THE MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY

CONCERNING THE

MOTION OF BODIES

BOOK ONE.

SECTION I.

Concerning the method of first and last ratios, with the aid of which the following are demonstrated.

LEMMA I.

Quantities, and so the ratios of quantities, which tend steadily in some finite time to equality, and before the end of that time approach more closely than to any given differences, finally become equal.

If you say no; so that at last they may become unequal, and there shall be a final difference *D* of these. Therefore they are unable to approach closer to equality than to the given difference a *D:* contrary to the hypothesis.

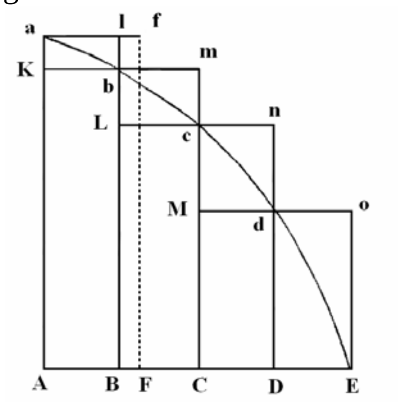
Q. E. D.

LEMMA II.

If in some figure AacE, with the right lines Aa, AE and the curve acE in place, some number of parallelograms are inscribed Ab, Bc, Cd, &c. with equal bases AB, BC, CD, &c. below, and with the sides Bb, Cc, Dd, &c. maintained parallel to the side of the figure Aa; & with the parallelograms aKbl, bLcm, cMdn, &c. filled in. Then the width of these parallelograms may be diminished and the number may be increased to infinity : I say that the final ratios which the inscribed figure AKbLcMdD, the circumscribed figure AalbmcndoE, and to the curvilinear figure AabcdE have in turn between each other, are ratios of equality.

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms *Kl, Lm, Mn, Do,* that is (on account of the equal bases) the rectangle under only one of the bases *Kb* and the sum of the heights *Aa*, that is, the rectangle *ABla.* But this rectangle, because with the width of this *AB* diminished indefinitely, becomes less than any given [rectangle] you please. Therefore (by lemma I) both the inscribed and circumscribed figures finally become equal, and much more [importantly] to the intermediate curvilinear figure.

Q. E. D.



LEMMA III.

Also the final ratios are the same ratios of equality, when the widths of the parallelograms* AB, BC, CD, &c. *are unequal, and all are diminished indefinitely.

For let *AF* be equal to the maximum width, and the parallelogram *FAaf* may be completed. This will be greater than the difference of the inscribed and of the circumscribed figure ; but with its own width *AF* diminished indefinitely, it is made less than any given rectangle.

Q. E. D.

Corol. I. Hence the final sum of the vanishing parallelograms coincides in every part with the curvilinear figure.

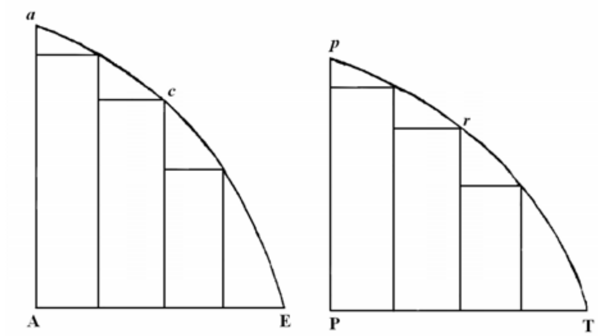
Corol. 2. And much more [to the point] a rectilinear figure, which is taken together with the vanishing chords of the arcs *ab, bc, cd,* &c., finally coincides with the curvilinear figure.

Corol. 3. And in order that the circumscribed rectilinear figure which is taken together with the tangents of the same arcs.

Corol. 4. And therefore these final figures (as far as the perimeter *ac*E,) are not rectilinear, but the curvilinear limits of the rectilinear [figures].

LEMMA IV.

If in the two figures* AacE, PprT, *there are inscribed (as above) two series of parallelograms, and the number of both shall be the same, and where the widths are diminished indefinitely, the final ratios of the parallelograms in the one figure to the parallelograms in the other, of the single to the single, shall be the same ; I say on which account, the two figures in turn* AacE, PprT *are in that same ratio.



And indeed as the parallelograms are one to one, thus (on being taken together) shall be the sum of all to the sum of all, and thus figure to figure; without doubt with the former figure present (by lemma III) to the first sum, and with the latter figure to the latter sum in the ratio of equality.

Q. E. D.

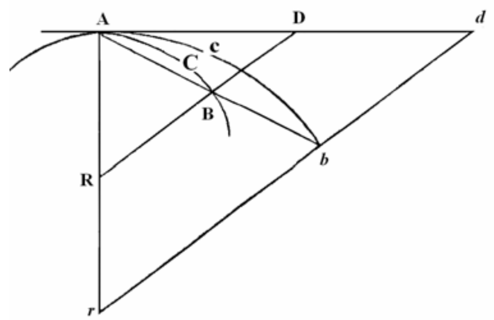
Corol. Hence if two quantities of any kind may be divided into the same number of parts in some manner; and these parts, where the number of these is increased and the magnitude diminished indefinitely, may maintain a given ratio in turn, the first to the first, the second to the second, and with the others in their order for the remaining : the whole will be in turn in that same given ratio. For if, in the figures of this lemma the parallelograms are taken as the parts between themselves, the sums of the parts always will be as the sum of the parallelograms ; and thus, when the number of parts and of parallelograms is increased and the magnitude is diminished indefinitely, to be in the final ratio of parallelograms to parallelograms, that is (by hypothesis) in the final ratio of part to part.

LEMMA V.

All the sides of similar figures, which correspond mutually to each other, are in proportion, both curvilinear as well as rectilinear: and the areas shall be in the squared ratio of the sides.

LEMMA VI.

If some arc in the given position* ACB *is subtended by the chord* AB*, and at some point* A*, in the middle of the continued curve, it may be touched by the right line* AD *produced on both sides*; *then the points* A, B *in turn may approach and coalesce ; I say that the angle* BAD*, contained within the chord and tangent, may be diminished indefinitely and vanishes finally.



For if that angle does not vanish, the arc *ACB* together with the tangent *AD* will contain an angle equal to a rectilinear angle, and therefore the curve will not be continuous at the point *A*, contrary to the hypothesis.

