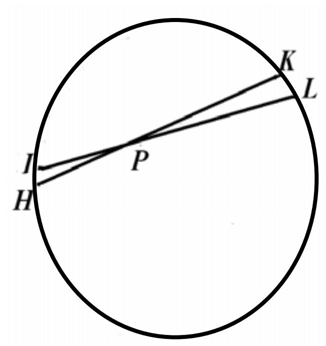
SECTION XII.
Concerning the attractive forces of spherical bodies.

PROPOSITION LXX. THEOREM XXX.
If the individual points of the surface of a some sphere may be drawn to the centre by equal forces inversely proportional to the square of the distance from the points : I say that a corpuscle within the surface agreed upon is not attracted by these forces in any direction.

Let *HIKL* be that spherical surface, and *P* a corpuscle placed within. Through *P* there may be drawn two lines to this surface *HK* and *IL*, the intercepting arcs *HI* and *KL* as minimum; and, on account of the similar triangles *HPI. LPK* (by Corol. 3, Lem. VII.), these arcs will be proportional to the distances *HP* and *LP* , and whatever the small parts of the spherical surface, proportional to *HI* and *KL*, with right lines drawn passing through the point *P* and terminating there on all sides, and they will be in that duplicate ratio. Therefore the forces of these particles acting on the body *P* are equal to each other. For they are directly as the particles, and inversely as the square of the distances. And these two ratios compound a ratio of equality. Therefore the attractions, acting equally on the opposite parts, mutually cancel each other out. And by a similar argument, all the attractions from the contrary parts for the whole spherical surface cancel each other by opposite attractions. Therefore the body *P* in no part is impelled by these attractions. *Q.E,D.*

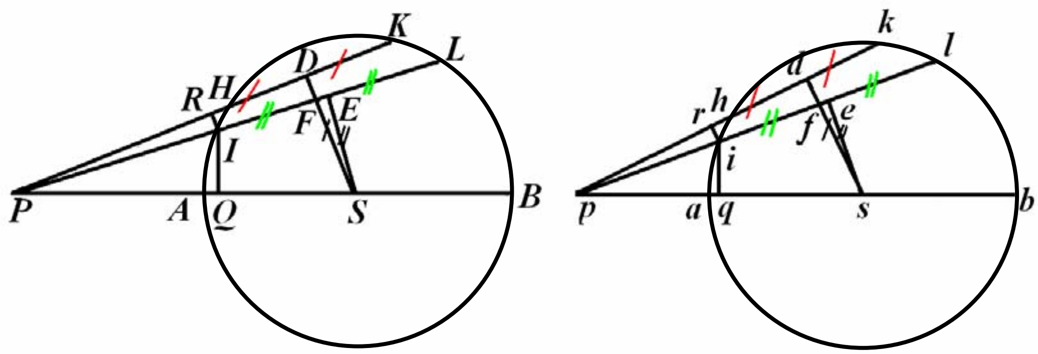


[Note 515 (o) L & J : For the angles *HPI* , *LPK* vertically opposite are equal; and the angles *HIL, LKH* resting on the same arc are equal, (by Prop. 27. Book 3, Eucl.) For the evanescent arcs IH, KL, can be taken for the chords themselves (by Cor. 3. Lem. 7) Whereby the arcs *HI, KL* with the distances *HP, LP* are proportional, and hence if an innumerable number of right lines may be understood to be drawn to the spherical surface through the point P to the smallest possible arcs as terminals *HI, KL,* these right lines will form similar solid figures – pyramids or cones, the bases of which will be similar on the surface of the sphere, and hence (by Lemma 5) they will be in the duplicate ratio of the sides *HI, HL,* or of the distances *HP, LP*. Hence the forces, etc.....]

PROPOSITION LXXI. THEOREM XXXI.

With the same in place, I say that a corpuscle put in place outside the surface of the sphere is attracted to the centre of the sphere, by a force inversely proportional to the square of its distance from the same centre.

Let *AHKB, abhk* be two spherical surfaces described, with centres *S, s*, diameters *AB*,



ab, & *P, p* corpuscles situated outside, on these diameters produced [in the case shown, $PA > pa$]. The lines *PHK, PIL, phk, pil* may be drawn from the corpuscles, taking the equal arcs *HK, hk* and *IL, il* from the great circles *AHB, ahb.* And the perpendiculars *SD, sd; SE, se; IR, ir;* may be sent to these of which *SD, sd* may cut *PL, pl* in *F* and *f:* also the perpendiculars *IQ, iq* may be sent to the diameters. The angles *DPE, dpe* may be vanishing*:* and on account of the equality *DS* and *ds, ES* and *es,* the lines *PE, PF* and *pe, pf* and the smallest lines *DF, df* may be considered as equal;
[We have taken the liberty of marking these equal chords and perpendiculars in different colours on both diagrams; unmarked diagrams are of course available in the Latin section of this file];
evidently the ultimate ratio of these is equality, with the angles *DPE, dpe* likewise evanescent. And thus with these put in place, *PI* to *PF* will be as *RI* to *DF,* & *pf* to *pi* as *df* or *DF* to *ri*

$$[i.e.\ \frac{PI}{PF} = \frac{RI}{DF};\ \frac{pi}{pf} = \frac{ri}{df}\ ];$$

and from the equality, $PI \times pf$ to $PF \times pi$ is as *RI* to *ri,*

$$[i.e.\ \frac{PI\times pf}{PF\times pi} = \frac{RI\times df}{DF\times ri} = \frac{RI}{ri} = \frac{HI}{hi};]$$

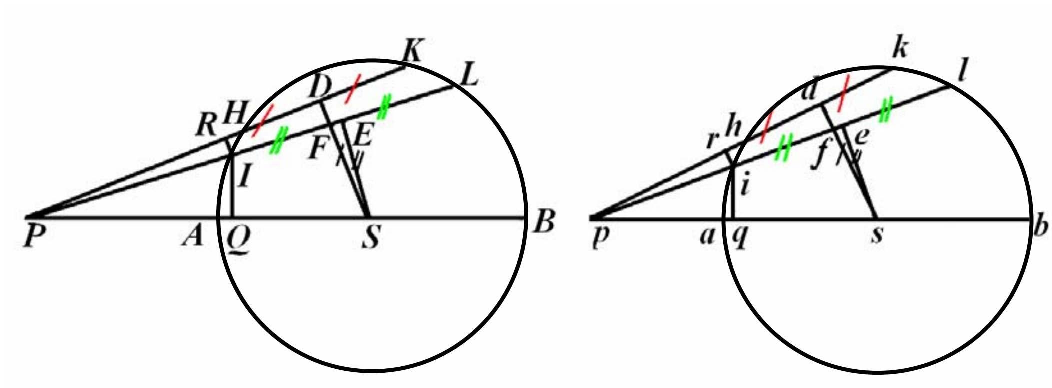
that is (by Corol. 3. Lem. VII.) as the arc *IH* to the arc *ih* [since $\frac{RI}{ri} = \frac{HI}{hi}$; from the similar triangles *RHI* and *rhi*] . Again, *PI* is to *PS* as *IQ* to *SE,* and *ps* to *pi* as *se* or *SE* to *iq;*

$$[i.e.\ \frac{PI}{PS} = \frac{IQ}{SE};\ \text{and}\ \frac{ps}{pi} = \frac{se\ \text{or}\ SE}{iq}\ ],$$

and from the equality $PI \times ps$ is to $PS \times pi$, as *IQ* to *iq;*

$$[i.e.\ \frac{PI\times ps}{PS\times pi} = \frac{IQ}{iq};\ ],$$

And with the ratios taken together $PI^2 \times pf \times ps$ to $pi^2 \times PF \times PS$, as $IH \times IQ$ to $ih \times iq$;



$$[i.e.\ \frac{PI\times ps}{PS\times pi}\times\frac{PI\times pf}{PF\times pi}=\frac{PI^2\times pf\times ps}{pi^2\times PF\times PS}=\frac{IH\times IQ}{ih\times iq};],$$

that is, as the circular surface, that the arc *IH* will describe by rotating about the diameter *AB* of the semicircle *AKB*, to the circular surface, that the arc *ih* will describe by rotating about the diameter *ab* of the semicircle *akb*. And the forces, by which these surfaces attract the corpuscles *P* and *p* along the lines tending towards themselves, are (by hypothesis) directly as the surfaces themselves, and inversely as the squares of the distances of the surfaces from the bodies, that is, as $pf\times ps$ to $PF\times PS$.

$$[\text{i.e. } \frac{IH\times IQ}{PI^2}:\frac{ih\times iq}{pi^2}=\frac{pf\times ps}{PF\times PS};\ ]$$

And these forces are to the oblique parts of these, which (by the resolution of the forces made according to Corol 2. of the laws by resolution of the forces) act along the lines *PS*, *Ps* to the centre, as *PI* to *PQ*, and *pi* to *pq;* that is (on account of the similar triangles *PIQ* and *PSF*, *piq* and *psf)* as *PS* to *PF* and *ps* to *pf*. Thus, from the equality, the attraction of the corpuscle *P* towards *S* shall be to the attraction of the corpuscle *p* towards *s*, as $\frac{PF\times pf\times ps}{PS}$ to $\frac{pf\times PF\times PS}{ps}$ that is, as ps^2 to PS^2.

$$[i.e.\ \frac{\text{horizontal attraction of left segment}}{\text{horizontal attraction of right segment}}=\frac{pf\times ps\times\cos PFS}{PF\times PS\times\cos pfs}=\frac{pf\times ps\times\frac{PF}{PS}}{PF\times PS\times\frac{pf}{ps}}=\frac{ps^2}{PF^2}.]$$

And by a similar argument the forces, by which the surfaces by the rotation of the described arcs *KL, kl* pull on the corpuscles, will be as ps^2 to PS^2. And it will be possible to distinguish the forces of all the circular surfaces in the same ratio in which each spherical surface will be, by always taking *sd* equal to *SD* and *se* equal to *SE*. And, by putting these together, the forces of all the spherical surfaces exercised on the bodies will be in the same ratio. *Q.E.D.*

[When this proof is understood, it is seen to be very simple, and this was probably the only way in which the problem could be solved at the time, without resorting to a full integration ; the distances of the corpuscles from the shells is the only variable, and other chords and perpendiculars, etc. maintain the same lengths in each shell, allowing the summation to proceed, and the final ratio required to emerge.]

PROPOSITION LXXII. THEOREM XXXII.

If equal centripetal forces attract particular points of some sphere decreasing in the inverse ratio of the distance from these points; and both the density of the sphere may be given, as well as the ratio of the diameter of the sphere to the distance of [such]a corpuscle from the centre of this : I say that the force, by which the corpuscle is attracted, will be proportional to the radius of the sphere.

For consider two separate corpuscles attracted by two spheres, one corpuscle by the one sphere, and the other corpuscle by the other sphere, and the distances of these from the centres of the spheres are proportional to the diameters of the spheres respectively, the spheres moreover are resolved into similar particles and similarly put in place with the corpuscles. And the attractions of the one corpuscle made towards the individual particles of the one sphere, will be to the attraction of the other corpuscle towards just as many particles of the other sphere, in a ratio compounded from the ratio of the particles directly, and in the ratio of the inverse squares of the distances. But the [number of the] particles are as the [volumes or masses of the] spheres, that is, in the triplicate [*i.e.* cubic] ratio of the diameters, and the distances are as the diameters ; and the first ratio directly together with the second ratio inversely twice is the ratio of the diameter to the diameter. *Q.E.D.*
[Consider two spheres A and B with radii R_A and R_B; corpuscles are situated at distances C_A and C_B from their respective spheres, then by hypothesis, $\frac{C_A}{C_B}=\frac{R_A}{R_B}$; let N_A and N_B be the number of particles or corpuscles in the respective spheres, then the respective F_A and F_B forces will be in the ratio $\frac{F_A}{F_B}=\frac{N_A}{N_B}\times\frac{C_B^2}{C_A^2}=\frac{M_A}{M_B}\times\frac{C_B^2}{C_A^2}=\frac{R_A^3}{R_B^3}\times\frac{R_B^2}{R_A^2}=\frac{R_A}{R_B}$.]

Corol. I. Hence if the corpuscles may be revolving in circles about the spheres constantly attracted equally by the matter ; and the distances from the centres shall be proportional to the diameters of the same: The periodic times shall be equal.
[Recall from previously, such S.H.M. motion has the period independent of the amplitude or radius.]

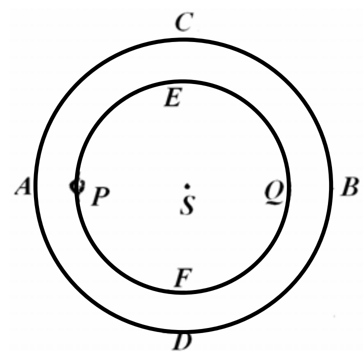
Corol. 2. And conversely, if the periodic times are equal; the distances shall be proportional to the diameters. These two are in agreement by Corol. 3. Prop. IV.

Corol. 3. If equal centripetal forces may attract the individual points [corpuscles] of any two similar solids of equal density, decreasing in the squared ratio of the distances from the points ; the forces, by which the corpuscles will be attracted by the same, by these two similar solids in place, will be in turn as the diameters of the solids.

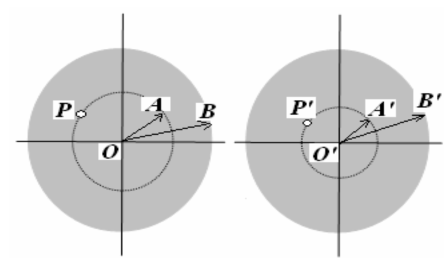
PROPOSITION LXXIII. THEOREM XXXIII.

If for some given sphere, equal centripetal forces attract particular points decreasing in the square ratio of the distances from the points : I say that a corpuscle within the sphere constituted is attracted by a force proportional to its distance from the centre.

A corpuscle *P* will be located in the sphere *ABCD* described with centre *S;* and from the same centre *S*, with the interval *SP*, consider the spherical interior *PEQF* to be described. It is evident, (by Prop. LXX.) that the concentric spherical surfaces, from which the difference of the spheres *AEBF* is composed, with their attractions destroyed by opposing attractions, do not act on the body *P*. Only the attraction of the interior sphere *PEQF* remains. And (by Prop. LXXII.) this is as the distance *PS*. *Q.E.D.*



[In the diagram constructed here, the original sphere is replaced by two spheres of the same material having radii *OB* and *O'B'*, while the corpuscles considered are at the positions *P* and *P'* with the radii *OA* and *O'A'* . From the first proposition in this section (no. 70), the corpuscles at P and P' experience no force from the outer particles beyond the radii *OB* and *OB'*, considering these outer particles as lying on concentric shells; meanwhile, from the second proposition in this section (no. 71), the inner particles lying in their concentric shells can be replaced by equivalent masses at the origins *O* and *O'*, exerting forces proportional to their respective masses, and inversely as the distances *OP* and *O'P'* squared in turn. The ratio of these forces in an obvious notation is given by :



$$\frac{F_{OP}}{F'_{O'P'}} = \frac{\rho R_A^3}{R_A^2} \div \frac{\rho R'^3_A}{R'^2_A} = \frac{R_A}{R'_A} = \frac{R_B}{R'_B}.]$$

Scholium.

