ON THE AMAZING CANON OF LOGARITHMS with their outstanding use in trigonometry.

***BOOK* II b.** [p. 39.]

Concerning non-quadrantal Spherical Triangles.

Cap. V.

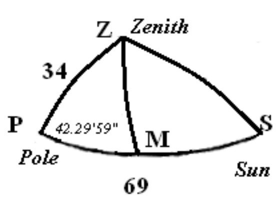
Up to this point, we have considered the principles of quadrantal spherical triangles, there now follows the principles of non-quadrantal spherical triangles.

1.*A non-quadrantal is a spherical triangle of which neither a side nor an angle is a right angle.*
2. *A non-quadrantal triangle can be reduced to two quadrantal triangles, if a perpendicular is sent from a vertex to the side opposite (which can be extended if the need arises), or a quadrant arc is sent across.*
3. *The perpendicular falls within the triangle, if the angles about the base are of the same kind* [i.e. both acute or both obtuse]***: and truly outside the triangle if they are of different kinds*** [e. g. one acute and one obtuse]***: and conversely.***
4. *A quadrant arc falls outside the triangle if both legs are of the same kind : and within if they are indeed of different kinds; and conversely.*
5.*From the six parts of a non-quadrantal triangle, three given only are sufficient to make known and to compare the remainder; except perhaps of the three given, of which one of the other kind is put in place, and the third shall be closer to being a quadrant, than the other given of the same kind. For in this case it is also required to be given the kind which is placed opposite the third part, in order that the remaining parts may be known.*[Some triangles require the supplement of the sine worked out.]
The fourth and sixth examples are of this kind.
6. ***The three parts given are either a mixture of the different kinds, or of the same kind.***
7. *Those are of a mixed kind, of which one is of a different kind from the remaining two.*
Since either two sides are given with some angle: or two angles are given with some side.
8.*For a spherical triangle with given mixed parts, if from that end of a given side the angle is given of the end of the remaining side,*[*i.e.* two sides and the included angle] ***the perpendicular may fall to the base or the quadrant subtending that given angle can be drawn, the triangle can be reduced from a non-quadrantal to two known quadrantals (by section 9 of Ch.4. of this book).***
Hence non-quadrantal parts (since they are in common with the parts of these quadrantals, or with the remaining parts in a semicircle) [p. 40] ***can easily become known, by finally being recognised from earlier sections 2, 3 and 4 of Ch.3 of this book, or by hypothesis from the kinds of the parts.***

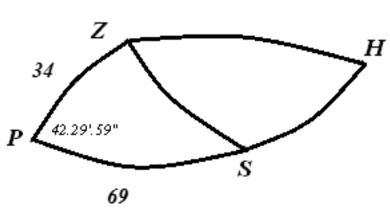
[**<u>A note regarding co-ordinates in spherical astronomy</u>** : Recall that the pole star is always present in the northern hemisphere, and it is the only object that is more or less fixed at the same location on the celestial sphere, even during the day, when of course it cannot normally be seen. The sun can be regarded as fixed in its apparent orbit around the earth, for the course of a given day, and thus it progresses around the pole star as do the fixed stars in the observer's reference frame, which thus rotate *en masse* aroung the celestial polar axis. The polar co-ordinates of the sun or a star in the observer's local reference frame are defined to be its altitude or angle from the horizontal, and azimuth or angle measured from the observer's meridian looking north, (the local meridian is the arc from south to north that passes through the pole star and the local zenith). The sun and stars rise in the east and reach a maximum altitude called the transit point as they pass through the observer's meridian, before falling and sinking in the west (some stars are always above the horizon in the northern and southern hemispheres; and of course we have lands of the midnight sun).

The corresponding coordinates on the celestial sphere, viewed as it were by an observer at the centre of the earth, are the declination from the celestial equator (in the plane perpendicular to the celestial polar axis), and the longitude or 'hour-angle' , the difference in angle of the sun's meridian from the transit point of the observer's meridian. There are simple relations between the coordinates in the two systems of coordinates. We need not delve further into the more specialised detail for the present work. See, *e.g. The VNR Concise Encyclopedia of Mathematics*. p. 277 (1977) for a good brief description of spherical astronomy, which also includes a good section on spherical trigonometry. The triangles dealt with in the *Descriptio* are now called nautical triangles : note that they are drawn on the surface of a sphere, and are not to be considered as 'flat'.]

An example of two given sides and the angle placed between them.



As an exercise in the use [of such spherical triangles], a non-quadrantal spherical triangle is described on the surface of the celestial sphere, P referring to the Pole, Z to the Zenth, and S to the sun : the six parts of which are, the side PZ, which is the interval between the pole and the zenith, or the complement of the elevation of the pole. The side ZS, the interval between the Zenith and the sun, or the complement of the altitude of the sun. The side PS, the interval between the pole and the sun, or the complement of the declination of the sun from the equator. The angle ZPS, the hour of the day, or the hours of time measured by the degrees from the celestial equator. The angle PZS, which is the azimuth, or the angle of the sun from north. The angle PSZ, which angle measured the position between the sun and the pole and the sun and the zenith.



A mixture of any three of these angles and sides can be given. For example, the hour angle ZPS of the 42.29'.59". (which is noted as 2 o'clock in the afternoon, 49 minutes, and 59.56''' seconds [recall that 15^0 of longitude is equivalent to 1 hour of time]) and the side PZ 34 degrees, is the complement of the elevation of the pole, and the side PS 69 degrees, is the complement of the elevation of the sun. From which, in order that the three remaining parts may be found, from the end Z of the given side PZ, the perpendicular ZM is sent, or, if you prefer, the quadrant ZH subtending the given angle ZPS, and reducing the non quadrantal oblate triangle PZS into two quadrantal triangles in the angle M, which are PMZ and ZMS, as in the first diagram: or (if variety pleases you) to two quadrantal triangles in the side ZH, which are ZHP and ZHS as in the second diagram:

All the parts of which quadrantal triangles you can find by [p. 41] section 9. of Ch.4 of this Book. For from the given PZ 34, and the angle ZPM, or ZPS 42.29'.59" you can find the perpendicular ZM 22.11'.47" and the angle PZM 52.46'.38" and the side PM 26.26'.29" : with which taken from PS 69, there remains MS 42.33'.31", from which and with the perpendicular ZM now known, by section 9 of Ch.4 of this Book, you can find the angle MSZ of this or the angle sought PSZ, 31.6'.5", and the side sought SZ 47, and the angle MZS 67.38'.11", from which by addition to PZM 52.46'.38", is the remaining angle sought PZS 120.24'.49". Thus these are the three parts sought with the help of the perpendicular ZM, of the first diagram.

You can also make the same use of the quadrant ZH in the second diagram. Indeed from PZ 34, and ZPS or ZPH 42.20'.59" given above, you can find the angle ZHP. 22.11'.47" and the angle PZH 142.46'.38" from the same section 9 of Ch. 4 of this Book and the side PH 116.26'.29". From which take awaw PS 69 and there remains SH 47.26'.29"; from which and from the angle about H 22.11'.47", that you now have, you can find by section 9 of Ch. 4 of this Book, the angle HSZ 148.53'.55", and the angle of this to the remaining angle sought PSZ to the semi-circle obviously 31.6'.5": and the side SZ sought 47. Hence by taking the angle HZS 22.21'.49" from from HZP 142.46'.38", there remains the angle sought of the remainder, PZS 120.24'.49", as above.

Summary.

By following this example, nine different kinds of problems can be solved for this and for any triangle. Indeed, from the given elevation of the pole, the hour of the day, and from the declination of the sun on that day, as above, can be found:
(1) the azimuth of the sun; (2) the altitude of the sun; and
(3) the angle of the position of the sun.

Likewise from the given sun's declination, the angle of the sun's position, and from the sun's altitude : (4) the azimuth of the sun is found; (5) the elevation of the pole;
(6), the hour of the sun.

From the given sun's altitude, the sun's azimuth, and elevation of the pole : we have (7) the hour of the day; (8) the declination of the sun; and then (9), the angle of the position of the sun.

[p. 42.]

A second example of two angles with the interposed side given.

From the given angles of the preceding diagrams, obviously the angle for the hours ZPS 42.29'.59", and for the azimuth of the sun PZS 120.24'.49" , with the complement of the elevation of the pole, with the interposed side PZ 34 of course. The three remaining sides are to be found. For you have firstly, as above, ZM 22.11'.47"; PM 26.26'.29", and with the angle PZM. 52.46'.38" taken from PZS. 120.24'.49", there remains the angle MZS 67.38'.11". And from ZM that you now know, you can finally find the length of the side sought ZS, 47, and ZSM, or the angle sought ZSP, 31.6'.5", and MS, 42.33'.31"; from which by adding to PM, the remaining side sought PS, 69 can be found. And these you have from the help of the perpendicular of the first diagram; and in the same way by means of the quadrant in the second diagram, you can acquire: (by section nine of the fourth chapter of this Book) from the given angles PHZ and PZH, and from this by subtracting the given PZS, there remains SZH, from which, and from the angle PHZ now known, all the other parts are made clear.

Summary.

By using this example, nine different kinds of questions can be solved, for this and for any other triangle. From the given hour of the day, as above, from the elevation of the pole, and from the sun's azimuth, can be found: (1) the declination of the sun;(2) the angle of the sun's position; and (3) the altitude of the sun.

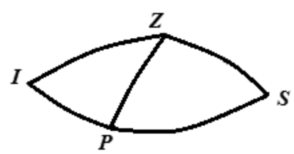
Likewise, from the given hour of the day, from the sun's declination, and from the angle of the sun's position; (4) the sun's altitude; (5) the sun's azimuth; (6) the elevation of the pole. Likewise from the given angle of the sun's position, with the sun's altitude, and with the azimuth of the sun , can be acquired : (7) the elevation of the pole; (8) the hour of the day; and (9), the declination of the sun.

[p. 43.]

The third example of two given sides, of which the side closer being a quadrant subtends a given angle.

The side PZ 34 and ZS 47, nearer to being a quadrant, are given in the preceding diagram, with that angle that it subtends ZPS 42. 29'.59". By section 9 of Ch.4 of this Book , ZX 22.11'.47", PZM 52.46'.38", and PM 26.26'.29" are found, and in a like manner, you will find ZSM, or the angle sought ZSP certainly here known by the minor quadrant, from the 2nd basic ideas of Ch. 3 of this Book, to be 31.6'.5" and not the supplement 148.53'.55". Also you find the angle MZS 67.38'.11", from which by adding to PZM 52.46'.38", the sought angle PZS 120.24'.49" is left. Then you have MS 42.33'.31" which added to PM. 26.26'.29", gives the required side PS, 69. In the same manner you can find the same quantities, if it pleases, with the aid of the quadrant ZH of the second diagram.

The fourth example of two given sides, of which the side further from being a quadrant subtends a given angle: moreover the side closer to being a quadrant subtends a given angle of such a kind.



The sides ZS, 47, and the side less close to being a quadrant PZ 34, with that angle which this subtends ZSP 31.6'.5", and being given that angle which ZS subtends (obviously the angle SPZ.) it is less than a quadrant: Thus by sending the perpendicular ZM from Z to the base PS (as done previously) or with the quadrant ZI (as here) subtending the given angle ZSP. The remaining parts are found by section 9, Ch.4 of this Book, as (for variation and as an exercise) from the quadrant ZI of this diagram the angle ZIS, 22.11'.47" is found, and IZS, 157.38'.11", and SI, 132.33'.31", and in a similar manner you can have the angle IPZ and as a consequence [p.44.] the angle sought SPZ, 42. 29'. 59". Since by hypothesis, it is expressly declared to be less than a quadrant, for to be given a kind otherwise, it would be uncertain (from the beginning of Ch.3, and the fifth section of this chapter) : for it could have otherwise 137.30'.1". You also thus will have the angle IZP, 37.13'.22", from which by taking IZS, 157.38'.11", the remaining angle sought PZS, 120.24'.49" is left. Finally you have both IP, 63.33'.31" , and from which is taken IS, 132.33'.31", and there remains the side PS, 69.

Also, you will get the same points, if you seek the computation of the parts of the triangle, with the help of the perpendicular Z.M of the first diagram.

Summary.

By following the third and fourth examples, 18 different kinds of questions relating to triangles can be solved. For from the given elevation of the pole, (as in the third example), from the altitude of the sun, and the said hour : (1) the azimuth of the sun; (2) the angle of the sun's position; and (3) the sun's declination can be found.

Likewise, from the elevation of the pole, the altitude of the sun, and the angle of the sun's position (as in the fourth example here), there may be found : (4) the sun's azimuth, (5) the hour of the day, (6) the sun's declination.

Likewise from the sun's altitude, its declination, and the hour of the day, there may be found : (7) the angle of the sun's position; (8) the sun's azimuth; and (9) the elevation of the pole.

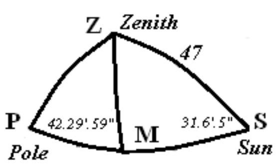
Likewise from the sun's given altitude, declination, and azimuth, there may be found : (10) the angle of the sun's position; (11) the hour of the day; and(12) the elevation of the pole.

Likewise, from the given sun's declination, the elevation of the pole, and the angle of the sun's position, there may be found : (13) the sun's azimuth, (14) the sun's altitude, and (15) the hour of the day.

Likewise, from the given sun's declination, elevation of the pole, and the sun's azimuth, there may be found : (16) the hour of the day;(17) the angle of the sun's position; and (18) the sun's altitude.

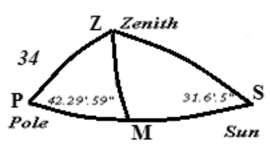
[p. 45.]

The fifth example of two given angles, of which the one closer to the quadrant subtends a given side.



In Triangle PZS in the first diagram, the angle PSZ 31.6'.5" is given, and the angle nearest to the quadrant SPZ 42. 29'. 59", thus subtended by the side ZS 47, are given. From which given PSZ and ZS can be found (by section 9 of Ch.4 of this Book) : the perpendicular ZM, 22.11'.47" and the other four parts of the quadrant S ZM, surely MZS, 67.38'.11", and MS, 42. 33'. 31". Thus from the perpendicular here with the given ZPS or with the angle ZPM, all the parts of ZMP are found from the quadrantal. In the first place, one may know the side PZ sought is here certainly known to be less than the quadrant (from basics 2 of Ch.3 of this Book), and is seen to be 34 degrees and not 146 degrees. Then PZM, 52.46'.38", can be found, which by adding to SZM, 67.38'.11", is the angle sought PZ S, 120.24'.49". Finally PM is found to be 26.26'.29", which on adding to MS, 42.33'.31", is the size of the remaining side sought PS, 69. These parts you can also acquire if you so wish otherwise from the two diagrams close together with the quadrantals ZIS and ZIP.

The sixth example of two given angles, of which the angle which is the less close to being a quadrant subtends the given side, but the angle closer to the quadrant subtends a given side of such a size.



In the triangle PZS of the first diagram, the angle ZPS, 42. 29'. 59" is given, and the angle furthest from being a quadrant ZSP, 31.6'.5" is given; with that angle being subtended by the side PZ, 34. And that angle ZPS being subtended, (truly by the side ZS) is less than a quadrant. From these given, the perpendicular is found, ZM, 22.11'.47", and all the parts of the quadrantal PZM, as it were, PZM, 52.46'.38", PM, 26.26'.29". In the same manner, from this perpendicular and with the given ZSM or ZSP, 31.6'.5", all the parts of the quadrantal ZMS are sought, surely in the first place the side ZS is chosen as ZS, 47 degrees, since by hypothesis it is declared expressly to be less than a quadrant [p. 46.], otherwise it would need to be 133 degrees. For (by 1 of Ch.3 and five of this) the kind is uncertain unless expressly given. Hence the angle MZS, 67.38'.11", which added to the angle MZP, 52.46'.38", gives the chosen angle PZS, 120.24'.49". And finally there is found SM, 42. 33'. 31". Which added to PM, 26.26'.29", is the chosen base PS, 69 degrees. You can also easily acquire also the same parts from the two quadrantals PHZ and SHZ of the second diagram.

Summary.

Form the preceding five and the present sixth example used as models, eighteen variations of this question on this and any other triangle can be solved. For from the given angle of the position of the sun, the hour of the day, and from the altitude of the sun (as in the fifth example) : (1) the elevation of the pole; (2) the sun's azimuth; and (3) the sun's declination, can all be found.
Likewise, from the given hour of the day, the angle of the sun's position, and the elevation of the pole, (as in this sixth example) : (4) the altitude of the sun; (5) the sun's azimuth; and (6) the declination of the sun, can all be found.
Likewise, from the given hour of the day, the sun's azimuth, and the altitude of the sun ; (7) the sun's declination; (8) the angle of the sun's position; and (9) the elevation of the pole, can all be found.
Likewise, from the given hour of the day, the sun's azimuth, and the sun's declination : (10) the altitude of the sun; (11) the angle of the sun's position; and (12) the elevation of the pole, can all be found.
Likewise from the given azimuth of the sun, the angle of the sun's position, and the declination of the sun : (13) the elevation of the pole; (14) the hour of the day; and (15) the altitude of the sun, can all be found.
Likewise from the given azimuth of the sun, the angle of the position of the sun, and the elevation of the pole : (16) the declination of the sun; (17) the hour of the day; and (18) the altitude of the sun, can all be found. And thus by the method of this canon alone, fifty four different questions of the same non-quadrantal triangle can be solved. The rest are solved below.

9. Thus from these it is apparent that of two angles and of their subtended sides, with three given, even the logarithm of the fourth can be found, with no mention of the quadrantal triangle. [p. 47.] ***For from the sum of the logarithms of the angles and the sides of the given adjacent parts, take the logarithm of the third given part, and there then arises the logarithm of the fourth part sought, and the fourth part itself is known, unless it is of an uncertain kind.***

As from the above examples, the third, fourth, fifth or sixth can be taken. Indeed from the bases of the angles ZPS and ZSP, and of their subtended legs ZS and ZP, three are given, which (for the sake of an example) are the legs ZS, 47 degrees, the logarithm of this is 3128580, and ZP, 34 degrees, and the logarithm of this is 5812606, with the angle adjacent to this, ZPS, 42.29'.50", and the logarithm of this 3921720, added to 5812606, is 9734326 (The logarithm surely of the silent and suppressed perpendicular ZM, or of the angle ZHS or ZIP) from which take 3128580 and there remains 6605746 , the logarithm of the fourth part sought ZSP. Thus the fourth part sought ZSP is itself 31.6'.5".Since by sect. 2. of Ch.3. it is proven to be less than the quadrant. But conversely, from the given ZP, 34 degrees, the logarithm of this is 55812606 and ZS, 47 degrees , and the logarithm of this is 3128580, with the angle adjacent to this ZSP, 31.6'.5" , to the logarithm of this, 6605746, add 3128580, and the sum is as above, 9734326, from which take 5812606, and there arrives 3921720, the logarithm of the fourth side sought, surely ZPS, and the arc of this is uncertain by sect.1 of Ch. 3 , either it is 42.29'.50" or 137.30'.1"; unless it is declared by hypothesis to be either larger or smaller than a quadrant.

Concerning Pure Non-quadrantal Triangles.

Ch. VI.

Up to this stage we have been concerned with given mixed parts: Those that follow are pure.

1. ***There are three pure parts of the same given kind. There are either three sides given, and the angles are sought; or there are three angles given, and the sides are sought.***

Note.

2. ***Although pure parts are in the first place simple, yet on account of the difficulty of the same it has been chosen to consider them here later.*** [p. 48.]

3. ***In spherical triangles, in the first place, the sum of the logarithms of the legs is to be taken from the sum of the logarithms, which in turn is the sum and difference of half the base and half the differences of the legs, and there remains twice the logarithm of half the vertical angle.***

[The half-angle formulae from plane trigonometry can be carries over into the spherical case, where the lengths of the sides of a plane triangle are replaced by the corresponding arcs. Thus, if the base is a and α is associated the vertical angle, while the legs are b and c, then

$$\sin^2\frac{\alpha}{2} = \frac{\sin\left(\frac{a+c-b}{2}\right)\sin\left(\frac{a+b-c}{2}\right)}{\sin b \sin c} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}$$

gives the above result, where we note that the logarithm of an angle above refers of course to the logarithm of the sine of that angle, according to the use in the tables; and so on with the cyclic permutation of the vertices and sides. Recall that $2s = a + b + c$.]

Since *Regiomontanus* taught in Book 5, Ch. 2. *De triangulis*, and others, as the rectangle made from the right sines of the legs, is to the square of the total sine : thus as the difference of the versed sines of the base and of the legs have to the versed sine of the vertical angle; moreover since as that difference of versed sines is to this versed sine, is thus as the rectangle made from the right sines of the sum and difference of the semibases and the legs, is to the square of the right sine of half the angle to the vertical (for the first rectangle is to that difference of versed sines , and this final square to that versed sine in the ratio 5000000 $^{\text{fold}}$,(five thousand thousand times) with the whole sine given as 10000000.

[For we can write $2R\sin^2\frac{\alpha}{2}/R^2 = \frac{2\sin\left(\frac{a+c-b}{2}\right)\sin\left(\frac{a+b-c}{2}\right)}{\sin b\sin c} = \frac{2\sin(s-b)\sin(s-c)}{\sin b\sin c}$, and

hence taking R = 10^7, we have $\sin^2\frac{a}{2}/5000000 = \frac{2\sin\left(\frac{a+c-b}{2}\right)\sin\left(\frac{a+b-c}{2}\right)}{\sin b\sin c} = \frac{2\sin(s-b)\sin(s-c)}{\sin b\sin c}$.]

