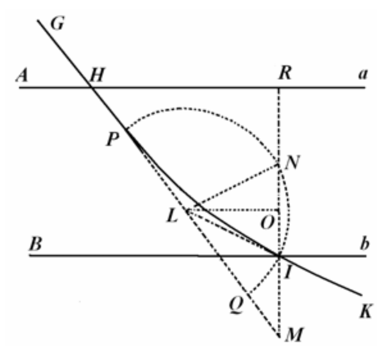
SECTION XIV.

Concerning the motion of the smallest bodies, which may be set in motion by attracting centripetal forces towards the individual parts of some great body.

PROPOSITION XCIV. THEOREM XLVIII.

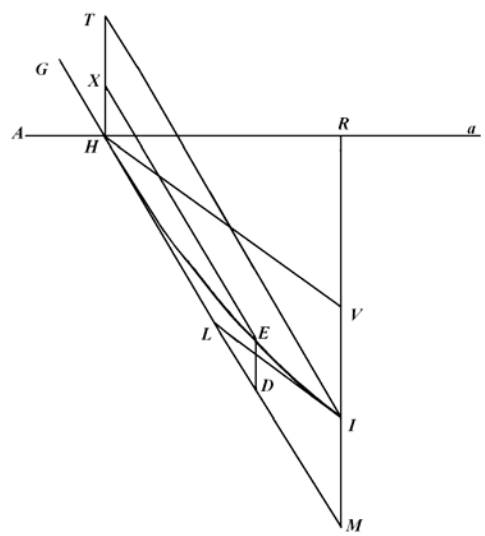
If two similar media may be distinguished in turn, with each space bounded by parallel planes, and a body in passing through this space is attracted or repelled perpendicularly towards either medium, and not set in motion or impeded by any other motion; moreover the attraction shall be the same at equal distances from each plane and taken in the same direction of each: I say that the sine of the incidence in each plane will be in a given ratio to the sine of the emergence from the other plane.



*Cas*e I. Take two parallel planes *Aa, Bb*. A body is incident on the first plane *Aa* along the line *GH,* and in its whole passage through the space within the medium it may be attracted or repelled towards the medium of incidence, and from that action will describe the curved line *HI,* and may emerge along the line *IK.* At the plane of emergence *Bb* there may be erected the perpendicular *IM,* crossing both the line of incidence *GH* produced in *M,* as well as the plane of incidence *Aa* in *R;* and the line of emergence *KI* produced crosses *HM* in *L.* A circle may be described with centre *L* and radius *LI,* cutting *HM* at both *P* and *Q,* as well as *MI* produced in *N;* and initially if a uniform attraction or impulse may be put in place, the curve will be the parabola *HI* (from *Galileo's* demonstration) [*i.e.* a constant vertical force acts along a diameter of the parabola *RI*, while no force acts along the horizontal direction between the plate surfaces. In the original formulation of the parabola by Apollonius – see below, the latus rectum *l*, not drawn on this diagram, multiplied by the ordinate *x*, or distance along some diameter of the parabola from a point on the curve to the mid-point of a diameter 2*y*, are related in skew coordinates by $l.x = y^2$, similar to our $y^2 = 4ax$.], a property of which is this : that the rectangle under the given latus rectum and the [ordinate]line *1M* shall be equal to HM^2; but also the line *HM* will be bisected in *L.* From which if a perpendicular *LO* may be sent to *MI, MO* and *OR* will be equal; and with the equal lines *ON* and *OI* added, the totals *MN* and *IR* become equal. Hence since *IR* may be given, *MN* is given also; and the rectangle *NM.MI* to the rectangle under the latus rectum by *IM* is in a given ratio to HM^2. But the rectangle *NM.MI* is equal to the rectangle *PM.MQ,* that is, to the difference of the squares ML^2 and PL^2 or LI^2; and HM^2 has the given ratio $\frac{ML^2}{4}$ *:* therefore the given ratio $ML^2 - LI^2$ to ML^2, and by converting the ratio LI^2 to ML^2, and with the square root taken, the ratio *LI* to *ML* is given. But in any triangle *LMI*, the sines of the angles are proportional to the opposite sides.

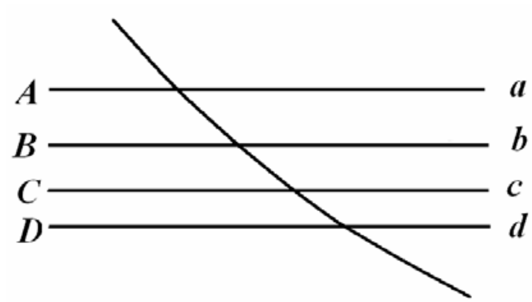
[i.e. $\frac{NM.MI}{HM^2}=\frac{PM.MQ}{HM^2}=\frac{ML^2-LI^2}{ML^2/4}=4-4\frac{LI^2}{ML^2}$ is in a given ratio, and thus $\frac{LI}{ML}$ is given.] Therefore the ratio of the sine of the angle of incidence *LMR* to the sine of the angle of emergence *LIR* is given. *Q. E. D.*

[Leseur & Janquire, note 551 (g) : With the diameter *HT* drawn through the point *H*, and with the right line *HV* the applied ordinate to the other diameter *IR*, and with *IT* the ordinate from the point *I* to the diameter *HT*, on account of the parallels *MI, HT* (by Theorem I *Apol. de Parabola*), and the parallels *MH, IT* (by Lem. 4, *Apol. de Conic*.), $MI = HT$ and $IT = MH$ (by 34. *Book I, Eu. Elem*.) ; but (by Theorem I. *de Parabola*), the square of the ordinate *TI* is equal to the rectangle under the given latus rectum of the diameter *HT* and with the abscissa *HT*, therefore the rectangle under the given latus rectum and the line *MI* is equal to the square *HM*. And since *HM* is a tangent to the parabola at *H* and thus (by Cor. 1, Lem. 5, *de Conic*.) $IM = VI$ and *HV* is parallel to *LI*, there will also be $HL = LM$.