The surfaces, from which the solids are composed, here are not purely mathematical, but thus tenuous shells [orbs], so that the thickness of these shall be equivalent to zero; without doubt the evanescent shells, from which the sphere finally will be composed when the number of these shells may be increased indefinitely and the thickness is minimised. Similarly the points, from which the lines, surfaces, and volumes are said to be composed, are understood to equal particles of insignificant magnitude.

PROPOSITION LXXIV. THEOREM XXXIV.

With the same in place, I say that a corpuscle constituted beyond the sphere is attracted by a force inversely proportional to the square of its distance from the centre of this sphere.

For the sphere may be separated into innumerable concentric spherical surfaces, and the attractions of the corpuscle arising from the individual surfaces will be inversely as the square of the distance from the centre (by prop. LXXI.) And on compounding, the sum of the attractions, that is the attraction of the corpuscle by the whole sphere, happens in the same ratio.
Q. E. D.

[Thus, the attractive force $\propto \frac{1}{\text{distance}^2}$]

Corol. I. Hence at equal distances from the centres of homogeneous spheres, the attractions are as the [masses of the] spheres. For (by Prop. LXXII.) if the distances are proportional to the diameters of the spheres, the forces will be as the diameters. The greater distance may be reduced into that ratio [of the radii]; and with the distances now made equal [to those of the respective spheres], the attraction will be increased in that ratio squared ; and thus it will be to the other attraction in that ratio cubed, that is, in the ratio of the spheres.
[See Chandrasekhar on this point : *i.e*. the forces are in the ratio of the cubes of the radii for equal distances from the spheres.]

Corol. 2. In which any attractions are as the [masses of the] applicable spheres to the squares of the distances.

Corol. 3. If a corpuscle, situated outside a homogenous sphere, is drawn by a force inversely proportional to the square of its distance from the centre itself, but the sphere may be constructed from attracting particles ; the force of each particle will decrease in the square ratio of the distance from the particle.

[As Chandrasakher points out on p. 278, this amounts to a statement of the universal law of gravitation, but applied to this circumstance.]

PROPOSITION LXXV. THEOREM XXXV.

If equal centripetal forces may extend to the individual particles of a given sphere, decreasing in the inverse square ratio of the distances from the points ; I say that any other similar sphere, by the same is attracted by a force inversely proportional to the square of the distance of the centres.

For the attraction of each particle is inversely as the square of its distance from the centre of the attracting sphere, (by Prop. LXXIV.) and therefore [the attraction of the particles in a sphere between themselves] is the same, as if the total attracting force may spring from a single corpuscle situated at the centre of this sphere. But this attraction is just as great, as would be the case as if in turn it were attracted by a force of the same size by all the other particles of the sphere. But the attraction of that sphere would be (by Prop. LXXIV.) inversely proportional to the square of its distance from the centre of the other sphere; and thus the attraction of the sphere is equal to this in the same ratio. *Q.E.D.*

Corol. I. The attractions of spheres, towards other homogeneous spheres, are as the applied attracting spheres to the squares of the distances of their centres from the centres of these which they attract.

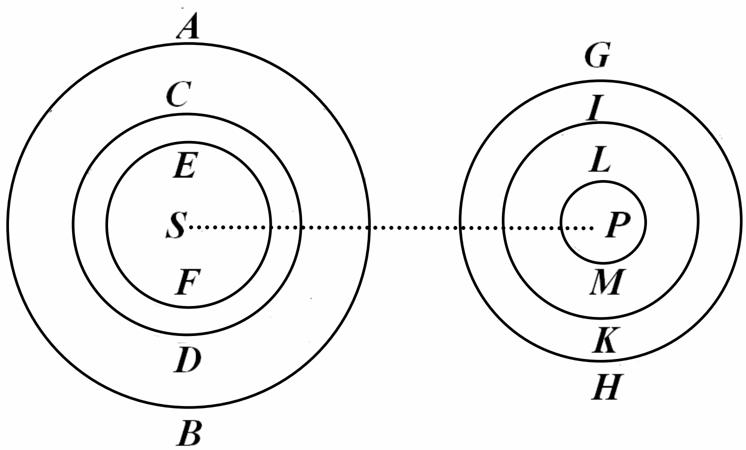
Corol.2. The same becomes apparent, when the attracted sphere also attracts. In as much as the individual points of this will attract the individual points of the other sphere with the same force, from which by these themselves in turn they are attracted; and thus as in any attraction both the attracting point, as well as the attracted point, may be acted on – by law 3, there will be a mutual pair of attracting forces with the proportions conserved.

Corol. 3. Everything prevails the same, which have been shown above concerning the motions of bodies about the focal points of conic sections, when the attracting sphere is located at the focus, and the bodies will be moving outside the sphere.

Corol. 4. That truly, which may be shown concerning the motion of bodies about the centre of a conic section, will prevail when the motions are taking place within the sphere.

PROPOSITION LXXVI. THEOREM XXXVI.

If spheres, in a progression from the centre to the circumference, dissimilar in some manner (as far as the density of the material and the attractive force are concerned), by progressing around, are all alike on all sides at a given distance from the centre; and he attractive force of each point decreases in the square ratio of the attracting body : I say that the total force, by which one sphere of this kind attracts another, shall be inversely proportional to the square of the distance from the centre.



Let *AB, CD, E F*, &c. be some similar concentric spheres the interiors of which added to the outsides put together a material denser towards the centre, or taken away leave behind a more tenuous medium; and these individual spheres (by Prop. LXXV.) will attract other individual similar concentric spheres *GH*, *IK*, *LM*, &c., with forces inversely proportional to the distance *SP*. And by adding together or by dividing apart, the sum of the forces of all these, or the excess of some above the others, that is the force by which the whole sphere, composed from any number of concentric spheres or differences *AB*, attracts the whole composed from some number of concentric spheres or differences *GH*, will be in the same ratio. Thus the number of concentric spheres may be increased indefinitely, to that the density of the material together with the attractive force, in progressing from the circumference to the centre, may increase of decrease following some law; and non attracting material may be added whenever the density is deficient, so that the spheres may acquire some optimum form; and the force, by which one of these may attract the other, even now will be by the above argument, in that inverse ratio of the square of the distance. *Q.E.D.*

Corol. 1. Hence if many spheres of this kind, themselves in turn similar through all, mutually may attract each other ; the accelerative attractions of individuals on individuals will be, at whatever equal distances of the attractions, as the attracting spheres.

Corol. 2. And with any unequal distances, as the applicable attracting spheres to the squares of the distances between the centres.

Corol. 3. Truly the motive attractions [i.e. the attractive forces], or the weights of the spheres on the spheres will be, at equal distances of the centres, as the attracting and attracted spheres conjointly, that is as the content within the spheres produced by multiplication.

Corol. 4. And in accordance with unequal distances, directly as that product and inversely as the square of the distances between the centres.

Corol.5. The same will emerge when the attraction arises on each sphere by virtue of the mutual attraction exercised by one sphere on the other. For with both forces the attraction forms a pair, with the proportion maintained.

Corol. 6. If some spheres of this kind may be revolving around others at rest, individual spheres around individual spheres; and the distances between the centres of the revolving and the resting spheres proportional to the resting diameters ; the periodic times will be equal.

Corol.7. And conversely, if the periodic times are equal ; the distances will be proportional to the diameters.

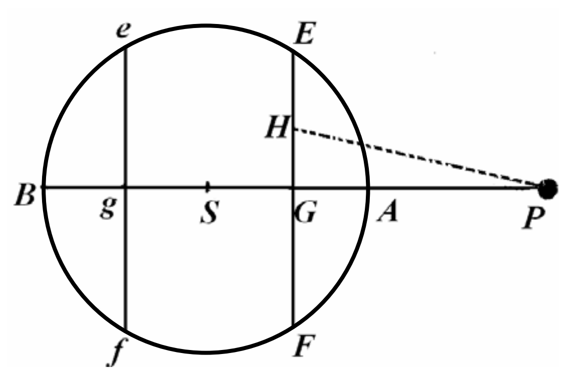
Corol. 8. Everything will be obtained the same, which have been shown above concerning the motion of bodies around the foci of conic sections ; where the attracting sphere, of the form and of each condition now described, may be located at the focus.

Corol. 9. And as also where the attracting spheres are rotating, with any of the conditions now described.

PROPOSITION LXXVII. THEOREM XXXVII.

If centripetal forces attract the individual points of the spheres proportional to the distances between the points from the attracting bodies : I say that the composite force, by which the two spheres will attract each other mutually, is as the distance between the spheres.

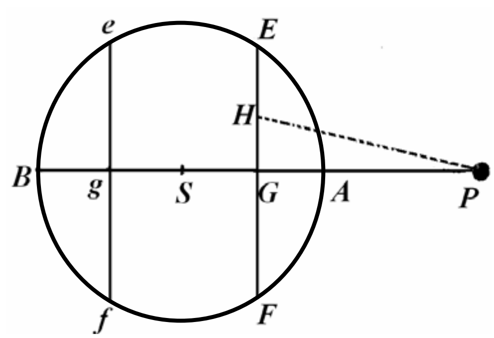
Case. 1. Let *AEBF* be the sphere; *S* the centre of this; *P* an attracted corpuscle, *PASB* the axis of the sphere passing through the centre of the corpuscle ; *EF, ef* two planes, by which the sphere may be cut, perpendicular to this axis, and hence thereupon equally distant from the centre of the sphere ; *G, g* the intersections of the planes and the axis ; and *H* some point in the plane *EF.* The centripetal force at the corpuscle *P* to the point *H,* acts along the line *PH* , is as the distance *PH;* and (by Corol. 2. of the laws) along the line *PG,* or towards the centre *S,* as the length *PG*. Therefore the force of all the points in the plane *EF,* that is of the whole plane, by which the corpuscle *P* is attracted towards *S,* is as the distance *PG* multiplied by the number of points, that is, as that solid contained under the plane *EF* itself and that distance *PG.* And similarly the force of the plane *ef*, by which the corpuscle *P* is attracted towards the centre *S,* is as that plane taken by its distance *Pg,* or equally as the plane *EF* taken by that distance *Pg;* and the sum of forces of each plane as the plane EF taken by the sum of the distances $PG + Pg$, that is, as that plane multiplied into twice the distance PS between the centre and the corpuscle, that is, as twice the plane *EF* multiplied into



the difference *PS*, or as the sum of the equal planes $EF+ef$ multiplied by the same distance. And by a similar argument, all the forces of the planes in the whole sphere, hence in this manner at equal distances from the centre of the sphere, as the sums of the planes multiplied by the distance *PS*, that is, as the whole sphere and as the distance *PS* conjointly. *Q.E.D.*

Case 2. Now the corpuscle *P* may attract the sphere *AEBF*. And by the same argument it will be approved that the force, by which that sphere is attracted, will be as the distance *PS*. *Q.E.D.*

Case 3. Now the other sphere may be put together from innumerable particles *P;* and because the force, by which any corpuscle whatever is attracted, is as the distance of the corpuscle from the centre of the first sphere and as the same sphere taken conjointly, and thus is the same, as if the whole may be produced from a single corpuscle at the centre of the sphere ; the whole force, by which all the corpuscles are attracted to the second sphere, will be the same as if that sphere may be attracted by the force produced from a single corpuscle at the centre of the first sphere, and therefore is proportional to the distance between the centres of the spheres. *Q.E.D.*



Case 4. The spheres may attract each other mutually, and the double force will maintain the previous proportion. *Q.E.D.*

Case 5. Now the corpuscle *p* may be located within the sphere *AEBF;* and because the force of the plane *ef* at the corpuscle is as the solid contained under that plane and the distance *pG;* and the force of the opposite plane *EF* is as the solid contained under that plane and the distance *pG;* the force composed from each will be as the difference of the solids, that is, as the sum of the equal planes multiplied by half the difference of the distances, that is, as half that multiplied into the distance *pS* of the corpuscle from the centre of the sphere. And by a similar argument, the attraction of all the planes *EF*, *ef* in the whole sphere, that is, the attraction of the whole sphere, is jointly as the sum of all the planes, or the whole sphere, and as the distance *pS* of the corpuscle from the centre of the sphere. *Q.E.D.*

[Note 517(s) Leseur & Jacquier : *the force composed from each will be as the difference of the solids, that is, as* $ef\times pg-EF\times pG$. But $Sg=SG$, and thus $pg-pG=pS+SG-pG=2pS;$ whereby since also there shall be $EF=ef$, there will be $ef\times pg-EF\times pG=ef\times\overline{pg-pG}=2ef\times pS=\overline{ef+EF}\times pS$. If the point *G* is placed between *p* and *S*, the total force will be as $ef\times pg+EF\times pG$, and because there is always $Sg = SG$, and in this case $pg + pG = pS + SG + pG = 2pS$, similarly the total force will be found as $\overline{ef + EF} \times pS$.]

Case 6. And if the new sphere may be composed from innumerable corpuscles *p*, situated within the first sphere *AEBF*; it will be approved as before that the attraction, either a simple one of one sphere towards the other, or mutually each in turn towards the other, will be as the distance of the centres *pS. Q.E.D.*

PROPOSITION LXXVIII. THEOREM XXXVIII.

If the spheres in the progression from the centre shall be dissimilar in some manner and unequal, but truly in progressing around, all at a given distance from the centre, they shall be the same on all sides; and the attractive force of each point shall be as the distance of the attracted body : I say that the total force by which two spheres of this kind mutually attract each other shall be proportional to the distance between the centres of the spheres.

This may be demonstrated from the previous proposition in the same manner, by which Proposition LXXVI was shown from Proposition LXXV.

Corol. Those matters have been shown above in Propositions X. and LXIV. concerning the motion of bodies about the centre of conic sections, where all the attractions emerge made from the force of spherical bodies under the conditions now described, and the attracted bodies are spheres of the same description.

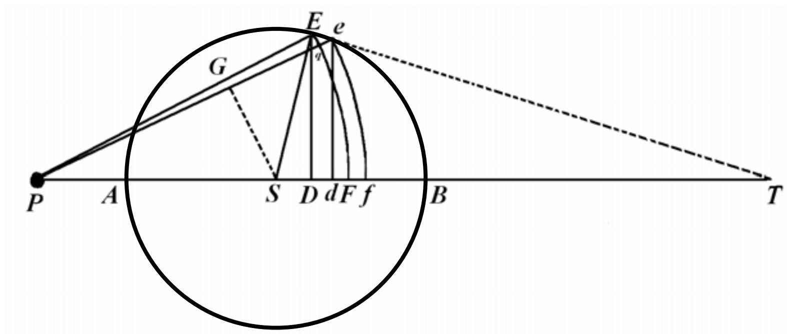
Scholium.

