SECTION XIII.

Concerning the attractive forces of non–spherical bodies.

PROPOSITION LXXXV. THEOREM XLII.

If the attraction of a body by another shall be far greater, when it is attracted in contact, than when the bodies may be separated by the smallest distance from each other: then the forces of the attracting particles in the distant parts of the attracting body, decrease in a ratio greater than the square of the distances from the particles.

For if the forces between the particles decrease in the inverse square ratio of the distances; the attraction towards a spherical body, [placed at the point of contact and drawn within the attracting body with the same curvature as the surface at this point], because that attraction (by Prop. LXXIV.) shall be inversely as the square of the distance of the attracted body from the centre of the sphere, will not be sensibly increased on being in contact; and it will be increased even less by the contact, if the attraction in the more distant parts from [the point of contact] of the attracting body may decrease in a smaller ratio. Therefore the proposition is apparent for the attraction of spheres. And there is the same reasoning for concave spherical shells attracting bodies externally. And the matter is agreed upon even more with hollow shells attracting bodies within themselves, since the attractions may be removed everywhere through the cavity by opposing attractions (by Prop. LXX), and thus are even zero at the contact point itself. For if from these spherical and spherical shells any parts may be taken from places remote from the point of the point of contact, and new parts may be added elsewhere : the shapes of these attracting bodies may be changed at will, yet the parts added or removed, since they shall be remote from the point of contact, will not notably increase the excess of the attraction which arises at the point of contact. Therefore the proposition may be agreed upon for all shapes of bodies. *Q. E. D.*

[The reader may wish to reflect on the electrostatic analogy of this theorem.]

PROPOSITION LXXXVI. THEOREM XLIII.

If the forces of the particles, from which the attracting body is composed, decrease in the remote parts of the attracting body in the triplicate or greater ratio of the distances from the particles : the attraction will be much stronger in contact, than when the attracted and attracting bodies may be separated in turn even at a minimal distance.

For it is agreed in the approach of an attracted corpuscle to an attracting sphere of this kind that the attraction increases indefinitely, by the solution of Problem XLI in the second and third example shown. Likewise, it is easily deduced from that example and Theorem XLI taken together, concerning the attractions of bodies towards concave-convex shells, either with the attracted bodies gathered together outside the shells, or within the cavities of these. But also by adding or taking away any attracting matter from these spheres and shells somewhere beyond the point of contact, so that the attracting bodies may adopt there some assigned figure, the proposition will be agreed upon generally for all bodies. *Q. E. D.*

PROPOSITION LXXXVII. THEOREM XLIV.

If two bodies similar to each other, and consisting of equally attracting material, may each attract corpuscles to themselves, with these corpuscles proportional and similar to themselves : the accelerative attractions of the corpuscles for the whole bodies shall be as the accelerative attractions of the corpuscles for all the proportional particles of these, similarly put in place in the whole bodies.

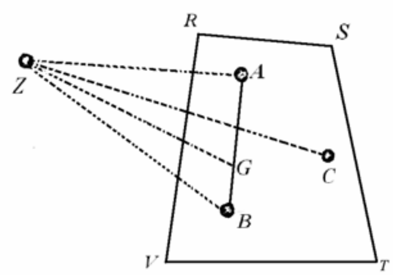
For if the bodies may be separated into particles which may be proportional to the whole, and similarly situated in the whole; there will be, as the attraction of some particle of one body to the attraction in the corresponding particle in the other body, thus the corresponding attractions in the individual particles of the first body to the individual particle attractions in the other ; and on adding these together, thus the attraction in the first whole body to the attraction in the second whole body.
Q. E. D.
[This proposition is an assertion that a body may be considered as consisting of particles, each of which, and collections of these, up to the whole body, behave in the same way under the influence of a mutual attraction with another body, similarly constituted: Newton's bodies are held together by gravitational attractions that extend to other bodies, wholly or in part. The same rules can be extended to other unspecified forces, obeying inverse power laws, that give rise to a scaling law, as set out by Chandrasekhar pp305-306 : Thus, if the force on a corpuscle at position *R* relative to an extended body, and summed over the constituent particles, each of which exerts a force, resolved along the axes, of which $dF_x(R) \propto density \times \frac{dv}{|R-r|^n}$ is a component, proportional to its mass or volume, and varying inversely as the n^{th} power of the distance, is given on replacing $r \to \alpha r; R \to \alpha R$, by $dF_x(\alpha R) \propto \alpha^{3-n} \times density \times \frac{dv}{|R-r|^n}$, which Newton now sets out for the certain cases $n = 2,3,4$.]

Corol. 1. Therefore if the attractive forces of the particles, with the distances of the particles increased, may decrease in the ratio of some power of any of the distances; the accelerative attractions in the whole bodies shall be as the bodies directly, and these powers of the distances, inversely. So that if the forces of the particles may decrease in the ratio of the squares of the distances from the corpuscles attracted, the bodies moreover shall be as $A^3 \& B^3$. and thus both the cubic sides of bodies, as well as the distances of the attracting bodies from the bodies, as *A* & *B:* the accelerative attractions in the bodies will be as $\frac{A^3}{A^2} \& \frac{B^3}{B^2}$, that is, as the sides from these cubic bodies *A* & *B.* If the forces of the particles may decrease in the triplicate ratio of the distances from the attracting bodies; the accelerative attractions in the whole bodies will be as $\frac{A^3}{A^3} \& \frac{B^3}{B^3}$, that is, equal. If the forces may decrease in the quadruple ratio; the attractions between the bodies will be as $\frac{A^3}{A^4}\&\frac{B^3}{B^4}$, that is , inversely as the cubic sides A & B. And thus for the others.

*Corol.*2. From which in turn, from the forces by which similar bodies attract corpuscles similarly put in place to themselves, the ratio of the decrease of the attractive forces of the particles can be deduced in the receding of the attracting corpuscles; but only if that decrease shall be directly or inversely in some ratio of the distances.

PROPOSITION LXXXVIlI. THEOREM XLV.

If the attractive forces of the equal particles of some body shall be as the distances of the places from the particles : the force of the whole body will tend towards the centre of gravity of this ; and it will be the same as with the force of a globe agreed upon from the same material and equal in all respects, and having the centre of this at the centre of gravity.



The particles *A*, *B* of the body *RSTV* may pull some corpuscle *Z* by forces, which, if the particles may be equal among themselves, shall be as the distances *AZ*, *BZ;* but if unequal particles may be put in place, the forces shall be as these particles and the distances of these *AZ, BZ* jointly, or (if I may say thus), as these particles respectively multiplied by their distances *AZ*, *BZ* . And these forces may be set out contained by these $A\times AZ\ \&\ B\times BZ$. *AB* may be joined and with that line cut in *G* so that there shall be *AG* to *BG* as the particle *B* to the particle *A;* and *G* will be the common centre of gravity of the particles *A* & *B* . The force $A\times AZ$ (by Corol. 2. of the laws) is resolved into the forces $A\times GZ$ and $A\times AG$ and the force $B\times BZ$ into the forces $B\times GZ$ and $B\times BG$. But the forces $A\times AG$ and $B\times BG$ are equal, on account of the proportionals *A* to *B* and *BG* to *AG*; therefore since they may be directed in opposite directions, the mutually cancel each other out. The forces remain $A\times GZ$ and $B\times Gz$. These tend from *Z* towards the centre *G*, and they compound the force $\overline{A+B}\times GZ$; that is, the same force. and as if the attracting particles *A & B* may constitute the components of the common centre of gravity *G* of these, there the components of a globe.

By the same argument, if a third particle *C* may be added and also it may be compounded with the force $\overline{A+B}\times GZ$ tending towards the centre *G*; hence the force arising tends towards the common centre of gravity of that globe at *G* and of the particle *C;* that is, to the common centre of gravity of the three particles *A*, *B*, *C;* and it will be the same as if the globe and the particle *C* may be present at that common centre, comprising a greater globe. And thus it may go on indefinitely. Therefore the total force of all the particles bodies of any kind is the same *RSTV*, as if that body, by maintaining the centre of gravity, may adopt the figure of a globe. *Q. E. D.*

Corol. Hence the motion of the attracted body *Z* will be the same, as if the attracting body *RSTV* were spherical : and therefore if that attracting body either were at rest, or progressing uniformly along a direction ; the attracted body will be moving in an ellipse having at its attracting centre the centre of gravity.

PROPOSITIO LXXXIX. THEOREMA XLVI.

If several bodies shall consist of equal particles, of which the forces shall be as the distances of the places from the individual particles : the force from all the forces added together, by which a certain corpuscle is attracted, will tend towards the common centre of gravitational attraction ; and it will be the same, as if the particles might group together and be formed into a globe by that attraction, with the centre serving as a common centre of gravity

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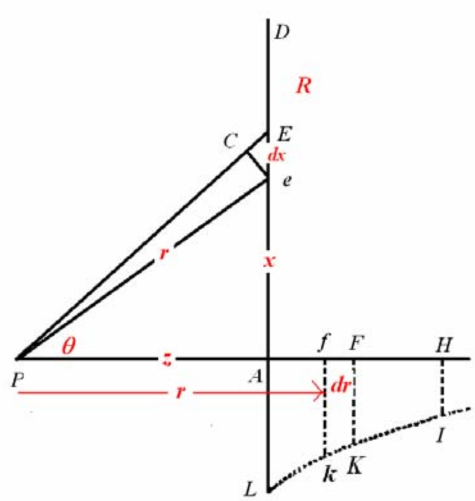
This may be demonstrated in the same manner as with the above proposition.

Corol. Therefore the motion of an attracted body will be the same, as if the attracting bodies, with a common centre of gravity maintained, may coalesce and be formed into a globe. And thus if the common centre of gravity of the attracting bodies may either be at rest, or progressing uniformly along a right line; an attracted body will be moving in an ellipse, by having the centre in that common attracting centre of gravity.

PROPOSITION XC. PROBLEM XLIV.

If equal centripetal forces may attract the individual points of any circle , increasing or decreasing in some ratio of the distances: to find the force, by which a corpuscle may be attracted at some position on a straight line, which remains perpendicular to the plane of the circle at its centre.

