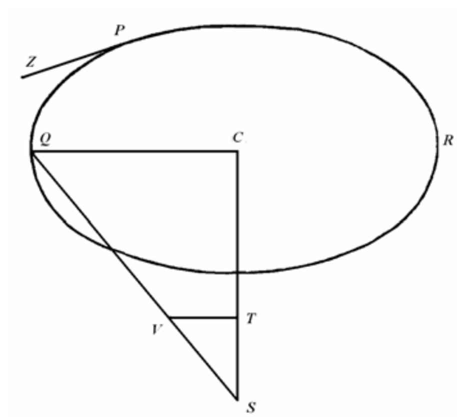
SECTION X.
Concerning the motion of bodies on given surfaces, and from that the repeating motions of string pendulums.

PROPOSITION XLVI. PROBLEM XXXII.

With some general kind of centripetal force in place, and with both a given centre of forces as well as some plane on which the body is revolving, and with the quadratures of curvilinear figures granted : the motion of a body is required starting out along a right line in that given plane from some given place, and with some given velocity,.

S shall be the centre of forces, *SC* the shortest distance of this centre from a given plane, *P* the body setting out from some place *P* along the right line *PZ*, *Q* the same body revolving in its trajectory, and *PQR* that trajectory described in the given plane, that it is required to find. *CQ* and *QS* are joined, and if on *QS*, *SV* may be taken proportional to the centripetal force by which the body is drawn towards the centre *S*, and *VT* may be drawn which shall be parallel to *CQ* and meeting *SC* in *T*. The force *SV* may be resolved (by Corol 2. of the laws) into the forces *ST, TV*; of which *ST* by acting on the body along a line perpendicular to the plane, will not change the motion of that body in this plane. But the other force *TV*, by acting the position of the plane, draws the body directly towards the point *C* in the given plane, and likewise it comes about, so that the body may be moving in this plane in the same manner, and if the force *ST* may be removed, and the body may be revolving about the centre *C* in free space acted on by the force *TV* alone. Moreover with the given centripetal force *TV* by which the body *Q* is revolving in free space about the given centre *C*, then both the trajectory *PQR* is given (by Prop. XLII.) that the body will describe, the place *Q*, in which the body will be rotating at some given time, as well as the velocity of the body at that place *Q*; and conversely. *Q E.1*.



[We should bear in mind here, that if the body travels in an elliptic orbit in a gravitational field, then the length *CS* cannot remain constant, and when the body is at the maximum distance *Q* from *C*, it has risen to its greatest height, being lowest at the minimum distance, assuming the length of the string remains unchanged. Thus Kepler's criterion of the motion of the body always being in the same plane is not satisfied, and hence one cannot use the Kepler criterion of equal areas in equal times ; or, there is an unbalanced torque acting which changes the angular momentum during the course of the orbit. Thus, Newton's Proposition XLVI relates to a zero gravity situation, or, as Newton states, *C* is the only centre of force present. Hence we are *not* looking at a conical pendulum, which only applies to circular motions.]

PROPOSITION XLVII. THEOREM XV.

Because a centripetal force may be put in place proportional to the distance of the body from the centre; all bodies in any planes revolving in some manner describe ellipses, and the ellipses are performed in equal times ; and those moving on right lines, and also running to and fro, may complete the individual coming and going motions in the same periods of time.

For, with which things in place from the above proposition, the force *SV*, by which the body *Q* rotating in some plane *PQR* is drawn towards the centre *S*, is as the distance *SQ*; and thus on account of the proportionals *SV* and *SQ* , *TV* and *C*Q the force *TV,* by which the body is drawn towards the point *C* given in the plane of the orbit, is as the distance *CQ*. Therefore the forces, by which bodies turning in the plane *PQR* are drawn towards the point *C*, on account of the distances, are equal to the forces by which any bodies are drawn in some manner towards the centre *S;* and therefore bodies are moving in the same times, in the same figures, in some plane *PQR* about the point *Q* and in the free space about the centre *S*; and thus (by Corol. 2, Prop. X. and Corol. 2, Prop. XXXVIII.) in equal times always they describe either ellipses in that plane about the centre *C,* or besides they will furnish periods by moving to and fro in right lines through the centre *C* drawn in that plane. *Q.E.D.*

Scholium.

The ascent and descent of bodies on curved surfaces are related to these. Consider curved lines described in a plane, then to be revolved around some axis given passing through the centre of the forces, and from that revolution to describe curved surfaces ; then bodies thus can move so that the centres of these may always be found on these surfaces. If bodies by ascending and descending these obliquely besides can run to and fro, the motions of these will be carried out in planes crossing the axis, and thus on curved lines, by the rotation of which these curves surfaces have arisen. Therefore for these it will suffice to consider the motion in these curved line cases.

[The two following propositions are handled by similar reasoning, on separate diagrams, in what follows. Newton calls all his curves cycloids or epicycloids (the evolute or epicycloid of any cycloid is a similar equal figure with its cusps translated through half the arc of the original curve).
According to Proctor, in his interesting book: *A Treatise on Cycloids*, (1878), which touches on some of the material in this sections, the best way to define such curves is as follows :

The *epicycloid/hypocycloid* is the curve traced out by a point on the circumference of a circle which rolls without sliding on a fixed circle in the same plane, the rolling circle touching the *outside/inside* of the fixed circle. Different values of the two radii give rise to different curves, some of which are well-known. Full descriptions of such curves can be found, e.g. in the *CRC Handbook of Mathematics*, and of course on the web.

We are interested in particular in the geometric method used by Newton in finding a geometric relation for such a curve, where he puts in place a finite figure derived from the geometry available, tangents, diameters, etc., and from this he constructs a similar figure composed of infinitesimal lengths both from linear and curvilinear increments, the vanishing ratio of which, for some chosen lengths, is equal to a fixed ratio in the macroscopic figure. Thus a differentiation has been performed, or fluxion found.

I have added some labels in red to Newton's diagram, to make reading a little easier; however, if you wish, you can look at the unadulterated diagram in the Latin section.]

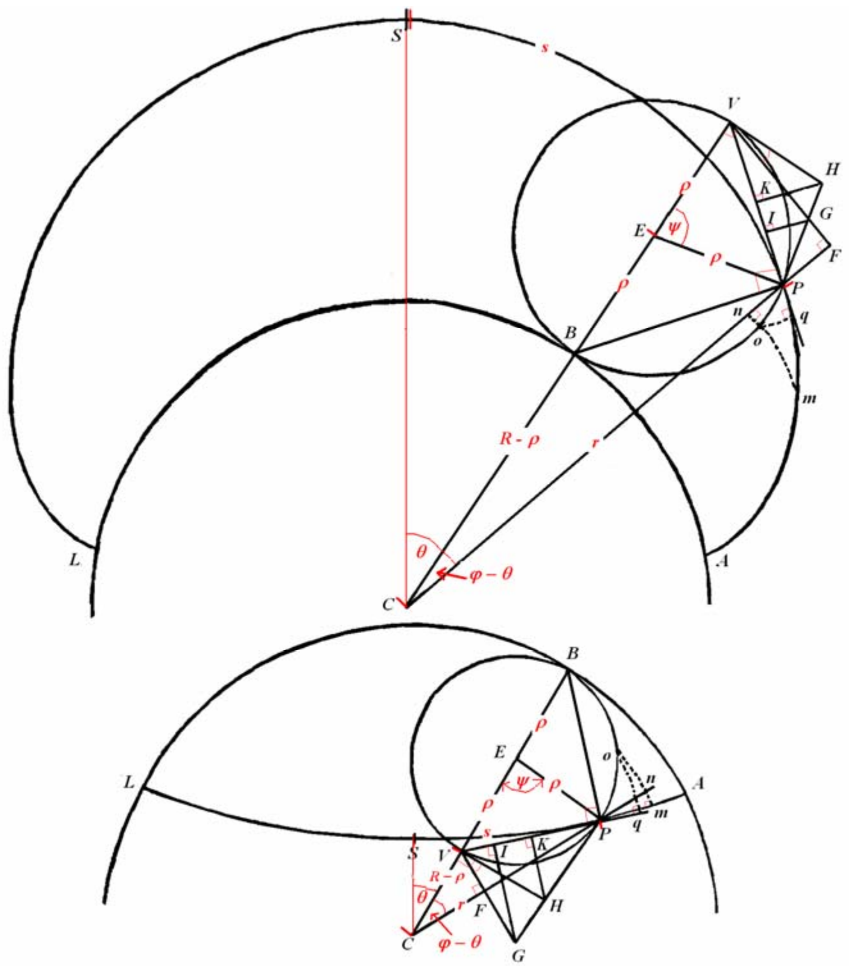
PROPOSITION XLVIII. THEOREM XVI.

If a wheel may stand at right angles on the outside of a sphere, and in the manner of rotation it may progress in a great circle ; the length of the curvilinear path, that some given point on the perimeter of the wheel made, from where it touched the sphere, (and which it is usual to call a cycloid or epicycloid) will be to double the versed sine of half the arc which it made in contact going between in this total time, as the sum of the diameters of the sphere and the wheel, to the radius of the sphere.

PROPOSITION XLIX. THEOREM XVII.

If a wheel may stand at right angles on the concave inside of a sphere and by rotating it may progress on a great circle; the length of the curved path that some given point on the perimeter of the wheel made, from where it touched the sphere during this total time, will be to twice the versed sine of half the arc which it made in contact going between in the whole time, as the difference of the diameters of the sphere and the wheel to the radius of the sphere.

ABL shall be the sphere, *C* the centre of this, *BPV* the wheel resting on this, *E* the centre of the sphere, *B* the point of contact, and *P* the given point on the perimeter of the wheel. Consider that the wheel to go on a great circle *ABL* from *A* through *B* towards *L*, and thus to rotate going between so that the arcs *AB* and *PB* themselves in turn will always be equal, and that point *P* given on the perimeter of the wheel meanwhile describes the curvilinear path *AP*. Moreover *AP* shall have described the whole curvilinear path from where the wheel touched the sphere at *A*, and the length *AP* of this



path to twice the versed sine of the arc $\frac{1}{2}PB$, shall be as 2*CE* to *CB*. For the right line *CE* (produced if there is a need) meets the wheel at *V*, and *CP*, *BP*, *EP* and *VP* may be joined, and the normal *VF* may be sent to *CP* produced. *PH* and *VH* may touch the circle at *P* and *V* meeting at *H*, and *PH* may cut *VF* in *G*, and the normals *GI* and *HK* may be sent to *VP*. Likewise from *C* and with some radius the circle *onm* may be drawn cutting the right line *CP* in *n*, the perimeter of the wheel *BP* in *o*, and the curved path *AP* in *m*; and with centre *V* and with the radius *Vo* a circle may be described cutting *VP* produced at *q*.