I have now explained the two most significant cases of the attractions ; without doubt where the centripetal forces decrease in the square ratio of the distances, or increase in the simple ratio of the distances ; in each case putting into effect that bodies may rotate in conic sections, and the component centripetal forces of spherical bodies by the same law, in receding from the centre, decreasing or increasing with these themselves : which is noteworthy. It would be tedious to run through the remaining cases one by one, which show less elegant conclusions. I am inclined to understand and to determine everything by a general method, as follows.

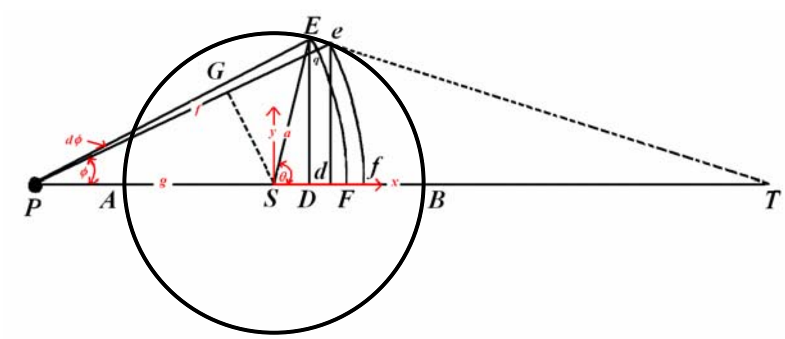
LEMMA XXIX.

If some circle AEB be described with centre S, and with centre P the two circular arcs EF, ef are described, cutting the first in E and e, and the line PS in F, f; and the perpendiculars ED and ed may be sent to PS: I say that, if the separation of the arcs EF and ef may be understood to be infinitely small, the final ratio of the vanishing line Dd to the vanishing line Ff shall be that, which the line PE has to the line PS.

For if the line *Pe* may cut the arc *EF* in *q;* and the right line *Ee*, which will coincide



with the vanishing arc *Ee*, produced meets the right line *PS* in *T;* and from *S* the normal *SG* may be sent to *PE* : on account of the similar triangles *DTE, dTe, DES; Dd* will be to *Ee,* as *DT* to *TE,* or *DE* to *ES;* and similar on account of the triangles *Eeq, ESG* (by Lem. VIII. & Corol 3. Lem. VII.), *Ee* will be to *eq* or *Ff* as *ES* to *SG;* and from the equality, *Dd* to *Ff* as *DE* to *SG;* that is (on account of the similar triangles *PDE, PGS*) as *PE* to *PS. Q.E.D.*
[Note initially, as was common practice for Newton, the increment in a length from a



point was given a small letter, such as *Ee,* etc. From similar triangles, $\frac{Dd}{Ee}=\frac{DT}{ET}=\frac{DE}{SE}$; $\frac{Ee}{eq}=\frac{Ee}{Ff}=\frac{ES}{SG}$. Thus, $\frac{Dd}{Ff}=\frac{DE}{SG}=\frac{PE}{PS}$. It is thus apparent that a ratio such as $\frac{Dd}{Ff}$ is one of vanishing quantities, where the sides of generating triangle tend towards zero and would be termed by us a differential ratio; it is also usual that such a vanishing ratio is set equal to some finite ratio found geometrically. We may also mention here again, that such quantities are considered by Newton as vanishing : not about to vanish, in which case they would be represented by finite difference, and not vanished, in which case they would all be the meaningless zero on zero, but in the act of vanishing together, in which circumstance they adopt the final ratio; which we might term now the limiting value of the ratio, which means the same thing in different words. In the annotated diagram, were *g* and *a* are constants, this becomes in modern terms : $\frac{dx}{df} = \frac{a\sin\theta}{g\sin\varphi} = \frac{f}{g}$.

To show this, we may revert to modern analysis, as the original calculation has not been supplied in this manner. Thus initially, we may consider the circle *AEB*, with *P* the origin of coordinates and the radius *a*; the coordinates of any point *E* on the circle are $x = g + a\cos\theta$ and $y = a\sin\theta$; and the implicit equation of the circle *AEB* is : $\left((x-g)^2 + y^2\right) = a^2$. However, from triangle *EPS*, we have $\frac{f}{\sin\theta} = \frac{a}{\sin\phi} = \frac{g}{\sin(\theta-\phi)}$, and hence $\frac{f}{g} = \frac{\sin\theta}{\sin(\theta-\phi)} = \frac{a\sin\theta}{a\sin(\theta-\phi)} = \frac{a\sin\theta}{g\sin\phi}$. Also,

$$f^2 = \left((g + a\cos\theta)^2 + a^2\sin^2\theta\right) = g^2 + 2ag\cos\theta + a^2 \text{ ; hence}$$

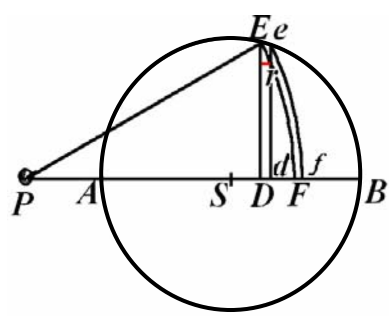
$$fdf = -ag\sin\theta d\theta = gd\left(a\cos\theta\right) = gdx \text{ , giving } \frac{dx}{df} = \frac{f}{g} = \frac{a\sin\theta}{g\sin\phi}\text{, as required and } \left(f,\phi\right)$$

may be taken as the independent variables for future use in integrations over the spherical volume. To turn the circle integrations into spherical integrations, it is necessary to multiply only to multiply by 2π . We will not do so, as this factor disappears in ratios. This derivation above is no more and no less than the original result for the Lemma obtained above by Newton, but written in more familiar terms.]

PROPOSITION LXXIX. THEOREM XXXIX.

If the surface EFfe is already vanishing on account of an infinite diminution of width, with the rotation of this around the axis PS, it may describe a concave-convex spherical solid, and equal centripetal forces are extended to the individual particles of this: I say that the force, by which that solid attracts a corpuscle situated on the axis at P, is in a ratio composed from the ratio of the solid $DE^2 \times Ff$ *, and in the ratio of that force by which the given particle at the place Ff may attract the same corpuscle.*

For if in the first place, we may consider the force of the spherical surface *FE*, which may be generated by the rotation of the arc *FE*, and which may be cut somewhere by the line *de* in *r;* the part of the surface of the ring, generated by the rotation of the arc *rE* , will be as the line increment *Dd*, with the radius of the sphere *PE* remaining constant (as *Archimedes* demonstrated in his book, *On the Sphere* & *Cylinder*.)



[Imagine *rE* coming out of the plane of the page as it rotates about the axis *PS* forming a ring of radius *DE* and width *Dd*, with *P* the apex of the twin cone; as we move towards *F*, concentric rings of width *Dd* are inscribed in the inner surface of the sphere centre *P*, radius *PE*, until an incremental circle is found at F; the diagram is a vertical section through this sphere and the spherical body *AEB*]

[Note 518(*x*) Leseur & Jacquier : The demonstration is easy. For because the angle *PEr* is right (from the nature of the circle) the angle *DEr* is equal to the angle DPE, on account of the sum of the angles $DPE + PED$ being equal to the right angle PEr . From which, if from the point *r* it may be considered to send a perpendicular (shown here in red) to the line *DE* , that is equal to the line increment *Dd*, there may be put in place a vanishing triangle similar to triangle *EPD*, and thus there will be $\frac{DE}{PE} = \frac{Dd}{Er}$ and $Er = \frac{PE \times Dd}{DE}$, but the circular region generated by the rotation of the arc *rE* is as the rectangle $Er \times DE$; whereby if in this rectangle in place of *Er* there may be substituted the value found in this manner, the region will be as $PE \times Dd$, that is, on account of the given radius *PE,* as *Dd*. Q.E.D.]

And the force of this element of area, exercised on the conical surface in place along the lines *PE* or *Pr*, shall be as this part [*i.e.* incremental area] of the annular surface itself; that is, as the line increment *Dd,* or, because it is likewise, as the rectangle under the sphere with the given radius *PE* and that line increment *Dd* : but along the line *PS* attracting towards the centre *S* in the smaller ratio *PD* to *PE,* and thus as $PD \times Dd$.

[Note 518(*z*) Leseur & Jacquier : For if the force acting along the direction *PE* may be set out by the line *PE*, the part of this force which acts along the direction *PS*, may be set out by the line *PD*; $\frac{PE}{PD} = \frac{PE \times Dd}{PD \times Dd}$, that hence will show the force along that direction *PD*, but the oblique forces *ED* from each part of the axis *PB* mutually destroy each other; end of L & J note. Thus, the force normal to the element $PE \times Dd$ is proportional to $PE \times Dd \times \frac{PD}{PE} = PD \times Dd$.]

Now the line *DF* may be considered to be divided into innumerable equal small parts, which one to the other may be called *Dd;* and the surface *FE* may be divided into an equal number of rings, the forces of which are as the sums of all the terms $PD \times Dd$, that is as $\frac{1}{2}PF^2 - \frac{1}{2}PD^2$, and thus as DE^2 .

[The force exerted on *P* due to this set of rings of successive width *dx* on the *PS* axis is the sum of forces parallel to the axis , any of which has the form $PD \times Dd$.]

[Note 519(*b*) Leseur & Jacquier : Clearly all *PD*, while they are being changed from *PD* into *PF* by increasing uniformly make an arithmetic progression, because all the small parts *Dd* by which the line *PD* may be increased are equal : therefore the sum of all *PD* will be in that ratio found from which the sums of the arithmetical progressions may be obtained, certainly by being multiplied jointly by the number of terms of the progression, and by taking half of the product. Truly the first term of this progression is *PD*, the final *PF* and *DF* the number of terms, if indeed *DF* is the sum of the equal vanishing increments of the line *PD*, therefore the sum of all the *PD* is $\frac{\overline{PF+PD} \times DF}{2}$ or (because *DF* is the difference of the lines *PF* and *PD*) the sum of all the PD is $= \frac{\overline{PF+PD} \times \overline{PF-PD}}{2}$; but (by 6.2. Eucl. *Elements*) the sum and the difference of two lines is equal to the difference

of the squares of these, hence $\frac{\overline{PF+PD}\times\overline{PF-PD}}{2}=\frac{1}{2}PF^2-\frac{1}{2}PD^2$, and the sum of all $=\overline{\frac{1}{2}PF^2-\frac{1}{2}PD^2}\times Dd$, but Dd is the increment which is assumed as constant in all these cases, therefore the forces of the whole surface FE which are as the sum of all the terms $PD\times Dd$,are as $\frac{1}{2}PF^2-\frac{1}{2}PD^2$ or as PF^2-PD^2; but $PF^2=PE^2$ by construction, and $PE^2-PD^2=DE^2$ (by 47.1 Eucl. *Elements*) hence the forces of the surface FE, are as DE^2.
The same otherwise: Let the given radius $PE=f$ (not to be confused with the increment Ff), the variable $FD=x$, the fluxion $Dd=dx$, and $PD=f-x$, and thus (the limits of integration, to the left, are $x=0$ and $x=DF$) : $PD\times Dd = fdx-xdx$, and with the fluents (integrals) taken on both sides : $Sum\ PD\times Dd = fx-\frac{1}{2}xx=\frac{2fx-xx}{2}=\frac{DE^2}{2}$, (Prop. 13. Book 6. Eucl. *Elements*). (For $f^2-(f-x)^2=2fx-x^2=ED^2$). Whereby the force of the surface arising by rotating the arc FE will be as DE^2 . Extra note : this integral with the limits 0 and x is understood to be prefixed by a force, which is constant under these circumstances, so that the original double integral can be expressed as the product of two integrals.]

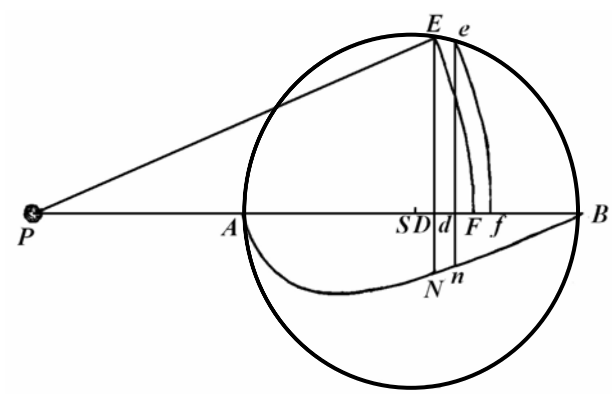
Now the surface FE may be multiplied by the [incremental] altitude Ff; and the force of the solid $EFfe$ exercised on the corpuscle P becomes as $DE^2\times Ff$: for example if Ff may be given as the force some small particle exerts on the corpuscle P at the distance PF . But if that force may not be given, the force of the solid $EFfe$ becomes as the solid $DE^2\times Ff$, and that force is not given conjointly. *Q.E.D.*

[i.*e*. we have not yet been given the force, just a very cunning way of evaluating its volume integral from a particular form of an increment of the sphere. Incidentally, Chandrasekhar seems to have missed the point here, as he sets out on a conventional double integration, which Newton has avoided by integrating over constant force shells in an almost trivial manner. It is a short step for us to consider these as equipotential surfaces; however, it cannot be assumed more than coincidence that such constant energy surfaces have appeared in Newton's calculations ; the idea of potential energy was just not present at the time, and when such energy related ideas arise. they must be considered merely as useful devices that ease calculations, rather than some fundamental insight. Chandrasekhar's book is full of such anachronisms, which does little to enhance one's understanding of the evolution of ideas that occurred at this time. One way to explain in ordinary language what Newton has done, is to consider a hypothetical onion, with uniform layers or shells; P lies at the centre of this onion; now imagine a device that can cut out a spherical shape from the body of the onion; retaining P at its original position and consider this dissected sphere, also at its original position: this now consists of layers or parts of shells each of which is equidistant from P, and an integration over the sphere can now be performed, so finding the total force using this unusual by highly effective way of dissecting the sphere into elements equidistant from P. Newton's originality continues to amaze. Later in this section, he introduces the idea of inversion, so that the forces due to all the points of the sphere can be found on some corpuscle located within the sphere.]

PROPOSITION LXXX. THEOREM XL.

If equal centripetal forces may be extended to the individual equal parts of a sphere ABE described with centre S, and to the axis AB of the sphere, on which some corpuscle may be placed at P, from individual points [such as] D perpendiculars DE may be erected crossing the sphere in E, and in these lengths [such as] DN may be taken, which shall be as the magnitude $\frac{DE^2 \times PS}{PE}$ *and the force, that the particles of the sphere situated on the axis at a distance PE may exercise on the corpuscle P, conjointly: I say that the total force, by which the corpuscle P is drawn towards the sphere, is as the area ANB taken under the axis of the sphere AB, and the curved line ANB, that always is a tangent at the point N.*

For indeed with everything in place which was constructed in the lemma and in the most recent theorem, consider the axis of the sphere *AB* to be divided into innumerable equal parts *Dd,* and the whole sphere to be divided into just as many concave – convex sheets *EFfe;* and the perpendicular *dn* may be erected.