With centre *A* and with some radius *AD,* a circle may be understood to be described in the plane, to which the right line *AP* is perpendicular ; and the force shall be required to be found, by which some corpuscle *P* on the same is attracted. A right line *PE* may be drawn from some point *E* of the circle to the attracted corpuscle *P*. On the right line *PA*, *PF* may be taken equal to *PE* itself, and the normal *FK* may be erected, which shall be as the force by which the point *E* attracts the corpuscle *P.* And let *IKL* be the curved line that always touches the point *K.* The same curved line may cross the plane of the circle at *L.* On *PA* there may be taken *PH* equal to *PD,* and a perpendicular to the aforementioned curve *HI* may be erected crossing at *I;* and the attraction of the corpuscle *P* to the circle will be as the area *AHIL* taken by the altitude *AP*. *Q. E. I.*



And indeed on *AE* the line *Ee* may be taken as minimal [i.e. as the increment *dx*]. *Pe* may be joined, and on *PE* and *PA* there may be taken *PC* and *Pf* themselves equal to *Pe*.

And since the force, by which the annulus with centre *A* and radius *AE* described in the aforementioned plane, attracts *P* to some point *E* of the body itself, is put to be as *FK*, and hence the force, by which that point attracts the body *P* towards *A*, is as $\frac{AP\times FK}{PE}$ and the force, by which the whole annulus attracts the body *P* towards *A* ,as the annulus and $\frac{AP\times FK}{PE}$ conjointly [for if $f\left(r\right)$ if the force due to the increment *Ee* along *Pe*, the normal force due to the whole annulus is $2\pi rf\left(r\right)cos\,\theta \propto$ area of annulis $\times FK\times\frac{AP}{PE}$]; but the annulus itself is as the rectangle under the radius *AE* and with the width *Ee*, and this rectangle (on account of the proportionals *PE* and *AE*, *Ee* and *CE)* is equal to the rectangle $PE\times CE$ or $PE\times Ff$ [for $AE\times Ee=PE\,sin\,\theta\times\frac{CE}{sin\,\theta}=PE\times CE=PF\times Ff$]; the force, by which the annulus itself attracts the body *P* towards *A*, will be as $PE\times Ff$ and $\frac{AP\times FK}{PF}$ jointly, that is, as contained by $Ff\times FK\times AP$, or as the area *FKkf* multiplied by *AP*. And therefore the sum of the forces, by which all the annuli in the circle, which is described with centre *A* and interval *AD*, attract the body *P* towards *A*, is as the total area *AHIKL* multiplied by *AP*. *Q.E.D.*

[We note that $Ff\times FK\rightarrow AP\times 2\pi\int f\left(r\right)dr$, where the area under the integral becomes *AHIL*.]

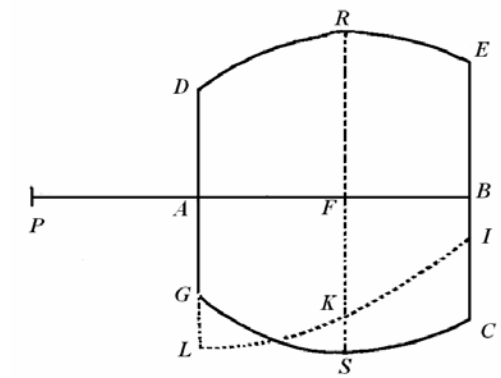
Corol. 1. Hence if the forces of the points decrease in the duplicate ratio of the distance, that is, if *FK* shall be as $\frac{1}{PF^2}$,and thus the area *AHIKL* as $\frac{1}{PA}-\frac{1}{PH}$; the attraction of the corpuscle *P* into the circle will be as $1-\frac{PA}{PH}$, that is, as $\frac{AH}{PH}$.

Corol.2. And universally, if the forces of the points at the distances *D* shall be reciprocally as some power of some distances D^n, that is, if *FK* shall be as $\frac{1}{D^n}$, and thus *AHIKL* as $\frac{1}{PA^{n-1}}-\frac{1}{PH^{n-1}}$; the attraction of the corpuscle P towards the circle shall be as $\frac{1}{PA^{n-2}}-\frac{1}{PH^{n-1}}$

Corol. 3. And if the diameter of the circle may be increased to infinity, and the number *n* shall be greater than unity ; the attraction of the corpuscle *P* in the whole infinite plane shall be as PA^{n-2}, because therefore the other term *PA* vanishes.

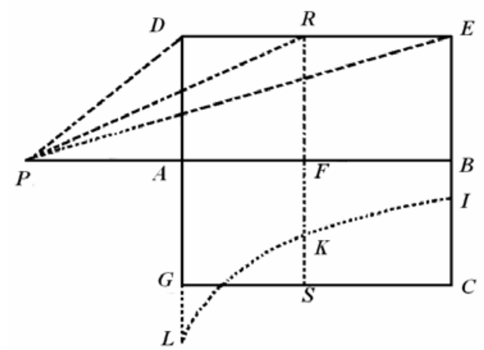
PROPOSITION XCI. PROBLEM XLV.

To find the attraction of a body situated on the axis of a solid of rotation, to the individual points of which equal centripetal forces attract in some decreasing ratio of the distances.



The corpuscle *P* is attracted towards the solid *DECG*, situated on the axis of this *AB*. This solid may be cut by some circle *RFS* perpendicular to its axis, and on the radius of this *FS*, by some other plane *PALKB* passing through the axis, the length *FK* may be taken for the force (by Prop. XC.), by which the corpuscle *P* is attracted proportionally in that circle. Moreover the point *K* may touch the line *LKI*, meeting the planes of the outermost circles *AL* and *BI* at *L* and *I* ; and the attraction of the corpuscle *P* towards the solid will be as the area *LABI. Q.E.I.*

Corol. 1. From which if the solid shall be a cylinder described by revolving the parallelogram *ADEB* about the axis *AB*, and the centripetal forces tending towards the individual points of this shall be inversely as the square of the distances from the points : the attraction of the corpuscle *P* will be towards this cylinder as $AB - PE + PD$. For the applied ordinate *FK* [*i.e.* the increment of the force] (by Corol. I. Prop. XC.) will be as $1 - \frac{PF}{PR}$ [for the force due to the incremental circle is proportional to $PF \times \left(\frac{1}{PF} - \frac{1}{PR}\right) = 1 - \frac{PF}{PR}$]. The part 1 which multiplied by the length *AB*, will describe the area $1 \times AB$ *:* and the other part $\frac{PF}{PR}$ multiplied by the length *PB*, will describe the area 1 by $\overline{PE - AD}$, that which can be easily shown from the quadrature of the curve *LKI* ;



and similarly the same part multiplied by the length *PA* will describe the area 1 by $\overline{PD - AD}$, and multiplied by the difference of *PB* and *PA*, *AB* will describe the difference 1 by $\overline{PE - AD}$ of the areas. From the first content $1 \times AB$ there may be taken away the latter 1 by $\overline{PE - AD}$, and there will remain the area *LABI* equal to 1 by $\overline{AB - PE + PD}$. Therefore the force, proportional to this area, is as $AB - PE + PD$.

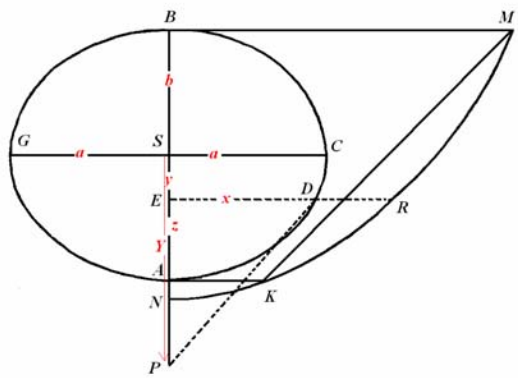
[for in a similar manner to the above : the incremental force *dF* due to the incremental cylinder circle with diameter *RS* and width *dr* from above will be equal to

$2\pi\int_{PF}^{PR} rdr\times\frac{z}{r}\times\frac{1}{r^2}=2\pi PF\int_{PF}^{PR}\frac{dr}{r^2}=2\pi PF\left(\frac{1}{PF}-\frac{1}{PR}\right)$; the sum of the increment forces due to these incremental cylinders will be as

$$2\pi\int_{PA}^{PB} zdz\left(\frac{1}{z}-\frac{1}{\sqrt{a^2+z^2}}\right)=2\pi\left(PB-PA\right)-\left(a^2+PB^2\right)^{\frac{1}{2}}+\left(a^2+PA^2\right)^{\frac{1}{2}}\propto AB-PE+PD\ ]$$

[There follows an extended note to this rather important following result.]

Corol. 2. Hence also the force will become known, by which a spheroid *AGBC* attracts some body *P*, situated on its axis outside *AB* ; *NKRM* shall be the conic section of which



the applied ordinate is *ER*, perpendicular to *PE* itself, that may always be equal to the length *PD*, which is drawn to that point *D*, at which the applied line may cut the spheroid. From the vertices of the spheroid *A* and *B* to its axis *AB*, the perpendiculars *AK* and *BM* may be erected equal to *AP* and *BP* respectively, and therefore crossing the conic section at *K* & *M;* and *KM* may be joined, taking *KMRK* from the same segment. Moreover let the centre of the spheroid be *S* and the greatest radius *SC*: and the force, by which the spheroid attracts the body *P*, will be to the force, by which a sphere with the diameter *AB* described cuts the same body, as $\frac{AS\times CS^2-PS\times KMRK}{PS^2+CS^2-AS^2}$ to $\frac{AS^3}{3PS^2}$. And from the same fundamentals it is possible by computation to find the forces of the spheroidal segment. [The red lines and variables have been added to aid the following integration.]