Because the wheel, by always moving is rotating about the point of contact *B*, it is evident that the right line *BP* is perpendicular to that curved line *AP* that the point of rotation *P* has described, and thus so that the right line *VP* may touch this curve at the point *P*. The radius of the [arc of the] circle *nom* , gradually increased or diminished is equal finally to the distance *CP;* and, because of the similarity of the vanishing figure *Pnomq* and the figure *PFGVI,* the final ratio of the vanishing line elements *Pm*, *Pn*, *Po*, *Pq*, that is, the ratio of the momentary changes of the curve *AP*, of the right line *CP*, of the circular arc *BP*, and of the right line *VP*, will be the same as of the lines *PV*, *PF*, *PG*, *PI* respectively. But since *VF* shall be perpendicular to *CF* and likewise *VH* to *CV*, and the angles *HVG* and *VCF* therefore equal ; and the angle *VHG* (on account of the right angles of the quadrilateral *HVEP* at *V* and *P*) is equal to the angle *CEP*, the triangles *VHG* and *CEP* are similar*;* and thence it comes about that *EP* to *CE* thus as *HG* to *HV* or *HP* and thus as *KI* to *KP*, and on adding together or separately, as *CB* to *CE* thus *PI* to *PK*, and on doubling in the following as *CB* to 2*CE* thus *PI* to *PV*, and thus *Pq* to *Pm.*

[*i.e.* $\frac{EP}{EC}=\frac{HG}{HV}=\frac{HG}{HP}=\frac{KI}{KP}$ (*HV* and *HP* are the common tangents from *H*) then $\frac{EP[=EB]}{EC}+1=\frac{KI}{KP}+1$, or $\frac{CB}{EC}=\frac{PI}{PK}$; hence $\frac{CB}{2EC}=\frac{PI}{PV}=\frac{Pq}{Pm}$.]

Therefore the decrement of the line *VP*, that is, the increment of the line *BV–VP* to the increment of the curve *AP* is in the given ratio *CB* to 2*CE,* and therefore (by the Corol. of Lem. IV.) the lengths *BV–VP* and *AP,* arising from these increments, are in the same ratio. But, with the interval *BV* present, *PV* is the cosine of the angle *BVP* or $\frac{1}{2}BEP$, and thus *BV–VP* is the versed sine of the same angle ; and therefore in this wheel, the radius of which is $\frac{1}{2}BV$, *BV–VP* will be twice the versed sine of the arc $\frac{1}{2}BP$. Therefore *AP* is to twice the versed sine of the arc $\frac{1}{2}BP$ as 2*CE* to *CB.*

[We may write this proportionality in the form : $\frac{d(-PV)}{d(AP)}=\frac{d(BV-VP)}{d(AP)}=\frac{CB}{2CE}$; giving $BV-VP=AP\times\frac{CB}{2CE}$; hence the arc length of the rectifiable curve $AP=\frac{2CE}{CB}\times(BV-VP)=\frac{2CE}{CB}\times BV\left(1-cos\frac{1}{2}BEP\right)=\frac{2R}{R+\rho}\times 2\rho\left(1-sin\frac{\psi}{2}\right)$, starting from *A*, as required (see Whiteside, note 275 Vol. VI, for the outline of a comparable, but far more complicated, analytical derivation).]

But we will call the line *AP* the cycloid outside the sphere in the first proposition and the cycloid inside the sphere in the following for the sake of distinction.

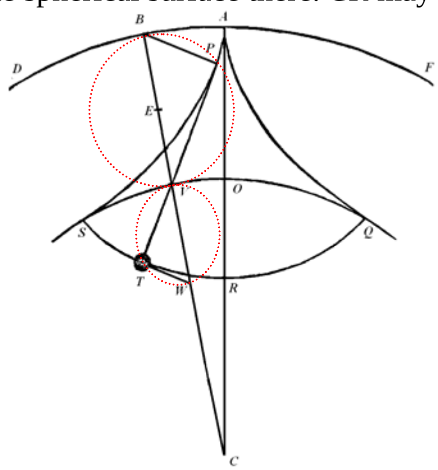
Corol. I. Hence if the whole cycloid *ASL* is described and that may be bisected at *S*, the length of the part *PS* to the length *VP* (which is twice the sine of the angle *VBP,* with the radius *EB* present) is as 2*CE* to *CB,* and thus in the given ratio.

Corol. 2. And the length of the semi-perimeter of the cycloid *AS* will be equal to the right line which is to the diameter of the wheel *BV* as 2*CE* to *CB.*

PROPOSITION L. PROBLEM XXXIII.

To arrange it so that the body of the pendulum may swing in a given cycloid.

Within a sphere *QVS*, described from the *C*, the cycloid *QRS* may be given bisected in *R* and with its end points *Q* and *S* hence meeting the spherical surface there. *CR* may be drawn bisecting the arc *QS* in *O*, and that may be produced to *A*, so that *CA* shall be to *CO* as *CO* to *CR*.



[Thus, if we let $R = CO$, and the radii of the two generating circles be given by $2r = AO \;\&\; 2\rho = OR$ the above terms introduced for the radii, this becomes $\frac{R+2r}{R} = \frac{R}{R-2\rho}$, in turn giving $\frac{2r}{R} = \frac{2\rho}{R-2\rho}$, $\frac{R+2r}{r} = \frac{R}{\rho}$; $\rho = \frac{Rr}{R+2r}$ and $\frac{\rho+r}{r} = \frac{2R+2r}{R+2r}$.]

With centre *C* and radius *CA*, the external sphere *DAF* may be described, and within this sphere by the rotation of a wheel, the diameter of which shall be *AO*, two semi-cycloids *AQ* and *AS* may be described, which touch the interior sphere at *Q* and *S* and meet the external sphere at *A*. From that point *A*, by a thread *AR* equaling the length *APT*, the body *Q* may be suspended and thus may swing between the semi-cycloids *AQ* and *AS*, so that as often as the pendulum is moving away from the perpendicular *AR*, with the upper part *AP* applied to that semi-cycloid *APS* towards which the motion is directed, and around that it may be wrapped as an obstacle, and with the remaining part *PT* to which the semi-cycloid has not yet got in the way, it may stretch out in a straight line; and the weight *T* is swinging on the given cycloid *QRS*. *Q.E.F.*

[The cycloid and the lower evolute cycloid obey the normal/tangent to normal relation at any points *T* and *P* on this pair of curves. We have included the generating circles in red, not present in the original figure.]

For the thread *PT* first may meet the cycloid *QRS* at *T*, and then the circle *QOS* at *V*, and *CV* may be drawn*;* and the perpendiculars *BP* and *TW* may be erected to the right line part of the thread *PT* from the end points *P* and *T*, crossing the right line *CV* in *B* and *W*. It is apparent, from the construction, and from the similar figures *AS* and *SR* arising, that these perpendiculars *PB* and *TW* cut off from *CV* the lengths *VB* and *VW* of the wheels with diameters equal to *OA* and *OR*. Therefore *TP* is to *VP* (twice the sine of the angle *VBP* multiplied by the radius $\frac{1}{2}BV$ present [Both Cohen and Whiteside have misunderstood this point in their translations: you cannot equate a length to the sine of an angle.]) as *BW* to *BV*, or $AO+OR$ to *AO*, that is (since *CA* to *CO*, *CO* to *CR*, and *AO* to *OR* separately shall be proportionals) as $CA+CO$ to *CA*, or, if *BV* may be bisected in *E*, as 2*CE* to *CB*.

[$\frac{TP}{VP} = \frac{BW}{BV}$; or, $\frac{BW}{BV} = \frac{AO+OR}{AO} = \frac{2r+2\rho}{2r} = \frac{CA+CO}{CA} = \frac{2R+2r}{R+2r} = \frac{2CE}{CB}$, as above.]

Hence (by Corol I. Prop. XLIX.) the length of the part of the right line of the thread *PT* is equal always to the arc *PS* of the cycloid, and the whole length [of the thread] *APT* is always equal to the arc *APS* of half the cycloid, that is (by Corol. 2. Prop. XLIX.), to the length *AR*. And therefore in turn if the thread always remains equal to the length *AR*, the point *T* will always be moving on the given cycloid *QRS*. *Q.E.D.*

Corol. The thread *AR* is equal to the semi-cycloid arc *AS*, and thus has the same ratio to the radius of the external sphere *AC* as that similar semi-cycloid *SR* has to the radius of the internal radius *CO*. [$\frac{AR}{AC} = \frac{SR}{CO}$]

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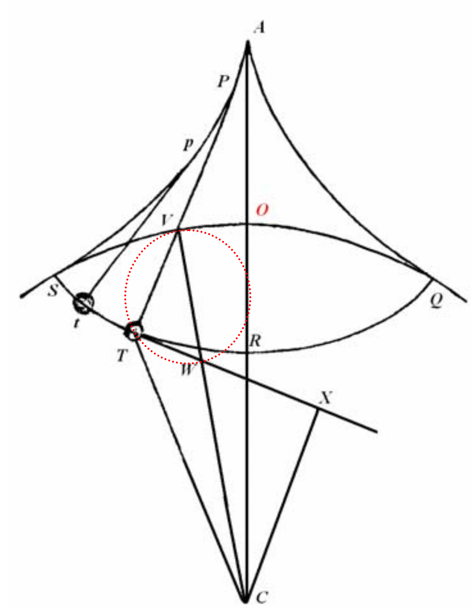
PROPOSITION LI. THEOREM XVIII.

If a centripetal force acting in any direction towards the centre C of the sphere shall be as the distance of this place from the centre, and the body T may be oscillating by this force acting alone (in the manner described just now) on the perimeter of the cycloid QRS: *I say that any whatever of the unequal oscillations are completed in equal intervals of time.*

For the perpendicular *CX* may fall on the tangent *TW* of the cycloid produced indefinitely and *CT* may be joined. Because the centripetal force by which the body *T* is impelled towards *C* is as the distance *CT*, and this (by Corol. 2 of the laws) is resolved into the parts *CX* and *TX*, of which *CX* by impelling the body directly from *P* stretches the thread *PT* and by the resistance of this it may cease to act completely, producing no other effect; but the other part *TX*, by acting on the body transversely or towards *X*, directly accelerates the motion of this on the cycloid; clearly because the acceleration of the body, proportional to this accelerating force, shall be as the length *TX* at individual instants, that is, on account of *CV* and *WV* given and *TX* and *TW* proportional to these [for we have the similar triangles *VTW* and *CXW* and $\frac{TX}{TW} = \frac{CV}{VW}$], as the length *TW*, that is (by Corol. I, Prop. XLIX.) as the length of the arc of the cycloid *TR*. Therefore with the two pendulums *APT* and *Apt* unequally drawn from the perpendicular *AR* and sent off at the same time,



the accelerations of these always will be as the arcs to be described *TR* and *tR*. But the parts described from the start are as the accelerations, that is, as the whole [arcs] to be described at the start, and thereupon the parts which remain to be described and the subsequent accelerations, from these proportional parts, also are as the total [parts to be described subsequently] ; and thus henceforth. Therefore both the accelerations and the velocities arising from the parts and the parts requiring to be described, are as the whole [arc remaining]; and thus the parts requiring to be described obey the given ratio, and to likewise vanish in turn, that is, the two oscillating bodies arrive at the perpendicular *AR* at the same time [i.e. both bodies arrive at *R* with the same speed and in the same time; a hall-mark of in-phase simple harmonic motion, where the period is independent of the amplitude]. Whenever in turn the ascents of the pendulums from the lowest place *R*, through the same cycloidal arcs made in a backwards motion, may be retarded separately by the same forces by which they were accelerated in the descent, it is apparent that the velocities of ascent and descent made through the same arcs are equal and thus for the times to become equal ; and therefore, since both parts of the cycloid *RS* and *RQ* lying on either side of the perpendicular shall be equal and similar, the two pendulums always complete their oscillations at the same times for a whole as well as for a half [oscillation]. *Q.E.D.*

Corol. The force by which the body *T* is accelerated or retarded at some place *T* on the cycloid, is to the whole weight of the same body in the place with the greatest altitude *S* or *Q*, as the arc *TR* of the cycloid to the arc of the same *SR* or *QR.*

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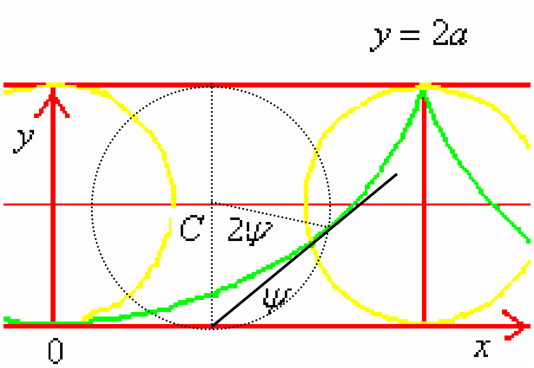
Digression.



