BOOK ONE.

CONCERNING THE COMPUTATIONS OF MAGNITUDES BY ALL KINDS OF COMMON LOGISTICS.

CHAPTER I.

PRIMARY CALCULATIONS.

Logistics is the science of performing skilful calculations.

A Calculation is the process or operation which finds the answer sought from several magnitudes, and from the given properties of the magnitudes.

But basic calculations are neither called by name, nor distinguished by graphical signs.

Hence in all cases, Logistics proceed initially with a name and a notation ; a calculation soon succeeds by using these.

[Napier looks at arithmetic from two points of view : in the first place, there is the simple process, of finding the sum or difference of two given numbers; one does this by a primitive calculation of some kind, such as a child counting on their fingers, from a memorized set of additions and subtractions, perhaps using an abacus, the calculi or counting stones of the ancients, etc. On the other hand, there are more sophisticated logical, or logistic processes, in the art of combining a set of such calculations in an optimum and skilful manner, such as we find in more complicated multiplications involving a number of simple multiplications, or in long division, root extractions of various orders, etc. : these form the subject matter of this book, and for which the simpler procedures are hopelessly inadequate.]

Moreover, a calculation is either simple or composite.

A simple calculation is one which finds a single third number from two given numbers, or one that is found by a single operation.

A simple calculation is either primary, or it has arisen from a sequence of primary computations.

A calculation is primary when one quantity is combined with another quantity only once.

And this calculation finds some given third quantity, from the wholes, parts, and remainder, of some two given numbers: Which will be shown soon by the following examples.

Hence moreover the calculation is either that of addition or subtraction.

Addition is a primary calculation in which several numbers are added together, and the total is produced.

For example, 3 and 4 may be added, and 7 is produced for the total; likewise 2, 3, and 4 may be added, and 9 will be produced for the total.

Subtraction also is a primary computation, in which the subtractor is taken away from that subtractand, and a remainder is produced.

So that by taking 4 from 9, 5 remains. Moreover it is said that on 4 being taken away, 9 is diminished, and 5 remains. Thus on removing 3 from 5, by subtraction there remains 2.

Subtraction is either of equal quantities, and nothing remains, or of unequal quantities.

Truly the unequal quantity is either by the subtraction of a smaller quantity from a larger one, and there remains a quantity greater than nothing, or of a larger quantity from a smaller one, and the remainder will be less than nothing.

So that by subtracting 5 from 5 nothing remains; on subtracting 3 from 5, 2 remains, certainly greater than nothing ; but on subtracting 7 from 5, 2 remains less than nothing, or nothing less by two.

Therefore from these it is agreed without doubt that defective quantities take their origin from the subtraction of a greater from a lesser quantity: by which action it is put into its proper place.

From what has been said already, it is clear that both addition and subtraction are related ; and thus the checking of one is related to the other.

Indeed we define only that examination which is in agreement by calculation for the whole as well as for the individual parts.

As above, for checking subtraction, or with 3 subtracted from 5 leaving 2, with 2 and 3 both added, 5 is put in place. And conversely, from the checking of addition, or 2 and 3 making 5, by subtracting 3 from 5, again 2 is produced in their place ; or again, with 2 subtracted from 5, 3 will be produced as before.

There is, besides, another check of subtraction of numbers between themselves, clearly by subtracting the remainder from the number to be diminished, as the first to be subtracted may be left.

So that for a check, or 3 subtracted from 5 leaves 2, by subtracting 2 from 5, again there is left 3.

And thus, out of the total of the parts and the remainder, with any two given, you have, the third, by addition or subtraction.

[Thus, we have talked about the *subtractor*, or the number to be taken away, that does the subtraction ; the *subtractand*, or the number to be diminished by subtraction, or from which the subtractor is taken; and the *remainder*, the number left when the operation of subtraction is performed. This is similar to the use of the words *divisor* and *dividend* used in describing division below. Later we make use of the *radicand*, the name given to the number for which we wish to find the root. Napier uses Latin phrases equivalent to : *that which is required to be subtracted* and *that which is required to be diminished* , etc.]

CHAPTER II.

CALCULATIONS ARISING FROM THESE PRIMARY OPERATIONS.

So far we have been concerned with primary calculations ; those that follow are derived from the primary calculations.

Derived [*i.e.* Arisen] quantities are those which calculate a quantity from a given quantity by acting a number of times. And these have a natural beginning coming from the former, being continued a number of times, [*i.e.* in iterative processes.]

Likewise derived quantities come either directly from the primary quantities or from those by continuation.

Derived quantities are from the primary kind, which find a third quantity, a part with a given name, from the whole and parts of some two given quantities. These will soon be shown from examples.

Moreover, derived quantities come from the primary parts by multiplication or continued addition, or by division or continued subtraction.

Therefore there is MULTIPLICATION, whenever either one or the other of the given quantities is required to be added to the other one ; and the process is said to produce that multiple.

So that e.g. to multiply 3 by 5 it is either by three added fives times consecutively, which makes 15 ; or by five added three times, which makes just as many also. And of these 3 and 5, the one is the multiplier, and the other the multiplicand, and truly the product 15 is called the multiple of the two.

In these themselves, one is to either the multiplier or the multiplied number as the other is to the multiple of both.

As in the above example, so that 1 to 3 is itself found, as 5 to 15; or thus 1 is to 5, as 3 to 15.

The kinds of multiplications are infinite ; as by doubling, which is the multiplication of a given quantity by 2; by tripling, which is multiplying by 3; quadrupling, which is multiplying by 4; and thus henceforth.

In these the multiplicands are said to be doubled, tripled, quadrupled, etc.; the multipliers being 2, 3, 4, etc.; the multiples are said to be the doubles, triples, quadruples, &c.

DIVISION is the subtraction of a portion from a number to be portioned out, continued until nothing remains ; and the number of subtractions is the quotient sought.

So that 15 may be portioned out by 5, then 5 parts shall be continued to be taken from 15 by subtraction until nothing remains, and the subtractions involved are three in number : therefore 3 is the quotient sought, 15 is the number to be shared out, and 5 the size of the portion.

In these 1 is to the quotient of either portion, as the other portion is to all the original portions.

As in the above example, thus 1 is to 3, as 5 to 15; or 1 to 5, as 3 to 15.

Moreover the partition is either uniform, and gives rise to unity, or unequal.

The inequality of division of the portions or parts on the one hand is either that of the lesser by the greater, and the quotient is a fraction, or a split quantity less than one ; or, on the other hand, of the greater by the lesser, and the quotient becomes a number greater than one.

So that if 10 shall be portioned out into 10 shares, precisely one will be produced for each portion; but in the second place, if 10 shall be shared out into 13 portions, ten thirteenth parts of unity are produced for each portion, and these are less than unity; in the third place, with 10 parts portioned out by parts of 5, 2 portions are produced greater than unity.

Again, the division of a number into larger portions by a lesser number either is perfect or imperfect.

These are perfect, when there is no remainder. In these the quotient is a whole number.

So that in the above example of 10 parts divided into portions of 5, 2 such portions precisely is produced, and without a remainder.

Indeed, it is the imperfect division which leaves behind a remainder.

As if 16 parts are to be offered, required in portions of 5, and thence 3 will be produced, and you will allow a single one to remain undivided, then it will be said to be an imperfect division.

Therefore it will be agreed from these, that fractions arise to be drawn out from the imperfect division both of lesser by greater, as well as from the imperfect division of greater by lesser: these we will discuss in their place.

In division, the magnitude which is offered for division is called the dividend ; the other given part the portion or divisor ; that which arises from the division is called the quotient ; and if anything remains, to be the remainder.

As in the preceding example, 16 is said to be required to be divided, 5 the divisor, 3 the quotient, and with a remainder of one left over.

The kinds of divisions are infinite; as by two parts [bipartite], which is the division of a given quantity into two equal parts ; tripartite, which is into three; quadripartite, which is into four; and thus henceforth.

In these the divisors are called the bisector, trisector, quadrisector, &c.; the divisors are 2, 3, 4, &c. ; but the quotients are called the half, third, quarter parts, &c.

From what has been put in place it is agreed that multiplication and perfect division are related, and we may check one or the other.

As if you may doubt that 3 multiplied by 5 gives 15, on an check of 15 shared into portions of 5, and from that 3 parts are returned, you know yourself to have multiplied correctly: or otherwise, to part 15 into portions of 3 and 5 will be returned, as at first. Likewise, if you may doubt that 16 divided per 5 gives 3, with one remaining undivided, on multiplying 3 by 5, 16 is not returned, but 15; one will be required to be added on inspection, and the division is proven to be imperfect by one.

Besides this division, there is the check of the other divisor into that dividend, without doubt divided by the former quotient, so that the former quotient may be returned thence as the divisor.

As by inspection that 15 divided by 5 gives 3, to divide 15 by 3, and 5 is given, as before.
And thus from the whole, the parts, and from a known kind of part, with any two given, you have the third part both by multiplication and division.

CHAPTER III.

EVALUATIONS DERIVED FROM THE FIRST CALCULATIONS : THE MULTIPLICATION AND DIVISION OF ROOTS [BASES].

So far the magnitudes arising have come from primary evaluations; those following come from these first calculations.
Magnitudes arising from the first calculations are those which, from any two given, of the radix and index, and of the square root, the third can be found. [Thus, for a given magnitude, the number uses in the division is called the radix (base), and the number of time it is applied to reduce the magnitude to 1 is the index; thus, $16 = 2^4$, so that in Napier's terminology, the radix 2 is applied 4 times to the initial magnitude 16 to reach 1; hence 4 is the index, and we will call 16 the radicand, in common with the usage of similar words elsewhere.]

The radix [base] is the magnitude that returns some magnitude to unity on dividing a number of times ; and the quotient from that division is called the index ; moreover the divisor is itself the radix.

As from these three terms, 32, 5, and 2, 32 is called the radicand [I have Anglicised Napier's Latin term, corresponding to the magnitude to which the division process is applied.], because that divided by 2 five times returns unity; it is evident that the first division gives 16, the second 8, the third 4, the fourth 2, and finally the fifth 1: And therefore of these, 5 is the index, and 2 the radix [base] .

Moreover the magnitudes arising from the first derived quantities are either the multiplication radicands arising from continued multiplication, or the division radicands, arising from the root extraction, or by continued division.

The MULTIPLICATION RADICAND is the whole radicand presented, arising by multiplication of the radix as many times as the index is a multiple of one ; and the radicand required is found.

So that 2 may be said to be the multiplied root-wise three times, since 8 thence is produced, because the third index holds unity three times ; and from the first by a single multiplication there arises 2, from the second 4, from the third 8, which indeed is the radicand sought.

There are an infinite number of kinds of multiplication radicands : So that doubling, which is equal in turn to a multiple of two, or two times the given magnitude in place ; tripling, which is three times the given magnitude in place, or of three given equal amounts.

In these the radicand is said to be multiplied by itself twice, three times, four times : The indices,– two, three, four: The roots, – the bipartite [square], the tripartite [cube], the quadripartite [fourth].

As on putting two for the radix, and two for the index, two of two makes 4 on doubling: And with the third index with the same two, evidently twice two doubled makes the triple 8 : Thus with the fourth index with just as many doubled, evidently twice two by twice two makes the quadruple 16. And thus indefinitely on putting two for the radix, as in the following table, the index is the first row of which, and the second is of the radicands.

	I.	II.	III.	IIII.	V.	VI.	VII.	&c.
1.	2.	4.	8.	16.	32.	64.	128.	&c.

The division of the radicand by the radix is to be continued so many times until one is reached, and the number of divisions is the index sought.

So that the radicand 8 may be continued to be divided by the radix 2, until one is reached, makes 4 in the first division, 2 in the second division, and 1 in the third. From which third division, clearly the number of divisions, is the index sought.

Here there are kinds of indices and radices, as in the multiplication of radicands above.

CHAPTER IV.

ON THE EXTRACTION OF ROOTS.

The extraction of a root, with a given index, is the quantity found which restores the given radicand by multiplication of the root; also the root divides the divided part of the radicand.

So that with the radicand 8 if the tripartite [*i.e.* the triple-divided or cube] root is sought, that, by the rules according to its place, it is found to be 2; for triplications of the root restores 2 by the first, 4 by the second, and the radicand 8 by the third: Likewise the radicand in the opposite sense is reduced to the radicand 4 by the first division, then to 2, and to 1 in the third.

The extraction of the root is either perfect or imperfect.

The extractions are perfect when nothing remains left over.

As in the above example.

Truly extractions of roots are imperfect, when some items are left unresolved.

As if the tripartite root were to be extracted from the radicand 9, as that shall be approximately 2, which approximate root in triplicate returns 8, and not 9; therefore with the remaining one not extracted, the extraction is said to be imperfect.

Here also the radicands are kinds of roots and indices, as they were above in the multiplication of radicands. And which after the extraction has been done, remainders will be left, are said to be irresolvable.

Because from the imperfect extraction a smaller term arises, to which if one is added it will be a larger term, between which the true and perfect term is contained and the root lies hidden.

As in the above example the imperfect root of nine arises consisting of two, to which if you add one, a root consisting of three comes about, between which the true and perfect triple-divided root of nine lies hidden.

Of course Geometers, with the greater accuracy of the studious, prefer to designate the radicand by an indicated sign, rather than to include the root between limits.

As in the above example the triple-divided root of nine they write thus as, $\sqrt{c}9$: that they call the cube root of nine. But this we designate thus, L9, and we call it the tripartite root of nine: concerning which we will talk about the signs further in their place.

Here the condensed numbers of the Geometers have their origin [Napier has called them so, presumably because they can always have something extra added on to make them more precise], which they call irrationals or surds.

In these calculations with roots, some indices are even, others are odd; again some are prime, that is, they can be divided by one only ; others are composite, that is, they can be divided perfectly by some number.

So that the indices 2, 4, 6, are even ; 3, 5, 7, odd: But the indices 2, 3, 5, 7, 11, are prime, nor to be divided by any number : but 4, 6, 8, 9, 10, &c. have been composed from numbers; certainly 4 from two by two ; 6 from two by three, or from three by two ; 8 from two by four, or from four by two ; 9 from three by three ; 10 from two by five.

Hence when the indices are composite the calculation can be done with the roots multiplied together, and a shortening of the extraction, for the indices are easier multiplied, or the roots extracted, one after the other through the components, than by the composite index.

For example, extracting the quadripartite root [*i.e.* so that the original number or radicand is reduced to four equal parts, or the fourth root, multiplied together] from a given radicand in a single operation is more difficult than if at first you extract the bipartite root [*i.e.* that consisting of two parts] and then perform another bipartite extraction. Thus the sextuplicate root, or the sextipartite root required to be extracted, shall not be as easy as if at first you extract the tripartite root, then you extract the bipartite root. Note that the same root is repeated by the same method. An example with numbers: The sextipartite [sixth] root is easily extracted from 64 by initially extracting the tripartite root, so that 4 arises, then the bipartite root is extracted, so that 2 arises ; or in the first place by extracting the bipartite root from 64, so that 8 arises, thence by extracting the tripartite root from 8, so that 2 is made, the sextipartite root sought.

It is gathered from the foregoing presentation of multiplication, divisions, and the extractions of roots, that two individual calculations have to be checked ; certainly multiplication is to be approved either by division or by root extraction ; division is to be approved either by multiplication or root extraction ; and root extraction, by either multiplication or division.

As in the above example of the root of 32, index 5, and the root 2, if you may doubt whether or not 32 shall be the five times multiple of two, divide 32 by 2 and into 5 equal roots, the first index ; or by extracting the quintipartite root of 32 and you come upon 2, the first root; from which it is augmented to be 32, the true radicand: Likewise if you doubt that 2 shall be the root, divide 32 root wise, and you come upon the index 5; or from the same quintuple of 2, you come upon 32: And then to test whether or not 5 shall be the true index, by extracting the quintipartite root of 32, you arrive at 2, or from the quintuple of the root 2, you arrive at 32.

And thus you have from any two given of the radicand, the index, and the root, the third follows by multiplication, division, and by extraction from the radicand.

CHAPTER V.

CONCERNING COMPOSITE CALCULATIOS.

Up to this stage computations have been simple ; those that follow are composite or regulated.

A calculation is composite which produces the answer sought from several given quantities, and by several different kinds of operations.

Composite calculations, or according to rules [or first principles], either are of proportionate, or of disproportionate quantities.

They are the rules of proportions, which evidently by single simple multiplications and divisions, find the quantity sought from several given quantities.

As if there may be sought, someone who proceeds four miles in three hours, how many miles will he proceed in six hours? Likewise if six oxen are fed with three measures of hay in four days, and it is asked what number of oxen are to be fed with five measures in two days ? Likewise 20 Scottish shillings make one pound, 2 pounds make three marks, 5 marks are worth a single crown; therefore how many shillings will be worth 9 crowns ? Questions of proportions are introduced without a single addition or subtraction. For multiplications and divisions are proportionals as a consequence of their definitions.

In these a situation and the operation [required to find the unknown] are considered.

Four situations [or examples] have been prepared: so that, in the first place, a line drawn is prepared for the quantity sought, with its collateral quantities located relative to the line as follows, for the above proposed examples. [Rule I.]

For the second and third example, it is a quicker and easier way, to join the smaller given numbers under the line together with the quantity sought; but whether or not this is a safer way is not yet clear.

First example.

6 hours, 4 miles;
3 hours, miles sought.

Second example.

6 oxen 5 measures 4 days.
how many oxen? 3 measures 2 days.

Third example.

$$\frac{\text{20 shillings 2 pounds 5 marks. 9 crowns}}{\text{how many shillings? 1 pound 3 marks 1 crown.}}$$

In the second place, for two quantities, of which the one by increasing decreases the other, the collaterals are put in place on the same side of the line. [Rule II.]