Q.e.d.
Here the latus rectum is a fixed length *l* not drawn on the diagram, for which $l.TH \propto TI^2; l.LH \propto XE^2;$ etc., where *TH, TI* etc. are oblique ordinates, in the original formulation of Apollonius, and which also has a place when the chord becomes a tangent to the parabola.]

Case 2. Now the body may pass successively through several spaces bounded by the planes *AabB, BbcC*, &c. and may be disturbed by a force which shall be uniform in every one apart, but which differ in different spaces; and now by the demonstration, the sine of incidence in the first plane *Aa* will be in a given ratio to the sine of emergence from the second plane *Bb* ; and this sine, which is the sine of incidence in the second plane *Bb*, will be in a given ratio to the line of emergence from the third plane *Cc*; and this sine will be in a given ratio to the sine of emergence from the fourth plane *Dd*, and thus indefinitely : and from the equation, the sine of incidence in the first plane to the sine of emergence from the final plane will be in a given ratio. Now the intervals between the planes may be minimised and the number may be increased indefinitely, so that from that attraction or from the action of the impulse, the following law assigned in some manner, will be

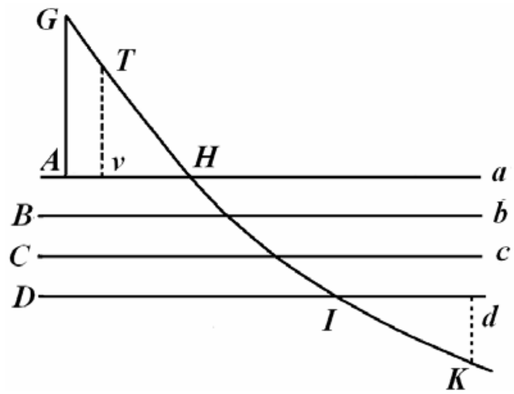


continually returned; and the ratio of the sine of incidence in the first plane to the line of emergence in the final plane, always proving to be given, also even now will be given. *Q. E. D.*

PROPOSITION XCV. THEOREM XLIX.

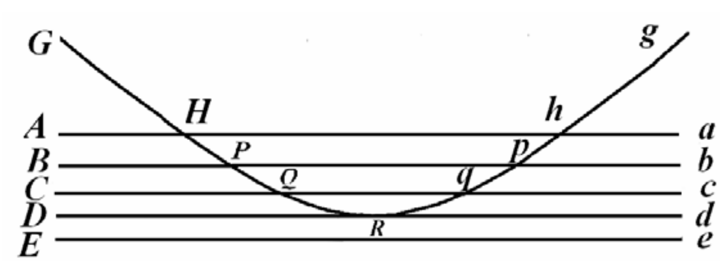
With the same in place; I say that the velocity of the body before incidence is to the velocity of the body after emergence, as the emergent sine to the incident sine.

AH and *Id* may be taken equal, and the perpendiculars *AG* and *dK* may be erected meeting the lines of incidence and emergence *GH* and *IK* in *G* and *K*. On *GH* there may be taken *TH* equal to *IK,* and *Tv* may be sent normally to the plane *Aa.* And (by Corol. 2 of the laws) the motion of the body may be distinguished into two parts, the one perpendicular to the planes *Aa, Bb, Cc,* &c., the other parallel to the same. The force of attraction or of impulses, by acting along the perpendicular lines, changes no motion along the parallels, and therefore the body may complete equal intervals in equal times following the parallels, which are between the line *AG* and the point *H,* and between the point *I* and the line *dK;* that is, the lines *GH* and *IK* are described in equal times. Hence the velocity before incidence is to the velocity after incidence as *GH* to *IK* or *TR,* that is, as *AR* or *Id* to *vH,* that is (with respect to the radii *TH* or *IK)* as the sine of emergence to the sine of incidence.



PROPOSITIO XCVI. THEOREMA L.

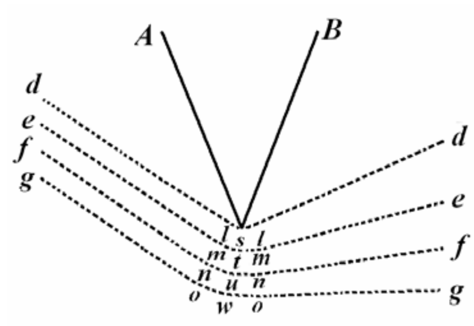
With the same in place, and because the motion before incidence shall be greater than after : I say that the body, on being inclined to the line of incidence, finally will be reflected, and the angle of reflection becomes equal to the angle of incidence.

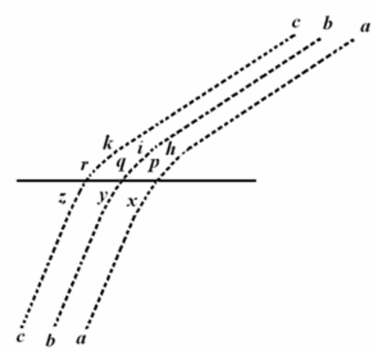


For consider the body to describe parabolic arcs between the parallel planes *Aa, Bb, Cc.* &c. as above, and those shall be the arcs *HP, PQ, QR,* &c. And that line of incidence *GH* shall be oblique to the first plane *Aa*, so that the sine of incidence shall be to the radius of the circle, of which it is the sine, in that same ratio as the sine of incidence to the sine of emergence from the plane *Dd,* in the space *DdeE* : and on account of the sine of emergence now made equal to the radius, the angle of emergence will be a right angle and thus the line of emergence will coincide with the plane *Dd.* The body may arrive at this plane at the point *R;* and because the line of emergence coincides with the same plane, it is evident that the body cannot progress further towards the plane *Ee.* But nor can it go on along the line of emergence *Rd,* as it is always attracted or repelled towards the medium of incidence. And thus it will be returned between the planes *Cc, Dd,* by describing the arc of the parabola QR^2 of which the principal vertex is at *R* (just as *Galileo* has shown)*;* and it will cut the plane *Cc* in the same angle at *q*, as before at *Q*; then by progressing in the parabolic arcs *qp, ph,* &c. by similar and equal arcs to the previous arcs *QP, PH,* it will cut the remaining planes in the same angles at *p, h,* &c. as before at *P, H,* &c. and finally it will emerge with the same obliquity at *h*, by which it began at *H.* Consider now the intervals between the planes *Aa, Bb, Cc, Dd, Ee,* &c. to be diminished indefinitely and the be increased in number indefinitely, so by that action of attraction or of impulse following some designated law it may be returned continually ; and the angle of emergence always arising equal to the angle of incidence, will remain even then equal to the same. *Q. E. D.*

Scholium.