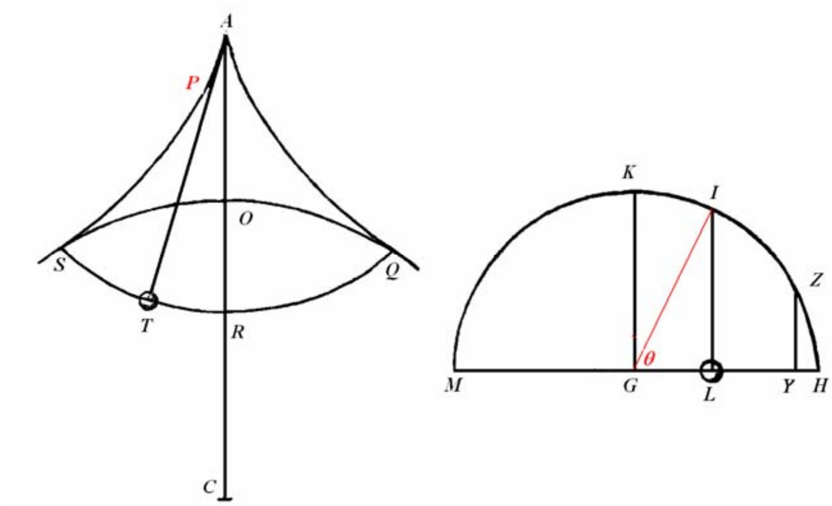
Note from earlier, we have shown that an arc length $AP = \frac{4\rho R}{R+\rho} \times \left(1 - sin\frac{\psi}{2}\right)$, where the angle ψ is subtended by the contact chord at the centre of the generating circle, this is also the angle between the tangent and the chord. We indicate here the common cycloid inverted with the angle ψ now as customarily shown – twice the above, rolling on the upper horizontal line, for which, with a generating circle of radius *a*, the coordinates are $x = a(2\psi + sin 2\psi); y = a(1 - cos 2\psi)$. The gradient at some point on the curve is $tan\psi$, and it is readily shown that the intrinsic equation of this curve is $s = 4a\, sin\psi$, taking $s = 0$ when $\psi = 0$. It is seen that the added complication of rotating the generating circle on or in another circle of greater radius *R* to produce an epicycloid changes the constant 4*a* in the arc length formula to become $s = \frac{4aR}{R \pm a} \times sin\psi$ in our definition of the angle; and the formula depends on where the origin has been chosen. Thus the length of a whole section of a simple cycloid from vertex to trough is 4*a*, with a similar formula for the epicycloid. We may consider such formulas in general to be of the form $s = k\, sin\psi$.

Note that a body *P* on a string acting as a pendulum drawn towards the point *C*, between the cusps of such a cycloid, of total length *k*, may have part of the string of length $s = k(1 - sin\psi)$ wrapped round the curve, while the remainder of length $s = k\, sin\psi$ is free, with corresponding results for Newton's epicycloids. We are interested in the velocities of points along and perpendicular to both curves : it is clear that the components of the velocity and acceleration of the point *P* along the tangent of the Newton's upper cycloid are equal and opposite to the components of the velocity and acceleration of the point *T* along the normal of the lower curve.]

PROPOSITION LII. PROBLEM XXXIV.

To define both the velocities of pendulums at individual places and the times in which both whole oscillations as well as individual parts of oscillations may be completed.

With some centre *G*, and with the radius *GH* equal to the arc of the cycloid *RS*, describe the semicircle *HKM* bisected by the radius *GK*.
[At this point we have to imagine the sphere *HKM*, of which we see a portion, to be endowed with an abstract absolute force field of a special kind, so that a body *L* has the same force acting on it as the body *T*, the abstract force acting along the radius *GH* (so performing pure S.H.M. with the simplest possible geometry), while the other acts along



the tangent at *T* on the cycloid; these centripetal forces are also equal on the periphery *MKH* of the circle and on the sphere *SOQ*. The idea being that both bodies will execute S.H.M. ; in addition, the lengths *GK*, *LI*, and *YZ* represent the velocities of a body released from *H* towards *G*. Such a situation might arise for a particle that could pass through a hypothetical uniform earth without hindrance, such as a mass dropped through a hole passing all the way through a diameter of the earth, affected only by gravity, which in this case varies directly as the distance from centre.]

And if a centripetal force proportional to the distances of the places from the centre, may tend towards the centre *G*, and let that force on the perimeter *HIK* be equal to the centripetal force on the perimeter of the sphere *QOS* tending towards the centre of this; and in the same time in which the pendulum *T* may be sent off from the highest place *S*, some body *L* may fall from *H* to *G*, because the forces by which the bodies may be acted on are equal from the beginning and with the intervals described *TR* and *LG* always proportional, and thus, if *TR* and *LG* may be equal at the places *T* and *L;* it is apparent that the bodies describe the equal intervals *ST* and *HL* from the beginning, and thus from that at once to be urged on to progress equally , and to describe equal intervals. Whereby (by Prop. XXXVIII.) the time in which the body will describe the arc *ST* is to the time of one oscillation, as the arc *HI*, the time in which the body *H* may arrive at *L*, is to the semi-perimeter *HKM*, the time in which the body *H* may arrive at *M*.

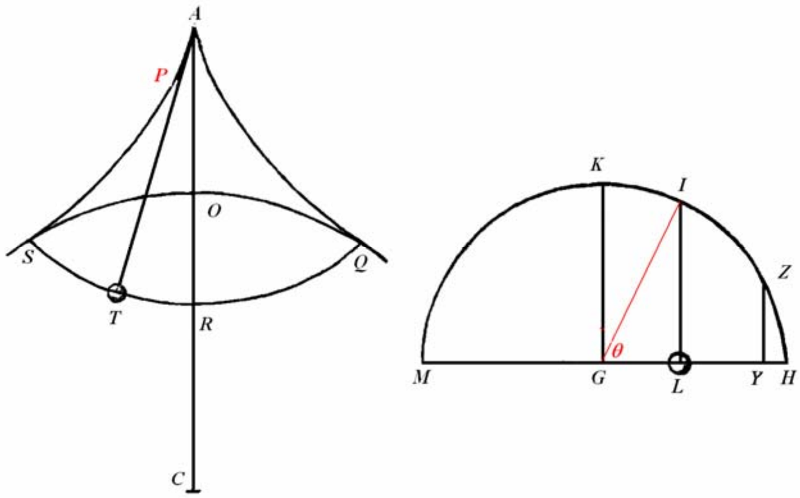
$$[\text{Thus}: \frac{T_{arcST}}{T_{SQ}} = \frac{T_{arcHI}}{T_{HKM}}; \frac{v_T}{v_R} = \frac{v_L}{v_G} = \frac{d(HL)/dt}{d(HG)/dt} = \frac{d(ST)/dt}{d(SR)/dt}.]$$

And the velocity of the body of the pendulum at the place *T* it to the velocity of this at the lowest place *R*, (that is, the velocity of the body *H* at the place *L* to the velocity of this at the place *G*, or the instantaneous increment of the line *HL* to the instantaneous increment of the line *HG*, with the arcs *HI* and *HK* increasing by equal fluxes) as the applied ordinate *LI* to the radius *GK*, or as $\sqrt{SR^2 - TR^2}$ to *SR*.

[We have already considered the accelerations along the tangent *PT* . The free straight length *PT* is equal to the arc *PS* at any instant, and the arc described by the body *P* , or *RT* can be given by $s = k\,sin\psi$, choosing the angle ψ as above, and we may take the velocity along the curve to be $\dot{s} = k\,cos\psi$, while the acceleration along the orbit is $\ddot{s} = -k\,sin\psi = -ks$. This can be written in terms of the tangential velocity *v* as :
$v\frac{dv}{ds} + ks = \frac{1}{2}\frac{d(v^2+ks^2)}{ds} = 0$, and hence $\frac{1}{2}\left(v^2 + ks^2\right) = \text{constant} = \frac{1}{2}kS^2$, since the velocity of *P* is zero when the arc $s = S$; we now regard first integration as the conservation of energy equation. Hence, $v = \frac{ds}{dt} = \sqrt{k\left(S^2 - s^2\right)} = LI.\sqrt{k}$; thus, the velocity at the point *T* has been found. The time to travel from *S* to *s*, is given by the indefinite integral:
$t = \frac{1}{\sqrt{k}}\int\frac{ds}{\sqrt{(S^2-s^2)}} = \frac{1}{\sqrt{k}}\int\frac{du}{\sqrt{(1-u^2)}} = \frac{1}{\sqrt{k}}arccos\left(\frac{s}{S}\right)$, where limits can be applied as needed, and $u = \frac{s}{S}$; (see Whiteside's note 286). Note that in the abstract force diagram, the radius $GH = S$ and we can write $s = S\,cos\sqrt{k}t$, in which case the angle $\theta = \sqrt{k}t$, and we can identify $\sqrt{k}$ as the angular frequency, from which the period of the oscillation is given by $T = \frac{2\pi}{\sqrt{k}}$, where we recall that $k = \frac{4rR}{R+r}$; note especially that the period is independent of the amplitude. Thus we have Newton's results; note also that the part of the string wrapped round the cycloid arc behaves as a store of gravitational potential energy, for as the weight falls, more kinetic energy is fed into the system by the weight being allowed to fall further, and vice versa when it rises, than by the weight moving in a circular arc; and again, the centre of forces is at a finite distance *C*, and so we do not have a uniform gravitational field.]

From which, since in the unequal oscillations, the arcs of the whole oscillations may be described in equal times proportional to the whole arcs of the oscillations; from the given times, both the velocities and the arcs may be had, to be described in all the oscillations. Which were to be found first.

Now bodies hanging from strings may be swinging in diverse cycloids described



between different spheres, of which the absolute forces are different also [*i.e.* those abstract forces giving rise to an S.H.M. above that do not specify the mechanism of the force]: and, if the absolute force of some sphere *QOS* may be called *V*, the accelerating force by which the pendulum is urged on the circumference of this sphere, where it begins to be moving directly towards the centre of this sphere, will be as the distance of that hanging body from that centre and the absolute force jointly on the sphere, that is, as $CO \times V$ [*i.e.* the original absolute force corresponding to *CO* is magnified by some factor *V*; now we have $dv = -k.(ds = HY)dt = -CO \times Vdt$]. And thus the incremental line *HY*; which shall be as this accelerating force $CO \times V$, described in the given time ; and if the normal *YZ* is erected to the circumference crossing at *Z*, the nascent arc *HZ* will denote that given time . But this nascent arc *HZ* is as $\sqrt{GH \times HY}$, [from similar triangles involving increments, $HY{:}ZH :: ZH{:}MH$, or $ZH^2 = 2Sds = 2GH.YH$ as the arc tends to zero] and thus the arc varies as $\sqrt{GH \times CO \times V}$. [As $dt \quad \alpha \quad d\theta \quad \alpha \quad \sqrt{GH \times CO \times V}$]. From which the time of a whole oscillation in the cycloid *QRS* (since it shall be as the semi-periphery *HKM*, which [angle] may denote the time for that whole oscillation directly ; and as the arc *HZ* directly, which similarly may denote the given time inversely) shall be as *GH* directly and as $\sqrt{GH \times CO \times V}$ inversely, that is, on account of the equal quantities *GH* and *SR*, as $\sqrt{\frac{SR}{CO \times V}}$ or (by the Coral. Prop. L.) as $\sqrt{\frac{AR}{AC \times V}}$.