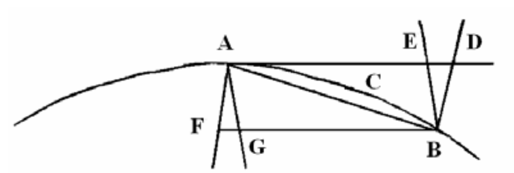
Q. E. D.

LEMMA VII.

With the same in place; I say that the final ratio of the arc, the chord, and of the tangent in turn is one of equality.

For while the point *B* approaches towards the point *A*, *AB* and *AD* are understood always to be produced to distant points *b* and *d*, and *bd* is drawn parallel to the section *BD.* And the arc *Acb* always shall be similar to the arc *ACB.* And with the points *A*, *B* coming together, the angle *dAb* vanishes, by the above lemma ; and thus the finite arcs *Ab, Ad* and the intermediate arc *Acb* coincide always, and therefore are equal. And thence from these always the proportion of the right lines *AB, AD,* and of the intermediate arc *ACB* vanish always, and they will have the final ratio of equality.

Q. E. D.



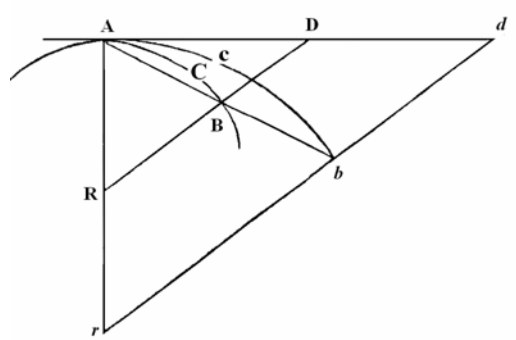
Corol. I. From which if through *B* there is drawn *BF* parallel to the tangent, some line *AF* is drawn passing through *A* always cutting at *F*, this line *BF* finally will have the ratio of equality to the vanishing arc *ACB*, because from which on completing the parallelogram *AFBD*, *AD* will have always the ratio of equality to *AD.*

Corol. 2. And if through *B* and *A* several right lines *BE*, *BD*, *AF*, *AG,* are drawn cutting the tangent *AD* and the line parallel to itself *BF;* the final ratio of the cuts of all *AD, AE, BF, BG,* and of the chords and of the arc *AB* in turn will be in the ratio of equality.

Corol. 3. And therefore all these lines can be taken among themselves in turn, in the whole argument concerning the final ratios.

LEMMA VIII.

If the given right lines* AR, BR, *with the arc* ACB, *the chord* AB *and the tangent* AD, *constitute the three triangles* RAB, RACB, RAD, *then the points* A *and* B *approach together: I say that the final form of the vanishing triangles is one of similitude, and the final ratio one of equality.

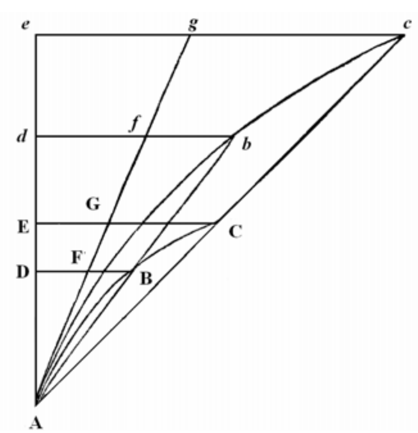


For while the point *B* approaches towards the point *A*, always *AB*, *AD*, *AR* are understood to be produced a great distance away to the points *b, d* and *r,* and with *rbd* itself to be made parallel to *RD*, and the arc *Acb* always shall be similar to the arc *ACB.* And with the points *A* and *B* merging, the angle *bAd* vanishes, and therefore the three finite triangles coincide *rAb*, *rAcb*, *rAd*, and by the same name they are similar and equal. From which and with these *RAB, RACB, RAD* always similar and proportional, finally become similar and equal to each other in turn.

Q. E. D.

Corol. And hence those triangles, in the whole argument about the final ratios, can be taken for each other in turn.

LEMMA IX.



***If the right line* AE *and the curve* ABC, *for a given position, mutually cut each other in the given angle* A, *and to that right line AE at another given angle,* BD *and* CE *may be the applied ordinates, crossing the curve at* B and C, *then the points* B *and* C *likewise approach towards the point* A: *I say that the areas of the triangles* ABD *and* ACE *will be in the final ratio in turn, in the square ratio of the sides* .**

And indeed while the points *B* and *C* approach towards the point *A,* it is understood always that the points *AD* and *AE* are to be produced to the distant points *d* and *e,* so that *Ad* and *Ae* shall be proportional to *AD* and *AE* themselves, and the ordinates *db* and *ec* are erected parallel to the ordinates *DB* and *EC* , which occur for *AB* and *AC* themselves produced to *b* and *c.* It is understood that there be drawn, both the curve *Abc* similar to *ABC* itself, as well as the right line *Ag,* which touches each curve at *A*, and which cuts the applied ordinates *DB, EC, db, ec* in *F, G, f, g.* Then with the length *Ae* remaining fixed, the points *B* and *C* come together at the point *A;* and with the angle *cAg* vanishing, the curvilinear areas *Abd* to *Ace* coincide with the rectilinear areas *Afd* to *Age*; and thus (by lemma V.) they will be in the square ratio of the sides *Ad* and *Ae* : But with these areas there shall always be the

proportional areas *ABD* to *ACE,* and with these sides the sides *AD* to *A E.* And therefore the areas *ABD* to *ACE* shall be in the final ratio as the squares of the sides *AD* to *AE.*
Q. E. D.

LEMMA X.

The finite distances which some body will describe on being pushed by some force, shall be from the beginning of the motions in the square ratio of the times, that force either shall be determined and unchanged, or the same may be augmented or diminished continually.

The times are set out by the lines *AD* to *AE,* and the velocities generated by the ordinates *DB* to *EC;* and the distances described by these velocities will be as the areas *ABD* to *ACE* described by these ordinates, that is, from the beginning of the motion itself (by lemma IX) in the square ratio of the times *AD* to *AE.*
Q. E. D.

Cor I. And hence it is deduced easily, that the errors of bodies describing similar parts of similar figures in proportional times, which are generated by whatever equal forces applied similarly to bodies, and are measured by the distances of bodies of similar figures from these places of these, to which the bodies would arrive in the same times with the same proportionals without these forces, are almost as the squares of the times in which they are generated.
[These corollaries examine the effect of small resistive forces on otherwise uniformly accelerated motion; the initial motion being free from such velocity-related resistance.]

