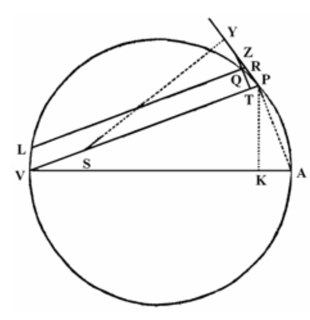
Corol. 3. Si orbis vel circulus est, vel circulum concentrice tangit, aut concentrice secat, id est, angulum contactus aut sectionis cum circulo quam minimum continet, eandem habens curvaturam eundemque radium curvaturae ad punctum *P;* & si *PV* chorda sit circuli huius a corpore per centrum virium acta: erit vis centripeta reciproce ut solidum $SYq \times PV$. Nam *PV* est $\frac{QPq}{QR}$.

Corol. 4. Iisdem positis, est vis centripeta ut velocitas bis directe, & chorda illa inverse. Nam velocitas est reciproce ut perpendiculum *SY* per corol. I. prop. I.

Corol. 5. Hinc si detur figura quaevis curviIinea *APQ*, & in ea detur etiam punctum *S*, ad quod vis centripeta perpetuo dirigitur, inveniri potest lex vis centripetae, qua corpus quodvis *P* a cursu rectilineo perpetuo retractum in figurae illius perimetro detinebitur, eamque revolvendo describet. Nimirum computandum est vel solidum $\frac{SPq \times QYq}{QR}$ vel solidum $SYq \times PV$ huic vi reciproce proportionale. Eius rei dabimus exempla in problematis sequentibus.

PROPOSITIO VII. PROBLEMA II.

Gyretur corpus in circumferentia circuli, requiritur lex centripetae tendentis ad punctum quodcunque datum.



Esto circuli circumferentia *VQPA;* punctum datum, ad quod vis ceu ad centrum suum tendit, *S;* corpus in circumferentia latum *P;* locus proximus, in quem movebitur *Q;* & circuli tangens ad cum priorem *PRZ.* Per punctum *S* ducatur chorda *PV;* & acta circuli diametro *VA*, iungatur *AP;* & ad *SP* demittatur perpendiculum *QT*, quod productum occurrat tangenti *PR* in *Z;* ac denique per punctum *Q* agatur *LR*, quae ipsi *SP* parallella fit, & occurrat tum circulo in *L*, tum tangenti *PZ* in *R*. Et ob similia triangula *ZQR, ZTP, VPA;* erit *RP quad.* hoc est *QRL* ad *QT quad.* ut *AV quad.* ad *PV quad.*. Ideoque $\frac{QRL \times PVquad.}{AVquad.}$ aequatur *QTquad.*

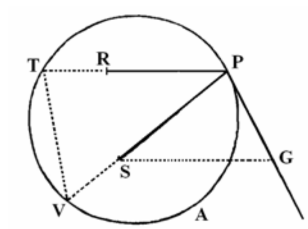
Ducantur haec aequalia in $\frac{SP\ quad.}{QR}$, & punctis *P* & *Q* coeuntibus scribatur *PV* pro *RL*.

Sic fiet $\frac{SP\ quad.\times PV\ cub.}{AV\ quad.}$ aequale $\frac{SP\ quad.\times QT\ quad.}{QR}$. Ergo (per corol. 1. & 5. prop. VI.) vis centripeta est reciproce ut $\frac{SP\ q\times PV\ cub.}{AV\ quad.}$ reciproce ut quadratum distantiae seu altitudinis *SP* & cubus chordae *PV* conjunctim. *Q. E. D.*

Idem aliter.