By the above theorem the force, by which the shell *EFfe* attracts the corpuscle *P,* is as $DE^2 \times Ff$ and as the force of the single particle *Dd* at the distance *PE* or *PF* exercised conjointly [with this mass or weight factor.] But (by the most recent lemma : $\frac{Dd}{Ff} = \frac{PE}{PS}$) *Dd* is to *Pf* as *PE* to *PS,* and thence *Ff* is equal to $\frac{PS \times Dd}{PE}$*;* and $DE^2 \times Ff$ equals $Dd \times \frac{DE^2 \times PS}{PE}$, and therefore the force of the shell *EFfe* is as $Dd \times \frac{DE^2 \times PS}{PE}$, and the force of the particle at the distance *PF* exercised conjointly, that is (by hypothesis) as $DN \times Dd$,[i.e. $Dd \times DN = Dd \times \frac{DE^2 \times PS}{PE}$], or the vanishing area *DNnd.* Therefore the forces of all the shells, exercised on the body *P*, are as all the areas *DNnd,* that is, the total force of the sphere is as the total area *ANB. Q.E.D.*

Corol. I. Hence if the centripetal force, attracting the individual particles, always remains the same at all distances, and *DN* becomes as $\frac{DE^2 \times PS}{PE}$ *;* the total force, by which the corpuscle is attracted to the sphere, shall be as the area *ANB.*

Corol. 2. If the centripetal force of the particles shall be inversely as the distances of the corpuscles attracting each other, and *DN* becomes as $\frac{DE^2 \times PS}{PE^2}$; the force, by which the corpuscle *P* is attracted to the whole sphere, shall be as the area *ANB.*

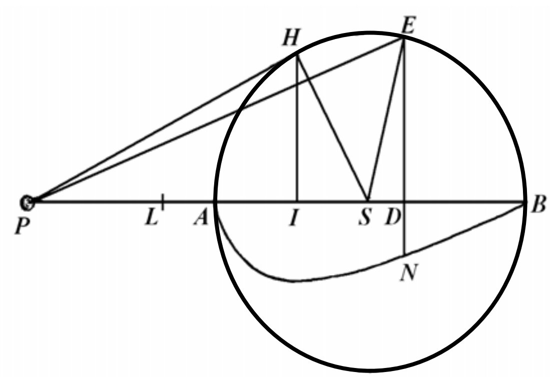
Corol. 3. If the centripetal force of the particles shall be inversely as the cube of the distances by which they attract each other, *DN* shall be as $\frac{DE^2 \times PS}{PE^4}$; and the force, by which the corpuscle is attracted by the whole sphere, shall be as the area *ANB.*

Corol. 4. And generally if the centripetal force drawing the individual particles of the sphere may be put to be inversely as the magnitude *V*, moreover *DN* becomes as $\frac{DE^2 \times PS}{PE \times V}$; and the force, by which the corpuscle is attracted to the whole sphere, is as the area *ANB.*

PROPOSITION LXXXI. PROBLEM XLI.

With everything remaining in place, it is required to measure the area ANB.

From the point *P* the right line *PH* may be drawn tangent at *H*, and the normal *HI* may be sent to the axes *PAB*, and *PI* may be bisected in *L;* and (by Prop. XII. Book 2. Eucl. *Elem*.) PE^2 will be equal to $PS^2 + SE^2 + 2PS.SD$. But SE^2 or SH^2 (on account of the similar triangles *SPH* and *SHI*) is equal to the rectangle *PS.SI* [*i.e.* $\frac{IS}{HS} = \frac{HS}{PS}$; or $\frac{IS}{a} = \frac{a}{g}$, giving $IS = \frac{a^2}{g}$, which is, of course, constant.] . Therefore PE^2 is equal to the rectangle contained by *PS* and *PS* +*SI*+2*SD,* that is, by *PS* and *2LS+2SD,* that is, by *PS* and *2LD.* Again DE^2 is equal to $SE^2 - SD^2$, or $SE^2 - LS^2 + 2SL.LD - LD^2$, that is, $2SL.LD - LD^2 - AL.LB$. For $LS^2 - SE^2$ or $LS^2 - SA^2$, (by Prop. VI. Book 2. Eucl. *Elem*.) is equal to the rectangle *AL.LB.* And thus



$$2SL.LD - LD^2 - AL.LB$$

may be written for DE^2 *;* and the magnitude $\frac{DE^2 \times PS}{PE \times V}$, which following the fourth corollary of the preceding proposition, which is as the length of the applied ordinate *DN*, may be resolved into three parts :

$$\frac{2SL\times LD\times PS}{PE\times V}-\frac{LD^{2}\times PS}{PE\times V}-\frac{AL\times LB\times PS}{PE\times V},$$

where if for *V* there may be written the centripetal force in the inverse ratio; and for *PE* the mean proportional between *PS* and 2*LD;* [for from above : $PE^2 = 2PS.LD$] these three parts produce the applied ordinates of just as many curved lines, the area of which become known by common methods.
Q.E.F.
[In more detail, for the first part : $PE^2 = (PS+SD)^2 + DE^2 = PS^2 + SE^2 + 2PS.SD$; but $SE^2 = PS.SI$ and hence $PE^2 = (PS+SD)^2 + DE^2 = PS^2 + PS.SI + 2PS.SD$,
or $PE^2 = PS(PS+SI+2SD)$; moreover, $PS+SI = 2IS+PI = 2IS+2LI = 2LS$; hence, $PE^2 = 2PS.LD$.

For the second part : $DE^2 = SE^2 - SD^2 = SE^2 - (LD-LS)^2$, giving $SE^2 - LD^2 - LS^2 + 2LD.LS$. But
$LS^2 - SE^2 = LS^2 - SB^2 = (LS-SA)(LS+SB) = LA.LB$;
hence, we may write

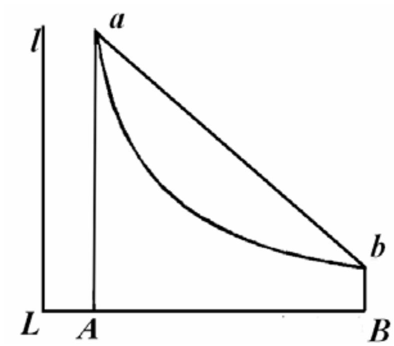
$$DE^2 = 2LD.LS - LD^2 - LA.LB;$$

which is very pretty, as all the line sections lie on the axis.
Algebraically, $PS = g; PI = g - \frac{a^2}{g} = g\left(1-\frac{a^2}{g^2}\right); PL = LI = \frac{g}{2}\left(1-\frac{a^2}{g^2}\right)$; and $LS = \frac{g}{2}\left(1+\frac{a^2}{g^2}\right)$;
$LD = LS + SD = \frac{g}{2}\left(1+\frac{a^2}{g^2}\right) + x$. While $LB = \frac{g}{2}\left(1+\frac{a^2}{g^2}\right) + a; LA = \frac{g}{2}\left(1+\frac{a^2}{g^2}\right) - a$. Hence,
$LB^2 - LA^2 = 2ag\left(1+\frac{a^2}{g^2}\right)$; while $LA.LB = \frac{g^2}{4}\left(1+\frac{a^2}{g^2}\right)^2 - a^2$. Also, we have
$LS = \frac{g}{2}\left(1+\frac{a^2}{g^2}\right)$.]

Example 1. If the centripetal force attracting the individual particles of the sphere shall be inversely as the distance; write the distance *PE* for *V;* then
$2PS\times LD$ for PE^2, and *DN* becomes as

$$SL - \tfrac{1}{2}LD - \frac{AL.LB}{2LD}.$$

Put *DN* equal to the double of this, or equal to $2SL - LD - \frac{ALB}{LD}$ *:* and the given [constant] part of the ordinate 2*SL* multiplied by the length *AB* [*i.e.* the sum of the increments *Dd*] describes the area of the rectangle $2SL\times AB$ *;* and the variable part *LD* multiplied normally into the same length [*Dd*] by a continued motion, as by that law requiring the motion to be either increasing or decreasing between [the limits] , may always be equal to the *LD*, and it will describe the area $\frac{LB^2-LA^2}{2}$, that is, the area $SL\times AB$ *;* which subtracted from the first area $2SL \times AB$ leaves the area $SL \times AB$. [Here we integrate $\int_{g-a}^{g+a} \frac{g}{2}\left(1+\frac{a^2}{g^2}\right)dx + \int_{-a}^{a} xdx = ag\left(1+\frac{a^2}{g^2}\right)$.]



But the third part $\frac{AL.LB}{LD}$, likewise with the same local motion multiplied normally into the same width, will describe the area under a hyperbola; which taken from the area $SL \times AB$ leaves the area sought *AN.NB.* From which will emerge the construction of such problems. Erect the perpendiculars *Ll, Aa, Bb,* at the points *L, A, B* of which *Aa* itself is equal to *LB,* and *Bb* to *LA.*

[Thus, the area is required under the rectangular hyperbola $y = \frac{1}{1+x}$ between the points (*A*,*LB*) and (*B*, *LA*) or ($g-a$, $\frac{g}{2}\left(1+\frac{a^2}{g^2}\right)+a$) and ($g+a$, $\frac{g}{2}\left(1+\frac{a^2}{g^2}\right)-a$) is required.]

With the asymptotes *Ll*, *LB*, the hyperbola *ab* is described through the points *a* and *b.* And with the chord *ba* drawn the area *ab.ba* enclosed [finally] is equal to the area sought *AN.NB.*

[Here the integration is performed over $\frac{AL.LB}{2LD}Dd$. This becomes:

$$\frac{LA.LB}{2LD}Dd \rightarrow \frac{\frac{g^2}{4}\left(1+\frac{a^2}{g^2}\right)^2-a^2}{g\left(1+\frac{a^2}{g^2}\right)+x}dx \rightarrow \left(\frac{A}{4}-\frac{a^2}{A}\right)\int_{g-a}^{g+a}\frac{dx}{1+\frac{x}{A}} \rightarrow \left(\frac{A^2}{4}-a^2\right)\int_{\frac{g-a}{a}}^{\frac{g+a}{a}}\frac{dx'}{1+x'}dx' = \frac{\left(g^2-a^2\right)^2}{2g^2}ln\left(\frac{g+a}{g-a}\right)$$

, where $A = g\left(1+\frac{a^2}{g^2}\right)$ and $\frac{A^2}{4}-a^2 = \frac{g^2}{4}\left(1+\frac{a^2}{g^2}\right)^2 - a^2 = \frac{\left(g^2-a^2\right)^2}{4g^2}$. The total force is then the sum of these component forces. Hence we have the total force *F* given by :

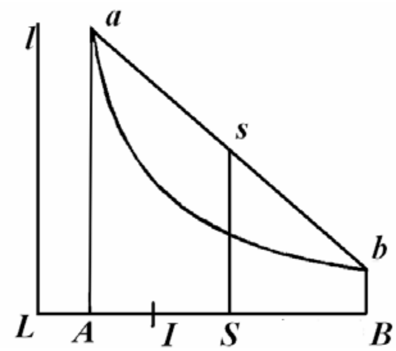
$$F = \frac{\pi}{4g^2}\left(2ag(a^2+\mathrm{g}^2)-\left(g^2-a^2\right)^2 ln\left(\frac{g+a}{g-a}\right)\right).]$$

Example 2. If the centripetal force attracting the corpuscle towards the individual particles shall be inversely as the cube of the distance, or (that is likewise) as that cube applied so some given plane ; write $\frac{PE^3}{2AS^2}$ for *V*, then as above write $2PS \times LD$ for PE^2*;* and *DN* becomes as $\frac{SL\times AS^2}{PS\times LD} - \frac{AS^2}{2PS} - \frac{AL.LB\times AS^2}{2PS\times LD^2}$, that is (on account of the continued proportionals *PS, AS, SI*) as $\frac{LS.SI}{LD} - \frac{1}{2}SI - \frac{AL.LB\times SI}{2LD^2}$.

[For recall that $SE^2 = SA^2 = PS.SI$ and hence *i.e.* $DN = \frac{AS^2}{PS}\left[\frac{SL}{LD} - \frac{1}{2} - \frac{AL.LB}{2LD^2}\right] = \left[\frac{SL\times SI}{LD} - \frac{SI}{2} - \frac{AL.LB\times SI}{2LD^2}\right]$

If the three parts of this may be multiplied by *AB*, the first $\frac{LS.SI}{LD}$,

[for $\frac{LS.SI}{LD}=\frac{a^2}{g}\times\frac{g}{2}\left(1+\frac{a^2}{g^2}\right)\times\frac{1}{\frac{g}{2}\left(1+\frac{a^2}{g^2}\right)+x}$, etc.]



will generate a hyperbolic area; the second $\frac{1}{2}SI$ the area $\frac{1}{2}AB\times SI$; the third $\frac{AL.LB\times SI}{2LD^2}$ will generate the area $\frac{AL.LB\times SI}{2LA}-\frac{AL.LB\times SI}{2LB}$, that is $\frac{1}{2}AB\times SI$. From the first there may be taken the sum of the second and the third, and the area sought *ANB* will remain. From such the construction of the problem emerges. Erect the perpendiculars *Ll, Aa, Ss, Bb* to the points *L, A, S, B,* of which *Ss* may be equal to *SI*, and through the point *s* to the asymptotes *Ll, LB* the hyperbola *asb* may be described crossing the perpendiculars *Aa, Bb* at *a* and *b;* and the rectangle 2*AS.SI* taken from the hyperbolic area *AasbB* leaves the area sought *ANB.*
[Thus, Newton fits a section of a hyperbola to the forces at the ends *A* and *B* of the diameter of the sphere, and uses this area as a known amount, without mentioning natural logarithms.]

Example 3. If the centripetal force, attracting the corpuscle to the individual particles of the sphere, decreases in the quadruple ratio of the distance from the particles ; write $\frac{PE^4}{2AS^3}$ for *V*, then $\sqrt{2PS\times LD}$ for *PE*, and *DN* becomes as

$$\frac{SI^2\times SL}{\sqrt{2SI}}\times\frac{1}{\sqrt{LD^3}}-\frac{SI^2}{2\sqrt{2SI}}\times\frac{1}{\sqrt{LD}}-\frac{SI^2\times AL.LB}{2\sqrt{2SI}}\times\frac{1}{\sqrt{LD^6}}\ ;$$

The three parts of which multiplied by the length *AB*, produces just as many areas, viz.

$$\frac{2SI^2\times SL}{\sqrt{2SI}}\text{ into }\frac{1}{\sqrt{LA}}-\frac{1}{\sqrt{LB}};\frac{SI^2}{\sqrt{2SI}}\text{ in }\sqrt{\sqrt{LB}-\sqrt{LA}};\text{ and }\frac{SI^2\times ALB}{2\sqrt{2SI}}\text{ into }\frac{1}{\sqrt{LA^3}}-\frac{1}{\sqrt{LB^3}}$$

And these after due reduction become $\frac{2SI^2\times SL}{LI}$, SI^2 ,& $SI^2+\frac{2SI^3}{3LI}$. Truly these emerge, with the latter taken from the former, $\frac{4SI^3}{3LI}$. Hence the total force, by which the corpuscle *P* is drawn to the centre of the sphere, is as $\frac{SI^3}{PI}$, that is, reciprocally as $PS^3\times PI$. *Q.E.I.*

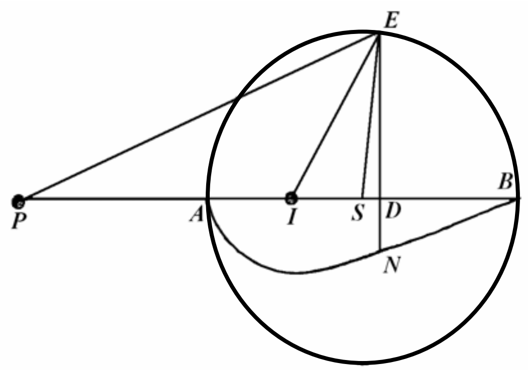
[Chandrasekhar provides equivalent modern integrations for Newton's results on pp.291-293 of his book. Leseur & Jacquier provide very longwinded derivations of these integrations, which have not been included here.]