[We acknowledge the work done by Chandrasekhar in solving this problem in modern terms, which is introduced and paraphrased here somewhat. The problem has also been solved by a Leseur and Janquier, which we subsequently present fully in translation. First of all, the oblate spheroid is an ellipse rotated about the major axis *GC*, while the minor axis is *AB*. Thus, we have the customary implicit equation of the ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, taking *S* as the origin, *SG* as the positive *x*-axis, and *SA* as the positive *y* axis. From

Newton's construction, we have given $PA = AK;\ PE = ER;\ PB = BM$; also,

$ED = x;\ SE = y;\ PS = Y$ and $EP = Y - y$; hence $PD = \sqrt{x^2 + (Y - y)^2}$. The incremental force due to an incremental slice of the oblate spheroid is given by Cor.1 above as $1 - \frac{PF}{PR} \to 1 - \frac{PE}{PD} = dz - \frac{zdz}{\sqrt{x^2+z^2}}$ in this case. The force due to the whole solid of revolution is then, ignoring the 2π factor, is then given by $F = \int_{Y-b}^{Y+b} \left(1 - \frac{z}{\sqrt{x^2+z^2}}\right) dz$; before this integral can be evaluated, we must write *x* in terms of *z* : from $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we have

$\frac{x^2}{a^2} + \frac{(b-z)^2}{b^2} = 1;\ x^2 = a^2 - \frac{a^2(b-z)^2}{b^2}$;

hence $F = \int_{Y-b}^{Y+b} \left(1 - \frac{z}{a\sqrt{\frac{z^2}{a^2} - \frac{(b-z)^2}{b^2}}}\right) dz = \int_{Y-b}^{Y+b} \left(1 - \frac{1}{\sqrt{1 - \frac{a^2(b-z)^2}{b^2 z^2}}}\right) dz = \int_{Y-b}^{Y+b} \left(1 - \frac{1}{\sqrt{1 - \frac{(b-z)^2}{(1-e^2)z^2}}}\right) dz$; recall that the eccentricity is related to *a* and *b* by the equation $b^2 = a^2\left(1 - e^2\right)$, hence

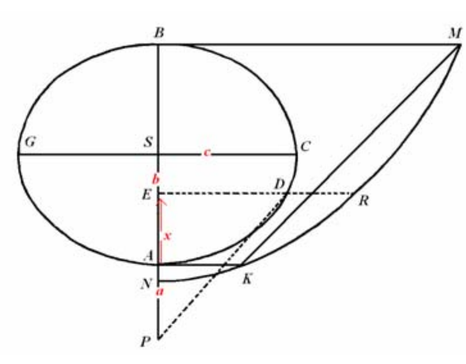
$F = \int_{Y-b}^{Y+b} \left(1 - \frac{1}{\sqrt{1 - \frac{(b-z)^2}{(1-e^2)z^2}}}\right) dz$; Chandrasekhar shows how an integral similar to this one is equivalent to Newton's result, on page 316 of his book, *Newton's Principia for the common reader*. Rather than presenting that derivation here, we will delve into the other presentation by *L* & *S*, which is more relevant to Newton's approach, and although rather long, is presumably similar to the manner in which Newton derived the result. The fact that an elliptic integral corresponding to the area *KRMK* arises, which Newton could not resolve by conventional means, is of some interest.]

ADDITION :

Leseur & Janquier derivation of Newton's result for the oblate spheroid:

[Note 542 part (*x*)] *NKRM shall be the conic section of which the applied ordinate is ER, perpendicular to PE itself, that may always be equal to the length PD, which is drawn to that point D, at which the applied line may cut the spheroid......*

Let $AP = a$, and $AS = b$ shall be the semi-axis of the given curve *ACB*, the rotation of which generates the spheroid, the other semi-diameter $SC = c,\ AE = x$, there will be $PE = a + x$, and from the equation of the ellipse [$\frac{(x-b)^2}{b^2} + \frac{ED^2}{c^2} = 1$],



$ED^2 = \frac{c^2}{b^2} \times \overline{2bx - x^2}$;

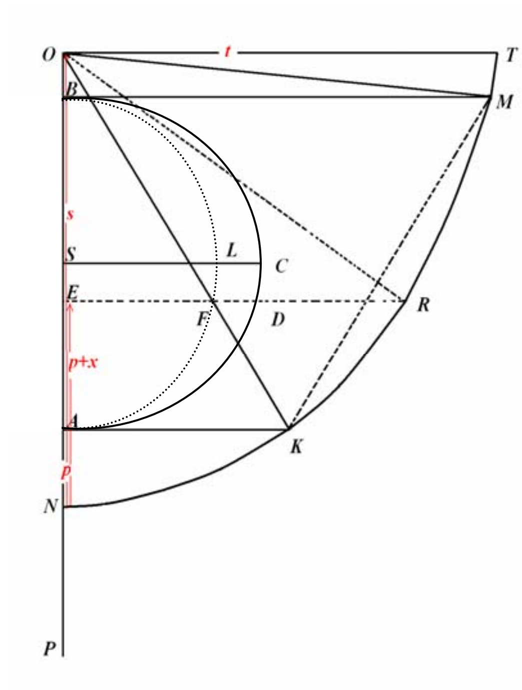
from which the square of the ordinate *ER* [= *PD*] to the curve *NKRM* is given by $PD^2 = PE^2 + ED^2 = a^2 + 2ax + x^2 + \frac{c^2}{b^2} \times 2bx - \frac{c^2}{b^2} x^2$; therefore, since this equation for the curve *NKRM* does not to rise beyond the second degree it is agreed that curve shall be one of the conic sections : moreover it will be an ellipse, if the quantity $x^2 - \frac{c^2}{b^2} x^2$ shall be negative, which comes about when *SC* or *c* is greater than *AS* or *b*; Truly it will be a parabola if that quantity may vanish, and thus if $c = b$, which eventuates when the curve *ACB* is a circle; and then it will be a hyperbola if that quantity is positive, that is, if *AS* is the longer semi-axis.

[Continuing, note 543 :] *ACB* shall be an ellipse the axis *CS* of which shall be greater than the axis *AS*, in which case the curve *NKRM* will be an ellipse [*i.e.* $c > b$ above], and from this ratio, the axes and vertex of this curve *NKRM* will be determined. The semi-axis of this ellipse *NKRM* may be called $ON = s$, and the other semi-axis *OT* may be called *t*, the distance of the vertex *N* from the vertex *A* of the curve *ACB* may be called *p*, the abscissa *NE* will be $= p + x$, and the square of the ordinate *ER* will be from the equation of the ellipse

$\frac{t^2}{s^2} \times \overline{2sp + 2sx - p^2 - 2px - x^2}$,

which from the hypothesis of the construction was found above

$= a^2 + 2ax + x^2 + \frac{c^2}{b^2} \times 2bx - \frac{c^2}{b^2} x^2$.



The homogeneous terms of these values may be brought together, evidently constants with constants, these which include one variable with similar ones, etc., and three equations arise, with the variable omitted:

$$a^2 = \frac{t^2}{s^2} \times \overline{2sp - p^2} \ ; \ a + \frac{c^2}{b} = \frac{t^2}{s^2} \times \overline{s - p} \ ; \ 1 - \frac{c^2}{b^2} = -\frac{t^2}{s^2} .$$

From that third equation, with each sign changed, with the first member reduced to a common denominator, and with the terms inverted, there becomes $\frac{s^2}{t^2} = \frac{b^2}{c^2 - b^2}$ and $s^2 = \frac{b^2 t^2}{c^2 - b^2}$. Then the second equation $a + \frac{c^2}{b} = \frac{t^2}{s^2} \times \overline{s - p}$, with the terms multiplied by $\frac{s^2}{t^2}$, with the reduction made of the first member to the same denominator, and with the substitution made of the above value of $\frac{s^2}{t^2}$ found, will become

$s-p=\frac{b}{c^2-b^2}\times\overline{ba+cc}$. And then, with the terms of the first equation $a^2=\frac{t^2}{s^2}\times\overline{2sp-p^2}$ multiplied $\frac{s^2}{t^2}$, each with the sign changed and with s^2 added, becomes finally $s^2-\frac{b^2}{c^2-b^2}a^2=s^2-2sp+p^2$, in which new equation the second term shall itself be the square of the quantity $s-p$, with the value of this substituted in the first value found, and in place of s^2 in the first term also with the value of this substituted, there becomes $\frac{b^2}{c^2-b^2}\times\overline{t^2-a^2}=\frac{b^2}{\left(c^2-b^2\right)}\times\overline{ba+c^2}$, and with each term divided by $\frac{b^2}{\left(c^2-b^2\right)}$ with a^2 transposed, and by reducing the second term to a common denominator, and with the common terms canceling each other, there shall be $t^2=\frac{c^2}{c^2-b^2}\times\overline{a^2+2ab+c^2}$, or because $PS=a+b$, thus, there becomes $t^2=\frac{c^2}{c^2-b^2}\times\overline{PS^2-b^2+c^2}$, certainly $OT^2=\frac{CS^2}{CS^2-AS^2}\times\overline{PS^2-AS^2+CS^2}$, all of which terms are given, therefore with this found the remaining terms pertaining to the ellipse can be conveniently found.

With regard to the following note, from these we will determine the value of the quantity $\frac{t^2+s^2-PO^2}{t^2}$ that is equal to the quantity $\frac{CS^2}{PS^2-AS^2+CS^2}$, which thus may be put in place with the above values found. From the third equation, there is $s^2=\frac{b^2t^2}{c^2-b^2}$, hence there will be $s^2+t^2=\frac{b^2t^2+c^2t^2-b^2t^2}{c^2-b^2}=\frac{c^2t^2}{c^2-b^2}$, and thus $\frac{s^2+t^2}{t^2}=\frac{c^2}{c^2-b^2}=\frac{CS^2}{CS^2-AS^2}$. Truly there will be $AO=s-p$, and $PO=PA+AO=a+s-p$, and since there shall be $s-p=\left(c^2-b^2\right)\times\left(ba+c^2\right)$, from the second equation, there is $PO=a+\frac{b}{c^2-b^2}\times\overline{ba+c^2}$, with which value reduced to a common denominator, and with terms canceling out, there is $PO=\frac{c^2}{c^2-b^2}\times\overline{a+b}$ or $=\frac{CS^2}{CS^2-AS^2}\times PS$, and since there shall be

$t^2=\frac{CS^2}{CS^2-AS^2}\times\overline{PS^2-AS^2+CS^2}$, there is $\frac{PO}{t^2}=\frac{PS}{PS^2-AS^2+CS^2}$ and

$\frac{PO}{t^2}=\frac{CS^2}{CS^2-AS^2}\times\frac{PS}{PS^2-AS^2+CS^2}$. From which there is finally

$\frac{s^2+t^2-PO^2}{t^2}=\frac{CS^2}{CS^2-AS^2}-\frac{CS^2}{CS^2-AS^2}\times\frac{PS^2}{PS^2-AS^2+CS^2}$, or $=\frac{CS^2}{CS^2-AS^2}\left(1-\frac{PS^2}{PS^2-AS^2+CS^2}\right)$, and on being reduced to the same denominator, with terms canceling,

$=\frac{CS^2}{CS^2-AS^2}\times\frac{-AS^2+CS^2}{PS^2-AS^2+CS^2}=\frac{CS^2}{PS^2-AS^2+CS^2}$ with the numerator and denominator divided by

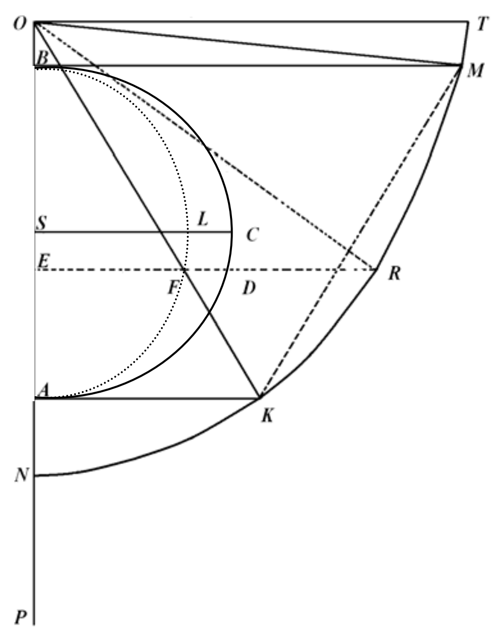
$CS^2 - AS^2$. Therefore there is $\frac{s^2+t^2-PO^2}{t^2} = \frac{CS^2}{PS^2-AS^2+CS^2}$. Q.e.d.