Ad tangentem *PR* productam demittatur perpendiculum *SY:* & ob similia triangula *SYP, VPA;* erit *AV* ad *PV* ut *SP* ad *SY* : ideoque $\frac{SP\times PV}{AV}$ aequale *SY*, & $\frac{SP\ quad.\times PV\ cub.}{AV\ quad.}$ aequale $SY\ quad.\times PV$. Et propterea (per. coral. 3. & 5. prop. VI.) vis centripeta est reciproce ut $\frac{SP\ q\times PV\ cub.}{AV\ q}$, hoc est, ob datam *AV* reciproce ut $SP\,q\times PV\ cub.$ *Q. E. D.*

Corol. I. Hinc si punctum datum *S*, ad quod vis centripeta semper tendit, locetur in circumferentia huius circuli, puta ad *V* erit vis centripeta reciproce ut quadrato-cubus altitudinis *SP*.



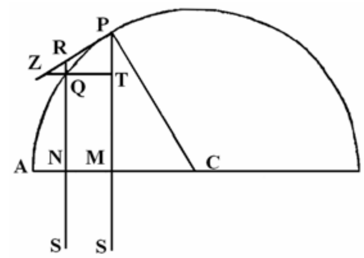
Corol. 2. Vis, qua corpus *P* in circulo *APTV* circum virium centrum *S* revolvitur, est ad vim, qua corpus idem *P* in eodem circulo & eodem tempore periodico circum aliud quodvis virium centrum *R* revolvi potest, ut $RP\,quad.\times SP$ ad cubum rectae *SG*, quae a primo virium centro *S* ad orbis tangentem *PG* ducitur, & distantiae corporis a secundo virium centro parallela est. Nam per constructionem huius propositionis vis prior est ad vim posteriorem ut $RPq\times PTcub.$ ad $SPq\times PVcub.$, id est, ut $SP\times RPq$ ad $\frac{SPcub.\times PVcub.}{PTcub.}$ sive (ob similia triangula *PSG, TPV*) ad *SG cub.*

Corol.3. Vis, qua corpus *P* in orbe quocunque circum virium centrum *S* revolvitur, est ad vim, qua corpus idem *P* in eodem orbe eodemque tempore periodico circum aliud quodvis virium centrum *R* revolvi potest, ut $SP\times RPq$, contentum utique sub distantia corporis a primo virium centro *S* & quadrato distantiae eius a secundo virium centro *R*, ad cubum rectae *SG*, quae a primo virium centro *S* ad orbis tangentem *PG* ducitur, & corporis a secundo virium centro distantiae *RP* parallela est. Nam vires in hoc orbe ad eius punctum quodvis *P* eaedem sunt ac in circulo eiusdem curvaturae.

PROPOSITIO VIII. PROBLEMA III.

Moveatur corpus in femicirculo* PQA : *ad hunc effectum requiritur lex vis centripetae tendentis ad punctum adeo longinquum* S, *ut lineae omnes* PS, RS *ad id ductae, pro parallelis haberi possint.

A semicirculi centro C agatur semidiameter *CA* parallelas istas perpendiculariter secans in *M* & *N*, & iungatur *CP*. Ob similia triangula *CPM, PZT & RZQ* est *CPq* ad *PMq* ut *PRq* ad *QTq*, & ex natura circuli *PRq* aequale est rectangulo $QR\times\overline{RN+QN}$, sive coeuntibus punctis *P* & *Q* rectangulo $QR\times 2PM$. Ergo est *CPq* ad *PM quad.* ut $QR\times 2PM$ ad *QT quad.* ideoque $\frac{QT\,quad.}{QR}$ aequae $\frac{2PM\,cub.}{CP\,quad.}$, & $\frac{QT\,quad.\times SP\,quad.}{QR}$ aequale $\frac{2PM\,cub.\times SP\,quad.}{CP\,quad.}$



Est ergo (per corol. I. & 5. prop. VI.) vis centripeta reciproce ut $\frac{2PM\,cub.\times SP\,quad.}{CP\,quad.}$,

hoc est (neglecta ratione determinata $\frac{2SP\,quad.}{CP\,quad.}$) reciproce ut *PMcub. Q. E. D.*

Idem facile colligitur etiam ex propositione praecedente.

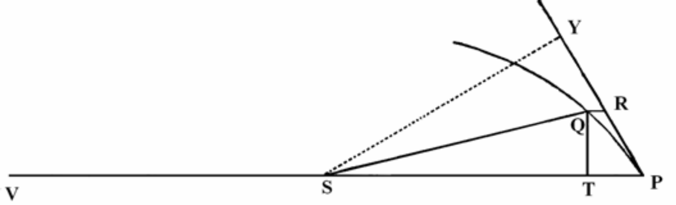
Scholium.

Et argumento haud multum dissimili corpus invenietur moveri in ellipsi, vel etiam in hyperbola vel parabolae, vi centripeta, quae sit reciproce ut cubus ordinatim applicatae ad centrum virium maxime longinquum tendentis.

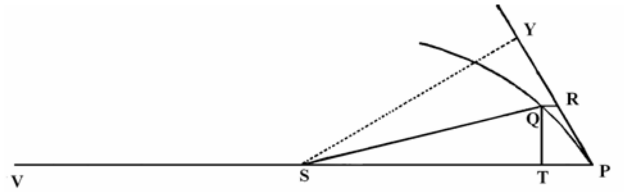
PROPOSITIO IX. PROBLEMA IV.

Gyretur corpus in spirali* PQS *secante radios omnes* SP, SQ, *&c. in angulo dato: requiritur lex vis centripeae tendentis, ad centrum spiralis.

Detur angulus indefinite parvus *PSQ*, & ob datos omnes angulas dabitur specie figura



SPRQT. Ergo datur ratio $\frac{QT}{QR}$ estque $\frac{QT\,quod.}{QR}$ ut *QT*, hoc est (ob datam specie figuram



illam) ut *SP*. Mutetur iam utcunque angulus *PSQ*, & recta *QR* angulum contactus *fLPR* subtendens mutabitur (per lemma XI.) in duplicata ratione ipsius *PR* vel *QT*. Ergo manebit $\frac{QT\,quod.}{QR}$ eadem quae prius, hoc est ut *SP*. Quare $\frac{QTq \times SPq}{QR}$ est ut *SP cub.* ideoque (per corol. I. & 5. prop. VI) vis centripeta est reciproce ut cubus distantiae *SP*. *Q. E. D.*

Idem aliter.

Perpendiculum *SY* in tangentem demissum, & circuli spiralem concentrice secantis chorda *PV* sunt ad altitudinem *SP* in datis rationibus; ideoque *SP cub.* est ut $STq \times PV$, hoc est (per corol. 3. & 5. prop.VI.) reciproce ut vis centripeta.

LEMMA XII.

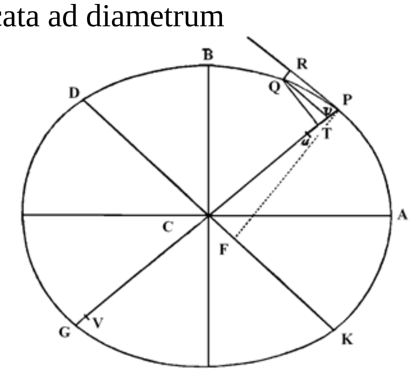
Parallelogramma omnia circa datae ellipseos vel hyperbolae diametros quasvis coniugatas descripta esse inter se aequalia.

Conftat ex conicis.

PROPOSITIO X. PROBLEMA V.

Gyretur corpus in ellipsi: requiritur lex vis centripetae tendentis ad centrum ellipseos.

Sumto *CA, CB* semiaxes ellipseos; *GP, DK* diametri aliae conjugatae; *P F*, *QT* perpendicula ad diametros; *Qv* ordinatim applicata ad diametrum *G P;* & si compleatur parallelogrammum *Qv*PR, erit (ex conicis) rectangulum *PvG* ad *Qv quad*. ut *PC quad.* ad *CD quad.* & (ob similia triangula *QvT*, *PCF*) *Qv quad.* ut *PC quad.* ad *PF quad.* & coniunctis rationibus, rectangulum *PvG* ad *QT quad.* ut *PC quad.* ad *CD quad.* & *PC quad.* ad *PF quad.* id est, vG ad $\frac{QTquad.}{Pv}$ ut $PCquad.$ ad $\frac{CDq \times PFq}{PCq}$. Scribe *QR*



pro *Pv*, & (per lemma XII.) $BC \times CA$ pro $CD \times PF$, nec non (punctis *P* & *Q* coeuntibus) 2*PC* pro *vG*, & ductis extremis & mediis in mutuo fiet $\frac{QT\,quad. \times PCq}{QR}$ aequale $\frac{2BCq \times CAq}{PC}$. Est ergo

(per coral. 5. prop.VI.) vis centripeta reciproce ut $\frac{2BCq \times CAq}{PC}$ id est (ob datum $2BCq \times CAq$) reciproce ut $\frac{1}{PC}$; hoc est, directe ut distantia *PC. Q. E. I.*

Idem aliter.

In recta *PG* ab altera parte puncti *T* sumatur punctum *u* ut *Tu* sit equalis ipsi *Tv;* deinde cape *uV*, quae sit ad *vG* ut est *DC quad.* ad *PC quad.* Et quoniam ex conicis est *Qv quad*. ad *PvG* ut *DC quad.* ad *PC quad.* erit *Qv quad.* aequale $PV \times uV$. Adde rectangulum *uPv* utrinque, & prodibit quadratum chordae arcus *PQ* equale rectangulo *VPv ;* ideoque circulus, qui tangit sectionem conicam in *P* & transit per punctum *Q*, transibit etiam per punctum *V*. Coeant puncta *P* & *Q*, & ratio *uV* ad *vG*, quae eadem est cum ratione *DCq* ad *PCq*, fiet ratio PV ad *PG* seu *PV* ad 2*PC*; ideoque *PV* equalis erit $\frac{2DCq}{PC}$.

Proinde vis, qua corpus *P* in ellipsi revolvitur, erit reciproce ut $\frac{2DCq}{PC}$ in *PFq* (per carol. 3. prop. VI) hoc est (ob datum 2*DCq* in *PFq)* directe ut *PC. Q. E. I.*

Scholium

Corol. I. Est igitur vis ut distantia corporis a centro ellipseos: & vicissim, si vis sit ut distantia, movebitur corpus in ellipsi centrum habente in centro virium, aut forte in circulo, in quem utique ellipsis migrare potest.

Corol.2. Et aequalia erunt revolutionum in ellipsibus universis circum centrum idem factarum periodica tempora. Nam tempora illa in eIlipsibus similibus aequalia sunt (per corol. 3. & 8. prop. iv.) in ellipsibus autem communem habentibus axem majorem sunt ad in vicem ut ellipscon areae totae directae, & arearum particulae simul descriptae inverse; id est, ut axes minores directe, & corporum velocitates in verticibus principalibus inverse; hoc est, ut axes minores directe, & ordinatim applicatae ad idem punctum axis communis inverse; & propterea (ob aequalitatem rationum directarum & inversarum) in ratione aequalitatis.

Scholium.

Si ellipsis centro in infinitum abeunte vertatur in parabolam, corpus movebitur in hac parabola; & vis ad centrum infinite distans iam tendens evadet requabilis. Hoc est theorema *Galilaei.*Et si coni sectio parabolica (inclinatione plani ad conum sectum mutata) vertatur in hyperbolam, movebitur corpus in huius perimetro vi centripeta in centrifugam versa. Et quemadmodum in circulo vel ellipsi si vires tendunt ad centrum figurae in abscissa positum; hae vires augendo vel diminuendo ordinatas in ratione quacunque data, vel etiam mutando angulum inclinationis ordinatarum ad abscissam semper augentur vel diminuuntur in ratione distantiarum a centro, si modo tempora periodica maneant aequalia; sic etiam in figuris universis si ordinatae augeantur vel diminuantur in ratione quacunque data, vel angulus ordinationis utcunque mutetur, manente tempore periodico ; vires ad centrum quodcunque in abscissa positum tendentes in singulis ordinatis augentur vel diminuuntur in ratione distantiarum a centro.