[Thus, $T_{QRS} \quad \alpha \quad \frac{GH}{\sqrt{GH \times CO \times V}} = \sqrt{\frac{GH}{CO \times V}} = \sqrt{\frac{SR}{CO \times V}} = \sqrt{\frac{AR}{AC \times V}}$, since $GH = SR$, and $\frac{AR}{AC} = \frac{SR}{CO}$]

And thus the oscillations on the spheres and with all the cycloids, made with whatever absolute forces, are in a ratio composed directly from the square root ratio of the lengths of the string, and inversely in the square root ratio of the distances between the point of suspension and centre of the sphere, and also inversely in the square root ratio of the force of the sphere. *Q.E.D.*

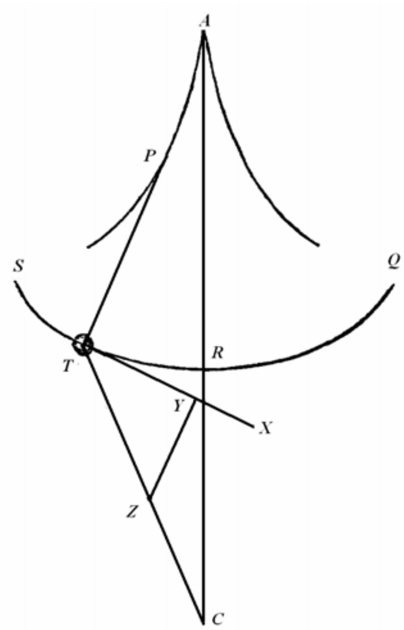
Corol. 1. Hence also the times of the oscillations, of the falling and of the revolutions of the bodies can be compared among themselves. For if for the wheel, by which the cycloid will be described between the spheres, diameter may be put in place equal to the radius of the sphere, the cycloid becomes a right line passing through the centre of the sphere, and the oscillation now will be a descent and accent on this right line. From which both the descent time from some place to the centre, as well as the time for this equally by which a body may describe the quadrant of an arc by revolving uniformly about the centre of the sphere at some distance. For this time (by the second case) is to the time of a semi-oscillation on some cycloid *QRS* as I to $\sqrt{\frac{AR}{AC}}$.

Corol. 2. Hence also these propositions lead to what *Wren* and *Huygens* had found concerning the common cycloid. For if the diameter of the sphere may be increased indefinitely: the surface of this will be changed into a plane, and the centripetal force will act uniformly along lines perpendicular to this plane, and our cycloid will change into a cycloid of the common kind. But in this case the length of the arc of the cycloid, between that plane and the describing point, will emerge equal to four times the versed sine of half the arc between the plane and the point of the wheel describing the same ; as *Wren* found*:* And the pendulum between two cycloids of this kind will be oscillating in a similar and equal cycloid in equal times, as *Huygens* demonstrated. And also the time of descent of the weight, in the time of one oscillation, this will be as *Huygens* indicated.

But the propositions from our demonstrations are adapted to the true constitution of the earth , just as wheels describe cycloids outside the sphere by going in great circles by the motion of nails fixed in the perimeters, and pendulums suspended lower in mines and caverns, must oscillate in cycloids within spheres, so that all the oscillations become isochronous. For gravity (as we will be teaching in the third book) decreases in progressing from the surface of the earth, upwards indeed in the square ratio of the distances from the centre of the earth, downwards truly in a simple ratio.

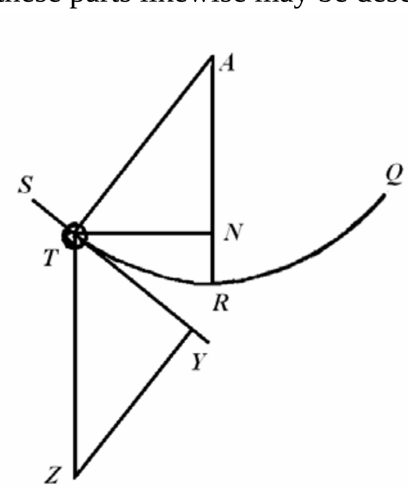
PROPOSITION LIII. PROBLEM XXXV.

With the quadratures of the curvilinear figures granted, to find the forces by which bodies on the given curves may perform isochronous oscillations.



The body *T* may be oscillating on some curved line *STRQ*, the axis of which shall be *AR* passing through the centre of forces *C.* The line *TX* may be drawn which may touch the curve at any place of the body *T*, and on this tangent *TX* there may be taken *TY* equal to the arc *TR.* For the length of that arc will be known from the quadrature of the figure by common methods. [Thus, Newton's criterion for isochronous motion is that the acceleration at *T* is in proportion to the length of the arc *TR*]. From the point *Y* there may be drawn the right line *YZ* perpendicular to the tangent. *CT* may be crossing that perpendicular in *Z*, and the centripetal force [parallel to *AT*] shall be proportional to the right line *TZ.*
Q E.I.

For if the force, by which the body is drawn from *T* towards *C*, may be represented by the right line *TZ* taken proportional to this, this may be resolved into the forces *TY, YZ;* of which *YZ* by drawing the body along the length of the thread *PT*, no motion of this changes, but the other force *TY* directly either accelerates or decelerates the motion of this body on the curve *STRQ*. Hence, since this path *TR* requiring to be described shall always be as the accelerations or retardations of the body to be described in the proportional parts of two oscillations, which shall always be as these parts (of the greater and lesser), and therefore [these accelerations and decelerations] may be made as these parts likewise may be described, [as in the modern equivalent S.H.M. view, the acceleration is proportional to the negative displacement; note that Newton considers an oscillation or swing to be a single motion clockwise or anticlockwise.] But bodies which at the same time always describe the proportional parts of the whole, likewise describe the whole. Q *E.D.*



Corol. 1. Hence if the body *T*, hanging by the rectilinear thread *AT* from the centre *A*, may describe the circular arc *STRQ*, [we must assume the angle *TAN* is small, so that isochronous motion occurs for this simple pendulum] and meanwhile it may be urged downwards along parallel lines in turn by a certain force, which shall be to the uniform force of gravity, as the arc *TR* to the sine of this *TN:* the times of the individual oscillations shall be equal. And indeed on account of the parallel lines *TZ, AR*,

the triangles *ATN* and *ZTY* are similar ; and therefore *TZ* will be to *AT* as *TY* to *TN;* that is, if the uniform force of gravity may be proposed by the given length *AT;* the force *TZ,* by which the isochronous oscillations may be produced, will be to the force of gravity *AT,* as the arc *TR* ,itself equal to *TY,* to the sine *TN* of that arc.
[Note that the isochronous accelerating force is proportional to the arc *TR,* which is almost equal to the semi-chord *TN* for small oscillations, while *AT* and *ZY* are almost vertically downwards, with the tension *AT* as the weight of the pendulum bob; hence

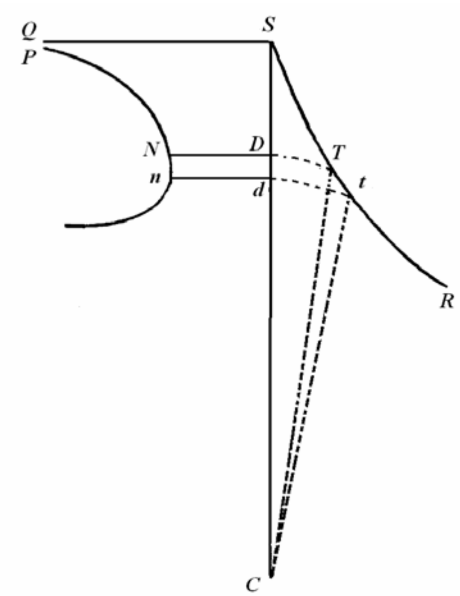
$$\frac{TZ}{TY} = \frac{AT}{TN} \quad \alpha \quad \frac{weight\ of\ bob}{restoring\ force}\ ]$$

Corol. 2. And therefore in clocks, if the forces impressed on the pendulum by the machinery to preserve the motion thus since with the force of gravity may be compared, so that the total force downwards always shall be as the line arising by dividing the multiple of the arc *TR* [$= TY$] and the radius *AR*[$= AT$] by the sine *TN,* all the oscillations will be isochronous.

PROPOSITION LIV. PROBLEM XXXVI.

With the quadrature of the curvilinear figures granted, to find the times, by which bodies acted on by some centripetal force on some curved lines, described in a plane passing through the centre of forces, may descend or ascend.

The body may descend from some place *S,* by a certain curve *STtR* in the plane passing through the given centre of forces *C*. Now *CS* may be joined and that divided into an innumerable number of equal parts, and *Dd* shall be some of these parts. With centre *C* and with the radii *CD, Cd* circles may be described, *DT, dt,* crossing the curved line *STtR* in *T* and *t.* And then from the given law of the centripetal force, and from the given height *CS* by which the body has fallen, the velocity of the body will be given at some other height *CT* (by Prop. XXXIX.).



[We are to consider a body to slide along the given curve without friction from rest at *S,* under the action of a radial force *f*(*r*) acting along *CT*, at the point *T,* so that the velocity at the distance r is given by along the curve in the line element *Tt* shall be $v = \left[-2\int_R^r f(r)dr \right]^{\frac{1}{2}}$,

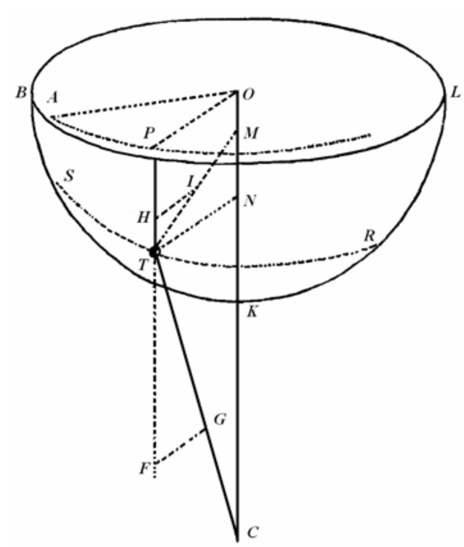
whatever the shape of the curve. Hence, this [energy] integral has to be evaluated to obtain the velocity at *T*. Consequently, the time to arrive at T is given by a second

integration, and $dt = \frac{ds}{v\cos tTC}$, where the component of the radial velocity down the slope is taken.]
But the time, in which the body will describe the line element *Tt*, is as the length of that element, that is, directly as the secant of that angle tTC ; and inversely as the velocity. The applied ordinate *DN* perpendicular to the right line *CS* through the point *D* shall be proportional to this time, and on account of *Dd* given, the rectangle $Dd \times DN$, that is the area *DNnd*, will be in the same proportion to the time. Therefore if *PNn* shall be that curved line that the point *N* always touches, and the asymptote of which shall be the right line *SQ*, standing perpendicularly on the right line *CS*: the area *SQPND* will be proportional to the time in which the body by falling has described the line; and therefore from that area found the time will given. *Q.E.I.*

PROPOSITION LV. THEOREM XIX.