[Continuing, note 544 :] Moreover, let the given curve *ACB* be a circle, thus so the spheroid arising from the rotation of this shall be a sphere, the curve *NKRM* then will be a parabola, for with everything that has been said remaining in place from the previous note, there will be as before, $PE = a + x$, and from the equation of the circle, $EP^2 = 2bx - x^2$, from which there shall be *PF* squared $= PE^2 + EF^2 = a^2 + 2ax + x^2 + 2bx - x^2 = a^2 + 2ax + 2bx$; since therefore the ordinate *ER* to the curve *NKRM* may be taken equal to *PF*, the square of this ordinate will be equal to the abscissa itself multiplied by a constant quantity, but not increasing beyond the first order, which is a property of the parabola. Therefore the latus rectum of this parabola may be called 1, the distance of the vertex *N* from the vertex *A* of the curve *ACB* may be called *p*, the abscissa *NE* will be $p + x$, and from the equation of the parabola, the square of the ordinate $ER = 1p + 1x$ this value may be united with the value of the same ER^2 found above, $a^2 + 2ax + 2bx$, the constant terms with constants, and those which include the variable with similar ones, become two equations $1p = a^2$, and $1 = 2a + 2b = 2PS$, and thus $p = \frac{a^2}{2a+2b} = \frac{PA^2}{2PS}$; and since from the equation of the parabola , there shall be $ER^2 = 1 \times \overline{p + x}$, there will be $p + x = NE = \frac{ER^2}{2PS}$; and since the parabolic area between the abscissa, the ordinate, and the curve intercepted shall be equal to two thirds of the rectangle of the abscissa by the ordinates, the area of the parabola $NER = \frac{\frac{2}{3}ER^3}{2PS} = \frac{ER^3}{3PS}$, and because, from the construction, the ordinates erected at *A* and *B* shall equal *PA* and *PB*, the parabolic area will be $NEK = \frac{PA^3}{3PS}$, and the parabolic area $NBM = \frac{PB^3}{3PS} = \frac{(PA+2AS)^3}{3PS}$, and the difference of these areas *AKRMB* corresponding to the axis of the sphere *AB*, will be $\frac{6PA^2 \times AS + 12PA \times AS^2 + 8AS^3}{3PS}$, and then with the trapezium *AKMB* removed, the segment of the remaining parabolic segment will be equal to $\frac{2PA^3}{3PS}$, for the trapezium *AKMB* is equal to $\frac{1}{2}AB \times \overline{AK + BM}$ or, since (because $\frac{1}{2}AB = AS$, $AK = PA$, and $BM = PB = PA + 2AS$) it is equal to $2AS \times PA + 2AS^2$, and by reduction to the denominator *3PS* or $3PA + 3AS$ it equals $\frac{6AS \times PA^2 + 12AS^2 \times PA + 6AS^3}{3PS}$, that taken from the area $AKRMB = \frac{6PA^2 \times AS + 12PA \times AS^2 + 8AS^3}{3PS}$ leaves $\frac{2AS^3}{3PS}$. Q.e.d.

[Continuing, note 545 :] *The force, by which the spheroid attracts the body P, will be to the force, by which a sphere with the diameter AB described cuts the same body, as* $\frac{AS \times CS^2 - PS \times KMRK}{PS^2 + CS^2 - AS^2}$ *to* $\frac{AS^3}{3PS^2}$.

Just as the solution of this problem may be put in place, the second curve described *AB*, the ordinates of which for the single point *E* shall be equal to the force by which the body *P* is attracted by a circle the radius of which is *ED*; that force is, by Cor. 1. Prop XC, as $1-\frac{PE}{PD}$, *OE* shall be the abscissa of this curve assumed from the point *O* (with the centre of the curve *NKRM* just noted 543 determined) and it may be called z, the fluxion [*i.e.* differential] of this will be *dz*, and thus the differential of the area of the curve which shows the force of the spheroid will be $dz-\frac{PE}{PD}dz$, and since there shall be $PE=PO-OE=PO-z$ and $PD=ER$, the ordinates of the curve *NKRM*, from the construction, and there shall be, as is easily deduced from note 543, $ER=\frac{t}{s}\sqrt{s^2-z^2}$, the differential of this area shall be $dz-\frac{POdz}{\frac{t}{s}\sqrt{s^2-z^2}}+\frac{zdz}{\frac{t}{s}\sqrt{s^2-z^2}}$. Of the positive terms $dz+\frac{zdz}{\frac{t}{s}\sqrt{s^2-z^2}}$ the fluent [i.e. the indefinite integral] is $z+\frac{s}{t}\sqrt{s^2-z^2}$ but as $z=OE$ and $\frac{s}{t}\sqrt{s^2-z^2}=\frac{s^2}{t^2}\times\frac{t}{s}\sqrt{s^2-z^2}=\frac{s^2}{t^2}ER$, the corresponding differential of the positive term is $OE-\frac{s^2}{t^2}ER$, and the total area of the corresponding line *OA* is $OA-\frac{s^2}{t^2}AK$, from which the corresponding second area of the part *OB* required to be taken away, as the curve which the force of the spheroid is not lead to express, and which is $OB-\frac{s^2}{t^2}BM$, and as by the construction $AK=AP$, and $BM=PB=BA+AP$, truly there will be the integral $OA-OB-\frac{s^2}{t^2}\times\overline{AP-BA-AP}=AB+\frac{s^2}{t^2}AB=AB\times\frac{s^2+t^2}{t^2}$.

The integral of the third term $\frac{POdz}{\frac{t}{s}\sqrt{s^2-z^2}}$ is found thus; the differential of the elliptic sector of the ellipse *TOK* $\frac{\frac{1}{2}stdz}{\sqrt{s^2-z^2}}$ is multiplied by $\frac{2PO}{t^2}$ and the proposed term arises $\frac{POdz}{\frac{t}{s}\sqrt{s^2-z^2}}$, from which the integral of the term proposed is that elliptic sector *TOK* multiplied by $\frac{2PO}{t^2}$, but because the area sought does not correspond to the whole area *OA*, but rather to the part *AB* of this, indeed the integral of the area sought found from the third term is the sector $TOK\times\frac{2PO}{t^2}$ with the sector $TOM\times\frac{2PO}{t^2}$ removed, or the sector $MOK\times\frac{2PO}{t^2}$. But the sector

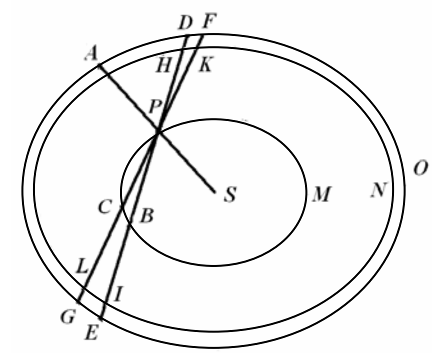


MOK is divided into the rectilinear figure *MOK* and the curved figure *MRK*; the triangle *MOK* becomes $\frac{1}{2}PO\times AB$, for the right line MK may be produced and will extend to P, because $PA = AK$ and $PB = BM$, the whole triangle certainly $OMP = \frac{1}{2}OP \times BM = \frac{1}{2}OP \times PB$, and the triangle $OKP = \frac{1}{2}OP \times AK = \frac{1}{2}OP \times AP$, from which with triangle OKP subtracted from triangle OMP, there remains the triangle $OMK = \frac{1}{2}OP \times \left(PB - AP\right) = \frac{1}{2}OP \times AB$. From which the integral sought of this third term is $\frac{2PO}{t^2} \times \frac{1}{2}OP \times AB + \frac{2PO}{t^2} \times MRK = \frac{PO^2}{t^2} \times AB + \frac{2PO}{t^2} \times MRK$, which taken from the integral of the positive terms $AB \times \frac{s^2+t^2}{t^2}$ becomes $AB \times \frac{s^2+t^2-PO^2}{t^2} - \frac{2PO}{t^2} \times MRK$, since there shall be therefore $\frac{s^2+t^2-PO^2}{t^2} = \frac{CS^2}{PS^2-AS^2+CS^2}$ and $\frac{PO}{t^2} = \frac{PS}{PS^2-AS^2+CS^2}$ by note 543, because $AB = 2AS$, the integral sought is $\frac{2AS \times CS^2 - 2PS \times MRK}{PS^2-AS^2+CS^2}$.

But if the curve *ACB* shall be a circle, truly the spheroid becomes a sphere, making $CS = AS$ and the segment MRK becomes $\frac{2AS^3}{3PS^2}$ and thus that formula is changed into this $\frac{2AS \times AS^2 - \frac{2PS \times 2AS^3}{3PS}}{PS^2-AS^2+AS^2} = \frac{2AS^3 - \frac{4}{3}AS^3}{PS^2} = \frac{2AS^3}{3PS^2}$, which expresses the force of the sphere; and thus with the expression of the force of the spheroid and the force of the sphere divided by the common multiple 2; *the force of the spheroid to the force of the sphere shall be as* $\frac{AS \times CS^2 - PS \times MRK}{PS^2-AS^2+CS^2}$ *to* $\frac{AS^3}{3PS^2}$. Q.e.d.