By the same method it is possible to determine the attraction of the corpuscle in place outside the sphere, but it will be more expeditious by the following theorem.

PROPOSITION LXXXII. THEOREM XLI.

In a sphere with centre S described with the radius SA, if the continued proportions SI, SA, SP may be taken: I say that the attraction of a corpuscle within the sphere, at some place I, to the attraction outside the sphere at the place P, is compounded in the ratio from the square root of the distances from the centre IS, PS, and in the square root ratio of the centripetal forces, at these places P and I, tending to the centre of attraction.

For, if the centripetal forces of the particles of the sphere shall be reciprocally as the



distances of the corpuscles attracting each other; the force, by which a corpuscle situated at *I* is attracted by the whole sphere, to that by which it is attracted at *P*, is compounded from the ratio of the square root of the distance *SI* to the distance *SP*, and from the square root ratio of the centripetal force at the position *I*, arising from some particle situated at the centre, to the centripetal force at the position *P* arising from the same particle at the centre, that is, in the square root ratio of the distances *SI*, *SP* reciprocally in turn.
[i.e. $\frac{F_I}{F_P} = \frac{\sqrt{SP}}{\sqrt{SI}} \times \sqrt{\frac{F_{PS}}{F_{IS}}}$ in an obvious notation.]

Then these square root ratios make a ratio of equality, and therefore the attractions at *I* and *P* from the whole sphere are made equal. By a similar computation, if the forces of the particles of the sphere are inversely in the square ratio of the distances, it may be deduced that the attraction at *I* shall be to the attraction at *P*, as the distance *SP* to the radius of the sphere *SA*: If these forces are inversely as the triplicate ratio of the distances, the attractions at *I* and *P* are in turn as SP^2 to SA^2 : If in the quadruplicate, as SP^3 ad SA^3. From which since the attraction at *P*, in this final case, were found inversely as $PS^3 \times PI$, the attraction at *1* will be inversely as $SA^3 \times PI$, that is (on account of SA^3) inversely as *PI*. And the progression to infinity is similar. Thus truly the Theorem is demonstrated.

Now with everything in place as before, and with a corpuscle present at some place *P*, the applied ordinate *DN* may be found, *i.e.* $DN = \frac{DE^2 \times PS}{PE \times V}$. Therefore if IE may be drawn, that ordinate for some other corpuscle in the place *I*, with everything changed requiring to be changed, in order that $\frac{DE^2 \times IS}{IE \times V}$ will emerge. Put the centripetal forces emanating from some point of the sphere *E*, to be in turn with the distances *IE*, *PE*, as

PE^n to IE^n (where the number *n* designates the index of the powers *PE* and *IE*) and these ordinates [of the forces] become as $\frac{DE^2 \times PS}{PE \times PE^n}$ and $\frac{DE^2 \times IS}{IE \times IE^n}$, the ratio of which in turn is as $PS \times IE \times IE^n$ to $IS \times PE \times PE^n$. Because on account of the continued proportionals *SI, SE, SP,* the triangles *SPE* and *SEI* are similar, hence there becomes *IE* to *PE* as *IS* to *SE* or *SA;* for the ratio *IE* to *PE* write the ratio *IS* to *SA;* and the ratio of the ordinates emerges $PS \times IE^n$ to $SA \times PE^n$.

[Thus, the ratio of the forces on like particles at *P* and *I* due to an incremental shell at E of the sphere at E, $\frac{F_{PE}}{F_{IE}} = \frac{DE^2 \times PS \times IE \times IE^n}{PE \times PE^n \times DE^2 \times IS} = \frac{IE}{PE} \times \frac{PS}{IS} \times \frac{IE^n}{PE^n} = \frac{IS}{SE=SA} \times \frac{PS}{IS} \times \frac{IE^n}{PE^n} = \frac{PS}{SA} \times \frac{IE^n}{PE^n}$;
but $\frac{IE}{PE} = \frac{SI}{SA} = \frac{SI}{\sqrt{SP.SI}}$; or $\frac{IE}{PE} = \frac{\sqrt{SI}}{\sqrt{SP}}$; and $\frac{SP}{SA} = \frac{SP}{\sqrt{SP.SI}} = \frac{\sqrt{SP}}{\sqrt{SI}}$ hence

$$\frac{F_{PE}}{F_{IE}} = \frac{DE^2 \times PS \times IE \times IE^n}{PE \times PE^n \times DE^2 \times IS} = \frac{IE}{PE} \times \frac{PS}{IS} \times \frac{IE^n}{PE^n} = \frac{\sqrt{SP}}{\sqrt{SI}} \times \frac{SI^{\frac{n}{2}}}{SP^{\frac{n}{2}}} = \frac{SI^{\frac{n-1}{2}}}{SP^{\frac{n-1}{2}}}.]$$

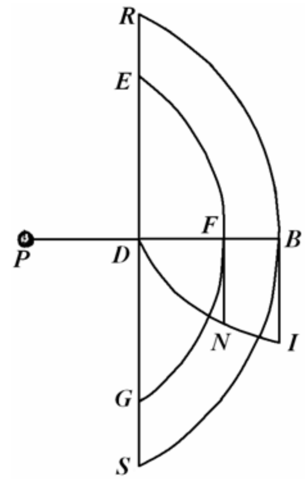
But *PS* to *SA* is as the square root ratio of the distances *PS to SI;* and the square root IE^n to PE^n (on account of the proportionals *IE* to *PE* as *IS* to *SA*) is the ratio of the forces at the distances *PS, IS.* Therefore the ordinates, and therefore the areas which the ordinates describe, and from these proportional attractions, are in a ratio composed from the square roots of these ratios. *Q.E.D.*

[The points *P* and *I* are inverse points with respect to the sphere of radius *a*; hence we have $a^2 = g \times (g-a) = SP \times SI$. From the first example, the force acting on *P* due to the whole sphere is $F_1 = \frac{\pi}{4g^2}\left(2ag(a^2+g^2) - \left(g^2 - a^2\right)^2 ln\left(\frac{g+a}{g-a}\right)\right)$ while the force F_2 acting on a similar corpuscle within the sphere at the inverse point is given by the same formula, and thus the forces at *P* and *I* are equal in this case. Chandrasekhar shows that such image forces at the image points R_1 and R_2 are related in general by $\left(\frac{F_1}{F_2}\right)^2 = \frac{R_1^{n+1}}{R_2^{n+1}}$, where $R_1 \times R_2 = a^2$, and the forces are proportional to the n^{th} power of the distance ; in this case $n = -1$, and thus the forces are equal; and the other cases Newton enumerates for higher powers are as given by this result. Thus, as Chandrasekhar points out, Newton was already familiar with the image method of inverse points applied to the gravitational cases of spheres as we will now find out, and later was discovered and used for solving electrostatics problems by William Thompson.]

PROPOSITION LXXXIII. PROBLEM XLII.

To find the force by which a corpuscle located at the centre of a sphere is attracted to any segment of this.

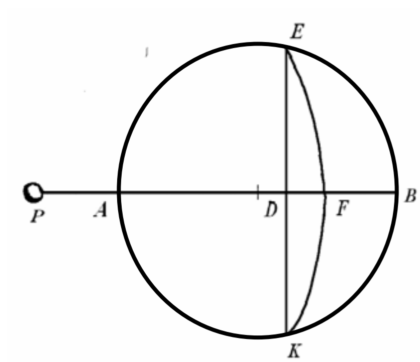
Let *P* be some body at the centre of the sphere, and *RBSD* a segment of this sphere contained by the plane *RDS* and by the spherical surface *RBS*. The spherical surface *EFG* described with centre *P* may cut *DB* in *F*, and the segment may be separated into the parts *BREFGS* and *FEDG*. But that surface shall not be a purely mathematical one, but a physical one, having a minimum depth. This thickness may be called *O*, and this will be the surface (demonstrated by *Archimedes)* as $PF \times DF \times O$. [we can think of *O* as the incremental thickness.] Besides we may put the attractive forces of the particles of the sphere to be inversely as that power of the distances of which the index is *n* ; and the force, by which the surface *EFG* pulls on the body *P*, will be (by Prop. LXXIX.) as $\frac{DE^2 \times O}{PF^n}$ that is, as $\frac{2DF \times O}{PF^{n-1}} - \frac{DF^2 \times O}{PF^n}$. [For $DE^2 = PF^2 - PD^2 = (2PF - DF) \times DF$.] The perpendicular *FN* multiplied by *O* shall be drawn proportional to this ; and the curved area *BDI*, that the applied ordinate *FN* [that is in modern terms, the *y* coordinate] will describe by the motion through the length *DB*, will be as the total force *RBSD* attracting the body *P*. *Q.E.I.*



PROPOSITION LXXXIV. PROBLEM XLIII.

To find the force, by which a corpuscle, beyond the centre of the sphere at some place on the axis of the segment, is attracted by the same segment.

The body *P* located on the axis of this *ADB* may be attracted by the segment *EBK*. With centre *P* and with the radius *PE* the spherical surface *EFK* is drawn, by which the segment may be separated into two parts *EBKSE* and *EBKDE*. The force of the first part may be found by Prop. LXXXI. and the force of the second part by Prop. LXXXIII ; and the sum of the forces will be the force of the whole segment *EBKDE*. *Q.E.I.*



Scholium.

With the attractions of spherical bodies explained, now it may be permitted to go on to the laws of attraction of certain bodies similarly constructed from attracting particles ; but to treat these a little less than it is considered customary. Certain more general proposition concerning the forces of bodies of this kind will suffice to be added, from which motions thence originated, on account of these being of some little use in philosophical matters.

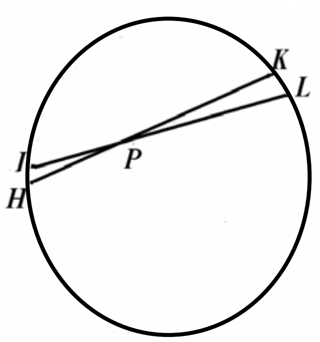
SECTIO XII.

De corporum sphaericorum viribus attractivis.

PROPOSITIO LXX. THEOREMA XXX.

Si ad sphaericae superficiei puncta singula tendant vires aequales centripetae decrescentes in duplicata ratione distantiarum a punctis : dico quod corpusculum intra superficiem constitutum his viribus nullam in partem attrahitur.

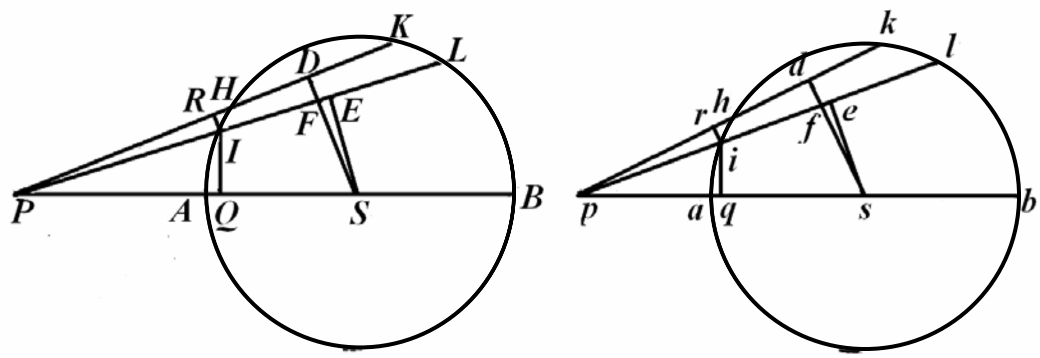
Sit *HIKL* superficies illa sphaerica, & *P* corpusculum intus constitutum. Per *P* agantur ad hanc superficiem lineae duae *HK, IL,*arcus quam minimos *HI, KL* intercipientes; &, ob triangula *HPI. LPK* (per corol. 3. lem. VII.) similia, arcus illi erunt distantiis *HP, LP* proportionales; & superficiei sphaericae particulae quaevis ad *HI* & *KL,* rectis per punctum *P* transeuntibus undique terminatae, ac erunt in duplicata illi ratione. Ergo vires harum particularum in corpus *P* exercitae sunt inter se aequales. Sunt enim ut particulae directe, & quadrata distantiarum inverse. Et hae duae rationes componunt rationem :equalitatis. Attractiones igitur, in contrarias partes aequaliter factae, se mutuo destruunt. Et simili argumento, attractiones omnes per totam sphrericam superficiem a contrariis attractionibus destruuntur. Proinde corpus *P* nullam in partem his attractionibus impellitur. *Q.E,D.*



PROPOSITIO LXXI. THEOREMA XXXI.