Thus it follows that, as the rectangle under the right sines of the legs is to the square of the total sine, thus the rectangle made by the right sines of the sum and difference of the half base and half the difference of the sides, is to the square of the sine of the half angle to the vertical: and as a consequence the right sine of the half angle of the vertical (from corol. to the definitions in Ch.1, and from Prop.4, Ch.2., and problem 3, Ch. 5, Book1.) The sum from the logarithms of the legs, taken from the logarithms of the sum and difference of the half-base and the half-difference of the legs, leaves twice the logarithm of the half-angle to the vertical, as above.

[Thus, the right-hand side of the ratios in proportion is simply sin$b$.sin$c$, while the right-hand side is the difference of the versed sine of the base and of the difference of the sides b and c. The versed sine of θ or versinθ = 1 – cosθ; hence the numerator of the right-hand side of the ratio is given by :
$\operatorname{ver}\sin a - \operatorname{ver}\sin(b-c) = 1-\cos a - 1 + \cos(b-c) = 2\sin\frac{a+b-c}{2}\sin\frac{a-b+c}{2}$; while the denominator is given by $\operatorname{ver}\sin\alpha = 1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$, from which the result follows. Note that the text uses a radius R of 10^7 rather than unity.]

4. *In the second place the sum from the logarithms of the legs taken from the sum from the logarithms of the sum and difference of the half-base and the half-difference of the legs, leaves the cosine of the half angle to the vertical.*

[In this case, $\cos^2\dfrac{\alpha}{2} = \dfrac{\sin\left(\dfrac{a+c+b}{2}\right)\sin\left(\dfrac{b+c-a}{2}\right)}{\sin b\sin c} = \dfrac{\sin(s)\sin(s-a)}{\sin b\sin c}$.]

For indeed none other is found from the sum of the logarithms of the sum and difference of the semibase and of the half sum of the legs of this proposition, to the similar sum and difference of the last proposition, than twice the cosine of half the angle to the vertical that is given here, to twice the logarithm of the sine of the half angle given above, which is demonstrated elsewhere. [p. 49.]

Notes.

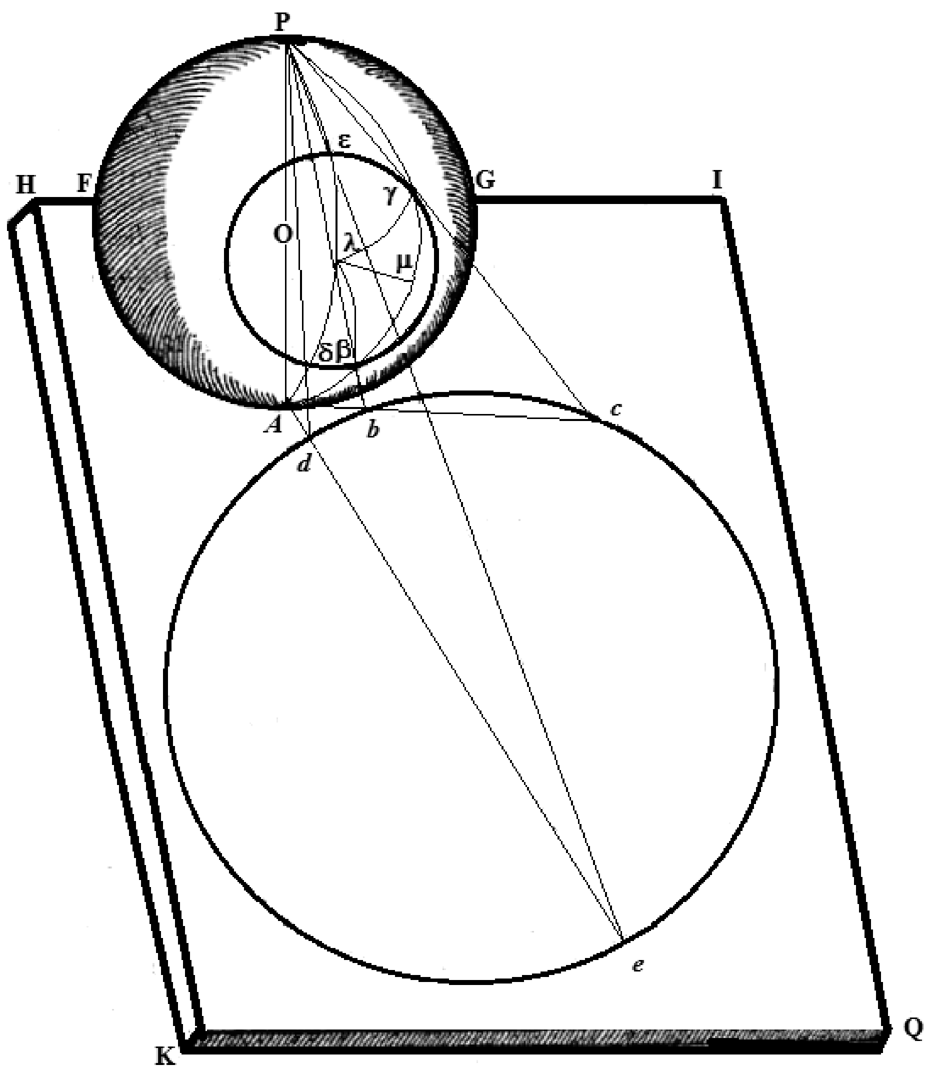
5. ***In Spherical trigonometry also, we take the true base and the other part in the same sense as with plane trigonometry; truly the one for the case of the sum of the logarithms of the sines and the other for the case of the logarithm of the difference of the sines.***[That is, the a is used in the formula with b and c in the argument of the sine or cosine, while α used in the formulae for tangents, etc.]

6. ***In the third place, the logarithm of the tangent of the true half base given, taken from the sum of the logarithms of the tangents of the half sum and the half difference of the legs, leaves the logarithm of the tangent of the other semibase.***

The ratio of this is fundamental, that as the tangent of the true half-base is to the tangent of the half-sum of the legs, thus as the tangent of the half-difference of the legs is to the tangent of the other half-base. $\frac{\tan a/2}{\tan(\frac{b+c}{2})} = \frac{\tan(\frac{b-c}{2})}{\tan\alpha/2}$. For the logarithms of tangents are the logarithms of the differences of their arcs, by sections 22 and 25 of Ch.3, Book 1.

Thus this ratio of the tangents follows that of their logarithms, or the equality of their difference by Prop.4 of Ch.2, Book I: Truly because of this fundamental ratio of the tangents, unknown until now, perhaps readers will require a demonstration by me, that therefore we will explain here, as much as this short compendium will bear.

Thus the sphere AFPG rests on the plane HIKQ so that they touch each other at the



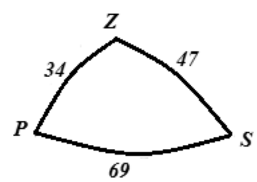
common point A, from which through the centre of the sphere O the line AOP is erected cutting the top of the hemisphere of the sphere at the point P [on the equator]. And AOP is perpendicular to the plane HIKQ. Then from the angle A on the surface of the sphere, the triangle Aλγ is described with γ acute, or Aλβ obtuse in β. The [longitude] semicircles AλP and AγP or AβP are drawn from the λ, taken as the pole, with the interval λγ and λβ equal [equal zenith angles] ; and the [latitude] circle εδβγ is drawn, cutting λP in ε and λA in Aβγ at the points β and γ. From the point λ to the arc [of the great circle] Aβγ, the arc λμ is sent down perpendicular. Thus, here the leg Aλ is the greater leg [*i. e.* the quadrant], and λγ or λβ the lesser leg of the spherical triangle, Aγ and Aβ the bases, the one true, and the other of the alternate kind [according as whether

the spherical triangle has acute angles to the base or an obtuse angle]; Aδ is the difference of the legs, and Aε is the sum of the legs, since λε and λδ from the construction, are equal to the lesser of the legs λγ or γβ. With these constructions completed, and by supposing P to be put in the place of the eye, and the same point to generate light rays which are sent from P to the plane HIKQ, the ray Pγ cutting the plane in *c*, and the ray Pβ cutting the plane in *b*: and since γβA is on the same plane or with the circle as the light from P, their shadows *cb*A are on the same straight line [i. e. they project onto a straight line on the plane HIKQ]. Similarly from the same point P, in the same plane, the ray P*ε* is sent cutting the plane in *e*, and the ray γδ cutting the plane in *d* , and since εδA are in the same plane with the circle as with the light from P : thus their shadows *ed*A lie on the same straight line. Besides, since POA, is [p. 51.]
in the orthogonal plane or at right angles, thus triangles PA*d* and PA*e,* and PA*b* and PA*c* are right angled at A, and thus also A*d* is a tangent of the angle APδ or AP*d,* and A*e* is a tangent of the angle APε or AP*e,* thus also A*b* is a tangent of the angle APβ or AP*b*, and A*c* is a tangent of the angle APγ or AP*c*, with the supposition of PA as the gnome or total sine, and since A*d* is a tangent of the angle APδ, and APδ is half of the angle AOδ by Prop.20, Book 3 of Euclid (since this angle is at the centre of a circle, and the other on the circumference of the same circle), therefore A*d* is the tangent of the half angle AOδ, or (as it is the same thing) of half the arc Aδ, which is the half-difference of the legs. Similarly, since A*e* is the tangent of the angle APε, moreover the angle APε on the circumference is half the angle AOε at the centre of the great circle, thus Aε is the tangent of half AOε, or half the arc Aε, which is the half-sum of the legs. In like manner, with the bases, the true and the alternate, A*b* is the tangent of the angle APβ, or of half the angle AOβ, or half the arc Aβ, which is the one half-base : and A*c* is the tangent of the angle APγ, or of the half angle AOγ, or of the half arc Aγ, which is the remaining half base. And since now it has been shown that A*b* is the tangent of the one half base, and A*c* the tangent of the remaining half base, and A*d* is the half difference of the legs, and A*e* the tangent of the half sum of the legs. I say that :
As A*b* the tangent of the true half base is itself to the tangent A*e* of the half sum of the legs, thus A*d* the tangent of the half difference of the legs is to A*c,* the tangent of the other half base : or conversely by interchanging the true and alternate bases ; as the tangent A*c* of the true half base is itself to the tangent A*e* of the half sum of the legs : Thus the tangent A*d* of the half difference of the legs is to the tangent A*b* of the other half base. Which I prove thus. For if the points *bcde* lie on the same circle, (by Prop.36, Book 3, and Prop.16, Book 6, Euclid [This follows from similar triangles, and is often stated in the form A*b*.A*c* = A*d*.A*e*.]), then as A*b* is to A*e*, thus A*d* is to A*c*, and conversely, etc., as we have now said. Truly the points *bcde* lie on the same circle : For all of the circle on the surface of the sphere is to be described by the shadow from the light on the same surface, which is not on the periphery of the same circle, proceeding in a perfectly round circle in the plane perpendicular to the line, which proceeds from the light through the centre of the sphere, which is apparent from optics, and from the construction of the astrolabe, and by Apollonius in Prop. 5, Book 1 of his book on Conic Sections. But this circle [p. 52.] αβγε is described on the surface of the sphere, and the light from P is beyond the circumference of the circle, and which from that the line proceeds through the centre (as it were POA) is at right angles to the plane. By necessity

therefore, the shadow of this circle, which is incident on the points *dbce*, is circular and perfectly round. Therefore as A*b* is to A*e* , thus A*d* is to Ac, and conversely, that is, as the tangent of the true half base is to the tangent of the half sum of the legs, thus the tangent of the half difference of the legs is to the tangent of the other half base : and as a consequence, the differential of the true half base taken from the sum of the differentials of the half sum and the half difference of the legs is equal to the differential of the half sum of the other base which we have undertaken to show.