The reflection and refraction of light are not much dissimilar to these attractions, made following a given ratio of the secants, as *Snell* found, and by the ratio of the sines as consequence, as set out by *Descartes.* Just as light can be propagated from the sun both successively from the start and through space in a time of seven or eight minutes to arrive at the earth, now agreed upon through the phenomena of the moons of Jupiter, confirmed from the observations of different astronomers. But the rays present in air (as *Grimaldi* found some time ago, with the light admitted through a hole into a darkened chamber, and that itself I have tried too) in passing close to either the edges of opaque or transparent bodies (such as are circles and rectangular edges of gold, silver, and brass coins, or of knives, or the fractured edges of stones or glass) may be curved around bodies, as if attracted to the same ; and with these rays, which in passing approach closer are curved more, as if attracted more, as that itself I have carefully observed also. And those which pass at greater distances are curved less; and at greater distances they are curved a little towards the opposite direction and form bands of three colours. In the figure *s* may designate the shape edge of a knife, or of some kind of wedge *AsB*; and *gowog, fnunf, emtme, dlsld* are the rays, with the arcs curved towards the knife edge; and that more or less with the distance of these from the knife edge. But since there is a lack of curvature of the rays in air without the knife edge, also the rays which are incident on the knife edge must not be curved in the air before they reach the knife. And the account is the same of rays incident on glass. Therefore refraction happens, not at the



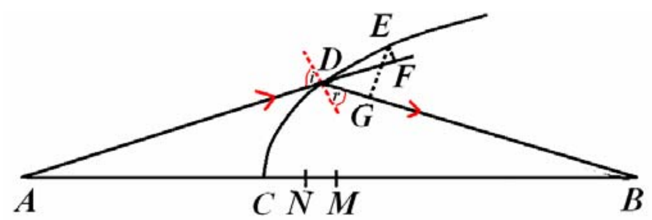


points of incidence, but by a little continuation of the rays, made partially in the air before they touch the glass, partially (lest I am mistaken) in the glass, after they have entered that: as with the incident rays *ckzc, biyb,ahxa* at *r, q, p,* and with the curvature traced out between *k* and *z, i* and *y, h* and *x*. Therefore on account of the analogy which there is between the propagation of rays of light and the progress of bodies, it is seen that the follow propositions be adjoined for optical uses; meanwhile concerning the nature of rays, (whether they may be bodies or not) nothing generally is disputed, but only that the trajectories of bodies are very alike to determining the trajectories of rays.

PROPOSITION XCVII. PROBLEM XLVII.

Because that sine of incidence placed in some surface shall be in a given ratio to the sine of emergence; and because of the in-curving nature of the path of bodies made in the shortest space near that surface, that may be possible to consider as a point: to determine the surface, which all the corpuscles successively arising from a given place may be able to converge to, at another given place.

Let *A* be the place from which the corpuscles diverge ; *B* the place at which they must converge [thus, a theory for a lens is presented here]; *CDE* the curved line which may describe the surface sought by rotating about the axis *AB* ; *D* and *E* any two points of this curve ; *EF* and *EG* perpendiculars sent to the paths *AD* and *DB* of the bodies . The point *D* may approach to the point *E;* and the final ratio of the line *DF,* by which *AD* may be increased, to the line *DG,* by which *DB* is being diminished, will be the same as the ratio which the sine of incidence has to the sine of emergence.



[For from the added normal line in the diagram, we have $\sin i = \frac{DF}{DE}$ and $\sin r = \frac{DG}{DE} : \frac{\sin i}{\sin r} = \frac{DF}{DG}$ as required.]

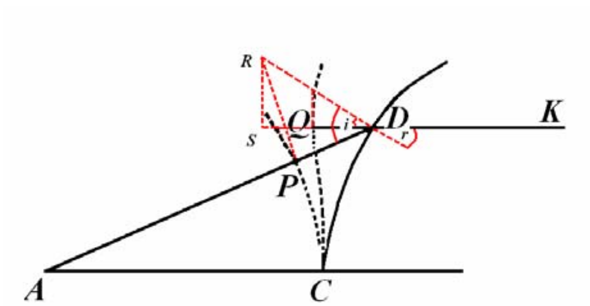
Therefore the ratio may be given of the increment of the line *AD* to the decrement of the line *DB* ; and therefore, if some point *C* may be taken on the axis *AB*, through which the curve *CDE* must pass, and the increment *CM* of *AC* itself may be taken, to the decrement of *BC* itself, *CN* in that given ratio, and with the centres *A* and *B,* and with the intervals *AM* and *BN* two circles may be described cutting each other mutually at *D;* that point *D* touches the curve sought *CDE,* and the same curve required to be touching everywhere will be determined. *Q. E. I.*

Corol. 1. But on requiring now that the point *A* or *B* may go off to infinity, or to be transported to other parts of the point *C,* all these figures will be had, that *Descartes* set out in his optics and geometry relating to refractions. The invention of which *Descartes* has concealed, may be seen to be explained in this proposition.