As in the first example, when there were so many more hours at first, indeed three, so much fewer will be the number of miles sought afterwards. Thus, (as in the second example), by increasing the number of oxen, the number of days will decrease in which they are fed with the fodder. From which there shall be 3 hours and the number of miles sought, as well as 6 hours and 4 miles; and likewise 6 oxen and 4 days, and the number of oxen sought and 2 days, each gathered together on the same side of the line.

In the third place, so that two quantities, likewise increasing or decreasing, are set up on opposite sides of the line. [Rule III.]

So that with a 3 hour increase, also a 4 mile increase will be necessary, and vice versa. Thus with the increase of 6 hours, the increase in the miles is asked for, and vice versa. Likewise with the increase in the number of oxen, it is necessary also that their fodder be increased, and vice versa, and to be diminished with less oxen. Thus to be increased with the number of days, and to be diminished with less days. In the third example, with the increase or decrease in the number of shillings, likewise also it is necessary that the former pounds, given equal to these, must be increased or decreased. Thus with the latter pounds, evidently 2 pounds is equal to the first 3 marks, increased or decreased by these ; and since with the latter, 5 marks makes one crown. And finally with 9 crowns, the question of the number of shillings likewise is by necessity increased or diminished. From which, so that the above amount can be determined, from these two individual quantities put in place below the line, and with the other quantities put in place above,.

In the fourth place, that two synonymous quantities shall always be separated by that line. [Rule IV.]

As in the above examples, separated by the interposed line ; in the first example: 3 hours to 6 hours after, and 4 miles to the sought number of miles ; and in the second example : 6 oxen to the number of oxen sought after, 5 measures of hay to 3 measures after, and 4 days to 2 days after; and in the third example 20 shillings to the number sought after, and 2 pounds to 1, and 5 marks to 3 marks, and 9 crowns to 1 crown.

From these observations, so that all questions of this kind may be addressed by a single general solution, by this precept of the operation :

Multiply the quantities above the line in turn, and likewise those below the line in turn, then divide the upper multiple by the lower multiple, and the quotient will be the amount sought satisfying the question.

As in the first example, multiply the upper numbers 6 and 4 in turn, making 24; which divided by the lower number 3, and there arises 8, the number of miles sought. In the second example, multiply the above 6, 5, and 4 in turn, and it makes 120; then multiply 8 by 2, and they make 6; divide 120 by which, and 20 will be produces, the number of oxen satisfying the second question. Likewise in the third question multiply 20, 2, 5, and by 9 in turn, they make 1800; then multiply 1, 3, and 1 in turn, they make only 3; by which 1800 is divided, there arises 600, the number of shillings of the value 9 crowns.

[Thus, quantities are gathered together above and under the line which are in inverse proportion to each other; quantities that are in direct proportion must go on opposite sides of the line. This for example if *d* is proportional to *a* and *b*, and inversely proportional to *c*, then $d = \frac{ab}{c}$ and $1 = \frac{ab}{cd} = \frac{a'b'}{xd'}$; then $x = \frac{a'b'cd}{d'ab}$. We see that the variables *a*, *b*, *c*, & *d* obey Napier's rules.]

And thus we understand all kinds of regular proportions by a single general method and operation.

From this limitless principle, – as the rules of three or the golden rule, directly or inversely for simple, five of six quantities, &c.,– the kinds and forms the authorities treat, but not yet triples, or other forms of multiples they touch on, all of which you have here briefly.

And these are all the rules of proportion ; the rules of the disproportionate will follow : But because these, besides calculating proportionates, also calculate additions and subtractions, and other calculations have upsetting mixed proportions, thus we cause all these to be dismissed, so that for all these a single one will take the place of algebra for us.

So that rules of the disproportionate shall be the greater part of all the rules of arithmetic, of alligations [*i.e.* mixing of proportions of herbs in a remedy, etc.], association, of false position, single, double and numerous others ; likewise of geometrical propositions, problems, theorems, etc. which, by confusing both by variety and multitude, disturb the memory ;– these therefore we abandon, requiring to be treated by algebra.

CHAPTER VI.

CONCERNING ABUNDANT AND DEFECTIVE QUANTITIES.

Up until now the calculations have been of magnitudes in general ; those that follow are of their own kinds.

First, therefore, magnitudes are either abundant or defective.
Abundant numbers are magnitudes greater than nothing, and by increasing carry themselves forward.
These, either with no sign, or with this + sign, which is said to increase the quantities by joining them together, is noted down before the number.
As if the wealth of 100 crowns is considered without a sign ; these may be expressed either as , 100 crowns, or thus, +100 crowns ; and thus they may be pronounced, as an increase of 100 crowns, always convenient for signifying money.

The calculations of these quantities are found both from what has gone before as well as from what follows.
Defective numbers are magnitudes smaller than nothing, and they themselves carry forwards a diminution.
These are always preceded by the sign – , which is said to be the bond of diminution [that is, the negative sign ; Napier uses the Latin word *copula* : usually meaning a string or connection of some kind, or some more abstract connection between two items, here designating a sign or coupling between two numbers, which we will simply call the + or – sign.]

So that if someone's wealth were valued, whose debts exceed his wealth by 100 crowns, the value of his wealth thus may be written down as –100 crowns; and thus his wealth may be expressed to be less than nothing by one hundred crowns, always by signifying loss and deficiency.

We have shown above that the origin and beginnings of defective numbers arise from the subtraction of a greater number from a lesser number.
Abundant and defective numbers are added, if the numbers to be joined are the same kind, by writing down the common sign from their aggregate.
So that by adding +3 and +2, they make +5: Likewise by adding –4 and –6, they make –10.
Truly if the numbers bound together are dissimilar, the differences of these are added by putting in place the sign of the greater part first.
So that by requiring to add +6 and –4, they make +2: Thus –6 and +4 added, make –2, the sign of the greater, evidently of six, always by setting down the sign of the greater difference first.

But the numbers are subtracted, if you change the sign of the number to be subtracted, and you may add that to the other given number by the given rules.

So that +5 is to be subtracted from +8; by changing +5 into –5, and by that presented before by adding –5 to +8, and there becomes +3 for the remainder sought from the subtraction: Likewise it is required to subtract +8 from –5; by changing +8 into –8, and by adding that to –5, also they make –13, the remainder sought: Thus by requiring to subtract –5 from +8; +13 is made ; and +5 from –8, makes –13; and –5 from –8, makes –3; and +8 from +5, becomes –3; and –8 from +5, becomes +13; and –8 from –5, makes +3.

Abundant and defective numbers are multiplied are multiplied and divided, if the signs are similar, by putting in place the sign of more [*i.e*. the positive sign] before some multiple or quotient ; and if the signs shall be dissimilar, by affixing the sign of the less [*i.e*. the negative sign].

So that if +3 is required to be multiplied by +2, or –3 by –2 the multiple +6 is produced; and if +6 shall be divided by +3, or –6 by –3, the quotient +2 is produced; truly if you wish to multiply +3 by –2 or –3 by +2, –6 will be produced, the multiple sought ; and if you should divide +6 by –3, or –6 by +3, the quotient –2 is produced.

The roots, both abundant as well as defective, multiplied by an even index, produce abundant radicands.

So that the root shall be +2, that you will multiply to the index 4, and they become : firstly +2, secondly +4, thirdly +8, fourthly + 16 likewise abundant ; similarly, –2 multiplied makes firstly –2, secondly +4, thirdly –8, fourthly the same abundant number +16 as above.

Hence it follows, that there are two equal roots for an even index of a positive radicand, the one positive and the other negative ; truly none for deficient or [negative] radicands.

As in the above example of the abundant radicand +16, both the abundant +2, as well as the defective –2, will be quadripartite roots, as from the above and will be apparent by checking each. Thence nothing remains, either abundant or defective, which shall be the quadripartite root of the defective radicand –16.

Abundant roots with an odd index return abundant radicands (by multiplication of the roots), and defective roots return defective radicands.

So that the abundant root +2, with the odd index 5, by multiplying the roots consecutively returns +32; evidently, firstly +2, secondly +4, thirdly +8, fourthly + 16, fifthly +32, the abundant radicand. Thus the defective root –2, with the index 5, multiplying the roots consecutively makes –32; evidently, firstly –2, secondly +4, thirdly –8, fourthly +16, finally the fifth –32, the radicand of the said defective root.

Hence it follows in a similar manner that the radicands of roots with an odd index have only a single root ; abundant from abundant ; and defective from defective.

As in the above example of the abundant radicand +32, with the index 5, it will have the abundant root +2. Thus the defective radicand –32, with the same index, will have the defective root –2, as in the preceding example, and each agrees clearly on checking.

The rule of proportion is not required to be repeated here, as that may be composed from multiplications and divisions, and is acquired by that presented already.

CHAPTER VII.

CONCERNING FRACTIONAL MAGNITUDES.

Until now we have considered the first division of the magnitudes ; the second now follows.

In the second division, the magnitudes are either whole numbers or fractions.

Here we call magnitudes whole, which either have a denominator of one, or they have no denominator. Moreover we have the calculations of whole numbers *per se* from the preceding; but the calculations of fractions indeed we have from the following.

We call numbers fractions which have a denominator different from unity and placed under the numerator.

The denominator is the magnitude placed below the line, which dividing by some number of parts shall indicate the total.

The numerator moreover is the magnitude placed above the line, which denotes from these parts how many shall be taken.

For example, this magnitude $3ab$, is a whole magnitude. Thus (since it is the same) $\frac{3ab}{1}$ is also whole, yet according to a kind of fraction. Likewise $\frac{3ab}{2bc}$, $\frac{5a}{2}$, and $\frac{3a}{2a}$, or since it is the same, $\frac{3}{2}$, are fractions, or fractured magnitudes, the upper terms of which are the numerators, and the lower are the denominators.

We have shown that fractional quantities greater than one have their origins in the imperfect divisions by the lesser magnitude, and fractional magnitudes less than unity, by the division of a minor by a greater magnitude.

As in dividing 9 by 2 , $4\frac{1}{2}$ is produced, or, if you prefer, $\frac{9}{2}$ greater than one. Likewise if you divide 3 by 5 , $\frac{3}{5}$ arises, as we have shown in the division above.

Hence the numerator bears all the magnitude to be divided; and the denominator truly, all of the magnitude required to divide that.

As in the above example $\frac{3ab}{2bc}$ likewise signifies that $3ab$ is divided by $2bc$; thus likewise $\frac{3a}{2a}$ gives rise to the same as $3a$ divided by $2a$, or shorter, 3 divided by 2; of if finally likewise it emerges that three parts of unity divided in two; thus $\frac{3}{4}$ are three quarters of one, or three parts of four, that is the same.

And the whole magnitude, having a numerator and denominator, is take for the fraction, and may be calculated as a fraction.

Hence sensibly we may put one for the denominator of whole numbers, so that whole numbers may be worked out as if they were fractions, with fractions.

But fractions are easier worked out if their terms are taken together and shortened, before they are increased by being operated on.

Moreover, they are abbreviated and collected together, by the dividing terms being increased by their greatest common divisor.

Moreover, the greatest common divisor is the greater given, by which each term given cannot be divide perfectly.

This may be found, by dividing the greater term by the lesser, for the first fraction, and then always by dividing the preceding divisor by its remainder, until at last nothing remains ; and that divisor finally (with the quotient taken) is the greatest common divisor sought. [The division algorithm.]

So that the maximum common divisor of the terms 55 and 15 may be obtained thus; divide 55 by 15, 10 will remain; divide 15 by 10, 5 will remain; divide 10 by 5 and nothing will remain : 5 therefore is the greatest common divisor, dividing15 by 3, and 55 by 11.

Truly if you come to a unit divisor, you cannot shorten the numbers, yet the discrete parts are the terms, or are had in turn as discrete magnitudes [*i.e.* they are relatively prime to each other.].
So that the terms may be 5*a* and 3*a;* with the part 5*a* divided by 3*a*, 2*a* remains*;* then with the part 3*a* divided by 2*a*, 1*a* remains*;* with 2*a* divided by which, nothing remains. Hence 5*a* et 3*a* have no greater divisor with unity, apart from 1*a;* if it is divided by which they will become between themselves as the discrete [*i.e.* prime] numbers 5 and 3, as will be discussed later in its proper place.

Truly here there is a great need to beware of the division of incommensurate magnitudes, of which there will never be an end [*i.e.* irrational numbers], so that in its place it is seen to be avoided.
As of ten, and of its square root, that may be called the square of the root, no common measure can ever be found ; much less that maximum divisor, as we discuss in its place later.

With the terms that cannot be abbreviated, so the shortening of these is by a defective operation.
By obtaining a maximum common divisor, and with each term divided by that, new abbreviated terms are generated ; and this operation is called abbreviation.

CHAPTER VIII.

CONCERNING THE RECKONING OF FRACTIONAL MAGNITUDES.

ADDITIONS and subtractions are of fractions of the same denomination.

If they are of different denominations, they are reduced to the same denomination.

But they are reduced, with the divisors of each by the denominators from their own greatest common divisor, with the quotients noted.

Thence by multiplying each term of the first by the quotient of the second denominator, and it becomes the new first term of the numerator; and the terms from each of the latter by the first quotient of the first denominator, and that becomes the new second term of the same denominator.

As the fractions $\frac{2}{3}$ and $\frac{7}{9}$ are required to be reduced to the same denomination ; 3 is the greatest common divisor of the denominators 3 and 9, with the denominators divided by which there arises 1 for the former, and 3 for the latter ; then, by multiplying each term of $\frac{2}{3}$, by the latter quotient 3, $\frac{6}{9}$ arises and for the new first term. Thus by multiplying $\frac{7}{9}$ by unity (clearly the quotient of the first), $\frac{7}{9}$ arises with the denomination of the same as $\frac{6}{9}$.

Now the new numerators of these, with the same denominations, are added or subtracted, on retaining the new and common denominator, and you will have the total of the addition, and the remainder of the subtraction.

As in the above example, the new numerators 6 and 7 are added, and 13 arises, which, with the common denominator 9, makes $\frac{13}{9}$ for the whole of the addition. Thus if you should subtract $\frac{6}{9}$ from $\frac{7}{9}$, $\frac{1}{9}$ will remain the remainder of the subtraction.

Fractions are also multiplied, by dividing the individual two, of which the one is the numerator, and of which the other is the denominator, into their maximum common divisor, with the final quotients of everything noted, then by multiplying the quotients of the numerators in turn, so that they make a new numerator ; and the quotients of the denominators in turn, and they make the new denominator of the denominator sought.

So that $\frac{18}{20}$ and $\frac{35}{231}$ are to be multiplied in turn ; first 18 and 20 are divided by their common maximum divisor 2, they become $\frac{9}{10}$ and $\frac{35}{231}$; then 10 and 35 can be divided by 5, they become 2 and 7, this is put in place, $\frac{9}{2}$ and $\frac{7}{231}$; then 9 and 231 are divided by 3, they become 3 and 77, this is put in place, $\frac{3}{2}$ $\frac{7}{77}$; and then 7 and 77 can be divided by their common maximum divisor 7, and they make 1 and 11, putting in place, $\frac{3}{2}$ $\frac{1}{11}$: With these acted on, take the product of the final numerators, the quotients 3 and 1 in turn, thus take also the product of the final denominators 2 and 11, these make 3, the others 22, in this situation, $\frac{3}{22}$ is the multiple sought. Likewise these irreducible fractions $\frac{2a}{3}$ and $\frac{4}{5}$ shall be the multiplied fractions taken in turn ; in the first place the numerators *2a* and 4 are multiplied in turn, and they make 8*a*, the new numerator; then the denominators 3 and 5 in turn, and they make 15, the new denominator; therefore $\frac{8a}{15}$ is the multiple sought.

By this multiplication the fractions of fractions, indeed, and the fractions of fractions of fractions again and again of fractions, are reduced to simple fractions.

So that while a fifth of three quarters, noted thus, $\frac{2}{5}$ from $\frac{3}{4}$, by the aforesaid becomes in the first place $\frac{1}{5}\frac{3}{2}$ by cancellation ; from that, by multiplication in turn of the numerators and by multiplication in turn of the denominators, becomes $\frac{3}{10}$, a single fixed fraction, the same emerging as the above fraction of a fraction. Thus three quarters of two thirds of one half, thus written, $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{1}{2}$, becomes initially be cancellation $\frac{1}{4}$ of $\frac{2}{1}$ of $\frac{1}{2}$, thence $\frac{1}{4}\frac{1}{1}\frac{1}{1}$ or $\frac{1}{2}\frac{1}{1}\frac{1}{2}$; finally with the above quotient produced in turn, it becomes $\frac{1}{4}$, the same value as $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{1}{2}$.

But by inverting the terms of the divisor, they are divided, and the inverted terms by dividing, always multiply as above in multiplication.

So that there shall be $\frac{3}{10}$ of the penultimate example to be divided by $\frac{3}{4}$; the terms of this divisor by inversion become $\frac{4}{3}$, which multiplied by $\frac{3}{10}$ make at first by cancellation $\frac{1}{10}\frac{4}{1}$, then $\frac{1}{5}\frac{2}{1}$, then by multiplication of the upper terms in turn, and of the lower terms in turn, they make $\frac{2}{5}$, the quotient chosen, and an example of the above multiplication.

But the extraction shall be made by extracting the indicated root both from the numerator as well as from the denominator, of from any terms of the same ratio, and they become the terms of the root sought.