Corol. 2. Moreover the errors which are generated by proportional forces similarly applied to similar parts of similar figures, are as the forces and the squares of the times jointly.

Corol. 3. The same is to be understood from that concerning any distances whatsoever will describe which bodies acted on by diverse forces. These are, from the beginning of the motion, as the forces and the squares of the times jointly.

Corol. 4. And thus the forces are described directly as the distances, from the start of the motion, and inversely as the squares of the times.

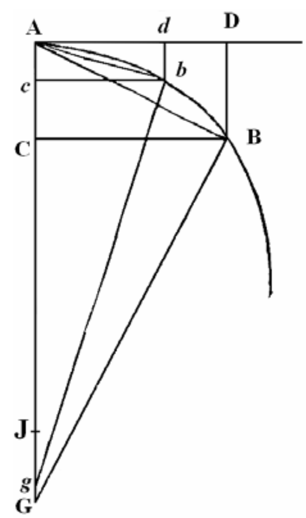
Corol.5. And the squares of the times are directly as the distances described and inversely with the forces.

Scholium.

If indeterminate quantities of different kinds between themselves are brought together, and of these any may be said to be as some other directly or inversely : it is in the sense, that the first is increased or decreased in the same ratio as the second, or with the reciprocal of this. And if some one of these is said to be as another two or more directly or inversely : it is in the sense, that the first may be increased or diminished in the ratio which is composed from the ratios in which the others, or the reciprocals of the others, are increased or diminished. So that if A may be said to be as B directly and C directly and D inversely: it is in the sense, that A is increased or diminished in the same ratio with $\mathrm{B} \times \mathrm{C} \times \frac{1}{\mathrm{D}}$, that is, that A and $\frac{\mathrm{BC}}{\mathrm{D}}$ are in turn in a given ratio.

LEMMA XI.

The vanishing subtense of the angle of contact, in all curves having a finite curvature at the point of contact, is finally in the square ratio of the neighbouring subtensed arcs.



Case I. Let that arc *AB* be [called] the subtensed arc [of the chord] *AB*, the subtense of the angle of contact is the perpendicular *BD* to the tangent. To this subtense *AB* and to the tangent *AD* the perpendiculars *AG* and *BG* are erected, concurring in *G;* then the points *D, B, G* may approach the points *d*,*b*,*g*, and let *J* be the intersection of the lines *BG*, *AG* finally made when the points *D* and *B* approach as far as to *A*. It is evident that the distance G*J* can be less than any assigned distance. But (from the nature of the circles passing through the points *ABG*, *Abg*) *AB* squared is equal to $AG \times BD$, and *Ab* squared is equal to $Ag \times bd$ *;* and thus the ratio *AB* squared to *Ab* squared is composed from the ratios *AG* to *Ag* and *BD* to *bd*. But because *GJ* can be assumed less than any length assigned, it comes about that the ratio *AG* to *Ag* may differ less from the ratio of equality than for any assigned difference, and thus so that ratio *AB* squared to *Ab* squared may differ less from the ratio *BD* to *bd* than for any assigned difference. Therefore, by lemma I, the final ratio *AB* squared to *Ab* squared is the same as with the final ratio *BD* to *bd*.

Q. E. D.

[Thus, in the limit, $BD{:}bd = AB^2{:}\, Ab^2$; the term *subtensed* is one not used now in geometry, and does not get a mention in the *CRC Handbook of Mathematics*, etc., and as it is used here, it means simply the chord *subtending* the smaller arc in a circle. The related *versed sine* or *sagitta* that we will meet soon is the maximum distance of the arc beyond the chord, given by $1 - cos\,\alpha = 2\,sin^2 \frac{\alpha}{2}$, where α is the angle subtended by the arc of the unit circle.]

Case 2. Now *BD* to *AD* may be inclined at some given angle, and always there will be the same final ratio *BD* as *bd* as before, and thus the same *AB* squared to *Ab* squared.

Q. E. D.

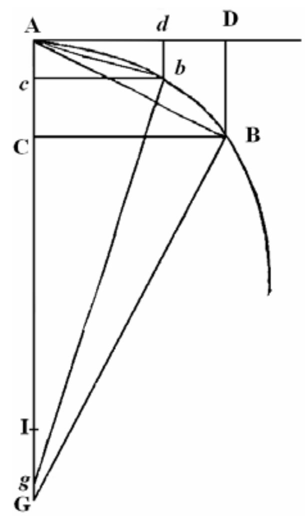
Case 3. And although the angle *D* may not be given, but the right line *BD* may converge to a given point, or it may be put in place by some other law; yet the angles *D*, *d* are constituted by a common rule and always will incline towards equality, and therefore approach closer in turn than for any assigned difference, and thus finally will be equal, by lemma I, and therefore the lines *BD* to *bd* are in the same ratio in turn and as in the prior proposition.

Q. E. D.

Corol. 1. From which since the tangents *AD* to *Ad*, the arcs *AB* to *Ab*, and the sines of these *BC* to *bc* become equal finally to the chords *AB* to *Ab* ; also the squares of these finally shall be as the subtenses *BD* to *bd*.

[Thus, $\frac{AD}{Ad} \to \frac{AB}{Ab}$; $\frac{arcAB}{arcAb} \to \frac{AB}{Ab}$; $\frac{BC}{bc} \to \frac{AB}{Ab}$; and $\frac{AB^2}{Ab^2} = \frac{BD}{bd}$. Care must be taken to note that some ratios are equal in the limit only, while others are true more generally.]

Carol. 2. The squares of the same also are finally as the versed sines [sagittae] of the arcs, which bisect the chords and they converge to a given point. For these versed sines are as the subtenses *BD* to *bd*.



Corol. 3. And thus the sagitta is in the square ratio of the time in which the body will describe the arc with a given velocity.

Carol. 4. The [areas of the] rectilinear triangles *ADB* to *Adb* are finally in the cubic ratio of the sides *AD* to *Ad*, and in the three on two ratio [of the powers] of the sides *DB* to *db;* as in the combined ratio of the sides *AD* and *DB* to *Ad* and *db* present. And thus the triangles *ABC* to *Abc* are finally in the triplicate ratio of the sides *BC* to *bc*. Truly I call the three on two ratio the square root of the cube, which is composed certainly from the simple cube and square root [ratios].