Iisdem positis, dico quod corpusculum extra sphaericam *superficiem constitutum attrahitur ad centrum sphaerae, vi reciproce proportionali quadrato distantiae suae ab eodem centro.*

Sint *AHKB, abhk* aequales duae superficies sphaericae, centris *S, s,* diametris *A B, a b* descriptae, & *P, P* corpuscula sita extrinsecus in diametris illis productis. Agantur a corpusculis lineae *PHK, PIL, phk, pil,* auferentes a circulis maximis *AHB, ahb,* aequales arcus *HK, hk* & *IL, il:* Et ad eas demittantur perpendicula *SD, sd; SE, se; IR, ir;* quorum *SD, sd* secent *PL, pl* in *F* & *f:* Demittantur etiam ad diametros perpendicula *IQ, iq.* Evanescant anguli *DPE, dpe:* & ob aequales *DS* & *ds, ES* & *es,* linem *PE, PF* & *pe, pf* &



lineola *DF, df* pro aequalibus habeantur; quippe quarum ratio ultima, angulis illis *DPE, dpe* simul evanescentibus, est aequalitatis. His itaque constitutis, erit *PI* ad *PF* ut *RI* ad *DF*, & *pf* ad *pi* ut *df* vel *DF* ad *ri;* & ex aequo $PI \times pf$ ad $PF \times pi$ ut *RI* ad *ri*, hoc est (per Corol. 3. Lem. VII.) ut arcus *IH* ad arcum *ih*. Rursus *PI* ad *PS* ut *IQ* ad *SE, & ps* ad *pi* ut *se* vel *SE* ad *iq;* & ex aequo $PI \times ps$ ad $PS \times pi$; ut *IQ* ad *iq*. Et conjunctis rationibus $PI\ quad. \times ps \times ps$ ad $pi\ quad. \times PF \times PS$, ut $IH \times IQ$ ad $ih \times iq$; hoc est, ut superficies circularis, quam arcus *IH* convolutione semicirculi *AKB* circa diametrum *AB* describet, ad superficiem circularem, quam arcus *ih* convolutione semicirculi *akb* circa diametrum *ab* describet. Et vires, quibus hae superficies secundum lineas ad se tendentes attrahunt corpuscula *P* & *p*, sunt (per hypothesin) ut ipsae superficies directe, & quadrata distantiarum superficierum a corporibus inverse, hoc est, ut $pf \times ps$ ad $PF \times PS$. Suntque hae vires ad ipsarum partes obliquas, quae (facta per legum corol 2. resolutione virium) secundum lineas *PS, Ps* ad centra tendunt, ut *PI* ad *PQ*, & *pi* ad *pq;* id est (ob similia triangula *PIQ* & *PSF, piq & psf)* ut *PS* ad *PF & ps* ad *pf*. Unde, ex aequa, fit attractio corpusculi hujus *P* versus *S* ad attractionem corpusculi *p* versus *s*, ut $\frac{PF \times pf \times ps}{PS}$ ad $\frac{pf \times PF \times PS}{ps}$ hoc est, ut *ps quad.* ad *PS quad.* Et simili argumento vires, quibus superficies convolutione arcuum *KL, kl* descriptae trahunt corpuscula, erunt ut *ps quad.* ad *PS quod.* inque eadem ratione erunt vires superficierum omnium circularium in quas utraque superficies sphaerica, capiendo semper *sd* aequalem *SD & se* aequalem *SE,* distingui potest. Et, per compositionem, vires totarum superficierum sphaericarum in corpuscula exercitae erunt in eadem ratione. *Q.E.D.*

PROPOSITIO LXXII. THEOREMA XXXII.

Si ad sphaerae, cuiusvis puncta singula tendant vires aequales centripetae decrescentes in duplicata ratione distantiarum a punctis; ac detur tum sphaerae densitas, tum ratio diametri sphaerae ad distantiam corpusculi a centro eius: dico quod vis, qua corpusculum attrahitur, proportionalis erit semidiametro sphaerae.

Nam concipe corpuscula duo seorsim a sphaeris duabus attrahi, unum ab una & alterum ab alterat & distantias eorum a sphaerarum centris proportionales esse diametris sphaerarum respective, sphaeras autem resolvi in particulas similes & similiter positas ad corpuscula. Et attractiones corpusculi uniust factae versus singulas particulas sphaerae unius, erunt ad attractiones alterius versus analogas totidem particulas sphaerae alterius. in ratione composita ex ratione particularum directe & ratione duplicata distantiarum inverse. Sed particulae sunt ut sphaerae, hoc est, in ratione triplicata diametrorum, & distantiae sunt ut diametri ; & ratio prior directe una cum ratione posteriore bis inverse est ratio diametri ad diametrum. *Q.E.D.*

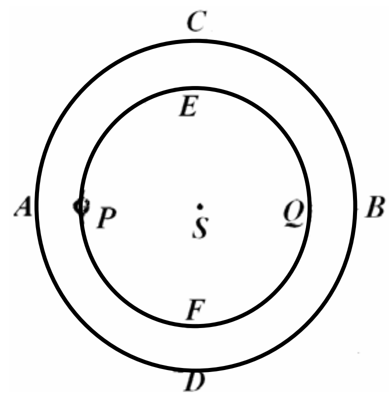
Corol. I. Hinc si corpuscula in circulis, circa sphaeras ex materia aequaliter attractiva constantes, revolvantur; sintque distantiae a centris sphaerarum proportionales earundem diametris: Tempora periodica erunt aequalia.

Corol. 2. Et vice versa, si tempora periodica sunt aequalia; distantiae erunt proportionales diametris. Constant haec duo per Corol. 3. Prop. IV.

*Corol.*3. Si ad solidorum duorum quorumvis, similium & aequaliter densorum, puncta singula tendant vires aequales centripetae, decrescentes in duplicata ratione distantiarum a punctis; vires, quibus corpuscula, ad solida illa duo similiter sita, attrahentur ab iisdem, erunt ad invicem ut diametri solidorum.

PROPOSITIO LXXIII. THEOREMA XXXIII.

Si ad sphaerae alicujus datae puncta singula tendant aequales vires centripetae decrescentes in duplicata ratione distantiarum a punttis: dico quod corpusculum intra sphaeram constitutum attrahitur vi proportionali distantiae suae ab ipsius centro.



In sphaera *ABCD,* centro *S* descripta, locetur corpusculum *P;* & centro eodem *S,* intervallo *SP,* concipe sphaeram interiorem *PEQF* describi. Manifestum est, (per Prop. LXX.) quod sphaericae superficies concentricae, ex quibus sphaerarum differentia *AEBF* componitur, attractionibus suis per attractiones contrarias destructis, nil agunt in corpus *P.* Restat sola attractio sphaerae interioris *PEQF.* Et (per Prop. LXXII.) haec est ut distantia *PS. Q.E.D.*

Scholium.

Superficies, ex quibus solida componuntur, hic non sunt pure mathematicae, sed orbes adeo tenues, ut eorum crassitudo instar nihili sit; nimirum orbes evanescentes, ex quibus sphaera ultimo constat ubi orbium illorum numerus augetur & crassitudo minuitur in infinitum. Similiter per puncta, ex quibus lineae, superficies, & solida componi dicuntur intelligendae sunt particulae aequales magnitudinis contemnendae.

PROPOSITIO LXXIV. THEOREMA XXXIV.

Iisdem positis, dico quod corpusculum extra sphaeram constitutum attrahitur vi reciproce proportionali quadrato distantiae suae ab ipsius centro.

Nam distinguatur sphaera in superficies sphaericas innumeras concentricas, & attractiones corpusculi a singulis superficiebus oriundae erunt reciproce proportionales quadrato distantiae corpusculi a centro (per prop. LXXI.) Et componendo fiet summa attractonum, hoc est attractio corpusculi in sphaeram totam, in eadem ratione. *Q. E. D.*

Corol. I. Hinc in aequalibus distantiis a centris homogenearum sphaerarum attractiones sunt ut sphaerae. Nam (per prop. LXXII.) si distantiae sunt proportionales diametris sphaerarum, vires erunt ut diametri. Minuatur distantia major in illa ratione; &, distantiis jam factis aequalibus, augebitur attractio in duplicata illa ratione; ideoque erit ad attractionem alteram in triplicata illa ratione, hoc est, in ratione sphaerarum.

Corol. 2. In distantiis quibusvis attractiones sunt ut sphaerae applicatae ad quadrata distantiarum.

Corol. 3. Si corpusculum, extra sphaeram homogeneam positum, trahitur vi reciproce proportionali quadrato distantiae suae ab ipsius centro, constet autem sphaera ex particulis attractivis; decrescet vis particulae cujusque in duplicata ratione distantiae a particula.

PROPOSITIO LXXV. THEOREMA XXXV.

Si ad sphaerae datae puncta singula tendant vires aequales centripetae, decrescentes in duplicata ratione distantiarum a punctis; dico quod sphaera quaevis alia similaris ab eadem attrahitur vi reciproce proportionali quadrato distantiae centrorum.

Nam particulae cuiusvis attractio est reciproce ut quadratum distantiae suae a centro sphaerae trahentis, (per Prop. *LXXIV.*) & propterea eadem est, ac si vis tota attrahens manaret de corpusculo unico sito in centro huius sphaerae. Haec autem attractio tanta est, quanta foret vicissim attractio corpusculi eiusdem, si modo illud a singulis sphaerae attractae particulis eadem vi traheretur, qua ipsas attrahit. Foret antem illa corpusculi attractio (per Prop. LXXIV.) reciproce proportionalis quadrato distantiae suae a centro sphaerae; ideoque huic aequalis attraccto sphaerae est in eadem ratione. *Q.E.D.*

Corol. I. Attractiones sphaerarum, versus alias sphaeras homogeneas, sunt ut sphaerae trahentes applicatae ad quadrata distantiarum centrorum suorum a centris earum, quas attrahunt.

Corol.2. Idem valet, ubi sphaera attractia etiam attrahit. Namque hujus puncta singula trahent singula alterius eadem vi, qua ab ipsis vicissim trahuntur; ideoque cum in omni attractione urgeatur (per legem 3.) tam punctum attrahens, quam punctum attractum, geminabitur vis attractionis mutuae, conservatis proportionibus.

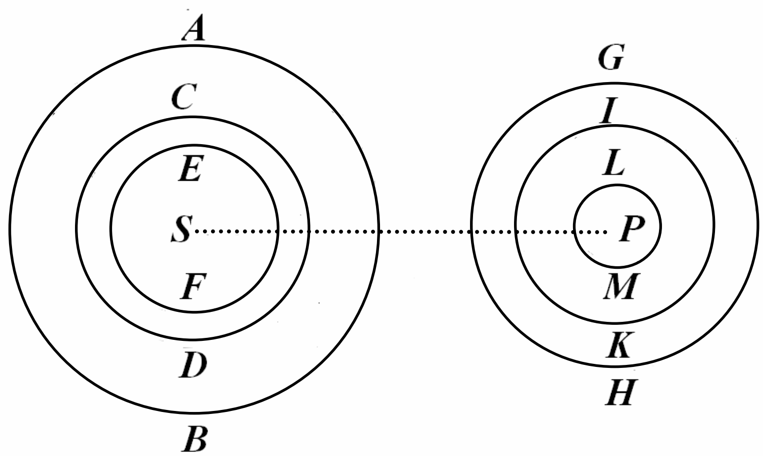
Corol. 3. Eadem omnia, quae superius de motu corporum circa umbilicum conicarum sectionum demonstrata sunt, obtinent, ubi sphaera attrahens locatur in umbilico, & corpora moventur extra sphaeram.

Corol. 4. Ea vero, quae de motu corporum circa centrum conicarum sectionum demonstrantur, obtinent ubi motus peraguntur intra sphaeram.

PROPOSITIO LXXVI. THEOREMA XXXVI.

Si sphaerae in progressu a centro ad circumferentiam (quoad materiae densitatem & *vim attractivam) utcunque dissimilares, in progressu vero per circuitum ad datam omnem a centro distantiam sunt undiquc similares;* & *vis attractiva puncti cuiusque decrescit in* *duplicata rationae distantiae corporis attracti: dico quod vis tota, qua huiusmodi sphaera una attrahit aliam, sit reciproce proportionalis quatrato distantiae centrorum.*

Sunto sphaerae quotcunque concentricae similares *AB, CD, E F,* &c. quarum interiores additae exterioribus componant materiam densiorem versus centrum, vel subductae relinquant tenuiorem; & hae (per prop. LXXV.) trahent sphaeras alias quotcunque concentricas similares *GH, IK, LM,* &c. singulae singulas, viribus reciproce



proportionalibus quadrato distantiae *SP*. Et componendo vel dividendo, summa virium illarum omnium, vel excessus aliquarum supra alias; hoc est, vis, qua sphaera tota, ex concentricis quibuscunque vel concentricarum differentiis composita *AB,* trahit totam ex concentricis quibuscunque vel concentricarum differentiis compositam *GH;* erit in eadem ratione. Augeatur numerus sphaerarum concentricarum in infinitum sic, ut materiae densitas una cum vi attractiva, in progressu a circumferentia ad centrum, secumdum legem quamcunque crescat vel decrescat ; &, addita materia non attractiva, compleatur ubivis densitas deficiens, eo ut sphaerae acquirant formam quamvis optatam; & vis, qua harum una attrahet alteram, erit etiamnum, per argumentum superius, in eadem illa distantiae quadratae ratione inversa. *Q.E.D.*

Corol. 1. Hinc si ejusmodi sphaecae complures, sibi invicem per omnia similes, se mutuo trahant ; attractiones acceleratrices singularum in singulas erunt, in aequalibus quibusvis centrorum distantiis, ut sphaerae attrahentes.

Corol. 2. Inque distantiis quibusvis inaequalibus, ut sphaerae attrahentes applicatae ad quadrata distantiarum inter centra.

Corol. 3. Attractiones vero motrices, seu pondera sphaerarum in sphaeras erunt, in aequalibus centrorum distantiis, ut sphaerae attrahentes & attractae conjunctim, id est, ut contenta sub sphaeris per multiplicationem producta.

Corol. 4. Inque distantiis inaequalibus, ut contenta illa directe & quadrata distantiarum inter centra inverse.

Corol.5. Eadem valent, ubi attractio oritur a sphaerae utriusque virtute attractiva mutuo exercita in sphaeram alteram. Nam viribus ambabus geminatur attractio, proportione servata.

Corol. 6. Si hujusmodi sphaerae aliquae circa alias quiescentes revolvantur, singulae circa singulas; sintque distantiae inter centra revolventium & quiescentium proportionales quiescentium diametris; aequalia erunt tempora periodica.

Corol.7. Et vicissim, si tempora periodica sunt aequalia; distantiae erunt proportionales diametris.

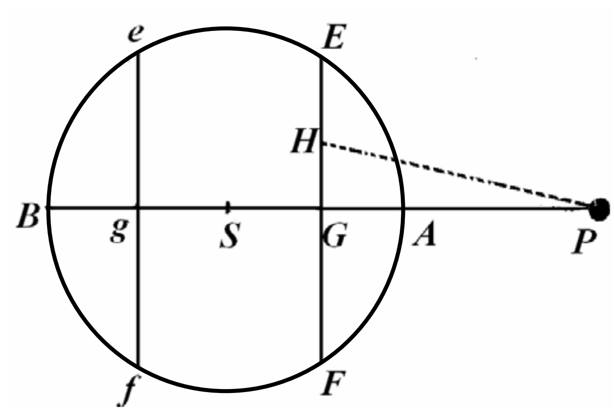
Corol. 8. Eadem omnia, quae superius de motu corporum circa umbilicos conicarum sectionum demonstrata sunt, obtinent; ubi sphaera attrahens, formae & conditionis cujusvis jam descriptae, locatur in umbilico.

Corol. 9. Ut & ubi gyrantia sunt etiam sphaerae attrahentes, conditionis cujusvis jam descriptae.

PROPOSITIO LXXVII. THEOREMA XXXVII.

Si ad singula sphaerarum puncta tendant vires centripetae proportionales distantiis punctorum a corporibus attractis: dico quod vis composita, qua sphaerae duae se mutuo trahent, est ut distantia inter centra sphaerarum.