7. *Thus from the three sides of a given spherical triangle, there are three ways in which the angles can be found as you please.*

8. *The first way is, that you set up some side for the base (taken closest to the quadrant* [i.e. the largest side]***). Then you add the half difference of the legs to the half base, and subtract this half difference from the half base; you add the logarithms of the sum and difference, hence you take the sum from the logarithms of the legs, giving twice the logarithm of half the arc angle, from which the angle to the vertical emerges, and thus for the others.***



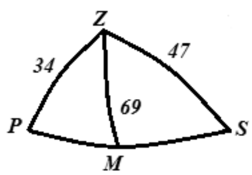
As of the repeated triangle PZS, the sides PZ 34 degrees and ZS 47 degrees, and SP 69 degrees. The angles are sought, and first the angle PZS closest to a quadrant, that SP 69 subtends (clearly the side nearest to being a quadrant). Hence this side SP 69 degrees is set up for the base. Hence the half difference of the legs, PZ and ZS, as one may see, is 6. 30'. And add this to the half base 34.30' and the sum becomes 41 degrees : and take from that, and the remainder is made 28 degrees. Add the logarithm of 41 degrees, obviously 4215044, and of 28 degrees, clearly 7561472 , and they make 11776516. Similarly the logarithms of the legs PZ, 34, and ZS, 47, which are 5812606 and 3128580 are added, the half logarithm of which is 1417665 obviously corresponding to the arc 60.12'.24$\frac{1}{2}$" , which doubled gives the angle 120.24'.49", the vertical angle PZS sought. The remaining angles can be found in the same way, if required : yet they may become known more easily by Sect.9. Ch.5 of this Book, since by basics.2, Ch.3, they are of a known kind. [p. 53.]

9. ***The second method is, with some side set up as the base (that especially close to the quadrant); you add the half base and the half sum of the legs, and also you subtract the same : you add the sum and difference of the logarithms, and hence you take the sum from the logarithms of the legs, and double the logarithm of the cosine*** [not called the cosine by Napier, which was so named later, but the complement of the angle, the logarithm of which he calls the antilogarithm] ***of the half arc emerges, and from this comes the vertical angle, and thus for the other angles.***

As of the same triangle PZS (set up as previously), add the half base 34.30' and the half sum of the legs 40.30', and they make 75, and from the same take away, and they become 6, the logarithms of 75 and of 6 degrees are 346683 and 2582951 are added, making 22929634. Hence take the sum from the logarithms of the legs, which as above is 8941186 and they make 13988448. Which halved, hence gives 6994224, the logarithm of the half the complement of the arc 60.12'. 24$\frac{1}{2}$", the double of which, 120.24'.49" is, as above, the sought vertical angle PZS. The rest also if it pleases, can be found in this way, yet the angles can be found easier by Sect.9 of Ch.5 of this Book.. They are also of a known kind, by basics.2 of Ch.3.

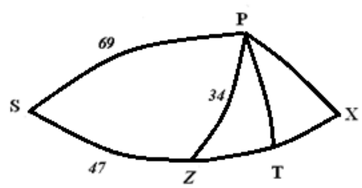
10. ***The third method is, as with some side proposed for the base, you add the differential of the half sum of the legs to the differential of the half difference of the legs*** [recall that the differential in Napier's tables is the log of the cosine taken from the log of the sine, and hence is the log of the tangent]***, and from the sum you take the differential of the true half base*** [the arc]***, and there arises thus the differential of the alternate half base*** [the angle] ***: of which the sum of the half bases is the major case, and the difference is the minor case for two right angled triangles, which render known both all their own parts and all the parts of the oblate triangle (by Sect.9 of Ch.4, and Sect.8, Ch.5 of this Book).***

As of the proposed triangle PZS, with the sides given as above, the angles ZPS and ZSP about the base are sought. The half sum of the sides PZ and ZS is 40.30'. The half difference of the legs is 6.30'. The differential of this is 1577296, of that truly 23298505. Hence take the differential of the true half base 34.30', 3750122, and there is remains 19548383, the differential of 8.3'.31", for the alternate half base, therefore add the half bases 34.30' and 8.3'.31" and hence they make 42.33'.31" for the greater case MS and by taking 8.3'.31" from 34.30'. there remains 26.26'.29" for the minor case PM. And thus on taking account of these cases you have now two right angled triangles [p. 54.]in M, as you can see, PMZ and SMZ: and for which the perpendicular ZM, and the vertical PZM and SZM, or if it is wished, PZS itself can be revealed (by Sect.9 of Ch.4, and Sect.9 of Ch.5 of this Book.) But with these omitted we can return to find the base angles ZPS and ZSP. The case PM, 26.26'.29" : now the differential is found 6985518 (by 9 of Ch.4,), add to the differential of the complement PZ, obviously to the differential 56, which is – 3937709, and there arises + 3047809, the logarithm of the complement of the angle ZPS, which complement is 47.30'. 1". Similarly for the case SM 42.33'.31" : now also the differential 853239 is to be found (by the same sect. 9) , add to the differential 853239 (by the same sect.9), add to the differential of the complement PZ , obviously to the differential of 43 degrees which is 698698, and there comes about +1551937, the logarithm of the complement of the angle ZSP, which complement is 58.53'.55". Remembering moreover this is not made from the parts P Z,34 and ZPS, or from PZ, 47, and ZSP, but from the complements, viz. 56 degrees and 47.30'.1", and 43 degrees and 58.53'.55", here called the circular parts by sect. 2 of Ch. 4 of this Book. And thus truly the angle sought ZPS is 42.29'.59", and ZSP is 31.6'.5", as also is apparent from sect.8 of Ch. 5 of this Book.



Another example of the same triangle.

With another construction from the same triangle PZS, SZ is the base, and with the sides given as above, the angle PZS is sought. The half sum of the legs SP 69 and PZ 34 is thus 51.30', and the differential of this is – 2288650 ; the half difference is indeed 17.30', and the differential of this is + 11542341. Add these differences and the sum is + 9253691, from which take the differential of the base SZ, as can be seen, the differential



of the half base SZ is 23.30'. which is 8328403, and there is left 925288, the differential of the arc 42.21'.11" for the other half base. Therefore add the half bases 42.21'.11" and 23.30', and there comes about 65.51'.11" for the greater case [p. 55.] ST, and then subtract 23.30' from 42.21'.11", and there is left 18.51'.11" for the lesser case TX, or TZ. Therefore, add the differential of this + 10745201, to the differential of the complement PZT, as can be agreed upon, to the differential of 56 degrees, which is – 3934409, and there then arises + logarithm of the complement of the angle PZT. Moreover the corresponding arc in the table for this logarithm 6807492 from the opposite, is 59.35'.11" for the angle PZT; of this angle PZT, when the angle sought is PZS, is the supplement (which always happens when the other base is the larger) and it is necessary that PZS is 120.24.min.49.sec.Otherwise the other base will be greater, and in this case the angles PZT coincide PZS and are equal.

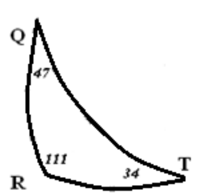
Notes.

Now you have three ways of finding the angles from the given sides, of which any one angle can be found in three ways, and so for the angles sought of any triangle. Indeed, from the given elevation of the pole, the altitude of the sun, and the declination of the sun, those who doubt are satisfied concerning the question as to whether the sun's azimuth, or whether the second or third hour of the day is sought.

Up to this stage, the angles have been found from the sides. There remains the task of finding the sides from the angles.

11. *In all spherical triangles the sides can be changed into angles and the angles into sides: yet first with the assumption of some single angle, and for the side subtending this, with the remainders to the semicircle.*

An Example.



Let the triangle be QRT, the angles of which are : Q, 47; R , 111, and T 34. In the first place we may take any angle, it is seen that for R 111, its remainder to the semicircle is 69 degrees. I say that the angles 47, 69 and 34 can be changed into sides, and becomes the above triangle PZS . In which PZ is 34 degrees, ZS is 47 degrees, and PS is 69 degrees, as also from the angles of this triangle PZS, the sides by changing become the angles. For the angle ZSP of this 31.6'.5", is the side QR of this triangle : and the angle ZPZ of this 42.29'.59", is the side RT of this : and of the third angle of this, which is 120.24'.49", the remainder to the semicircle, which is 59.35'.11", is the side QT of this triangle. The demonstration of this idea is shown by authors such as *Bartholomæus Pitiscus, Adrianus Metius,* and others. I think therefore there should only be a minimal repetition in this volume.

12. ***Thus from the three given angles of a spherical triangle, the sides are easily acquired by conversion.***

Since the angles are given of the preceding triangle QRT : Q 47; R 111; and T 34. Moreover the sides are sought. For some given angle, for example as above, for R, 111, the remainder of its angle to the semicircle is 69 degrees. Hence with 47, 69, and 34 put in place for the sides, in order that with the above triangle PZS made, by any of the three ways described above, whereby you can find the angles of this : opposite the side 47, the angle 42.29'.59" ; and opposite the side 34, the angle 31.6'.5"; and opposite the side 69, (which we have taken for 111) you find the angle 120.24'.49". Thus, in the oblate triangle QRT, for the side RT, where the angle Q, 47, is to be subtended, put 42.29'.59". And for the side QR, to be subtending the angle T, 34, put 31.6'.5"; and surely for the side QT to be subtending the angle R 111, put 59.35'.11"; which before for 111 you took its supplement, surely 69. And thus from the angles by conversion you acquire the sides.

Notes.

From this, by finding the three sides of this triangle from the given angles, which are solved in various ways, and thus questions concerning any triangle. As in the triangle PZS. From the given, hour of the day, azimuth of the sun, and from the angle of position of the sun's situation, this preceding method satisfies the question from which in the first place either the elevation of the pole, or secondly [p. 57.] the altitude of the sun, or thirdly the declination of the sun is sought. Thus from sect.8 of the preceding Ch. 5 and sect.7 and 12 of this chapter, you have 60 different solutions of the questions that can occur in any triangle. Neither are more variations able to arise from the multiplicity of any three parts from the composition of these. Therefore you have a perfect and complete doctrine for solving both plane and spherical triangles.

CONCLUSION.