[For the areas of the triangles are as

$$AD.DB{:}Ad.db = AD{:}Ad \times BD{:}bd$$

$$= AD{:}\ ad \times AB^2{:}Ab^2 = AB^3{:}Ab^3 = AD^3{:}ad^3.$$

Also, $AD.DB{:}Ad.db = AD{:}ad \times BD{:}bd = BD{:}bd \times \sqrt{BD{:}bd} = BD^{\frac{3}{2}}{:}bd^{\frac{3}{2}}$.]

Corol.5. And because *DB* to *db* finally are parallel and in the square ratio of *AD* to *Ad :* the final curvilinear areas *ADB* to *Adb* (from the nature of parabolas) are two thirds of the parts of the rectilinear triangles *ADB* to *Adb;* and the segments *AB* to *Ab* one third parts of the same triangles. And thence these areas and these segments will be in the cubic ratio both of the tangents *AD* to *Ad;* as will as of the chords and of the arcs *AB* to *Ab.*

Scholium.

Moreover in all these we suppose the angle of contact neither to be infinitely greater to the angles of contact which circles maintain with their tangents, nor with the same infinitely small ; that is, the curvature at the point *A*, neither to be infinitely small nor infinitely great, or the interval *AJ* to be of finite magnitude. For *DB* can taken as AD^3*:* [Thus far, as the angle DAB becomes very small, $AB \to AD$, and $BD{:}bd \to AD^2{:}\,Ad^2$.] as in that case no circle can be drawn through the point *A* between the tangent *AD* and the curve *AB*, and hence the angle of contact will be infinitely smaller than with circles. And by a similar argument, if *DB* becomes successively as AD^4, AD^5, AD^6, AD^7,&c. there will be had a series of contact angles going on to infinity, of which any of the latter is infinitely smaller then the first. And if *DB* is made successively as AD^2, $AD^{\frac{3}{2}}$,$AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$,$AD^{\frac{7}{6}}$,&c. [*i.e.* powers between 1 and 2] another infinite series of contact angles will be had, the first of which is of the same kind as with circles, the second infinitely greater, and any later infinitely greater with the previous. But between any two from these angles a series of intermediate angles to be inserted can go off on both sides to infinity, any latter of which will be infinitely greater of smaller with the previous. As if between the terms AD^2 and AD^3 the series $AD^{\frac{13}{6}}$,$AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{6}{4}}$,$AD^{\frac{7}{3}}$,$AD^{\frac{5}{2}}$,$AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$,$AD^{\frac{17}{6}}$,&c. is inserted. And again between any two angles of this series a new series of intermediate angles can be inserted in turn with infinite intervals of differences. And nor by nature will it know a limit

Everything which have been shown fully about curved lines and surfaces , easily may be applied to the surface curves of solids and the curves within. Truly I have presented these lemmas, so that I might escape the tedium of deducing long demonstrations *ad absurdum,* in the custom of the old geometers. For they are rendered more contracted by the method of indivisibles. But because the hypothesis of indivisibles is harder, and therefore that method is less recommended geometrically; I have preferred the demonstrations of the following things, the sums and ratios of the first arising and for the final of the vanishing quantities, that is, to deduce the limits of sums and ratios; and therefore demonstrations of these limits which I have been able to present briefly. Because indeed with these the same may be done better by the method of indivisibles ; and with the principles now demonstrated we may use them without risk. Hence in the following, if when I have considered quantities as it were from [being] small and unchangeable, or if I have used small curved lines for right lines ; I may not wish indivisibles to be understood, but vanishing divisibles, not the sums and ratios of parts but the limits of the sums and ratios of parts being determined always to be understood ; and the strength of such demonstrations always to be recalled to the method of the preceding lemmas.

To the accusation, that the final proportion of vanishing quantities shall be nothing; obviously which, before they will have vanished, it is not the final, where they will have vanished, and there is nothing. But by the same argument it may be equally contended the velocity is to be zero for a body at a certain place, when the motion may have finished, you may arrive at the final velocity : for this not to be the final, before the body has reached the place, and when it has reached there it is zero. And the easy response is : By the final velocity that is to be understood, by which the body is moving, and neither before it has reached the final place and the motion has ceased, nor afterwards, but then at the instant it touches that place; that is, that velocity itself by which the body reaches the final place and from which motion it ceases. And similarly by the ultimate ratio of vanishing quantities, the ratio of the quantities is required to be understood, not before they vanish, not after, but with which they vanish. And equally the first ratio arising is the ratio by which they are generated. And the first sum and final sum [in turn] is to be, by which they begin or cease (either to be increased or decreased). The limits stand out to which the velocity is able to reach at the end of the motion, but not to be transgressed. This is the final velocity. And in a like manner is the ratio of the limit of a quantity and of the proportions of all beginnings and endings. And since here the limit shall be certain and defined, the problem is truly the same geometrical one to be determined. Truly everything geometrical has been legitimately used in all the geometrical determinations and demonstrations.

It may be contended also, that if the final vanishing ratios of vanishing quantities may be given, and the final magnitudes will be given: and thus a quantity will be constructed entirely from indivisibles, contrary to what *Euclid* demonstrated concerned with incommensurables, in Book X of the Elements. Truly this objection is supported by a false hypothesis. These final ratios actually vanishing from any quantities, are not the ratios of the final quantities, but the limits to which the ratios of quantities always approach by decreasing without bounds ; and which can be agreed they approach nearer than for whatever differences, at no time truly to be transgressed, nor at first to be considered as quantities being diminished indefinitely. The matter is understood more clearly with the infinitely great. If two quantities of which the difference has been given may be increased indefinitely, the final ratio of these is given, without doubt the ratio of equality, nor yet thus will the final quantities of this be given or the maxima of which that is the ratio. In the following, therefore if when I discuss quantities, in advising about things to be considered easily, either as minimas, vanishing or final ; you may understand the quantities are determined with great care, but always to be thought of as diminishing without limit. [In the sense that they are finite and getting smaller, and keep on doing so.]