It is also possible to determine the force of the sphere, by this calculation, as first there shall be $PA = a, AB = 2b,$ the abscissa $AE = x, PF = v,$ and there will be $PE^2 = a^2 + 2ax + x^2$, and $EF^2 = 2bx - x^2$, from the equation of the circle, and thus PF^2 or $v^2 = a^2 + 2ax + 2bx$, from which there is found $x = \frac{v^2-a^2}{2\times(a+b)}$ and $dx = \frac{2vdv}{2\times(a+b)} = \frac{vdv}{a+b}$, and $PE = \frac{a^2+2ab+v^2}{2\times(a+b)}$ and $\frac{dx}{PF} = \frac{dv}{a+b}$. And thus, with the derivative of the area expressed, the force of the sphere shall be by Cor. I Prop. XC, as $dx - \frac{PEdx}{PF}$, this differential [or fluxion] will be as $dx - \frac{a^2+2ab+v^2}{2\times(a+b)^2}dv$, of which the integral [or fluent] is $x - \frac{a^2v+2abv+\frac{1}{3}v^3}{2\times(a+b)^2} + Q$, where Q is a constant, which must vanish on putting $x = 0$ and $x = a$, and thus $-\frac{a^3+2a^2b+\frac{1}{3}a^3}{2\times(a+b)^2} + Q = 0,$ and $Q = \frac{\frac{4}{3}a^3+2a^2b}{2\times(a+b)^2}$; but the force of the whole sphere may be obtained if there becomes $x = AB$ or $2b$ and $v = PB$ or $a + 2b$, and thus there is $2b + \frac{\frac{4}{3}a^3+2a^2b-a^3-4a^2b-\frac{1}{3}a^3-2a^2b-4ab^2-\frac{8}{3}b^3}{2\times(a+b)^2}$ which reduces to $2b \times \frac{\frac{2}{3}b^2}{2\times(a+b)^2}$, or on putting *AS* for *b* and *PS* for $a + b$, the total force of the sphere is $\frac{2AS^3}{3PS^2}$. Q.e.i.

Corol. 3. But if the corpuscle may be placed within the spheroid on the axis, the attraction will be as its distance from the centre. This may be easily proven by this argument, whether the particle shall be on the axis, or on some other given diameter. Let *AGOF* be the attracting spheroid, *S* its centre, and *P* the attracted body. Through that body *P* there are drawn both a semi-diameter *SPA,* as well as any two right lines *DE* and *FG,* hence crossing the spheroid at *D* and *E, F* and G; *PCM* and *HLN* shall be the surfaces of two interior spheroids, similar and concentric with the exterior, the first of which may pass through the body *P*, and may cut the right lines *DE* and *FG* in *B* and *C*, the latter may cut the same right lines in *H* and *I* and in *K* and *L*. But all the spheroids have a common axis, and hence the parts of the right lines thus intersected are mutually equal to each other : *DP* and *BE, FP* and *CG, DH* and *IE, FK* and *LG* ; therefore as the right lines *DE*, *PB* and *HI* may be bisected at the same point, as well as the right lines *FG, PC* and *K L*. Now consider *DPF* and *EPG* to designate opposing cones, described with infinitely small vertical angles *DPF* and *EPG*, and the lines *DH* and *EI* to be infinitely small also; and the small parts cut of the spheroidal surfaces *DHKF* and *GLIE,* on account of the equality of the lines *DH* and *EI,* will be in turn as the squares of their distances from the corpuscle *P*, and therefore attract that corpuscle equally. And by the same reason, if the spaces *DPF* and *EGCB* may be divided into small parts by the surfaces of innumerable similar concentric spheroids having a common axis, all these each attract the body *P* equally on both sides in the opposite directions. Therefore the forces of the cone *DPF* and of the segment of the cone *EGCB* are equal, and by opposing each other cancel out. And the ratio of all the forces of the material beyond the inner spheroid *PCBM.* Therefore the body *P* is attracted only by the inner spheroid *PCBM,* and therefore (by Corol.3. Prop. LXXII.) the attraction of this is as the force, by which the body *A* is attracted by the whole spheroid *AGOD,* as the distance *PS* to the distance *AS. Q.E.D.*



PROPOSITION XCII. PROBLEM XLVI.

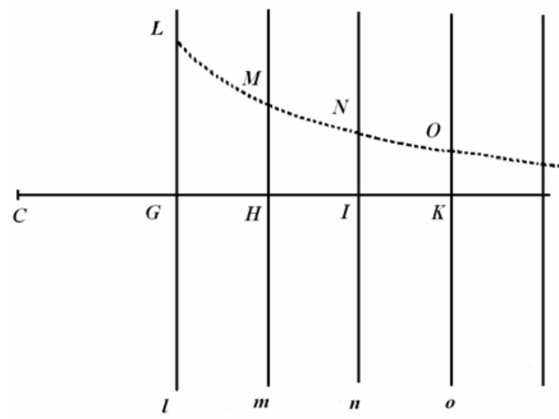
For a given attracted body, to find the ratio of the decrease of the attraction of the centripetal forces at the individual points of this.

From some given body either a sphere, a cylinder, or some other regular body is required to be formed, of which the law of attraction can be found, for any decrease of the ratio you please (by Prop. LXX, LXXXI, & XCI.). Then from the experiment performed the force of attraction at different distances can be found, and the law of attraction thence revealed for the whole body will give the ratio of the decrease of the forces of the individual parts, as was required to be found.

[Such experiments were eventually performed by Henry Cavendish a century later.]

PROPOSITI XCIII. THEOREM XLVII.

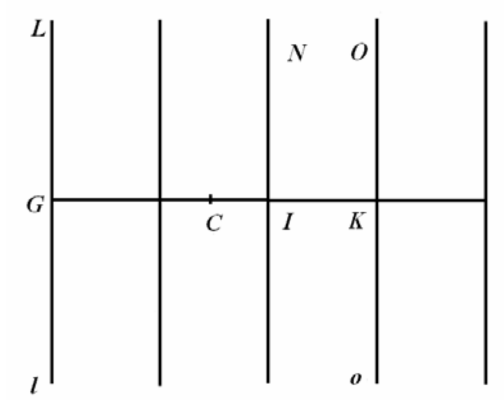
If a solid may consist of a single plane surface [part], but with the remaining parts from an infinitude of planes, with equal particles attracting equally, the forces of which in the depths of the solid decrease in some ratio of the power greater than the square, and a corpuscle may be attracted by the force of the whole solid, by each of the plane parts put in place : I say that the attractive force of the solid, receding away from the plane surface of this, decreases in the ratio of some power, the root [i.e. base] of this power is the distance of the corpuscle from the plane, and the index by three less than the index of the power of the distances.



Case 1. Let *LGl* be the plane in which the solid is terminated. But the solid may be put in place from plane parts of this towards *I*, and resolved into innumerable planes *mHM, nIN, oKO*, &c. parallel to *GL* itself. And in the first place the attracted body may be put in place at *C* outside the solid. But *CGHI* may be drawn from these innumerable perpendicular planes, and the attractive forces of the points of the solid may decrease in a ratio of the powers of the distances, the index of which shall be the number *n* not less than 3. Hence (by Corol. 3. Prop. XC.) the force, by which some plane *mHM* attracts the point *C*, is inversely as CH^{n-3}. In the plane *mHM*, the length *HM* may itself be taken reciprocally proportional of CH^{n-2}, and that force will be as *HM*. Similarly in the individual planes *lGL, nIN, oKO,* &c. the lengths may be taken *GL, IN, KO,* &c. reciprocally proportional to $CG^{n-2}, CI^{n-2}, CK^{n-2}$, &c.; and the forces of the same planes will be as the lengths taken, and thus the sum of the forces as the sum of the lengths, that is, the whole force is as the area *GLOK* produced towards *OK* indefinitely. But that area (by the known methods of quadratures) is reciprocally as CG^{n-3}, and therefore the force of the whole solid is inversely as CG^{n-3}. *Q. E. D.*

[Note from L & J : Let $CH = x,$ then *MH* will be as $\frac{1}{x^{n-2}}$ by hypothesis, and the element of the area *GLMH* – corresponding to an element of the force- will be as $\frac{dx}{x^{n-2}}$, and thus the area itself will be as $Q - \frac{1}{(n-3)x^{n-3}}$, for some constant Q, which vanishes when $x = CG$. Whereby $Q = \frac{1}{(n-3)CG^{n-3}}$ and the area *GLMH*, becomes as $\frac{1}{(n-3)CG^{n-3}} - \frac{1}{(n-3)CH^{n-3}}$. But since *CH* becomes infinite, the term $\frac{1}{(n-3)CH^{n-3}}$ vanishes and the area *GLOK* becomes infinite as $\frac{1}{(n-3)CG^{n-3}}$, or on account of the given $n-3$, inversely as CG^{n-3}.]

Case 2. Now the corpuscle *C* may be placed in that part of the plane *l*G*L* within the solid, and take the distance *CK* equal to the distance *CG.* And the part of the solid *LGloKO,* terminated by the parallel planes *lGL*, *oKO,* the corpuscle *C* situated in the middle will not be attracted by any part, with the counter actions of the opposing points mutually being removed from the equality. On that account the corpuscle *C* is attracted only by the force of the solid situated beyond the plane *OK.* But this force (by the first case) is inversely as CK^{n-3}, that is (on account of the equality of *CG* and *CK)* inversely as CG^{n-3}. *Q. E. D.*



Corol. I. Hence if the solid *LGIN* may be bounded by the two infinite parallel planes *LG* and *IN* on each side ; the attractive force of is known, by taking away from the attractive force of the whole infinite solid *LGKO*, the attractive force of the part beyond *NIKO,* produced from *KO* indefinitely.

Corol. 2. If the ulterior part of this infinite solid, when the attraction of this taken with the attraction of the nearer part is hardly of any concern, it may be rejected : the attraction of that closer part may decrease in increasing the distance approximately in the ratio of the power CG^{n-3}.

Corol. 3. And hence if some finite body, with a single plane part, may attract a corpuscle from the middle region of this plane, and the distances between the corpuscle collated with the dimensions of the attracting body shall be extremely small, it may be agreed moreover that the body be attracting with homogeneous particles, the attractive forces of which decrease in some ratio of the power greater than the square of the distances; the attractive force of the whole body will decrease approximately in the ratio of the powers, the base of which shall be that very small distance, and the index by three less than the index of the former power. Concerning a body constructed from particles, the attractive forces of which may decrease in the triplicate ratio of the distances, the assertion is not valid; because hence, in that case, the attraction of these further parts of the infinite body in the second corollary, is always infinitely greater than the attraction of the closer parts.

Scholium.

If some body may be attracted perpendicularly towards a given plane, and the motion of the body may be sought from some given law : the problem may be solved by seeking (by Prop. XXXIX) the right motion of the body descending to this plane, and (by Corol. 2 of the laws) compounding with that uniform motion, following lines made parallel to the same plane. And conversely, if the law of attraction of the body towards the plane along lines made perpendicular may be sought, so that the attracted body may be moving along some given curve from that condition, the problem may be solved by working according to the example of the third problem.