[Note (b) L & J relating to this corollary : Which lines indeed *Descartes* calls *A*5, *A*6, or *A*7, *A*8 in his *Geometry.* page 50 et seq., and these are called here by Newton *CM, CN,* and with the others the construction is as by that first author. (The interested reader may wish to examine the Dover edition in translation of : *The Geometry of Rene Descartes*, *circa* p.110) From which it is evident, if the point *C*, between the points *A* and *B*, and the point *N* between *C* and *M*, the first Cartesian shall be situated to be described by the Newtonian construction; if for the remaining points *A, C, B, M*, the point *N* may be located between *C* and *A*, the second Cartesian oval will be obtained; truly if the point *B* may be moved to other regions of the point *C* beyond *A*, and the point *C* shall be between *A* and *N*, and *M*, the third Cartesian oval will be obtained, and with the same positions, if the point *N* shall be between *C* and *A*, the fourth Cartesian oval will be set out. Again, if the point *A* or *B* may go off to infinity so that the incident or refracted ray are parallel, then through the point *M* or *N* a perpendicular will be erected, that will cut a circle to be described with centre *B* or *A*, and with radius *BN* or *AM* , at some point D, of the curve CDE, which shall be either an ellipse of hyperbola, as may be apparent from an easy calculation, and these are the figures for which Descartes has described the use in optics in Chapter 8. (We may note here that the usual modern approach to establishing such curves, ideal of course for designing lenses free from spherical aberration, is to use Fermat's Principle of least time, where all the path lengths of the rays are equal, so that above we would have $n_A \times AD = n_B \times DB$; from which of course we have at once $n_A \times DF = n_B \times DG$); the different ovals which arise both for reflection and refraction are conic sections.)]

Corol. 2. If a body incident on some surface *CD*, along a right line *AD*, acted on by some law, may emerge along some other right line *DK*, and the curved lines *CP* and *CQ* may be understood to be drawn from the point *C* always perpendicular to *AD* and *DK* themselves: the increments of the lines *PD* and *QD*, and thus these lines themselves arising from these increments *PD* and *QD*, will be as the sine of incidence and of emergence inversely: and conversely.

[In this case the path length increases are again equal, or, the increase in one is reduced by the decrease in the other to equality. The added lines and letters *R* and *S* show how this can be proven, using the cyclic points *RSDP* and similar triangles.



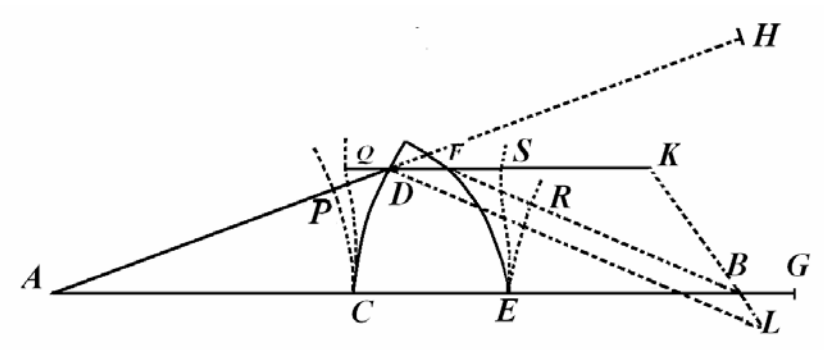
The *Schaum Outline* book *Optics* by Eugene Hecht is particularly illuminating on Descartes Ovoids at an elementary level. One wonders why Newton did not make use of Fermat's Principle, which would apply to small bodies as well as waves.]

PROPOSITION XCVIII. PROBLEM XLVIII.

With the same in place; and *some attractive surface CD may be described around the axis AB, regular or irregular, through which bodies leaving from the given place A are able to pass through: to find another attractive surface EF, by which that body may be made to converge to the given place B.*

With *AB* joined it may cut the first surface in *C* and the second in *E,* at some assumed point.

[The idea being that the refracting surface *CD* is present already, and a new element is to be added to the surface *EF*, to focus the particles travelling along *AP* at *B* : thus, the position of F is required to be found.]



And on putting the sine of incidence on the first surface to the sine of emergence from the same, and with the sine of emergence from the second surface to the sine of incidence in the same, as some given quantity *M* to another given *N*: then produce *AB* to *G,* so that there shall be *BG to CE* as $M-N$ to *N*; as well as produce *AD* to *H,* so that *AH* shall be equal to *AG,* as well also *DF* to *K,* so that there shall be *DK* to *DH* as *N* to *M*. Join *KB,* and with centre *D* and with radius *DH* describe the circle meeting *KB* produced in *L,* and draw *BF* itself parallel to *D L :* and the point *F* may touch the line *EF* [the emergent ray, but not a tangent at *F*], which rotated about the axis *AB* will describe the surface sought. *Q. E. F.*

[So far, we have been introduced to the ratio $\frac{M}{N}$ equal to the ratio of the sines of incidence and emergence, so that we may write $\frac{1}{M}\sin\theta_i = \frac{1}{N}\sin\theta_e$, then corresponding to the Snell's law, we have $M = \frac{1}{n_i} \propto v_i$ and $N = \frac{1}{n_e} \propto v_e$, where n_i, n_e and v_i, v_e are the refractive indices and velocities in the mediums *i* and *e*. The time to traverse the distance *CE* with the lower speed *N* is proportional to $\frac{CE}{N}$, which otherwise without the lens would be traversed in a time proportional to $\frac{CE}{M}$; the extra time is given by $\frac{CE}{N} - \frac{CE}{M} = \frac{CE}{NM} \times (M-N)$, and is equivalent to an extra distance *BG* and an extra time $\frac{BG}{M}$ in the first medium, so that $\frac{CE(M-N)}{MN} = \frac{BG}{M}$; or $\frac{CE}{N} = \frac{BG}{M-N}$. Thus, *AH* and *AG* correspond to the equal lengths that would be traversed in the actual equal times taken to traverse the system by any path, if no change in the medium occurs. Optically, it is the equivalent vacuum path distance.