[Translator's Note: At this stage, we are made aware by Newton of the two methods of doing calculations, apart from the ponderous *reducti ad absurdum* type geometrical methods of the ancient Greeks, *e.g.* the approach of Archimedes to solving certain problems, and as used by Huygens in his *Horologium* in deriving the isochronous property of his cycloidal pendulum : The method of the first and last ratio, which appears to be none other than extracting a limit from first principles by seeking closer and closer upper and lower bounds indefinitely; and the method of indivisibles, which is akin to the modern calculus. At present, for the sake of economy and to avoid further controversy it would seem regarding vanishing quantities, only the first of the two has been used and shown in some detail.

Newton goes to great pains to construct a geometrical method which embodies the ideas both of integration and differentiation, but which avoids directly the forming of integrals and derivatives as we know them now; instead, the idea of a limit is set out initially, which incidentally demolishes Zeno' s Paradox in a sentence, and which in words is more or less the present definition of such, the difference between the limiting value and nearby values can be made as small as it pleases without end, without actually being made equal. In the method of first and last sums and ratios, a geometrical method is established for carrying out this process, and so for treating integration and differentiation.

At first we look at slightly greater and slightly smaller rectangles enclosing a curve, which approach each other and the segment of the area under the curve closer and closer as their number is increased and the bases diminished, which on the whole seems highly credible. Thus a formulation of integration is obtained. In the second a parallelogram is presented in which a small arc of the curve lies near the diagonal, crossing at the ends, and one side of the parallelogram is a tangent at one vertex; as the parallelogram is diminished in size, the diagonal, the curve, and the tangent finally merge together closer and closer; on elaboration, we are to follow points on lines in diminishing similar triangles approaching an ultimate point at their intersection, where the two similar triangles vanish, but their finite ratios of corresponding sides is maintained by two other trustworthy similar triangles which remain finite all along, this method also seems intuitively ok, as we are told that the final ratio is to be evaluated from the finite similar triangles, at least in the case of the curve being a circle, which follows readily from elementary geometry, while of course Newton rightly asserts that the dreaded zero on zero is never used.

Thus the objections of the doubters were allayed, or at least they had less to argue about. This approach is and was not enough for the pure mathematicans at the time, at least those on the continent; we must remember that Newton was (in my view) essentially a physicist doing mathematics, at which he was extraordinarily adept, as well as an experimentalist cum alchemist cum theologian, rather than a pure mathematician. The method considers points moving along lines in time, which we now would consider as a mere parameter, and we are asked to retrace these positions into the past to arrive at the starting point, or very close to it. Thus, Newton's calculus in the *Principia* is about the rates of change of quantities with time.]

PHILOSOPHIAE NATURALIS

PRINCIPIA MATHEMATICA

DE

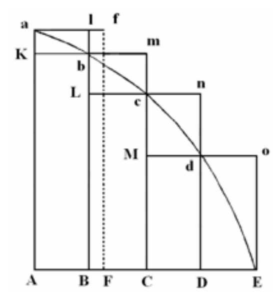
MOTU CORPORUM

LIBER PRIMUS.

SECTIO I.

De methodo rationum primarum & ultimarum, cujus ope sequentia demonstrantur.

LEMMA I.



Quantitates, ut & quantitatum rationes, quae ad aequalitatem tempore quovis finito constanter tendunt, & ante finem temporis illius propius ad invicem accedunt quam pro data quavis differentia, fiunt ultimo aequales.

Si negas; fiant ultimo inaequales, & fit earum ultima differentia *D.* Ergo nequeunt propius ad aequalitatem accedere quam pro data differentia *D:* contra hypothesin.

LEMMA II.

***Si in figura quavis* AacE, *rectis* Aa, AE & *curva* acE *comprehensa, inscribantur parallelogramma quotcunque* Ab, Bc, Cd, &c. *sub basibus* AB, BC, CD, &c. *aequalibus, & lateribus* Bb, Cc, Dd, &c. *figurae lateri* Aa *parallelis contenta; & compleantur parallelogramma* aKbl, bLcm, cMdn, *&c*. *Dein horum parallelogrammorum latitudo minuatur, & numerus augeatur in infinitum: dico quod ultimae rationes quas habent ad se invicem figura inscripta* AKbLcMdD, *circumscriptae* AalbmcndoE, *& curvilinea* AabcdE, *sunt rationes aequalitatis*.**

Nam figurae inscripae & circumscriptae differentia est summa parallelogrammorum *Kl*, *Lm*, *Mn*, *Do*, hoc est (ob aequales omnium bases) rectangulum sub unius basi *Kb* & altitudinum summa *Aa*, id est, rectangulum *ABla*. Sed hoc rectangulum, eo quod latitudo eius *AB* in infinitum minuitur, fit minus quovis dato. Ergo (per lemma I) figura inscripta & circumscripta & multo magis figura curvilinea intermedia fiunt ultimo aequales. *Q. E. D.*

LEMMA III.

Eaedem rationes ultimate sunt etiam rationes aequalitatis, ubi parallelogrammorum latitudines* AB, BC, CD, &c. *sunt inaequales, & omnes minuuntur in infinitum.

Sit enim *AF* aequalis latitudini maximae, & compleatur parallelogrammum *FAaf*. Hoc erit maius quam differentia figurae inscriptae & figurae circumscriptae; at latitudine sua *AF* in infinitum diminuta, minus fiet dato quovis rectangulo. *Q. E. D.*

Corol. I. Hinc summa ultima parallelogrammorum evanescentium coincidit omni ex parte cum figura curvilinea.

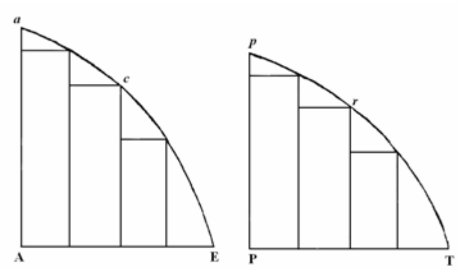
Corol. 2. Et multo magis figura rectilinea, quae chordis evanescentium arcuum *ab, bc, cd,* &c. comprehenditur, coincidit ultimo cum figura curvilinea.

Corol. 3. Ut & figura rectilinea circumscripta quae tangentibus eorundem arcuum comprehenditur.

Corol. 4. Et propterea hae figurae ultimae (quoad perimetros *ac*E,) non sunt rectilineae, sed rectilinearum limites curvilinei.

LEMMA IV.