Moreover the operations are accustomed to be drawn together by resolving the applied ordinates in converging series. Just as if to the base A [*i.e.* the x coordinate] the applied ordinate may be given a length B at some given angle, which shall be as some power of the base $A^{\frac{m}{n}}$ [i.e. $B = A^{\frac{m}{n}}$ or $A = B^{\frac{n}{m}}$]; and the force by which the body shall be moving along some curved line, following the position of the applied ordinate, either attracted to or repelled from the plane base, so that it will always touch the upper end of the applied ordinate [*i.e.* the y coordinate]: I may suppose the base to be increased from the minimum O, and the applied ordinate $\left(A+O\right)^{\frac{m}{n}}$ to be resolved into an infinite series $A^{\frac{m}{n}} + \frac{m}{n}OA^{\frac{m-n}{n}} + \frac{mm-mn}{2nn}OOA^{\frac{m-2n}{n}}$ *&c.* and the term of this, in which O is of two dimensions, I suppose to be proportional to the force, that is, the term $\frac{mm-mn}{2nn}OOA^{\frac{m-2n}{n}}$.

Therefore the force sought is as $\frac{mm-mn}{nn}A^{\frac{m-2n}{n}}$, or what is moreover, as $\frac{mm-mn}{nn}B^{\frac{m-2n}{m}}$. So that if the applied ordinate may touch a parabola, with $m = 2,$ and $n = 1$ arising: the force becomes as given $2B^0$, and thus may be given [as constant]. Therefore with a given force a body will be moving in a parabola, just as *Galilio* showed. But if the applied ordinate may touch a hyperbola, with $m = 0-1,$ and $n = 1$ arising ; the force becomes as $2A^{-3}$ or $2B^3$: and thus the force, which shall be as the cube of the applied ordinate, will cause the body to move in a hyperbola. But with propositions of this kind dismissed, I go on to certain other kinds of motion, which I shall only touch on.

SECTIO XIII.
De corpora non sphaericorum viribus attractivis.

PROPOSITIO LXXXV. THEOREMA XLII.

Si corporis attracti, ubi attrahenti contiguum est, attractio longe fortior sit, quam cum vel minimo intervallo separantur ab invicem: vires particularum trahentis, in recessu corporis attracti, decrescunt in ratione plusquam duplicata distantiarum a particulis.

Nam si vires decrescunt in ratione duplicata distantiarum a particulis; attractio versus corpus sphaericum, propterea quod (per Prop. LXXIV.) sit reciproce ut quadratum distantiae attracti corporis a centro sphaerae, haud sensibiliter augebitur ex contactu; atque adhuc minus augebitur ex contactu, si attractio in recessu corporis attracti decrescat in ratione minore. Patet igitur propositio de sphaeris attractivis. Et par est ratio orbium sphaericorum concavorum corpora externa trahentium. Et multo magis res constat in orbibus corpora interius constituta trahentibus, cum attractiones passim per orbium cavitates ab attractionibus contrariis (per Prop. LXX) tollantur, ideoque vel in ipso contactu nullae sunt . Quod si sphaeris hisce orbibusquc sphaericis partes quaelibet a loco contactus remotae auferantur, & partes novae ubivis addantur: mutari possint figurae horum corporum attractivorum pro lubitu, nec tamen partes additae vel subductae, cum sint a loco contactus remotae, augebunt notabiliter attractionis excessum, qui ex contactu oritur. Constat igitur propositio de corporibus figurarum omnium. *Q. E. D.*

PROPOSITIO LXXXVI. THEOREMA XLIII.

Si particularum, ex quibus corpus attractivum componitur, vires in recessu corporis attracti decrescunt in triplicata vel plusquam triplicata ratione distantiarum a particulis : : attractio longe fortior erit in contactu, quam cum attrahens & attractum intervallo vel minimo separantur ab invicem.

Nam attractionem in accessu attracti corpusculi ad hujusmodi sphaeram trahentem augeri in infinitum, constat per solutionem Problematis XLI in exemplo secundo ac tertio exhibitam. Idem, per exempla illa & Theorema XLI. inter se collata, facile colligitur de attractionibus corporum versus orbes concavo-convexos, sive corpora attracta collocentur extra orbes, sive intra in eorum cavitatibus. Sed & addendo vel auferendo his sphaeris & orbibus ubivis extra locum contactus materiam quamlibet attractivam, eo ut corpora attractiva induant figuram quamvis assignatam, constabit propositio de corporibus universis. *Q. E. D.*

PROPOSITIO LXXXVII. THEOREMA XLIV.

Si corpora duo sibi invicem similia, & *ex materia aequaliter attractiva constantia, seorsim attrahant corpuscula sibi ipsis proportionalia* & *ad se similiter posita: attractiones acceleratrices corpusculorum in corpora tota erunt ut attractiones acceleratrices corpusculorum eorum particulas totis proportionales,* & *in totis similiter positas.*

Nam si corpora distinguantur in particulas, quae sint totis proportionales, & in totis similiter sitae; erit, ut attractio in particulam quamlibet unius corporis ad attractionem in particulam correspondentem in corpore altero, ita attractiones in particulas singulas primi corporis ad attractiones in alterius particulas singulas correspondentes; & componendo, ita attractio in totum primum corpus ad attractionem in totum secundum.
Q. E. D.

Corol. 1. Ergo si vires attractivae particularum, augendo distantias corpusculorum attractorum, decrescant in ratione dignitatis cujusvis distantiarum; attractiones acceleratrices in corpora tota erunt ut corpora directe, & distantiarum dignitates illae. inverse. Ut si vires particularum decrescant in ratione duplicata distantiarum a corpusculis attractis, corpora autem sint ut *A cub.* & *B cub.* ideoque tum corporum latera cubica, tum corpusculorum attractorum distantiae a corporibus, ut *A* & *B:* attractiones acceleratrices in corpora erunt ut $\frac{A\,cub.}{A\,quad.}\,\&\,\frac{B\,cub.}{B\,quad.}$, id est, ut corporum latera illa cubica *A* & *B.* Si vires particularum decrescant in ratione triplicata distantiarurn a corpusculis attractis; attractiones acceleratrices in crorpora tota erunt ut $\frac{A\,cub.}{A\,cub.}\,\&\,\frac{B\,cub.}{B\,cub.}$, id est, aequales. Si vires decrescant in ratione quadruplicata; attractiones in corpora erunt ut $\frac{A\,cub.}{A\,qq.}\,\&\,\frac{B\,cub.}{B\,qq.}$, id est, reciproce ut latera cubica *A* & *B.* Et sic in caeteris.

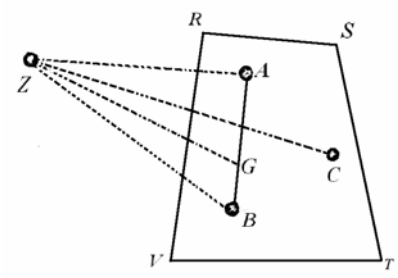
*Corol.*2. Unde vicissim, ex viribus, quibus corpora similia trahunt corpuscula ad se similiter posita, colligi potest ratio decrementi virium particularum attractivarum in recessu corpusculi attracti; si modo decrementum illud sit directe vel inverse in ratione aliqua distantiarum.

PROPOSITIO LXXXVIlI. THEOREMA XLV.

Si particularum aequalium corporis cuiusquncue vires attractivae ,sint ut distantiae locorum a particulis: vis corporis totius tendet ad ipsius centrum gravitatis; & eadem erit cum vi globi ex materia consimili & aequali constantis, & centrum habentis in eius centro gravitatis.

Corporis *RSTV* particulae *A*, *B* trahant corpusculum aliquod *Z* viribus, quae, si particulae aequantur inter se, sint ut distantiae *AZ, BZ;* sin particulae statuantur inaequales, sint ut hae particulae & ipsarum distantiae *AZ, BZ* conjunctim, sive (si ita loquar) ut hae particulae in distantias suas *AZ, BZ* respective ductae. Et exponantur haes vires per contenta illas $A\times AZ\ \&\ B\times BZ$. Iungatur *AB &* secetur ea in *G* ut sit *AG* ad *BG* ut particula *B* ad particulam *A;*& erit *G* commune centrum gravitatis particularum *A* & *B* . Vis $A\times AZ$ (per legum corol. 2.) resolvitur in vires $A\times GZ\ \&\ A\times AG$

& vis $B \times BZ$ in vires $B \times GZ$ & $B \times BG$. Vires autem $A \times AG$ & $B \times BG$, ob proportionales *A* ad *B* & *BG* ad *AG*, aequantur; ideoquae cum dirigantur in partes contrarias, se mutuo destruunt. Restant vires $A \times GZ$ & $B \times Gz$. Tendunt hae ab *Z* versus centrum *G*, & vim $\overline{A+B} \times GZ$ componunt ; hoc est, vim eandem ac si particulae attractivae *A* & *B* consisterent in eorum communi gravitatis centro *G*, globum ibi componentes.



Eodem argumento, si adjungatur particula tertia ac & componatur hujus vis cum vi $\overline{A+B} \times GZ$ tendente ad centrum *G*; vis inde oriunda tendet ad commune centrum gravitatis globi illius in *G* & particulae *C;* hoc est, ad commune centrum gravitatis trium particularum *A, B, C;* & eadem erit, ac si globus & particula *C* consisterent in centro illo communi, globum majorem ibi componentes. Et sic pergitur in infinitum. Eadem est igitur vis tota particularum omnium corporis cujuscunque *RSTV,* ac si corpus illud, servato gravitatis centro, figuram gIobi indueret. *Q. E. D.*

Corol. Hinc motus corporis attracti *Z* idem erit, ac si corpus attrahens *RSTV* esset sphaericum: & propterea si corpus illud attrahens vel quiescat, vel progrediatur uniformiter in directum; corpus attractum movebitur in ellipsi centrum habente in attrahentis centro gravitatis.

PROPOSITIO LXXXIX. THEOREMA XLVI.