In the same way, a similar ratio results for the ray *DF*: $\frac{DK}{DH} = \frac{N}{M}$ or $\frac{DK}{N} = \frac{DH}{M}$, or, the time to traverse *DK* at the reduced speed *N* is the same as the time to traverse the free space with the speed *M*. In turn, this means that the ratio of the sides of the respective triangle are as the ratio of the speeds, and so of angles of incidence and emergence. Finally, from the construction, *DL* is the equivalent free space distance from *D* to *H*. Hence, if we move the line DL up parallel to itself, a path *DF* in the slower medium and a path *FB* in the faster medium replaces the original path *DL*; when the ratio *FK* to *FR* has the required value, the appropriate point *F* has been reached. We now give Newton's explanation, in terms of ratios only; note that Newton often uses added or separated ratios to achieve his ends: that is , if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}=\frac{a\pm c}{b\pm d}$; thus, if for some *k* other than 1, $c=ka$ and $d=kb$, then $\frac{a\pm c}{b\pm d}=\frac{a\pm ka}{b\pm kb}=\frac{a}{b}$, or the original result follows simply by cross-multiplying.]

For consider the lines *CP* and *CQ* with respect to *AD* and *DF* themselves, and the lines *ER*, *ES* with *FB*, *FD* themselves to be everywhere perpendicular, and thus *QS*, is always equal to *CE* itself;
[*i.e.* the times to traverse the sections *CE* and the composite section *QS* are always equal; for along *ACEB*, the angles of incidence and refraction are all 90^0, and the argument in the above note can be used to determine the length *BG*, from which the 'time of flight' can be found. The lengths in the following ratios have physical significance : thus, *DL* is the equivalent length in the first medium to travel from *D* to *L*, etc. These hypothetical lengths are shown dotted. Thus $\frac{M}{N}=\frac{DL-FB}{FQ-QD}$ arises from the same time to traverse the numerator distance with speed *M* as does the denominator with speed *N*, etc.]
and there will be (by Corol. 2. Prop. XCVII.), *PD* to *QD* as *M* to *N*, and thus as *DL* to *DK* or *FB* to *FK;*

$$[i.e. \frac{PD}{QD}=\frac{M}{N}=\frac{DL}{DK}=\frac{FB}{FK}\text{; the last step by similar triangles, }]$$

and on separating, as $DL-FB$ or $PH-PD-FB$ to *FD* or $FQ-QD$;

$$[i.e. \frac{DL-FB}{DK-FK}=\frac{DL-FB}{FD}=\frac{PH-PD-FB}{FD} \text{ or } \frac{M}{N}=\frac{DL-FB}{FQ-QD}=\frac{PH-PD-FB}{FQ-QD};\quad]$$

and on adding, as $PH-FB$ to *FQ*, that is (on account of *PH* and *CG*, *QS* and *CE* being equal) $CE+BG-FR$ to $CE-FS$.

$$[i.e.\ \frac{M}{N}=\frac{PH-FB}{FQ}=\frac{CE+BG-FR}{CE-FS}]$$

Truly (on account of the proportionals *BG* to *CE* and $M-N$ to *N*) also there is $CE+BG$ to *CE* as *M* to *N*; and thus separated *FR* to *FS* as *M* to *N*; and therefore (by Corol. 2. Prop. XCVII.) the surface *EF* collects the body, incident on that second line itself *DF* , to go along in the line *FR* to the place *B.*
Q. E. D.

[Essentially, all the paths have the same time of traversal, or, they obey Fermat's Principle of least time; clearly the straight–through path is such a minimum, on which others can be gauged.]

Scholium.

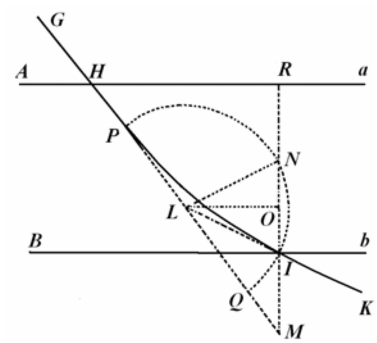
It is permissible to go on by the same method to three or more surfaces. But spherical figures are the most convenient for use in optics. If spyglasses with objective and eyepiece may be constructed from two spherical glass figures, and water enclosed between them; it can come about that the refraction errors, which are present in the extreme surfaces of the glasses, may be corrected well enough by the refraction of the water. But such glass objectives are to be preferred than ellipses or hyperbolas, not only because they are they are easier and more accurate to be made, but also the pencils of rays beyond the axis of the glass in place may refract more accurately. But this is impeded from perfection by the diverse refractions of the different kinds of rays, by which the optics either by spherical or other figures is less than perfect. Unless the errors hence arising shall be corrected, all the labour involved in correcting the other errors will be to no avail.

SECTIO XIV.

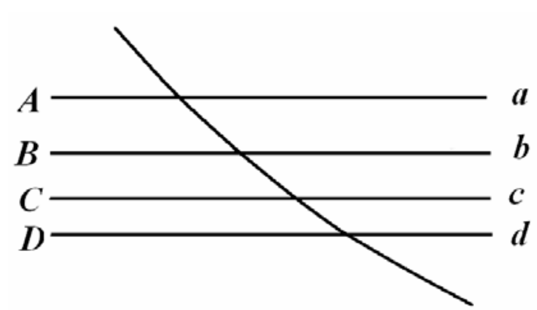
De motu corporum minimorum, quae viribus centripetis ad singulas magni alicuius corporis partes tendentibus agitantur.

PROPOSITIO XCIV. THEOREMA XLVIII.

Si media duo similaria, spatio planis parallelis utrinque terminato, distinguantur ab invicem, & corpus in transitu per hoc spatium attrahatur vel impellatur perpendiculariter versus medium alterutrum, neque ulla alia vi agitetur vel impediatur; sit autem attractio, in aequalibus ab utroque plano distantiis ad eandem ipsius partem captis, ubique eadem: dico quod sinus incidentiae in planum alterutrum erit ad sinum emergentiae ex plano altero in ratione data.