Si in duabus figuris* AacE, PprT, *inscribantur (ut supra) duae parallelogrammorum series, sitque idem amborum numerus, & ubi latitudines in infinitum diminuuntur, rationes ultimae parallelogrammorum in una figura ad parallelogramma in altera, singulorum ad singula, sint eaedem; dico quod figurae duae* AacE, PprT, *sunt ad invicem in eadem illa ratione.



Etenim ut sunt parallelogramma singula ad singula, ita (componendo) sit summa omnium ad summam omnium, & ita figura ad figuram; existente nimirum figura priore (per lemma III) ad summam priorem, & figura posteriore ad summam posteriorem in ratione aequalitatis. *Q. E. D.*

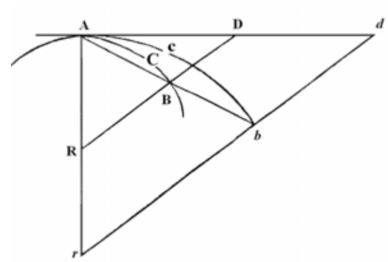
Corol. Hinc si duae cujuscunque generis quantitates in eundem partium numerum utcunque dividantur; & partes illae, ubi numerus earum augetur & magnitudo diminuitur in infinitum, datam obtineant rationem ad invicem, prima ad primam, secunda ad secundam, caeteraeque suo ordine ad caeteras: erunt tota ad invicem in eadem illa data ratione. Nam si in lemmatis hujus figuris sumantur parallelogramma inter se ut partes, summae partium semper erunt ut summae parallelogrammorum; atque ideo, ubi partium & parallelogrammorum numerus augetur & magnitudo diminuitur in infinitum, in ultima ratione parallelogrammi ad parallelogrammum, id est (per hypothesin) in ultima ratione partis ad partem.

LEMMA V.

Similium figurarum latera omnia, quae sibi mutuo respondent, sunt proportionalia, tam curvilinea quam rectilinea: & areae sint in duplicata ratione laterum.

LEMMA VI.

Si arcus quilibet positione datus* AcB *subtendatur chorda* AB, *& in puncto aliquo* A, *in medio curvae continuae, tangatur a recta utrinque producta* AD ; *dein puncta* A, B *ad invicem accedant & coëant ; dico quod augulus* BAD, *sub chorda* & *tangente contentus, minuetur in infinitum ac ultimo evanescet.

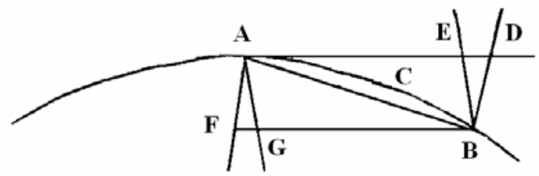


Nam si angulus ille non evanescit, continebit arcus *AcB* cum tangente *AD* angulum rectilineo aequalem, & propterea curvatura ad punctum *A* non erit continua, contra hypothesin.

LEMMA VII.

Iisdem positis; dico quod ultima ratio arcus, chordae, & tangentis ad invicem est ratio aequalitatis.

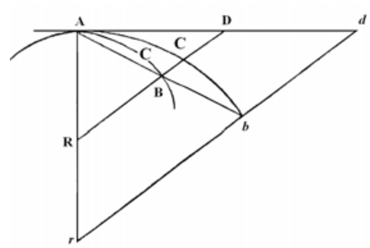
Nam dum punctum *B* ad punctum *A* accedit, intelligantur semper *AB* & *AD* ad puncta longinqua *b* ac *d* produci, & secanti *BD* parallela agatur *bd.* Sitque arcus *Acb* semper similis arcui *A CB.* Et punctis *A, B* coeuntibus, angulus *dAb*, per lemma superius, evanescet; ideoque rectae semper finitae *Ab, Ad,* & arcus intermedius *Acb* coincident, & propterea aequales erunt. Unde & hisce semper proportionales rectae *AB, AD,* & arcus intermedius *ACB* evanescent, & rationem ultimam habebunt aequalitatis. *Q. E. D.*



Corol. I. Unde si per *B* ducatur tangenti parallela *BF,* rectam quamvis *AF* per *A* transeuntem perpetuo secans in *F,* haec *BF* ultimo ad arcum evanescentem *ACB* rationem habebit aequalitatis, eo quod completo parallelogrammo *AFBD* rationem semper habet aequalitatis ad *AD.*
Corol. 2. Et si per *B* & *A* ducantur plures rectae *BE, BD, AF, AG,* secantes tangentem *AD* & ipsius parallelam *BF;* ratio ultima abscissarum omnium *AD, AE, B F, BG,* chordaeque & arcus *AB* ad invicem erit ratio aequalitatis.
Corol. 3. Et propterea hae omnes lineae, in omni de rationibus ultimis argumentatione, pro se invicem usurpari possunt.

LEMMA VIII.

Si rectae datae* AR, BR *cum arcu* ACB, *chorda* AB & *tangente* AD, *triangula tria* RAB, RACB, RAD *constituunt, dein puncta* A, B *accedunt ad invicem: dico quod ultima forma triangulorum evanescentium est similitudinis, & ultima ratio aequalitatis.

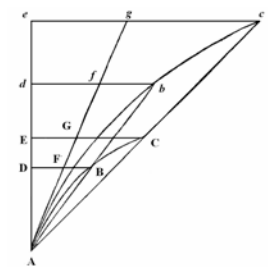


Nam dum punctum *B* ad punctum *A* accedit, intelligantur semper *AB, AD, AR* ad puncta longinqua *b, d* & *r* produci, ipsique *RD* parallela agi *rbd,* & arcui *ACB* similis semper sit arcus *Acb.* Et coeuntibus punctis *A, B,* angulus *bAd* evanescet, & propterea triangula tria semper finita *rAb, rAcb, rAd* coincident, suntque eo nomine similia & aequalia. Unde & hisce semper similia & proportionalia *RAB, RACB, RAD* fient ultimo sibi invicem similia & aequalia. *Q. E. D.*

Corol. Et hinc triangula illa, in omni de rationibus ultimis argumentatione, pro se invicem usurpari possunt.

LEMMA IX.

Si recta* AE *& curva* ABC *positione datae se mutuo secent in angulo dato* A, *& ad rectam illam in alio dato angulo ordinatim applicentur* BD, CE, *curvae occurrentes in* B, C, *dein puncta* B, C *simul accedant ad punctum* A: *dico quod areae triangulorum* ABD, ACE *erunt ultimo ad invicem in duplicata ratione laterum.