If a body may be moving on some curved surface, the axis of which passes through the centre of the forces, and a perpendicular is sent from the body to the axis, and from some point a line parallel and equal to this line is drawn: I say that parallel line will describe an area proportional to the time.

BKL shall be the curved surface, *T* the body revolving on that, *STR* the trajectory, that the body will describe on the same, *S* the start of the trajectory, *OMK* the axis of the curved surface, *TN* the right line perpendicular to the axis from the body, *OP* drawn equal and parallel to this from the point *O*, which is given on the axis ; *AP* the track of the trajectory described by the point *P* by the winding of *OP* in the plane *AOP* ; *A* corresponding to the start of the trace from the point *S* ; *TC* a right line drawn from the body to the centre ; *TG* the proportional part of the centripetal force *TC*, by which the body is urged to the centre *C*; *TM* a right line perpendicular to the curved surface; *TI* the proportional part of this pressing force, by which the body may be acted on in turn by the surface towards *M*; *PTF* a line passing through the body parallel to the axis, and *GF* and *IH* parallel right lines sent perpendicularly from the points *G* and *I* on that parallel line *PRTF*.



Now I say, that the area *AOP*, described from the start of the motion by the radius *OP*, shall be proportional to the time. For if the force *TG* (by Corol. 2. of the laws) is resolved into the forces *TF* and *FG;* and the force *TI* into the forces *TH* and *HI:* But the forces *TF* and *TH* acting along the line *PF* perpendicular to the plane *AOP* in as much as they only change the motion of the body perpendicular to this plane. And thus in as much as the motion of this is body is made along the position of this plane; that is, the motion of the point *P*, by which the trace of the trajectory *AP* is described in this plane, is the same as if the forces *TF* and *TR* may be removed, and the body is being acted on by the forces *FG* and *HI* only ; and that is, likewise if the body in the plane *AOP* may describe the curve

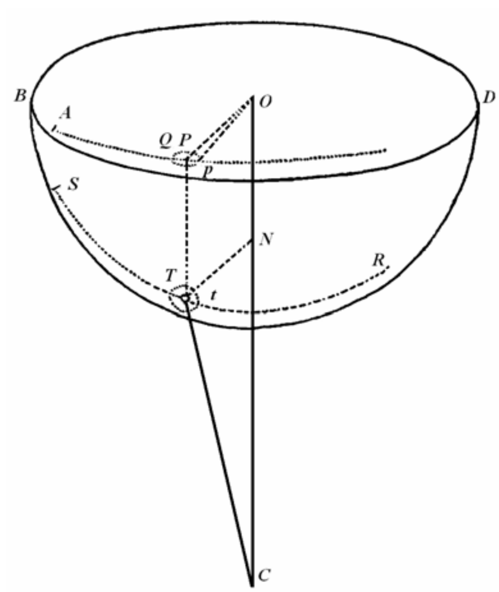
AP, by a centripetal force tending towards the centre *O* and equal to the sum of the forces *FG* and *HI*,. But such a force will describe the area *AOP* (by Prop. I.) proportional to the time. Q *E.D.*

Corol. By the same argument if the body, acted on forces tending towards two or more centres on the same given right line *CO*, may describe in free space some curved line *ST;* the area *AOP* always becomes proportional to the time.

PROPOSITION LVI. PROBLEM XXXVII.

With the quadrature of the curvilinear figure given, and with the law of the centripetal force tending towards the centre given, as well as the curved surface whose axis passes through that centre ;the trajectory is required to be found that the body describes on that same surface, advancing from a given place with a given velocity in a given direction on the surface.

With everything in place which have been constructed in the above proposition, the body *T* may emerge from a given place *S* following a given right line in place in a trajectory required to be found *STR*, the trace of which in the plane *BDO* [*D* is called *L* in the previous diagram, and is used here in the original text] shall be *AP*. And from the given velocity of the body at the height *SC*, the velocity of this will be given at some other height *TC*. Since with that velocity, in the shortest time given, the body may describe a small part of its trajectory *Tt*, and let *Pp* be the trace of this described in the plane *AOP*. *Op* may be joined, and with the centre *T* of a small circle with the radius *Tt* of the trace described on the curved surface, in the plane *AOP* the ellipse *pQ* shall be described. And on account of the given circle with magnitude *Tt*, and the given distance *TN* or *PO* of this from the axis *CO*, that ellipse *pQ* will be given in kind and magnitude, and so in place according to the right line *PO*. And since the area *POp* shall be proportional to the time, and thus the angle *POp* may be given from the given time. And thence the common intersection *p* of the ellipse and of the right line *OP* will be given, together with the angle *OPp* in which the trace of the trajectory *APp* cuts the line *OP*. Hence truly (on bringing together Prop. XLI. with its Corol. 2.) an account of determining the curve *APp* may be readily apparent. Moreover from the individual points *P* of the trace, by raising perpendiculars *PT* to the plane *AOP* of the surface of the curve meeting in *Q*, the individual points *T* of the trajectory will be given. *Q.E.I.*



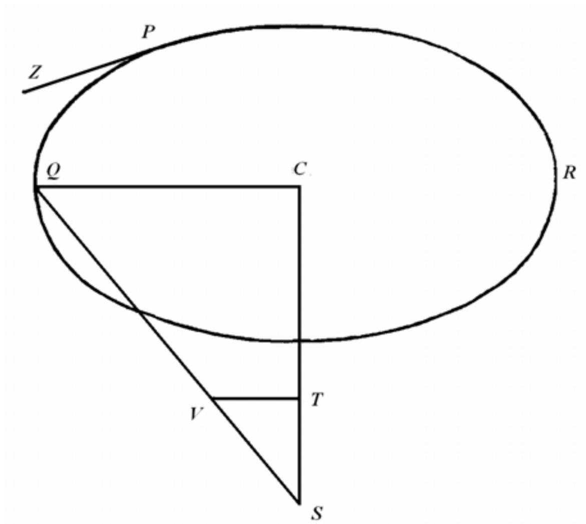
SECTIO X.

De motu corporum in superficiebus datis, deque funipendulorum motu reciproco.

PROPOSITIO XLVI. PROBLEMA XXXII.

Posita cujuscunque generis vi centripeta, datoque tum virium centro tum plano quocunque in quo corpus revolvitur, & concessis figurarum curvilinearum quadraturis: aequiritur motus corporis de loco dato, data cum velocitate, secundum rectam in plano illo datam egressi.

Sit *S* centrum virium, *SC* distantia minima centri hujus a plano dato, *P* corpus de loco *P* secundum rectam *PZ* egrediens, *Q* corpus idem in trajectoria sua revolvens, & *PQR* trajectoria illa, in plano data descripta, quam invenire oportet. Jungantur *CQ*, *QS*, & si in *QS* capiatur *SV* proportionalis vi centripetae qua corpus trahitur versus centrum *S*, & agatur *VT* quae sit parallela *CQ* & occurrat *SC* in *T:* Vis *SV* resolvetur (per legum Corol 2.) in vires *ST, TV*; quarum *ST* trahendo corpus secundum lineam plano perpendicularem, nil mutat motum ejus in hoc plano. Vis autem altera *TV,* agendo secundum positionem plani, trahit corpus directe versus punctum *C* in plano datum, ideoque efficit, ut corpus illud in hoc plano perinde moveatur, ac si vis *ST* tolleretur, & corpus vi sola *TV* revolveretur circa centrum *C* in spatio libera. Data autem vi centripeta *TV*qua corpus *Q* in spatio libero circa centrum datum *C* revolvitur, datur (per prop. XIII;) tum trajectoria *PQR*, quam corpus describit, tum locus *Q*, in quo corpus ad datum quodvis tempus versabitur, tum denique velocitas corporis in loco illo *Q*; & contra. *Q E.1.*



PROPOSITIO XLVII. THEOREMA. XV.

Posito quod vis centripeta proportionalis sit distantiae corporis a centro; corpora omnia in planis quibuscunque revolventia describent elliples, & *revolutiones temporibus aequalibus peragent; quaeque moventur in lineis rectis, ultro citroque discurrendo, singulas eundi* & *redeundi periodos iisdem temporibus absolvent.*

Nam, stantibus quae in superiore propositione, vis *SV*, qua corpus *Q* in plano quovis *PQR* revolvens trahitur versus centrum *S*, est ut distantia *SQ*; atque ideo ob proportionales *SV* & *SQ* , *TV& C*Q vis *TV,* qua corpus trahitur versus punctum *C* in orbis plano datum, est ut distantia *CQ*. Vires igitur, quibus corpora in plano *PQR* versantia trahuntur versus punctum *C*, sunt pro ratione distantiarum aequales viribus quibus corpora undiquaque trahuntur versus centrum *S;* & propterea corpora movebuntur iisdem temporibus, in iisdem figuris, in plano quovis *PQR* circa punctum *Q* atque in spatiis liberis circa centrum *S*; ideoque (per Corol. 2, Prop. X. & Corol. 2, Prop. XXXVIII.) temporibus semper aequalibus vel describent ellipses in plano illo circa centrum *C,* vel periodos movendi ultro citroque in lineis rectis per centrum *C* in plano illo ductis, complebunt.
Q. E.D.

Scholium.

His affines sunt ascensus ac descensus corporum in superficiebus curvis. Concipe lineas curvas in plano describi, dein circum axes quosvis datos per centrum virium transeuntes revolvi, & ea revolutione superficies curvas describere; tum corpora ita moveri ut eorum centra in his superficiebus perpetuo reperiantur. Si corpora illa oblique ascendendo & descendendo currant ultro citroque; peragentur eorum motus in planis per axem transeuntibus, atque ideo in lineis curvis, quarum revolutione curvae illae superficies genitae sunt. Istis igitur in casibus sufficit motum in his lineis curvis considerare.

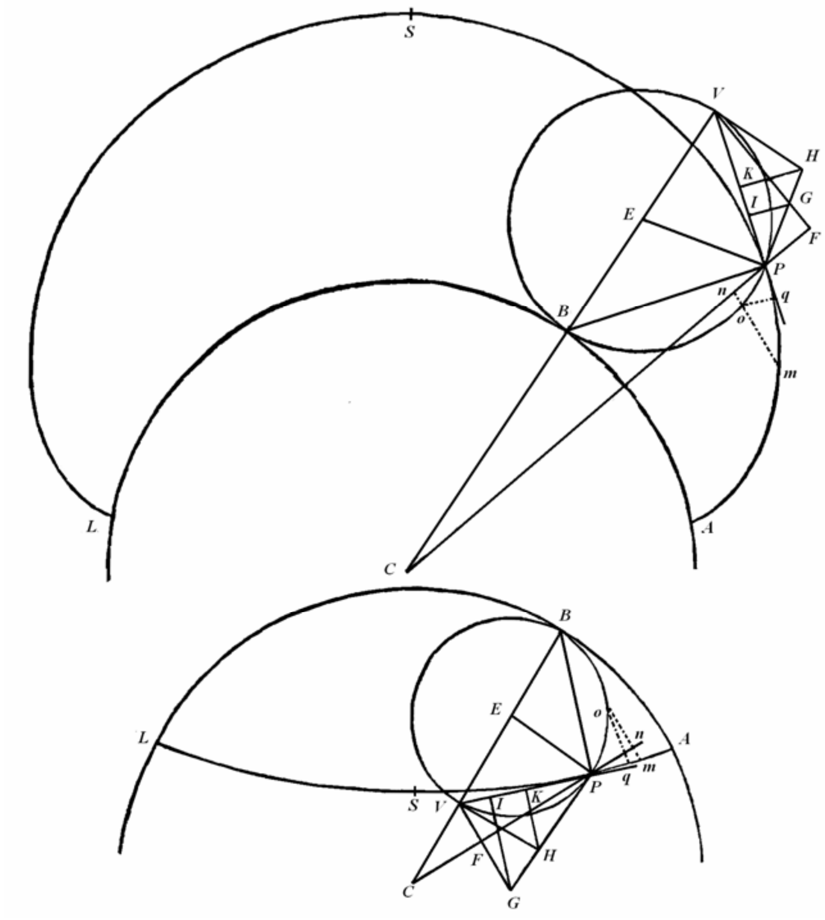
PROPOSITIO XLVIII. THEOREMA XVI.