Si corpora sint plura ex particulis aequalibus constantia, quarum vires sint ut distantiae locorum a singulis: vis ex omnium viribus composita, qua corpusculum quodcunqne trahitur, tendet ad trahentium commune centrum gravitatis ; & *eadem erit, ac si trahentia illa, servato gravitatis centro communi , coirent & in globum formarentur.*

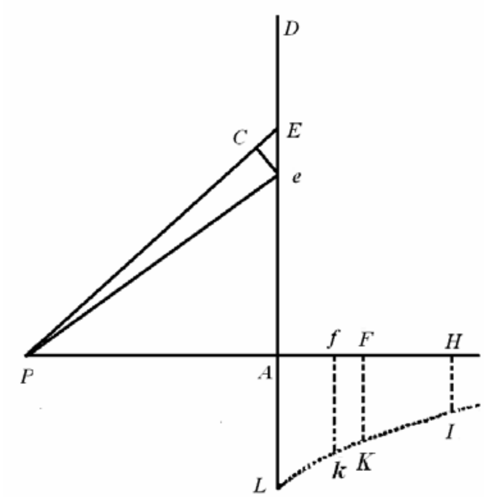
Demonstratur eodem modo, atque propositio superior.

Corol. Ergo motus corporis attracti idem erit, ac si corpora trahentia, servato communi gravitatis centro, coirent & in globum formarentur. Ideoque si corporum trahentium commune gravitatis centrum vel quiescit, vel progreditur uniformiter in linea recta; corpus attractum movebitur in ellipsi, centrum habente in communi illo trahentium centro gravitatis.

PROPOSITIO XC. PROBLEMA XLIV.

Si ad singula circuli cuiuscunque puncta tendant vires aequales centripetae, crescentes vel decrescentes in quacunque distantiarum ratione: invenire vim, qua corpusculum attrahitur ubivis positum in recta, quae plano circuli ad centrum eius perpendiculariter insistit.

Centro *A* intervallo quovis *AD*, in plano, cui recta *AP* perpendicularis est, describi intelligatur circulus; & invenienda sit vis, qua corpusculum quodvis *P* in eundem attrahitur. A circuli puncto quovis *E* ad corpusculum attractum *P* agatur recta *PE.* In recta *PA* capiatur *PF* ipsi *PE* aequalis, & erigatur normalis *FK,* quae sit ut vis qua punctum *E* trahit corpusculum *P.* Sitque *IKL* curva linea quam punctum *K* perpetuo tangit. Occurrat eadem circuli plano in *L.* In *PA* capiatur *PH* aequalis *PD,* & erigatur perpendiculum *HI* curvae praedictae occurrens in *I;* & erit corpusculi *P* attractio in circulum ut area *AHIL* ducta in altitudinem *AP, Q. E. I.*



Etenim in *AE* capiatur linea quam minima *Ee.* Jungatur *Pe,* & in *PE, PA* capiantur *PC, Pf* ipsi *Pe* aequales, Et quoniam vis, qua annuli centro *A* intervallo *AE* in plano praedicto descripti punctum quodvis *E* trahit ad se corpus *P,* ponitur esse ut *FK,* & inde vis, qua punctum illud trahit corpus *P* versus *A*, est ut $\frac{AP\times FK}{PE}$ & vis, qua annulus totus trahit corpus *P* versus *A* ,ut annulus & $\frac{AP\times FK}{PE}$ conjuntim ; annulus autem iste est ut rectangulum sub radio *AE* & latitudine *Ee,* & hoc rectangulum (ob proportionales *PE* & *AE, Ee* & *CE)* aequatur rectangulo $PE\times CE$ seu $PE\times Ff$ *;* erit vis, qua annulus iste trahit corpus *P* versus *A*, ut $PE\times Ff$ *&* $\frac{AP\times FK}{PF}$ conjunctim, id est, ut contentum $Ff\times FK\times AP$, sive ut area *FKkf* ducta in *AP*. Et propterea summa virium, quibus annuli omnes in circulo, qui centro *A* & interval *AD* describitur, trahunt corpus *P* versus *A,* est ut area tota *AHIKL* ducta in *AP. Q.E.D.*

Corol. 1. Hinc si vires punctorum decrescunt in duplicata distantiarum ratione, hoc est, si sit *FK* ut $\frac{1}{PF\ quad}$,atque ideo area *AHIKL* ut $\frac{1}{PA}-\frac{1}{PH}$; erit attractio corpusculi *P* in circulum ut $1-\frac{PA}{PH}$, id est, ut $\frac{AH}{PH}$.

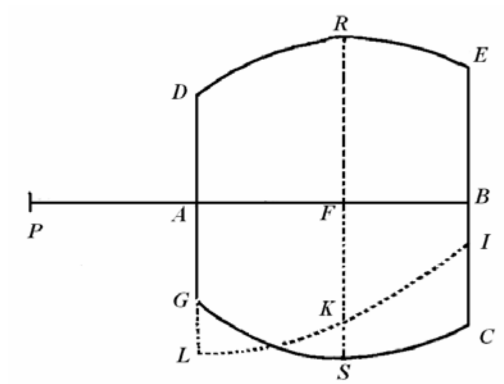
Corol.2. Et universaliter, si vires punctorum ad distantias *D* sint reciproce ut distantiarum dignitas quaelibet D^n , hoc est. si sit *FK* ut $\frac{1}{D^n}$, ideoque *AHIKL* ut $\frac{1}{PA^{n-1}}-\frac{1}{PH^{n-1}}$; erit attractio corpusculi P in circulum ut $\frac{1}{PA^{n-2}}-\frac{1}{PH^{n-2}}$

Corol. 3. Et si diameter circuli augeatur in infinitum,. & numerus n sit unitate major; attractio corpusculi P in planum totum infinitum erit reciproce ut PA^{n-2}, propterea quod terminus alter PA evanescet.

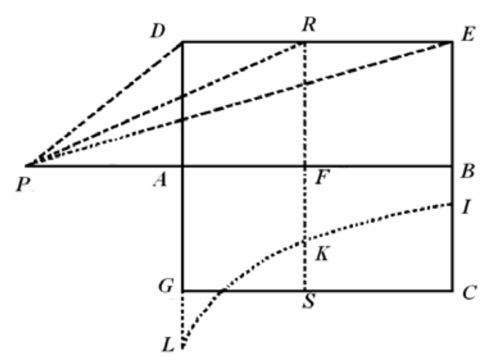
PROPOSITIO XCI. PROBLEMA XLV.

Invenire attractionem corpusculi siti in axe solidi rotundi, ad cujus puncta singula tendunt vires aequales centripetae in quacunque distantiarum ratione decrescentes.

In solidum *DECG* trahatur corpusculum *P*, situm in eius axe *AB.* Circulo quolibet *RFS* ad hunc axem perpendiculari secetur hoc solidum, & in eius semidiametro *FS*,in plano aliquo *PALKB* per axem transeunte, capiatur (per Prop. XC.) longitudo *FK* vi, qua corpusculum P in circulum illum attrahitur, proportionalis. Tangat autem punctum *K* curvam lineam *LKI*, planis extimorum circulorum *AL* & *BI* occurrentem in *L* & *I* ; & erit attractio corpusculi *P* in solidum ut area *LABI. Q.E.I.*

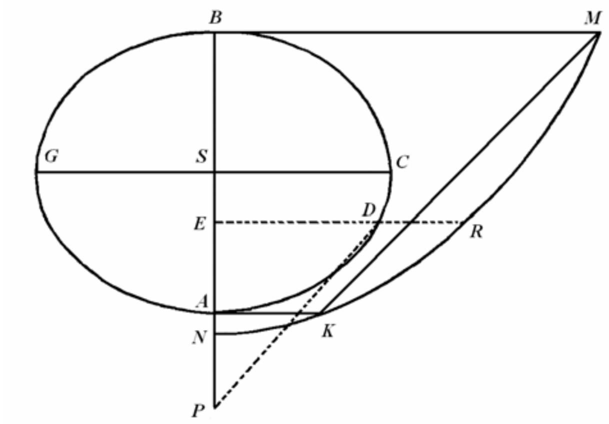


Corol. 1. Unde si solidum cylindrus sit, parallelogrammo *ADEB* circa axem *AB* revoluto descriptus, & vires centripetae in singula eius puncta tendentes sint reciproce ut quadrata distantiarum a punctis: erit attractio corpusculi *P* in hunc cylyndrum ut $AB - PE + PD$. Nam ordinatim applicata *FK* (per corol. I. Prop. XC.) erit ut $1-\frac{PF}{PR}$. Huius pars 1 ducta in longitudinem *AB*, describit aream $1\times AB$: & pars altera $\frac{PF}{PR}$ ducta in longitudinem *PB*, describit aream 1 in $\overline{PE-AD}$, id quod ex curvae *LKI* quadratura facile ostendi potest ; & similiter pars eadem ductai in longitudinem *PA* describit aream I in $\overline{PD-AD}$, ductaque in ipsarum *PB, PA* differentiam *AB* describit arearum differentiam 1 in $\overline{PE-AD}$. De contento primo $1\times AB$ auferatur contentum postremum 1 in $\overline{PE-AD}$, & restabit area *LABI* aequalis 1 in $\overline{AB-PE+PD}$. Ergo vis, huic areae proportionalis, est ut $AB - PE + PD$.



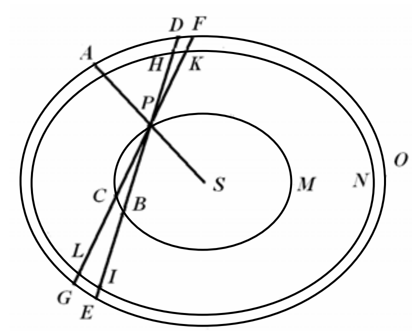
Corol. 2. Hinc etiam vis innotescit, qua sphaerois *AGBC* attrahit corpus quodvis *P,* exterius in axe suo *AB* situm. Sit *NKRM* sectio conica cuius ordinatim applicata *ER*, ipsi *PE* perpendicularis, aequetur semper longitudini *PD,* quae ducitur ad punctum illud

D, in quo applicata ista sphaeroidem secat. A sphaeroidis verticibus *A*, *B* ad eius axem *AB*



erigantur perpendicula *AK*, *BM* ipsis *AP*, *BP* aequalia respective, & propterea sectioni conicae occurrentia in *K* & *M;* & jungatur *KM* auferens ab eadem segmentum *KMRK.* Sit autem sphaeroidis centrum *S* & semidiameter maxima *SC:* & vis, qua sphaerois trahit corpus *P*, erit ad vim, qua sphaera diametro *AB* descripta trahit idem corpus, ut $\frac{AS\times CSq-PS\times KMRK}{PSq+CSq-ASq}$ ad $\frac{AS\ cub.}{3PS\ quad.}$. Et eodem computandi fundamento invenire licet vires segmenrorum sphaeroidis.