Etenim dum puncta *B, C* accedunt ad punctum *A*, intelligatur semper *AD* produci ad puncta longinqua *d* & *e*, ut sint *Ad, Ae* ipsis *AD, AE* proportionales, & erigantur ordinatae *db*, *ec* ordinatis *DB, EC* parallelae quae occurrant ipsis *AB, AC* productis in *b & c*. Duci intelligatur, tum curva *Abc* ipsi *ABC* similis, tum recta *Ag*, quae tangat curvam utramque in *A*, & secet ordinatim applicatas *DB, EC, db, ec* in *F, G, f, g*. Tum manente longitudine *Ae* coeant puncta *B, C* cum puncto *A;* & angulo *cAg* evanescente, coincident areae curvilineae *Abd, Ace* cum rectilineis *Afd*, *Age*; ideoque (per lemma v.) erunt in duplicata ratione laterum *Ad, Ae* : Sed his areis proportionales semper sunt areae *ABD, ACE, &* his lateribus latera *AD, A E.* Ergo & areae *ABD, ACE* sunt ultimo in duplicata ratione laterum *AD, AE. Q. E. D.*

LEMMA X.

Spatia quae corpus urgente quacunque vi finita describit, sive vis illa determinata & immutabilis fit, sive eadem continuo augeatur vel continuo diminuatur, sunt ipso motus initio in duplicata ratione temporum.

Exponantur tempora per lineas *AD, AE,* & velocitates genitae per ordinatas *DB, EC;* & spatia his velocitatibus descripta, erunt ut areae *ABD, ACE* his ordinatis descriptae, hoc est, ipso motus initio (per lemma IX) in duplicata ratione temporum *AD, A E. Q. E. D.*

Cor I. Et hinc facile colligitur, quod corporum similes similium figurarum partes temporibus proportionalibus describentium errores, qui viribus quibusvis aequalibus ad corpora similiter applicatis generantur, & mensurantur per distantias corporum a figurarum similium locis illis, ad quae corpora eadem temporibus iisdem proportionalibus sine viribus illis pervenirent, sunt ut quadrata temporum in quibus generantur quam proxime.

Corol. 2.Errores autem qui viribus proportionalibus ad similes figurarum similium partes similiter applicatis generantur, sunt ut vires & quadrata temporum conjunctim.

Corol. 3. Idem intelligendum ea de spatiis quibusvis quae corpora urgentibus diversis viribus describunt. Haec sunt, ipso motus initio, ut vires & quadrata temporum conjunctim.

Corol. 4. Ideoque vires sunt ut spatia, ipso motus initio, descripta directe & quadrata temporum inverse.

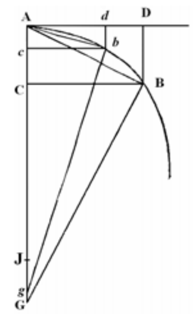
Corol.5. Et quadrata temporum sunt ut descripta spatia directe & vires inverse.

Scholium.

Si quantitates indeterminatae diversorum generum conferantur inter se, & earum aliqua dicatur esse ut est alia quaevis directe vel inverse: sensus est, quod prior augetur vel diminuitur in eadem ratione cum posteriore, vel cum ejus reciproca. Et si earum aliqua dicatur esse ut sunt aliae duae vel plures directe vel inverse: sensus est, quod prima augetur vel diminuitur in ratione quae componitur ex rationibus in quibus aliae vel aliarum reciprocae augentur vel diminuuntur. Ut si A dicatur esse ut B directe & C directe & D inverse: sensus est, quod A augetur vel diminuitur in eadem ratione cum $\mathrm{B} \times \mathrm{C} \times \frac{1}{\mathrm{D}}$ hoc est, quod A & $\frac{\mathrm{BC}}{\mathrm{D}}$ sunt ad invicem in ratione data.

LEMMA XI.

Subtensa evanescens anguli contactus, in curvis omnibus curvaturam finitam ad punctum contactus habentibus, est ultimo in ratione duplicata subtensae arcus contermini.



Cas. I. Sit arcus ille *A B*, tangens eius *AD*, subtensa anguli contactus ad tangentem perpendicularis *BD*, subtensa arcus *AB*. Huic subtensae *AB* & tangenti *AD* perpendiculares erigantur *AG*, *BG*, concurrentes in *G;* dein accedant puncta *D*, *B*, *G*, ad puncta *d*,*b*,*g*, sitque *J* intersectio linearum *BG*, *AG* ultimo facta ubi puncta *D*, *B* accedunt usque ad *A*. Manifestum est quod distantia G*J* minor esse potest quam assignata quaevis. Est autem (ex natura circulorum per puncta *ABG*, *Abg* transeuntium) *AB quad.* aequale $AG \times BD$, & *Ab quad.* aequale $Ag \times bd$ *;* ideoque ratio *AB quad.* ad *Ab quad.* componitur ex rationibus *AG* ad *Ag* & *BD* ad *bd.* Sed quoniam G*J* assumi potest minor longitudine quavis assignata, fieri potest ut ratio *AG* ad *Ag* minus differat a ratione aequalitatis quam pro differentia quavis assignata, ideoque ut ratio *AB quad.* ad *Ab quad.* minus differat a ratione *BD* ad *bd* quam pro differentia quavis assignata. Est ergo, per lemma I, ratio ultima *AB quad.* ad *Ab quad.* eadem cum ratione ultima *BD* ad *bd. Q. E. D.*

Cas. 2. Inclinetur iam *BD* ad *AD* in angulo quovis dato, & eadem semper erit ratio ultima *BD* ad *bd* quae prius, ideoque eadem ac *AB quad.* ad *Ab quad. Q. E. D.*

Cas. 3. Et quamvis angulus *D* non detur, sed recta *BD* ad datum punctum convergat, vel alia quacunque lege constituatur; tamen anguli *D*, *d* communi lege constituti ad aequalitatem semper vergent & propius accedent ad invicem quam pro differentia quavis assignata, ideoque ultimo aequales erunt, per lem. I, & propterea linaea *BD, bd* sunt in eadem ratione ad invicem ac prius. *Q. E. D.*

Corol. 1. Unde cum tangentes *AD, Ad,* arcus *AB, Ab*, & eorum sinus *BC, bc* fiant ultimo chordis *AB*, *Ab* aequales; erunt etiam illorum quadrata ultimo ut subtensae *BD, bd.*

Carol. 2. Eorundem quadrata sunt etiam ultimo ut sunt arcuum sagittae, quae chordas bisecant & ad datum punctum convergunt. Nam sagittae illae sunt ut subtensae *BD.bd.*

Corol. 3. Ideoque sagitta est in duplicata ratione temporis quo corpus data velocitate describit arcum.