Cas. 1. Sit *AEBF* sphaera; *S* centrum ejus; *P* corpusculum attractum, *PASB* axis sphaerae per centrum corpusculi transiens; *EF, ef* plana duo, quibus sphaera secatur, huic axi perpcndicularia, & hinc inde aequaliter distantia a centro sphaerae; *G, g* intersectiones planorum & axis; & *H* punctum quodvis in plano *EF*. Puncti *H* vis centripeta in corpusculum *P,* secundum lineam *PH* exercita, est ut distantia *PH;* & (per legum corol 2.) secundum lineam *PG,* seu versus centrum *S,* ut longitudo *PG*. Igitur punctorum omnium in plano *EF,* hoc est plani totius vis, qua corpusculum *P* traihitur versus centrum *S,* est ut distantia *PG* multiplicata per numerum punctorum, id est, ut solidum quod continetur sub plano ipso *EF* & distantia illa *PG.* Et similiter vis plani *ef,* qua corpusculum *P* trahitur versus centrum *S,* est ut planum illud ductum in distantiam suam *Pg,* sive ut huic aequale planum *EF* ductum in distantiam illam *Pg;* & summa virium plani utriusque ut planum *EF* ductarum in summam distantiarum $PG+Pg$, id est, ut planum illud ductum in duplam centri & corpusculi distantiam *PS,* hoc est, ut duplum planum *EF* ductum in distantiam *PS,* vel ut summa aequalium planorum $EF+ef$ ducta in distantiam eandem. Et simili argumento, vires omnium planorum in sphaera tota, hinc inde aequaliter a centro



sphaerae distantium, sunt ut summa planorum ducta in distantiam *PS*, hoc est, ut sphaera tota & ut distantia *PS* conjunctim. *Q.E.D.*

Cas. 2. Trahat jam corpusculum *P* sphaeram *AEBF*. Et eodem argumento probabitur quod vis, qua sphaera illa trahitur, erit ut distantia *PS. Q.E.D.*

Cas: 3. Componatur jam sphaera altera ex corpusculis innumeris *P;* & quoniam vis, qua corpusculum unumquodque trahitur, est ut distantia corpusculi a centro sphaerae primae & ut sphaera eadem conjunctim, atque ideo eadem est, ac si prodiret tota de corpusculo unico in centro sphaerae; vis tota, qua corpuscula omnia in sphaera seeunda trahuntur, hoc est, qua sphaera illa tota trahitur, eadem erit ac si sphaera illa traheaetur vi prodeunte de corpusculo unico in centro sphaerae primae, & propterea proportionalis est distantiae inter centra sphaerarum. *Q.E.D.*

Cas. 4. Trahant sphaerae se mutuo, & vis geminata proportionem priorem servabit. *Q.E.D.*

*Cas.*5. Locetur jam corpusculum *p* intra sphaeram *AEBF; &* quoniam vis plani *ef* in corpusculum est ut solidum contentum sub plano illo & distantia *pG;* & vis contraria plani *EF* ut solidum contentum sub plano illo & distantia *pG;* erit vis ex utraque composita ut differentia solidorum, hoc est, ut summa aequalium planorum ducta in semissem differentiae distantiarum, id est, ut semissema illa ducta in *pS* distantiam corpusculi a centro sphaerae. Et simili argumento, attractio planorum omnium *EF, ef* in sphaera tota, hoc est, attractio sphaerae totius, est conjunctim ut summa planorum omnium, seu sphaera tota, & ut *pS* distantia corpusculi a centro sphaerae. *Q.E.D.*

Cas. 6. Et si ex corpusculis innumeris *p* componatur sphaera nova, intra sphaeram priorem *AEBF* sita; probabitur ut prius quod attractio, sive simplex sphaerae unius in alteram, sive mutua utriusque in se invicem, erit ut distantia centrorum *pS. Q.E.D.*

PROPOSITIO LXXVIII. THEOREMA XXXVIII.

Si sphaerae in progressu a centro ad circumferentiam sint utcunque dissimilares & inaequabiles in progressu vero per circuitum ad datam omnem a centro distantiam sint undique similares; & vis attractiva puncti cujusque sit ut distantia corporis attracti: dico quod vis tota qua hujusmodi sphaerae duae se mutuo trahant sit proportionalis distantia inter centra sphaerarum.

Demonstratur ex propositione praecedente eodem modo, quo Propositio LXXVI. ex Propositione LXXV. demonstrata fuit.

Corol. Quae superius in Propositionibus X. & LXIV. de motu corporum circa centra conicarum sectionum demonstrata sunt, valent ubi attractiones omnes fiunt vi corporum sphaericorum conditionis jam descriptae, & attracta corpora sunt sphaerae conditionis ejusdem.

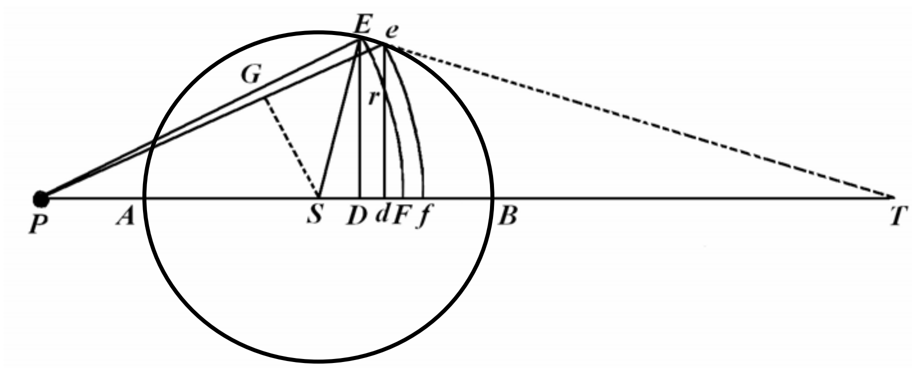
Scholium.

Attractionum carus duos insigniores jam dedi expositos; nimirum ubi vires centripetae decrescunt in duplicata distantiarum ratione, vel crescunt in distantiarum ratione simplici; efficientes in utroque casu ut corpora gyrentur in conicis sectionibus, & componentes corporum sphaericorum vires centripetas eadem lege, in recessu a centro, decrescentes vel crescentes cum seipsis: Quod est notatu dignum. Casus caeteros, qui conclusiones minus elegantes exhibent, figillatim percurrere longum esset. Malim cunctos methodo generali simul comprehendeae ac determinare, ut sequitur.

LEMMA XXIX.

Si describantur centro S circulus quilibet AEB, & centro P circuli duo EF, ef , *secantes priorem in E, e, lineamque PS in F, f*; *& ad PS demittantur perpendicula ED, ed*: dico *quod, si distantia arcuum EF, ef in infinitum minui intelligatur, ratio ultima lineae evanescentis Dd ad lineam evanescentem Ff ea sit, quae linae PE ad lineam PS*.

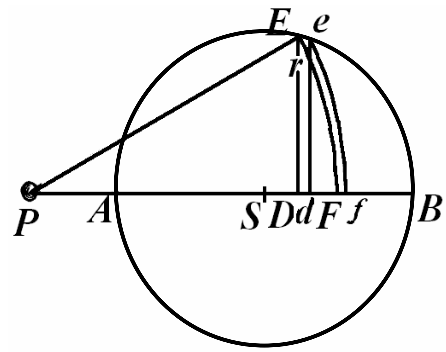
Nam si linea *Pe* secet arcum *EF* in *q;* & recta *Ee,* quae cum arcu evanescente *Ee* coincidit, producta occurrat rectae *PS* in *T;* & ab *S* demittatur in *PE* normalis *SG*: ob similia triangula *DTE, dTe, DES;* erit *Dd* ad *Ee,* ut *DT*ad *TE,* seu *DE* ad *ES;* & ob triangula *Eeq, ESG* (per Lem. VIII. & Corol 3. Lem. VII.) similia, erit *Ee* ad *eq* seu *Ff* ut *ES* ad *SG;* & ex aequo, *Dd* ad *Ff* ut *D E* ad *S G;* hoc est (ob similia triangula *PDE, PGS*) ut *PE* ad *PS. Q.E.D.*



PROPOSITIO LXXIX. THEOREMA XXXIX.

Si superficies ob latitudinem infinite diminutam jamjam evanescens EFfe, convolutione sui circa axem PS, describat solidum sphaericum concavo-convexum, ad cujus particulas singulas aequales tendant aequales vires centripetae: dico quod vis, qua solidum illud trahit corpusculum situm in P, est in ratione composita ex ratione solidi $DE\,q \times Ff$, *& ratione vis qua particula data in loco Ff traheret idem corpusculum.*

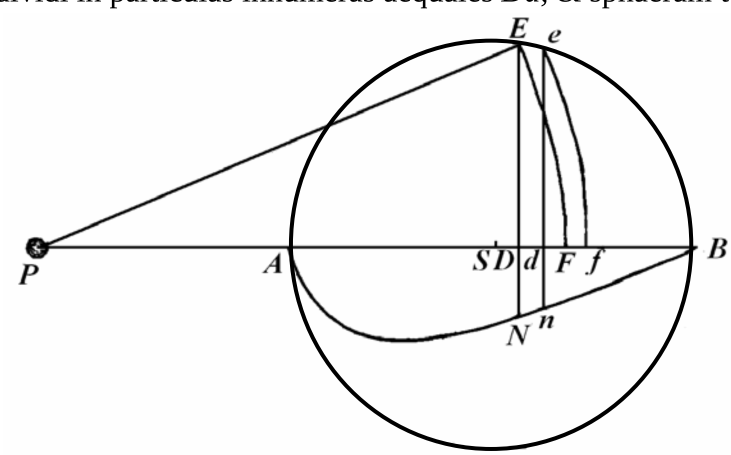
Nam si primo consideremus vim superficiei sphaericae *FE*, quae convolutione arcus *FE* generatur, & a linea *de* ubivis secatur in *r;* erit superficiei pars annularis, convolutione arcus *rE* genita, ut lineola *Dd,* manente sphaerae radio *PE* (uti demonstravit *Arcbimedes* in lib. de *Sphaera* & *Cylindro.*) Et hujus vis, secundum lineas *PE* vel *Pr* undique in superficie conica sitas exercita, ut haec ipsa superficiei pars annularis ; hoc est, ut lineola *Dd*, vel, quod perinde est, ut rectangulum sub dato spharrar radio *PE* & lineola illa *Dd* : at secumdum lineam *PS* ad centrum *S* tendentem minor in ratione *PD* ad *PE*, ideoque ut $PD \times Dd$. Dividi jam intelligatur linea *DF* in particulas innumeras aequales, quae singulae nominentur *Dd;* & superficies *FE* dividetur in totidem aequales annulos, quorum vires erunt ut summa omnium $PD \times Dd$, hoc est, ut $\frac{1}{2}PF\, q - \frac{1}{2}PD\, q$, ideoque ut *DE quad.* Ducatur jam superficies *FE* in altitudinem *Ff;* & fiet solidi *EFfe* vis exercita in corpusculum *P* ut $DE\, q \times Ff$ *:* puta si detur vis quam particula aliqua data *Ff* in distantia *PF* exercet in corpusculum *P*. At si vis illa non detur, fiet vis solidi *EFfe* ut solidum $DE\, q \times Ff$ & vis illa non data conjunctim. *Q.E.D.*



PROPOSITIO LXXX. THEOREMA XL.

Si ad sphaerae alicujus ABE, centro S descriptae, particulas singulas aequales tendant aequales vires centripetae, & *ad sphaerae axem AB, in quo corpusculum aliquod P locatur, erigantur de punctis singulis D perpendicula DE, sphaerae occurentia in E,* & *in ipsis capiantur longitudines DN, quae sint ut quantitas* $\frac{DE\, q \times PS}{PE}$ & *vis, quam sphaerae particula sita in axe ad distantiam PE exercet in corpusculum P, conjunctim: dico quod vis tota, qua corpusculum P trahitur versus sphaeram, est ut area ANB comprehensa sub axe sphaerae AB,* & *linea curva ANB, quam punctum N perpetuo tangit.*

Etenim stantibus quae in lemmate & theoremate novissimo consiructa sunt , concipe axem sphaerae *AB* dividi in particulas innumeras aequales *Dd*, & sphaeram totam dividi



in totidem laminas sphaericas concavo - convexas *EFfe;* & erigatur perpendiculum *dn.*

Per theorema superius vis, qua lamina *EFfe* trahit corpusculum *P*, est ut $DEq \times Ff$ & vis particulae *Dd* unius ad distantiam *PE* vel *PF* exercita conjunctim. Est autem (per lemma novissimum) *Dd* ad *Pf* ut *PE* ad *PS*, & inde *Ff* aequalis $\frac{PS \times Dd}{PE}$; & $DEq \times Ff$ aequale *Dd* in $\frac{DEq \times PS}{PE}$, & propterea vis laminae *EFfe* est ut *Dd* in $\frac{DEq \times PS}{PE}$ & vis particulae ad distantiam *PF* exercita conjunctim, hoc est (ex hypothesi) ut $DN \times Dd$, seu area evanescens *DNnd*. Sunt igitur laminarum omnium vires, in corpus *P* exercitae, ut areae omnes *DNnd*, hoc est, sphaerae vis tota ut area tota *ANB*. *Q.E.D.*

Corol. I. Hinc si vis centripeta, ad particulas singulas tendens, eadem semper maneat in omnibus distantiis, & fiat *DN* ut $\frac{DEq \times PS}{PE}$; erit vis tota, qua corpusculum a sphaera attrahitur, ut area *ANB*.

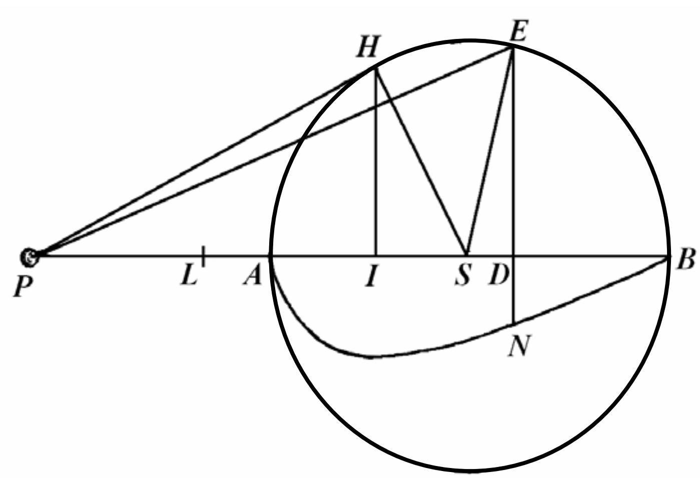
Corol. 2. Si particularum vis centripeta sit reciproce ut distantia corpusculi a se attracta, & fiat *DN* ut $\frac{DEq \times PS}{PEq}$; erit vis, qua corpusculum *P* a sphaera tota attrahitur, ut area *ANB*.

Corol. 3. Si particularum vis centripeta sit reciproce ut cubus distantiae a se attracti, & fiat *DN* ut $\frac{DEq \times PS}{PEqq}$; erit vis, qua corpusculum a tota sphaera attrahitur, ut area *ANB*.

Corol. 4. Et universaliter si vis centripeta ad singulas sphaerae particulas tendens ponatur esse reciproce ut quantitas *V*, fiat autem *DN* ut $\frac{DEq \times PS}{PE \times V}$; erit vis, qua corpusculum a sphaera tota attrahitur, ut area *ANB*.