Now therefore, it has been shown well enough what logarithms can be, why they should be, and what their uses can be : Indeed without the benefit of these for multiplications, divisions, or troublesome root extractions, the arithmetical solution of any geometrical question can be promptly shown, moreover we have Cleary established the proofs, and we have given instructions by examples from both kinds of trigonometry. And thus as promised, you have the admirable table of logarithms, of the greatest use: which if, from the more learned, that may arise from the gratitude of your letters, I may have it in mind to add to the tables the method of their construction, that should be put in place. Meanwhile take advantage of this short work, and to God the great inventor and guider of all good works, give the greatest praise and glory.

The Tables or Canon of Logarithms follows.

De Canonis mirifici LOGARITHMORUM
præclaro usu in Trigonometria.

LIBER IIb.

De non quadrantalibus mixtis.

Cap. V.

Hactenus quadrantalium, sequitur triangularum Sphæricorum non quadrantialim doctrina.
1.Non quadrantale est triangulum Sphæricum, cujus nec latus, nec angulus quadrans est.
2. Non quadrantale reducitur ad bina quadrantalia, Si à vertice ad ejus basim (prout opus fuerit extensam) dimittatur perpendicularis, quadrans arcus.
3. Perpendicularis cadit intra triangulum, si anguli apud basim sint ijusdem speciei : extrà verò si diversæ : & contrà.
4. Quadrans arcus cadit extra triangulum, si crura sint ejusdem speciei : Intrà verò si diversa : & contrà.
5. Ex non quadrantalis sex partibus, tres datæ solum sufficiunt ad reliquarum scientiam comparandam : nisi forsan trium datarum, quarum una alteri opponatur, tertia sit propinquior, quam altera ejusdem generis data. In hoc enim casu requiritur etiam dari species partis quæ tertiæ opponitur, ut reliquæ sciantur.

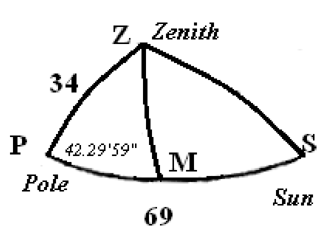
Hujus casus exempla sunt quartum & sextum exemplum.

6.Partes tres datæ aut miscellaneæ sunt, aut puræ.
7. Miscellaneæ sunt, quarum una est diversi generis à reliquis duabus.

Ut cùm dantur duo latera, & angulos aliquis: aut duo anguli cum latere aliquo.

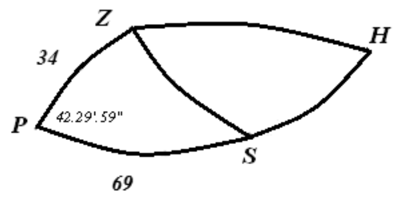
8. In partibus miscellaneis datis, si ab illo termino lateris dati in cuius reliquo termino sit angulus datus, cadat ad basem perpendicularis aut quadrans datum illum angulum subtendens, reducetur non quadrantale ad bina quadrantalia (per nonam sect.cap.4. huius) scibilia.

Unde & non quadrantalia partes (quia cum horum quadrantalium partibus, aut partium reliquis communes sunt) [p. 40] ***facilè-innotescunt, cognitis tamen priùs per 2.3.& 4.sect.cap.3.hujus, aut ex hypothesi partium speciebus.***



Exemplum duorum laterum, & anguli interpositi datorum.

Ut sit (usus & exercitii gratia) Triangulum Sphæricum non quadrantale in superficie primi mobilis descriptum P.Z.S. polum, Zenth, & solem referens : cujus sex partes sunt, latus P.Z., quod est interstitium poli & Zenith, seu complementum elevationis poli. Latus Z.S., interstitium Zenith & solis, seu



complementum altitudinis solis. Latus P.S. interstitium poli & solis, seu complementum declinationis solis ab æquatore. Angulus Z.P.S. hora diei, seu tempora horaria æquatoris. Angulus P.Z.S., quæ plaga est, seu azimuth solis à septentrione. Angulus P.S.Z., qui angulus est situs & positionis solis ad polum & zenith. Harum sex partium dentur tres quæcunque mistellaniæ. Verbi gratia, angulus horarius Z.P.S. 42.29'.59". (qui horam secendam 49.59'.56" pomeridianam notat) & latus P.Z. 34. complementum elevationis poli. atque latus. P.Z. 69. complementum elevationis solis. Ex quibus, ut acquirantur tres reliquæ partes, ab Z termino lateris PZ. dati, dimittatur perpendicularis Z.M, aut, si mavis, quadrans Z.H. angulum datum Z.P.S. subtendens, reducensque non quadrantale oblatum P.Z.S. ad duo triangula in angulo M. quadrantalia, quæ sunt P.M.Z. & Z.M.S, ut in primo schemate : vel (si varietate delecteris) ad duo triangula in latere Z. H. quadrantalia, quæ sunt Z.H.P, & Z.H.S. ut secundo schemate : Quorum quadrantalium omnes partes [p. 41] per 9.sect.cap.4. hujus acquires. Nam ex datis PZ 34. atque ZPM, seu ZPS 42.29'.59". invenies perpendicularem ZM 22.11'.47" & angulum PZM 52.46'.38" & latus PM 26.26'.29" : quo ablato à PS 69. restat MS 42.33'.31". quo & perpendiculari ZM jam cognitis, invenies per 9.sect.cap.4. hujus angulum MSZ seu quæsitum PSZ, 31.6'.5". & latus quæsitum SZ 47, atque angulum MZS 67.38'.11". quo ad PZM 52.46'.38" addito, sit angulus reliquus quæsitus PZS 120.24'.49". Tres itaque habes partes quæsitas officio perpendicularis ZM, primi schematis. Easdem quoque officia quadrantis ZH secundi schematis venari poteris. Ex datis enim, ut supra, PZ 34. & ZPS seu ZPH 42.20'.59". invenies per eandem 9.sect.cap.4. hujus angulus ZHP. 22.11'.47". & angulum PZH 142.46'.38"., & latus PH 116.26'.29". ex quo aufer PS 69. restat SH 47.26'.29". quo & angulo apud H 22.11'.47". jam habitis, invenies per 9.cap.4.huius, angulum HSZ 148.53'.55". ejusque ad semicirculum reliquum scilicet 31.6'.5" angulum PSZ quæsitum : atque latus quæsitum S. Z. 47. Denique angulum HZS 22.21'.49". quo ex HZP 142.46'.38". ablato restat angulus reliquus quæsitus PZS 120.24'.49". prorsus ut supra.

Admonitio.

Hujus exempli imatatione novem variæ solvuntur hujus & cujusque trianguli quæsitiones. Ex datis enim elevatione poli, hora diei, & declinatione solis illius diei, habetur (ut supra) primo azimuth seu plaga solis, secundò altitudino solis, Tertiò angulus positionis solis. Item datis declinatione solis, angulo positionis solis, & altitudine solis, habetur quartò plaga solis, quintò elevatio poli, sextò hora solis. Idem datis altitudine solis, plagi solis, & elevatione poli, habetur septimò hora diei, octavò declinatio solis, nonò denique angulus positionis solis.

[p. 42.]

Secundum exemplum duorum angulorum, & lateris interpositi, datorum.

Præcedentium schematum datis angulis, horario scilicet Z.P.S 42.29'.59". & plagæ solis P.Z.S 120.24'.49". cum complemento elevationis poli, latere scilicet interposito P.Z. 34. Tres cæteræ partes exquiruntur. Nam habitis primò (ut supra) Z.M 22.11'.47". & P.M.

26.26'.29", & angulo P.Z.M. 52.46'.38", quo ex P.Z.S. 120.24'.49". ablato relictoque angulo M.Z.S. 67.38'.11". ex hoc atque Z. M. jam notis, invenietur tandem latus quæsitum Z.S. 47. & Z. S. M, sive angulus quæsitus Z. S. P. 31.6'.5". atque M.S. 42.33'.31". quo ad P. M. additio, sit reliquum latus quæsitum P. S. 69. Hasque beneficio perpendicularis primi schematis habes, nec secus easdem officio quadrantis secundi schematis acquirere poteris : acquiruntur enim (per nonam quarti hujus) ex datis, anguli P. H.Z. & P.Z. H, & ex hoc subducto P. Z.S. dato, restat S. Z. H. quo, & angulo P. H. Z. jam notis propalantur cæteræ omnes partes.

Admonitio.

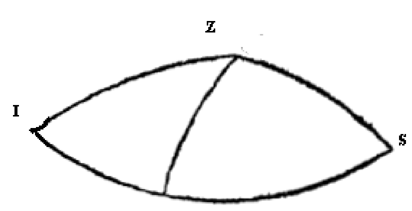
Hujus exempli imatatione novem variæ solvuntur hujus, & cujusque trianguli quæstiones. Ex datis enim (ut supra) hora diei, elevatione poli, & plaga solis, habetur primò declinatiò solis, secundò angulus positionis solis, tertiò altitudino solis. Item, datis hora diei, declinatione solis, & angulo positionis solis, habetur quartò altitude solis, quintò plaga solis, sextò elevatio poli. Item datis angulo positionis solis, altitudine solis, & plaga solis acquiretur septimò elevatio polì octavò hora dici, nono declinatio solis.

[p. 43.]

Tertium exemplum duorum datorum lateram, quorum quadranti propinquius subtendit angulum datum.

Præcedentium schematum dentur latera P. Z. 34, & eo quadranti propinquius Z.S. 47. cum eo quem hoc subtendit angulo Z. P. S. 42. 29'.59" acquirantur per 9.sec.cap.4. hujus Z.X. 22.11'.47". & P.Z.M. 52.46'.38". & P.M. 26.26'.29" & simili modo habebis Z.S.M. seu quæsitum angulum Z.S.P. certissimè enim scitur hic per 2.sent.cap.3. hujus, minor quadrante, scilicet esse 31.6'.5". & non esse 148.53'.55". Habebis etiam angulum M.Z.S. 67.38'.11". quo ad P.Z.M. 52.46'.38". addito, sit reliquus quæsitus angulus P.Z.S. 120.24'.49". Habebis denique & M.S. 42.33'.31" quo ad P.M. 26.26'.29". addito, sit quæsitum latus P.S., 69. Nec secus eadem acquirere poteris (si libet) officio quadrantis Z.H. : secundi schematis.

Quartum exemplum duorum datorum lateram, quorum quadranti minus propinquum subtendit angulum datum: magis autem propinquum subtendit angulum datæ tantùm speciei.



Dentur latera ZS, 47, & eo quadranti minus propinquum PZ 34, cum eo quem hoc subtendit angulo ZSP 31.6'.5". deturque quod quem ZS subtendit (angulus scilicet SPZ.) sit specie minor quadrante : Dimisso itaque ab Z ad basim P.S. perpendiculari Z,M. (ut prius) aut quadrante ZI (ut hic) subtendente angulum datum Z.S.P. Acquirantur per 9.sect.4.hujus cæteræ partes, ut (exercitii & varietatis gratia) ex hujus schematis quadrante Z.I. acquires angulum ZI.S. 22.11'.47", & I. Z. S. 157.38'.11". & S. I. 132.33'.31", & simili modo habebis angulum I.P.Z & per [p.44.] consequens angulum quæsitum S. P. Z. 42. 29'. 59". Quia ex hypothesi expressè quadrante minor declaratur, alioquin nisi ejus daretur species, foret (ex prima cap.3, & quinta sect.hunus cap.) incertus : potuit enim aliter suisse 137.30'.1". Habebis etiam sic angulum I.Z.P. 37.13'.22". quo ex I.Z.S. 157.38'.11". ablato, reliquitur reliquus quæsitur angulus P.Z.S. 120.24'.49". Habebis denique & I.P. 63.33'.31". quo ex I.S. 132.33'.31". ablato, remanetquæsitum latus P.S. 69.