Si rota globo extrinsecus ad angulos rectos insistat, & *more rotarum revolvendo progrediatur in circulo maximo ; longitudo itineris curvilinei, quod punctum quodvis in rotae perimetro datum, ex quo globum tetigit, confecit, (quodque cycloidem vel epicycloidem nominare licet) erit ad duplicatum sinum versum arcus dimidii qui globum ex eo tempore inter eundum tetigit, ut summa diametrorum globi & rotae ad semidiametrum globi.*

PROPOSITIO XLIX. THEOREMA XVII.

Si rota globo concavo ad rectos angulos intrinsecus insistat & *revolvendo progrediatur in circulo maximo; longitudo itineris curvilinei quod punctum quodvis in rotae perimetro datum, ex quo globum tetigit, confecit, erit ad duplicatum sinum versum arcus dimidii qui globum toto hoc tempore inter eundem tetigit, ut differentia diametrorum globi* & *rotae ad semidiametrum globi.*

Sit *ABL* globus, *C* centrum ejus, *BPV* rota ei insistens, *E* centrum rotae, *B* punctum contactus, & *P* punctum datum in perimetro rotae. Concipe hanc rotam pergere in circulo maximo *ABL* ab *A* per *B* versus *L,* & inter eundum ita revolvi ut arcus *AB, PB* sibi invicem semper aequentur, atque punctum illud *P* in perimetro rotae datum interea describere viam curvilineam *AP.* Sit autem *AP* via tota curvilinea descripta ex quo rota globum tetigit in *A,* & erit viae hujus longitudo *AP* ad duplum sinum versum arcus $\frac{1}{2}PB$, ut 2*CE* ad *CB.* Nam recta *CE* (si opus est producta) occurrat rotae in *V*, junganturque *CP, BP, EP, VP,* & in *CP* productam demittatur normalis *VF.* Tangant *PH,*



VH circulum in *P* & *V* concurrentes in *H,* fecetque *PH* ipsam *VF* in *G,* & ad *VP* demittantur normales *GI, H K.* Centro item *C* & intervallo quovis describatur circulus *nom* secans rectam *CP* in *n*, rotae perimetrum *BP* in *o*, & viam curvilineam *AP* in *m*; centroque *V* & intervallo *Vo* describatur circulus secans *VP* productam in *q*.

Quoniam rota eundo semper revolvitur circa punctum contactus *B*, manifestum est quod recta *BP* perpendicularis est ad lineam illam curvam *AP* quam rotae punctum *P* describit, atque ideo quod recta *VP* tanget hanc curvam in puncto *P*. Circuli *nom* , radius sensim auctus vel diminutus aequetur tandem distantiae *CP;* &, ob similitudinem figurae evanescentis *Pnomq* & figurae *PFGVI*, ratio ultima lineolarum evanescentium *Pm*, *Pn*, *Po*, *Pq*, id est, ratio mutationum momentanearum curvae *AP*, rectae *CP*, arcus circularis *BP*, ac rectae *VP*, eadem erit quae linearum *PV*, *PF*, *PG*, *PI* respective. Cum autem *VF* ad *CF* & *VH* ad *CV* perpendiculares sint, angulique *HVG*, *VCF* propterea aequales; & angulus *VHG* (ob angulos quadrilateri *HVEP* ad *V* & *P* rectos) angulo *CEP* aequalis est, similia erunt triangula *VHG*, *CEP;* & inde fiet ut *EP* ad *CE* ita *HG* ad *HV* seu *HP* & ita *KI* ad *KP*, & composite vel divisim ut *CB* ad *CE* ita *PI* ad *PK*, & duplicatis consequentibus ut *CB* ad 2*CE* ita *PI* ad *PV*, atque ita *Pq* ad *Pm*. Est igitur decrementum lineae *VP*, id est, incrementum lineae *BV–VP* ad incrementum lineae curvae *AP* in data ratione *CB* ad 2*CE*, & propterea (per Corol. Lem. IV.) longitudines *BV–VP & AP*, incrementis illis genitae, sunt in eadem ratione. Sed, existente *BV* radio, est *PV* cosinus anguli *BVP* seu $\frac{1}{2}BEP$, ideoque *BV–VP* sinus versus est ejusdem anguli ; & propterea in hac rota, cuius radius est $\frac{1}{2}BV$, erit *BV–VP* duplus sinus versus arcus $\frac{1}{2}BP$. Ergo *AP* est ad duplum sinum versum arcus $\frac{1}{2}BP$ ut 2*CE* ad *CB*. *Q.E.D.*

Lineam autem *AP* in proposition *Q* priore cycloidem extra globum, alteram in posteriore cycloidem intra globum distinctionis gratia nominabimus.

Corol. I. Hinc si describatur cyclois integra *ASL* & bisecetur ea in *S*, erit longitudo partis *PS* ad longitudinem *VP* (quae duplus est sinus anguli *VBP*, existente *EB* radio) ut 2*CE* ad *CB*, atque ideo in ratione data.

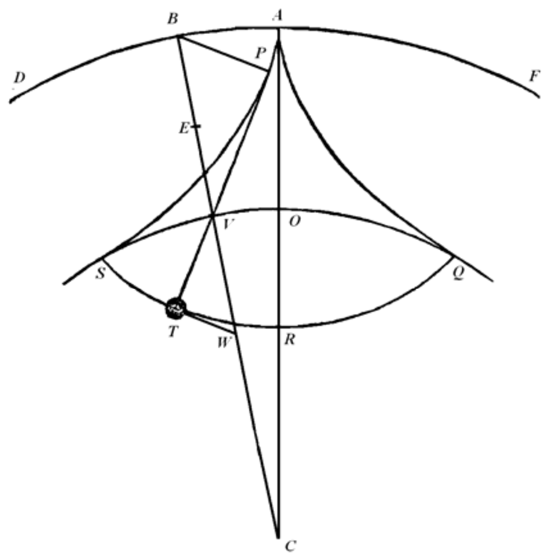
Corol. 2. Et longitudo semiperimetri cycloidis *AS* aequabitur linem rectae, quae est ad rotae diametrum *BV* ut 2*CE* ad *CB*.

PROPOSITIO L. PROBLEMA XXXIII.

Facere ut corpus pendulum oscilletur in cycloide data.

Intra globum *QVS*, centro *C* descriptum, detur cyclois *QRS* bisecta in *R* & punctis suis extremis *Q* & *S* superficiei globi hinc inde occurrens. Agatur *CR* bisecans arcum *QS* in *O*, & producatur ea ad *A*, ut sit *CA* ad *CO* ut *CO* ad *CR*. Centro *C* intervallo *CA* describatur globus exterior *DAF*, & intra hunc globum a rota, cujus diameter sit *AO*, describantur duae semicycloides *AQ*, *AS*, quae globum interiorem tangant in *Q* & *S* & globo exteriori occurrant in *A*. A puncto illo *A*, filo *APT* longitudinem *AR* aequante, pendeat corpus *Q* & ita intra semicycloides *AQ*, *AS* oscilletur, ut quoties pendulum digreditur a perpendiculo *AR*, filum parte sui superiore *AP* applicetur ad semicycloidem illam *APS* versus quam peragitur motus, & circum eam ceu obstaculum flectatur, parteque reliqua *PT* cui semicyclois nondum objicitur, protendatur in lineam rectam; & pondus *T* oscillabitur in cycloide data *QRS*. *Q.E.F.*

Occurrat enim filum *PT* tum cycloidi *QRS* in *T*, tum circulo *QOS* in *V*, agaturque *CV;* & ad fili partem rectam *PT,* e punctis extremis *P* ac *T,* erigantur perpendicula *BP, TW*, occurrentia rectae *CV* in *B* & *W.* Patet, ex constructione & genesi similium figurarum *AS, SR*, perpendicula illa *PB*, *TW* abscindere de *CV* longitudines *VB*, *VW* rotarum diametris *OA*, *OR* aequales. Est igitur *TP* ad *VP* (duplum sinum anguli *VBP* existente $\frac{1}{2}BV$ radio) ut *BW* ad *BV,* seu $AO+OR$ ad *AO,* id est (cum sint *CA* ad *CO*, *CO* ad *CR* & divisim *AO* ad *OR* proportionales) ut $CA+CO$ ad *CA*, vel, si bisecetur *BV* in *E*, ut 2*CE* ad *CB.* Proinde (per Corol I. Prop. XLIX.) longitudo partis rectae fili *PT* aequatur semper cycloidis arcui *PS*, & filum totum *APT* aequatur semper cycloidis arcui dimidio *APS*, hoc est (per Corol. 2. Prop. XLIX.) longitudini A*R.* Et propterea vicissim si filum manet semper aequale longitudini *R* movebitur punctum *T* in cycloide data *QRS. Q.E.D.*



Corol. Filum *AR* aequatur semicycloidi *AS,* ideoque ad globi exterioris semidiametrum *AC* eandem habet rationem quam similis illi semicyclois *SR* habet ad globi interioris semidiametrum *CO.*

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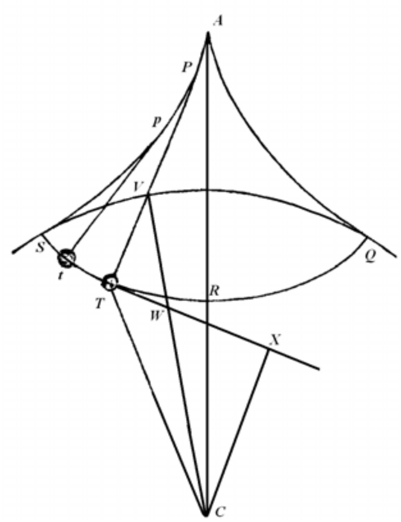
PROPOSITIO LI. THEOREMA XVIII.