Cas.I. Sunto *Aa*, *Bb* plana duo parallela. Incidat corpus in planum prius *Aa* secundum lineam *GH*, ac toto suo per spatium intermedium transitu attrahatur vel impellatur versus medium incidentiae, eaque actione describat lineam curvam *HI, &* emergat secundum lineam *IK.*Ad planum emergentiae *Bb* erigatur perpendiculum *IM,* occurrens tum lineae incidentiae *GH* productae in *M,* tum plano incidentiae *Aa* in *R;* & linea emergentiae *KI* producta occurrat *HM* in *L.* Centro *L* intervallo *LI* describatur circulus, secans tam *HM* in *P & Q,* quam *MI* productam in *N;* & primo si attractio vel impulsus ponatur uniformis, erit (ex demonstratis *Galilaei)* curva *HI* parabola, cujus haec est proprietas, ut rectangulum sub dato latere recto & linea *1M* aequale sit *HM* quadrato; sed & linea *HM* bisecabitur in *L.* Unde si ad *MI* demittatur perpendiculum *LO,* aequaIes erunt *MO, OR;* & additis aequalibus *ON, OI,* fient totae aequales *MN, 1R.* Proinde cum *IR* detur, datur etiam *MN;* estque rectangulum *NMI* ad rectangulum sub latere recto & *IM*, hoc est, ad *HMI* in data ratione. Sed rectangulum *NMI* aequale est rectangulo *PMQ,* id est, differentiae quadratorum *MLq,* & *PLq* seu *LI q;* & *HMq* datam rationem habet ad sui ipsius quartam partem *MLq:* ergo datur ratio *M Lq-LIq* ad *MLq,* & convertendo ratio *LIq* ad *MLq*, & ratio dimidiata *LI* ad *ML.* Sed in omni triangulo *LMI,* sinus angulorum sunt proportionales lateribus oppositis. Ergo datur ratio linus anguli incidentiae *LMR* ad sinum anguli emergentiae *LIR. Q. E. D.*

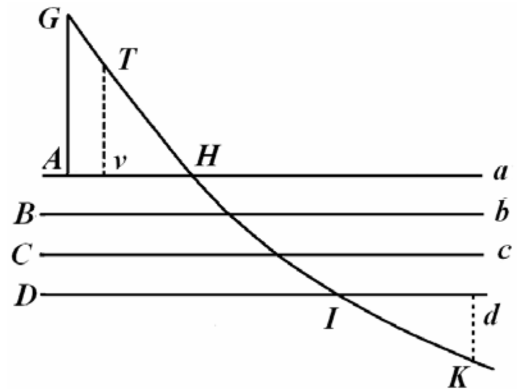


Cas. 2. Transeat jam corpus Successive per spatia plura parallesis planis terminata, *AabB, BbcC,* &c. & agitetur vi quae sit in singulis separatim uniformis, at in diversis diverfsa; & per jam demonstrata, sinus incidentiae in planum primum *Aa* erit ad sinum emergentiae ex plano secundo *Bb*, in data ratione; & hic sinus, qui est sinus incidentiae in planum secundum *Bb*, erit ad linum emergentiae ex plano tertio *Cc*, in data ratione; & hic sinus ad sinum emergentiae ex plano quarto *Dd*, in data ratione; & sic in infinitum: & ex aequo, sinus incidentiae in planum primum ad sinum emergentiae ex plano ultimo in data ratione. Minuantur iam planorum intervalla & augeatur numerus in infinitum, eo ut attractionis vel impulsus actio, secundum legem quamcunque assignatam, continua reddatur; & ratio sinus incidentiae in planum primum ad linum emergentiae ex plano ultimo, semper data existens, etiamnum dabitur. *Q. E. D.*

PROPOSITIO XCV. THEOREMA XLIX.

Iisdem positis; dico quod velocitas corporis ante incidentiam est ad eius velocitatem post emergentiam, ut sinus emergentiae ad sinum incidentiae.

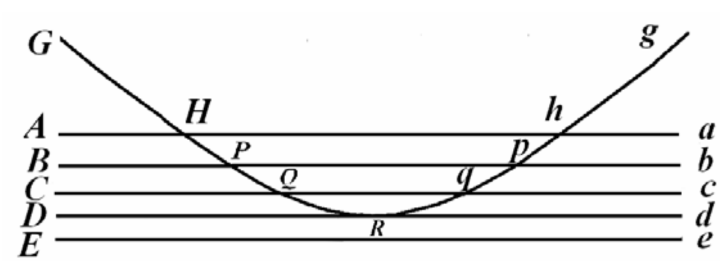
Capiantur *AH*, *Id* aequales, & erigantur perpendicula *AG*, *dK* occurrentia lineis incidentiae & emergentiae *GH*, *IK*, in *G* & *K*. In *GH* capiatur *TH* aequalis *IK*, & ad planum *Aa* demittatur normaliter *Tv*. Et (per legum. Corol. 2.) distinguatur motus corporis in duos, unum planis *Aa*, *Bb*, *Cc*, &c. perpendicularem, alterum iisdem parallelum. Vis attractionis vel impulsus, agendo secundum lineas perpendiculares, nil mutat motum secundum parallelas, & propterea corpus hoc motu conficiet aequalibus temporibus aequalia illa secundum parallelas intervalla, quae sunt inter lineam *AG* & punctum *H*, interque punctum *I* & lineam *dK*; hoc est, aequalibus temporibus describet lineas *GH*, *IK*. Proinde velocitas ante incidentiam est ad velocitatem post emergentiam, ut *GH* ad *IK* vel *TR*, id est, ut *AR* vel *Id* ad *vH*, hoc est (respectu radii *TH* vel *IK)* ut sinus emergentiae ad sinum incidentiae.



PROPOSITIO XCVI. THEOREMA L.

Iisdem posttis, & *quod motus ante incidentiam velocior sit quam postea : dico quod corpus, inclinando lineam incidentiae, reflectetur tandem,* & *angulus reflexionis fiet aequalis angulo incidentiae.*

Nam concipe corpus inter parallela plana *Aa*, *Bb*, *Cc*. &c. describere arcus parabolicos.