Easdem etiam metas attinges si beneficio perpendicularis Z.M. primi schematis, partium logisticum quæsiveris.

Admonitio.

Præcedentis tertii & hujus quarti exemplorum imitatione, octodecim variæ solvuntur hujus & cujusque trianguli quæstiones. Ex datis enim (ut in tertio exemplo) elevatione poli, altitudine solis, & hora dici, habetur primò plaga solis, secundò angulus positionis solis, tertiò declinatio solis. Item datis (ut in hoc quærto exemplo) elevatione poli,altitudine solis, & angulo positionis solis, habetur quartò plaga solis, quintò hora diei, sextò declinatio solis. Item datis altitudino solis, declinatione solis, & hora diei, habetur septimò angulus positionis solis, octavò plaga solis, nonò elevatio poli. Item datis altitudine solis, declinatione solis, & plaga solis, habetur decimiò angulus positionis solis, undecimò hora diei, duodecimò elevatio poli. Item datis declinatione solis, elevatione poli, & angulo positionis solis, habetur decimotertiò plaga solis,decimoquartò altitudino solis, decinquintò hora diei. Item datis declinatione solis, elevatone poli, & plaga solis, habetur decimosextò hora diei, decimeseptimò angulus positions solis, & decimo-octavò altitudo solis.

[p. 45.]

Quintum exemplum duorum datorum angulorum, quorum quadranti propinquiorum latus datum subtendit.

Trianguli P.Z.S. primi schematis, dentur anguli P.S.Z. 31.6'.5". & eo quadranti propinquior S.P.Z. 42. 29'. 59". cum latere Z. S. 47. hunc subtendente. Ex quibus P.S.Z. & Z.S. habetur (per nonam quarti huius) perpendicularis Z.M. 22.11'.47". & cæteræ partes quadrantalis S. Z. M. scilicet M. Z. S 67.38'.11". & M. S. 42. 33'. 31". Sicut & ex perpendiculari hoc cum dato Z. P. S. seu Z.P.M. angulo, habentur partes omnes quadrantalis Z. M. P. Scilicet primò latus qæsitum P. Z. certissimè enim scitur hoc (per 2.sentent.cap.3.hujus) minus quadrante, videlicet, esse 34. non autem esse 146. Deinde habetur P. Z. M. 52.46'.38". quo ad S. Z. M. 67.38'.11". addito, sit quæsitus angulus P. Z. S. 120.24'.49". Ultimò habetur P. M. 26.26'.29". quo ad M. S. 42.33'.31". addito, sit reliquum latus quæsitum P.S. 69. Has etiam ipsas partes aliter (si mavis) ex duobus proximè præcedentis Schematis quadrantalibus Z. I. S. & Z. I. P. acquirere poteris.

Sextum exemplum duorum datorum angulorum, quorum quadranti minus propinquum subtendit latus datum, magis autem propinquum subtendit latus datæ tantum speciei.

Trianguli P.Z.S. primi Schematis, dentur anguli Z. P. S. 42. 29'. 59". & eo quadranti minus propinquus Z. S. P. 31.6'.5". cum eum subtendente latere P. Z. 34. Deturque quod angulum Z. P. S. subtendens, (scilicet latus Z. S.) sit speciè minus quadrante. Ex his datis quæratur perpendicularis Z. M. 22.11'.47". & cæteræ quadrantalis P. Z. M. partes, scilicet P. Z. M. 52.46'.38".& P. M. 26.26'.29". Sicut & ex perpendiculari hoc cum dato Z. S. M. seu Z. S. P. 31.6'.5". quæratur partes omnes quadrantalis Z. M. S. scilicet primò latus optatum Z. S. 47. quia ex hypothesi expressè quadrante minus [p. 46.] declaratur, alioquin potuit suisse 133. Nam (per 1.cap.3 & quintam hujus) incertum est nisi ejus expressè detur species. Deinde angulus M. Z. S. 67.38'.11". quo ad M. Z. P. 52.46'.38". addito, sit optatus angulus P. Z. S. 120.24'.49". Denique habetur S. M. 42. 33'. 31". Quo ad P. M. 26.26'.29". addito, sit optata basis P. S. 69. Easdem etiam partes ex duobus quadrantalibus P. H. Z. & S. H. Z. secundi schematis, quam facillimè acquirere poteris.

Admonitio.

Præcedentis quinti & hujus sexti exemplorum imatatione, octodecim variæ solvuntur hujus, cujusque trianguli quæstiones. Ex datis enim (ut in quinto exemplo) angulo positionis solis, hora diei, & altitudine solis, habetur primò elevatio poli, secundò plaga solis, tertiò declinatio solis. Item datis (ut in hoc sexto exemplo) hora diei, angulo positionis solis, & elevatione poli, habetur quartò altitudo solis, quintò plaga solis, sextò declinatio solis. Item datis hora diei, plaga solis, & altitudine solis, habetur septimò declinatio solis, octavò angulus positionis solis, nonò elevatio poli. Item datis hora diei, plaga solis, & declinatione solis, habetur decimò altitudino solis, undecimò angulus positionis solis, duodecimò elevatio poli. Item datis plaga solis, angulo positionis solis, & declinatione solis, habetur decimotertiò elevatio poli, decimoquartò horae diei, decimquintò altitudo solis. Item datis plaga solis, angulos positionis solis, & elevatione poli, habetur decimosetò declinatio solis, decimoseptimò hora diei, decimo-octavò altitudino solis. Atque ita hujus solius canonis methodo, quinquaginta quatuor variæ solvuntur quæstiones ejusdem trianguli non quadrantalis. Cæteræ inferius solventur.

9. Ex his itaque patet quod duorum angulorum & suorum subtendentium laterum tribus datis, quarti saltem Logarithmus innotescet, tacita [p. 47.] ***etiam quadrantalium descriptione. Ab aggregato enim ex Logarithmis anguli & lateris sibi adjacentis datorum, aufer Logarithmum tertii dati, & proveniet inde Logarithmus quarti quœsiti, ipsumque quartum, nisi sit incertœ speciei. innotescet.***

Ut ex superioribus tertio, quarto, quinto, & sexto examplis percipi potest. Angulorum enim basis Z. P. S. & Z. S. P. & suorum subtendentium crurum Z. S. & Z. P. dentur tria,quæ (verbi gratia) sunt crura Z. S. 47, ejusque Logarithmus 3128580, & Z. P. 34, ejusque Logarithmum 5812606, cum huic adjacente angulo Z. P. S. 42.29'.50". cujus Logarithmus 3921720 adde ad 5812606, sit 9734326 (Logarithmus scilicet taciti & suppressi perpendicularis Z. M. vel anguli Z. H. S. seu Z. I. P.) à quo aufer 3128580 remanet 6605746 Logarithmus quarti Z. S. P. quæsiti. Ipsum itaque quartum Z. S. P. erit 31.6'.5".Quoniam per 2. sect.cap.3. minus quadrante arguitur. Contra autem datis Z. P. 34. ejusque Logarithmo 55812606.& Z. S. 47, ejusque Logarithmo 3128580, cum huic adjacente angulo Z. S. P. 31.6'.5" ad cujus Logarithmum 6605746, adde 3128580, sit aggretatum (ut supra) 9734326. à quo aufer 5812606, provenient 3921720 Logarithmus quarti quæsiti, scilicet Z. P. S. , cujus arcus per 1.sect.cap.3. incertus est an sit 42.29'.50" an 137.30'.1". nisi declaret hypothesis majorne, an minor sit quadrante.

De non Quadrantalibus puris.

Cap. VI.

Hactenus de partibus miscellaneis datis : Sequuntur puræ.

1. ***Puræ sunt tres partes ejusdem generis datæ. Sunt aut tria latera data, & quæruntur anguli : aut tres anguli dati, & quaruntur latera.***

Admonitio.

2. ***Puræ quamvis simplicitate priores, ob difficultatem tamen earundem meritò hic posteriorem sortiuntur locum.*** [p. 48.]

3. *In triangulis Sphæricis primò summa ex Logarithmis crurum subducti à summa Logarithmis aggregato & differentiæ semibasis & semidifferentiæ crurum, relinquit duplum Logarithmi dimidis anguli verticalis.*

Quia docent Regiomontanus libro 5.cap.2. de triangulis, & alii, ut rectangulum comprehensum sub sinibus rectis crurum, se habet ad quadratum sinus totius : Ita differentiam sinuum versorum basis & differentiæ crurum se habere ad sinum versum anguli verticalis : quum autem ut illa differerntia ad hunc sinum versum, ita rectangulum factum ex sinibus rectis aggregati & differentiæ semibasis & semidifferentiæ crurum, se habet ad quadrantum sinus recti dimidi anguli verticalis (est enim novissimum hoc rectangulum ad illam differentiam sinuum versum, & hoc ultimum quadratum ad illum sinum versum in ratione 5000000 $^{\text{cupla}}$,(quinquis millies millecupla) existente sinu toto 10000000.) Ideo sequetur quod, ut rectangulum sub sinibus rectis crurum se habet ad quadratum sinus totius, ita rectangulum factum ex sinibus rectis aggregati & differerentiæ simibasis & demidifferentiæ crurum, se habebit ad quadratum sinus recti dimidii anguli verticalis : & per consequens sinus recti dimidii anguli verticalis : & per consequens (ex corol.def.cap.1. & prop.4.cap.2.& probl.3.cap.5.lib.1.) Summa ex Logarithmis crurum, subducta ex Logarithmis aggregati & differentiæ semibasis & semidifferentiæ crurum, relinquit duplum Logarithmi dimidii anguli verticalis, ut supra.

4. *Secundò, Summa ex Logarithmis crurum subducta à summa ex Logarithmis aggregati & differerentiæ semibasis & semiaggregati crurum, relinquit duplum antilogarithmi dimidii anguli verticalis.*

Non enim alter se habet summa ex Logarithmis aggretati & differentiæ semibasis & semiaggregati crurum hujus propositionis, ad summam ex Logarithmis aggregati & differentiæ semibasis & semidifferentiæ crurum præcedentis propositionis, quam duplam antilogarithmi dimidii anguli verticalis hic, ad duplum Logarithmi ejusdem dimidii anguli verticalis superius, quod alterius loci est demonstrare. [p. 49.]