Si vis centripeta tendens undique ad globi centrum C *sit in locis singulis ut distantia loci cujusque a centro,* & *hac sola vi agente corpus T oscilletur (modo jam descripto) in perimetro cycloidis QRS*: *dico quod oscillationum utcunque inaequalium aequalia erunt tempora.*



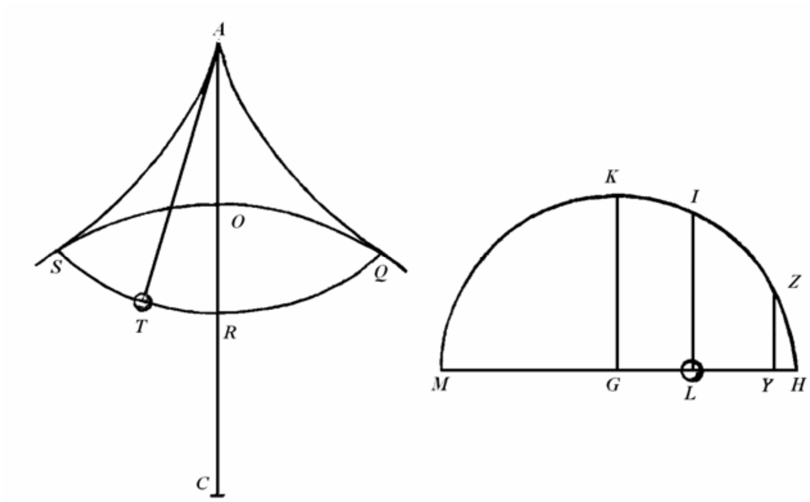
Nam in cycloidis tangentem *TW* infinite productam cadat perpendiculum *CX* & jungatur *CT.* Quoniam vis centripeta qua corpus *T* impellitur versus *C* est ad distantia *CT,* atque haec (per legum corol. 2.) resolvitur in partes *CX*, *T X*, quarum *CX* impellendo corpus directe a *P* distendit filum *PT* & per ejus resistentiam tota cessat, nullum alium edens effectum; pars autem altera *TX*, urgendo corpus transversim seu versus *X*, directe accelerat motum ejus in cycloide; manifestum est quod corporis acceleratio, huic vi acceleratrici proportionalis, sit singulis momentis ut longitudo *TX*, id est, ob datas *CV*, *WV* iisque proportionales *TX*, *TW*, ut longitudo *TW*, hoc est (per corol.I, prop. XLIX.) ut longitudo arcus cycloidis *TR*. Pendulis igitur duobus *APT*, *Apt* de perpendiculo *AR* inaequaliter deductis & simul dimissis, accelerationes eorum semper erunt ut arcus describendi *TR*, *tR*. Sunt autem partes sub initio descriptae ut accelerationes, hoc est, ut totae sub initio describendae, & propterea partes quae manent describendae & acceIerationes subsequentes, his partibus proportionales, sunt etiam ut totae; & sic deinceps. Sunt igitur accelerationes, atque ideo velocitates genitae & partes his velocitatibus descriptae partesque describtae, semper ut totae; & propterea partes describendae datam servantes rationem ad invicem simul evanescent, id est, corpora duo oscillantia simul pervenient ad perpendiculum *AR*. Cumque vicissim ascensus perpendiculorum de loco infimo *R*, per eosdem arcus cycloidales motu retrogrado facti, retardentur in locis singulis a viribus iisdem a quibus descensus accelerabantur, patet velocitates ascensuum ac descensuum per eosdem arcus factorum aequales esse, atque ideo temporibus aequalibus fieri ; & propterea, cum cycloidis partes dum *RS* & *RQ* ad utrumque perpendiculi latus jacentes sint similes & aequales, pendula duo oscillationes suas tam totas quam dimidias iisdem temporibus semper peragent. *Q. E.D.*

Corol. Vis qua corpus *T* in loco quovis *T* acceleratur vel retardatur in cycloide, est ad totum corporis ejusdem pondus in loco altitudimo *S* vel *Q*, ut cycloidis arcus *TR* ad ejusdem arcum *SR* vel *QR*.

PROPOSITIO LII. PROBLEMA XXXIV.

Definire & velocitates pendulorum in locis singulis, & tempora quibus tum oscillationes totae, tum singulae oscillationum partes peraguntur.

Centro quovis G, intervallo *GH* cycloidis arcum *RS* aequante, describe semicirculum *HKM* semidiametro *GK* bisectum. Et si vis centripeta, distantiis locorum a centro proportionalis, tendat ad centrum *G*, sitque ea in perimetro *H I K* aequalis vi centripetae in perimetro globi *QOS* ad ipsius centrum tendenti; & eodem tempore quo pendulum *T* dimittitur e loco supremo *S*, cadat corpus aliquod *L* ab *H* ad *G*, quoniam vires quibus corpora urgentur sunt aequales sub initio & spatiis describendis *TR*, *LG* semper



proportionales, atque ideo, si aequantur *TR* & *LG*, aequales in locis *T* & *L;* patet corpora illa describere spatia *ST*, *HL* aequalia sub initio, ideoque subinde pergere aequaliter urgeri, & aequalia spatia describere. Quare (per Prop. XXXVIII.) tempus quo corpus describit arcum *ST* est ad tempus oscillationis unius, ut arcus *HI*, tempus quo corpus *H* perveniet ad *L*, ad semiperipheriam *HKM*, tempus quo corpus *H* perveniet ad *M*. Et velocitas corporis penduli in loco *T* est ad velocitatem ipsius in loco infima *R*, (hoc eit, velocitas corporis *H* in loco *L* ad velocitatem ejus in loco *G*, seu incrementum momentaneum linere *HL* ad incrementum momentaneum linere *HG*, arcubus *HI*, *HK* aequabili fluxu crescentibus) ut ordinatim applicata *LI* ad radium G*K*, sive ut $\sqrt{SRq - TRq}$ ad *SR*. Unde cum, in oscillationibus inaequalibus, describantur aequalibus temporibus arcus totis oscillationum arcubus proportionales ; habentur, ex datis temporibus, & velocitates & arcus descripti in oscillationibus universis. Quae erant primo invenienda. Oscillentur jam funipendula corpora in cycloidibus diversis intra globos diversos, quorum diversae sunt etiam vires absolutae, descriptis: &, si vis absoluta globi cujusvis *QOS* dicatur *V*, vis acceleratrix qua pendulum urgetur in circumferentia hujus globi, ubi incipit directe versus centrum ejus moveri, erit ut distantia corporis penduli a centro illo & vis absoluta globi conjunctim, hoc est, ut $CO \times V$. Itaque lineola *HI*; quae sit ut haec vis acceleratrix $CO \times V$, describetur dato tempore; &, si erigatur normalis *YZ* circumferentiae occurrens in *Z*, arcus nascens *HZ* denotabit datum illud tempus. Est autem arcus hic nascens *HZ* in subduplicata ratione rectanguli *GHY*, ideoque ut $\sqrt{GH \times CO \times V}$. Unde tempus oscillationis integrae in cycloide *QR S* (cum sit ut semiperipheria *HKM*, quae oscillationem illam integram denotat, directe; utque arcus *HZ*, qui datum tempus similiter denotat, inverse) fiet ut *GH* directe & $\sqrt{GH \times CO \times V}$ inverse, hoc est, ob aequales *GH* & *SR*, ut $\sqrt{\frac{SR}{CO \times V}}$ sive (per Coral. Prop. L.) ut $\sqrt{\frac{AR}{AC \times V}}$. Itaque oscillatationes in globis & cycloidibus omnibus, quibuscunque cum viribus absolutis factae, sunt in ratione quae componitur ex subduplicata ratione longitudinis fili directae, & subduplicata ratione distantiae inter punctum suspensionis & centrum globi inverse, & subduplicata ratione vis absolutae globi etiam inverse. *Q. E.D.*

Corol. 1. Hinc etiam oscillantium, cadentium & revolventium corporum tempora possunt inter se conferri. Nam si rotae, qua cyclois intra globum describitur, diameter constituatur aequalis semidiametro globi cyclois evadet linea recta per centrum globi transiens, & oscillatio jam erit descensus & subsequens ascensus in hac recta. Unde datur tum tempus descensus de loco quovis ad centrum, tum tempus huic aequale quo corpus uniformiter circa centrum globi ad distantiam quamvis revolvendo arcum quadrantalem describit. Est enim hoc tempus (per casum secundum) ad tempus semioscillationis in cycloide quavis *QRS* ut 1 ad $\sqrt{\frac{AR}{AC}}$.

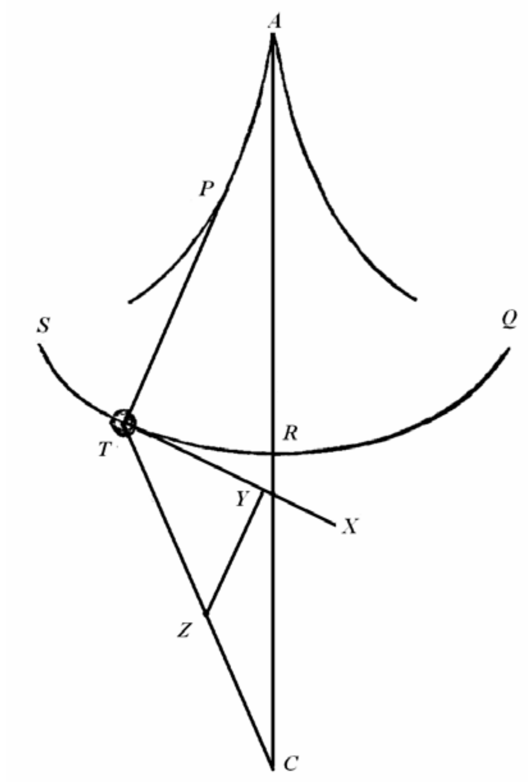
Corol. 2. Hinc etiam confectantur quae *Wrennus* & *Hugenius* de cycloide vulgari adinvenerunt. Nam si globi diameter augeatur in infinitum: mutabitur ejus superficies sphaerica in planum, visque centripeta aget uniformiter secundum lineas huic plano perpendiculares, & cyclois nostra abibit in cycloidem vulgi. Isto autem in casu longitudo arcus cycloidis, inter planum illud & punctum describens, aequatis evadet quadruplicato sinui verso dimidii arcus rotae inter idem planum & punctum describens ; ut invenit *Wrennus:* Et pendulum inter duas ejusmodi cycloides in simili & aequali cycloide temporibus aequalibus oscillabitur, ut demonstravit *Huygenius*. Sed & descensus gravium, tempore oscillationis unius, is erit quem *Huygenius* indicavit.

Aptantur autem propositiones a nobis demonstratae ad veram constitutionem terrae, quatenus rotae eundo in ejus circulis maximis describunt motu clavorum, perimetris suis infixorum, cycloides extra globum; & pendula inferius in fodinis & cavernis terrae suspensa, in cycloidibus intra globos oscillari debent, ut oscillationes omnes evadant isochronae. Nam gravitas (ut in libro tertio docebitur) decrescit in progressu a superficie terrae, sursum quidem in duplicata ratione distantiarum a centro ejus, deorsum vero in ratione simplici.

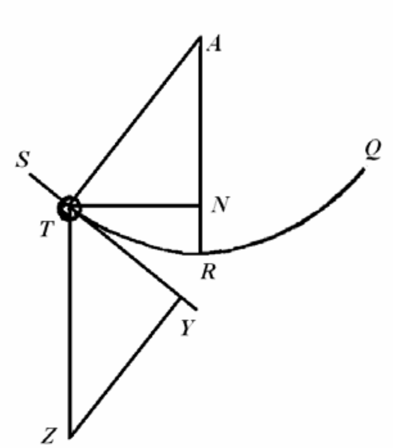
PROPOSITIO LIII. PROBLEMA XXXV.

Concessis figurarum curvilinearum quadraturis, invenire vires quibus corpora in datis curvis lineis oscillationes semper isochronas peragent.