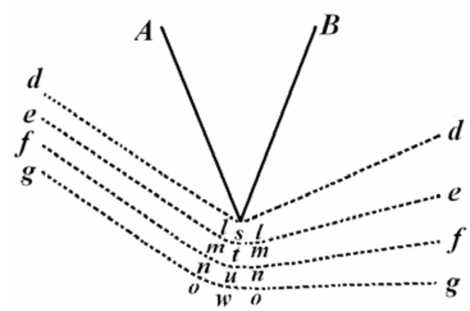


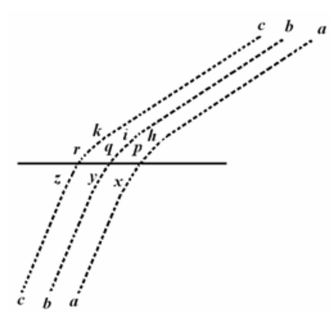
ut supra; sintque arcus illi *HP*, *PQ*, *QR*, &c. Et sit ea lineae incidentiae *GH* obliquitas ad planum primum *Aa*, ut sinus incidentiae sit ad radium circuli, cujus est sinus, in ea ratione quam habet idem sinus incidentiae ad sinum emergentiae ex plano *Dd*, in spatium *DdeE* :

& ob sinum emergentiae iam factum aequalem radio, angulus emergentiae erit rectus, ideoque linea emergentiae coincidet cum plano *Dd.* Perveniat corpus ad hoc planum in puncto *R;* & quoniam linea emergentiae coincidit cum eodem plano, perspicuum est quod corpus non potest ultra pergere versus planum *Ee.* Sed nec potest idem pergere in linea emergentiae *Rd,* propterea quod perpetuo attrahitur vel impellitur versus medium incidentiae. Revertetur itaque inter plana *Cc, Dd,* describendo arcum parabolae *QRq* cuius vertex principalis (iuxta demonstrata *Galilaei*) est in *R;* secabit planum *Cc* in eodem angulo in *q,* ac prius in *Q*; dein pergendo in arcubus parabolicis *qp, ph,* &c. arcubus prioribus *QP, PH* similibus & aequalibus, secabit reliqua plana in iisdem angulis in *p, h,* &c. ac prius in *P, H,* &c. emergetque tandem eadem obliquitate in *h,* qua incidit in *H.* Concipe iam planorum *Aa, Bb, Cc, Dd, Ee,* &c. intervalla in infinitum minui & numerum augeri, eo ut actio attractionis vel impulsus secundum legem quamcunque assignatam continua reddatur; & angulus emergentia: semper angulo incidentiae aequalis existens, eidem etiamnum manebit aequalis. *Q. E. D.*

Scholium.

Harum attractionum haud multum dissimiles sunt lucis reflexiones & refractiones, factae secundum datam secantium rationem, ut invenit *Snellius,* & per consequens secundum datam sinuum rationem, ut exposuit *Cartesius.* Namque lucem successive propagari & spatio quasi septem vel octo minutorum primorum a sole ad terram venire, jam constat per phaenomena satellitum *Jovis*, observationibus diversorum astronomorum confirmata. Radii autem in aere existentes (uti dudum *Grimaldus,* luce per foramen in tenebrosum cubiculum admissa, invenit, & ipse quoque expertus sum) in transitu suo prope corporum vel opacorum vel perspicuorum angulos (quales sunt nummorum ex auro, argento & aere cusorum termini rectanguli circulares, & cultrorum, lapidum aut fractorum vitrorum acies) incurvantur circum corpora, quasi attracti in eadem; & ex his radiis, qui in transitu illo propius accedunt ad corpora incurvantur magis, quasi magis attracti, ut ipse etiam diligenter observavi. Et qui transeunt ad majores distantias minus incurvantur; & ad distantias adhuc majores incurvantur aliquantulum ad partes contrarias, & tres colorum fascias efformant. In figura *s* designat aciem cultri vel cunei cujusvis *AsB*; & *gowog, fnunf, emtme, dlsld* sunt radii, arcubus, versus cultrum incurvati; idque magis vel minus pro distantia eorum a cultro. Cum autem talis incurvatio radiorum fiat in aere extra cultrum, debebunt etiam radii, qui incidunt in cultrum, prius incurvari in aere quam cultrum attingunt. Et par est ratio incidentium in vitrum. Fit igitur refractio, non in puncto incidentiae, sed paulatim per continuam incurvationem radiorum, factam partim in aere antequam attingunt vitrum, partim (ni fallor) in vitro, postquam illud ingressi sunt: uti in radiis *ckzc, biyb,ahxa* incidentibus ad *r, q, p,* & inter *k* & *z, i* & *y, h* & *x* incurvatis,



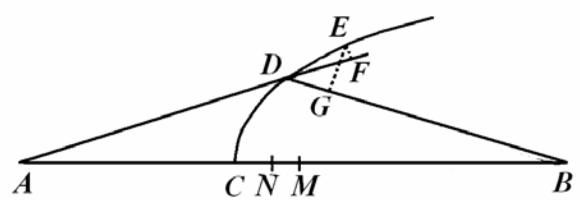


delineatum est. Igitur ob analogiam quae est inter propagationem radiorum lucis & progressum corporum, visum est propositiones sequentes in usus opticos subiungere; interea de natura radiorum (utrum sint corpora necne) nihil omnino disputans, sed trajectorias corporum traiectoriis radiorum persimiles solummodo determinans,

PROPOSITIO XCVII. PROBLEMA XLVII.

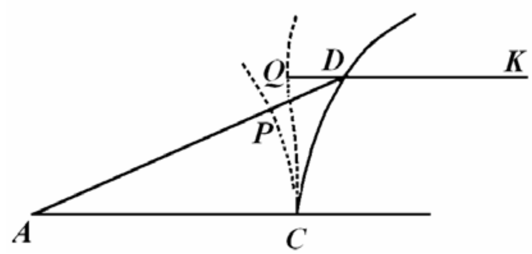
Posito quod sinus incidentiae in superficiem aliquam sit ad sinum emergentiae in data ratione; quodque incurvatio viae corporum jucta superficiem illam fiat in spatio brevissimo, quod ut punctum considerari possit: determinare superficiem, quae corpuscula omnia de loco dato successive manantia convergere faciat ad alium locum datum.