So that if the bipartite root shall be required to be extracted from the fraction $\frac{16}{25}$; the bipartite root of 16 may be extracted, and it is 4; thence the bipartite root of 25 is 5; from that the numerator becomes 4, from this the denominator 5, in that place, $\frac{4}{5}$ becomes the bipartite root of $\frac{16}{25}$ wished. Likewise if the bipartite root of $\frac{3}{4}$ shall be required to be extracted, or, what is better, from $\frac{48}{64}$; for just as the amounts of this same ratio are greater, so much more exact the roots are, unless they can be extracted perfectly. Therefore extract the bipartite root from 48, which cannot be had ; therefore from 49, and that will be 7; thus the root can be extracted from 64, and it is 8; therefore $\frac{7}{8}$ is the bipartite root of $\frac{3}{4}$, truly as an approximation.

The multiplications of roots, and the divisions, and the rule of proportion, which are nothing other than repeated multiplications and divisions, we may refer to the above, from which they are easily obtained.

With these calculations completed, mixed fractions are required to be restored, clearly the numerator of which exceeds the denominator, to their whole and fraction parts. Moreover this restoration shall be made, by dividing the numerator by the denominator, and the whole magnitude arises in the whole quotient, and the remainder left will be the numerator, and the divisor will be the denominator, of that mixed fraction and to be adjoined.

So that if from a complete calculation there may come about $\frac{11}{4}$; 11 can be divided by 4, and 2 is made in the quotient, and 3 remains ; from which the whole number 2, and three quarters of one, in this case $2\frac{3}{4}$, shall be the same as the above $\frac{11}{4}$, transformed and clearer.

THE END OF BOOK I.

SECOND BOOK.

CONCERNING ARITHMETICAL CALCULATIONS.

CHAPTER I.

THE NAMES AND NOTATION OF WHOLE NUMBERS.

Until now the calculations have been of all kinds of common magnitudes ; the following are more individual.

Thus, the calculations are concerned with numbers in three ways either with true numbers, or with fictitious or hypothetical magnitudes. Thus reckoning is either with true numbers, with which Book II and Book III are concerned ; or with fictitious or algebraic magnitudes, with which Book IV is concerned.

Magnitudes called real are magnitudes defined with true names, by which the reckoning process shall be explained, for however many numbers there shall be, or with whatever magnitude.

Real magnitudes are either discrete given by a discrete number, or composite with a composite number for a name.

Thus, the reckoning [logistics] of real numbers is either of discrete magnitudes, which is called Arithmetic, about which Book II is concerned here ; or of composite magnitudes, which is called Geometry, and about which Book III is concerned.

ARITHMETICA, therefore, is the reckoning of discrete magnitudes by discrete numbers.

A number is discrete, that is measured by its own single indivisible number.

A discrete number is either whole [i.e. an integer], or a fraction; thus Arithmetic is the science of integers and fractions.

Integers are those magnitudes, by which it is measured by the number of individual ones enumerated.

Language peculiarities supplies each of the spoken names of the integers; as in Latin,– unum, duo, tria, quatuor, &c.

Moreover these nine signify the written names of integers, or these noted down : 1 one, 2 two, 3 three, 4 four, 5 five, 6 six, 7 seven, 8 eight, 9 nine.

These, in different positions, signify different numbers.

Besides these nine noted, or figures, there is the circle 0, which nowhere signifies any of the places, but is set aside to fill empty positions.

The series of places is considered from the right to the left, in the first of which, the figure is called by its value ; for the second, by tenfold ; for the third, by a hundredfold ; the fourth, by a thousand fold; the fifth, by ten times a thousand fold ; the sixth, by a hundred times a thousand fold ; in the seventh place, by a thousand times a thousand fold ; in the eighth, by ten thousand times a thousand fold. And thus henceforth indefinitely, progressing always by tenfold.

As 7 shall be seven ; but 70 shall be seventy ; 700, truly shall be seven hundred : Thus 8000, eight thousand; 60000, moreover, sixty thousand. From which 68777 indicates sixty eight thousand, seven hundred and seventy seven. Likewise 90680, ninety thousand, six hundred and thirty: And thus for the rest.

Hence the naming of the majority of numbers shall be easily done, if after each third number a point is put in place, the first point you call the thousand ; the second, a thousand of thousands; the third, a thousand thousand of thousands ; the fourth, a thousand thousand thousand of thousands ; and thus for the rest of the points. The figures, truly, from the first place of the point put in place, is called by its own value ; with the second indeed, from the position of the point, by a tenfold value ; and then the third is called by its hundredfold value.

So that here the number, 4734986205048205, is marked with points thus, 4. 784. 986. 205. 048. 205. And this will be the name : four thousand times a thousand by the thousand thousands of thousands, seven hundred and thirty four thousand by the thousand thousands of thousands, nine hundred and eighty six by the thousand thousands of thousands, two hundred and five thousands of thousands, forty eight thousand, two hundred and five: And thus with the rest.

CHAPTER II.

THE ADDITION AND SUBTRACTION OF INTEGERS.

Up to this point we have described the names and notation of integers ; in what follows we consider calculations with integers ; and in the first place with addition and subtraction.

In addition the place of the figure and the operation or practice is to be considered.

130
105
90
70
65
162
65
187
182
600
———
1656

It is the place that the numbers are to be written with the numbers beneath ; thus so that, by beginning from the left, the figures of the first, of the second following, and of the remaining for the rest are put in place directly below, with a line drawn under the lowest number.

There are three precepts of the operation.

The first, that the figures of the first places, are all gathered together into one sum, and of this sum only the first figure from these, is written below the line ; for the remainder, if which there shall be, you bear in mind.

The second precept, so that these others, stored away in the mind, together with the figures of the second position, are added together in a single sum and, also the first figure of this sum from these, is written on the line below ; with the remaining figures of this sum, if which there shall be, you hold in the mind: And this operation is required to be repeated in the last figure of all.

The third precept, so that the figures of the final place, with the latest held in the mind, (if which there shall be) be gathered into a single sum, completed in the last place to the left.

As the fifth chapter of Genesis shows 130 years from the creation of Adam and of the world to Sheth; hence 105 to Enosch, hence 90 to Kenan, hence 70 to Mehalalelem, 65 years to Jered, hence 162 to Henoch, thence 65 to Methuselach, 187 to Lamech, hence 182 to the birth of Noah, from Noah to the start of the flood 600. There is sought, from these added together, the sum of the years from the making of the earth to the flood? Therefore all the years are set up in the correct place, that is, from the margin; thence, initially, the rightmost or first places of the figures 5, 5, 2, 5, 7, 2 are added, and they make 26; 6 is written down below immediately, but 2 is remembered; in the second place, you add this 2 together with the figures of the second place 3, 9, 7, 6, 6, 6, 8, 8 and they will make 55, the first of which 5 you write, keeping the latter in mind ; in the third place, this five held on to, together with 1, 1, 1, 1, 1, 6, with the figures of the final place, you add, and they make 16, which are put in the leftmost places, and the total sum becomes 1656.

Yet an addition properly is of two numbers, so that a third is produced thence.

9754862
<u>863556</u>
10618418

As 9754862 and 868556 may be required to be added; these, from the above precepts, produce the total 10618418, as put in place by the margin. You have an check of this in the following subtraction.

In subtraction there is the place and the operation to be considered.

The place, as in addition, starts from the right; the greater number is placed in the middle, but the lesser number to be subtracted is placed below the number to be lessened, and the remaining number is placed on top, with a line interposed between the upper and the middle numbers ; thus so that from the right the first figure with the first, the second with the second, and the remaining figures with the rest, are written directly above.

The operation, on the other hand, may start most conveniently from the left, or if you wish, from the right, with each served by a single precept in some manner :

So that evidently some lower figure, either nothing when nothing is present below, is subtracted from that figure placed above with that not being smaller, or subtracted with 10 added to the smaller, if smaller ; and with the remaining integer written above that, but only if the upper is less, hence from the extension of the right, it does not exceed the sum of that written above ; otherwise, if it does exceed, then the remaining number put in place before written above is diminished by one, if it shall not be 0, or increased by nine, if it shall be 0.

<u>2690997393</u>
2738154098
47156705

So that 47156705 may be subtracted from 2738154098, the number to be diminished : Let the position of this number above, and of that below, be as shown in the margin, and the most right figures correspond to the most right: Hence initially nothing is taken away from the left 27 (for nothing is present below), 27 remains to be diminished by one, because 4, etc. written below exceeds 3, etc. written above ; and thus 26, the remainder, is to be written for the former 27; then take 4 from 3, yet with 10 added first to this, so that 4 thence can be taken away from 13, and 9 is placed above ; the following three numbers [*i.e.* 715 from 815] may be written above without being diminished, because 7 is less than 8 ; then take 7 from 8, and there remains one with one to be diminished, and thus 0 is to be written above, because of the leftmost 156, etc. is greater than the 154, etc. written above ; then 1 is taken from 1, and 0 remains ; but because the following 56, etc. exceed the 54, etc. written above, and thus for 0, 9 is to be written above; similarly take 5 from 5, and 0 remains, therefore 9 is to be written above, because 6 exceeds the 4 written above; thence 6 is to be taken from 14, and 8 remains, yet write 7, because the figure 7 placed below exceeds 0 above that ; then take 7 from 10, and 3 remains, to be written above correctly, because 0 placed below is less than 9 written above ; then take 0 from 9, and 9 remains, required to be written correctly above, because 5 below is less than 8 above; finally take 5 from 8, and 3 remains, required to be written above correctly, because nothing follows after the outermost figure placed below, that exceeds that placed above : And thus you have the remainder sought, 2690997893.

The checking of this operation can be made from by adding 47156705 to 2690997898, and 288154098 is restored; or by subtracting 2690997893 from 288154098, and 47156705 is restored: And thus in other examples.

Another example, and a check of the above addition.

It is required to subtract 868556 from 10618418, the remaining number will be 9754862, as in the addition above. Likewise it comes about if you work on the two rightmost numbers.

If a greater number may be offered to be subtracted from a smaller number, the lesser number must always be subtracted from the greater, you subtract by the method given ; indeed, the remainder connected by the first noted, and thence a defective number will arise.

So that if 10618418 is to be subtracted from 868556, it becomes, from previously, the subtraction of one from the other, and 9754862 arises (as in the preceding example), which converted into defective number –9754862, is the remainder of the number sought; as in general we have said above in Chapter 6 of Book I about defective magnitudes.

CHAPTER III.

THE MULTIPLICATION OF INTEGERS.

Multiplication is either by a single, or by several figures.
Multiplication by as single figure, either is of one or of several figures.

The multiplication of one figure by another single figure, or of individuals figures by individual figures from memory as most readily, is required to be learned from the table in place.

	9	8	7	6	5	4	3	2	1
1	9	8	7	6	5	4	3	2	1
2	18	16	14	12	10	8	6	4	
3	27	24	21	18	15	12	9		
4	36	32	28	24	20	16			
5	45	40	35	30	25				
6	54	48	42	36					
7	63	56	49						
8	72	64							
9	81								

So that if, what may arise from 7 and 8 multiplied in turn, is asked? Whereby the greater 8, clearly on the upper line, and the smaller 7 in the left line, and you offer 56 in the common angle, the chosen multiple.

The product of the multiple forgotten from any two greater figures, the difference of these from ten multiplied in turn, and the figure of the furthest right will be provided; then the difference from the lesser figure, or the greater from the greater, on being taken away, the leftmost figure will remain, completing the number sought.

$$\begin{array}{c} 8 \times 2 \\ \underline{7 \quad 3} \\ 5 \quad 6 \end{array}$$

As if you may have forgotten how many seven times 8, or (what is the same) eight times 7, make, the difference of these from ten, 3 and 2 multiplied in turn, and 6 comes about, the right figure; then take 2 from 7, or 3 from 8, and 5 will remain, the leftmost figure. And thus 56 is the sought multiple from 7 and 8.

[For, $7 \times 8 = (10-3) \times (10-2) = 100 - 10 \times (2+3) + 6 = 80 - 30 + 6$, or $70 - 20 + 6$.]

The operation of multiplication of several figures by a single figure begins from the right, both the operation as well as the place, and by proceeding in succession to the left ; and with the multiplicand also, put in place either above or before the multiplication, and under each a line is drawn. The multiplication of several figures by a single figure has three precepts in the operation.

The first precept is, that the furthest right figure be multiplied by the given single figure, and of the multiple by a single figure, and the right be noted down (if there shall be two) ; but the left, if which there shall be, is kept in mind.

The second precept is, that this figure, kept in mind (if which there shall be), is added to the multiple of the following figure, in the single multiplication ; the sum, truly by the single figure, or the most right (if there shall be two), is noted down, but the more left figure (if which there shall be) may be kept in mind, being added as before ; and this operation is repeated as far as the most left or to the furthest figure.

The third precept is, that the multiple of the final figure, since it is the most recent held in mind (if which there shall be), be collected into a single sum, with the whole noted in the leftmost places.

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 865091372
         5
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4325456860

 865091372(5
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4325456860
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As 865091372 shall be multiplied by 5, or the quintuple found: The places of these, as everywhere, shall be on the margin: Firstly, and thus the rightmost 2, is multiplied in the single multiplication by 5 ; 10 is made (two figures), of which I mark down the rightmost 0, but 1 I bear in mind; then I draw, or multiply, 7 by 5, they make 35; to which the number reserved is added, making 36; of which the rightmost 6 I note down, with the three kept in mind: I go on ;– three times 5, or five times 3, are 15, and 3, kept in mind, make 18, I note down 8, and 1 I remember; five times 1 follows, making 5, and the1 kept in mind make 6, with the single figure noted, with nothing to be remembered ; then, five times 9 are 45; I write down 5, but with 4 born in mind ; five times 0 follows, which certainly is nothing; this, with the 4 reserved, makes 4 to be noted down : I keep going —five times 5 are 25; with 5 noted, I reserve 2; thence I multiply 6 by 5, 30 is produced, and with the 2 reserved they make 32; I write down 2, I reserve 3; finally, I multiply 8 into 5, they make 40, with which with the reserved 3 added, now with the last of all, and they make 43, which indeed I write correctly in all the most left places. Thus the total, the multiple sought, is 432545860.

Now, there remains, that the multiplication shall be of several figures by several more.

The multiplication of several figures by more figures have three precepts, besides what has been provided before.

In the first place, so that the whole required to be multiplied (by the precepts now presented) may be multiplied in some figure of the multiplicand, free to begin either from the right or from the left.

In the second place, so that the multiple of this figure may have it rightmost figure directly noted under the figure multiplied, and the rest following in order to the left.

The third precept is, so that for these particular multiples another line may be drawn underneath, and all the multiples may be added into a single sum, that certainly will be the total multiple, and the product sought of the whole multiplication.

So that the preceding number 865091372 may be multiplied into 92105: They may be arranged together, with the places on whatever side you please, as from the margin ; then begin from some term of the multiplication, in order either from the right or from the left; *e.g.*, from the right: And thus perform the whole multiplication by 5; it becomes, by the foregoing, 4325456860, which thus may be put in place so that the rightmost figure 0 is placed directly under the figure from the multiplier 5, with the remainder following towards the left: then, the same required to be multiplied by 0, which is nothing, may be multiplied, and nothing arises ; nothing therefore of this multiplication has a need to be noted,: it follows that multiplication by 1 is to be considered; the same arises from the same, *viz*. 865091372, of which the figures, the rightmost 2 by multiplying, is placed under the figure 1, with the rest following to the left : then the same multiplication may be duplicated, or taken by 2 , and it becomes 1730182744, the place of which begins