Corol. 3. Quod si corpusculum intra sphaeroidem in axe collocetur ; attractio erit ut ipsius distantia a centro. Id quod facilius hoc argumento colligitur, sive particula in axe sit, sive in alia quavis diametro data. Sit *AGOF* sphaerois attrahens, *S* centrum eius, & *P* corpus attractum. Per corpus illud *P* agantur tum semidiameter *SPA*, tum rectae duae quaevis *DE, FG* sphaeroidi hinc inde occurrentes in *D* & *E, F* & G; sintque *PCM, HLN* superficies sphaeroidum duarum interiorum, exteriori similium & concentricarum, quarum prior transeat per corpus *P*, & secet rectas *DE* & *FG* in *B* & *C*, posterior secet easdem rectas in *H, I* & *K, L.* Habeant autem sphaeroides omnes axem communem, & erunt rectarum partes hinc inde interceptae *DP* & *BE, FP* & *CG, DH* & *IE, FK* & *LG* sibi mutuo aequales; propterea quod rectae *DE, PB* & *HI* bisecantur in eodem puncto, ut & rectae *FG, PC* & *K L.* Concipe jam *DPF, EPG* designare conos oppositos, angulis verticalibus *DPF, EPG* infinite parvis descriptos, & lineas etiam *DH, EI* infinite parvas esse; & conorum particulae sphaeroidum superficiebus abscissae *DHKF, GLIE,* ob aequalitatem linearum *DH*, *EI,* erunt ad invicem ut quadrata distantiarum suarum a corpusculo *P*, & propterea corpusculum illud aequaliter trahent. Et



pari ratione, si superficiebus sphaeroidum innumerarum similium concentricarum & axem communem habentium dividantur spatia *DPF, EGCB* in particulas, hae omnes utrinque aequaliter trahent corpus *P* in partes contrarias. Aequales igitur sunt vires coni *DPF* & segmenti conici *EGCB,* & per contrarietatem se mutuo destruunt. Et par est ratio virium materiae omnis extra sphaeroidem intimam *PCBM.* Trahitur igitur corpus *P* a sola sphaeroide intima *PCBM,* & propterea (per corol.3. Prop. LXXII.) attractio eius est ad vim, qua corpus *A* trahitur a sphaeroidc tota *AGOD,* ut distantia *PS* ad distantiam *AS. Q.E.D.*

PROPOSITIO XCII. PROBLEMA XLVI.

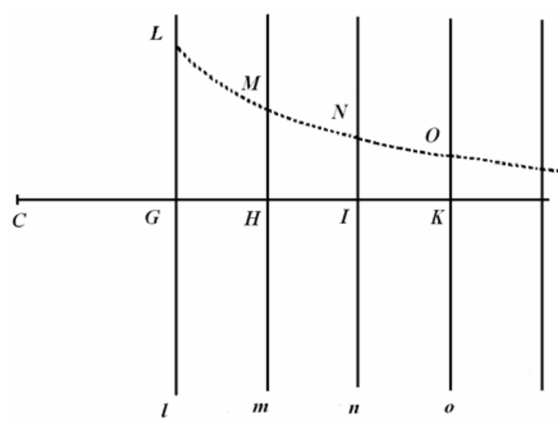
Dato corpore attractivo, invenire rationem decrementi virium centripetarum in eius puncta singula tendentium.

E corpore dato formanda est sphaera vel cylindrus aliave figura regularis, cuius lex attractionis, cuivis decrementi rationi congruens (per Prop. LXX, LXXXI, & XCI.) inveniri potest. Dein factis experimentis invenienda est vis attractionis in diversis distantiis, & lex attractionis in totum inde patefacta dabit rationem decrementi virium purtium singularum, quam invenire oportuit.

PROPOSITIO XCIII. THEOREMA XLVII.

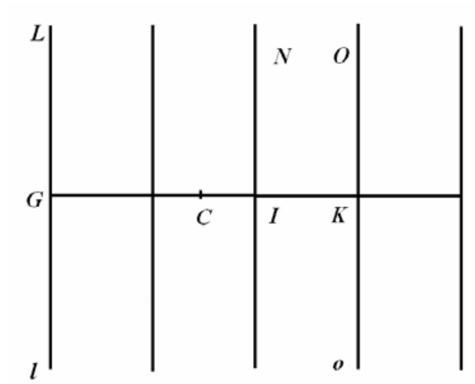
Si solidum ex una parte planum, ex reliquis autem partibus infinitum, constet ex particulis aequalibus aequaliter attractivis, quarum vires in recessu a solido decrescunt in ratione potestatis cuiusvis distantiarum plusquam quadraticae & vi solidi totius corpusculum ad utramvis plani partem constitutum trahatur: dico quod solidi vis illa attractiva, in recessu ab ejus superficie plana, decrescet in ratione potestatis, cuius latus est distantia corpusculi a plano, & *index ternario minor quam index potestatis distantiarum.*

Cas: 1. Sit *LGl* planum quo solidum terminatur. Iaceat solidum autem ex parte plani hujus versus *I*, inque plana innumera *mHM, nIN, oKO,* &c. ipsi *GL* parallela resolvatur. Et primo collocetur corpus attractum *C* extra solidum *CGHI* planis illis innumeris perpendicularis, & decrescant vires attractivae punctorum solidi in ratione potestatis distantiarum, cuius index sit numerus *n* ternario non minor. Ergo (per Corol. 3. Prop. XC.) vis, qua planum quodvis *mHM* trahit punctum *C*, est reciproce ut CH^{n-3}. In plano *mHM* capiatur longitudo *HM* ipsi CH^{n-2} reciproce proportionalis, & erit vis illa ut *HM.* Similiter in planis



singulis *lGL, nIN, oKO,* &c. capiantur longitudines *GL, IN, KO,* &c. ipsis $CG^{n-2}, CI^{n-2}, CK^{n-2}$, &c. reciproce proportionales; & vires planorum eorundem erunt ut longitudines captae, ideoque summa virium ut summa longitudinum, hoc est, vis solidi totius ut area *GLOK* in infinitum versus *OK* producta. Sed area illa (per notas quadraturarum methodos) est reciproce ut CG^{n-3}, & propterea vis solidi totius est reciproce ut CG^{n-3}. *Q. E. D.*

Cas. 2. Collocetur jam corpusculum *C* ex parte plani *lGL* intra solidum, & capiatur distantia *CK* aequalis distantiae *CG.* Et solidi pars *LGloKO,* planis parallelis *lGL, oKO,* terminata, corpusculum *C* in medio situm nullam in partem trahet, contrariis oppositorum punctorum actionibus se mutuo per aequalitatem tollentibus. Proinde corpusculum *C* sola vi solidi ultra planum *OK* siti trahitur. Haec autem vis (per casum primum) est reciproce ut CK^{n-3}, hoc est (ob aequales *CG, CK)* reciproce ut CG^{n-3}. *Q. E. D.*



Corol. I. Hinc si solidum *LGIN* planis duobus infinitis parallelis *LG, IN* utrinque terminetur; innotescit ejus vis attractiva, subducendo de vi attractiva solidi totius infiniti *LGKO* vim attractivam partis ulterioris *NIKO,* in infinitum versus *KO* productae.

Corol. 2. Si solidi hujus infiniti pars ulterior, quando attractio ejus collata cum attractione partis citerioris nullius pene est momenti, rejiciatur: attractio partis illius citerioris augendo distantiam decrescet quam proxime in ratione potestatis CG^{n-3}.

Corol. 3. Et hinc si corpus quodvis finitum & ex una parte planum trabat corpusculum e regione medii illius plani, & distantia inter corpusculum & planum collata cum dimensionibus corporis attrahentis perexigua sit, constet autem corpus attrahens ex particulis homogeneis, quarum vires attractivae decrescunt in ratione potestatis cujusvis plusquam quadruplicatae distantiarum; vis attractiva corporis totius decrescet quamproxime in ratione potestatis, cujus latus sit distantia illa perexigua, & index ternario minor quam index potestatis prioris. De corpore ex particulis constante, quarum vires attractivae decrescunt in ratione potestatis triplicatae distantiarum, assertio non valet; propterea quod, in hoc casu, attractio partis illius ulterioris corporis infiniti in corollario secundo, semper est infinite major quam attractio partis citerioris.

Scholium.

Si corpus aliquod perpendiculariter versus planum datum trahatur, & ex data lege attractionis quaeratur motus corporis: solvetur problema quaerendo (per Prop. XXXIX) motum corporis recta descendentis ad hoc planum, & (per legum corol. 2.) componendo motum istum cum uniformi motu, secundum lineas eidem plano parallelas facto. Et contra, si quaeratur lex attractionis in planum secundum lineas perpendiculares factae, ea conditione ut corpus attractum in data quacunque curva linea moveatur, solvetur problema operando ad exemplum problematis tertii.

Operationes autem contrahi solent resolvendo ordinatim applicatas in series convergentes. Ut si ad basem *A* in angulo quovis dato ordinatim applicetur longitudo *B*, quae sit ut basis dignitas quaelibet $A^{\frac{m}{n}}$; & quaeratur vis qua corpus, secundum positionem ordinatim applicatae, vel in basem attractum vel a basi fugatum, moveri possit in curva linea, quam ordinatim applicata termino suo superiore semper attingit: Suppono basem augeri parte *m* quam minima *O*, & ordinatim applicatam $\overline{A+O}^{\frac{m}{n}}$ resolvo in seriem infinitam $A^{\frac{m}{n}} + \frac{m}{n}OA^{\frac{m-n}{n}} + \frac{mm-mn}{2nn}OOA^{\frac{m-2n}{n}}$ *&c.* atque hujus termino in quo *O* duarum est dimensionum, id est, termino $\frac{mm-mn}{2nn}OOA^{\frac{m-2n}{n}}$ vim proportionalem esse suppono. Est igitur vis quaesita ut $\frac{mm-mn}{nn}A^{\frac{m-2n}{n}}$, vel quod perinde est, ut $\frac{mm-mn}{nn}B^{\frac{m-2n}{m}}$. Ut si ordinatim applicata parabolam attingat, existente $m=2, \& \ n=1$: fiet vis ut data $2B^0$, ideoque dabitur. Data igitur vi corpus movebitur in parabola, quemadmodum *Galilaeus* demonstravit. Quod si ordinatim applicata hyperbolam attingat, existente $m=0-1, \& \ n=1$; fiet vis ut $2A^{-3}$ seu $2B^3$: ideoque vi, quae sit ut cubus ordinatim applicatae, corpus movebitur in hyperbola. Sed missis hujusmodi propositionibus, pergo ad alias quasdam de motu, quas nondum attigi.