Sit *A* locus a quo corpuscula divergunt; *B* locus in quem convergere debent; *CDE* curva linea quae circa axem *AB* revoluta describat superficiem quaesitam; *D, E* curvae illius puncta duo quaevis; & *EP, EG* perpendicula in corporis vias *AD, DB* demissa. Accedat punctum *D* ad punaum *E;* & lineae *DF,* qua *AD* augetur, ad lineam *DG,* qua *DB* diminuitur, ratio ultima erit eadem, quae sinus incidentiae ad sinum emergentiae. Datur ergo ratio incrementi lineae *AD* ad decrementum lineae *DB;* & propterea si in axe *AB*sumatur ubivis punctum *C,* per quod curva *CDE* transire debet, & capiatur ipsius *AC* incrementum *CM* ad ipsius *BC* decrementum *CN* in data illa ratione, centrisque *A, B,* & intervallis *AM, BN* describantur circuli duo se mutuo secantes in *D;* punctum illud *D* tanget curvam quaesitam *CDE,* eandemque ubivis tangendo determinabit. *Q. E. I.*



Corol. 1. Faciendo autem ut punctum *A* vel *B* nunc abeat in infinitum, nunc migret ad alteras partes puncti *C*, habebuntur figurae illae omnes, quas *Cartesius* in optica & geometria ad refractiones exposuit. Quarum inventionem cum *Cartesius* celaverit, visum fuit hac propositione exponere.

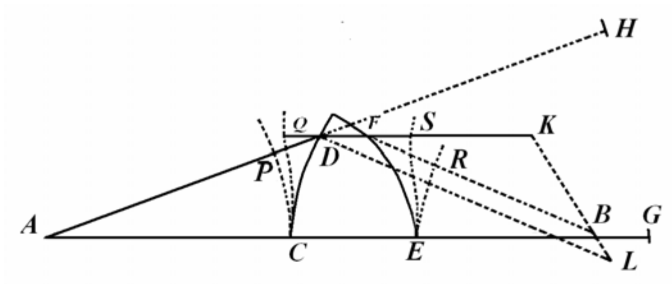
Corol. 2. Si corpus in superficiem quamvis *CD,* secundum lineam rectam *AD,* lege quavis ductam incidens, emergat secundum aliam quamvis rectam *DK,* & a puncto *C* duci intelligantur lineae curvae *CP, CQ* ipsis *AD, DK* semper perpendiculares: erunt incrementa linearum *PD, QD,* atque ideo lineae ipsae *P D, QD,* incrementis istis genitae, ut sinus incidentiae & emergentiae ad invicem: & contra.



PROPOSITIO XCVIII. PROBLEMA XLVIII.

Iisdem positis; & *circa axem* A*B descripta superficie quacunque attractiva CD, regulari vel irregulari, per quam corpora de loco dato A exeuntia transire debent: invenire superficiem secundam attractivam EF, quae corpora illa ad locum datum B convergere faciat.*

Iuncta *AB* secet superficiem primam in *C* & secundam in *E*, puncto utcunque assumpto. Et posito sinu incidentiae in superficiem primam ad sinum emergentiae ex eadem, & sinu emergentiae e superficie secunda ad sinum incidentiae in eandem, ut quantitas aliqua data *M* ad aliam datam *N*: produc tum *AB* ad *G*, ut sit *BG* ad *CE* ut $M-N$ ad *N*; tum *AD* ad *H*, ut sit *AH* aequalis *AG*, tum etiam *DF* ad *K*, ut sit *DK* ad *D H* ut *N* ad *M*. Iunge *KB*, & centro *D* intervallo *DH* describe circulum occurrentem *KB* productae in *L*, ipsique *D L* parallelam age *BF:* & punctum *F* tanget lineam *EF*, quae circa axem *AB* revoluta describet superficiem quasitam. *Q. E. F.*



Nam concipe lineas *CP*, *CQ* ipsis *AD*, *DF* respective, & lineas *ER*, *ES* ipsis *FB*, *FD* ubique perpendiculares esse, ideoque *QS*, ipsi *CE* semper aequalem; & erit (per Corol. 2. prop. XCVII.) *PD* ad *QD* ut *M* ad *N*, ideoque ut *DL* ad *DK* vel *FB* ad *FK; &* divisim ut $DL-FB$ seu $PH-PD-FB$ ad *FD* seu $FQ-QD$; & composite ut $PH-FB$ ad *FQ*, id est (ob aequales *PH* & *CG*, *QS* & *CE)* $CE+BG-FR$ ad $CE-FS$. Verum (ob proportionales *BG* ad *CE* & $M-N$ ad *N*) est etiam $CE+BG$ ad *CE* ut *M* ad *N*; ideoque divisim *FR* ad *FS* ut *M* ad *N*; & propterea (per Corol. 2. Prop. XCVII.) superficies *EF* cogit corpus, in ipsam secundum lineam *DF* incidens, pergere in linea *FR* ad locum *B*.
Q. E. D.

Scholium.

Eadem methodo pergere liceret ad superficies tres vel plures. Ad usus autem opticos maxime accommodatae sunt figurae sphaericae. Si perspicillorum vitra obiectiva ex vitris duobus sphaeriae figuratis & aquam inter se claudentibus conflentur; fieri potest ut, a refractionibus aquae errores refractionum, quae fiunt in vitrorum superficiebus extremis, satis accurate corrigantur. Talia autem vitra objectiva vitris ellipticis & hyperbolicis praeferenda sunt, non solum quod facilius & accuratius formari possint, sed etiam quod penicillos radiorum extra axem vitri sitos accuratius refringant. Veruntamen diversa diversorum radiorum refrangibilitas impedimento est, quo minus optica per figuras vel sphaericas vel alias quascunque perfici possit. Nisi corrigi possint errores illinc oriundi, labor omnis in caeteris corrigendis imperite collocabitur.