Carol. 4. Triangula rectilinea *ADB, Adb* sunt ultimo in triplicata ratione laterum *AD, Ad,* inque sesquiplicata laterum *DB, db;* utpote in composita ratione laterum *AD & DB, Ad* & *db* existentia. Sic & triangula *ABC, Abc* sunt ultimo in triplicata ratione laterum *BC, bc.* Rationem vero sesquiplicatam voco triplicatae subduplicatam, quae nempe ex simplici & subduplicata componitur.

Corol.5. Et quoniam *DB, db* sunt ultimo parallelae & in duplicata ratione ipsarum *AD, Ad :* erunt areae ultimae curvilinaea *ADB, Adb* (ex natura parabolae) duae tertiae partes triangulorum rectilineorum *ADB, Adb;* & segmenta *AB, Ab* partes tertiae eorundem triangulorum. Et inde hae areae & haec segmenta erunt in triplicata ratione tum tangentium *AD, Ad;* tum chordarum & arcuum *AB,Ab.*

Scholium.

Caeterum in his omnibus supponimus angulum contactus nec infinite majorem esse angulis contactuum, quos circuli continent cum tangentibus suis, nec iisdem infinite minorem; hoc est, curvaturam ad punctum *A*, nec infinite parvam esse nec infinite magnam, seu intervallum *AJ* finitae esse magnitudinis. Capi enim potest *DB* ut AD^3*:* quo in casu circulus nullus per punctum *A* inter tangentem *AD* & curvam *AB* duci potest, proindeque angulus contactus erit infinite minor circularibus. Et simili argumento fi fiat *DB* successive ut AD^4, AD^5, AD^6, AD^7 ,&c. habebitur series angulorum contactus pergens in infinitum, quorum quilibet posterior est infinite minor priore. Et si fiat *DB* successive ut AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$,&c. habebitur alia series infinita angulorum contactus, quorum primus est eiusdem generis cum circularibus, secundus infinite major, & quilibet posterior infinite major priore. Sed & inter duos quosvis ex his angulis potest series utrinque in infinitum pergens angulorum intermediorum inseri, quorum quilibet posterior crit infinite major minorve priore. Ut si inter terminos AD^2 & AD^3 inferatur series $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{6}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$,&c. Et rursus inter binos quosvis angulos hujus seriei inseri potest series nova angulorum intermediorum ab invicem infinitis intervallis differentium. Neque novit natura limitem.

Quae de curvis lineis deque superficiebus comprehensis demonstrata sunt, facile applicantur ad solidorum superficies curvas & contenta. Praemisi vero haec lemmata, ut effugerem taedium deducendi longas demonstrationes, more veterum geometrarum, ad absurdum. Contractiores enim redduntur demonstrationes per methodum indivisibilium. Sed quoniam durior est indivisibilium hypothesis, & propterea methodus illa minus geometrica censetur; malui demonstrationes rerum sequentium ad ultimas quantitatum evanescentium summas & rationes, primasque nascentium, id est, ad limites summarum & rationum deducere; & propterea limitum illorum demonstrationes qua potui brevitate praemittere. His enim idem praestatur quod per methodum indivisibilium; & principiis demonstratis jam tutius utemur. Proinde in sequentibus, siquando quantitates tanquam ex particulis constantes consideravero, vel si pro rectis usurpavero lineolas curvas; nolim indivisibilia, fed evanescentia divisibilia, non summas & rationes partium determinatarum, sed summarum & rationum limites semper intelligi ; vimque talium demonstrationum ad methodum praecedentium lemmatum semper revocari.

Objecto est, quod quantitatum evanescentium nulla sit ultima proportio; quippe quae, antequam evanuerunt, non est ultima, ubi evanuerunt, nulla est. Sed & eodem argumento aeque contendi posset nullam esse corporis ad certum locum, ubi motus finiatur, pervenientis velocitatem ultimam: hanc enim, antequam corpus attingit locum, non esse ultimam, ubi attingit, nullam esse. Et responsio facilis est : Per velocitatem ultimam intelligi eam, qua corpus movetur, neque antequam attingit locum ultimum & motus cessat, neque postea, sed tunc cum attingit; id est, illam ipsam velocitatem quacum corpus attingit locum ultimum & quacum motus cessat. Et similiter per ultimam rationem quantitatum evanescentium, intelligendam esse rationem quantitatum, non antequam evanescunt, non postea, sed quacum evanescunt. Pariter & ratio prima nascentium est ratio quacum nascuntur. Et summa prima & ultima est quacum esse (vel augeri aut minui) incipiunt & cessant. Extat limes quem velocitas in fine motus attingere potest, non autem transgredi. Haec est velocitas ultima. Et par est ratio limitis quantitatum & proportionum omnium incipientium & cessantium. Cumque hic limes sit certus & definitus, problema est vere geometricum eundem determinare. Geometrica vero omnia in allis geometricis determinandis ac demonstrandis legitime usurpantur.

Contendi etiam potest, quod si dentur ultimae quantitatum evanescentium rationes, dabuntur & ultimae magnitudines: & sic quantitas omnis constabit ex indivisibilibus, contra quam *Euclides* de incommensurabilibus, in libro decimo elementorum, demonstravit. Verum haec objectio falsae innititur hypothesi. Ultimae rationes illae quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum, sed limites ad quos quantitatum sine limite decrescentium rationes semper appropinquant; & quas propius assequi possunt quam pro data quavis differentia, nunquam vero transgredi, neque prius attingere quam quantitates diminuuntur in infinitum. Res clarius intelligetur in infinite magnis. Si quantitates duae quarum data est differentia augeantur in infinitum, dabitur harum ultima ratio, nimirum ratio aequalitatis, nec tamen ideo dabuntur quantitates ultimae seu maximae quarum ista est ratio. In sequentibus, igitur siquando facili rerum conceptui consulens dixero quantitates quam minimas, vel evanescentest vel ultimas; cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite.