SECTION XI.

Concerning the motion of bodies with centripetal forces mutually attracting each other.

Up to the present, I have explained the motion of bodies attracted to a fixed centre of force, yet scarcely such a force is extant in the nature of things. For the attractions are accustomed to be for bodies ; and the actions of pulling and attracting are always mutual and equal, by the third law: thus, if there shall be two bodies, so that neither shall it be possible to be attracting or to be attracted and to be at rest, but both shall be rotating around the common centre of gravity (by the fourth corollary of the laws), as if by mutual attraction : and if there shall be several bodies, which either may be attracted by a single body, and which likewise they may attract, or all may mutually attract each other; thus these must be moving among themselves, so that the common centre of gravity may be at rest, or may be moving uniformly in a direction. From which reason I now go on to explain the motion of bodies mutually attracting each other, by considering the centripetal forces as attractions, although perhaps more truly they may be called impulses, if we may speak physically. For now we may turn to mathematics, and therefore, with the physical arguments dismissed, we use familiar speech, by which we shall be able to be understood more clearly by mathematical readers.

[The stand adopted by Newton has been explained in detail by Cohen in the introduction to his translation; essentially the physical world is to be understood from mathematical laws and considerations, and forces such as gravity, acting at a distance with no visible means of communicating forces, are to be understood by the mathematical relations they satisfy, rather than from some physical model involving unseen fluids and vortices, as in Descartes' model.]

PROPOSITION LVII. THEOREM XX.

Two bodies attracting each other in turn describe similar figures, both around the common centre of gravity and mutually around each other.

For the distances of the bodies from the common centre of gravity are inversely proportional to the bodies [Newton means masses of bodies when he refers to bodies]; and thus in a given ratio one to the other, and on being put together in a given ratio to the total distance between the bodies. But these distances are carried around their common end [i.e. the centre of mass or gravity] by an equal angular motion, so that they do not therefore change their mutual inclination, always lying on a line. But right lines, which are reciprocally in a given ratio, and which are carried around their ends by an equal angular motion, describe completely similar figures about these same ends in planes, which together with these ends either are at rest, or may be moving in some non angular motion [*i.e.* the linear motion of the centre of mass.] Hence these figures which are described from the distances being turned through are similar.
Q.E.D.

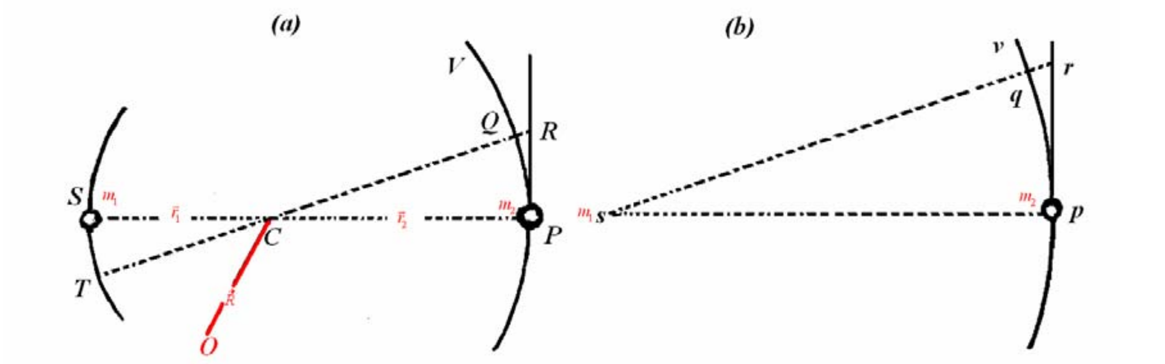
[The analytical solutions of the two – body problem now presented in this section are solved to some extent in modern texts on dynamics, and Chandrasekhar gives proofs in his work on Newton, from p. 207, onwards. Chandrasekhar thus solves the dynamical problem in the reference frame of a body *P*, and finds that the original force is augmented, for what Newton states depends on the bodies being in an inertial reference frame in which the forces and accelerations remain the same (thus, a kinematic change of viewpoint can be adopted) ; what we find in non-inertial frames, of course, is not surprising, as the rotating body *P* below is itself accelerating, and so 'fictitious' forces must be added to *P* to accommodate the correct motion, if we are to consider *S* as an inertial frame : for *S*'s frame likewise is an accelerating reference; Chandrasekhar goes on to state that, ' Newton's proof, again couched in words, it is essentially the same....' .

Thus, it seems appropriate to add some variables in red to Newton's diagrams that follow, and to indicate briefly the mathematical origins of the statements made and not demonstrated fully by Newton. We may note in modern terms that the left-hand diagram (*a*) refers either to an inertial frame in which the centre of mass of the system is at rest or moving uniformly in a straight line (note that Newton refrains from talking about straight lines, and discusses only directions, because otherwise it begs the question of the first law of motion, as discussed in the definitions and axioms of this work), while the right-hand diagram (*b*) considers *s* as a reference frame at rest. So we begin....]

PROPOSITION LVIII. THEOREM XXI.

If two bodies attract each other by forces of some kind, and meanwhile they rotate about their common centre of gravity: I say of the figure, which the bodies describe around each other mutually by moving thus, that it shall be similar and equal to the figure, around either body at rest, to be described by the same forces.

The bodies *S* and *P* are revolving around the common centre of gravity *C*, by going from *S* to *T*; and from *P* to *Q* . From a given point *s*, *sp* and *sq* may always be drawn equal and parallel to *SP* and *TQ* themselves ; and the curve *pqv*, that the point *p* will describe by revolving around the fixed point *s*, will be similar and equal to the curves, which the bodies *S* and *P* describe around each other mutually: and hence (by Theorem XX) similar to the curves *ST* and *PQV*,



which the same bodies will describe around the common centre of gravity *C:* and that because the proportions of the lines *SC*, *CP*, and *SP* or *sp* may be given in turn.

[Note from (*a*) that the orbits of the conic sections *S* and *P* generally are of differing sizes (here we will concentrate on ellipses), and the mass m_1 in the case shown is greater than the mass m_2 : if the force is proportional to the distance from the centre, then the two orbits will be concentric ellipses with the centre at the centre of gravity. In the inverse square case, the centre of mass is a common focus, the ellipses may intersect, but the bodies themselves must always be at opposite ends of a line such as *TQ*, intersecting both orbits. In both cases, the areas swept out by a radius are proportional to the times. In (*b*), according to C, the orbit of *p* is similar to that of *P*, but viewed from the stationary reference frame *S*, and a similar remark can be made about the orbit of *s* relative to *p*.

If *O* is an arbitrary origin in (*a*), then the positions of *S* and *P* are given by $\vec{R}+\vec{r}_1$ and $\vec{R}+\vec{r}_2$, where we consider $OC=\vec{R}$, $CS=\vec{r}_1$, and $CP=\vec{r}_2$ as the vectors shown; and the centre of gravity or mass is given by $\vec{R}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$. If the centre of gravity is at rest, then $\dot{\vec{R}}=0$ and $0=m_1\dot{r}_1+m_2\dot{r}_2$, dispensing with vector notation, as the bodies lie on a straight line. We also have $0=m_1\ddot{r}_1+m_2\ddot{r}_2$, on differentiating again, so that the forces acting on *S* and *P* are equal and opposite, as required by the third law. If we choose C to be the origin, then we can write $m_1r_1=m_2r_2$, dispensing with signs : a useful relation. The force between the masses meanwhile may be represented by *F*, and this is a symmetric function of the masses, and depends on the distance between them. Hence we may write $F=m_1\ddot{r}_1$ and $F=-m_2\ddot{r}_2$, acting along the line joining *SP* in an attractive manner, positive going from left to right, so that *S* is urged forwards by a positive force, while *P* is urged backwards by an equal and opposite negative force at the instant shown. We also observe that though the forces are equal and opposite, the acceleration of the body varies inversely with its mass, and thus the larger mass accelerates less and produces the smaller orbit than the smaller mass.

Now if we wish to consider the motion of *P* relative to *S*, then the relative displacement can be written as $\vec{r}_2-\vec{r}_1$, and once more dispensing with vectors, we have the acceleration of *P* relative to *S* given by $\ddot{r}_2-\ddot{r}_1$, and this can be written as $a_{SP}=a_{sp}=\ddot{r}_2-\ddot{r}_1=-\frac{F}{m_2}-\frac{F}{m_1}=-\frac{m_1+m_2}{m_1m_2}F$, acting towards *S*. Hence, the force acting on *P* relative to *S* in (*a*), or as *p* relative to *s* in (*b*), which is seen to be reduced to the previous immoveable cases treated, is given by $F_{PS}=F_{ps}=m_2\left(\ddot{r}_2-\ddot{r}_1\right)=-\frac{m_1+m_2}{m_1}F$ towards *s* : that is, the force *F* for the motion relative to *C*, has been augmented by the factor $\frac{m_1+m_2}{m_2}$ for the motion of *s* relative to *p*. In a similar manner, we can write $F_{SP}=m_1\left(\ddot{r}_1-\ddot{r}_2\right)=\frac{m_1+m_2}{m_2}F$, which is the augmented force on *S* due to *P*, now considered at rest. Notice that these forces on *P* and *p* in diagrams (*a*) and (*b*) are no longer equal to each other, the one is given by $F_{CP}=-m_2\ddot{r}_2=F$ and the other by $F_{ps}=-\frac{m_1+m_2}{m_1}F$; hence the ratio of the accelerations on the same mass *P* is given by $\frac{a_{PC}}{a_{ps}} = \frac{m_1}{m_1+m_2}$; while the nascent distances are given in the ratio $\frac{RQ}{rq} = \frac{m_1(\Delta t_1)^2}{(m_1+m_2)(\Delta t_2)^2} = \frac{CP}{sp}$, and the velocities are in the ratio $\frac{v_P}{v_p} = \frac{m_1(\Delta t_1)}{(m_1+m_2)(\Delta t_2)}$, where we consider the forces to act for different times.

Following Newton's arguments presented in case 1 below, due to the similar figures, we have : $\frac{RQ}{rq} = \frac{CP}{sp} = \frac{r_2}{r_1+r_2}$; hence *if* we had the forces in this ratio [corresponding to forces proportional to distance from the centre of the ellipses], then we would have : $\frac{F_{CP}}{F_{sp}} = \frac{r_2}{r_1+r_2} = \frac{CP}{SP} = \frac{CP}{sp}$ acting for the same small interval of time Δt , giving rise to the same centripetal accelerations, either $\frac{v_P^2}{CP}$ or $\frac{v_p^2}{sp}$, from which $\frac{v_P}{v_p} = \sqrt{\frac{CP}{sp}}$, giving rise to the similar curves *pqv* and *PQV*, and the revolutions would be completed in the same time.

But according to Newton, the forces are considered to be the same in each case, [we may presume that he means in the centre of mass inertial frame], so that the accelerations produced on *P* and *p* are equal, and the distances through which the body is drawn inwards is *RQ* in one case, and *rq* in the other, where $rq > RQ$, and hence the same acceleration must act for a longer interval to produce these nascent or infinitesimal displacements *QR* and *qr* ; thus we have, since $QR = \frac{1}{2}a\left(\Delta t_1\right)^2$ and $qr = \frac{1}{2}a\left(\Delta t_2\right)^2$, then $\frac{\sqrt{QR}}{\sqrt{qr}} = \frac{\Delta t_1}{\Delta t_2} = \frac{\sqrt{CP}}{\sqrt{sp}} = \sqrt{\frac{r_2}{r_1+r_2}} = \frac{\sqrt{m_1}}{\sqrt{m_1+m_2}}$. In this case, we have the similar curves *pqv* and *PQV* produced, but not in the same times; as in this case, the time for one body to orbit the other *t* is not the same as the time *T* for both bodies to orbit around the centre of mass *C*.

We can continue to apply the dynamic analysis to the situation with modified forces, and retaining Newton's insightful way of dealing with the nascent distances, to consider the ratio of the forces acting on *P* and *p*, which is the same as the ratio of the accelerations, augmented as discussed above : $\frac{v_P}{v_p} = \frac{a_{CP}\Delta t_1}{a_{ps}\Delta t_2}$, where

$\frac{a_{CP}}{a_{ps}} = \frac{m_1}{m_1+m_2}$ and $\frac{QR}{qr} = \frac{CP}{ps} = \frac{a_{CP}(\Delta t_1)^2}{a_{ps}(\Delta t_2)^2} = \frac{m_1(\Delta t_1)^2}{(m_1+m_2)(\Delta t_2)^2}$ and thus giving $\frac{\sqrt{(m_1+m_2)CP}}{\sqrt{m_1 ps}} = \frac{\Delta t_1}{\Delta t_2}$; and $\frac{v_P}{v_p} = \frac{a_{CP}\Delta t_1}{a_{ps}\Delta t_2} = \frac{m_1}{m_1+m_2} \times \frac{\sqrt{(m_1+m_2)QR}}{\sqrt{m_1 qr}} = \frac{\sqrt{m_1^2}}{\sqrt{(m_1+m_2)^2}} = \frac{m_1}{m_1+m_2} = \frac{r_2}{r_1+r_2}$; which Newton deduces below, and thus it seems that Newton was using this formulation of the problem after all.]

Case 1. That common centre of gravity *C*, by the fourth corollary of the laws, either is at rest or moving uniformly in a direction. We may put that initially to be at rest, and the two bodies may be located at *s* and *p*, with the immobile body at *s*, the mobile one at *p*, with the bodies *S* and *P* similar and equal to the bodies *s* and *p*. Then the right lines *PR* and *pr* may touch the curves *PQ* and *pq* in *P* and *p*, and *CQ* and *sq* may be produced to *R* and *r*. And on account of the similitude of the figures *CPRQ* and *sprq*, *RQ* will be to *rq* as *CP* to *sp*, and thus in a given ratio [indeed, $\frac{RQ}{rq} = \frac{CP}{sp} = \frac{r_2}{r_1+r_2}$]. Hence if the force, by which the body *P* is attracted towards the body *S*, and thus towards the intermediate centre *C*, should be in that same given ratio to the force, by which the body *p* is attracted towards the centre *s* [i.e. the above ratio $\frac{F_{CP}}{F_{sp}} = \frac{r_2}{r_1+r_2}$] ; these forces in equal times always attract the bodies from the tangents *PR* and *Pr* to the arcs *PQ* and *pq* by intervals proportional to *RQ* and *rq* themselves, and thus the latter force effects, that the body *p* may rotate in the curve *pqv*, which shall be similar to the curve *PQV*, in which the former force effects that the body *P* may be revolving; and the revolutions may be completed in the same time.

But since these forces are not inversely in the ratio *CP* to *sp*, rather (on account of the similitude and equality of the bodies *S* and *s*, *P* and *p*, and the equality of the distances *SP* and *sp*) but mutually equal to each other [as discussed above]; the bodies will be drawn equally in equal times from the tangents: and therefore, so that the latter body *p* may be drawn by the greater interval *rq*, a greater time is required, and that in the square root ratio of the intervals; therefore (by the tenth lemma) because the distances have been described in the squares ratio of the times from the beginning of the motion itself. Therefore we may put the velocity of the body *p* to be as the velocity of the body *P* in the square root ratio of the distance *sp* to the distance *CP*, thus so that in the times, which shall be in the same square root ratio, the arcs *pq* and *PQ* may be described, which are in a whole [ordinary] ratio : And the bodies *P* and *p* always attracted by equal forces describe the similar figures *PQV, pqv*, around the centres *C* and *s* at rest, of which the latter *pqv* is similar and equal to the figure, that the body *P* will describe about the mobile centre *S*. *Q.E.D.*

Case 2. Now we may consider that the common centre of gravity, together with the distance in which the bodies may be moving among themselves, is progressing uniformly along a direction; and (by the law of the sixth corollary) all the motions may advance in this space as before, and thus the bodies describe the same figures around each other as at first, and therefore to the similar and equal figure *pqv*. *Q.E.D.*

Corol. I. Hence two bodies with the forces proportional to their distances attracting each other mutually, (by Prop. X.) describe concentric ellipses both around the common centre of gravity and around each other, and vice versa, if such figures are described, then the forces are proportional to the distances.

Corol. 2. And two bodies, with forces inversely proportional to the square of the distance, describe (by Prop. XI. XII. XIII.) both around the common centre of gravity and about each other, conic sections having the focus at the centre, around which the figures are described. And conversely, if such figures are described, the centripetal forces are reciprocally as the squares of the distance.

Corol. 3. Any two bodies rotating about the common centre of gravity, with the radii drawn both to that centre and between themselves, describe areas proportional to the times.

PROPOSITION LIX. THEOREM XXII.

The periodic time of two bodies S and P, rotating about their common centre of gravity C, is to the periodic time of rotation of either of the bodies P, rotating about the other stationary body S, and with the figures, which the bodies describe mutually around each other, similar and equal to the figure described, as the square root ratio of [the mass of] the other body S, to the sum of the [masses of the] bodies $S+P$.

Indeed, from the demonstration of the above proposition, the times, in which any similar arcs *PQ* and *pq* are described, are in the square root ratio of the distances *CP* and *SP* or *sp,* that is, in the square root ratio of the body *S* to the sum of the bodies $S+P$. And on adding together, the sum of all the times in which all the similar arcs *PQ* and *pq* are describes, that is, the total time, in which the whole similar figures may be described, are in the same square root ratio. *Q.E.D.*
[See notes above.]

PROPOSITION LX. THEOREM XXIII.

If two bodies S and P, with forces inversely proportional to the square of their distance, mutually attract each other about the common centre of gravity: I say that ellipse, which one of the bodies P will describe about the other ellipse in this motion S, the principal axis is to that principal axis, which the same body P will describe around the other body S at rest, in the same periodic time, shall be as the sum of the two bodies $S+P$ *to the first of the two mean proportionals between this sum and that other body S.*

For if the ellipses were to be described equal to each other, the periodic times (by the above theorem) would be in the square root ratio of the body *S* to the sum of the bodies $S+P$. The periodic time of the latter ellipse may be diminished in this ratio, and the periodic times may become; [i.e. $\frac{t}{T}=\frac{\sqrt{S}}{\sqrt{P+S}}$, and $\frac{t}{T\times\frac{\sqrt{S}}{\sqrt{P+S}}}=1$] ; but the principal axis of the ellipse (by Prop. XV.) will be diminished in a ratio, which is in the three on two ratio of this, that is in a ratio which is the triplicate of S to $S+P$
[For $T^2 : t^2 = S+P : S$ and $T^2 : t^2 = A^3 : X^3$, whereby by Kepler III, $A^3 : X^3 = S+P : S$; now if two mean proportional *B* and *C* are taken between $S+P$ and *S*, then $\frac{S+P}{B}=\frac{B}{C}=\frac{C}{S}$

hence $S+P$ will be in the triplicate ratio to *B*, that is $S+P:S=(S+P)^3:B^3$, and hence $A^3:X^3=(S+P)^3:B^3$, and thus $A:X=(S+P):B$ or $X:A=B:(S+P)$];

and thus the principal axis will be to the principal axis of the other ellipse, as the first of the two mean proportionals between $S+P$ and *S* to $S+P$. And conversely, the principal axis of the ellipse described about the mobile body will be to the principal axis described about the stationary ellipse, as $S+P$ to the first of the two mean proportions between $S+P$ and *S*. *Q.E.D.*

PROPOSITION LXI. THEOREM XXIV.

If two bodies may be mutually attracted by any forces, and neither disturbed nor impeded by any others, in whatever manner they may be moving ; the motions of these thus may be had, and, as if they may not attract with each other mutually, but each may be attracted by the same forces by some third body established at the common centre of gravity. And the law of the attracting forces will be with respect to the distance of the bodies from that common centre, and with respect to the whole distance between the bodies.

For these forces, by which the bodies mutually pull on each other, stretch towards the common intermediate centre of gravity ; and thus they are the same, as if they spring from an intermediate body. *Q.E.D.*

And since the ratio of the distance of whichever body from that common centre to the distance between the bodies may be given, the ratio will be given of any power of one distance to the same power of the other distance; and so that the ratio of whatever quantity, which may be derived from one distance and with whatever quantities given, to another quantity, which from the other distance and from just as many given quantities, and that given ratio of the distances to the former had similarly may be derived. Hence if the force, by which one body is pulled by the other, shall be directly or inversely as the distance of the bodies in turn ; either as any power of this distance ; or finally so that some quantity may be derived in some manner from this given distance and given quantities: the force will be the same, by which the body likewise is drawn to the common centre of gravity, likewise directly or inversely as the distance of attraction from that common centre, either as the same power of this distance, or finally as a quantity similarly derived from this distance and with similar given quantities. That is, the force of attraction will be the same law with respect to each distance. *Q.E.D.*

PROPOSITION LXII. PROBLEM XXXVIII.
To determine the motion of two bodies, which mutually attract each other with forces inversely proportional to the square of their distances, and which are sent off from given places.

Bodies (by the latest theorem) may be moved in the same way, as if they may be attracted by a third body put in place at the common centre of gravity ; and that centre from its own initial motion by hypothesis remains at rest; and therefore (by Corol. 4 of the laws) will always remain at rest. Therefore the motions of the bodies are required to be determined (by Prob. XXV.) in the same manner as if they may be urged by attracting forces at the centre, and the motions of the bodies mutually attracting each other will be found. *Q.E.D.*

PROPOSITION LXIII. PROBLEM XXXIX.
To determine the motion of two bodies which attract each other with forces inversely proportional to the square of their distance, and with the places given, and the bodies leave along given straight lines with given velocities.

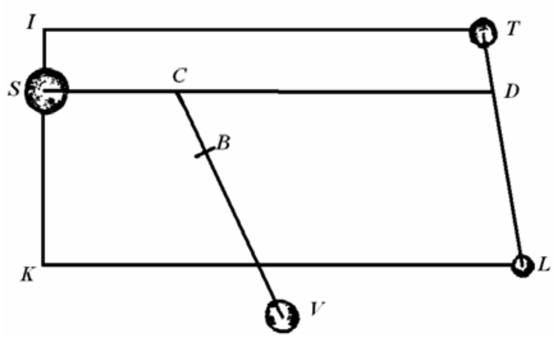
At the beginning with the given motions of the bodies, the uniform motion of the common centre of gravity is given, and so that the motion of the space, which together with this centre is moved uniformly in a direction, without any motion of the bodies with respect to this space. But the subsequent motions (by the fifth corollary of the laws, and the latest theorem) happen in this space in the same way, as if the space itself together with that common centre of gravity were at rest, and the bodies are not attracting each other mutually, but were attracted by a third body situated at that centre. Therefore the motion of either body is to be determined (by Problems 9 and 26) in this moving space, from the place given, along a given right line, with the given departure speed, and acted on by the centripetal force tending towards that centre: and likewise the motion of the other body about the same centre will be known. Since to this motion it is required to add the uniform motion of the space, and in that space the progressive rotational motion of the bodies found above, and the absolute motions of the bodies will be known in immobile space.
Q.E.D.

[Recall that Newton believes in the existence of a universal rest frame, that of the fixed stars, and relative to which all motions are absolute. Newton now proceeds from the generally soluble two-body problem to the only-soluble n-body problem.]

PROPOSITION LXIV. PROBLEM XL.

The motions of several bodies among themselves are required, with the forces by which the bodies mutually attract each other increasing in a simple ratio from the centres.

At first two bodies *T* and *L*, having a common centre of gravity *D*, may be put in place. These describe ellipses (by the first corollary of Theorem XXI.) having centres at *D*, the magnitude of which becomes known from Problem X.



Now a third body *S* may attract the first two *T* and *L* with the accelerating forces *ST* and *SL*, and it may be attracted by these in turn. The force *ST* (by Corol. 2. of the laws) is resolved into the forces *SD* and *DT;* and the force *SL* into the forces *SD* and *DL*. But the forces *DT* and *DL*, which are as their sum *TL*, and thus as the accelerating forces by which the bodies *T* and *L* are attracted mutually, add to these the forces of the bodies *T* and *L*, each to the other in turn, composing forces proportional to the distances *DT* and *DL*, as at first, but with the forces greater than with the former forces; and thus (by Prop. X, Corol. I. and Prop. IV, Corol's.1. and 8) have the effect that these bodies will describe ellipses as before, but with a faster motion. The remaining accelerating forces *SD* and *SD* [contributed from the forces *ST* and *SL*], from the motive actions $SD \times T$ and $SD \times L$,

[*i.e.* in an analytical approach, such forces between the bodies 1 and 2 may be given by a generalised Hooke's law type formula : motive $\text{force} \propto -\text{displacement}$, provided there are only two bodies present, where the masse M_1 and M_2 may be incorporated into the constant of proportionality; or by considering a motive $\text{force} \propto -M_3 \times \text{displacement}$ for forces that involve either body 1 or 2 but always body 3, some such scheme Newton has adopted, as this constant of proportionality must change if a third mass M_3 is present, unless the masses are equal, etc.]

which are as the bodies, by attracting these bodies equally and along the lines *TI, LK*, themselves parallel to *DS*, and in turn do not change the situation of these, but act so that they accelerate equally to the line *IK;* that taken drawn through the middle of the body *S*, and perpendicular to the line *DS*. But that access to the line *IK* may be impeding by arranging so that the system of bodies *T* and *L* from one side, and the body *S* from the other, with the correct velocities, may be rotating around the common centre of gravity *C*. From such a motion the body *S*, because with that the sum of the motive forces $SD \times T$ and $SD \times L$, of the proportional distance *CS*, tends towards the centre C, will describe an ellipse around the same *C;* and the point *D*, on account of the proportionals *CS, CD*, describes a similar ellipse out of the region. But the bodies *T* and *L* attracted by the motive forces $SD \times T$ and $SD \times L$, the first body by the first force, the second by the second, equally and along the parallel lines *TI* and *LK*, as it has been said, proceed to

describe their own ellipses around the moving centre *D* (by the 5th and 6th corollaries of the laws), as at first.
Q.E.D.

Now a fourth body *V* may be added, and by a similar argument it may be concluded that this body and the point *C* describe ellipses about the common centre of gravity of all the bodies *B* ; with the former motions of the bodies *T*, *L* and *S* about the centres *D* and *C* remaining, but with an acceleration. And by the same method more bodies are allowed to be added. *Q.E.D.*

Thus these may be themselves considered, as if the bodies *T* and *L* attract each other with greater or less accelerations than by which the remaining bodies for a [given] ratio of the distances. Let all the mutual accelerative attractions be in turn as the distances by the [masses of the] bodies attracted, and from the proceeding it may be readily deduced that all the bodies describe different ellipses in equal periodic times, around the common centre of gravity *B*, in a fixed plane. *Q.E.D.*

[On page 217, Chandrasekhar produces the analytical solution that Newton must have worked through, by reducing the motion of each mass to an attraction about the common centre of mass, without any other accelerations needed, and these motions for each body are *S.H.M.*'s of the form :

$$\text{acceleration of body} \propto -\text{total mass of all bodies} \times \text{displacement of body}.$$

Thus the basis is set for treating a three or more body problem where the forces follow an inverse square law, having established the reliability of a method of adding the forces.]

PROPOSITION LXV. THEOREM XXV.

Several bodies, the forces of which decrease in the inverse square ratio of the distance from their common centre of gravity, can move in ellipses amongst themselves ; and with the radii drawn to the focus describe areas almost proportional to the times.

In the above proposition the case has been shown where several bodies are moving forwards precisely in ellipses. From which the more the law of the forces departs from the law put in place there, from that the more the bodies mutually disturb the motions ; nor can it happen, following the law put in place here that bodies by mutually attracting each other, can be moving precisely in ellipses, unless by obeying in turn a certain proportion of the distances. But in the following cases it will not differ much from ellipses.

Case: I. Put several smaller bodies to revolve around some large body at various distances from that body, and the bodies are attracted by the same body according to proportional individual absolute forces. And because the common centre of gravity of all is either at rest or may be moving in a direction uniformly (by the fourth Corollary of the laws), we may set the bodies in place to be rather small, so that the large body at no time departs sensibly from this centre : and the large body may be at rest, or moving uniformly in a direction, without sensible error; but the smaller bodies may be revolving around this large body in ellipses, and with radii drawn to the same, they describe areas proportional to the times ; unless in so far as errors are induced, either by the large body receding from that common centre of gravity, or by the actions of the smaller bodies mutually between each other. But the smaller bodies can be diminished , until this error and the mutual interactions themselves, shall be less than in any given amount ; and thus the orbits can be made to agree with ellipses, and the areas may correspond to the times, with no tangible errors. *Q.E.O*.

Case 2. Now we may put in place a system of smaller bodies revolving around some very large body in the manner now described, or some other system of two bodies revolving around each other progressing uniformly in direction, and meanwhile to be acted on laterally by the force of another by far greater body at a great distance. And because the equal accelerating forces do not act by changing the positions of the bodies in turn among themselves, with the motions of the parts maintained between themselves, but affect the system as a whole, by which the small bodies may be acted on along parallel lines, and may be moved together ; it is clear that, by the attractions from the great body, no change in the motion of the attraction of the bodies between themselves may arise, unless either from the inequality of the attracting accelerations or from the inclination of the lines in turn, along which the attractions happen. Therefore put all the attractive accelerations due to the great body to be inversely as the square of the distances between themselves ; and by increasing the distance of the great body, until the differences of the right lines drawn from that body to the rest of the bodies in respect of the length of these, and the inclinations in turn, may be made smaller than any given amount; then the motions of the parts of the system may persevere among themselves with minimal errors given, which may not be given any smaller. And because, on account of the smallness of the parts of these in turn with the great distance, the whole system is attracted in the manner of a single body ; and will be moved likewise by this attraction in the manner of a single body ; that is, it may describe with its centre of gravity some conic section about the great body (*viz*. a hyperbola or a parabola from a weaker attraction, an ellipse from a stronger attraction) and with a radius drawn to the great body, areas will be described proportional to the times, without any errors, except those arising from the distances of the parts, reasonably small, and to be minimized as it pleases. *Q.E.O.*

It is permitted to proceed indefinitely to more composite cases.

Corol. 1. In the second case, in which the greatest body of all approaches closer to the system of two or more bodies, the motions of the parts of the system among themselves are more disturbed by that ; because now the inclination of the lines drawn from the great body to these in turn is greater, and the greater the inequality of the proportion.

Corol.2. Moreover the small bodies will be disturbed the most, on being put in place, in such a way that the attractive accelerations of the parts of the system towards the greatest body of all shall not be inversely in turn as the squares of the distances from that great body ; especially if the inequality of this proportion shall be greater than the inequality of the proportions of the distances from the great body. For if the accelerative force [of the great body], by acting equally along parallel lines, disturbs nothing in the motion between themselves, it is necessary that a perturbation may arise from the inequality of the action, either it shall be a smaller disturbance for a greater body, or one of greater inequality for a lesser body. The excess of the impulses of the greater body, by acting on some bodies and by not acting on others, by necessity will change the position of these amongst themselves. And this perturbation added to the perturbation, which arises from the inclination and inequality of the lines, may return the whole major perturbation.

Corol. 3. From which if the parts of this system may be moving in ellipses or in circles without significant disturbance ; it is evident, that the same bodies on attracting other bodies by accelerative forces, either may not to be acted on unless they are the lightest, or to be acted on equally, and approximately along parallel lines.

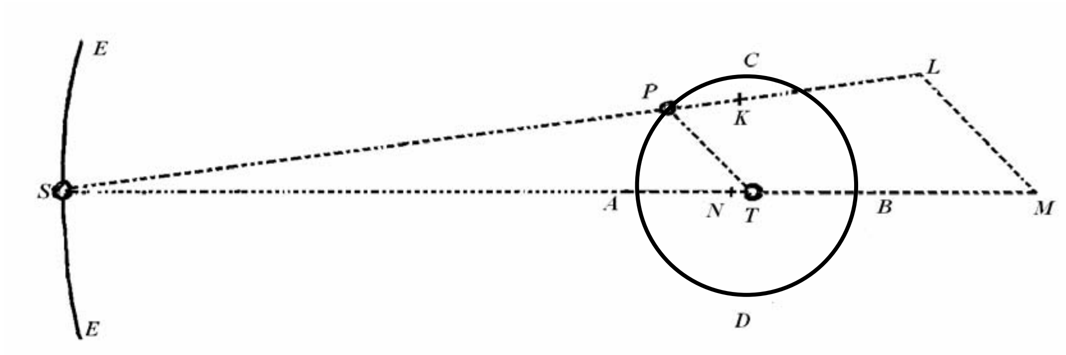
PROPOSITION LXVI. THEOREM XXVI.

If three bodies, the forces of which decrease in the square ratio of the distances, mutually attract each other; and the accelerative attractions of any two on the third shall be reciprocally as the square of the distances; moreover with the smaller ones revolving around the greatest : I say that the inner of the two bodies revolving about the innermost and greatest body, by the radii drawn to the innermost itself, describes areas more proportional to the times, and a figure of an elliptic form, by having more of the radii meeting at the focus, if the greatest body may be disturbed by these attractions, than it would if that greatest body either was at rest and not attracted by the smaller bodies, or if it were attracted much more or much less, or disturbed much more or much less.

It may almost be evident from the demonstration of the second corollary of the foregoing proposition ; but it may be established thus by a more widely compelling and distinct argument.

Case 1 The smaller bodies *P* and S may be revolving in the same plane about the greatest body *T*, of which *P* may be described in the inner orbit *PAB*, and *S* in the outer orbit *ESE*. [Note that *S* may be presumed to be the sun, and is small only because of the great distance, while *P* is the moon revolving around the earth *T* in an almost circular orbit; also, point masses are assumed for these bodies.]
Let *SK* be the mean distance of the bodies *P* and *S;* and the attractive acceleration of the body *P* towards *S*, at that average distance, may be expressed by the same line *SK*;

[Thus, as in all of Newton's dynamics, some lines are geometrical lengths, others are forces or attractive accelerations, and some act as both, we will prefix some line sections by a word or short phrase not present in the original text, containing the word *line* or *attractive acceleration* or simply *force* if there is confusion ; in this case the line *SK* defines the unit of attractive acceleration, or force per unit mass.]



The ratio of the attractive accelerations *SL* to *SK* may be taken in the same square ratio *SK* to *SP*, and *SL* will be the attractive acceleration of the body *P* towards *S* at some distance *SP*.

[i.e. the attractive accelerations on the body *P* due to *S* are : $\frac{|\overrightarrow{SL}|}{|\overrightarrow{SK}|} = \frac{SK^2}{SP^2}$, so that when *P* is at the average distance, the force $\overrightarrow{SL}$ is equal to the force $\overrightarrow{SK}$; if the distance $SP < SK$, the force $\overrightarrow{SL}$ is greater than the average and if *P* is more distant, i.e. if $SP > SK$, then the force $\overrightarrow{SL}$ is less than the average.]

Join *PT*, and the line *LM* acts parallel to that line crossing *ST* in *M*, [which may need to be extended as here when $SP < SK$]; and the attractive acceleration *SL* may be resolved (by Corol. 2 of the laws) into the attractions $\overrightarrow{SM}$ and $\overrightarrow{LM}$. And thus the body *P* will be urged by three accelerative forces.

[*i.e.* $\overrightarrow{LS} = \overrightarrow{LM} + \overrightarrow{MS}$ gives two of the forces, and $\overrightarrow{PT}$ is the other.]

One force [$\overrightarrow{PT}$] attracts *P* towards *T*, and arises from the mutual attraction of the bodies *T* and *P*. By this force alone the body *P* must describe equal areas in proportional times with the radius *PT* about the body *T*, either fixed, or disturbed by this attraction, and in an ellipse the focus for which is at the centre of the body *T*. This is apparent from Prop. XI, and the Corollaries 2 and 3 of Theorem XXI.

The second force is by the attraction of the force *LM*, which because it attracts from *P* to *T*, may be added onto the first force and will coincide with that [in direction], and thus might bring about that the areas even still may be described in proportional times by Corol. 3 of Theorem XXI. But because it is not inversely proportional to the square of the distance *PT*, this force adding together with the first force differs from that proportion, and this with a greater variation; by which the proportion of this force is greater than the first force, with all else being equal.

[Note 495 adapted, L & J. : For from the construction, $\frac{SK^2}{SP^2} = \frac{|\overrightarrow{SL}|}{|\overrightarrow{SK}|} = \frac{SL}{SK}$, and thus $\frac{SK^3}{SP^3} = \frac{SL \times SK}{SK \times SP} = \frac{SL}{SP}$. But on account of the similar triangles *MLS, TPS* : $\frac{SL}{SP} = \frac{LM}{PT}$; hence $\frac{LM}{PT} = \frac{SK^3}{SP^3}$, and therefore the force $\overrightarrow{LM}$ is as $\frac{SK^3}{SP^3} \times \overrightarrow{PT}$, or with *SK* given, as $\frac{1}{SP^3} \times \overrightarrow{PT}$; from which with the distance *PT* increased, so the force $\overrightarrow{LM}$ will increase]

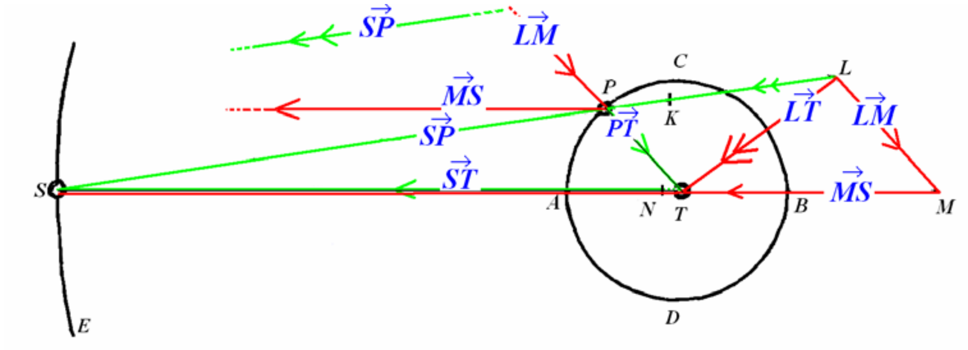
Hence since (by Prop. XI, and by Corol. 2 of Theorem XXI.) the force, by which an ellipse will be described about a focus *T* must attract towards that focus, and to be in the inverse square ratio of the distance *PT*; but that composite force, erring from that proportion, so makes the orbit *PAB* err from the elliptic form having a focus at *T;* and with that the more, so that from which the departure is greater from that proportion ; and thus also by how much greater is the proportion of the second force *LM* to the first force, with all else being equal.

Now indeed the third force *SM*, by attracting the body *P* along a line parallel to the line *ST* itself, together with the previous forces comprises a force, which is no longer directed from *P* to *T;* and which may differ so much more from this determination, when the proportion of this third force to the former forces is greater, with all else being equal : and thus which will cause the areas no longer to be described by the body *P* in proportional times, with the radius *TP* ; and as the aberration from this proportionality may be so much greater, when the proportion of this third force to the other forces is greater. Truly the third force will increase the aberration of the orbit *PAB* from the previous elliptic form in a two fold manner, because that force is not directed from *P* to *T*, and also because it shall not be inversely proportional to the square of the distance *PT*.

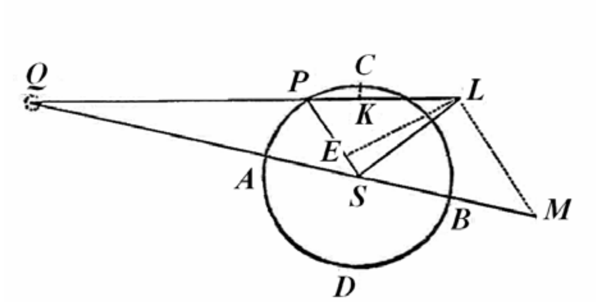
[Note 496, L & J. : For *PT* is to *ST* as the force *LM* is to the force *SM*, but from the previous note, the force *LM* is as $\frac{SK^3 \times PT}{SP^3}$, and hence the force *SM* is as $\frac{SK^3 \times ST}{SP^3}$. Whereby the force *SM*, with *SK* and *ST* given, is as $\frac{1}{SP^3}$.]

With which understood, it is evident, that the areas are made maximally in proportion to the times when the third force shall be a minimum, with the rest of the forces remaining constant; and so that the orbit *PAB* then can approach maximally to the previous elliptic form, where both the second force as well as the third, but particularly the third force, shall be a minimum, with the first force remaining.

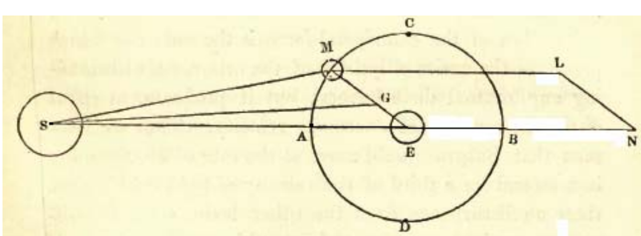
[Extra diagram and notes for this *T* based system : Black lines are geometric lines; green lines are forces exerted by *S* on *T* and on *P*; red lines are components of the excess or deficient force of *S* on *P*, resolved along *PT* and *SN*; *PT* and *ST* are overlapping lines.



Here we have detached the components of the *PS* force acting on *P* in our coloured diagram, showing the three forces considered acting on *P* ; we note that *T*(*erra*) the earth, is acted on by the *S*(*un*) directly at a great distance, by the force $\overrightarrow{SN}$, moving slowly at a great distance, and so slowly pulling the two-body system slowly through a complete circle; and by the moon directly at *P* by the force variable force $\overrightarrow{PT}$, while the force of the sun on the moon at *P*, (with the average force $\overrightarrow{SK}$), gives rise on resolution, during its orbit, to the indirect forces acting on the earth, $\overrightarrow{LM}$ and $\overrightarrow{SN}$. These are the three forces per unit mass acting on *P*, and so really are accelerations, or gravitational field strengths. Also, the perturbing force on *P* at the position shown is $\overrightarrow{LT} = \overrightarrow{LS} - \overrightarrow{TS}$, the difference of the moon-sun and the earth-sun accelerations ;and note that *T* is the centre of gravity of the body, while *N* is the position of the focus of the elliptical orbit of *P*.

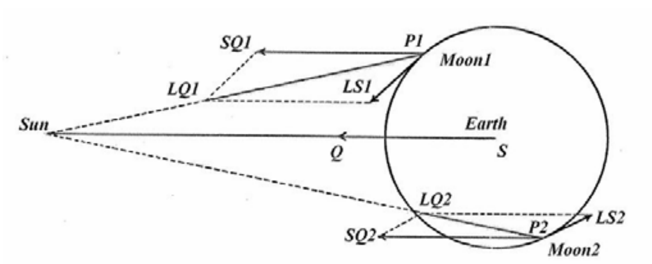


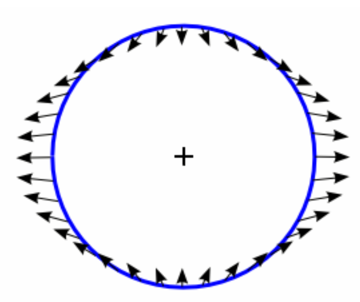
It is interesting to observe the progression of Newton's figure, of which he must have been very proud, as it provides so much information, and it expresses one of these happy moments in physics where a lot of ideas suddenly tumble out into the light from some obscure place, and at least a qualitative understanding of a situation can be grasped. Above is the diagram from the first edition of the Principia, where *SL* is the perturbing force.



And here on the left is the diagram from Brougham and Routh's book, where the original quadrilateral *PLSM* has become the parallelogram *MLNE*. However, they have not shown the resultant perturbing force.

And here is a later diagram from a 19th century book, that I have borrowed from the useful article on Lunar Theory, from that wonderful resource Wikipedia, for which we must thank the anonymous provider. Note that two opposite positions have now been put in place, one with the gravitational acceleration of the sun on the moon greater, and other less, than the gravitational acceleration of the earth.

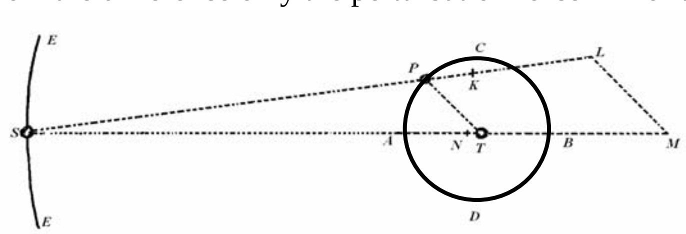




Finally, here is a modern diagram from the same source, showing essentially the tidal forces on a particle in orbit or on the earth's surface at various positions.

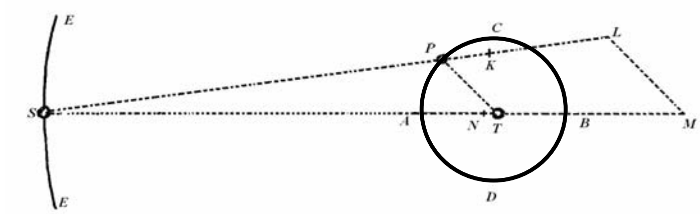
End of translator's note. Back to Newton]

The attractive acceleration of the body *T* towards *S* may be represented by the line *SN;* and if the accelerative attractions *SM* and *SN* were equal to each other; these, by attracting the bodies *T* and *P* equally along parallel lines, would make no change in the positions of these relative to each other. Now the motions of these bodies would be the same between each other (by Corol. VI of the laws) as if these attractions were removed. And by similar reasoning, if the attraction *SN* were less than with the attraction *SM*, that part of the attractive force *SN* of *SM* itself may be removed, and only the part *MN* may remain, by which the proportions of the times and of the areas, and by which the elliptic form may be disturbed. And similarly if the attraction *SN* should be greater than the attraction *SM*, there may arise from the difference only the perturbation force *MN* of the proportionality of the orbit.



Thus, by the attraction *SN*, the attraction of the third body always reduces the third attraction *SM* above to the attraction *MN*, with the first and second attractions completely unchanged [thus, the common part of the attraction can be ignored, leaving only the part corresponding to the force *MN*] : and therefore the proportionality for the areas and the times also remain unchanged, and the orbit *PAB* then approaches as close as possible to the previous elliptic form, when the attraction *MN* either is zero, or that shall be made as small as possible ; or, when the accelerative attractions of the bodies *P* and *T*, made towards the body *S* approach as close as possible to being equal ; *i.e.*, when the attraction *SN* is not zero, nor smaller than the minimum attraction of all *SM*, but as if it were a mean between all of the maximum and minimum attractions of *SM*, not much greater nor much less than the attraction *SK. Q.E.D.*

Case 2. Now the smaller bodies *P* and *S* may rotate about the largest body *T* in different planes [or, the new plane of the ellipse *PAB* is inclined to the above plane *STP*, which we can imagine as fixed]; and the force *LM*, by acting along the line *PT* situated in the plane of the orbit *PAB*, will have the same effect as before, nor will the body *P* be disturbed from its orbital plane [by this force]. [That is, the force *SP* is now resolved into components in the new plane *SPT* .] But the other force *NM*, by acting along a line which shall be parallel to *ST* itself, is (and therefore when the body *S* may be situated beyond the line of the nodes, to be inclined to the orbital plane *PAB*) besides the perturbation of the motion in longitude that now has explained before, may lead to a perturbation of the motion in latitude, by drawing the body *P* from its own orbital plane. [Thus, the forces $\overrightarrow{ST}$ and $\overrightarrow{MN}$ no longer lie in the same plane, and an extra perturbation arises changing the angle of this plane to the fixed plane]. And this perturbation, in some given situation of the bodies *P* and *T* to one another, will be as that force generating *MN*, and thus may emerge the minimum when *MN* is minimal, that is (as I have just explained) where the attraction *SN* is not much greater, nor much less than the attraction *SK. Q.E.D.*



[The first ten corollaries that follow give qualitative arguments about the kinds of motions that can arise between the three bodies. People normally agree that the explanations supplied by Newton are quite inadequate for a proper understanding of the subject, and that he retained the accompanying analytical derivations that he had worked out; see Chandrasekhar on this point, at least until Book III. We offer here as part of this qualitative understanding, some of the notes supplied by Leseur & Janquier in their edition of the *Principia*. We will indicate briefly what each corollary sets out to establish if this is not apparent at once ; the whole theory can be applied to the moon as *P*, *S* as the sun, and *T* as the earth, but is presented in a more general way. Most of the technical terms used in the following can be found on line in the 1911 eleventh edition of the Encyclopaedia Britannica (Cambridge University Press), and from various astronomy websites.]

Corol. 1. [This corollary indicates that the inner body is affected less by the disturbing force; Newton show in Prop. XXVI of Book III that the components of this force acting along and perpendicular to the radius at a given point, for each body in orbit around *T*, are in fact proportional to this radius.]

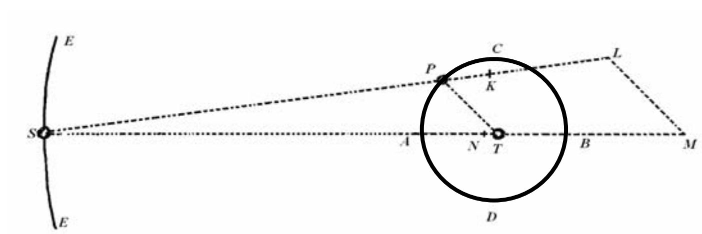
From these it is easily deduced, that if several smaller bodies *P*, *S*, *R*, &c. may be revolving around the greatest body *T*, the motions of the innermost body *P* will be disturbed minimally by the attractions of the exterior bodies, where the greatest body *T* with all other things being equal, for a given ratio of the accelerative forces, is attracted and disturbed by the others, and as are the lesser bodies mutually between themselves.

*Corol.*2. [This corollary indicates that the 'constant' areas are in fact slightly more at the syzygies and slightly less by the same amount at the quadratures.]

For in a system of three bodies *T, P, S,* if the accelerative attractions of any two on the third shall be to each other inversely as the square of the distances ; the body *P*, with the radius *PT,* describes an area about the body *T* faster on account of the nearby conjunction *A* and the opposition *B,* than near the quadratures [*i.e*. the perpendicular positions on the diagram above] C, *D.* For all the force by which the body *P* may be acted on, (and the body *T* is not acted on, and which does not act along the line *PT*), either accelerates or retards the description of the areas, thus as the bodies come together or are moving apart. Such is the force *NM.* This, in the passage of the body *P* from *C* to *A,* attracts as a forward motion, and the motion accelerates; then as far as to *D* in moving apart, the motion is retarded; then acting together as far as to *B,* and finally by acting in opposition in passing from *B* to *C.*

[Note 498 from L & J : *Such is the force NM*.... If we may suppose the orbit *CADB* to be almost a circle, and the distance *SD* a maximum with respect to the radius *PT*, there will be almost $SC = SK = ST = SN$, and hence $NM = TM$. Again with the body *P* at the quadratures *C* and *D*, there is $SC = SP = SK$; whereby since there shall be $\left|\frac{\overrightarrow{SL}}{\overrightarrow{SK}}\right| = \frac{SK^2}{SP^2}$,

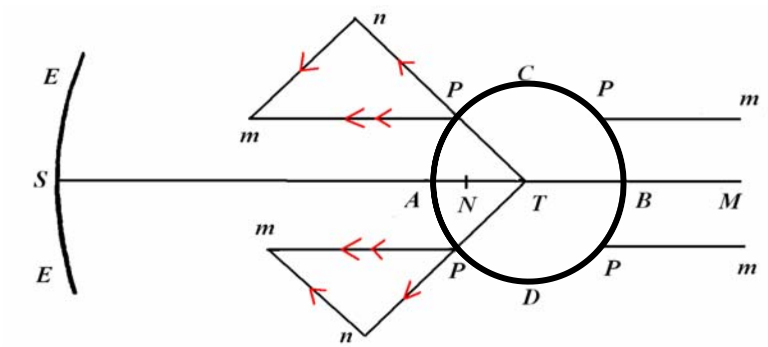
by the construction of Prop.66, , there will be $SL = SK = SC$ at the quadratures and *LM* coincides with *CT* or *PT*, and thus *TM* or *NM* vanishes. Therefore there



will be no difference in the forces *SM* and *SN* at the quadratures, and thus the body *P* is disturbed by the remaining forces, and is attracted towards the centre *T*, describes there by the radius drawn, areas proportional to the times : [we can think of both *T* and *P* accelerating equally towards *S* at these points, and the non-central force corresponding to *LM* is taken as negligible.] Moreover, when the body *P* is in the hemisphere *CAD* beyond the quadratures, the force *SM* is greater than the force *SN* and the body *P* is drawn by the difference of the forces along a direction parallel to *TS* itself.

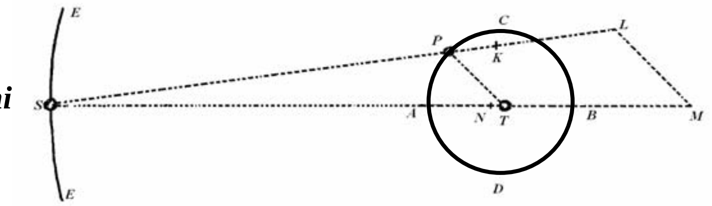
Let *Pm* be equal and parallel to *NM* itself, and by sending from *m* a perpendicular to the radius *TP* produced, the force *Pm*, or *NM*, is resolved into the two forces *Pn* and *nm*, the former of which *Pn* by drawing along the direction of the radius *TP*, the motion of the body *P* does not change in longitude – *i.e*. the rate at which the angle *P* rotates through in its orbit, nor will it disturb the equality of the area described; truly the latter force *nm*, by attracting along the direction of the line *nm*, perpendicular to the radius *TP*, that is, along the direction of the tangent at *P*, accelerates the motion in the longitude in the first quadrant *CA* and retards it in the second quadrant *AD*. [Assuming the northern hemisphere view as an anticlockwise rotation of *P* about *T*.]

In the other hemisphere *DBC*, the force *SM* is less than the force *SN*, because the body *P* is at a greater distance from the body *S* than the body *T*, from which if the perturbing forces may be considering only on the body *P*, the difference of the forces *SM* and *SN* will be negative or taken away, or because it is the same force, acting in the opposite direction. For both the bodies *T* and *P* may be considered to be urged by the force *SN* equal and parallel to itself everywhere, and they can move among themselves as if all that force were absent, by Cor. 6. of the laws of motion; The body *P* may be drawn by the force *NM* along the opposite direction to the force *SN*, and by that action the motion of the bodies will be changed between themselves; but also by that action the force *SN* which was being considered to draw the body *P*, has been reduced to the force *SM* , which is the reverse force acting while the force *SN* acts on *T*. Therefore if the motion of the bodies *T* and P may be judged between themselves, so that the body P may be urged by the difference of the forces *NM*, acting in the opposite direction in the hemisphere, the true changes in the motions of the bodies *T* and *P* between themselves will be obtained. arising from the actions *SN* and *SM*. Finally, the body *P* may be considered in the



hemisphere *DBC* as if urged by the force *NM* along the direction *Pm* parallel to *NM* itself by acting from *P* towards *m*; and thus, if the force *Pm* may be resolved into two forces as has been done in the other hemisphere, it is evident the longitudinal motion will be accelerated in the quadrant *DB*, and retarded in the quadrant *BC*.]

Corol.3. And by the same argument it is apparent that the body *P*, with all else being equal, will be moving faster in conjunction and in opposition than at quadrature. [But as we have seen above, in opposite directions.]



Corol. 4. For the orbit of the body *P*, with all else being equal, is more curved at quadrature than in conjunction and opposition. For swifter bodies are deflected less from a straight path. And in addition the force *KL*, or *NM*, in conjunction and in opposition is opposite to the force, by which the body *T* attracts the body *P;* and thus that force is decreased; but the body *P* is deflected less from a right path, where it is less urged towards the body *T*.

[Note 499, L & J : *And besides the force KL*......With everything in place as above, the right lines *SL* and *SM* are almost parallel, and hence $TM = PL$ and $LM = PT$ approximately; whereby *P* coincides with *A* and *K* with *T*, there becomes $LM = AT = PK$, and *NM* or $TM = PL + AT + KL$, and $NM - LM = KL$, that is, the whole disturbing force by which the body *P* in conjunction with *A* is withdrawn from the body towards *S*, is as *KL* approximately ; for the force *LM* attracting *P* towards T and by the force *NM* is withdrawn from the body *T* towards *S*. Likewise it may be shown in the same manner, with the body *P* in opposition to the position *B*.]

Corol. 5. From which the body *P*, with all else being equal, departs further from the body *T* at the quadratures, than at conjunction and at opposition [*i.e.* the elliptical shape is made slightly prolate from perturbation]. These thus are considered with the exclusion of the eccentricity from the motion. For if the orbit of the body *P* shall be exocentric, the eccentricity of that (as will be shown in Corol. 9 of this work soon) emerges the maximum when the apsides are at the syzygies ; and thus it is possible to happen that the body *P*, calling at the greater apside, may be further from the body *T* at the syzygies than at the quadratures.

[Note 499*q*, L & J : *From which the body P*, ... For since the orbit of the body P shall be more curved at the quadratures *C* or *D* than at the syzygies *A* and *B* (by Corollary 4), it is necessary, with all else equal, that at the syzygies *A* and *B* shall be more squeezed than at the quadratures *C* and *D* to the image of the ellipse, the centre of which shall be *T*, *CD* the major axis, and *AB* the minor axis. Thus, these may be found if , with the exclusion of the perturbing forces, the orbit of the body *P* were a circle of which the centre were *T*.]

Corol. 6. [In which changes in the Kepler Law III are accounted for in terms of perturbing factors. This is really a 'second order' effect : see Chandrasekhar p. 245 for a detailed account.]

Because the centripetal force of the central body *T*, by which the body *P* may be held in its orbit, is increased at the quadratures by the addition of the force *LM*, and diminished at the syzygies by taking away the force *KL*, and on account of the magnitude of the force *KL* being greater than *LM*, it is more diminished more [at *A* and *B*]than increased [at *C* and *D*]; but that centripetal force (by Corol. 1. Prop. IV.) is in a ratio compounded from the simple ratio of the radius *TP* directly and from the ratio of the inverse of the square of the time [*i.e.* $C.F. \propto \frac{\text{radius } PT}{\text{period}^2}$]: it is apparent that the compounded ratio be diminished by the action of the force *KL;* and thus the periodic time, if the radius of the orbit *TP* may remain, to be increased, and that in the square root ratio, by which that centripetal force is diminished: and thus by diminishing or increasing this radius, the periodic time becomes greater, or to be diminished less than in the three on two power of this radius (by Corol. VI. Prop. IV.) If that force of the central body were to become gradually weaker, the body *P* attracted always less and less continually recedes further from the centre *T;* and on the other hand, if that force were increased it would draw closer. Therefore if the action of the distant body *S,* by which that force is diminished, were increased and diminished in turn: likewise the radius *TP* will be increased and diminished in turn; and the periodic time will be increased and diminished in a ratio composed from the three on two power of the radius, and from the square root ratio by which that centripetal force of the central body *T;* by increasing or decreasing the action of the distant body *S,* is diminished or increased.

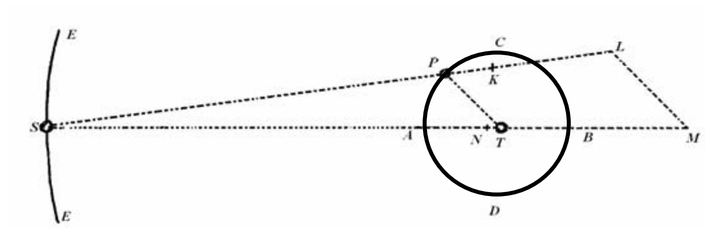
[Note 500, L & J : And *because the magnitude of the force KL,* ...... If the mean distance *SK* or *ST* were made large with respect to the radius of the orbit *TP* of the orbit *PAB*, in some position of the body *P*, the force *LM* will be approximately to the force *NM* as the whole sine to the triple sine of the angle of the distance of the body *P* from the quadrature. For on account of the increase of the distance of the body *S* (by hypothesis) the lines *SL* and *SM* are almost parallel and hence $LM = PT$, NM or $TM = PL$, and $SP = SK$; and since *ST* shall be perpendicular to the line of the quadratures *C*D, also *SK* will be normal to the same line, and with the radius *PT* present, *PK* will be the sine of the angle *PTC,* that is, the sine of the of the angle of the distance of the body *P* from the quadrature *C* approximately. Again (by Prop. 66) $\frac{SL}{SK} = \frac{SK^2}{SP^2}$, and thus $\frac{SL-SK}{SK} = \frac{SK^2-SP^2}{SP^2}$, that is, $\frac{KL}{SK} = PK \times SK + \frac{SP}{SP^2} = PK \times \frac{2SP}{SP^2} = \frac{2PK}{SP} = \frac{2PK}{SK}$, on account of $SK = SP$, and $SK + SP = 2SP$. Whereby there will be $KL = 2PK$, and *PL* or $NM = 3PK$, that is, the force *LM* or *PT* to the force *NM* or *PL* as the whole sine *PT* to 3*PK* the triple sine of the angle of the distance of the body *P* from the nearby quadrature.

Extra *L & J* Corollary: The (extra) force *KL* at the conjunction *A*, is to the similar force *K'L'* at the opposition *B* –*K'* and *L'* are not shown, almost as *AT* to *TB*; i.e. $\frac{KL}{K'M} = \frac{AT}{TB}$, [for the extra force F_A at *A* is proportional to $\frac{1}{SA^2} - \frac{1}{ST^2} = \frac{AT(ST+SA)}{SA^2.ST^2}$, while the extra force F_B at B is proportional in the same way to $\frac{1}{SB^2} - \frac{1}{ST^2} = \frac{BT(ST+SB)}{SB^2.ST^2}$ in the opposite direction ; hence $\frac{F_A}{F_B} = \frac{\frac{AT(ST+SA)}{SA^2.ST^2}}{\frac{BT(ST+SB)}{SB^2.ST^2}} = \frac{TA}{TB} \times \frac{\left(1+\frac{ST}{SA}\right)}{\left(1+\frac{ST}{SB}\right)} \times \frac{SB}{SA} \approx \frac{TA}{TB}$, as *SA, ST*, and *ST* are almost equal.] and if the orbit *PAB* were circular or almost circular, the force *KL* at the syzygies will be almost twice as great as the force *LM* at the quadratures. For with the body P turning about the syzygies, there shall be $PK = AT = PT = LM$, and hence *NM* or *PL* becomes $= 3LM$, and $KL = 2LM$. Yet with the same positions, the force *NM* is a maximum at the syzygies, because there PK becomes a maximum or it emerges $= AT$, and $NM = 3AT$.

From which on account of the magnitude of the force *KL*, the centripetal force of the central body *T* is more diminished than augmented, and thus is to be thought to be diminished by the action of the body *S*.]

Corol. 7. [In which the detailed motion of the apsidal line is discussed.]

Also from what has been presented it follows, that the axis of the ellipse described by the body *P*, or the line of the apsides, as far as the angular motion is concerned, in turn moves forwards and backwards, but yet it progresses more forwards than backwards, and by the excess of the progression in succession it is carried forwards. For the force which acts on the body *T* at the quadratures, where the force *MN* vanishes, is composed from the force *LM* and the centripetal force, by which the body *T* attracts the body *P*. The former force *LM*, if the distance *PT* may be increased, may be increased in almost the same ratio with this distance, and the latter force decreases in that square ratio, and thus the sum of the two forces decreases in less than in the square ratio of the distance *PT*, and therefore (by Corol. I. Prop. XLV.) effects that the upper or greater apside, may be regressing. Truly in conjunction and in opposition the force, by which the body *P* is urged towards the body *T*, is the difference between the forces, by which the body *T* attracts the body *P*, and the force *KL* ; and that difference, because that force *KL* may be increased approximately in the ratio of the distance *PT*, decreases in more than the inverse square of the distance *PT*, and thus (by Corol. I Prop. XLV.) has the effect that the greater apside progresses. In the places between the syzygies [or conjunctions] and the quadratures the motion may depend on increases from both causes taken together, and thus so that either from the excess of one or the other it may progress or regress. From which since the force *KL* at the syzygies shall be as if twice as large as the force *LM* at the quadratures, the excess will belong to the force *KL*, and an increase will be carried forwards in succession. But the truth of this and of the preceding corollary may be understood more easily by considering the system of the two bodies *T* and *P* with several bodies *S, S, S,* &c. in place in the orbit *ESE*, surrounded on every side. In as much as the action of T with the actions of these may reduce the action of *T* on both sides, and it may decrease in a ratio more than the square of the distance.



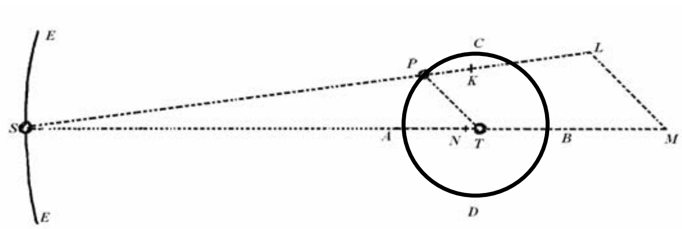
Corol. 8. [In which the motion of the apsidal line is discussed at the syzygies and at the quadratures].

But since the progression or regression of the apsides depends on the decrease of the centripetal force, that is, being made in either a greater or lesser ratio than in the square ratio of the distance *TP*, in the transition of the body from the lower apside [*i.e.* closer to the focus of the ellipse] to the higher apside; and thus it shall be a maximum when the proportion of the force at the higher apside to the force at the lower apside departs in the inverse square of the distances ; it is evident that the apsides in the conjunctions of this, by removing the force *KL* or $NM - LM$, to be progressing faster, and at their quadratures to recede slower, by the addition of the force *LM*. Truly on account of the long period of time, in which the velocity of the progression or the slow regression may be continued, this inequality becomes maximally long.

Corol. 9. [In which the variation of the eccentricity is discussed].

If some body, by some force inversely proportional to the square of its distance from the centre, may be revolving about this centre in an ellipse ; and soon, in descending from the upper apside or by increasing from the lower apside, that force may be increasing to a new force by always approaching in a ratio diminishing more than the square of the distance: it is evident that the body, by always approaching towards the centre by the impulse of that new force, may be inclined more towards the centre, than if it were acted on only by the decrease in the square ratio of the distance ; and thus it may describe a more inner elliptic orbit, and at the inner apside it may accelerate closer to the centre than before. Therefore this orbit by the influence of this new force, may become more exocentric. If now the force, in the recession of the body from the lower apside to the upper apside, may decrease in the same steps by which before it had increased, may return the body to the former distance, and thus if the force may decrease in a greater ratio, now the body attracted less may ascend to a greater distance and thus the eccentricity of the orbit will be increased even more at this stage. Whereby if the ratio of the increase and decrease of the individual centripetal force may be increased in the rotations, the eccentricity will always be increased; and on the other hand, that will be decreased the same, if that ratio may decrease.

Now truly in a system of bodies *T, P, S,* when the apsides of the orbit *PAB* are at the quadratures, that ratio of the increase and decrease is a minimum, and it shall be a maximum when the apsides are in conjunction [syzygies]. If the apsides may be set up at the quadratures, the ratio near the apsides is smaller, and near the syzygies greater than the square of the distances, and from that increased ratio a direct motion or the line of the apsides arises, as had been said just now. But if the ratio of all the increase or decrease may be considered in the motion between the apsides, this is less than the square of the distances. The force at the lower apside is to the force at the upper apside in a ratio less than the square of the distance of the upper apside from the focus of the ellipse to the distance of the lower apside from the same focus: and conversely, when the apsides may be put in place at the syzygies, the force at the lower apside is to the force at the upper in a ratio greater than the square of the distances. For the forces *LM* added at the quadratures to the forces of the body *T* compound forces in a smaller ratio, and the forces *KL* at the syzygies taken from the forces of the body *T* leave forces in a greater ratio. Therefore the ratio of the whole decrease and increase, in passing between the apsides, is a minimum at the quadratures and a maximum at the syzygies: and therefore it will always be increased in the passing of the apsides from the quadratures to the syzygies, and the ellipse increases in

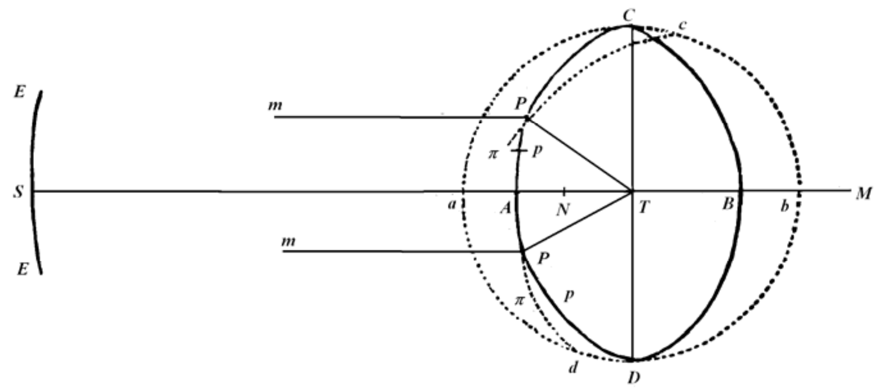


eccentricity; and in the transition from the syzygies to the quadratures the ratio will always be diminished, and the eccentricity diminished.

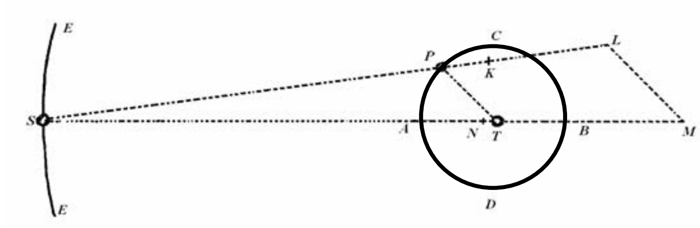
*Corol.*10. [In which the variation of the inclination is discussed].

In order that we may enter into the error [i.e. the nature of the disturbing forces] in the latitude, we may imagine the plane of the orbit *EST* to remain at rest; and the cause of the errors is evident from the exposition, because from the forces *NM* and *ML*, which are that whole cause, the force *ML* by always acting along the plane of the orbit *PAB*, at no time disturbs the motion in the latitude ; each force *NM*, when the nodes are at the syzygies, also may be acting along the same plane of the orbit, and so does not disturb this motion either ; truly when the bodies are at the quadratures, this force disturbs these motions greatly, and the body *P* is always attracted from the plane of its orbit, it diminishes the inclination of the plane in the passage of the body from the quadratures to the syzygies, and it may augment the same in turn in the passage from the syzygies to the quadratures. From which it may be that with the body present at the syzygies the smallest inclination of all may emerge, and it may be returned to the former magnitude almost, when the body approaches close to a node. But if the nodes may be put in place at the octants after the quadrants, that is between *C* and *A*, *V* and *H*, it may be understood from the manner shown, that in the transition of the body *P* from either body thus to 90^0, the inclination of the plane is continually diminished; then in a transition through approximately 45^0, as far as to the nearest quadrant, the inclination will be increased, and again it is diminished after passing through another transition of 45^0 , as far as to the nearest node. And thus the inclination is diminished more than it is increased, and therefore it is always less in the subsequent node than at the preceding one. An by similar reasoning, the inclination is increased more than it is diminished, when the nodes are in the alternate octants between *A* and *D*, *B* and *C*. Therefore the greatest inclination of all is when the nodes are at the syzygies. In the passage of these from the syzygies to the quadratures, the inclination of the body is diminished by the individual influences to the nodes ; and it shall be the least all when the nodes are at the quadrants, and the body at the syzygies: then the inclination increases by the same steps by which previously it decreased; and under the influences [of the other bodies acting] it reverts to the original magnitude with the nodes at the nearby syzygies.

[Note 507 L & J : If the nodes of the body *P* are situated at the quadratures *C* and *D*, the angle of inclination of the orbit to the fixed plane EST is always diminished in the



passage of the body from the quadratures to the syzygies, truly to be enlarged in the passage of the body from the syzygies to the quadratures, and in each passage the nodes are regressed. For let *CAD* be the part of the orbit *PAB* raised higher above the fixed plane of the orbit *EST*, and truly *CBD* may be understood to be the part depressed below ; through the place of the body *P* the right line *Pm* acts parallel to the line *TS*, showing the direction of the force *NM*, and the body *P* may be carried first from a node or quadrature to the conjunction *A*, and because the body *P* is urged by the force of rotation through the arc *Pp*, in an element of time, by the force put in place, it will describe the line element $P\pi$ which is not in the plane *CPT*, but curved away from that towards Pm, and thus the body is moving in the plane $TP\pi$ that produced with the plane *EST* does not cross at *C* but beyond *C* towards the opposition *B*. With centre *C* and interval *TP* the circle *CaDb* is described in the plane *EST*, in the plane *CPD* the arc of the circle *PC*, and in the plane $TP\pi$ the arc *Pc* crossing the circle *CaDb* at *c*. And because the force *NM* is a minimum with respect to the force of rotation of the body *P*, the angle *CPc*, of the inclination of the planes *CPT* and *cPT* is the smallest possible or infinitesimal, and the arc *Pc* only differs from the arc *PC* by an infinitesimal quantity; whereby since, by hypothesis, the arc *PC* differs from the quadrant *CA* by a finite amount *PA*, the sum of the arcs *PC* and *Pc* is less than a semicircle, and hence in the spherical triangle *CPC*, the exterior angle *PCa* (by Prop.13 of the Sphericorum of *Menelaus*, or by the Sphericorum of *Wolf*) is greater than the internal angle *PcC*, that is, the inclination of the plane *cPT* to the plane *ES*T is less than the inclination of the plane *CPT* to the same plane *EST*. Therefore in the passage of the body *P* from the quadrature *C* to the conjunction *A* the inclination of the orbit is always diminished, and because the node *C* is transferred to *c*, there becomes because of the way in which the rotation of the body, the nodes are returned. In the same way the diminution of the inclination and the nodes to be returned in the passage of the body from the quadrature *D* to the opposite B is demonstrated. Now the body may be carried from the conjunction A to the vicinity of the quadrature *D*, and at some place *P*, by the squared force, certainly it is urged by the force of rotation through the arc *Pp* and by the force *NM* along the right line *Pm*, and thus it will describe the line element $P\pi$, which leans towards *Pm* from the arc *Pp*. Whereby if from the centre *T* and with the radius *TP*, three



arcs *PD, aD*, and *Pd* are described, in the same way it will be shown that the node *D* to be transferred to the previous at *d*, and the angle *Pda* to be greater than the angle the internal angle *PDd*, that is, the inclination of the orbit is to be augmented in the transition of the body *P*, from the conjunction to the nearby quadrature, and in the same way it is shown to happen in the transition from the opposition *B* to the quadrature *C*. Q.e.d.]

Corol. 11. [In which the variation of the ascending node is discussed].

Because the body *P,* when the nodes are in the quadratures, are perpetually attracted from the plane of its orbit, and that in part towards *S* in its transition from the node *C* through the conjunction *A* to the node *D* ; and in the opposite part in the transition from the node *D* through the opposition *B* to the node *C* : it is evident, that in its motion from the node *C* the body constantly recedes from the orbit of its first plane *CD,* until then it has arrived at the nearest node; and thus at this node, with the greatest distance from that first plane *CD,* it passes through the plane of the orbit *EST* not in its plane with the other node *D,* but at some point thence it turns in the direction of the body *S,* each hence is turning towards the new position of a node. And by a similar argument the nodes go on to recede in the transition of the body from this node to the neighbouring node. Therefore the nodes continually recede from the quadratures put in place; the nodes are at rest at the syzygies, where nothing in the latitude disturbs the motion; at the intermediate positions, participants of each condition, the nodes recede more slowly: and thus, always either by receding, or by stationary individual revolutions, they are carried to the preceding nodes.

*Corol.*12. All these errors described in these corollaries are a little greater at the conjunction of the bodies *P* and *S,* than at the opposition of these; and that on account of the greater generating forces *NM* and *ML.*

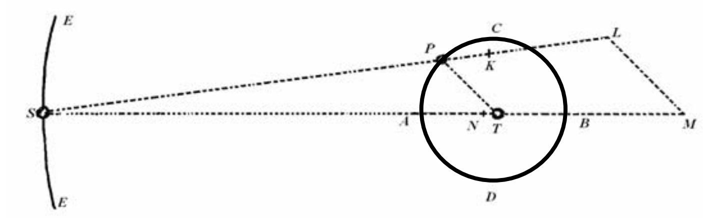
Corol. 13. [In which the motion of the body *S*, so far ignored, is discussed].

Whenever ratios of these corollaries do not depend on the magnitude of the body *S,* all the preceding will be obtained, when only the magnitude of the body *S* is put in place, so that the system of the two bodies *T* and *P* may be rotating about the centre of this. And from the increase in the body *S,* and thus with the increase of the centripetal force, from which the errors of the body *P* arise, all these errors emerge, with the distances greater in this case than in the other, where the body *S* is revolving around the system of the bodies *P* and *T.*

*Corol.*14. [In which the formula for S's perturbation is introduced].

But since the forces *NM* and *ML*, when the body *S* is remote, shall be approximately as the force *SK* and the ratio *PT* to *ST* conjointly, that is, if there may be given both the distance *PT*, as well as the absolute force of the body *S*, inversely as *ST* cubed ; but these forces *NM* and *ML* the cause of all the errors and effects, which have been acted on in the preceding corollaries: it is evident, that these have affected everything, with the system of the bodies *T* and *P* in place, and only with the distance *ST* chanced and with the absolute force of the body *S*, shall be composed approximately in the ratio from the direct ratio of the absolute force of the body *S*, and inversely in the triplicate ratio of the inverse distance *ST*. From which if the system of bodies *T* and *P* may be revolving about the distant body *S;* these forces *NM* and *ML*, and the effects of these will be will be (by Corol. 2. and 6, Prop. IV.) reciprocally in the square ratio of the periodic times. And thence also, if the magnitude of the body *S* shall be the proportional of the absolute force itself, these forces *NM* and *ML* and the effects of these, shall be directly as the cube of the apparent diameter of the distant body *S* observed from the body *T*, and conversely. For these ratios are the same, and composed from the above ratio.

[Note 512, *L* & *J* : On account of the great distance of the body *S*, *LS* will be almost parallel to *MS*, and $SN = ST = SK$, and $ML = PT$; and because *NM* at the syzygies is as *ML* at the quadratures. If increased or diminished by the action of the body *S*, the orbit *CADB* hence together with the depending lines *PT*, *NM*, *ML* may be increased or diminished (by Cor.6 of this Prop. 66) , these three lines will be increased or diminished in almost the same ratio between themselves (with all else being equal). But the force *ML* to the force *SK* is as the right line *ML* to the right line *SK*, or approximately as *PT* to ST ; whereby the force ML (and thus also the force *NM*) is nearly as the force *SK* and the ratio *PT* to *AD*, jointly that is, if the accelerative force *SK* may be called *A*, as $\frac{A\times PT}{ST}$. Again with the given absolute force of the body *S*, the accelerative force *A* at the distance *SK* or *ST* is as $\frac{1}{ST^2}$ (by hypothesis). Whereby the forces *NM, ML*, with the given absolute force of the body *S*, are as $\frac{PT}{ST^3}$; that is (if the distance PT may be given), inversely as ST^3. Indeed if the variable shall be the absolute force *V* of the body *S*, the accelerative force *A* will be as directly as the absolute force and inversely as the square of the distance *ST*, (for with the absolute force of the body S, the accelerative force is inversely as ST^2, and with the distance *ST* remaining, the accelerative force is as the absolute force directly, and therefore likewise with the variations of the absolute force and the distance, the accelerative force is as the absolute force directly and the square of the distance inversely). Whereby if in place of the accelerative force *A* that ratio composed from the factor $\frac{A\times PT}{ST}$ may be put in place, the forces *NM, ML* will be approximately as $\frac{V\times PT}{ST^3}$, or with *PT* given, as $\frac{V}{ST^3}$, that is in a ratio composed from the direct ratio of the absolute force of the body *S*, and from the inverse cube of the distance *ST*. But the absolute force of the body *S*, is as shown in the ratio composed of the accelerative force *A* and the square of the distance *ST*, and the accelerative force *A* at the distance *ST* is (by Corol.2 Prop. 4) in a ratio composed from the direct ratio of the distance *ST* and in the inverse square ratio of the periodic time of the body *T* around *S* to the distance *ST* of the circle described, and thus the absolute force of the body *S* is as the cube of the distance ST directly, and the square of the periodic time of the body *T* inversely. Whereby the forces *NM*, *ML* (and the effects of these) which are directly as the absolute force, and inversely as the cube of the distances, are inversely in the square ratio of the periodic times of the body T. End of note.]



Corol. 15. [In which Newton infers the proportionality of perturbing force to the radius of the orbit of *P* about *T*.]

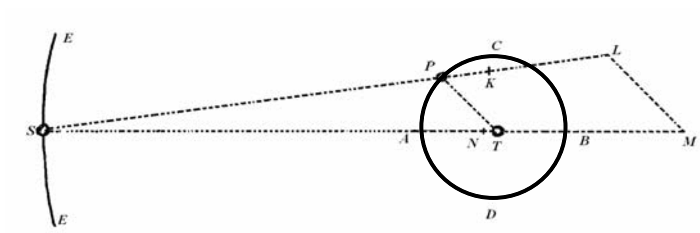
And because if, with the forms of the orbits *ESE* and *PAB*, with the proportions and inclination to each other remaining constant, the magnitude of these may be changed, and if the forces on the bodies *S* and *T* may either remain or be changed in some given ratio; these forces (that is, the force of the body *T,* by which the body may be deflected from a straight path into the orbit *PAB*, and the force of the body *S*, from which the same body *P* is forced to deviate from that) always act in the same way, and in the same proportion: it is necessary that all effects shall be similar and the times of the effects to be proportional; that is, so that all linear errors shall be as the diameters of the orbits, truly the angles shall be as at first, and the times of similar linear errors, or equal angular errors, shall be as the periodic times of the orbits.

[Note from L & J : That is, if the absolute forces of the bodies *S* and *T* either may remain or be changed in some given ratio, and the magnitude thus may change of the orbits *ESE* and *PAB*, so that it may always remain similar to the orbit *ESE* itself, thus also so that of the orbit *PAB* itself, and the inclination of these orbits may not change, nor the proportion of the ratio of the axis of one orbit to the axes of another or of any lines whatever in one orbit to the homologous lines in another orbit. End of L & J note.
Translator's note : these are not errors as we now understand the word, but disturbances by forces that alter the orbits in some way.]

*Corol.*16. [In which the mean motion of the increase of the line of apses and the mean regression of the line of ascending nodes are to be the same, apart from sign].

From which, if the forms of the orbits and the inclination to each other may be given, and the magnitudes, forces and distances of the bodies may be changed in some manner ; from the given errors and from the errors in the times in one case, the errors and the error in the time can be deduced closely : but this is shorter by this method. The forces *NM* and *ML*, with the rest remaining, are as the radius *TP*, and the periodic effects of these (by Corol. 2, Lemma X.) as the forces and the squares of the periodic times of the body *P* taken together. These are the linear errors [disturbing the orbit] of the body *P;* and hence the angular errors have been viewed from the centre *T* (that is, both the motion of the apsides and of the nodes, as well as all the errors appearing from the longitude and latitude), in whatever revolution of the body *P,* as the square of the time of revolution approximately. These ratios may be taken together with the ratios of Corollary XlV and in some system of the bodies *T, P, S,* where *P* is revolving around the vicinity of *T* itself, and *T* around the distant body *S*, the angular errors of the body *P*, appearing from the centre *T*, will be, in the individual revolutions of that body *P*, as the squares of the periodic times of the body *P* directly and the square of the periodic time of the body *T* inversely. And thence the mean motion of the apsides will be in a given ratio to the mean motion of the nodes ; and each motion will be as the periodic time of the body *P* directly and the square of the periodic time of the body *T* inversely. By increasing or lessening the eccentricity and the inclination of the orbit *PAB*, the motion of the apsides and of the nodes are not changed appreciably, unless where the same are exceedingly large.

Corol. 17. [In which the gravitational forces of S and T may be compared with their distances and periods].



But since the line *LM* may now be greater, then less than the radius *PT*, the mean force *LM* may be expressed by that radius *PT;* and [from the similar triangles *SPT* and *SLM*] this will be to the mean force *SK* or *SN* (that is allowed to express by *ST* [recall that *N* is the focus of the elliptical orbit of *P*, *T* the centre of gravity of the fixed body *T*]) as the length *PT* to the length *ST*.

$$[i.e. \tfrac{LM}{SK} = \tfrac{PT}{ST}\ ]$$

But the mean force *SN* or *ST*, by which the body *T* is held in its orbit around *S*, to the force, by which the body *P* is held in its orbit around in *T*, is in a ratio compounded from the ratio of the radius *ST* to the radius *PT*, and the square ratio of the periodic time of the body *P* around *T* [which confusingly I have called *T*], to the periodic time of the body *T* [*t*] around *S*.

$$[i.e. \frac{|\overrightarrow{ST}|}{|\overrightarrow{PT}|} \propto \tfrac{ST}{PT} \times \left(\tfrac{T}{t}\right)^2 .]$$

And from the equality, the mean force *LM* to the force, by which the body *P* in held in its orbit around *T* (or by which the same body *P*, in the same periodic time may be able to revolve around some fixed point *T* at a distance *PT*) is in that square ratio of the periodic times. Therefore with the given periodic times together with the distance *PT*, the mean force *LM* may be given*;* and with that given, the force *MN* is also given approximately by the analogy of the lines *PT* and *MN*.

Corol. 18. [Some relevant outcomes of the theory are now discussed, relating to the tides and associated matters, in the final corollaries. These are of considerable interest].

From the same laws, by which the body *P* is revolving about the body *T*, we may imagine many fluid bodies to be moving around the same body *T* at equal distances from that; then from these made touching each other a ring of fluid may be formed, round and concentric to the body *T*; and the individual parts of the ring, on completing all their motion according to the law of the body *P*, approach closer to the body *T*, and faster at the conjunction and opposition of these and the body *S*, than at the quadratures. And the nodes of this annulus, or the intersections of this with the plane of the orbiting body *S* or *T*, are at rest at the syzygies ; truly beyond the syzygies they will be moving as before, and most quickly indeed at the quadratures, more slowly at other places. Also the inclination of the annulus will vary in inclination, and the axis of this will oscillate in the individual revolutions, and with a complete revolution completed it will return to the former position, unless as far as it will be carried around by the precession of the nodes.

Corol. 19. We may now consider the globe of the body *T;* agreed not to be fluid matter, to be made bigger and to be extended as far as to this annulus, and with the hollow evacuated around to contain water, and with the same motion to revolve with the same period around its axis uniformly. This liquid will be accelerated and retarded (as in the above corollary) in turn ; at the syzygies it will be faster, and at the quadratures slower than the surface of the globe, and thus it will flow to and fro in the manner of the sea. Water, by revolving around the centre of a globe at rest, if the attraction of the body *S* may be taken away, will acquire no motion of flowing backwards and forwards. The reasoning is the same for a globe progressing uniformly in a direction, and meanwhile turning about its centre (by Corol. 5 of the laws) and as of a globe drawn uniformly from its course (by Corol. 6. of the laws). But the body *S* may approach, and from the inequality of this attraction soon the water will be disturbed. And also the water to be attracted more nearer than the body, lesser with that more remote. Moreover the force *LM* pulls the water down at the quadratures, and will make that descend as far as the syzygies ; and the force *KL* draws the same water up at the syzygies, and stop the descent of this, and will make that ascend as far as the quadratures: unless in so far as the motion of flux and reflux of the water may be directed by the channel, and may be retarded a little by friction.

Corol. 20. If the annulus now may be rigid, and the globe may be made smaller, the flow to and fro will stop ; but that inclination of the motion and of the precession of the nodes will still remain. The globe may have the same axis with the annulus, and the gyrations may be completed in the same times, and the annulus may touch the interior itself with its own surface, and may adhere to that; and by sharing the motion of this each structure will be oscillating, and the nodes will be regressing. For the globe, as soon will be said [Cor. XXII], may be indifferent to all forces being accepted. The maximum angle of the inclination of the annulus with the globe of the orbit is when the nodes are at the syzygies. Thence in the progression of the nodes to the quadratures the annulus tries to minimize its own inclination, and by that trial it impresses the motion on the whole globe. The globe retains the impressed motion, until the annulus then by attempting motion in the opposite sense may hence remove that motion, and a new motion is impressed in the opposite direction: And on this account the maximum motion of the decrease of the inclination shall be at the quadrature of the nodes, and the smallest angle of inclination in the octants after the quadratures; then the maximum motion of re-inclination at the syzygies, and the maximum angle in the next octant. And the same account is given of the globe with the ring removed, which is a little higher at the equatorial or higher regions than near the poles, or is made from a slightly denser material. For this excess of the material in the equatorial regions supplies that of the ring. And nevertheless, with any increase of the centripetal force of this globe, all the parts of this are supposed to be attracted downwards, according to the manner of the gravitating parts of the earth, yet the phenomena of this and of the preceding corollary hence scarcely are changed, except that the places of the maxima and minima height of the water will be different. For now the water will be sustained and remain in its own orbit, not by its centrifugal force, but by the hollow in which is flows. And in addition the force LM pulls the water downwards maximally at the quadratures, and the force KL or $NM - LM$ draws the same upwards maximally at the syzygies. And these forces taken together stop the water being drawn downwards and begin to draw the water upwards in the octants before the syzygies, and they stop drawing the water upwards and begin to draw the water downwards at the octant after the syzygies. And hence the maximum height of the water can come about in the octants after the syzygies, and the minima in the octants after the quadrutures approximately; unless as it were the motion of rising or falling impressed by these forces, for the water to persist with a little daily motion, either by a force in place or by the hindrance of some channel it might have stopped a little quicker.

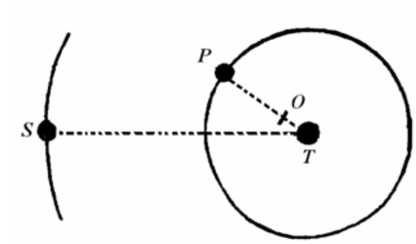
Corol. 21. By the same account, by which the excess matter of the globe lying around the equator caused the nodes move backwards [or regress], and thus by the increase of this matter this regression in turn may be increased, again truly with diminution it is diminished, and on being removed is completely removed; if more than the excess matter may be removed, that is, if the globe near the equator is either further lowered, or rarer than around the poles, the motion of the nodes will arises in succession.

Corol. 22. And thus in turn, the constitution of the globe is understood from the motion of the nodes. Without doubt if the same poles of the globe are maintained constantly, and the motion shall be as previously [*i.e.* retrograde], there is an excess of matter near the equator ; if it is in succession, there is deficiency. Put a uniform and perfectly round globe first at rest in free space; then propelled from its place by some oblique impulse on its surface, and the motion thence considered partially to be circular and partially along a direction. Because this globe has all axis passing through its centre indifferently, there is no propensity for one axis or any other axis to be in place, it is evident that neither this particular axis, nor the inclination of the axis, at any time will be changed by its own force. Now the globe may be impelled obliquely, in the same part of the surface, as the first, by some new impulse; and since the effect of impulses coming sooner or later changes nothing, it is evident that these two successive impulses impressed in succession produce the same motion, as if they were impressed together, that is , the same [motion is produced], as if the globe had been struck by a simple force composed from each impulse (by Corol. 2 of the laws), and thus a simple force about the given inclination of the axis.

And the reasoning is the same of a second impulse made at some other place on the equator of the first motion ; or of the first impulse made at some place on the equator of the motion generated by the second impulse without the first ; and of both impulses made at any places : these will generate the same circular motion as if they were impressed once simultaneously at the intersection of the equators of their motions, which they would generate by themselves. Therefore a homogeneous and perfect globe will not retain several distinct motions, but adds together all the impressed forces and reduces that to one, and just as within itself, it always rotates in a simple and uniform motion about a single axis, always with an invariable inclination given. Moreover neither will a centripetal force be able to change the inclination of the axis, nor the velocity of the rotation. If a globe may be considered to be divided into two hemispheres by some plane through its centre, and in which a force is directed passing through its centre ; that force will always urge each of the hemispheres equally, and therefore the globe, by which the motion of rotation, will be inclined in no direction. Truly there may be added at some place between the pole and the equator new matter in the form of raised mountains, and these, by always trying to be receding from the centre of its motion, will disturb the motion of the globe, and it may happen that the poles of this globe may wander on the surface, and they may always describe circles about themselves and about the points opposite to themselves. Nor may the immensity of these wanderings be themselves corrected, except by locating that mountain either at one pole or the other, in which case (by Corol. XXI.) the nodes of the equator will be progressing ; or at the equator, by which account the nodes will be regressing (by Corol. XX.) ; or finally by adding new matter to the other part of the axis, by which the mountain will be balanced in its motion, and with this agreed upon the nodes either will be progressing, or receding, provided that the mountain and these new matters are either nearer to the pole or to the equator.

PROPOSITION LXVII. THEOREM XXVII.

With the same laws of attraction in place, I say that a body beyond S, around the common centre of gravity O between P and T, by radii drawn to that centre, will describe areas more proportional to the times and the orbit approaching more to the form of an ellipse having the focus at the centre, than can be described around the innermost and greatest body T, with radii drawn to that itself.



For the attractions of the body *S* towards *T* and *P* are composed of this absolute attraction, which is directed more to the common centre of gravity *O* of the bodies *T* and *P* , than to the greatest body *T* , with every square of the distance *SO* more inversely proportional, than to the square of the distance *ST:* as the matter on examination will be readily agreed upon.

PROPOSITION LXVIII. THEOREM XXVIII.

With the same laws of attraction in places , I say that the external body S, with radii drawn to that centre, will describe areas more proportional to the times, about the common centre of gravity O of the interior bodies P and T, and an orbit more approaching to the form of an ellipse having the focus at the same centre, if the innermost body and greatest body and the others thus may be disturbed by these attractions, than if that either were at rest and not being attracted, or much less or much more attracted, and consequently either much more or much less disturbed.

This may be shown in almost the same manner as Prop. LXVI, by a more lengthy argument, which therefore I shall pass by. It will suffice to consider the matter thus. From the demonstration of the latest proposition it is apparent that the centre, towards which the body *S* is urged by the forces conjointly, is very close to the common centre of gravity of these two. If this centre should coincide with that common centre, and the common centre of gravity of the three bodies may remain at rest; accurate ellipses are described by the body from the one side, and the common centre of gravity of the other two from the other side, about the common centre of gravity of everything at rest. This may be apparent from the second corollary of Proposition LVIII, taken with the demonstrations in Prop. LXIV. & LXV. This elliptical motion may itself be disturbed a little by the distance of the centre of the two from the centre, to which the third body *S* is attracted. Besides a motion may be given to the common centre of the three, and it will be increased by the perturbation, Hence the perturbation is a minimum, when the common centre of the three is at rest ; that is, when the innermost and largest body *T* is attracted by the law of the others: and the perturbation shall always be the largest, when that common centre of the three, by the smallest decrease in the motion of the body *T* , begins to move, and it is disturbed more and more henceforth.

Corol. And hence, if several small bodies were revolving around the largest, it is possible to deduce that the orbits described approach closer and closer to ellipses, and the descriptions of the areas become more equal, if all the bodies by accelerative forces, which are as the absolute forces of these directly and inversely as the square of the distances, and may mutually attract and perturb each other, and also the focus of each orbit may be deduced to be at the common centre of gravity of all the interior bodies (without doubt the focus of the first and innermost at the centre of gravity of the first and largest body; that of the second orbit, at the common centre of gravity of the two innermost bodies; that of the third, at the common centre of gravity of the three innermost , and thus henceforth), as if the innermost body were at rest and may be situated at the common focus of all the orbits.

PROPOSITION LXIX. THEOREM XXIX.

In a system of several bodies A, B, C, D, &., if some body A shall attract all the others B, C, D, &c. by accelerative forces which are inversely as the squares of the distances from the attracting body; *and another B also attracts the others A, C, D, &c. by forces which are inversely as the squares of the distances from the attracting body: the absolute forces of attraction of the bodies A, B to each other, as are the bodies themselves A, B, of which they are the forces.*

For the accelerative attractions of all the bodies *B, C, D* towards *A,* with equal distances, by hypothesis are equal to each other; and similarly the acclerative attractions of all the bodies towards *B,* with equal distances, in turn are equal to each other. Moreover the absolute force of attraction of the body *A* is to the absolute force of attraction of the body *B,* as the accelerative attraction of all the bodies towards *A* to the accelerative attraction of all the bodies towards *B,* with the distances equal; and thus is the accelerative attraction of the body *B* towards *A,* to the accelerative attraction of the body *A* towards *B.* But the accelerative attraction of the body *B* towards *A* is to the accelerative attraction of the body *A* towards *B,* as the mass of the body *A* to the mass of the body *B* ; therefore because the motive forces, which (by the second, seventh and

eighth definitions) are as the accelerative forces and the bodies attracted jointly, here they are (by the third law of motion) in turn equal to each other. Therefore the absolute attractive force of the body *A* is to the absolute attractive force of the body *B,* as the mass of the body *A* to the mass of the body *B.*
Q. E. D.

Corol. I. Hence if the individual bodies of the system *A, B, C, D,* &c. seen to attract all the rest separately by accelerative forces, which are inversely as the squares of the distances from the attracting body ; the absolute forces of all the bodies will be in turn as the bodies themselves.

Corol. 2. By the same argument, if the individual bodies of the system *A, B,C, D,* &c. separately observed attract all the others with accelerative forces, which are either inversely, or directly in the ratio of any powers of the distances from the attracting body, or which are defined to be following some common law by attracting each other from the distances ; it is agreed that the absolute forces of these bodies are as the bodies.

Corol.3. In a system of bodies, the forces of which decrease in the square ratio of the distances, if the lesser may be revolving around the largest one in ellipses, having the common focus of which at the centre of the largest of these, as it can happen that they are revolving most accurately ; and with the radii drawn to that largest body, areas may be described exactly proportional to the times : the absolute forces of these bodies will be in turn, either accurately or approximately, in the ratio of the bodies; and conversely. This is apparent from the Corollary of Prop. LXVIII, taken with the corollary I of this Prop.

Scholium.

We are led by these propositions to an analogy between the centripetal forces, and the central bodies, to which these forces are accustomed to be directed. For it is agreeable to the reason, that the forces which are directed towards the bodies, depend on the nature and quantity of the same, as happens in experiments with magnets. And as often as cases of this kind arise, the attractions of the bodies are to be considered, by designating appropriate forces to the individual members of these, and by gathering together the sum of the forces. I may talk about this attraction generally by taking some known attempt of the bodies to approach each other: either that attempt comes about itself from the action of the bodies, or they reach towards each other mutually, or by some agent [spirit in the original] of agitation sent between themselves in turn; or this arises from the action of the ether, or of the air, or by some medium whatsoever, originating either from bodies or not from bodies, by impelling the floating bodies in some manner towards each other in turn.

In the same general sense used I may talk not about the kind of forces and physical quantities. but the amounts and mathematical proportions of these set out in this tract; as I have explained in the definitions. In mathematics it is required to investigate the magnitudes of forces and the ratios of these , which follow from some conditions put in place: then, when we descend to the level of physics, we compare these ratios with the phenomena; so that it may become known which conditions may be agreed upon with the kinds of the individual attractive forces of the bodies. And then finally concerning the kinds of bodies, it will be allowed to argue without risk about the causes and reasons from physics. Therefore we may see by which forces spherical bodies, now agreed upon in the manner established from the attraction of small bodies, must act mutually on each other ; and thus what kind of motions they may thence follow.

SECTIO XI.

De motu corporum viribus centripetis se mutuo petentium.

Hactenus exposui motus corporum attractorum ad centrum immobile, quale tamen vix extat in rerum natura. Attractiones enim fieri solent ad corpora; & corporum trahentium & attractorum actiones semper mutuae sunt & aequales, per legem tertiam: adeo ut neque attrahens possit quiescere neque attractum, si duo sint corpora, sed ambo (per legum corollarium quartum) quasi attractione mutua, circum gravitatis centrum commune revolvantur: & si plura sint corpora, quae vel ab unico attrahantur, & idem attrahant, vel omnia se mutuo attrahant; haec ita inter se moveri debeant, ut gravitatis centrum commune vel quiescat, vel uniformiter moveatur in directum. Qua de causa jam pergo motum exponere corporum se mutua trahentium, considerando vires centripetas tanquam attractiones, quamvis fortasse, si physice loquamur, verius dicantur impulsus. In mathematicis enim jam versamur; & propterea, missis disputationibus physicis, familiari utimur sermone, quo possimus a lectoribus mathematicis facilius intelligi.

PROPOSITIO LVII. THEOREMA XX.

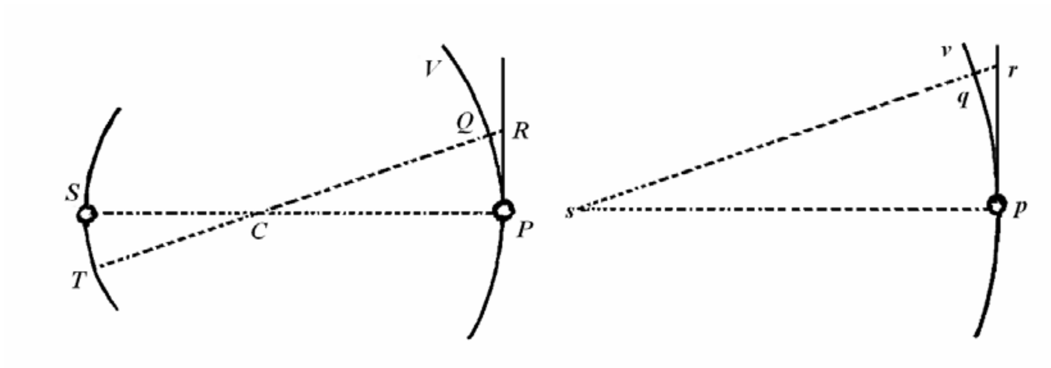
Corpora duo se invicem trahentia describunt & *circum commune centrum gravitatis, & circum se mutuo, figuras similes.*

Sunt enim distantiae corporum a communi gravitatis centro reciproce proportionales corporibus; atque ideo in data ratione ad invicem, & componendo in data ratione ad distantiam totam inter corpora. Feruntur autem hae distantiae circum terminum suum communem aequali motu angulari, propterea quod in directum semper jacentes non mutant inclinationem ad se mutuo. Lineae autem rectae, quae sunt in data ratione ad invicem, & aequali motu angulari circum terminos suos feruntur, figuras circum eosdem terminos in planis, quae una cum his terminis vel quiescunt, vel motu quovis non angulari moventur, describunt omnino similes. Proinde similes sunt figurae, quae his distantiis circumactis describuntur.
Q.E.D.

PROPOSITIO LVIII. THEOREMA XXI.

Si corpora duo viribus quibusvis se mutuo trahunt, & interea revolvuntur circa gravitatis centrum commune: dico quod figuris, quas corpora sic mota describunt circum se mutuo, potest figura similis & *aequalis, circum corpus alterutrum immotum, viribus iisdem describi.*

Revolvantur corpora *S*, *P* circa commune gravitatis centrum *C*, pergendo de *S* ad *T*; deque *P* ad *Q* . A dato puncto *s* ipsis *SP*, *TQ* aequales & parallelae ducantur semper *sp*, *sq;* & curva *pqv*, quam punctum *p* revolvendo circum punctum immotum *s* describit, erit similis & aequalis curvis, quas corpora *S*, *P* describunt circum se mutuo: proindeque (per Theor. XX.) similis curvis *ST* & *PQV*, quas eadem corpora describunt circum commune gravitatis centrum *C:* idque quia proportiones linearum *SC, CP,* & *SP* vel *sp* ad



invicem dantur.

Cas. I. Commune illud gravitatis centrum. *C,* per legum corollarium quartum, vel quiescit, vel movetur uniformiter in directum. Ponamus primo, quod id quiescit, inque *s* & *p* locentur corpora duo, immobile in *s*, mobile in *p*, corporibus *S* & *P* similia & aequalia. Dein tangant rectae *PR* & *pr* curvas *PQ* & *pq* in *P* & *p*, & producantur *CQ* & *sq* ad *R* & *r*. Et ob similitudinem figurarum *CPRQ, sprq* erit *RQ* ad *rq* ut *CP* ad *sp,* ideoque in data ratione. Proinde si vis, qua corpus *P* versus corpus *S*, atque ideo versus centrum intermedium *C* attrahitur, esset ad vim, qua corpus *p* versus centrum *s* attrahitur, in eadem illa ratione data; hae vires aequalibus temporibus attraherent semper corpora de tangentibus *PR, Pr* ad arcus *PQ, pq* per intervalla ipsis proportionalia *RQ, rq,* ideoque vis posterior efficeret, ut corpus *p* gyraretur in curva *pqv,* quae similis esset curvae *PQV*, in qua vis prior efficit, ut corpus *P* gyretur; & revolutiones iisdem temporibus completentur. At quoniam vires illae non sunt ad invicem in ratione *CP* ad *sp,* sed (ob similitudinem & aequalitatem corporum *S* & *s, P* & *p, &* aequalitatem distantiarum *SP, sp*) sibi mutuo aequales ; corpora aequalibus temporibus aequaliter trahentur de tangentibus: & propterea, ut corpus posterius *p* trahatnr per intervallum majus *rq,* requiritur tempus majus, idque in subduplicata ratione intervallorum; propterea quod (per lemma decimum) spatia ipso motus initio descripta sunt in duplicata ratione temporum. Ponatur igitur velocitas corporis *p* esse ad velocitatem corporis *P* in subduplicata ratione distantiae *sp* ad distantiam *CP*, eo ut temporibus, quae sint in eadem subduplicata ratione, describantur arcus *pq, PQ* qui sunt in ratione integra: Et corpora *P, p* viribus aequalibus semper attracta describent circum centra quiescentia *C* & *s* figuras similes *PQV, pqv,* quarum posterior *pqv* similis est & aequalis figurae, quam corpus *P* circum corpus mobile *S* describit. *Q.E.D.*

Cas: 2. Ponamus jam quod commune gravitatis centrum, una cum spatio in quo corpora moventur inter se, progreditur uniformiter in directum; & (per legum corollarium sextum) motus omnes in hoc spatio peragentur ut prius, ideoque corpora describent circum se mutuo figuras easdem ac prius, & propterea figurae *pqv* similes & aequales. *Q.E.D.*

Corol. 1. Hinc corpora duo viribus distantiae suae proportionalibus se mutuo trahentia, describunt (per prop. X.) & circum commune gravitatis centrum, & circum se mutuo, ellipses concentricas; & vice versa, si tales figurae describuntur, sunt vires distantiae proportionales.

Corol. 2. Et corpora duo, viribus quadrato distantiae suae reciproce proportionalibus, describunt (per prop. XI. XII. XIII.) & circum commune gravitatis centrum, & circum se mutuo, sectiones conicas umbilicum habentes in centro, circum quod figurae describuntur. Et vice versa, si tales figurae describuntur, vires centripetae sunt quadrato distantiae reciproce proportionales.

Corol. 3. Corpora duo quaevis circum gravitatis centrum commune gyrantia, radiis & ad centrum illud & ad se mutuo ducis, describunt areas temporibus proportionales.

PROPOSITIO LIX. THEOREMA XXII.

Corporum quorum S ac P, circa commune gravitatis centrum C *revolventium, tempus periodicum esse ad tempus periodicum corporis alterutrius P, circa alterum immotum S gyrantis, & figuris, quae corpora circum se mutuo describunt, figuram similem ac aequalem describentis, in subduplicata ratione corporis alterius S, ad summam corporum* $S+P$.

Namque, ex demonstratione superioris propositionis, tempora, quibus arcus quivis similes *PQ* & *pq* describuntur, sunt in subduplicata ratione distantiarum *CP & SP* vel *sp*, hoc est, in subduplicata ratione corporis *S* ad summam corporum $S+P$. Et componendo, summae temporum quibus arcus omnes similes *PQ* & *pq* describuntur, hoc est, tempora tota, quibus figurae totae similes describuntur, sunt in eadem subduplicata ratione. *Q.E.D.*

PROPOSITIO LX. THEOREMA XXIII.

Si corpora duo S & P, viribus quadrato distantiae suae reciproce proportionalibus, se mutuo trahentia, revolvuntur circa gravitatis centrum commune: dico quod ellipseos, quam corpus alterutrum P hoc motu circa alterum S describit, axis principalis esse ad axem principalem ellipseos, quam corpus idem P circa alterum quiestens S eadem tempore periodico describere possit, ut summa corporum duorum $S+P$ *ad primum duorum medie proportionalium inter hanc summam & corpus illud alterum S.*

Nam si descriptae ellipses essent sibi invicem aequales, tempora periodica (per theorema superius) forent in subduplicata ratione corporis *S* ad summam corporum $S+P$. Minuatur in hac ratione tempus periodicum in ellipsi posteriore, & tempora periodica evadent aequalia; ellipseos autem axis principalis (per Prop. XV.) minuetur in ratione, cujus haec est sesquiplicata, id est in ratione, cujus ratio S ad $S+P$ est triplicata; ideoque erit ad axem principalem ellipseos alterius, ut primum duorum medie proportionalium inter $S+P$ *& S* ad $S+P$. Et inverse, axis principalis ellipseos circa corpus mobile descriptae erit ad axem principalem descriptae circa immobile, ut $S+P$ ad primum duorum medie proportionalium inter $S+P$ *& S. Q.E.D.*

PROPOSITIO LXI. THEOREMA XXIV.

Si corpora duo viribus quibusvis se mutuo trahentia, neque alias agitata vel impedita, quomodocunque moveantur ; motus eorum perinde se habebunt, ac,si non traherent se mutuo, sed utrumque a corpore tertio in communi gravitatis centro constituto viribus iisdem traheretur: Et virium trahentium eadem erit lex respectu distantiae corpontm a centro illo communi atqua respectu distantae totius inter corpora.

Nam viaes illae, quibus corpora se mutuo trahunt, tendendo ad corpora, tendunt ad commune gravitatis centrum intermedium ; ideoque eaedem sunt, ac si a corpore intermedio manarent. *Q.E.D.*

Et quoniam datur ratio distantiae corporis utriusvis a centro illo communi ad distantiam inter corpora, dabitur ratio cujusvis potestatis distantiae unius ad eandem potestatem distantiae alterius; ut & ratio quantitatis cujusvis, quae ex una distantia & quantitatibus datis utcunque derivatur, ad quantitatem aliam, quae ex altera distantia & quantitatibus totidem datis, datamque illam distantiarum rationem ad priores habentibus similiter derivatur. Proinde si vis, qua corpus unum ab altero trahitur, sit directe vel inverse ut distantia corporum ab invicem ; vel ut quaelibet hujus distantiae potestas ; vel denique ut quantitas quaevis ex hac distantia & quantitatibus datis quomodocunque derivata: erit eadem vis, qua corpus idem ad commune gravitatis centrum trahitur, directe itidem vel inverse ut corporis attracti distantia a centro illo communi, vel ut eadem distantiae hujus potestas, vel denique ut quantitas ex hac distantia & analogis quantitatibus datis similiter derivata. Hoc est, vis trahentis eadem erit lex respectu distantiae utriusque. *Q.E.D.*

PROPOSITIO LXII. PROBLEMA XXXVIII.

Corporum duorum, quae viribus quadrato distantiae suae reciproce proportionalibus se mutuo trahunt, ac de locis datis demittuntur, determinare motus.

Corpora (per theorema novissimum) perinde movebuntur, ac si a corpore tertio in communi gravitatis centro constituto traherentur; & centrum illud ipso motus initio quiescet per hypothesin; & propterea (per legum Corol. 4.) semper quiescet. Determinandi sunt igitur motus corporum (per Prob. XXV.) perinde ac si a viribus ad centrum illud tendentibus urgerentur, & habebuntur motus corporum se mutuo trahentium. *Q.E.D.*

PROPOSITIO LXIII. PROBLEMA XXXIX.

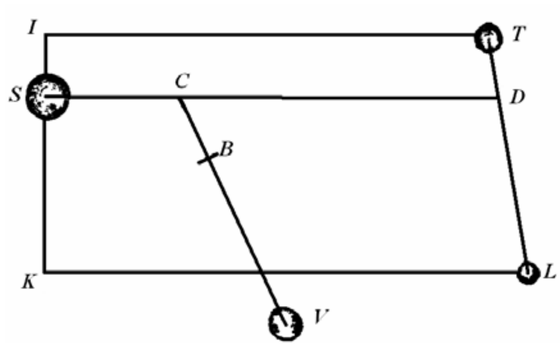
Corporum duorum quae viribus quadrato distantiae suae reciproce proportionalibus se mutuo trabunt, deque locis datis, secundum datas rectas, datis cum velocitatibus exeunt, determinare motus.

Ex datis corporum motibus sub initio, datur uniformis motus centri communis gravitatis, ut & motus spatii, quod una cum hoc centro movetur uniformiter in directum, nec non corporum motus initiales respectu hujus spatii. Motus autem subsequentes (per legum corollarium quintum, & theorema novissimum) perinde fiunt in hoc spatio, ac si spatium ipsum una cum communi illo gravitatis centro quiesceret, & corpora non traherent se mutuo, sed a corpore tertio sito in centro illo traherentur. Corporis igitur alterutrius in hoc spatio mobili, de loco dato, secundum datam rectam, data cum velocitate exeuntis, & vi centripeta ad centrum illud tendente correpti, determinandus est motus per problema nonum & vicesimum sextum: & habebitur simul motus corporis alterius circum idem centrum. Cum hoc motu componendus est uniformis ille systematis spatii & corporum in eo gyrantium motus progressivus supra inventus, & habebitur motus absolutus corporum in spatio immobili.

Q.E.D.

PROPOSITIO LXIV. PROBLEMA XL.

Viribus quibus corpora se mutuo trahunt crescentibus in simplici ratione distantiarum a centris: requiruntur motus plurium corporum inter se.



Ponantur primo corpora duo *T* & *L* commune habentia gravitatis centrum *D*. Describent haec (per corollarium primum theorematis XXI.) ellipses centra habentes in *D*, quarum magnitudo ex problemate X innotescit.

Trahat jam corpus tertium *S* priora duo *T* & *L* viribus acceleratricibus *ST, SL,* & ab ipsis vicissim trahatur. Vis *ST* (per legum Corol. 2.) resolvitur in vires *SD, DT;* & vis *SL* in vires *SD, DL.* Vires autem *DT, DL,* quae sunt ut ipsarum summa *TL,* atque ideo ut vires acceleratrices quibus corpora *T* & *L* se mutuo trabunt, additae his viribus corporum *T* & *L,* prior priori & posterior posteriori, componunt vires distantiis *DT* ac *DL* proportionales, ut prius, sed viribus prioribus majores; ideoque (per Corol. 1. prop.X & Corol.l. &.8, Prop.IV.) efficiunt ut corpora illa describant ellipses ut prius, sed motu celeriore. Vires reliquae acceleratrices *SD* & *SD,* actionibus motricibus $SD \times T$ & $SD \times L$, quae sunt ut corpora, trahendo corpora illa aequaliter & secundum lineas *TI, LK,* ipsi *DS* parallellas, nil mutant situs eorum ad invicem, sed faciunt ut ipsa aequaliter accedant ad lineam *IK;* quam ductam concipe per medium corporis *S,* & lineae *DS* perpendicularem. Impedietur autem iste ad lineam *IK* accessus faciendo ut systema corporum *T* & *L* ex una parte, & corpus *S* ex altera, justis cum velocitatibus, gyrentur circa commune gravitatis centrum *C.* Tali motu corpus *S,* eo quod summa virium motricium $SD \times T$ & $SD \times L$, distantiae *CS* proportionalium, tendit versus centrum C, describit ellipsin circa idem *C;* & punctum *D,* ob proportionales *CS, CD,* describet elliptin consimilem e regione. Corpora autem *T* & *L* viribus motricibus $SD \times T$ & $SD \times L$, prius priore, posterius posteriore, aequaliter & secundum lineas parallelas *TI* & *LK,* ut dictum est, attracta, pergent (per legum corollarium quintum & sextum) circa centrum mobile *D* ellipses suas describere, ut prius.

Q.E.D.

Addatur jam corpus quartum *V*, & simili argumento concludetur hoc & punctum *C* ellipses circa omnium commune centrum gravitatis *B* descrbere; manentibus motibus priorum corporum *T*, *L* & *S* circa centra *D* & *C*, sed acceleratis. Et eadem methodo corpora plura adjungere licebit. *Q.E.D.*

Haec ita se habent, edi corpora *T* & *L* trahunt se mutuo viribus acceleratricibus majoribus vel minoribus quam quibus trahunt corpora reliqua pro ratione distantiarum. Sunto mutua omnium attractiones acceleratrices ad invicem ut distantiae ductae in corpora trahentia, & ex praecedentibus facile deducetur quod corpora omnia aequalibus temporibus periodicis ellipses varias, circa omnium commune gravitatis centrum *B*, in plano immobili describunt. *Q.E.D.*

PROPOSITIO LXV. THEOREMA XXV.

Corpora plura, quorum vires decrescunt in duplicata ratione distantiarum ab eorundem centris, moveri posse inter se in ellipsibus; & radiis ad umbilicos ductis areas describere temporibus proportionales quam proxime.

In propositione superiore demonstratus est casus ubi motus plures peraguntur in ellipsibus accurate. Quo magis recedit lex virium a lege ibi posita, eo magis corpora perturbabunt mutuos motus; neque fieri potest, ut corpora, secundum legem hic positam se mutuo trahentia, moveantur in ellipsibus accurate, nisi servando certam proportionem distantiarum ab invicem. In sequentibus autem calibus non multum ab ellipsibus errabitur.

Cas: I. Pone corpora plura minora circa maximum aliquod ad varias ab eo distantias revolvi, tendantque ad singula vires absolutae proportionales iisdem corporibus. Et quoniam omnium commune gravitatis centrum (per legum Corol. quartum) vel quiescit vel movetur uniformiter in directum, fingamus corpora minora tam parva esse, ut corpus maximum nunquam distet sensibiliter ab hoc centro: & maximum illud vel quiescet, vel movebitur uniformiter in directum, sine errore sensibili; minora autem revolventur circa hoc maximum in ellipsibus, atque radiis ad idem ductis describent areas temporibus proportionales; nisi quatenus errores inducuntur, vel per errorem maximi a communi illo gravitatis centro, vel per actiones minorum corporum in se mutuo. Diminui autem possunt corpora minora, usque donec error iste, & actiones mutuae sint datis quibusvis minores ; atque ideo donec orbes cum ellipsibus quadrent, & areae respondeant temporibus, sine errore, qui non sit minor quovis dato. *Q.E.O*.

Cas. 2. Fingamus jam systema corporum minorum modo jam descripto circa maximum revolventium, aliudve quodvis duorum circum se mutuo revolventium corporum systema progredi uniformiter in directum, & interea vi corporis alterius longe maximi & ad magnam distantiam siti urgeri ad latus. Et quoniam aequales vires acceleratrices, quibus corpora secundum lineas parallelas urgentur, non mutant situs corporum ad invicem, sed ut systema totum, servatis partium motibus inter se, simul transferatur, efficiunt: manifestum est quod, ex attractionibus in corpus maximum, nulla prorsus orietur mutatio motus attractorum inter se, nisi vel ex attractionum acceleratricum inaequalitate, vel ex inclinatione linearum ad invicem, secundum quas attractiones fiunt. Pone ergo attractiones omnes acceleratrices in corpus maximum esse inter se reciproce ut quadrata distantiarum; & augendo corporis maximi distantiam, donec rectarum ab hoc ad reliqua ductarum differentiae respectu earum longitudinis, & inclinationes ad invicem minores sint, quam datae quaevis; perseverabunt motus partium systematis inter se sine erroribus, qui non sint quibusvis datis minores. Et quoniam, ob exiguam partium illarum ab invicem distantiam, systema totum ad modum corporis unius attrahitur ; movebitur idem hac attractione ad modum corporis unius; hoc est, centro suo gravitatis describet circa corpus maximum sectionem aliquam conicam (*viz*. Hyperbolam vel parabolam attractione languida, ellipsin fortiore) & radio ad maximum ducto describet areas temporibus proportionales. sine ullis erroribus, nisi quas partium distantiae, perexiguae sane & pro lubitu minuendae, valeant efficere. *Q.E.O.*

Simili argumento pergere licet ad casus magis compositos in infinitum.

Corol. 1. In casu secundo, quo propius accedit corpus omnium maximum ad systema duorum vel plurium, eo magis turbabuntur motus partium systematis inter se; propterea quod linearum a corpore maximo ad has ductarum jam major est inclinatio ad invicem, majorque proportionis inaequalitas.

Corol.2. Maxime autem turbabuntur, ponendo quod attractiones acceleratrices partium systematis, versus corpus omnium maximum, non sint ad invicem reciproce ut quadrata distantiarum a corpore illo maximo; praesertim si proportionis hujus inaequalitas major sit quam inaequalitas proportionis distantiarum a corpore maximo. Nam si vis acceleratrix, aequaliter & secundum lineas parallelas agendo, nil perturbat motus inter se, necesse est, ut ex actionis inaequalitate perturbatio oriatur, majorque sit, vel minor pro majore, vel minore inaequalitate. Excessus impulsuum majorum, agendo in aliqua corpora & non agendo in alia, necessario mutabunt situm eorum inter se. Et haec perturbatio addita perturbationi, quae ex linearum inclinatione & inaequalitate oritur, majorem. reddet perturbationem totam.

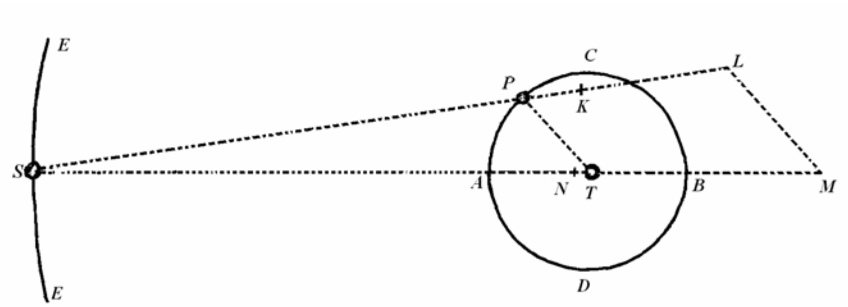
Corol. 3. Unde si systematis hujus. partes in ellipsibus, vel circulis sine perturbatione insigni moveantur; manifestum est, quod eaedem a viribus acceleratricibus, ad alia corpora tendentibus, aut non urgentur nisi levissime, aut urgentur aequaliter, & secundum lineas parallelas quamproxime.

PROPOSITIO LXVI. THEOREMA XXVI.

Si corpora tria, quorum vires decrescunt in duplicata ratione distantiarum, se mutuo trahant; & attractiones acceleratrices binorum quorumcunque in tertium sint inter se reciproce ut quadrata distantiarum; minora autem circa maximum revolvantur: dico quod interius circa intimum & maximum, radiis ad ipsum ductis, describet areas temporibus magis proportionales, & figuram ad formam ellipseos umbilicum in concursu radiorum habentis magis accedentem ; si corpus maximum his attractionibus agitetur; quam si maximum illud vel a minoribus non attractum quiescat, vel multo minus vel multo magis attractum, aut multo minus auti multo magis agitetur.

Liquet fere ex demonstratione corollarii secundi propositionis raecedentis; sed argumento magis distincto & latius cogente sic vincitur.

*Cas.*1 Revolvantur corpora minora *P* & S in eodem plano circa maximun *T,* quorum *P* describat orbem interiorem *PAB,* & *S* exteriorem *ESE.* Sit *SK* mediocris distantia corporum *P* & *S; &* corporis *P* versus *S* attrahio acceleratrix, in mediocri illa distantia, exponatur per eandem. In duplicata ratione *SK* ad *SP* capiatur *SL* ad *SK*, & erit *SL* attractio acceleratrix corporis *P* versus *S* in distantia quavis *SP*. Junge *PT,* eique



parallelam age *LM* occurrentem *ST* in *M;* & attractio *SL* resolvetur (per legum Corol. 2.) in attractiones *SM*, *LM.* Et sic urgebitur corpus *P* vi acceleratrice triplici. Vis una tendit ad *T,* & oritur a mutua attractione corporum *T* & *P*. Hac vi sola corpus *P* circum corpus *T,* sive immotum, sive hac attractione agitatum, describere deberet & areas, radio *PT*, temporibus proportionates, & ellipsio cui umbilicus est in centro corporis *T.* Patet hoc per prop. XI. & corollaria 2. & 3.Theor. XXI. Vis altera est attractionis *LN,* quae quoniam tendit a *P* ad *T,* superaddita vi priori coincidet cum ipsa, & sic faciet ut areae etiamnum temporibus proportionales describantur per Corol. 3. Theor. XXI. At quoniam non est quadrato distantiae *PT* reciproce proportionalis, componet ea cum vi priore vim ab hac proportione aberrantem, idque eo magis; quo major est proportio hujus vis ad vim priorem, caeteris paribus. Proinde cum (per prop. XI. & per corol. 2.. theor. XXI.) vis, qua ellipsis circa umbilicum *T* describitur, tendere debeat ad umbilicum illum, & esse quadrato distantiae *PT* reciproce proportionalis; vis illa composita, aberrando ab hac proportione, faciet ut orbis *PAB* aberret a forma ellipseos umbilicum habentis in *T;* idque eo magis, quo major est aberratio ab hac proportione; atque ideo etiam quo major est proportio vis secundae *LM* ad vim primam, caeteris paribus. Jam vero vis tertia *SM,* trahendo corpus *P* secundum lineam ipsi *ST* parallelam, componet cum viribus prioribus vim, quae non amplius dirigitur a *P* in *T;* quaeque ab hac determinatione tanto magis aberrat, quanto major est proportio hujus tertim vis ad vires priores, caeteris paribus: atque ideo quae faciet ut corpus *P*, radio *TP,* areas non amplius temporibus proportionales describat; atque ut aberratio ab hac proportionalitate tanto major sit, quanto major est proportio vis hujus tertiae ad vires caeteras. Orbis vera *PAB* aberrationem a forma elliptica praefata haec vis tertia duplici de causa adaugebit, tum quod non dirigatur a *P* ad *T*, tum etiam quod non sit reciproce proportionalis quadrato distantiae *PT*. Quibus intelleclis, manifestum est, quod arem temporibus tum maxime fiunt proportionales, ubi vis tertia, manentibus viribus caeteris, sit minima; & quod orbis *PAB* tum maxime accedit ad praefatam formam ellipticam. ubi vis tam secunda quam tertia, sed praecipue vis tertia sit minima, vi prima manente.

Exponatur corporis *T* attractio acceleratrix versus *S* per lineam *SN;* & si attractiones acceleratrices *SM, SN* aequates essent; hae, trahendo corpora *T* & *P* aequaliter & secundum lineas parallelas, nil mutarent situm eorum ad invicem. Iidem jam forent corporum illorum motus inter se (per legum CoroI. VI.) ac si hae attractiones tollerentur. Et pari ratione si attractio *SN* minor esset attractione *SM*, tolleret ipsa attractonis *SM* partem *SN*, & maneret pars sola *MN*, qua temporum & arearum proportionalitas & orbite forma illa elliptica perturbaretur. Et similiter si attractio *SN* major esset attractone *SM*, orietur ex differentia sola *MN* perturbatio proportionalitatis & orbitae. Sic per attractionem *SN* reducitur semper attractio tertia superior *SM* ad attractionem *MN*, attractione prima & secunda manentibus prorsus immutatis: & propterea areae ac tempora ad proportionalitatem, & orbita *PAB* ad formam praefatam ellipticam tum maxime accedunt, ubi attractio *MN* vel nulla est, vel quam fieri possit minima; hoc est, ubi corporum *P* & *T* attractiones acceleratrices, factae versus corpus *S*, accedunt quantum fieri potest ad aequalitatem; id est, ubi attractio *SN* non est nulla, neque minor minima attractionum omnium *SM*, sed inter attractionum omnium *SM* maximam & minimam quasi mediocris, hoc est, non multo major neque multo minor attractione *SK. Q.E.D.*

Cas. 2. Revolvantur jam corpora minora *P, S* circa maximum *T* in planis diversis; & vis *LM*, agendo secundum lineam *PT* in plano orbitae *PAB* sitam, eundem habebit effectum ac prius, neque corpus *P* de plano orbitae suae deturbabit. At vis altera *NM*, agendo secundum lineam que ipsi *ST* parallela est (atque ideo quando corpus *S* versatur extra lineam nodorum, inclinatur ad planum orbitae *PAB)* praeter perturbationem motus in longitudinem jam ante expositam, inducet perturbationem motus in latitudinem, trahendo corpus *P* de plano suae orbitae. Et hae perturbatio, in dato quovis corporum *P* & T ad invicem situ, erit ut vis illa generans *MN*, ideoque minima evadet ubi *MN* est minima, hoc est (uti jam exposui) ubi attractio *SN* non est multo major, neque multo minor attractione *SK. Q.E.D.*

Corol. 1. Ex his facile colligitur, quod, si corpora plura minora *P, S, R,* &c. revolvantur circa maximum *T,* motus corporis intimi *P* minime perturbabitur attractionibus exteriorum, ubi corpus maximum *T* pariter a caeteris, pro ratione virium acceleratricum, attrahitur & agitatur, atque caetera a se mutua.

*Corol.*2. In systemate vero trium corporum *T, P, S,* si attractiones acceleratrices binorum quorumcunque in tertium sint ad invicem reciproce ut quadrata distantiarum; corpus *P*, radio *PT*; aream circa corpus *T* velocius describet prope conjunctionem *A* & oppositionem *B,* quam prope quadraturas C, *D.* Namque vis omnis qua corpus *P* urgetur & corpus *T* non urgetur, quaeque non agit secundum lineam *PT* accelerat vel retardat descriptionem areae, perinde ut ipsa in consequentia vel in antecedentia dirigitur. Talis est vis *NM.* Haec in transitu corporis *P* a *C* ad *A* tendit in consequentia, motumque accelerat; dein usque ad D in antecedentia, & motum retardat; tum in consequentia usque ad *B,* & ultimo in antecedentia transeundo a *B* ad *C.*

*Corol.*3. Et eodem argumento patet quod corpus *P,* caeteris paribus, velocius movetur in conjunctione & oppositione quam in quadraturis.

Corol. 4. Orbita corporis *P,* caeteris paribus, curvior est in quadraturis quam in conjunctione & oppositione. Nam corpora velociora minus deflectunt a recto tramite. Et praeterea vis *KL,* vel *NM,* in conjunctione & oppositione contraria est vi, qua corpus *T* trahit corpus *P;* ideoque vim illam minuit; corpus autem *P* minus deflectet a recto tramite, ubi minus urgetur in corpus *T.*

Corol. 5. Unde corpus *P,* caeteris paribus, longius recedet a corpore *T* in quadraturis, quam in conjunctione & oppositione. Haec ita se habent excluso motu excentricitatis. Nam si orbita corporis *P* excentrica sit, excentricitas ejus (ut mox in hujus Corol. 9. ostendetur) evadet maxima ubi apsides sunt in syzygiis; indeque fieri potest ut corpus *P,* ad apsidem summam appellans, absit longius a corpore T in syzygiis quam in quadraturis.

Corol. 6. Quoniam vis centripeta corporis centralis *T;* qua corpus *P* retinetur in orbe suo, augetur in quadraturis per additionem vis *LM,* ae diminuitur in syzygiis per ablationem vis *KL,* & ob magnitudinem vis *KL,* magis diminuitur quam augetur; est autem vis illa centripeta (per Corol. 1. Prop. IV.) in ratione composita ex ratione simplici radii *TP* directe & ratione duplicata temporis periodici inverse: patet hanc rationem compositam diminui per actionem vis *KL;* ideoque tempus periodicum, si maneat orbis radius *TP,* augeri, idque in subduplicata ratione, qua vis illa centripeta diminuitur: auctoque ideo vel diminuto hoc radio, tempus periodicum augeri magis, vel diminui minus quam in radii hujus ratione sesquiplicata (per Corol. VI. Prop. IV.) Si vis illa corporis centralis paulatim languesceret, corpus *P* minus semper & minus attractum perpetuo recederet longius a centro *T;* & contra, si vis illa augeretur, accederet propius. Ergo si actio corporis longinqui *S,* qua vis illa diminuitur, augeatur ac diminuatur per vices: augebitur simul ac diminuetur radius *TP* per vices; & tempus periodicum augebitur ae diminuetur in ratione composita ex ratione sesquiplicata radii, & ratione subduplicata, qua vis illa centripeta corporis centralis *T;* per incrementum vel deerementum actionis corporis longinqui *S,* diminuitur vel augetur.

Corol. 7. Ex praemissis consequitur etiam, quod ellipseos a corpore *P* descriptae axis, seu apsidum linea, quoad motum angularem, progreditur & regreditur per vices, sed magis tamen progreditur, & per excessum progressionis fertur in consequentia. Nam vis qua corpus *P* urgetur in corpus *T* in quadraturis, ubi vis *MN* evanuit, componitur ex vi *LM* & vi centripeta, qua corpus *T* trahit corpus *P.* Vis prior *LM,* si augeatur distantia *PT,* augetur in eadem fere ratione cum hac distantia, & vis posterior decrescit in duplicata illa ratione, ideoque summa harum virium decrescit in minore quam duplicata ratione distantiae *PT,* & propterea (per Corol. I. Prop. XLV.) efficit ut aux, seu apsis summa, regrediatur. In conjunctione vero & oppositione vis, qua corpus *P* urgetur in corpus *T,* differentia est inter vim, qua corpus *T* trahit corpus *P,* & vim *KL;* & differentia illa, propterea quod vis *KL* augetur quamproxime in ratione distantiae *PT,* decrescit in majore quam duplicata ratione distantiae *PT,* ideoque (per Corol. I. Prop. XLV.) efficit ut aux progrediatur. In locis inter syzygias & quadraturas pendet motus augis ex causa utraque conjunctim, adeo ut pro hujus vel alterius excessu progrediatur ipsa vel regrediatur. Unde cum vis *KL* in syzygiis sit quasi duplo major quam vis *LM* in quadraturis, excessus erit penes vim

KL, transferetque augem in consequentia. Veritas autem hujus & praecedentis corollarii facilius intelligetur concipiendo systema corporum duorum *T, P* corporibus pluribus *S, S, S*, &c. in orbe *ESE* consistentibus, undique cingi. Namque horum actionibus actio ipsius *T* minuetur undique, decrescetque in ratione plusquam duplicata distantiae.

Corol. 8. Cum autem pendeat apsidum progressus vel regressus decremento vis centripetae facto in majori vel minori quam duplicata ratione distantiae *TP*, in transitu corporis ab apside ima ad apsidem summam ; ut & a simili incremento in reditu ad apsidem imam; atque ideo maximus sit ubi propostio vis in apside summa ad vim in apside ima maxime recedit a duplicata ratione distantiarum inversa: manifestum est quod apsides in syzygiis suis, per vim ablatitiam *KL* seu $NM - LM$, progredientur velocius, inque quadraturis suis tardius recedent per vim addititiam *LM*. Ob diuturnitatem vero temporis, quo velocitas progressus vel tarditas regressus continuatur, fit haec inaequalitas longe maxima.

Corol. 9. Si corpus aliquod, vi reciproce proportionali quadrato distantiae suae a centro, revolveretur circa hoc centrum in ellipsi; & mox, in descensu ab apside summa fseu auge ad apsidem imam, vis illa per accessum perpetuum vis novae augeretur in ratione plusquam duplicata distantiae diminutae: manifestum est quod corpus perpetuo accessu vis illius novae impulsum semper in centrum, magis vergeret in hoc centrum, quam si urgeretur vi sola crescente in duplicata ratione distantiae diminutae; ideoque orbem describeret orbe elliptico interiorem, & in apside ima propius accederet ad centrum quam prius. Orbis igitur, accessu hujus vis novae, fiet magis excentricus. Si jam vis, in recessu corporis ab apside ima ad apsidem summam, decresceret iisdem gradibus quibus ante creverat, rediret corpus ad distantiam priorem, ideoque si vis decrescat in majori ratione, corpus jam minus attractum ascendet ad distantiam majorem & sic orbis excentricitas adhuc magis augebitur. Quare si ratio incrementi & decrementi vis centripetae singulis revolutionibus augeatur, augebitur semper excentricitas; & contra, diminuetur eadem, si ratio illa decrescat.

Jam vero in systemate corporum *T, P, S*, ubi apsides orbis *PAB* sunt in quadraturis, ratio illa incrementi ac decrementi minima est, & maxima sit ubi apsides sunt in syzygiis. Si apsides constituantur in quadraturis, ratio prope apsides minor est & prope syzygias major quam duplicata distantiarum, & ex ratione illa majori oritur augis motus directus, uti jam dictum est. At si consideretur ratio incrementi vel decrementi totius in progressu inter apsides, haec minor est quam duplicata distantiarum. Vis in apside ima est ad vim in apside summa in minore quam duplicata ratione distantiae apsidis summae: ab umbilico ellipteos ad distantiam apsidis imae ab eodem umbilico: & contra, ubi apsides constituuntur in syzygiis, vis in apside ima est ad vim in apside summa in majore quam duplicata ratione distantiarum. Nam vires *LM* in quadraturis additae viribus corporis *T* componunt vires in ratione minore, & vires *KL* in syzygiis subductae a viribus corporis *T* relinquunt vires in ratione majore. Est igitur ratio decrementi & incrementi totius in transitu inter apsides, minima in quadraturis, maxima in syzygiis; & propterea in transitu apsidum a quadraturis ad syzygias perpetuo augetur, augetque excentricitatem ellipseos; inque transitu a syzygiis ad quadraturas perpetuo diminuitur, & excentricitateae diminuit.

*Corol.*10. Ut rationem ineamus errorum in latitudinem, fingamus planum orbis *EST* immobile manere; & ex errorum exposita causa manifestum est, quod ex viribus *NM*, *ML*, quae sunt causa illa tota, vis *ML* agendo semper secundum planum orbis *PAB*, nunquam perturbat motus in latitudinem; quodque vis *NM*, ubi nodi sunt in syzygiis, agendo etiam secundum idem orbis planum, non perturbat has motus ; ubi vero sunt in quadraturis, eos maxime perturbat, corpusque *P* de plano orbis sui perpetuo trahendo, minuit inclinationem plani in transitu corporis a quadraturis ad syzygias, augetque vicissim eandem in transitu a syzygiis ad quadraturas. Unde sit ut corpore in syzygiis existente inclinatio evadat omnium minima, redeatque ad priorem magnitudinem circiter, ubi corpus ad nodum proximum accedit. At si nodi constituantur in octantibus post quadraturas, id est, inter *C* & *A*, *V* & *H*, intelligetur ex modo expositis, quod, in transitu corporis *P* a nodo alterutro ad gradum inde nonagesimum, inclinatio plani perpetuo minuitur; deinde in transitu per proximos 45 gradus, usque ad quadraturam proximam, inclinatio augetur, & postea denuo in transitu per alios 45 gradus, usque ad nodum proximum, diminuitur. Magis itaque diminuitur inclinatio quam augetur, & propterea minor est semper in nodo subsequente quam in praecedente. Et simili ratiocinio, inclinatio magis augetur, quam diminiuitur, ubi nodi sunt in octantibus alteris inter *A* & *D*, *B* & *C*. Inclinatio igitur ubi nodi sunt in syzygiis est omnium maxima. In transitu eorum a syzygiis ad quadraturas, in singulis corporis ad nodos appulsibus, diminuitur; sitque omnium minima, ubi nodi sunt in quadraturis, & corpus in syzygiis: dein cresdt iisdem gradibus, quibus antea decreverat; nodisque ad syzygias proximas appulsis, ad magnitudinem primam revertitur.

Corol. 11. Quoniam corpus *P*, ubi nodi sunt in quadraturis, perpetuo trahitur de plano orbis sui, idque in partem versus *S* in transitu suo a nodo *C* per conjunctionem *A* ad nodum est ; & in contrariam partem in transitu a nodo *D* per oppositionem *B* ad nodum c: manifestum est, quod in motu suo a nodo *C* corpus perpetuo recedit ab orbis sui plano primo *CD*, usque dum perventum est ad nodum proximum; ideoque in hoc nodo, longissime distans a plano illo primo *CD*, transit per planum orbis *EST* non in plani illius nodo altero *D*, sed in puncto quod inde vergit ad partes corporis *S*, quodque proinde novus est nodi locus in anteriora vergens. Et simili argumento pergent nodi recedere in transitu corporis de hoc nodo in nodum proximum. Nodi igitur in quadraturis constituti perpetuo recedunt; in syzygiis, ubi motus in latitudinem nil perturbatur, quiescunt; in locis intermediis, conditionis utriusque participes, recedunt tardius: ideoque, semper vel retrogradi, vel stationarii singulis revolutionibus feruntur in antecedentia.

*Corol.*12. Omnes illi in his corollariis descripti errores sunt paulo majores in conjunctione corporum *P*, *S*, quam in eorum oppositione; idque ob majores vires generantes *NM* & *ML*.

Corol. 13. Cumque rationes horum corollariorum non pendeant a magnitudine corporis *S*, obtinent praecedentia omnia, ubi corporis *S* ut tanta statuitur magnitudo, ut circa ipsum revolvatur corporum duorum *T* & *P* systema. Et ex aucto corpore *S*, auctaque ideo ipsius vi centripeta, a qua errores corporis *P* oriuntur, evadent errores illi omnes, paribus distantiis, majores in hoc casu quam in altero, ubi corpus *S* circum systema corporum *P* & *T* revolvitur.

*Corol.*14. Cum autem vires *NM, ML,* ubi corpus *S* longinquum est, sint quamproxime ut vis *SK* & ratio *PT* ad *ST* conjunctim, hoc est, si detur tum distantia *PT*, tum corporis *S* vis .absoluta, ut *STcub.* reciproce ; sint autem vires illae *NM, ML* causae errorum & effectuum omnium, de quibus actum est in praecedentibus corollariis: manifestum est, quod effectus illi omnes, stante corporum *T* & *P* systemate, & mutatis tantum distantia *S T* & vi absoluta corporis *S,* sint quamproxime in ratione composita ex ratione directa vis absolutae corporis *S,* & ratione triplicata inversa distantiae *ST.* Unde si systema corporum *T* & *P* revolvatur circa corpus longinquum *S;* vires ilIae *NM, ML,* & earum effectus erunt (per Corol. 2. & 6. Prop. IV.) reciproce in duplicata ratione temporis periodici. Et inde etiam, si magnitudo corporis *S* proportionalis sit ipsius vi absolutae, erunt vires illae *NM, ML,* & earum effectus directe ut cubus diametri apparentis longinqui corporis *S* e corpore *T* spectati, & vice versa. Namque hae rationes eaedem sunt, atque ratio superior composita.

Corol. 15. Et quoniam si, manentibus orbium *ESE* & *PAB* forma, proportionibus & inclinatione ad invicem, mutetur eorum magnitudo, & si corporum *S* & *T* vel maneant, vel mutentur vires in data quavis ratione; hae vires (hoc est, vis corporis *T,* qua corpus *P* de recto tramite in orbitam *PAB* deflectere, & vis corporis *S,* qua corpus idem *P* de orbita illa deviare cogitur) agunt semper eodem modo, & eadem proportione: necesse est ut similes & proportionales sint effectus omnes, & proportionalia effectuum tempora; hoc est, ut errores omnes lineares sint ut orbium diametri, angulares vero iidem, qui prius, & errorum linearium similium, vel angularium aequalium tempora ut orbium tempora periodica.

*Corol.*16. Unde, si dentur orbium formae & inclinatio ad iuvicem, & mutentur utcunque corporum magnitudines, vires & distantiae; ex datis erroribus & errorum temporibus in uno casu, colligi possunt errores & errorum tempora in alio quovis, quam proxime: sed brevius hac methodo. Vires *NM, ML,* caeteris stantibus, sunt ut radius *TP,* & harum effectus periodici (per Corol. 2. lem. X.) ut vires, & quadratum temporis periodici corporis *P* conjunctim. Hi sunt errores lineares corporis *P;* & hinc errores angulares e centro *T* spectati (id est, tam motus augis & nodorum, quam omnes in longitudinem & latitudinem errores apparentes) sunt, in qualibet revolutione corporis *P,* ut quadratum temporis revolutionis quam proxime. Conjungantur hae rationes cum rationibus Corollarii XlV & in quolibet corporum *T, P, S* systemate. ubi *P* circum *T* sibi propinquum, & *T* circum *S* longinquum revolvitur, errores angulares corporis *P,* de centro *T* apparentes, erunt, in singolis revolulionibus corporis illius *P,* ut quadratum temporis periodici corporis *P* directe & quadratum temporis periodici corporis *T* inverse. Et inde motus medius augis erit in data ratione ad motum medium nodorum; & motus uterque erit ut tempus periodicum corporis *P* directe & quadratum temporis periodici corporis *T* inverse. Augendo vel minuendo excentricitatem & inclinationem orbis *PAB* non mutantur motus augis & nodorum sensibiliter, nisi ubi eaedem sunt nimis magnae.

Corol. 17. Cum autem linea *LM* nunc major sit nunc minor quam radius *PT,* exponatur vis mediocris *LM* per radium illum *PT;* & erit haec ad vim mediocrem *SK* vel *SN* (quam exponere licet per *ST*) ut longitudo *PT* ad longitudinem *ST*. Est autem vis mediocris *SN* vel *ST,* qua corpus *T* retinetur in orbe suo circum *S,* ad vim, qua corpus *P* retinetur in orbe suo circum *T*, in ratione composita ex ratione radii *ST* ad radium *PT,* & ratione duplicata temporis periodici corporis *P* circum *T* ad tempus periodicum corporis *T* circum *S.* Et ex aequo, vis mediocris *LM* ad vim, qua corpus *P* retinetur in orbe suo circum *T* (quave corpus idem *P,* eodem tempore periodico, circum punctum quodvis immobile *T* ad distantiam *PT* revolvi posset) est in ratione illa duplicata periodicorum temporum. Datis igitur temporibus periodicis una cum distantia *PT*, datur vis mediocris *L M;* & ea data, datur etiam vis *M N* quamproxime per analogiam linearum *PT, MN.*

Corol. 18. Iisdem legibus, quibus corpus *P* circum corpus *T* revolvitur, fingamus corpora plura fluida circum idem *T* ad aequales ab ipso distantias moveri; deinde ex his contiguis factis conflari annulum fluidum, rotundum ac corpori *T* concentricum; & singulae annuli partes, motus suos omnes ad legem corporis *P* peragendo, propius accedent ad corpus *T,* & celerius movebuntur in conjunctione & oppositione ipsarum & corporis *S,* quam in quadraturis. Et nodi annuli hujus, seu intersectiones ejus cum plano orbitae corporis *S* vel *T,* quiescent in syzygiis; extra syzygias vero movebuntur in antecedentia, & velocissime quidem in quadraturis, tardius aliis in locis. Annuli quoque inclinatio variabitur, & axis ejus singulis revolutionibus oscillabitur, completaque revolutione ad pristinum situm redibit, nisi quatenus per praecessiohem nodorum circumfertur.

Corol. 19. Fingas jam globum corporis *T;* ex materia non fluida constantem, ampliari & extendi usque ad hunc annulum, & alveo per circuitum excavato continere aquam, motuque eodem periodico circa axem suum uniformiter revolvi. Hic liquor per vices aeceleratus & retardatus (ut in superiore corollario) in syzygiis velocior erit, in quadraturis tardior quam superficies globi, & sic fluet in alveo refluetque ad modum maris. Aqua, revolvendo circa globi centrum quiescens, si tollatur attractio corporis *S,* nullum acquiret motum fluxus & refluxus. Par est ratio globi uniformiter progradientis in directum, & interea revolventis circa centrum suum (per legum Corol. V.) ut & globi de cursu rectilineo uniformiter tracti (per legum Corol. 6.) Accedat autem corpus *S,* & ab ipsius inaequabili attractione mox turbabitur aqua. Etenim major erit attracti aquae propioris, minor ea remotioris. Vis autem *LM* trahet aquam deorsum in quadraturis, facietque ipsam descendere usque ad syzygias ; & vis *KL* trahet eandem sursum in syzygiis, sistetque descensum ejus, & faciet ipsam ascendere usque ad quadraturas: nisi quatenu motus fluendi & refluendi ab alveo aquae dirigatur, & per frictionem aliquatenus retardetur.

Corol. 20. Si annulus jam rigeat, & minuatur globus, cessabit motus fluendi & refluendi; sed oscillatorius ille inclinationis motus & praecessio nodorum manebunt. Habeat globus eundem axem cum annulo, gyrosque compleat iisdem temporibus, & superficie sua contingat ipsum interius, eique inhaereat; & participando motum ejus compages utriusque oscillabitur, & nodi regredientur. Nam globus ut mox dicetur, ad suscipiendas impressiones omnes indifferens est. Annuli globo orbati maximus inclinationis angulus est, ubi nodi sunt in syzygiis. Inde in progressu nodorum ad quadraturas conatur is inclinationem suam minuere, & isto conatu motum imprimit globo toti. Retinet globus motum impressum, usque dum annulus conatu contrario motum hunc tollat, imprimatque motum novum in contrariam partem: Atque hac ratione maximus decrescentis inclinationis motus sit in quadraturis nodorum, & minimus inclinationis angulus in octantibus post quadraturas; dein maximus reclinationis motus in syzygiis, & maximus angulus in octantibus proximis. Et eadem est ratio globi annulo nudati, qui in regionibus aequatoris vel altior est paulo quam juxta polos, vel constat ex materia paulo densiore. Supplet enim vicem annuli iste materiae in aequatoris regionibus exessus. Et quanquam, aucta utcunque globi hujus vi centripeta, tendere supponantur omnes ejus partes deorsum, ad modum gravitantium partium telluris, tamen phaenomena hujus & praecedentis corollarii vix inde mutabuntur; nisi quod loca maximarum & minimarum altitudinum aquae diversa erunt. Aqua enim jam in orbe suo sustinetur & permanet, non per vim suam centrifugam, sed per alveum in quo fluit. Et praeterea vis LM trahit aquam deorsum maxime in quadraturis, & vis KL seu $NM-LM$ trahit eandem sursum maxime in syzygiis. Et hae vires conjunctae desinunt trahere aquam deorsum & incipiunt trahere aquam sursum in octantibus ante syzygias, ae desinunt trahere aquam sursum incipiuntque trahere aquam deorsum in octantibus post syzygias. Et inde maxima aquae altitudo evenire potest in octantibus post syzygias, & minima in octantibus post quadraturas circiter; nisi quatenus motus ascendendi vel descendendi ab his viribus impressus vel per vim insitam aquae paulo diutius perseveret, vel per impedimenta alvei paulo citius sistatur.

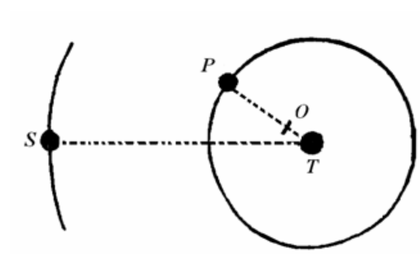
Corol. 21. Eadem ratione, qua materia globi juxta aequatorem redundans efficit ut nodi regrediantur, atque ideo per hujus incrementum augetur iste regressus, per diminutionem vero diminuitur, & per ablationem tollitur; si materia plusquam redundans tollatur, hoc est, si globus juxta aequatorem vel depressior reddatur, vel rarior quam juxta polos, orietur motus nodorum in consequentia.

Corol. 22. Et inde vicissim, ex motu nodorum innotescit constitutio globi. Nimirum si globus polos eosdem constanter servat, & motus sit in antecedentia, materia juxta aequatorem redundat ; si in consequentia, deficit. Pone globum uniformem & perfecte circinatum in spatiis liberis primo quiescere; dein impetu quocunque oblique in superficiem suam facto propelli, & motam inde concipere partim circularem, partim in directum. Quoniam globus iste ad axes omnes per centrum suum transeuntes indifferenter se habet, neque propensior est in unum axem, unumve axis situm, quam in alium quemvis; perspicuum est, quod is axem suum, axisque inclinationem vi propria nunquam mutabit. Impellatur jam globus oblique, in eadem illa superficiei parte, qua prius, impulsu quocunque novo; & cum citior vel serior impulsus effectum nil mutet, manifestum est, quod hi duo impulsus successive impressi eundem producent motum, ac si simul impressi fuiisent, hoc est, eundem, ac si globus vi simplici ex utroque (per legum Corol. 2.) composita impulsus fuisset, atque ideo simplicem, circa axem inclinatione datum. Et par est ratio impulsus secundi facti in locum alium quemvis in aequatore motus primi ; ut & impulsus primi facti in locum quemvis in aequatore motus, quem impulsus secundus sine primo generaret ; atque ideo impulsuum amborum factorum in loca quaecunque: generabunt hi eundem motum circularem ac si simul & semel in locum intersectionis aequatorum motuum illorum, quos seorsim generarent, fuissent impressi. Globus igitur homogeneus & perfectus non retinet motus plures distinctos, sed impressos omnes componit & ad unum reducit, & quatenus in se est, gyratur semper motu simplici & uniformi circa axem unicum, inclinatione semper invariabili datum. Sed nec vis centripeta inclinationem axis, aut rotationis velocitatem mutare potest. Si globus plano quocunque, per centrum suum & centrum in quod vis dirigitur transeunte, dividi intelligatur in duo hemisphaeria; urgebit semper vis illa utrumque hemisphaerium aequaliter, & propterea globum, quoad motum rotationis, nullam in partem inclinabit. Addatur vero alicubi inter polum & aequatorem materia nova in formam montis cumulata, & haec, perpetuo conaru recedendi a centro sui motus, turbabit motum globi, facietque ut poli ejus errent per ipsius superficiem, & circulos circum se punctumque sibi oppositum perpetuo describant. Neque corrigetur ista vagationis enormitas, nisi locando montem illum vel in polo alterutro, quo in casu (per Corol. XXI.) nodi aequatoris progredientur; vel in aequatore, qua ratione (per Corol. XX.) nodi regredientur; vel denique ex altera axis parte addendo materiam novam, qua mons inter movendum libretur, & hoc pacto nodi vel progredientur, vel recedenr, perinde ut mons & haecce nova materia sunt vel polo vel aequatori propiores.

PROPOSITIO LXVII. THEOREMA XXVII.

Positis iisdem attractionum legibus, dico quod corpus exterius S, circa interiorum P, T commune gravitatis centrum O, radiis ad centrum illud ductis, destribit areas temporibus magis proportionales & orbem ad formam ellipseos umbilicum in centro eodem hobentis magis accedentem, quam circa corpus intimum & maximum T, radiis ad ipsam ductis, describere potest.

Nam corporis *S* attractiones versus *T* & *P* componunt ipsius attractionem absolutam, quae magis dirigitur in corporum *T* & *P* commune gravitatis centrum *O*, quam in corpus maximum *T* , quaeque quadrato distantiae *SO* magis est proportionalis reciproce, quam quadrato distantiae *ST:* ut rem perpendenti facile constabit.



PROPOSITIO LXVIII. THEOREMA XXVIII.

Positis iisdem attractionum legibus, dico quod corpus exterius S, circa interiorum P & T commune gravitatis centrum O, radiis ad centrum illud ductis, describit areas temporibus *magis proportionales, & orbem ad formam ellipseos umbilicum in centro eodem habentis magis accedentem, si corpus intimum & maximum his attractionibus perinde atque caetera agitetur, quam si id vel non attractum quiescat, vel multo magis aut multo minus attractum aut multo magis aut multo minus agitetur.*

Demonstratur eodem fere modo cum Prop. LXVI. sed argumento prolixiore, quod ideo praetereo. Sufficeret rem sic aestimare. Ex demonstratione propositionis novissimae liquet centrum, in quod corpus *S* conjunctis viribus urgetur, proximum esse communi centro gravitatis duorum illorum. Si coincideret hoc centrum cum centro illo communi, & quiesceret commune centrum gravitatis corporum trium; describerent corpus ct ex una parte, & commune centrum aliorum duorum ex altera parte, circa commune omnium centrum quiescens, ellipses accuratas. Liquet hoc per corollarium secundum propositionis LVIII. collatum cum demonstratis in Prop. LXIV. & LXV. Perturbatur iste motus ellipticus aliquantulum per distantiam centri duorum a centro, in quod tertium *S* attrahitur, Detur praeterea motus communi trium centro, & augebitur perturbatio, Proinde minima est perturbatio, ubi commune trium centrum quiescit; hoc est, ubi corpus intimum & maximum *T* lege caeterorum attrahitur: sitque major semper, ubi trium commune illud centrum, minuendae motum corporis *T* ; moveri incipit, & magis deinceps magisque agitatur.

Corol. Et hinc, si corpora plura minora revolvantur circa maximum, colligere licet quod orbitae descriptae propius accedent ad ellipticas, & arearum descriptiones fient magis aequabiles, si corpora omnia viribus acceleratricibus, quae sunt ut eorum vires absolute directe & quadrata distantiarum inverse, se mutuo trahant agitentque, & orbitae cujusque umbilicus collocetur in communi centro gravitatis corporum omnium interiorum (nimirum umbilicus orbitae primae & intimae in centro grivitatis corporis maximi & intimi; ille orbitae secundae, in communi centro gravitatis corporum duorum intimorum; iste tertiae, in communi centro gravitatis tritiae interiorum; & sic deinceps) quam si corpus intimum quiescat & statuatur communis umbilicus orbitarum omnium.

PROPOSITIO LXIX. THEOREMA XXIX.

In systemate corporum plurium A, B, C, D, &. si corpus aliquod A trabit caetera omnia B, C, D, &c. viribus acceleratribus quae sunt reciproce ut quadrata distantiarum a trahente; & *corpus aliud B trahit etiam caetera A, C, D, &c. viribus quae funt reciproce ut quadrata distantiarum a trahente: erunt absolutae corporum trahentium A, B vires ad invicem, ut sunt ipsa corpora A, B, quorum sunt vires.*

Nam attractiones acceleratrices corporum omnium *B, C, D* versus *A*, paribus distantiis, sibi invicem aequantur ex hypothesi; & similiter attractiones acceleratrices corporum omnium versus *B*, paribus distantiis, sibi invicem aequantur. Est autem absoluta vis attractivam corporis *A* ad vim absolutam attractivam corporis *B*, ut attractio acceleratrix corporum omnium versus *A* ad attractionem acceleratricem corporum omnium versus *B*, paribus distantiis; & ita est attractio acceIeratrix corporis *B* versus *A*, ad attractionem acceleratricem corporis *A* versus *B.* Sed attractio acceleratrix corporis *B* versus *A* est ad attractionem acceleratricem corporis *A* versus *B*, ut massa corporis *A* ad massam corporis

B ; propterea quod vires motrices, quae (per definitionem secundam, septimam & octavam) sunt ut vires acceleratrices & corpora attraha conjunctim, hic sunt (per motus legem tertiam) sibi invicem aequales. Ergo absoluta vis attractiva corporis *A* est ad absolutam vim attractivam corporis *B,* ut massa corporis *A* ad massam corporis *B.* *Q. E. D.*

Corol. I. Hinc si singula systematis corpora *A, B, C, D,* &c. seorsim spectata trahant caetera omnia viribus acceleratricibus, quae sunt reciproce ut quadrata distantiarum a trahente; erunt corporum illorum omnium vires absolutm ad invicem ut sunt ipsa corpora.

Corol. 2. Eodem argumento, si singula systematis corpora *A, B,C, D,* &c. seorsim spectata trahant caetera omnia viribus acceleratricibus, quae sunt vel reciproce, vel directe in ratione dignitatis cujuscunque distantiarum a trahente, quaeve secundum legem quamcunque communem ex distantiis ab unoquoque trahente definiuntur; constat quod corporum illorum vires absolutae sunt ut corpora.

Corol.3. In systemate corporum, quorum vires decrescunt in ratione duplicata distantiarum, si minora circa maximum in ellipsibus, umbilicum communem in maximi illius centro habentibus, quam fieri potest accuratissimis revolvantur; & radiis ad maximum illud ductis describant areas temporibus quam maxime proportionales: erunt corporum illorum vires absolutae ad invicem, aut accurate aut quamproxime, in ratione corporum; & contra. Patet per corol. Prop. LXVIII. collatum cum hujus corol.I.

Scholium.

His proportionibus manuducimur ad analogiam inter vires centripetas, & corpora centralia, ad quae vires illae dirigi solent. Rationi enim consentaneum est, ut vires quae ad corpora diriguntur, pendeant ab eorundem natura & quantitate, ut fit in magneticis. Et quoties hujusmodi casus incidunt, aestimandae erunt corporum attractiones, assignando singulis eorum particulis vires proprias, & colligendo summas virium. Vocem attractionis hic generaliter usurpo pro corporum conatu quocunque accedendi ad invicem: sive conatus iste fiat ab actione corporum, vel se mutua petentium, vel per spiritus emissos se invicem agitantium; sive is ab actione aetheris, aut aeris, mediive cujuscunque seu corporei seu incorporei oriatur corpora innatantia in se invicem utcunque impellentis. Eadem sensu generali usurpo vocero impulsus, non species virium & qualitates physicas. sed quantitates & proportiones mathematicis in hoc tractatu expendens; ut in definitionibus explicui. In mathesi investigandae funt virium quantitates & rationes illae, quae ex conditionibus quibuscunque positis consequentur: deinde, ubi in physicam descenditur, conferendae sunt hae rationes cum phaenomenis; ut innotescat quaenam virium conditiones singulis corporum attractivorum generibus competant. Et tum demum de virium speciebus, causis & rationibus physicis tutius disputare licebit. Videamus igitur quibus viribus corpora sphrerica, ex particulis modo jam exposito attractivis constantia, debeanr in se mutua agere; & quales motus inde consequantur.