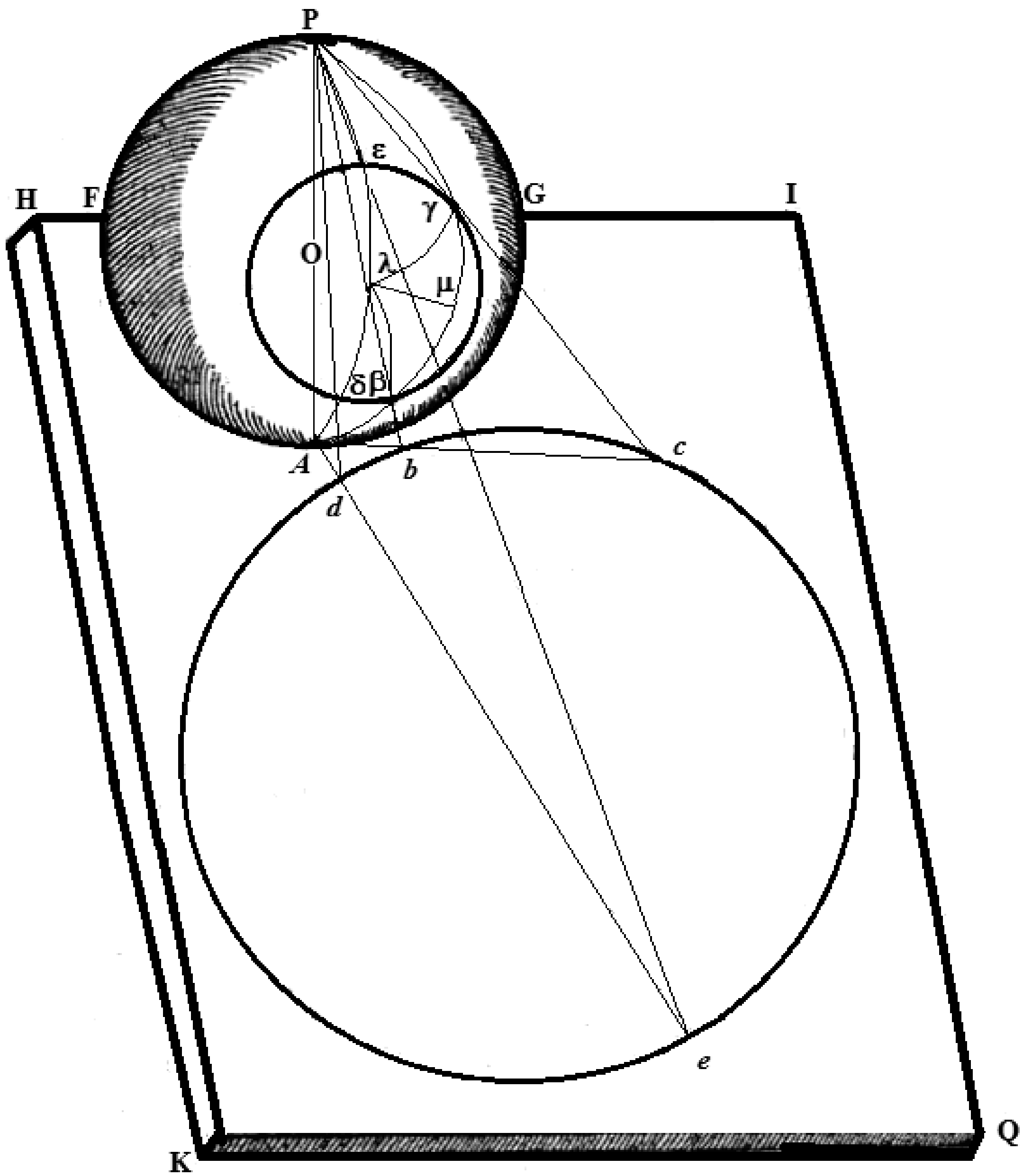
Admonitio.

5. ***In Sphœrics etiam, bases veram & alteram, eodem sensu capimus, quo in rectilineis, nimirum alteram pro aggregato, alteram pro differentiæ casuum.***

6. ***Tertio differentialis semibasis veræ datæ, subductus ex summa differentialum semiaggregato & semidifferentiæ crurum, relinquit differentialem semibasis alternæ.***

Hujus ratio fundamentalis est, quod ut tangens semibasis veræ se habet ad tangentem semiaggregati crurum, ita tangens semidifferentiæ crurum se habeat ad tangentem semibasis alternæ. Tangentium enim Logarithmi sunt suorum arcuum differentiales per sect.22 & 25.cap.3.lib.1. Unde hanc tangentium analogiam sequetur illa suorum logarithmorum, seu differentialium æqualitas per prop.4.cap.2.lib.1: Verum quia hujus analogiæ tangentium fundamentalis, hactenus ignotæ, demonstrationem à me fortè requirent Lectores, eam ideo, quantum hujus compendii brevitas patitur, hic explicabimus.

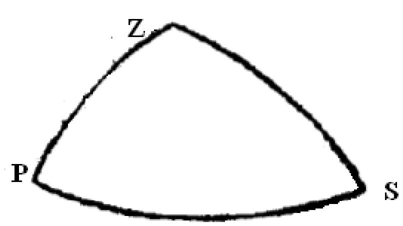
Sphæra itaque AFPG incumbat plano HIKQ ut se invicem tangant in communi puncto A, à quo per Sphæræ centrum $\oplus$ erigatur recta $A\oplus P$ secans surpremum Sphæræ Hemisphærium in puncto P. eritque $A\oplus P$ perpendiculis plano HIKQ. deinde angulo A describatur in Sphæræ superficie triangulum Aλγ in γ acutum, aut Aλβ in β obtusam, & protractis semicirculis A λP, & AγP. seu AβP, polo λ, intervallo λγ, seu ei æquali λβ ducatur circulis εδβγ, secans λP in ε & λA in Aβγ in punctis β & γ. Ex puncto λ in arcum Aβγ dimittatur perpendicularis arcus λμ. Erunt itaque hic Aλ crus magis, λγ vel λβ crus minus, Aγ & Aβ bases, altera vera, reliqua alternæ, Aδ differentia crurum, & Aε aggregatum crurum, quia λε, & λδ ex constructione sunt æqualia minori cruri λγ seu γβ. His peractis, & supposito P vicem gerere oculi aut lucidi cujuspiam, ab eodem P in subjectum planum HIKQ dimittantur, radius Pγ secans planum in c, & radius Pβ secans planum in b: & quia γβA in eodem plano seu circulo sunt cum lucido P, erunt suæ umbræ cbA in eadem recta. Similiter ab eodem puncto P, in idem planun dimittantur radius Ps secans planum in e, & radius γδ secans planum in d & quia εδA in eodem sunt plano & circulo cum lucido P : ideo suæ umbræ edA erunt in eadem rect. Præterea quia $P\oplus A$, est [p. 51.]



est plano orthogonalis seu rectangula, ideo triangula P.A.d. & P.A.e. atque P.A.b., & P.A.e. sunt in A rectangula, atque ideo etiam A d est tangens anguli A.P.δ., seu A.P.d, & A. e. est tangens anguli APε vel APe sic etiam A b est tangens anguli APβ vel APb, & Ac est tangens anguli APγ vel APc, posito gnomene seu sinu toto PA, & quia Ad est tangens anguli APδ, & APδ est dimidium anguli A⊕δ per 20.prop.3.Eucl. (quod hoc sit in centro, ille in circumferentia) ideo Ad est tangens dimidi anguli A⊕δ, seu (quod idem est) dimidi arcus Aδ, quod est semidifferentia cruruum. Similiter quia A e est tangens anguli APε, angulus autem APε in circumferentia sit dimiium anguli A⊕ε in centro, ideo Aε est tangens dimidii A⊕ε, seu dimidii arcus, Aε, quod est semiaggregatum crurum. Simili modo in basibus vera & alterna erit Ab tangens anguli APβ, seu dimidii anguli A⊕β, seu dimidii arcus Aβ, quod est altera semibasis : atque A c erit tangens anguli APγ, seu dimidii anguli A⊕γ, seu dimidii arcus Aγ, quod est reliqua semibasis. Quumque jam ostendum sit quod Ab sit tangens alterius semibasis, & Ac tangents reliquæ semibasis, atque Ad sit tangens semidifferentiæ crurum, & Ae tangens semiaggregati crurum. Dico quod ut Ab tangens semibasis veræ se habet ad Ae tangentem semiaggregati crurum, ita Ad tangens semidifferenciæ crurum ad Ac tangentem semibasis alternæ : vel contra ex alterna veram faciendo; ut Ac tangens semibasis veræ se habeat ad Ae tangentem semiaggregati crurum : Ita Ad tangens semidifferentiæ crurum ad Ab tangentem semibasis alternæ. Quod sic probo. Si puncta bcde sint in eodem circulo, erit (per 36.prop.3.& 16.prop.6.Euclid.), ut Ab ad Ae, Ita Ad ad Ac . & contra, &c. ut jam diximus. Verum puncta bcde cadunt in eodem circulo : Omnes enim circuli in superficie Sphæræ descripti umbra à lucido in eadem superficie quod non est in circuli peripheria procedens circulum facit perfectè rotundum in plano orthogono ad rectam, quæ à lucido per centrum Sphæræ progreditur, ut ex Opticis, & astrolabi fabrica patet. At hic circulus [p. 52.] αβγε in sphæræ superficie describitur, & lucidù P est extra ciruli peripheriam, quæque ab eo procedit recta per centrum (videlicet P⊕A) est ad planum orthogona. Necessatiò ergo ejus ciruli umbra, quæ in puncta d.b.c.e. incidit, circularis est, & perfectè rotunda. Ergo ut se habent A.b. ad A.e. Ita A.d. ad A.c. & contra, id est, ut tangens semibasis veræ ad tangentem semiaggregati cruruum, ita tangens semidifferentiæ crurum ad tangentem semibasis alternæ : & per consequens, differentialis semibasis veræ subductus ex summa differentialium semiaggregati, & semidifferentiæ crurum æquatur differentiali semibasis alternæ quæ demonstranda suscepimus.

7. *Unde trianguli Sphœrici datis tribus lateribus habetur triplici modo angulorum quivis.*

8. *Primus modus est, ut latus quodvis (præcipuè quadrantis proximum) pro basi statuas. Inde semidifferentiam crurum, & ad semibasim addas, & à semibasi subtrahas: producti & residui Logarithmos addas, hinc auferas aggregatum ex Logarithmis crurum, reliqui bipartiti Logarithmi arcum duplicis, & proveniet angulum verticalis, atque ita cœteri.*



Ut trianguli PZS repetiti, dentur latera PZ 34 gr. & ZS 47 gr. & SP 69 gr. Quærantur anguli, primóque quadranti proximus PZS angulus, quem SP 69 (latus scilicet quadranti proximum) subtendit. Hoc itaque SP 69 pro basi statuatur. Inde semi differentiam crurum PZ, & ZS, videlicet 6. 30'. Et adde ad semibasim 34.30'. fientque 41 aggregatum : & subtrahe ab ea, fientque 28. residiuum. Logarithmos graduum 41, scilicet 4215044, & graduum 28, scilicet 7561472 adde, fient 11776516. Similiter crurum P.Z. 34, & ZS, 47. Logarithmos 5812606, & 3128580 adde, cujus dimidio Logar. 1417665. respondentem arcum, videlicet 60.12'.24$\frac{1}{2}$" duplica, provenient 120.24'.49" angulus verticalis P.Z.S. quæsitus. Nec secus angulos reliquos, si libet, invenire poteris : facilius tamen per 9.cap.5.hujus innotescent, quia per 2.sentent.cap.3.sunt certæ speciei. [p. 53.]

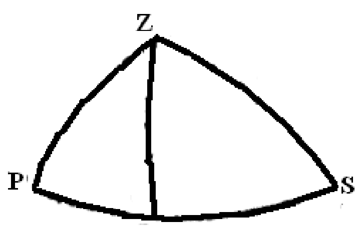
9. ***Secundus modus est, ut latere quovis (præcipuè quadranti proximo)pro basi statuto, Semi-basim & semi-aggregatum crurum addas, & ab eodem subtrahas : producti &***

residui Logarithmos addas, & hinc auferas aggregatum ex logarithmis crurium, relique bipartiti antilogarithmi arcum duplices, & proveniet inde angulus verticalis : atque ita cæteri.

Ut ejusdem trianguli P.Z.S. (constituti ut in præmissa) semibasim 34.30'. & ad semiaggregatum crurum 40.30'. adde, fientque 75. & ab eodem subtrahe, fientque 6, quorum 75. & 6 graduum logarithmos 346683, & 2582951 adde, fientque 22929634. Hinc aufer aggregatum ex Logarithmis crurum, quod (ut supra) est 8941186, fientque 13988448. Quæ bipartire, fient inde 6994224 antilogarithmus conveniens arcui 60.12'. $24\frac{1}{2}$", cujus duplum 120.24'.49". est (ut supra) quæsitus angulus PZS verticalis. Cæteros licet etiam hoc modo, facilius tamen per 9.cap.5.hujus invenies angulos. Sunt emim per 2. sentent.cap.3. notæ speciei.

10. ***Tertius modum est, ut latere quovis probasi posito, differentialem semi-aggregati crurum ad differentialem semi-differentiæ crurum addas, & à producto auferas differentialem semi-basis veræ, & proveniet inde differentialis semi-basis alternæ : quærum semi-basium summa est casus major, & differentia casus minor, duo distinguentes rectangula, quæ & suas, & ipsius oblati trianguli partes omnes (per nonam cap.4.& octavam cap.5. hujus) notas reddunt.***

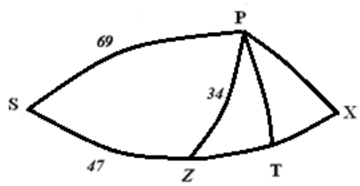
Ut propositi trianguli P.Z.S. datis lateribus ut supra, quærantur anguli apud basim ZPS & ZSP. Semi-aggregatum crurum PZ & ZS est 40.30'. Semidiffererna crurum est 6.30'. Illius differentialis est 1577296, hujus verò est 23298505. Hinc aufer semibasis veræ 34.30'. differentialem 3750122m remanent 19548383, differentialis 8.3'.31". pro semibasi alterna adde ergo semibases 34.30'. & 8.3'.31". fient inde 42.33'.31" pro major casu MS & subtrahe 8.3'.31".à 34.30'. relinquentur 26.26'.29" pro minor casu PM. horum itaque casuum



officio habes duo [p. 54.] jam rectangula in M scilicet PMZ, & SMZ: quæ & perpendicularem ZM, & angulos verticales PZM, & SZM, aut,si libet, ipsum PZS, patefaciunt (per nonam cap.4, & octavam cap.5.hujus.) Sed his omissis ad quæsitos basis angulos ZPS, & ZSP, redeamus. Casus P M, 26.26'.29". jam acquisiti differentialem 6985518 (per 9.cap.4,) adde ad differentialem complementi PZ, scilicet ad differentialem 56, qui est – 3937709, provenient + 3047809 Logarithmus complementi anguli ZPS, quod complementum est 47.30'. 1". Similiter casus SM 42.33'.31" jam etiam acquisiti differentialem 853239 (per eandem nonam sect.) adde ad differentialem 853239 (per eandem nonam sect.) adde ad differentialem complementi PZ scilicet ad differentialem 43 gr.qui est 698698, provenient +1551937 Logarithmus complementi anguli ZSP, quod complementum est 58.53'.55". Memor autem hic fis non ipsas partes P Z.34. & ZPS. aut PZ. 47, & ZSP, sed sua complementa, viz. 56 gr. & 47.30'.1". & 43 gr. & 58.53'.55". circulares partes hic dici per secundam cap.4.hujus. Verus itaque angulus quæsitus ZPS est 42.29'.59". & ZSP est 31.6'.5". ut etiam ex sect.octava.cap.5. hujus patet.

Aliud ejusdem trianguli exemplum.

Eodem triangulo P. Z. S. alio situ constituto, sit S. Z. basis, & datis lateribus ut supra,



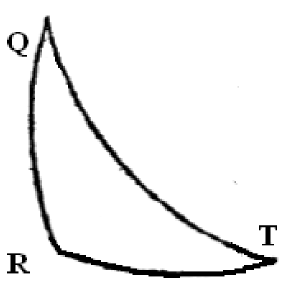
quæratur angulus P. Z. S. Crurum itaque S.P. 69. & P. Z. 34. semi-aggragatum est 51.30', ejusque differentialis – 2288650 : semi-differentia verò est 17.30', ejusque differentialis est + 11542341. Quos differentiales adde, erit summa + 9253691. à qua aufer differentialem dimidii basis S. Z. , differentialem dimidii basis S Z videlicet differentialem 23.30'. qui est 8328403, remanebit 925288 differentialis arcus 42.21'.11". pro semi-basi alterna. Adde ergo semi-bases 42.21'.11". & 23.30'. provenient 65.51'.11" pro majore
[p. 55.] casu ST, & tunc subtrahe 23.30'. à 42.21'.11". remanent 18.51'.11". pro minor casu TX, vel TZ, Hujus ergo differentialem + 10745201, adde ad differentialem complem. PZT, scilicet ad differerentialem grad. 56. qui est – 3934409, & provenient inde + Logarithmus complementi anguli PZT. Arcus autem in tabula respondens huic Logarithmo 6807492 ex adverso, est graruum 59.35'.11" pro angulo PZT, cujus anguli PZT, quum angulus quæsitus PZS, sit ad semi-circulum reliquus (quod semper occurrit quum basis alterna est major vera) erit necesse PZS. esse graduum 120.24.min.49.sec. alioquin si basis vera alternam superaverit, coincident anguli PZT, & PZS, & æquales erunt.

Admonitio.

Tres jam habes veros modos invenniendi angulos ex datis lateribus, quorum unoquoque tres variæ solvuntur hujus, & cujusque trianguli quæstiones. Ex datis enim elevatione poli,altitudine solis, & declinatione solis, dubitantibus satisfit ad quæstionem qua vel plaga solis, vel secundò, vel tertiò hora diei quæritur.

Huc usque ex lateribus invenimus angulos. Superest ex angulis invenire latera.

11. ***In omni triangulo sphœrica mutari possunt latera in angulos, & anguli in latera: assumptus tamen prius pro unico quovis angulo, & suo subtendente latere suis ad semicirculum reliquis.***



Exempli gratia.

Esto triangulum QRT, cujus sint anguli Q 47. R 111. & T 34. Sumamus primò pro angulo quovis, videlicet pro R 111 suum ad semicirculum reliquum, quod est 69. grad. Dico hos angulos 47. 69. & 34. mutari posse in latera, & fiet superius triangulum PZS. In quo PZ est 34 grad. ZS. est 47 grad. & PS est 69 grad. ut etiam ex illius trianguli, PZS, angulis, fient hujus mutuo latera. Nam ZSP angulus grad. 31.6'.5" illius, est latus QR hujus : & angulus ZPZ grad. 42.29'.59". illius, est latus RT hujus : & tertii anguli illius, qui est grad. 120.24'.49". reliquum ad semi-circulum, quod est 59.35'.11". est latus QT hujus. Cujus rei demonstrationem exhibent Bartholomæus Pitiscus, Adrianus Metius, & alii. Eam igitur hac epitome minimè repetendam censeo.

12. ***Unde trianguli Sphærici datis tribus angulis, facili conversione acquirantur latera.***
Ut prædedentis trianguli QRT dentur anguli Q 47. R 111. & T 34. Quærantur autem latera. Pro angulo quovis unico, verbi gratia (ut supra) pro R 111. sumatur suum ad semicirculum reliquum 69. grad. Inde positis 47.69. & 34. pro lateribus, ut triangulo superiore PZS factum est, per quemvis ex tribus modis suprascriptis, quare illius angulos, & invenies contra latus 47, angulum 42.29'.59" : & contra latus 34, angulum 31.6'.5". & contra latus 69 (quod pro 111 posuimus) reperies angulum 120.24'.49". Ideo in triangulo oblato QRT, pro latere RT. subtendere angulum Q 47, pone 42.29'.59". Et pro latere QR, subtendere angulum T. 34, pone 31.6'.5". verum pro latere QT subtendere angulum R 111, pone 59.35'.11". quæ prius pro 111 sumpseras suum ad semi-circlulum reliquum, scilicet 69. Et ita ex angulis per conversionem acquires latera.

Admonitio.

Ex hac laterum per angulos datos, inventione tres variæ solvuntur hujus, & cujuscunque trianguli quæstiones. Ut in triangulo PZS. Ex datis, hora diei, plaga solis, & angulo situs positionis solis, hæc præcedens satisfacit quæstioni, qua vel primò elevatio poli, vel secundò [p. 57.] altitudo solis, vel tertiò declinatio solis quæratur. Ex octava itaque sect.præcedentis cap.5. & septima ac duodecima hujus, sexaginta habes variarum quæstionum solutiones, quæ in quodque triangulum cadunt. Nec his plures ex multiplici trium quarumlibet partium compositione oriri possunt variationes. Perfectam igitur habes, & absolutam triangulorum tam Sphæricorum, quam planorum doctrinam.

CONCLUSIO.

Satis ergo jam ostensum est quod sint, quid sint, & cuius usus sint Logarithmi : Eorum enim beneficio absque multiplicationis, divisionis, aut radicum extractionis molestia, omnis Geometricæ quæstionis solutionem logisticam promptissimè exhibiri, tum apodeicticè demonstravimus, tum exemplis utruiusque Trigonometriæ docuimus. Promissum itaque mirificium Logarithmorum canonem habetis, eiusque amplissimum usum: quæ si vobis eruditioribus grata fore ex rescriptis vestris intellexero, animus mihi addetur, ad tabulæ condendæ methodum in lucem etiam proferendam. Interim hoc brevi opusuclo fruamini, Deoque opifici summo, omniumque operum bonorum opitulatori laudem summam & gloriam tribuite.

Sequitur Tabula seu canon Logarithmorum.