Oscilletur corpus *T* in curva quavis linea *STRQ*, cujus axis sit *AR* transcens per virium centrum *C.* Agatur *TX* quae curvam illam in corporis loco quovis *T* contingat, inque hac tangente *TX* capiatur *TY* aequalis arcui *TR*. Nam longitudo arcus illius ex figurarum quadraturis, per methodos vulgares, innotescit. De puncto *Y* educatur recta *YZ* tangenti perpendicularis. Agatur *CT* perpendiculari illi occurrens in *Z,* & erit vis centripeta proportionalis rectae *TZ. Q E.I.*



Nam si vis, qua corpus trahitur de *T* versus *C*, exponatur per rectam *TZ* captam ipsi proportionalem, resolvetur haec in vires *TY, YZ;* quarum *YZ* trahendo corpus secundum longitudinem fili *PT*, motum ejus nil mutat, vis autem altera *TY* motum ejus in curva *STRQ*, directe accelerat vel directe retardat. Proinde cum haec sit ut via describenda *TR*, accelerationes corporis vel retardationes in oscillationum duarum (majoris & minoris) partibus proportionalibus describendis, erunt semper ut partes illae, & propterea facient ut partes illae simul describantur. Corpora autem quae partes totis semper proportionales simul describunt, simul describent totas. Q *E.D.*



Corol. I. Hinc si corpus *T;* filo rectilineo *AT* a centro *A* pendens, describat arcum circularem *STRQ*, & interea urgeatur secundum lineas parallelas deorsum a vi aliqua, quae sit ad vim uniformem gravitatis, ut arcus *TR* ad ejus sinum *TN:* aequalia erunt oscillationum singularum tempora. Etenim ob parallelas *TZ, AR*, similia erunt triangula *ATN, ZTY;* & propterea *TZ* erit ad *AT* ut *TY* ad *TN;* hoc est, si gravitatis vis uniformis exponatur per longitudinem datam *AT;* vis, *TZ*, qua oscillationes evadent isochronae, erit ad vim gravitatis *AT*, ut arcus *TR* ipsi *TB* aequalis ad arcus illius sinum *TN*.

Corol. 2. Et propterea in horologiis, si vires a machina in pendulum ad motum conservandum impressae ita cum vi gravitatis componi possint, ut vis tota deorsum semper sit ut linea quae oritur applicando rectangulum sub arcu *TR* & radio *AR* ad sinum *TN*, oscillationes omnes erunt isochronae.

PROPOSITIO LIV. PROBLEMA XXXVI.

Concessis Concessis figurarum curvilinearum quadraturis, invenire tempora, quibus corpora vi qualibet centripeta in lineis quibuscunque curvis, in plano per centrum virium transeunte descriptis, descendent et ascendent.

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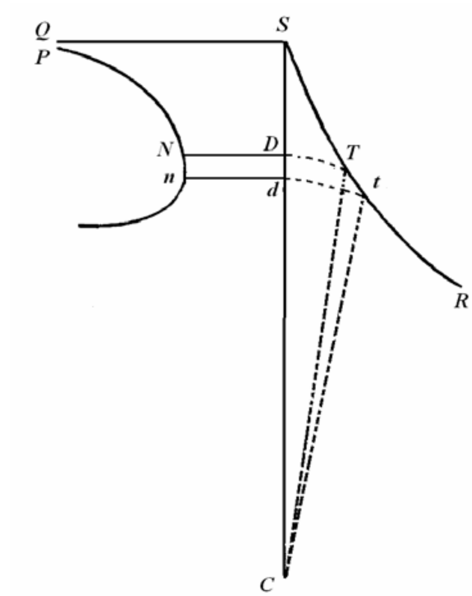
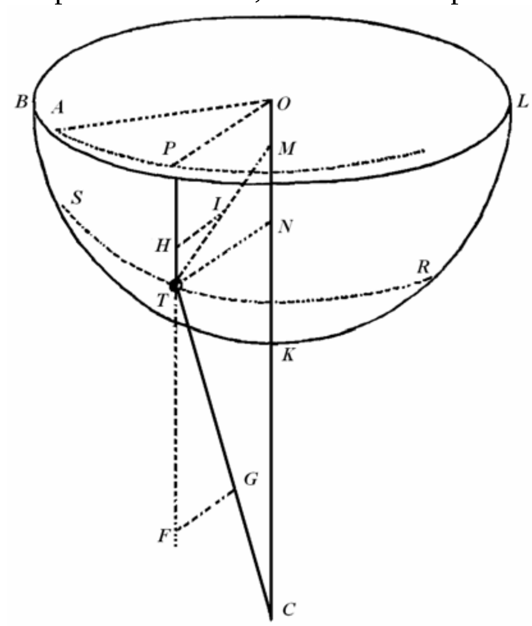
Descendat corpus de loco quovis *S*, per lineam quamvis curvam *STtR* in plano per virium centrum *C* transeunte datam. Jungatur *CS* & dividatur eadem in partes innumeras aequales, sitque *Dd* partium illarum aliqua. Centro *C* intervallis *CD, Cd* describantur circuli, *DT*, *dt*, lineae *STtR* occurrentes in *T* & *t*. Et ex data tum lege vis centripetae, tum altitudine *CS* de qua corpus cecidit; dabitur velocitas corporis in alia quavis altitudine *CT* (per Prop. XXXIX.). Tempus autem, quo corpus describit lineolam *Tt*, est ut lineolae hujus longitudo, id est, ut secans anguli *tTC* directe; & velocitas inverse. Tempori huic proportionalis sit ordinatim applicata *VN* ad rectam *CS* per puntum *D* perpendicularis, & ob datam *Vd* erit rectangulum $Dd \times DN$, hoc est area *DNnd,* eidem tempori proportionale. Ergo si *PNn* sit curva illa linea quam punctum *N* perpetuo tangit, ejusque asymptotos sit recta *SQ* rectae *CS* perpendiculariter insistens: erit area *SQPND* proportionalis tempori quo corpus descendendo descripsit lineam est; proindeque ex inventa illa area dabitur tempus. *Q.E.I.*



PROPOSITIO LV. THEOREMA XIX.

Si corpus movetur in superficie quacunque curva, cuius axis per centrum virium transit, & a corpore in axem demittatur perpendicularis, eique parallela & aequalis ab axis puncto quovis data ducatur: dico quod parallela illa aream tempori proportionalem describet.

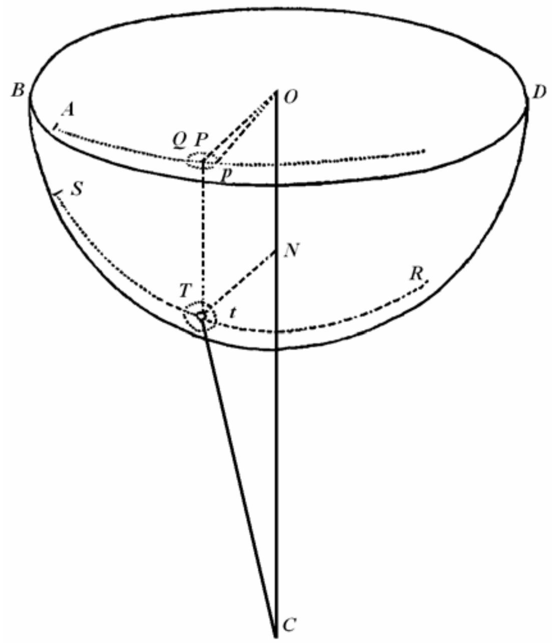
Sit *BKL* superficies curva, *T* corpus in ea revolvens, *STR* trajectoria, quam corpus in eadem describit, *S* initium trajectorim, *OMK* axis superficiei curvae, *TN* recta a corpore in axem perpendicularis, *OP* huic pa,al1ela & aequalis a puncto *O*, quod in axe datur, educta; *AP* vestigium trajectoriae a puncto *P* in lineae volubilis *OP* plano *AOP* descriptum; *A* vestigii initium puncto *S* respondens; *TC* recta a corpore ad centrum ducta; *TG* pars ejus vi centripetae, qua corpus urgetur in centrum *C*, proportionalis ; *TM* recta ad superficiem curvam perpendicularis; *TI* pars ejus vi pressionis, qua corpus urget superficiem vicissimque urgetur versus *M* a superficie, proportionalis; *PTF* recta axi parallela per corpus transiens, & *GF,IH* rectae a punctis *G* & *I* in parallelam *PRTF* perpendiculariter demissae. Dico jam, quod area *AOP*, radio *OP* ab initio motus descripta, sit tempori proportionalis. Nam vis *TG* (per legum Carol. 2.) resolvitur in vires *TF, FG;* & vis *I* in vires *TH, HI:* Vires autem *TF, TH* agenda secundum lineam *PF* plano *AOP* perpendicularem mutant solummodo motum corporis quatenus huic plano perpendicularem. Ideoque motus ejus quatenus secundum positionem plani factus; hoc est, motus puncti *P*, quo trajeaoriae vestigium *AP* in hoc plano describitur, idem est ac si vires *TF*, *TR* tollerentur, & corpus solis viribus *FG*, *HI* agitaretur ; hoc est, idem ac si corpus in plano *AOP*, vi centripeta ad centrum *O* tendente & summam virium *FG* & *HI* aequante, describeret curvam *AP.* Sed vi tali describitur area A*OP* (per Prop. I.) tempori proportionalis. Q *E.D.*



Corol. Eodem argumento si corpus, a viribus agitatum ad centra duo vel plura in eadem quavis recta *CO* data tendentibus, describeret in spatio libero lineam quamcunque curvam *ST;* foret area *AOP* tempori semper proportionalis.

PROPOSITIO LVI. PROBLEMA XXXVII.

Concessis figurarum curvilinearum quadratis, datisque tum lege vis centripetae ad centrum datum tendentis, tum superficie curva cujus axis per centrum illud transit; invenienda est trajectoria quam corpus in eadem superficie describet, de loco dato, data cum velocitate, versus plagam in superficie illa datam egressum.



Stantibus quae in superiore propositione constructa sunt, exeat corpus *T* de loco dato *S* secundum rectam positione datam in trajectoriam inveniendam *STR*, cujus vestigium in plano *BLO* sit *AP*. Et ex data corporis velocitate in altitudine *SC*, dabitur ejus velocitas in alia quavis altitudine *TC*. Ea cum velocitate dato tempore quam minimo describat corpus trajctoriae suae particulam *Tt*, sitque *Pp* vestigium ejus in plano *AOP* descriptum. Jungatur *Op*, & circelli centro *T* in intervallo *Tt* in superficie curva descripti vestigium in plano *AOP* sit ellipsis *pQ*. Et ob datum magnitudine circellum *Tt*, datamque ejus ab axe *CO* distantiam *TN* vel *PO*, dabitur ellipsis illa *pQ* specie & magnitudine, ut & positione ad rectam *PO*. Cumque area *POp* sit tempori proportionatis, atque ideo ex dato tempore detur, dabitur angulus *POp*. Et inde dabitur ellipseos & rectae *OP* intersectio communis *p*, una cum angulo *OPp* in quo trajectoriae vestigium *APp* secat lineam *OP*. Inde vero (conferendo Prop. XLI. cum Corol. suo 2.) ratio determinandi curvam *APp* facile apparet. Tum ex singulis vestigii punctis *P*, erigendo ad planum *AOP* perpendicula *PT* superficiei curvae occurrentia in *Q* dabuntur singula trajectorim puncta *T*. *Q.E.I.*