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     865091372
         92105
    ----------
    4325456860
   865091372
 1730182744
7785822348
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79679240818060
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865091372(92105
---------------
7785822348
 1730182744
   865091372
    4325456860
--------------
79679240818060
```

under its multiplier 2, and then by progressing to the left: then the whole is to be multiplied by 9; 7785822348 arises, which, having its starting place under the 9, hence may be noted down in order towards the left: Finally, with all the particular terms multiplied in turn, you add these numbers between the including lines drawn, which are put in place, and the sum becomes 79679240818060, for the complete multiple found from the multiplication. Neither does it come about otherwise from the second situation, without doubt by beginning the multiplication by the leftmost multiplying figure 9, as is apparent from the second derivation. Similarly also with the similar parts.

You have a check of this in the following division, and as of that this is a check.

The multiplication of three or of more numbers, is performed by multiplying the first and the second, and the multiple of these into a third, and the multiple of these again into a fourth ; and thus until the final.

So that 5, 4, 2, and 3 shall be required to be multiplied in turn ; take 5 by 4, they make 20; by the second, take 20 by 2, they make 40; by the third, take 40 by 8, they make 120, the multiple of everything.

CHAPTER IV.

THE DIVISION OF INTEGERS.

The division of a smaller by a greater number shall not be other than by a line being interposed, between the dividend above and the divisor below, and the whole quotient here is a fraction less than one.

As 3 shall be required to be divided by 5, the quotient shall be $\frac{3}{5}$, which is pronounced, three fifths of unity, or three divided by five ; and it is a fraction.

From which hence the fractional numbers have their origin, as we have taught in Chapter 7, concerning fractional quantities in general. The second part of the Arithmetic will treat these.

We have said above, that an equal divided by an equal, in all magnitudes, produces one.

Yet in the division of a greater magnitude by a lesser, a quotient is produced greater than one always; and hence here division is examined properly.

In this division of a greater by a lesser magnitude, the place and operation is given consideration, each of which begins from the left, tending to the right. The most appropriate position being, that the leftmost figure of the divisor be put in place under the leftmost part of a dividend smaller than that, or before a leftmost divisor greater than that, with the rest of the figures following in order to the right, where the quotient being resolved begins immediately after a bracket. And a line is drawn under everything.

There are four rules [or precepts] of the operation:

In the first place, so that the quotient may be sought out carefully, the number of times the divisor can be taken from the greatest of these leftmost figures is defined in the place of the empty quotient of the nearby bracket: with that place taken usually by a conjecture of this quotient, with the first place being related to the first conjecture ; from which, so that the figure noting this approximate quotient can be put in place in the empty space after the dividend and the bracket.

In the second place, so that the whole divisor may be multiplied into this figure of the quotient recently acquired, and its multiple must be put in place, as in Ch. 3.

In the third place, so that the recently acquired multiple may be subtracted from the upper figures put in place directly above that, these figures also are deleted, both from the subtracted and from those from which they were subtracted, and the remainders may be noted above, by Ch. 2 of this book.

The fourth rule being that these three operations may be repeated, until the whole dividend may be removed, and the residue or the part remaining, either is nothing or gives rise to smaller parts that remain to be divided; and also provided no places of the quotient remain, all the way to the rightmost figure to be divided, empty spaces are left; and finally after the quotient on the right, a line may be drawn, to which the remainder is placed above and the divisor put in place below, they will denote a fraction of the quotient with one.

An example : The 366 days of a leap year, shall be divided by the 7 days in the week, so that it may be known how many weeks there shall be in a leap year: They are put in place as from the margin ; then first it may be enquired how many times 7 can be taken

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  3 6 6
  7 (5
  -----
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      0
    1
  3 6 6
7 (5 2 2/7
  -------
  3 5
    1 4
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from 36 (for 36 are these leftmost figures placed above, the end of which, as one can see, the figure 6, ceases above the empty place of the quotient, close to the bracket), and you may find five times 7 in 36; therefore put 5 in place in the first empty space of the quotient ; in the second place, multiply 7 by 5, they make 35 in the corresponding place, (clearly 5 under 5, and 3 the most left) ; 0 in the third place, take 35 from the 3 and 6 written above, and, with these deleted, one is left to be written above.

Therefore the remainder 16 follows, and to the above sequence of empty places of the quotient, from which 16, (by repeating the work as at first), recount how many times 7 shall be taken away, and you will observe this can happen twice, with 2 left over; therefore in the following and for the final place of the quotient, 2 is noted; then 2 is taken by 7, they make 14, which duly arranged fall under 16, from which take away 14, and with both then deleted, 2 days will be left, or $\frac{2}{7}$ of a week, required to be attached to the quotient, so that the true quotient is made, of 52 and $\frac{2}{7}$ weeks, evidently of two sevenths of a week in a leap year.

Another example.

861094 is required to be divided by 432: as set up in the margin ; clearly, 4 before 8, because 4 is less than 8; then consider how many times 432 may be taken away from 861, guess by taking from 4 and 8, or from 43 and 86, and you find that 4 can be taken twice from 8, and 3 twice from 6 ; but yet 2 cannot be taken from 1; from which (with this conjecture failing) you will put all to be taken only once from everything; therefore 1 is written, for the first figure of the quotient, by which 432 is multiplied, and there is made 432, required to be taken from 861, and 429 will remain. And thus, from 4290 you search how many times you can take 432, with the conjecture made from 4, which is had 9 times into 42, and 6 remains ; 69 also contains 3 nine times, and much more in addition will remain, by which 2 may be contained ; therefore 9 is put in the following empty place for the second figure of the quotient ; then, with 432 taken by 9, 3888 arises, required to be taken from 4290, and 402 remains. And thus, search for how many times 432 can be taken from 4029, similar to the first conjecture, and you find again this can be done nine times also; therefore by another 9, added on to the quotient, multiply 432, and there arises, as above, 3888 requiring to be subtracted from 4029, and to be in excess by 141. Consider, therefore, how many times 432 can be taken away from 1414, and that you find can come about three times.

```
          1 1 8
      1 4 1
    4 0 2
  4 2 9
  8 6 1 0 9 4
4 3 2(1 9 9 3 118/432
  4 3 2
  3 8 8 8
    3 8 8 8
      1 2 9 6
```

Therefore 3 may be put in the final place of the quotient, by which multiply 432, making 1296, to be taken from 1414 put in place above, and finally the remainder 118 is left over, or $\frac{118}{432}$, which may be adjoined to the quotient, and the finished quotient $1993\frac{118}{432}$ will be returned.

A third example, and a test of the above multiplication.