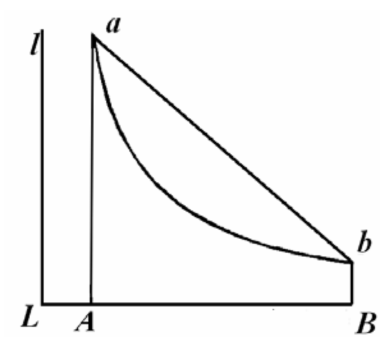
PROPOSITIO LXXXI. PROBLEMA XLI.
Stantibus jam positis, mensuranda est area ANB.

A puncto *P* ducatur recta *PH* sphaeram tangens in *H*, & ad axem *PAB* demissa normali *HI*, bisecetur *PI* in *L;* & erit (per Prop. XII. Lib. 2. elem.) *PEq* aequale



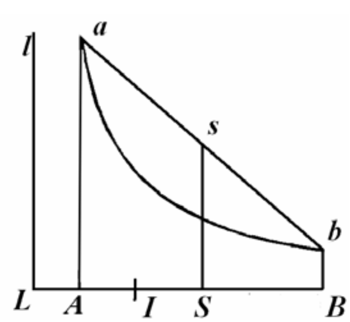
$PSq + SEq + 2PSD$. Est autem *SEq* seu *SHq* (ob similitudinem triangulorum *SPH, SHI*) aequale rectangulo *PSI*. Ergo *PEq* aequale est contento sub *PS & PS +SI*+2*SD,* hoc est, sub *PS* & *2LS+2SD,* id est, sub *PS* & *2LD.* Porro *DE quad.* aequale est $SEq - SDq$, seu $SEq - LSq + 2SLD - LDq$, id est, $2SLD - LDq - ALB$. Nam $LSq - SEq$ seu $LSq - SAq$, (per Prop. VI. Lib. 2. elem.) aequatur rectangulo *ALB.* Scribatur itaque $2SLD - LDq - ALB$ pro DEq; & quantitas $\frac{DEq \times PS}{PE \times V}$, quae secundum corollarium quartum propositionis praecedentis est ut longitudo ordinatim applicatae *DN*, resolvet in tres partes $\frac{2SLD \times PS}{PE \times V} - \frac{LDq \times PS}{PE \times V} - \frac{ALB \times PS}{PE \times V}$ ubi si pro *V* scribatur ratio inversa vis centripetae; & pro *PE* medium proportional,: inter *PS & 2LD;* tres illae partes evadent ordinatim applicatae linearum totidem curvarum, quarum area per methodos vulgatas innotescunt
Q.E.F.

Exempl. 1. Si vis centripeta ad singulas sphaerae particulas tendens sit reciproce ut distantia; pro *V* scribe distantiam *PE;* dein $2PS \times LD$ pro PEq, & fiet *DN* ut $SL - \frac{1}{2}LD - \frac{ALB}{2LD}$.



Pone *DN* aequalem ejus duplo $2SL - LD - \frac{ALB}{LD}$ *:* & ordinatae pars data 2*SL* ducta in longitudinem *AB* describet aream rectangulam $2SL \times AB$*;* & pars indefinita *LD* ducta normaliter in eandem longitudinem per motum continuum, ea lege ut inter movendum crescendo vel decrescendo aequetur semper longitudini *LD*, describet aream $\frac{LBq - LAq}{2}$, id est, aream $SL \times AB$*;* qui subducta de area priore $2SL \times AB$ relinquit aream $SL \times AB$.

Pars autem tertia $\frac{ALB}{LD}$, ducta itidem per motum localem normaliter in eandem longitudinem; describet aream hyperbolicam; quae subducta de area $SL \times AB$ relinquet aream quaesitam *ANB.* Unde talis emergit problematis constructio. Ad puncta *L, A, B* erige perpendicula *Ll, Aa, Bb,* quorum *Aa* ipsi *LB,* & *Bb* ipsi *LA* aequetur. Asymptotis *Ll, LB,* per puncta *ab* describatur hyperbola *ab.* Et acta chorda *ba* claudet aream *aba* areae quaesitae *ANB* aequalem.



Exempt. 2. Si vis centripeta ad singulas sphaerae particulas tendent sit reciproce ut cubus distantiae, vel (quod perinde est) ut cubus ille applicatus ad planum quovis datum; scribe $\frac{PE\,cub.}{2ASq}$ pro *V*, dein $2PS \times LD$ pro PEq*;* & fiet *DN* ut $\frac{SL \times ASq}{PS \times LD} - \frac{ASq}{2PS} - \frac{ALB \times ASq}{2PS \times LDq}$, id est (ob continue proportionales *P S, AS, S I*) ut $\frac{LSI}{LD}-\frac{1}{2}SI-\frac{ALB\times SI}{2LDq}$. Si ducantur huius partes tres in longitudinem *AB*, prima $\frac{LSI}{LD}$ generabit aream hyperbolicam; secunda $\frac{1}{2}SI$ aream $\frac{1}{2}AB\times SI$; tertia $\frac{ALB\times SI}{2LDq}$ aream $\frac{ALB\times SI}{2LA}-\frac{ALB\times SI}{2LB}$, it est $\frac{1}{2}AB\times SI$. De prima subducatur summa secundae & tertiae, & manebit area quaesita *ANB.* Unde talis emergit problematis constructio. Ad puncta *L, A, S, B* erige perpendicula *Ll, Aa, Ss, Bb*, quorum *Ss* ipsi *SI* aequetur, perque punctum *s* asymptotis *Ll, LB* describatur hyperbola *asb* occurrens perpendiculis *Aa, Bb* in *a* & *b;* & rectangulum 2*ASI* subductum de area hyperboica *AasbB* relinquet aream quaesitam *ANB.*

Exempt. 3. Si vis centripeta, ad singulas sphaerae particulas tendens, decrescit in quadruplicata ratione distantim a particulis ; scribe $\frac{PE\,qq}{2AS\,cub.}$ pro *V*, dein $\sqrt{2PS\times LD}$ pro *PE,* & fiet *DN* ut

$$\frac{SI\,q\times SL}{\sqrt{2SI}}\times\frac{1}{\sqrt{LDc}}-\frac{SI\,q}{2\sqrt{2SI}}\times\frac{1}{\sqrt{LD}}-\frac{SI\,q\times ALB}{2\sqrt{2SI}}\times\frac{1}{\sqrt{LDqc}}\ ;$$

Cuius tres partes ductae in longitudinem *AB,* producunt areas totidem, viz.

$$\frac{2SI\,q\times SL}{\sqrt{2SI}}\text{ in }\frac{1}{\sqrt{LA}}-\frac{1}{\sqrt{LB}};\frac{SI\,q}{\sqrt{2SI}}\text{ in }\overline{\sqrt{LB}-\sqrt{LA}};\text{ et}\frac{SI\,q\times ALB}{2\sqrt{2SI}}\text{ in }\frac{1}{\sqrt{LA\,cub.}}-\frac{1}{\sqrt{LB\,cub.}}$$

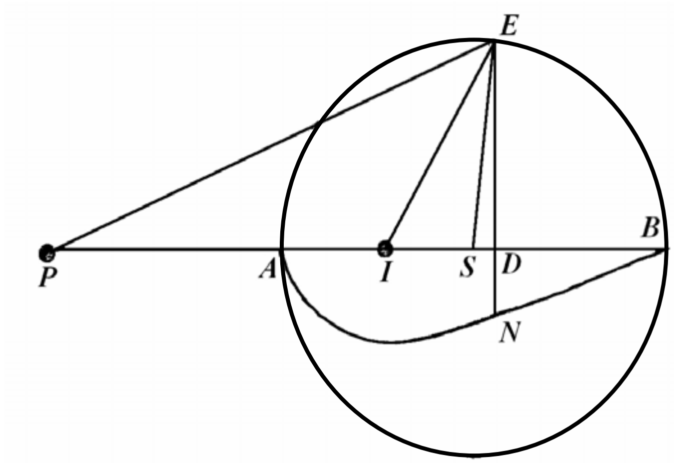
Et hae post debitam reductionem fiunt $\frac{2SI\,q\times SL}{LI}$,$SI\,q$,&$SI\,q+\frac{2SI\,cub.}{3LI}$. Hae vero, subductis posterioribus de priore, evadunt $\frac{4SI\,cub.}{3LI}$. Proinde vis tota, qua corpusceulum *P* in sphaerae centrum trahitur, est ut $\frac{SI\,cub.}{PI}$, id est, reciproce ut $PS\,cub.\times PI$. *Q.E.I.*

Eadem methodo determinari potest attractio corpusculi siti intra sphaeram, sed expeditius per theorema sequens.

PROPOSITIO LXXXII. THEOREMA XLI.

In sphaera centro S intervallo SA descripta, si capiantur SI, SA, SP continue proportionales: dico quod corpusculi intra sphaeram, in loco quovis I, attractio est ad attractionem ipsius extra sphaeram, in loco P, in ratione composita ex subduplicata ratione distantiarum a centro IS, PS, & *subduplicata ratione virium centripetarum, in locis illis P & I, ad centrum tendentium.*

Ut, si vires centripetae particularum sphaerae sint reciproce ut distantiae corpusculi a se attracti; vis, qua corpusculum situm in *I* trahitur a sphaera tota, erit ad vim, qua trahitur in *P,* in ratione composita ex subduplicata ratione distantiae *SI* ad distantiam *SP,* & ratione subduplicata vis centripetae in loco *I*, a particula aliqua in centro oriundae, ad vim centripetam in loco *P* ab eadem in centro particula oriundam, id est, ratione subduplicata distantiarum *SI, SP* ad invicem reciproce. Hae dum rationes subduplicatae componunt rationem aequalitatis, & propterea attractiones in *I* & *P* a sphaera tota factae aequantur. Simili computo, si vires particularum sphaerae sunt reciproce in duplicata ratione distantiarum, colligetur quod attractio in *I* sit ad attractionem in *P*, ut distantia *SP* ad sphaerae semidiametrum *SA:* Si vires illae sunt reciproce in triplicata ratione distantiarum, attractiones in *I* & *P* erunt ad invicem ut *S:P quad.* ad *SA quad.:* Si in



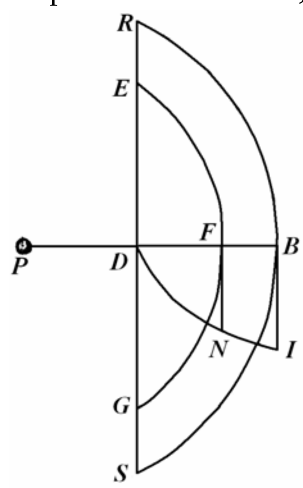
quadruplicata, ut *SP cub.* ad *SA cub.* Unde cum attractio in *P*, in hoc ultimo casu, inventa fuit reciproce ut $PS\,cub.\times PI$, attractio in *1* erit reciproce ut $SA cub.\times PI$, id est (ob datum *SA cub.)* reciproce ut *PI.* Et similis est progressus in infinitum. Theorema vero sic demonstratur.

Stantibus jam ante constructis, & existente corpusculo in loco quovis P, ordinatim applicata *DN* inventa fuit $\frac{DE\,cub.\times PS}{PE\times V}$. Ergo si agatur *IE*, ordinata illa pro alia quovis corpusculi loco *I*, mutatis mutandis, evadet ut $\frac{DE\,cub.\times IS}{IE\times V}$. Pone vires centripetas, e sphaerae puncto quovis *E* manantes, esse ad invicem in distantiis *IE, PE,* ut PE^n ad IE^n (ubi numerus *n* designet indicem potestatum *PE* & *IE*) & ordinatae illae fient ut $\frac{DE\,q\times PS}{PE\times PE^n}$ & $\frac{DE\,q\times IS}{PE\times IE^n}$ quarum ratio ad invicem est ut $PS\times IE\times IE^n$ ad $IS\times PE\times PE^n$. Quoniam ob continue proportionales *SI, SE, SP,* similia sunt triangula *SPE, SEI,* & inde fit *IE* ad *PE* ut *IS* ad *SE* vel *SA;* pro ratione *IE* ad *PE* scribe rationem *IS* ad *SA;* & ordinatarum ratio evadet $PS\times IE^n$ ad $SA\times PE^n$. Sed *PS* ad *SA* subduplicata est ratio distantiarum *PS, SI;* & IE^n ad PE^n (ob proportionales *IE* ad *PE* ut *IS* ad *SA*) subduplicata est ratio virium in distantiis *PS, IS.* Ergo ordinatae, & propterea areae quas ordinatae describunt, hisque proportionales attractiones, sunt in ratione composita ex subduplicatis illis rationibus. *Q.E.D.*

PROPOSITIO LXXXIII. PROBLEMA XLII.

Invenire vim qua corpusculum in centro sphaerae locatum ad eius segmentum quodcunque attrahitur.

Sit *P* corpus in centro sphaerae, & *RBSD* segmentum ejus plano *RDS* & superficie sphaerica *RBS* contentum. Superficie sphaerica *EFG* centro *P* descripta secetur *DB* in *F*, ac distinguatur segmentum in partes *BREFGS, FEDG.* Sit autem superficies illa non pure mathematica, sed physica, profunditatem habens quam minimam. Nominetur ista profunditas *O*, & erit haec superficies (per demonstrata *Archimedis)* ut $PF \times DF \times O$. Ponamus praeterea vires attractas particularum sphaerae esse reciproce ut distantiarum dignitas illa, cujus index est *n* ; & vis, qua superficies *EFG* trahit corpus *P*, eritt (per Prop. LXXIX.) ut $\frac{DE\,q \times O}{PF^{n}}$ id est, ut $\frac{2DF \times O}{PF^{n-1}} - \frac{DF^{2} \times O}{PF^{n}}$. Huic proportionale sit perpendiculum *FN* ductum in O; & area curvilinea *BDI*, quam ordinatim applicata *FN* in longitudinem *DB* per motum continuum ducta describit, erit ut vis tota qua segmentum totum *R B SD* trahit corpus *P*. *Q.E.I.*



PROPOSITIO LXXXIV. PROBLEMA XLIII.

lnvenire vim, qua corpusculum, extra centrum sphaerum in axe segmenti cuiusvis locatum, attrahitur ab eodem segmento.

A segmento *EBK* trahatur corpus *P* in ejus axe *ADB* locatum. Centro *P* intervallo *PE* describatur superficies sphaerica *EFK*, qua distinguatur segmentum in partes duas *EBK SE & EBKDE* quaeratur vis partis prioris per Prop. LXXXI. & vis partis posterioris per Prop. LXXXIII ; & summa virium eric vis segmenti totius *EBKDE. Q.E.I.*

Scholium.

Explicatis attractionibus corporum sphaericoram, jam pcrgere liceaet ad leges attractionum aliorum quorundam ex particulis attractivis similiter constantium corporum; sed ista particulatim tractare minus ad institutum spectat. Suffecerit propositiones quaedam generaliores de viribus hujusmodi corporum, deque motibus inde oriundis, ob earum in rebus philosophicis aliqualem usum, subjungere.