```
                          0
      4 3 2 5 4 5 6 8
    9 0 8 8 4 5 9 4
  1 8 2 1 0 1 7 3 3
7 9 6 7 9 2 4 0 8 1 8 0 6 0
8 6 5 0 9 1 3 7 2(9 2 1 0 5
---------------------------
7 7 8 5 8 2 2 3 4 8
  1 7 3 0 1 8 2 7 4 4
    8 6 5 0 9 1 3 7 2
      4 3 2 5 4 5 6 8 60
```

Let 79679240818060 be divided by 865091372; they may be put in place as from the margin ; then consider how often 8650 may be taken from 79679, or how often 8 from 79, and you see that it can come about nine times, also sufficient for as many times for 6 and 5 and the other numbers following, also taken away nine times ; then with 9 in the place of the quotient, multiply the divisor 865091872 by 9, and it makes 7785822348, noted in the place below, and from these numbers above to be taken away, and 1821017338 will remain requiring to be divided by the offered divisor 86509, &c., and you see that hence you can take this away as many as twice from these (without doubt 8 from 18, and the others from the remaining); therefore with the quotient in place, multiply the divisor by 2, and it becomes 1730182744, to be taken away from the remainder above, and there will remain 908345940, from which, that given divisor can be taken away only once ; therefore with one placed in the quotient, multiply the divisor by 1, and from that there arises the same divisor itself 865091372; with which taken from 908345940, there will remain 432545686, from which the divisor 865091372 indeed cannot be taken once ; thus, by putting 0 into the quotient, you will proceed, and you ask, how many times that divisor may be taken from 4325456860, and you find it can be done five times (with the conjecture made through 8, which can be taken away from 43 five times); therefore with 5 added to the quotient, multiply the divisor by 5, and 4325456860 is produced, also to be taken away from these above equal magnitudes, and nothing is left over of the remainder. Therefore this division has been perfect, and 92105 is the whole quotient ; and it confirms the penultimate example of multiplication, and from that approved, thus it is confirmed for the rest.

CHAPTER V.

MISCELLANEOUS SHORT CUTS IN MULTIPLICATION AND DIVISION.

MULTIPLICATIONS by 10, 100, 1000, or by others constructed from unity and some number of zeros, are easily brought about by adding zeros only: just as many zeros are added to the right as the multiplier has.

So that 865091372 may be multiplied by ten, and it becomes 8650913720; or by a hundred, and it becomes 86509137200; or to be multiplied by 10000000, and it becomes 8650913720000000.

On the other hand, division by 10, 100, 1000, or by other numbers constructed from one and some number of zeros, is easily done by taking from the total, from the rightmost figures divided, as many zeros as the divisor: again the figures cut off are put in place above a line drawn, and the divisor below the line, and with the quotient added on to the fraction of the quotient noted.

So that 865091372 may be divided by 100, it becomes $8650913\frac{72}{100}$ for the quotient. Likewise 8650913720000089 may be divided by 10000000, and the quotient $865091372\frac{89}{10000000}$ is made.

Hence it follows, that multiplications and divisions by any numbers increased by zeros are easily made, with the zeros first removed in the multiplication, then the multiplier and the multiplicand of the remaining figures are evaluated in turn ; and from that with just as many zeros restored to the multiple ; and for division by taking away just as many zeros from the divisor and from the dividend ; and the remaining dividend to be divided by the remaining divisor, and the remainder with the part removed, with a line put between these, added on.

~~1~~ ~~9~~ 8
~~6~~ ~~5~~ ~~2~~ |94
23 (2 8
~~4~~ ~~6~~
~~1~~ ~~8~~ ~~4~~

So that by multiplying 65294 by 2300, first there becomes 1501762 (from multiplying 65294 by 23), then by adding 00 it becomes 150176200, for the whole multiplication. Likewise 65294 may be divided by 2300, as put in place in the margin, thus so that for the 00 cut off by the part indicated 94, written in this manner |94 ; then 652 is divided by 23, and the quotient is made, with the fragment of the remainder, $28\frac{894}{2300}$, or (which is the same) $28\frac{447}{1150}$.

First to consider a number presenting itself easily for dividing by two.

So that 65294 may be divided by two ; who would be unaware in the first place that 3 is the half of 6, and 2 of five, with 1 remaining, by putting in place the following two numbers 12, of which the half is 6; then 9 follows, of which the half is 4, with 1 left (evidently for ten); therefore the final 14 of everything follows, of which the half is 7; and from all 3, 2, 6, 4, 7, 32647 is made, half of the number presented, I say can be found quicker, as set out in the margin.

66294
32647

Hence the fivefold multiple follows most easily, without doubt, from half of the tenfold.

So that 865091372 shall be multiplied by 5; tenfold this, (by the first section of this) is 8650913720, of which the half 432456860 is the fivefold easily found of the number presented.

And on the other hand, it is easier to find the fifth part, from twice the tenth part.

So that the fifth part of 4325456860 is required ; first it is doubled, and it becomes 8650913720; then (from section 2 of this) it is divided by 10, and it becomes 865091372, the fifth part sought from the presented number.

Nor with difficulty will the nine fold be found, by taking the number from its tenfold; or the quadruple, by taking a number from its fivefold ; or the six fold, by adding a number to its fivefold ; or from twice a number, a number equal to itself is to be added. If indeed these multiplications, are performed the most easily by addition or subtraction.

And again from these, likewise facilitated, from the triplicate, the sevenfold and the eightfold, and thus by multiplying by all the nine figures.

In place of examples, there shall be the preceding and the following.

But where the multiplier should include all the nine figures, or the greater part of these, or the quotient of the divisor is seen to be large, it is agreed that the multiplications of the individual terms, either from established tables, or by continued addition, or most easily of all, to have by the bones of our Rhabdology in readiness, and thus with these acquired, to put together the whole multiplication or division.

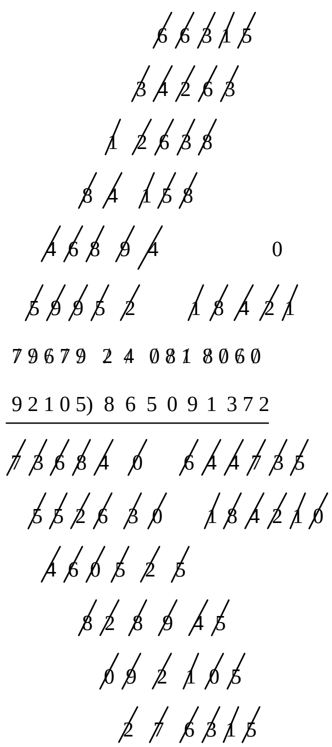
Corresponding terms.	
092105	times 1
184210	times 2
276315	times 3
368420	times 4
460525	times 5
552630	times 6
644735	times 7
736840	times 8
828945	times 9
921050	times 10

So that, to examine the final example in the chapter on multiplication, we may look at what 92105 multiplied by 865091372 produces: Therefore take 92105 by 2, 3, 4, 5, 6, 7, 8, 9, either by the given shorthand multiplication, or by continued addition of the simple number to the simple number, to the double, treble, quadruple, and the rest up to the tenth fold multiple ; and the multiples are made as at the margin. Thus yet, so that the individual multiples may be present with the same number of figures, by putting in place a lacking 0 at the furthest left; then, for the set of multiplications, these single multiples are collected together from that, which the multiple 865091372 shows, clearly, under 8 the eightfold starts from the little table, under 6 the six fold, under 5 the fivefold, and thus so that the rest may be put in place ;

865091 372
736840 | 276315
552630 | 644735
460525 | 184210
8289 45
092105
7967 9240818060

from which finally with everything added, they come to 79679240818060, the multiple sought, and that in agreement with what was found above.

Similarly for a check of the above, both of multiplication as well as division, 79679240818060 shall be divided by 92105: They are put in place by the margin (by Ch. 4 of this Book) ; then take away from 796792, the number of the table closest to the smaller, *viz.* 736840, and for the quotient put the figure corresponding to that figure in the table, *viz.* 8, and there will remain 599524; from which take away the six fold of the divisor, clearly that number of the table closest the smaller number, which is 552630, and 6 may be attached to the quotient, with the remainder 468940; from which take away 460525, the fivefold of the divisor, and the quotient now becomes 865, with the remainder 84158, from which the simple divisor 092105 cannot be taken away even once. Thus now the quotient will be 8650, and now the remainder will be left 841581; from which take away 828945, the nine fold table of the divisor, and now the quotient becomes 86509, with the remainder 126368*;* from which take away 092105, clearly the single and simple number divisor; and now with the quotient composed 865091, 342630 remain; from which take away 876315, clearly the triple of the divisor, and now the quotient will be 8650913, and 663156 will remain; from which take away the sevenfold from the table 644735, and from the quotient 86509137 there remains 184210; from which take away twice the table of the divisor, and nothing remains, with the complete and integral divisor agreeing with the above quotient 865091372.



Up to this point, we have considered simple multiplication and division, with their shortened forms : in the following we will consider the multiplication and division of roots.

CHAPTER VI.

THE MULTIPLICATION AND DIVISION OF INTEGER RADICALS.

The practice of multiplication of radicals [*i.e*. base or root numbers] is apparent from the definition of the same. For if you should multiply a radical by one, it becomes the radical itself ; so that if you multiply the radical by itself, it becomes the duplicand of the radical [*i.e.* square] ; which if you multiply three times, the duplicand by the radical, it becomes the triplicand [*i.e.* cube]; which indeed, if you multiply by the radical four times, thence it gives rise to the quadruplicand of the radical, then the quintuplicand, sextuplicand, etc., following the nature of the index.

So that 235 is to be multiplied to the index 4, or to be quadruplicated : First multiply one by the radicand 235, and it becomes 235*;* which is multiplied twice by the root 235, and it becomes 55225, the duplicand of the radical presented; and which on multiplication by the radical three times, becomes 12977875, the triplicand of the radical present ; so that on multiplying four times in turn at this stage by the radical 235, it becomes 3049800625, the wished for quadruplicand of the radical present.

Thus, a radical multiplied from unity a number of times, is the same as all the equal roots to be multiplied in turn.

So that in the above example, the radix 235 multiplied by four simple multiplications starting from unity, becomes the quadruplicand 3049800625 ; and if you multiply 235, 235, 235, and 235, in turn, the same radicand or quadruplicand arises ; and thus with the others.

The indices of composite radicals, are able to be multiplied and extracted more easily by their components, rather than by the composite indices, as we have said above.

As in the above example, it is easier to duplicate 235 radically [*i.e.* not by multiplying the number by two, but by multiplying the number by itself or squaring as we now call the process, so that it occurs twice in the product], and it becomes 55225; and to duplicate this again, or to multiply 55225 , and it becomes 3049800625, the above quadruplicand : for to quadruplicate is the same as to duplicate the duplicate. Likewise10 shall be sextuplicated radically ; that comes about continually from unity by multiplication, – firstly by 10; secondly by 100, the duplicand; thirdly the triplicand, 1000; fourthly the quadruplicand, 10000; fifthly the quintuplicand, 100000; sixthly, finally, the sextuplicand, 1000000. Truly, by the said shortcut, if you were to duplicate the triplicand 100, or if you duplicated the triplicand 1000, likewise the sextuplicand 1000000 arises; as has been set out a little more fully before.

For it is the same to sextuplicate radically as to triplicate the duplicand, or to duplicate the triplicand of the radicals. For twice three, or three times two, is the same as six.

Also, there is another way of duplicating radicals in practice, rather than by continued multiplication ; evidently, by dividing the radical to be multiplied into two parts, and the two parts to be multiplied, which is done by multiplying the former into the latter twice, together with the duplicands of the two parts, added together make the duplicand sought.

For if the radical presented 35 is to be duplicated radically, it may be cut into two parts: the former 30, the latter 5; take 30 into 5, they make 150, twice which is 300; to which add the duplicand of 30 and the duplicand of 5, which is the sum of 900 and 25, and they make 1225, the duplicand of the whole number 35.

The same arises from 35 multiplied by 35. Likewise if the radical may be extended, let the number be 352 ; the first part shall be 350, the latter 2; multiply these in turn, and they make 700, the double of which is 1400; to which add the duplicand of 2, which is 4, and the duplicand of 350 found above, which is 122500 ; and from these three take the sum of 122500, 1400, and 4, to be added, they make l23904, the duplicand sought of the radical 3522 Test by multiplying 352 by 352, and you arrive at the same. In the same manner the operation may be extended as far as the fourth figure, and let the number be 3521. Now with l2390400 obtained from the first part 3520 duplicated, and 1 is obtained from the second part 1 duplicated, then 12397441 will be obtained, the duplicand of the whole number, or of the radical 3521.

For also there is another procedure for the triplication of radicals ; clearly, by dividing the radical into two parts, and by multiplying any part three times into the duplicand of the other part, and with the individual parts triplicated radically; for from these four sums added, the triplicand is produced of the radical presented.

So that the radical presented 35 may be tripled radically, it is divided up as above into the parts 30 and 5; multiply 5 three times, that is 15, into 30 duplicated, which is 900; and triple 39, which is 90, into 5 duplicated, which is 25; and the former makes 13500, the latter 2250; then 30 is triplicated radically, and it becomes 27000; also 5 is triplicated radically and it becomes 125; from these therefore with the four sums added : 13500, 2250, 27000, and 125, they make 42875, the triplicand of the radix presented.

You test by the continued multiplication of 35, 35, and 35 in turn, and you come upon the same result. Similarly if the radical may be extended, and it shall be 351; it may be divided into the first part 350, and the second part 1, from which, in the manner established, you will have these four sums : 42875000, 367500, 1050, and 1; with which added, 43243551 is made, the triplicand sought of the radical 351 presented.

Indeed there are particular practices for finding the radicals of each of the indices of quadrupling, quintupling, sextupling, and of the other radicals of multiplication. But because both the general manner has been described in the first section by continued multiplication, and that is easy enough, then from these particular rules it is possible to have the rules of extraction from the converse, and we will present these here.

Also the practice of division is apparent from the same definition of this. For if the radicand were divided by the radical until unity is reached, the number of divisions is the quotient of the radical, or the index sought.

So that 55225 shall be the radicand to be divided by the radical 285, the first quotient becomes 285, with which divided by 285, 1 emerges, the second and final quotient. Therefore the index of this division is of the order 2, and the radical is the bipartite divisor of the radicand. Likewise if 12977875 shall be the radicand required to be divided by the radical 235, 55225 appears in the first division, 285 in the second, and 1 in the third. From which 235 is the tripartite divisor, and the index is of the third order, and the radical is called tripartite.

The number of the index, or the nature of the radical, is obtained, both by descending from the radicand to unity by division, as well as by ascending from unity to the radicand by multiplication : For in each, the index is the number of operations, and the nature of the radical.

As in the above example, just as the radicand 12977875, divided by the radical 235 in the third division shows one; thus, from unity by continued multiplication of the radical 235, falls on the same radicand by the third multiplication ; from which in each way the index is proved to be three. Hence it happens, that the division of radicals shall be of occasional use in computations, since also by its service it may perform a multiplication. But they are not useful in these cases, with the multiplication of radicals from unity to a radicand or in division, which do not fall precisely on unity from a radicand. For they prove instead that the presented radical is not a perfect radical of the radicand.

CHAPTER VII.

FINDING THE RULES FOR THE EXTRACTION OF ROOTS.

The rule of extracting each and every radical [*i.e.* root] is individual and particular to that one.

Each rule of extraction is consistent with the resolution of the radicand into its supplementary parts : The supplement is the difference of two radicands of the same kind.

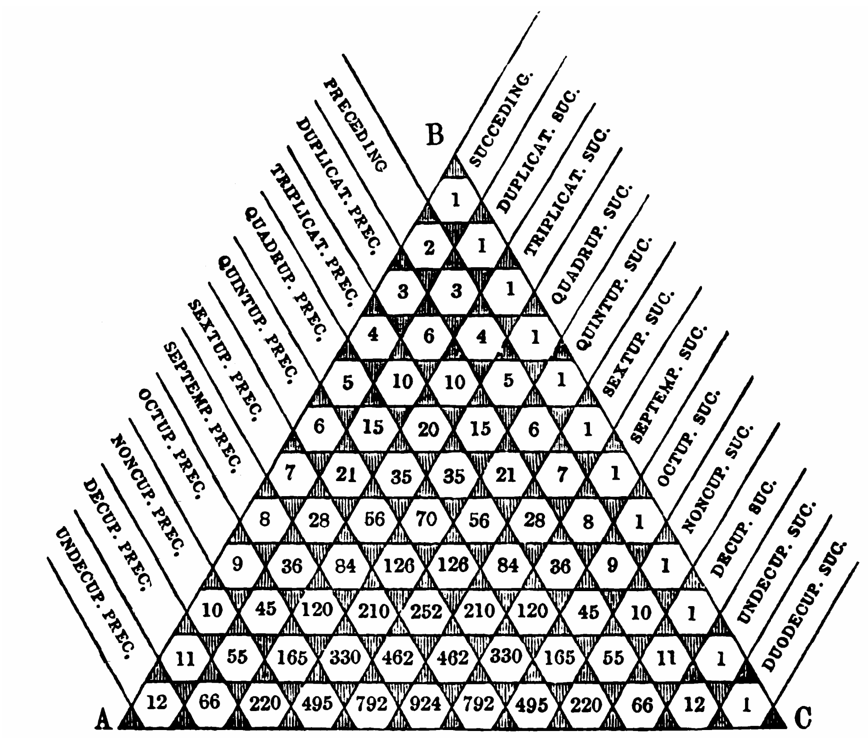
So that if 100 and 144 shall be radicands of the same kind ; clearly, the one the duplicate of [the radical] 10, and the other the duplicate of 12; the difference of the duplicands 100 and 144 is 44, which is the true supplement of the radicands just mentioned.

Therefore the supplements can be varied, provided the kind of radicands and radicals can be changed. For one rule is found for the supplements of duplicands, and for the extraction of the bipartite radicals, another for triplicands, and for the extraction of tripartite radicals, another for quadruplicands, and for the extraction of the quadrupartite radicals; and thus of all the others.

Moreover our little triangular table, filled with hexagonal areas the rules, demonstrates the rule for finding all the radicands and supplements of radicands, with the furthest right inscribed with one only, and indeed with the numbers furthest left increasing from one at *B* by unit increments, descending from the vertex; and that each of the individual areas it has a number, and which is equal to the sum of the two nearest numbers put in place above.

Let *ABC* be the triangle, having the angles *A* on the left, *B* the vertical, and *C* to the right. Moreover whatever kinds of radicands you would like the table to handle, divide each side by twice the number of parts plus one ; e.g., for the extraction of twelve kinds in continuation, each side is divided into 25 equal parts ; and, by beginning from the base *AC*, through the individual alternate points of the lines, twelve lines are drawn parallel to the base within the triangle ; in a like manner you begin from the side *AB*, and twelve parallel lines to *AB*, are drawn from the individual alternate points of the base, through the individual alternate points of the side *BC*, both within the triangle as well as beyond the line *BC*, extending by the distance of an inch ; again in the same manner, from the side *BC*, twelve lines are drawn parallel to *BC* through the individual alternate points of the base, and the individual alternate points of the side *BA*, extending a distance of an inch. Hence you have filled the triangle with hexagonal areas, of which the 12 areas to the right, and closest to the line *BC*, are inscribed with 12 units, one by one ; the most left indeed are inscribed with the numbers 1, 2, 3, 4, 5, etc., as far as to 12, in order, by descending from the vertex *B* to the left angle *A* ; then each of the empty interior hexagons is inscribed with the sum of the above two numbers closest to that ; so 3 may be written under 2 and 1; 6 under 3 and 3; 4 under 3 and 1; and thus as far as to the corner of the table.

TABLE OF THE SUPPLEMENTS.



Finally, there may be written for the title of the most left '[of the] preceding', above the second hexagon 2; above the third 3 write, ' duplicand [of the] preceding'; above the fourth write, 'triplicand [of the] preceding' ; and thus with the remaining radicands as far as to the twelfth : truly of the rightmost write above the first of the hexagons '[of the] succeeding'; above the second, '[duplicand of the] succeeding' ; above the third, '[triplicand of the] succeeding' ; and thus with the remaining radicands succeeding, as far as to the thirteenth, as you have described in the diagram of its table.

The two parts of the radical [i.e. root] correspond to each supplement : with the one part agreeing with one or several figures now found to the left, which is called the preceding part ; the other agreeing with a single right figure now to be found approximately, which is called the succeeding part. And these parts of the root and the supplement are placed together mutually and are used in turn in the composition; as will be apparent later.

[The reader needs to be aware that Napier is handling algebraic equations as word statements, on the assumption that intended readers at the time would understand these better than algebraic equations, which were then quite a new development. See for example, Harriot's *Praxis*, and the use of a related table by Briggs in his *Trigonometria Britannica*, both translated in this series of translations; as well of course as the final book on algebra to be presented here. We can understand the preceding *n*-tuplet as some single number *a* raised to the same power *n*, while the succeeding or following *n*-tuplet is the sum of the two numbers *a* and *b* raised to the same power ; the supplement in a row of the triangular table is the difference of these *n*-tuplets, *i.e.* $(a+b)^n - a^n$. Thus, we almost have Pascal's triangle, lacking the final right-left symmetry; and of course the purpose of the table is different : it is really a means of performing an interpolation. Thus, the preceding number *a* and the following number *b* for any situation, are not present in this table, which is concerned with the necessary coefficients to be multiplied by the appropriate powers of these numbers, which are then added together.]

So the bipartite radical [i.e. square root] of the duplicand 144 is 12, the two parts of which, 1 and 2, correspond to the supplement 44, and the first 1 is called 'preceding,' the second truly 2, 'succeeding' ; but if you should take the duplicand 15129, the bipartite radical of which is 123, the first part now becomes that just found, clearly 12, is called the 'preceding,' and with the single figure 3, the 'succeeding.' [The associated zeros have been omitted; for we have $12 = 10 + 2$, $123 = 120 + 3$, etc.]

Thus from this table, the rule for any supplement required may be deduced, and so put in place. Initially, the number of the index of the proposed radical is sought next to the left leg ; for this index, together with the remaining numbers in the areas on the same line following directly in order, provide the information for the supplement to be put in place ; according to the oblique title of the index, together with the individual terms of these summed and multiplied in turn, derived from the sum of both the left-hand and right-hand numbers in place above.

For example :–The rule is sought from the table, for finding the duplicand supplement and for the extraction of bipartite radicals. This is expressed on the second line by the two numbers 2 and 1: and with [the actual preceding and subsequent numbers] multiplied together, according to the index, by the preceding 2 ; and the following number duplicated by 1, which thus are to be added together, and which are pronounced [in words] as follows: the supplement of the duplicand consists of the preceding number multiplied twice by the following, and once by the following number itself duplicated ; as will become apparent from the following examples. Thus for the supplement of the triplicand, on the third line you will find the three numbers, 3, 3, 1, which may be summed with the inscribed values. The supplement of the triplicand is put in place from the three numbers ; the first, the duplicate of the preceding number multiplied by three times the following number; the second is the preceding number multiplied by three times the duplicate of the second number; the third is the triplicate of the second number itself.

Similarly, for the supplement of the quadruplicand, or of the extraction of the quinquipartite radical : On the fifth line you will find these five numbers, 5, 10, 10, 5, 1, which, from their contents, indicate the supplement of the quintuplicand, or of the extraction of the quintipartite radical, to be constructed from the five parts ; the first of which is the quadruplication of the preceding number multiplied by five times the following number ; the second is the triplicate of the preceding by ten times the duplicate of the following number; the third the preceding duplicated by ten times the following triplicated; the fourth is the preceding multiplied by five times the following quadruplicated ; the fifth is the following itself quintuplicated: And thus you will be able to find the supplements of the quadruplicand, sextuplicand, and the supplements of the remaining from this table.

The radical emerges fully extracted from the radicand of the leftmost figures as the greatest possible from the preceding, and with the individual figures of the supplement collected together into one figure. [Thus, *a* above is made as large as possible, while *b* accommodates the remainder of the radicand.]

Moreover you have now the supplements of all the radicals described ; therefore it remains now to discuss the extraction of the same.

CHAPTER VIII.

THE EXTRACTION OF ROOTS.

The root of any radicand is either of one or of several figures.

All the roots of a single figure, below the thirteenth, are shown in the following table.

So that if you seek the fifth root of 16809 : In the line of the quintuplicates the number present is found, or one a little less nearby perhaps, 16807; and in the place directly on the top you will find the figure sought, surely, the fifth root of the radicand presented, or to one presented close to it, which is 7, and 2 left over, with the extraction irresolvable.

TABLE OF RADICANDS AND ROOTS.

[n]	2	3	4	5	6	7	8	9
Duplicate [n^2]	4	9	16	25	36	49	64	81
Triplicate [n^3]	8	27	64	125	216	343	512	729
Quadruplicate [n^4]	16	81	256	625	1296	2401	4096	6561
Quintuplicate [n^5]	32	243	1024	3125	7776	16807	32768	59049
Sextuplicate [n^6]	64	729	4096	15625	46656	117649	262144	531441
Septemplicate n^7]	128	2187	16364	78125	279936	823643	2097162	4782969
Octuplicate [n^8]	256	6561	65536	390625	1679616	5764801	16777216	43046721
Noncuplicate [n^9]	512	19683	262144	1953125	10077696	40353607	134217728	387420489
Decuplicate [n^{10}]	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401
Undecuplicate [n^{11}]	2048	1771471	4194304	48828125	362797056	1977326743	8589934592	31381059609
Duodecuplicate [n^{12}]	4096	531141	16777216	244140625	2176782336	13841287201	68719476736	282429536481

This table is arranged by continued multiplications, through the individual nine figures with the series descending from the same figures. As at the top of the table, you have the figures, from the eight multiples 2, 3, 4, 5, &c. as far as 9 ; and under each figure, its duplicate, triplicate, quadruplicate, etc., as far as the duodecuplicate [*i.e.*12] : so that, under 2 you have 4, 8, 16, 32, 64, 128, 256, 512, 1004, 2048, 4096, with its inscriptions from the leftmost margin ; evidently, the duplicate, triplicate, quadruplicate, &c. and finally the duodecuplicate ; under 3, moreover, you have its radicands 9, 27, 81, 243, &c. by descending in the ratio continued; under 4, likewise you have, 16, 64, 256, &c.: and thus up to the nines; under which you have 81, 729, 6561, &c.; so that they can be seen in the table itself.

In the extraction of roots with several figures agreed on, the places are to be marked and the operation to be considered.

The positions to be marked are found as follows : as under the radicand presented two parallel lines are drawn, the interval of which shall be the number of places of the root sought, between which, under the rightmost figures , a point is marked; but at this stage to the left, under some second figure for the bipartite root, and under some third for the tripartite, and under some fifth for the quintupartite, &c., points are marked. The individual figures of the roots sought fall at these points, between the lines in place.

Moreover, there are two precepts of the operation :

The first rule: so that from the figures, ending in the left point, you take away the root of the proposed kind, found in the second table, as large as you are able to take away, with the remainder noted above, and in place of this most left point the figure of the root assumed in the table is put in place, for the first figure of the root.

The second rule: so that, with a conjecture made according to the first numbers of the supplement, a new recently made figure may succeed at the following point, or at a period of this kind, the supplement of which of this proposed kind, shall be as large as possible, yet shall not be larger than these figures above ending in that period itself ; from which likewise the supplement shall be taken away, from the remainder noted above. And this second operation may be repeated in turn just as often as there are empty spaces towards the right ; and the figures placed into the periods are the root sought.

So that, for example, the bipartite root [*i.e.* the root of the second part, or the square root in modern terms] shall be extracted from 55225 :

55225

1
5 5225
2
4

The numbers are put in place as from the margin ; then in the first place, from 5, of the left most period, take away the greatest duplicand of the table that it is possible to take away, clearly 4, and 1 remains, with the 5 deleted above and the four below 4; the bipartite root of four in the table, which is 2, may be put in place of the first period ; therefore 2 is called the preceding figure ; the following to that is sought, and to be put in place in the second period, the supplement of this shall not exceed the figures 152 written above, with the conjecture made from the first part of the supplement, evidently from the 2 now found (which is 20 with this in place) doubled and multiplied by that required to be found, that is, of 40 multiplied by the succeeding number. But if the following number had a value of four, the first part of the supplement becomes 160, which exceeds 152; therefore with four rejected, we select a value of three to be taken up at this place; the supplement of this duplication, from the first table, 129 (clearly three times 40 and three times 3); by which the supplement from the presented 152, with this in place leaves behind a remainder of 23, or to the right as far as 2325, and the preceding part of the root now is 23, or 230 ; therefore the second operation is repeated, towards arriving at the following new figure, to be put in the place of the rightmost period ; the supplement of this new succeeding number, whatever it shall be, agrees (by the first table) with twice the preceding 280, evidently 460, multiplied by the following number and with the following number duplicated ; but if you multiply 6 by 460, the produced exceeds 232; therefore with 6 rejected, the supplement of the fifth is calculated, and it will be, from the tables, 2325, (clearly five times 460 and five times 5) with which taken from the same amount above, nothing remains ; and the bipartite root 235, now satisfied and perfect, appears out of the number presented 55225.

20
2
40 3
3 3
120. First part of the first supplement. 9. Second part of the first supplement.
129. Total first supplement.

```
   123
 55225
 2 3
 4
 129
```

[Note that in the following examples, where there is considerable arithmetic involved, operation has been split into two parts, the one in totaling up the contributions to successive supplements, while the other uses these results to start the next operation.]

230
2
460 5
5 5
2300. First part of the second supplement. 25. Second part of the second supplement.
2325. Total second supplement.

```
      0
    123
  55225
  2 3 5
```

Another example of the extraction of a bipartite root.

```
 164860
  4 0
 16
```

400 6
2 6
800 36.The latter part of this supplement.
6
4800. The first part of the supplement. 4836. The whole supplement.

```
 0   24
164860
4 0 6
16
  4836
```

You arrange the bipartite root to be extracted from 164860, as shown from the margin: The root of the first period of the figures, clearly 16, is 4, to be located in that period; the duplicand of which 4 is 16, to be taken away from the upper 16, and 48 are left over as far as the second period, and the preceding figures are 40 ; now if perhaps you select one for the succeeding, 81 will be the supplement of this, which cannot be taken away from 48 ; therefore the following and new figure required to be put in place in the second period will be 0, the supplement of which is 0; with which taken from 48, 4860 are left in the final period : and now 40 are the preceding figures found; and a new following figure is sought, to be put in place of the final period ; that by necessity is 6, the supplement of which is 4836 (clearly six times 400 doubled, and six times 6 added on) ; with which taken from 4860, 24 remains, the unresolved remainder ; and the bipartite root sought, still not exact, is 406, of the radicand 164860 presented.

Example of the extraction of a tripartite root.

Let the root of the third part [*i.e.* the tripartite root] be extracted from 12977875: as from the points and places collected together by the margin ; thence, take the maximum triplicand of the table not exceeding 12, clearly 8, from12, and 4 remains, to be written above, with 12 crossed out; moreover, the cube root of eight, evidently 2, is put in place of the first period, and 20 is now the preceding number; the following of which, with its supplement, thus is required : the supplement of the tripartite root, (from the first table) depends on three parts, the first of which is made from the triple of the preceding square multiplied by the succeeding ; from which it follows, if 4977 written above, should be divided three times the square of the preceding, clearly by the triple of 400, which is 1200, the quotient will give an estimate for the succeeding number ; for the following number is generally either equal to this quotient, or less than that by one ; and so if 4977 were divided by 1200, 4 becomes the quotient and the following number; but the supplement of four [*i.e.* altogether] is 5824, which exceeds 4977; therefore with 4 rejected, we may select 3, the supplement of which is 4167 (for 1200 multiplied by 3, and three times 3 multiplied by three times 20, but the triplicand 3, makes the sum 4167); from these with 4977 taken away, this here 810 remains, and in the third period 810875, and 230 now are the preceding figures. The succeeding remaining number is sought located at the rightmost point : that may be found, in a similar manner to what has preceded, and it is 5, the supplement of which is 810875 (for three times the duplicate of 230 multiplied by 5, clearly 793500, and the tripartite root 230 multiplied by 25, evidently 17250, and the triplicate 5, which is 125, make 810875)*;* therefore from these put in place with these taken away from the equal figures above, nothing remains ; and the tripartite root sought, 235, complete and whole, if produced from the presented triplicand 12977875. And thus you will be able to operate in all the extractions of roots, where the indices are single and not composite, as in the quintupartites, and sextupartites, &c.

4
~~1~~ ~~2~~ 977875
2
8

20
20
400
3
1200
3
3600. First part of the first supplement.

20 3
3 3
60 9
9
540. Second part of the first supplement.

3
3
9
3
27. Third part of the first supplement.

3600
540
27
4167. Total of the first supplement.

~~4~~ 8 1 0
~~1~~ ~~2~~ ~~9~~ ~~7~~ ~~7~~ 875
2 3
8
~~4~~ ~~1~~ ~~6~~ ~~7~~

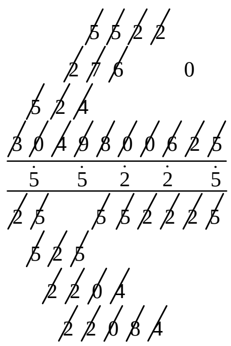
230
230
69
46
52900
3
158700
5
793500. First part of final supplement.

230
3
690
25
345
138
17250. Second part of final supplement.

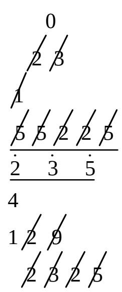
5
5
25
5
125. Third part of final supplement.

793500
17250
125
810875. Total final supplement.

Example of the extraction of a root where the indices are composite.



The quadrupartite root [*i.e.* root of the fourth part] may be extracted from 3049800625; because this can be made less encumbered, by extracting the bipartite root twice, than by extracting the quadrupartite root once, consult Ch. 4 Book. I. Therefore the bipartite root of this number may be extracted, by the appropriate places and operation, as from the margin, and 55225 is produced, the bipartite root of the number presented ; again extract now the second bipartite root of this bipartite root, as you see, also from the margin, and you will find the bipartite root of the bipartite root to be 235, which is the quadrupartite root sought of the radicand presented 3049800625. And thus in the extraction of the sextupartite [*i.e.* 6th root], octupartite [*i.e.* 8th root], nonupartite [*i.e.* 9th root], and of any other composite roots you will work on.



50
2
100 5
5 5
500 25. First supplement.

550
2
1100 2
2 2
2200 4. Second supplement.

5520
2
11040 2
2 2
22080 4. Third supplement.

55220
2
110440 5
5 5
552200 25. Fourth supplement.

20
2
40 3
3 3
120 9. First supplement.

230
2
460 5
5 5
2300 25. Second supplement.

CHAPTER IX.

A METHOD FOR CORRECTING INCOMPLETE EXTRACTIONS.

The remainders which are left behind after imperfect extractions, however great the unresolved part may be as regards the total, yet are able to be transformed or partially from the part remaining.

But these remainders left over, unless they may be transformed from the part remaining or partially, generally bear a sensible error.

The detectable error of a root and of its remainders can be amended in two particular ways.

The first method consists of drawing a line after the imperfect root, above that the remainders are written, and truly below the supplement of one, a whole number for the lesser term, and with one less for the greater ; for the true magnitude of the root lies hidden between these bounds, which cannot be defined by a number.

So that the bipartite root may be extracted from 164860, in the second of the preceding examples you will find the imperfect bipartite root of the same to be 406, and a remainder 24 present. Therefore put in place a line after 406, and above the line 24 is written, as in the margin ; thence from the preceding 406 now found, look for the supplement corresponding to the duplicate of the succeeding one; this whole number will be 813, and diminished by one it becomes 812, of which that first number may be written for the lesser, while this second number may be written for the major limit both written under the line. Therefore the lesser limit of the sought bipartite root will be $406\frac{24}{813}$, and the major $406\frac{24}{812}$, between which limits the precise bipartite root of the number presented 164860 is contained and lies hidden. Thus, so that without sensible error (especially in mechanics), the bipartite root of 164860 may be said to be $406\frac{24}{813}$, or $406\frac{24}{812}$.

$406\frac{24}{813}$

$406\frac{24}{812}$

406	
2	
812	1
1	1
812	1. Supp. of unity.

An example of a tripartite root being corrected.

The tripartite root of the of the number 998 shall be extracted, hence (from the preceding table) it is taken to be almost 9, and still 269 of the remainder present is to be corrected. With that 9 before the line, this 269 above that line, as may be put in place from the margin. But from the figure 9, from the preceding found, the supplement of the tripartite of unity is found to be the integer 271, and diminished by one 270, as can be seen from the margin, which is written under the line. And thus the tripartite root of the number 998 presented is established to lie between the smaller limit $9\frac{269}{271}$, and the greater limit $9\frac{269}{270}$; so thus for that, hence either one may be taken without sensible error.

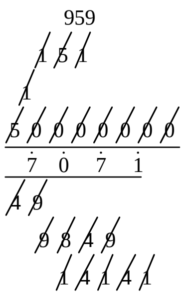
$9\frac{269}{271}$

$9\frac{269}{270}$

9	9	1	
9	3	1	
81	27	1.	Third part of supp.
3	1		
243	27.	27.	Second part.
1			
243		243.	First part.
		271.	Total supp. of unity.

The second method is, that the whole radicand presented of some kind may be multiplied by the radicand of some number chosen of the same kind ; moreover the root produced by the radicand of the same kind (with the remainder removed) may be divided by that chosen number. For the quotient, with the line for its distinct fractions, will be the lesser limit ; and if to the numerators of the same fractions you add one, it will be the greater limit, between which the true root may be held.

So that the bipartite root extracted from the number 50 certainly may be improved on : the selected number shall be 1000, the duplicand of this will be1000000, which multiplied by 50, makes the product 50000000 ; of which the bipartite root of the same kind is sought, from the placement and operation by the margin ; and this bipartite root will be 7071 (with the remainder 959 ignored), to be divided by the chosen 1000, and thence the quotient becomes, with the fractions, $7\frac{71}{1000}$, for the lesser limit, and $7\frac{72}{1000}$ for the greater limit. Thus so that either may be taken for the bipartite root of the number 50 presented, without perceptible error.

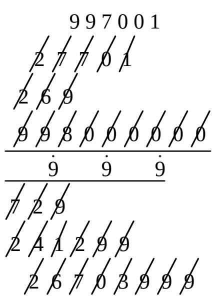


700
2
1400 7
7 7
9800 49. First supplement.
7070
2
14140 1
1 1
14140 1. second supplement.

The example of the preceding tripartite root corrected in this manner.

The tripartite root to be extracted from the number 998 is to be amended.

Let the other number chosen be 100; the triplicate of which will be 1000000 ; this multiplied by 998, becomes 998000000, the tripartite root of which can be extracted, as by the margin, and from that there is produced as an approximation 999, which (with the remainder 997001 removed) may be divided by the chosen number 100; thence there arises $9\frac{99}{100}$, for the lesser limit, and (with the numerator increased by one) $9\frac{100}{100}$, which is 10, for the greater limit. And thus, without perceptible error, $9\frac{99}{100}$ or 10, can be said to be the tripartite root of the number 998 presented; to be included between these closest terms.



90 | 90 |
90 | 3 |
8100 | 270 | 81
3 | 81 | 9
24300 | 270 | 729
9 | 216 |
 | 21870 |

218700. Supp. of first part.
21870. Second part.
729. Third part.
241299. Total first suppl.

990 | 990 |
990 | 3 |
89100 | 2970 | 81
891 | 81 | 9
980100 | 2970 | 729
3 | 2376 |
2940300 | 240570 |
9 | |

26462700. Posterior supp. first part.
240570. Second part.
729. Third part.
26703999. Total posterior suppl.

These methods, because they do not render imperfect roots perfect, but exceedingly imperfect, are more useful to Mechanics than to Mathematicians. As we have said in Ch. 4 Book I.

Geometers therefore with these roots for numbers, not having the roots, must put a sign of the root in front. From which thus from these signs arises the first kind of geometrical numbers, that they call by a single name. [Napier's Latin word *uninomia* does not seem to exist, and it appears to be an invented word.]

So that of the above duplicands 164860, and 50, neither do they have bipartite roots extracted (because none are had in precise numbers), nor can the extracted roots be amended, but the sign of the root to be extracted, that they call the square, they prefix to the number in this manner, $\sqrt{Q}164860$, $\sqrt{Q}50$; or thus, $\sqrt{q}164860$, and $\sqrt{q}\,50$; which they pronounce thus, the square root of the number 164860, – and, the square root of the number 50. But we may call these thus $\sqcup$164860, and $\sqcup$50; and thus we may pronounce, the bipartite root 164860 – and the bipartite root by $\sqcup$50. Thus, the tripartite root of the number 998 neither is extracted (because it shall not be in [finite] finite numbers) nor may be amended, but thus may be indicated, $\sqrt{c}\,998$, and thus it may be pronounced the cube root of 998. But we may denote it thus , L998; moreover we prefer thus, the tripartite root L998, as we will discuss further in its proper place. Hence, by whatever single or mean names these may be called, and they are the foundation of Geometric Logistics : they will be treated in the following book; but hence their coming about has been well enough forewarned.

Thus far we have investigated the simple computations of whole numbers ; the rules of composite computations follow.

CHAPTER X.

THE RULES FOR THE PROPORTIONS OF WHOLE NUMBERS.

The rules and precepts for the proportions of whole numbers are explained profusely enough according to the general method of quantities.

You will be able to discern the rules of proportion, both from the single general method of solution, as well as in the three exemplary questions proposed in Ch.5 Book I, regarding simple, duplicate, and triplicate proportions.

Yet particular examples of numbers have specific abridged methods by which they may be evaluated more easily : If indeed with somewhat greater numbers, some divisor may have zeros put in place towards the right, then you will be able, thanks to the abridgment, to carry just as many places to the right with empty spaces, or to be replaced by zeros, by ordering the multiplication from the left.

[Thus, the result finally has the same number of significant figures as those forming the product ; there is no point in working out the less significant figures, except perhaps one or two of them for rounding purposes, as they are not present finally, on division by the appropriate multiple of ten, as in figures taken from the table of sines to be investigated next. Napier does not do this, and quite ruthlessly cuts of the fractional parts.]

So that (with an example from sines) if 10000000 should give 9925461, how much does 7986354 give ? But hence because nearly all the numbers are included in the multiplicand, thus, by Ch. 5 Book II. it will save time for 9925461 to be multiplied by the individual nine figures, as has been done in the table by the margin; then the multiplication of 7986854 can begin under the individual figures, and thence the number proceeds according to that corresponding figure of the table, yet with the six of the rightmost places of all the figures omitted, as therefore seven of these zero divisors 10000000 can be cut off, if they were expressed and not omitted:

09925461	1
19850922	2
29776383	3
39701844	4
49627305	5
59552766	6
69478227	7
79403688	8
89329149	9
99254610	10.

```
7986354
69478227
 8932914 .
  794036 . .
   59552 . . . .
    2977 . . . . .
     496 . . . . . .
      39 . . . . . . . . .
7926824|1 . . . . . . . . .
     10000000
```

Therefore this particular multiple, besides its six empty places to the right, added into one number becomes 79268241...... from which, both with the places as with the rightmost figures, if seven of the zero divisors located in unused parts may be cut off (as the contraction of the division removes), there will remain 7926824, of the corresponding number sought. Therefore when 10000000 gives 9925461, it follows that 7986854 will give 7926824.

Moreover because, in the contraction of this multiplication, figures towards the right are omitted, even if they should all be of nines present, they do not increase by a single answering one. Therefore deservedly all these are able to be ignored from these somewhat larger numbers in which indeed an error of the whole and integral part by one is not noticeable.

```
.
. .
. . .
. . . .
. . . . .
. . . . . .
-----------
5888889
```

Indeed let these empty points be of nines, put in place from the margin (because it is possible above), nevertheless they may increase only to 5888889, which added to 79268241...... shall make 79268246888889, with which divided by 10000000 parts they come to the sum $7926824\frac{6888889}{10000000}$, which cannot be increased to 7926825, nor exceed the above product by one. And thus with the largest numbers, this abridgment of the rule of three is worthy of the greatest praise.

There is another contraction of this rule, without the omission of figures ; as we have taught in the general method, Ch. 5 Book I, by putting in place all the numbers of the investigation present in the places above or below a line ; then two individual numbers, of which one is the upper and as it were the numerator, and the other as it were the denominator, may be divided by their greatest common divisor, then the individual numerators to the individual denominators would be in the first or minimum ratio to each other, with the final quotients of all noted ; finally, a multiple of all the upper quotients is divided by a multiple of all the lower quotients, and this quotient will be the answer sought satisfying the question.

Builders.	Feet	Spans.	Days.
4	9	50	42
5	6	48	How many days?

So 4 builders have constructed a wall 6 feet high, 48 spans long, in 42 days; it is sought, in how many days will 5 builders construct a wall 9 feet high, 50 spans long? By Ch. 5 Book I. on putting in place all the numbers, and so that they stand by the margin; then the upper numerator 4, and the lower 6, abbreviated by 2, the maximum divisor of these, and they become $\frac{2}{3}$ with this in place, $\frac{2\cdot 9\cdot 50\cdot 42}{5\cdot 3\cdot 48}$; then divide the upper 2 and the lower 48, by the common divisor 2, and they become 1 and 24, with this in place, $\frac{1\cdot 9\cdot 50\cdot 42}{5\cdot 3\cdot 24}$ then with the upper 9 and the lower 3, by 3, and they become 3 above, and 1 below, with this in place, $\frac{1\cdot 3\cdot 50\cdot 42}{5\cdot 1\cdot 24}$; then 50 and 5, are divided by 5, and they become 10 above and 1 below, with this in place, $\frac{1\cdot 3\cdot 10\cdot 42}{1\cdot 1\cdot 24}$; then with 10 above, and 24 below, by their maximum common divisor 2, and they become 5 above, and 12 below, with this in place, $\frac{1\cdot 3\cdot 5\cdot 42}{1\cdot 1\cdot 12}$; finally, the upper 42 and the lower 12 are divided by their maximum divisor 6, and they become 7 upper and 2 lower, with this in place, $\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 1\cdot 2}$. Now behold you have the familiar and tractable numbers 1, 3, 5, 7, and 1, 1, 2, to be multiplied in turn, instead of the rather larger numbers presented. Therefore in turn 1, 3, 5, 7, are multiplied and they make 105; likewise the lower numbers are multiplied 1, 1, 2, in turn, and they make 2; by which 105 is divided, and the quotient $52\frac{1}{2}$ comes about, the number of days satisfying the question, without greater or more labourous multiplication or division operations.

This rule of proportions has several unmentioned and hidden kinds, not to be passed over, which also profit by the benefit of this contraction, and are expedited by that.

So that the rule reducing fractions of fractions of fractions over and over again (by which generally the upper and particularly the lower parts will be treated), is a kind of rule of this proportion, from the definition of the rules of proportion. From which by that same contraction, by which fractions hence may be contracted, as has been shown in the examples of Ch. 8 Book I, and will be shown below.

Enough has been said about the arithmetic of whole numbers ; there remains the arithmetic of fractured numbers or fractions.

CHAPTER XI.

CONCERNING FRACTIONS, OR FRACTURED NUMBERS.

A fraction or fractured number is that smallest and indivisible part, clearly it is measured by a single numerator of one.

As in astronomy the degree is taken for the unit, and 7 first scruples [i.e. minutes] is a fraction of this ; for in this situation a single first scruple is the smallest and indivisible part of this, by which that is measured, and from that the numerator is multiplied by 7. Thus 5 shillings is a fraction of a pound, because here 1 shilling is the smallest and indivisible part of one pound, by which fraction it is measured, and from that by a numerator of 5. Thus 17 thirteenth parts of unity is a fraction ; because a single thirteenth with this in place is the smallest and indivisible part of unity, by which 17 thirteenths is measured, and with these it is counted seventeen times.

Of fractions, some are called vulgar, others physical.
The vulgar fractions are those which have varied and free denominations.
As one half, two thirds, four elevenths, which have arisen in division, Ch. 4 Book II.
The name and operation are considered in the arithmetic of vulgar fractions.
The name, by which it happens to be called, is termed the word pronunciation ; but by which it is written, the notation.
The naming of vulgar fractions is expressed by two terms, the numerator and the denominator.
It is the denominator which nominates into how many equal parts the fraction shall be distributed.
It is the numerator truly, which counts how many from these parts are required to be added together.
And this may be pronounced first and by a cardinal number, –that the latter an ordinal number; this first also above the line and that following to be noted below the line.
So that if unity, or some single number be proposed, were divided into some number of parts, as for example into 11 parts, and of these 5 parts were to be taken in place – 11 is the denominator of this fraction, and 5 the numerator of the same ; of these this is mentioned first, and pronounced by the cardinal number (five) ; and that latter 11 by the ordinal number (eleventh) ; in this manner, 'five eleventh parts.' Truly this 5 is marked above the line, and that 11 below, in this fashion, $\frac{5}{11}$; and thus with the others.
For which it is agreed that a fraction, or a broken number, shall be the part, of parts of unity divided into some number of parts.
It is agreed also that the fraction likewise prevails as the number of the numerator divided by the number of the denominator.
As in the above fraction $\frac{5}{11}$, likewise it can be called , 'five elevenths of unity,' because generally we have said, 'five divided by 11', as in Ch. 7 Book I.
And there are certain improper fractions, which are not parts of one, or expressly parts, but which are parts of fractions ; and these are called fractions of fractions ; which we note by the interposed words 'out of, ', [or 'of'] ; others note these by the omission of the latter line, or lines.
So that two fifths parts , of three quarter parts of unity, is not properly nor immediately a fraction of one, but a fraction of a fraction of one ; for it signifies one to be divided into four parts, and of these three parts are again to be divided into 5 parts, of which finally two are to be taken. Therefore we may note thus, $\frac{2}{5}$ of $\frac{3}{4}$, others write thus, $\frac{2}{5}\frac{3}{4}$; and we thus call, 'two fifths out of three quarters,' or, 'two fifths of three quarters.'
The named operations of vulgar fractions follow, and the practices of the computations.
Therefore, according to practice, we reduce whole numbers to a kind of fraction, by substituting one for the denominator, and we may shorten the terms of the fractions when they may have increased, by dividing those by their maximum common divisor, that also we come upon with the continual division of the parts by their remainders, then nothing will remain ; and the benefit of this is that we reduce different denominations to the same ; finally we add and subtract the reduced fractions, in everything and by everything, just as with the rules of quantities written in general down in Ch. 7, Book I.
So that the whole number of two, is reduced a kind of fraction when it becomes $\frac{2}{1}$. Thus 3 to $\frac{3}{1}$, 4 to $\frac{4}{1}$, and 5 to $\frac{5}{1}$, indeed likewise prevail. Also we may abbreviate the terms of

this fraction, $\frac{6}{10}$, by dividing both the numerator and the denominator by their maximum common divisor 2, and it becomes $\frac{3}{5}$. Thus $\frac{35}{49}$, divided by their maximum common divisor 7, becomes $\frac{5}{7}$. Also that maximum divisor thus you will find thus in that place : divide 49 by 35, 14 remains; divide 35 per 14, 7 remains; divide 14 by 7, nothing remains ; 7 therefore is the maximum common divisor of the terms 35 and 49. Thus also by these rules, two fractions of different denominations, as $\frac{11}{132}$ and $\frac{7}{128}$, you reduce to the same denomination in this manner : divide 132 and 128 by their maximum common divisor, evidently by 4, and they become 33 and 32; take 7 by 33, and they make 231; and take 128 by 33, and they make 4224, in this case, $\frac{231}{4224}$ for the fraction given $\frac{7}{128}$: Similarly, take 22 by 11, and 32 by 132, and they become 352 and 4224, if this place, $\frac{352}{4224}$ for the fraction $\frac{11}{132}$ presented. Therefore you have the two fractions given, $\frac{11}{132}$ and $\frac{7}{128}$, reduced to these of the same denominator, $\frac{352}{4224}$ and $\frac{231}{4224}$, by the general rules laid down. With which now reduced, we add the numerators by the same said rules, and they make, with the common denominator, $\frac{583}{4224}$, for the whole addition. Thus, we subtract the new numerator 231 from the new 352, and they leave behind, with the new denominator, $\frac{121}{4224}$, for the residue of the subtraction. Likewise, $\frac{4}{12}$ and $\frac{7}{15}$ may be reduced to the same denomination : in the first place they are shortened, and they become $\frac{1}{3}$ and $\frac{7}{15}$; then 3 and 15 may be divided by the greatest common divisor 3, and they become 1 and 5; multiply both the numerator and the denominator $\frac{1}{3}$ by 5, and they become $\frac{5}{15}$; thus take $\frac{7}{15}$ by 1, and $\frac{7}{15}$ remains of the same denominator with $\frac{5}{15}$; with which now added, they make $\frac{12}{15}$, or by abbreviation, $\frac{4}{5}$, for the sum of the addition. In a similar manner, for the remainder of subtraction, take $\frac{5}{15}$ from $\frac{7}{15}$ and there remains $\frac{2}{15}$, the remainder of the subtraction. Likewise, 66 and $\frac{2}{3}$ shall be added or rather united : initially they become $\frac{66}{1}$ and $\frac{2}{3}$, by the writing of one below ; then, by the reduction to a common denominator, they become $\frac{198}{3}$ and $\frac{2}{3}$; finally, on adding the numerators, they make $\frac{200}{3}$. Likewise, $\frac{1}{2}$ and $\frac{2}{3}$ and $\frac{3}{4}$ shall be fractions to be added : these, by reduction to the same denomination, become initially $\frac{6}{12}$ and $\frac{8}{12}$ and $\frac{9}{12}$; then, by writing the sum of the numerators on the top, and by writing the common denominator below, the sum becomes $\frac{23}{12}$, which is 1 and $\frac{11}{12}$. Likewise, $\frac{8}{12}$ may be taken from $\frac{11}{12}$: initially there remains by subtraction, $\frac{3}{12}$; then, by cancellation, it becomes $\frac{1}{4}$, the remainder sought.

CHAPTER XII.

CONCERNING THE MULTIPLICATION AND SIMPLE DIVISION AND ROOTS OF FRACTIONAL NUMBERS.

Fractions are multiplied, and also fractions of fractions in turn again are reduced to simple fractions, and which have a single numerator and a single denominator, by Ch.7 Book I, on being cancelled thus so that nothing may remain between those in a composite ratio. Thence, with new numbers multiplied above in turn, a new numerator is made; and with the new numbers multiplied in turn below, the product becomes a new denominator, or of the fraction sought.

As $\frac{1078}{1768}$ and $\frac{3705}{1449}$ and $\frac{1455}{2090}$ shall be multiplied in turn: the above 1078, and the lower 1768, may be cancelled down by 2, and it becomes 539 above, and 884 below. Likewise, 3705 above can be cancelled down with1449 below, by 3, and there becomes 1235 above, and 483 below. Thence, 1455 and 2090 can be shortened on dividing by 5, and there arises 291 above, and 418 below; with this in place, $\frac{539}{884}\frac{1235}{483}\frac{291}{418}$: then the fractions may be cancelled down to 539 above, and 483 below, by their common divisor 7, and they make 77 above, and 69 below ; then with 1235 above cancelled with 418 below, by their common divisor 19, and they make 65 above, and 22 below ; then abbreviate 291 above, and 884 below, by their maximum common divisor 17, and the fraction becomes 23 above, and 52 below; with this in place, $\frac{77}{52}\frac{65}{69}\frac{23}{22}$: With these made, abbreviate 77 above and 22 below, by their common maximum divisor 11, and there arises 7 above and 2 below; thus with 65 above and 52 below cancelled by their maximum common divisor 13, and the fraction becomes 5 above and 4 below; finally, abbreviate 23 above and 69 below by their common maximum divisor, clearly by 23, and the fraction becomes 1 above and 3 below, with this in place, $\frac{7\cdot5\cdot1}{4\cdot3\cdot2}$: thus multiply in turn these easy and tractable numbers above 7, 5, 1, and they produce 35 for the new numerator ; similarly, the lower numbers 4, 3, 2 are multiplied in turn, and they produce 24 for the new denominator: Therefore $\frac{35}{24}$ is the fraction sought of all the fractions presented multiplied. Likewise, these fractions that cannot be cancelled down $\frac{3}{4}$ and $\frac{5}{7}$ are to be multiplied together : hence 3 by 5, and 15 becomes the new numerator; also the lower numbers may be multiplied 4 by 7, and 28 becomes the new denominator: From which $\frac{15}{28}$ is the multiplied fraction sought. Likewise, $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{6}{7}$ of $\frac{8}{9}$ shall be reduced to the same simple fraction : first the upper 6 and the lower 9 are cancelled by their common maximum divisor 3, and the fraction becomes 2 above and 3 below, with this in place , $\frac{2}{3}\frac{4}{5}\frac{2}{7}\frac{8}{3}$; for there is no further cancellation between these upper with any of the lower : therefore finally the upper numbers are multiplied 2, 4, 2, 8, in turn and they become 128; likewise, the lower 3, 5, 7, 8 may be multiplied in turn, and they make become 315; from these the new denominator is made, from the others, truly the numerator, with these in place, $\frac{128}{315}$ is the simple fraction sought, the fraction likewise that prevails of the given $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{6}{7}$ of $\frac{8}{9}$ presented.

But for division the terms of the divider are transposed, and the transposed terms multiplied by everything effecting the division, as we have taught in Ch.7 Book I.

So that if $\frac{4}{5}$ shall be divided by $\frac{2}{3}$: with the former retained, the latter inverted, and thus by multiplying $\frac{4}{5}$ and $\frac{3}{2}$; in the first place, by cancellation, there becomes $\frac{2}{5}$ and $\frac{3}{1}$; then there comes about $\frac{6}{5}$ by multiplication, or which is the same, $1\frac{1}{5}$ for the quotient sought.

Another example of mixed fractions.

$66\frac{2}{3}$ shall be divided by $2\frac{1}{5}$, or, which is the same, by reduction $\frac{200}{3}$ is divided by $\frac{11}{5}$ *:* this inverted in this manner, $\frac{5}{11}$ and multiplied by that, $\frac{200}{3}$, in turn, and $\frac{1000}{33}$ arises, which is $10\frac{10}{33}$, for the quotient sought.

Another example for testing multiplication.

There shall be, from the preceding examples, $\frac{15}{28}$ divided by $\frac{3}{4}$: hence it may be inverted and it becomes $\frac{4}{3}$ and the other remains $\frac{15}{28}$; they are cancelled down, and at first they become $\frac{5}{28}$ and $\frac{4}{1}$, then they become $\frac{5}{7}$ and $\frac{1}{1}$; multiply both the numerators and denominators in turn, and the fraction becomes $\frac{5}{7}$; as above by multiplication.

The multiplication of roots [*i.e.* a number as a base with its powers] with fractional numbers, just as with whole numbers, is brought about by continued multiplication : For if you will multiply a fraction into itself, or into a fraction equal to itself, it becomes duplicated ; if hence you multiply the duplicand into the same first fraction, it becomes triplicated; as if now you multiply the triplicand into the first, thence the quadruplicate of the root and of the first fraction arises : And thus henceforth the quintuplicate, sextuplicate, etc. of the roots are composed.

So that if the root shall be $\frac{2}{3}$, it can be multiplied radically to the index 5, or quintuplicated: Initially, multiply $\frac{2}{3}$ into $\frac{2}{3}$ and it becomes $\frac{4}{9}$ for the duplicate ; then multiply this duplicate $\frac{4}{9}$ by the same $\frac{2}{3}$ making $\frac{8}{27}$, the triplicate of the fraction put in place; hence the triplicate $\frac{8}{27}$, multiplied by the same $\frac{2}{3}$ and the quadruplicate of the root is presented $\frac{16}{81}$; which at last multiplied by $\frac{2}{3}$, produces $\frac{32}{243}$.

Also the division of fractional roots, is brought about from simple continued division, by the above rules, and in the manner of integers, by dividing the fractional root proposed by its given root, continually, as far as into unity ; and the number of divisions will give the index, as has been said above many times.

There shall be, as an example and for testing the preceding, $\frac{32}{243}$ divided by $\frac{2}{3}$: that becomes, initially, $\frac{16}{81}$; secondly, $\frac{8}{27}$; thirdly, $\frac{4}{9}$; fourthly, that itself gives the root $\frac{2}{3}$; fifthly, it comes to unity: Therefore, these five divisions show the index five for the root of the quotient sought, and it proves the preceding quintuplication to be legitimate.

CHAPTER XIII.

CONCERNING THE EXTRACTION OF ROOTS FROM FRACTIONAL NUMBERS.

If each term of a fraction will have had roots of a proposed kind reduced to the smallest terms, these are extracted in the manner of whole numbers, and become the numerator and denominator of the root sought.

But if each term of the fraction may not have such a root, then the root of this fraction has to be extracted in the manner of geometry or mechanically.

So that the tripartite root can be extracted from $\frac{54}{16}$, the smallest terms of which are $\frac{27}{8}$; in each term of which clearly the tripartite root of the numerator is agreed to be of the number 3, and of the denominator 2; therefore $\frac{3}{2}$ is the tripartite root sought of the offered $\frac{54}{16}$, or $\frac{27}{8}$.

Another example.

On the other hand let the fraction offered be $\frac{290}{40}$, the smallest terms of which are $\frac{29}{4}$*;* the bipartite root is to be extracted from these : I say, because each of the terms, 29 and 4, do not have a bipartite root, (although the one, clearly 4, does) thus, a perfect root cannot be extracted arithmetically from the fraction present $\frac{290}{40}$ or $\frac{29}{4}$, but either geometrically or mechanically [*i.e.* numerically].

The geometrical method is, to put in front of the root of the whole fraction the sign of the root to be extracted, or the sign of each term.

As in the fraction of the above example $\frac{290}{40}$, or $\frac{29}{4}$ that does not have a true bipartite root ; thus, the sign of the whole fraction or of that required to be placed in front is the sign of the root, in this manner, $\sqcup\frac{29}{4}$ or of each term, in this manner $\frac{\sqcup 29}{\sqcup 4}$; or finally, the same root is certainly included mechanically between the limits, as follows.

But the mechanical way is, the root sought, which cannot be expressed by a number, is included between greater and lesser limits as closely as possible.

By how much greater the limit of the denomination may be, and by how much less the difference might be from the lesser limit, by so much more will the root become defined more closely and precisely.

Therefore so that the limits shall be of the chosen denomination, a number of the chosen denomination is multiplied radically, according to the index and the kind of root sought; thence, the radicand of this is multiplied by the numerator of the fraction presented, the product to be divided by the denominator of the same ; the root of the proposed quotient, will be close both to the lesser as well as to the greater limit, these may be extracted in the manner of an integer, and subscribed to each denominator selected: Indeed they will be the limits chosen, the one smaller, the other greater, enclosing the root sought.

So that the bipartite root of the above fraction $\frac{290}{40}$ or $\frac{29}{4}$ shall be extracted: Let the chosen denominator be of 200 parts; of which the radicand of the proposed kind (clearly the duplicate) is 40000; therefore multiply 40000 by 29, the numerator present, and divide the product by 4, the denominator of the same quotient, and the quotient becomes

290000 ; its bipartite root, without doubt close to the lesser limit, is 538, and 539 is close to the maximum limit, (from the above rules of integers) from which by writing the chose denominator 200 below, the lesser limit becomes $\frac{538}{200}$, and the greater limit $\frac{539}{200}$, the differences only by one in two hundred, and thus enclosing ⊔ $\frac{29}{4}$ closely enough, or the bipartite root of this fraction sought $\frac{29}{4}$.

An example of the tripartite root.

Likewise, let the tripartite root be extracted from $\frac{2}{3}$ or rather (because this root cannot be set out by a number) the greater and lesser limits of this are sought, having a denomination as you wish, for example, of 1000 parts ; between which the tripartite root of two thirds may be contained, or L $\frac{2}{3}$. Therefore the triplicate of a thousand parts, which is 1000000000, multiplied by 2, becomes 2000000000; which is divided by 3, and it becomes $66666666\frac{2}{3}$; the tripartite root of which is 873, and the greater is 874, from which the chosen denominator 1000 is written in the denominator, and the smaller limit of the tripartite root of two thirds becomes $\frac{873}{1000}$, and the greater limit $\frac{874}{1000}$, or of L $\frac{2}{3}$ sought.

Last of all, the fractions, of which the numerator exceeds the denominator, are required to be restored to whole numbers by dividing the numerator by the denominator, as has been explained in full, at the end of Book One.

So that if from the preceding works the fraction $\frac{562}{18}$ may arise, this fraction may be contained by integers by which the final answer is required to be given rather than by fractions. Therefore 562 is divided by 18, and 31 arises in the quotient, and 4 is left over, which is $\frac{4}{18}$, or $\frac{2}{9}$; from which, from the total reduction $31\frac{2}{9}$ arises, which is of the integer 31, and two ninths of one.

CHAPTER XIV.

THE RULES OF PROPORTIONAL FRACTIONS.

In the rules of fractions in proportion, the place and operation are to be observed.

In these, the place of the numerators is the same as that of the magnitudes generally of Ch. 5 of Book One ; truly the place of denominators is the opposite ; thus so that if a numerator will fall beneath the line by this law, the denominator will be inverted, and falls above the line ; and on the contrary, if the former lies above, the latter (as is equally the case) falls below.

So if it is proposed that a measure of $\frac{3}{7}$ parts water flows out of some cistern in $\frac{5}{12}$

.5.5
3.

hours; moreover it is asked in what time a measure of $2\frac{1}{2}$, or $\frac{5}{2}$ parts flows out ? Initially, draw a line as from the margin, the number 5 of the first time can be put in place (by this as well as the general precept of Ch. 5 Book I.), because $\frac{5}{12}$ and the number of hour as well as the number of hours sought are named alike, and rise or fall at the same time : Thus $\frac{5}{2}$ measures increase and decrease with the sought number of hours ; therefore, the numerator 5 of that is placed on the upper line as well : but for the first parts of the measure $\frac{3}{7}$, with the increased, the parts of the time sought decrease, and for decreases they increase ; therefore the numerator 3 of these is situated below the line. On the other hand, truly all the denominators, to wit, 7, 12, and 2, of the individual by their individual numerators are put opposing on the opposite side, as from the margin.

$\frac{7\cdot5\cdot5}{3\cdot12\cdot2}$

The operation thus is performed. Multiply the upper numbers in turn, and the numerator is produced ; likewise also the lower numbers in turn, and the denominator is produced, dividing the aforesaid numerator, and showing the wished for product. So the aforementioned upper numbers of the example 7, 5, 5, are to be multiplied in turn, and they make the numerator 175; and the lower numbers 3, 12, 2, multiplied in turn, produce the denominator 72; from which $\frac{175}{72}$, or (by dividing 175) $2\frac{31}{72}$ hours is the time sought, by which without doubt $2\frac{1}{2}$ measures of water flow out.

$\frac{7\cdot5\cdot5}{3\cdot12\cdot2}$ $\frac{175}{72}$

Here also you may recall another of the shortcuts of Ch. 10. Book II. whenever a composite ratio which can be cancelled down falls between some upper and some lower fraction.

So that with the terms of the above example $\frac{7}{3}$ $\frac{5}{12}$ and $\frac{5}{2}$, which are unable to be cancelled down, if these terms are taken : $\frac{7}{6}$ and $\frac{10}{21}$ and $\frac{15}{8}$, which have composite ratios, and cancellations indeed are possible : Thus, with the terms gathered together as by the margin, the upper 7 and the lower 21 are divided, by the common maximum measure 7 and the fraction becomes $\frac{1}{3}$: with this in place, $\frac{1\cdot10\cdot15.}{6\cdot3\cdot8.}$; then, in a similar manner the upper 10 and the lower 6 are cancelled, and there is made in place of these, 5 above and 3 below; with this in place, $\frac{7\cdot5\cdot15}{3\cdot3\cdot8}$; finally 15 above and 3 below may be cancelled, and 5 arises above and 1 below; with this in place, $\frac{1\cdot5\cdot5}{3\cdot1\cdot8}$ or this, $\frac{1\cdot5\cdot5}{1\cdot3\cdot8}$, or otherwise $\frac{1\cdot5\cdot5}{6\cdot1\cdot4}$; of which with the uppers multiplied in turn 1, 5, 5, there becomes 25; then, similarly with the numbers below, 3, 1, 8, or 6, 1, 4, multiplied in turn , and there becomes 24: Thus, the whole fraction satisfying the question, will be $\frac{25}{24}$, or on cancellation, $1\frac{1}{24}$. And thus in all similar cases requiring the working to be shortened.

$\frac{7\cdot10\cdot15}{6\cdot21\cdot8}$

CHAPTER XV.

CONCERNING PHYSICAL FRACTIONS.

Now we have completed all the computations of vulgar fractions ; physical fractions are to follow in order.

Physical fractions are the part or parts of the whole to be divided by statute, and with the generally accepted share of the denominator in place, imposed by their authorities.

So that with our money dealers it pleases that the pound of money be divided into none other than 20 parts, and to impose the denomination of the shilling on these, and to subdivide the shilling into 12 parts, which are denominated pence. Thus physicians have divided the pound weight into 12 parts, which they call ounces, and the ounce into 8 drachmas, and the drachmas into 3 scruples, etc. Timekeepers, have distributed the year into 12 months, the month into 30 days (or around that), the day into 24 hours, &c. Astronomers have divided the degree into 60 scruples or first minutes, the first into 60 seconds, the seconds into 60 thirds, and thus henceforth.

With practical fractions, this is common with vulgar fractions, that as often as they have increased their denominator, they augment the upper place by one ; and as often as a greater fraction shall be divided by that lesser, the outstanding amount is to that of the upper place of unity according to the amount deficient.

So that if 14 hours were to be added to19 hours, which constitutes 33 hours, for which there is not written 33 but, under the title of days and hours there is written 1 day and 9 hours ; because 9 hours have accrued beyond the 24 hours in the day. Thus, for 7 and 9 pence to be added, there is not written 16 pence, but 4 pence, and for the accrued 12 pence, 1 may be written under the title shillings. Likewise, for 48 first scruples to be added to 15 first scruples, there is not written 63 first scruples, but only 3, and 1 degree for the remaining 60 scruples. Thus, it is accustomed to be done not at all otherwise, but exactly as with whole numbers and the vulgar fractions of numbers ; as with five hundred to be added to eight hundred, 13 hundred is not written nor said, but one thousand three hundred, in this manner, 1300. And with vulgar fractions to be added, just as six sevenths or $\frac{6}{7}$, and five sevenths or $\frac{5}{7}$ are to be added, and duly abbreviated, they are not said to be eleven sevenths, but four sevenths, and for the remaining seven sevenths, one has to be written, in this place, $1\frac{4}{7}$. Thus, in subtraction, if 14 hours were to be subtracted, in the above example, from one day and 9 hours,– because 14 hours exceeds 9 hours, thus that single day has to be resolved into 24 hours, which with 9 hours make 33 hours; from which with 14 hours taken away, there are left, as above, the 19 hours sought. Thus, from 1 shilling and 4 pence, if 9 pence shall be taken away, which exceed 4 pence, and thus with 4 pence outstanding there is one shilling or 12 pence, so that thence there becomes 16 pence, from which now with 9 pence taken away, there remains as above, the 7 pence sought. Thus, if from a degree and 3 minutes there were subtracted 15 minutes, the one degree and three minutes may be reduced to 63 minutes, so that thence 15 may be taken away, and 48 minutes remain. And not otherwise with vulgar fractions with integers, the outstanding units of the preceding place is to the following place according to the defect of its supplement. So that if from one thousand three hundred there may be taken five hundred, with three hundreds outstanding there is one thousand, so that hence they make thirteen hundred, from which now take five hundred, eight hundred will remain, as above. Thus, from one and four sevenths, six sevenths are to be taken away; so that this becomes, the unity being reduced to seven sevenths, which are before the four sevenths, so that they become eleven sevenths ; from which with six sevenths taken away, five sevenths remain, as above. And thus you see the common agreement of all of these.

Indeed the reasoning is one of equality and respect to their denominations, both of fractions as well as of whole numbers, both by increasing as well as decreasing.

So that the unity of whole numbers may be raised upwards to their tens, hundreds, and thousands, or lowered to their tenths, hundredths, thousandths downwards. Thus, the unity of minutes may be raised on dividing by sixty to the unity of sixties (which are degrees), and to units of sixty degrees (which have two signs), by ascending ;

and to their sixtieth parts (which are the second scruples [*i.e.* seconds]), and to the second scruples of the sixtieth scruples by descending (which are the third scruples [*i.e.* sixtieth parts of seconds of arc]). Likewise, shillings may be raised to their twentieth multiple by ascending (which are called pounds [sterling]), and to their twelfth parts below by decreasing (which are called pence). Parity is respected, and in all cases by a similar calculation.

Hence it will come about that there may be leisure and time left over, to carry out a particular calculation with these, since the computation of these can be set out more easily by common integers and fractional numbers, than by their own particular forms of calculation, especially of astronomy, for which also an extended table with sixties is required, together with the inconvenience of double figures for each denomination of the subjects, where common arithmetic requires only a single figure for each in place.

As becomes apparent with experience: For an astronomical calculation can be performed quicker and easier in the manner of arithmetic, and resolved, than by the astronomical calculation itself ; besides also the tedium of the tables mentioned before, and the doubling of figures, under any denomination. In as much as common arithmetic sets up in the first place single figures, and in the second place tens, and in the third place hundreds, and only with a single place for the remaining figures. But an astronomical calculation in its individual places, both for ascending as well as descending, thus with the notations, *' '' ''' ''''*, etc. and it is performed generally subject to two figures, which gives rise to every confusion.

From which, with all fractions except vulgar fractions now set aside, we impose an end to this Logistic Arithmetic.
[Henry Briggs, in his *Trigonometria Britannica*, subsequently tried to establish the decimal degree as the measure of choice; however, Vlaq subsequently published the tables using the traditional measures, and guess what, they are still with us.]

Moreover, to the Best and Greatest God, and to all His numbers Infinite, Immense, and Perfect, may all praise, honour, and glory be attributed for every.

Amen.

End of Book Two.

LIBER PRIMUS.

DE COMPUTATIONIBUS QUANTITATUM OMNIBUS LOGISTICAE SPECIES COMMIUM.

CAPUT I.

DE COMPUTATIONIBUS PRIMIS.

Logistica est ars bene computandi.

COMPUTATIO est actio seu operatio quae ex pluribus quantitatibus, etquantitatum proprietatibus datis, quaesita invenit.
Dantur autem aut vocali nominatione, aut graphica notatione.
Unde in omni Logistica primo procedunt nominatio et notatio; mox cum eis succedit computatio.

Computatio autem est simplex vel composita.
Simplex est computatio, quae ex duabus datis tertiam unica aut unimoda operatione invenit.
Simplex computatio vel est prima vel orta.
Prima est computatio, quae quantitatem cum quantitate semel tantum computat.
Atque haec, ex totius, partis, et residae, duabus quibuscunque datis, tertiam quamcunque invenit: Quod mox patebit exemplis subsequentibus.

Hinc autem vel est additio vel substractio.
ADDITIO est computatio prima qua plures quantitates adduntur, et producitur tota.
Exempli gratia., addantur 3 et 4, et producentur 7 pro tota.
item addantur 2, 3, et 4, et producentur 9 pro tota.

SUBSTRACTIO est computatio prima qua substrahendum a minuendo aufertur, et producitur residuum.
Ut auferendo 4 a 9 remanent 5. Dicuntur autem 4 auferendum, 9 minuendum, et 5 residuum. Sic ablatis 3 a 5 remanent per substractionem 2.

Substractio aut est aequalium, et nihil remanet, aut inaequalium.
Inaequalium vero est aut quantitatis minoris a majore, et remanet quantitas major nihilo, aut quantitatis majoris a minore, et residuum erit minus nihilo.
Ut substractis 5 ex 5 remanet nihil; substractis 3 a 5 remanent 2, majores quidem nihilo; substractis autem 7 a 5 relinquentur 2 minores nihilo, seu nihil minutum duobus.
Ex his ergo constat defectivas quantitates hinc originem trahere, ex substractione nimirum majoris a minore: De quibus suo loco agetur.

Ex praemissis clarum est additionem et substractionem relata esse ; atque ideo alteram alterius examen.

Examina enim definimus ea tantummodo quae tum omnibus tum solis recte computatis conveniunt.

Ut supra, pro examine substractionis, an 3 substracta ex 5 relinquant 2, adde 2 et 3 et restituent 5. Et contra, pro examine additionis, an 2 et 3 faciant 5, substrahe 3 ex 5, et prodibunt 2 restituta; vel aliter, substrahe 2 ex 5, et prodibunt 3 ut prius.

Est et praeter hinc aliud examen substractionis in se, substrahendo nimirum residuum ex minuendo ut relinquatur prius substrahendum.

Ut pro examine an 3 substracta ex 5 relinquant 2, substrahe 2 ex 5, et restituentur 3. Habes itaque, ex totius, partis, et residum, duabus quibuscunque datis, tertiam, per additionem et substractionem.

CAPUT II.

DE COMPUTATIONIBUS ORTIS EX IPSIS PRIMIS.

Hucusque de primis computationibus egimus; sequuntur ortae a primis.

Ortae sunt quae quantitatem cum quantitate pluries computant. Atque hae ex prioribus aliquoties continuatis naturalem originem ducunt.

Ortae item vel ex primis, vel ex primo ortis continuatione oriuntur.

Ortae ex primis sunt, quae ex totius, partis, et partem cognominantis duabus quibuscunque datis tertiam inveniunt. Exemplis mox patebunt haec.

Ortae autem ex primis sunt multiplicatio ex continuata additione, et partitio ex continuata substractione.

Est ergo MULTIPLICATIO, alterutrius datarum toties continuata additio quoties est in altera unitas; et quod producitur multiplum dicitur.

Ut multiplicare 3 per 5 est alterutrius v.g. trium quinquies continuata additio, quae 15 efficit; vel quinarii ter additio, quae totidem etiam efficit. Et horum 3 et 5, alterum multiplicans, et alterum multiplicandum, producta vero 15 multiplum dicuntur.

In his se habet unitas ad multiplicantis et multiplicandi alterutrum, ut alterum ad multiplum.

Ut in superiore exemplo, ita se habet 1 ad 3, ut 5 ad 15; seu ita 1 ad 5, ut 3 ad 15.

Multiplicationis species infinitae sunt; ut duplatio, quae est multiplicatio quantitatis oblatae per 2; triplatio, quae per 3; quadruplatio, quae per 4; et ita deinceps.

In his multiplicanda dicuntur, duplandum, triplandumt quadruplandum, &c.; multiplicantia sunt 2, 3, 4, &c.; multipla–dupla, tripla, et quadrupla, &c. dicuntur.

PARTITIO est partientis apartiendo substractio in nihilum usque continuata; et numerus substractionum est quotus quaesitus.

Ut sint partienda 15 per 5, auferantur 5 ex 15 continua substractione donec nihil remanserit, et fient substractiones numero tres: 3 ergo sunt quotus quaesitus, 15 sunt partiendum, et 5 partiens.

In his se habet unitas ad quoti et partientis alterutrum, ut alterum ad partiendum.

Ut superiore exemplo, ita se habet 1 ad 3, ut 5 ad 15; vel 1 ad 5, ut 3 ad 15.

Partitio aut est aequalium, et producitur unitas, aut inaequalium.

Inaequalium vero aut est minoris per majorem, et quotus est fractio, seu fracta quantitas unitate minor; aut majoris per minorem, et fit quotus unitate major.

Ut sint partienda 10 per l0, producitur praecise unitas; at 10 divisis per 13, producuntur decem decimaetertiae partes unitatis et unitate minores; tertio, partitis10 per 5, producuntur 2 unitate majores.

Partitio rursus majoris per minorem, aut est perfecta aut imperfecta.

Perfecta, ubi nullae sunt reliquiae. In his quotus est integrorum.

Ut superiore exemplo partitis 10 per 5, producuntur 2,praecise absque reliquiis.

Imperfecta, vero, quae reliquias impartitas relinquit.

Ut si partienda offerantur 16 per 5, et inde produxeris 3, et unitatem indivisam remaneae permiseris, imperfecta dicetur partitio.

Ex his ergo constat tam ex partitione minoris per majorem, quam ex imperfecta partitione majoris per minorem, fractiones originem trahere :

De quibus suo loco.

In partitione, quantitas quae partienda offertur partiendum dicitur; datarum altera, partiens aut partitor; quae provenit quotus dicitur; et si quid indivisum remanserit, reliquiae.

Ut praecedente exemplo, 16 dicuntur partiendum, 5 partiens, 3 quotus, et unitas superstans reliquiae.

Partitionis species infinitae sunt; ut bipartitio, quae est partitio quantitatis oblatae in duo aequalia; tripartitio, quae in tria; quadripartitio, quae in quatuor; et ita deinceps.

In his partienda dicuntur bipartiendum, tripartiendum, quadripartiendum, &c.; partientes sunt 2, 3, 4, &c.; at quoti dicuntur partes dimidia, tertia, quarta, &c.

Ex praemissis constat multiplicationem et perfectam partitionem relata esse, atque alteram alterius examen.

Ut si dubites an 3 multiplicata in 5 producant 15, pro examine partire 15 per 5, et cum inde redeant 3, scis te recte multiplicasse: vel aliter, partire 15 per 3 et redeant 5, ut prius. Item, si dubites an 16 divisa per 5 producant 3, relicta unitate indivisa, multiplicato 3 in 5, non redeunt 16, sed 15; ideo addenda erit unitas examini, et partitio arguetur unitate imperfecta esse.

Est et praeter haec aliud partitionis examen in se, nimirum partiendo partiendum per quotum, ut pristinus inde redeat partitor.

Ut pro examine an 15 partita per 5 producant 3, partire 15 per 3, et redibunt 5, ut prius.

Habes itaque ex totius, partis, et partem cognominantis, duabus quibuscunque datis, tertiam per multiplicationem et partitionem.

CAPUT III.

DE COMPUTATIONIBUS ORTIS EX PRIMO ORTIS: RADICALIS MULTIPLICATIO ET PARTITIO.

HACTENUS ortae ex ipsis primis; sequuntur ortae ex primo ortis.
Ortae ex primo ortis sunt computationes quae, ex radicati, indicis, et radicis, duabus quibuscunque datis, tertiam inveniunt.

Radicatum est quod aliquoties partitum per quantitatem aliquam in unitatem redit; et quotus ille partitionum index dicitur; quantitas autem partiens est ipsa radix.

Ut ex his tribus terminis, 32, 5, et 2, 32 dicuntur radicatum, quia ea partita quinquies per 2 in unitatem redeunt; scilicet prima partitione fiunt 16, secunda 8, tertia 4, quarta 2, quinta denique 1: Atque igitur horum 5 sunt index, et 2 sunt radix.

Ortae autem ex primo ortis aut sunt radicalis multiplicatio ex continuata multiplicatione; aut radicalis partitio, et radicis extractio, ex continuata partitione.

RADICALIS MULTIPLICATIO est radicis oblatae toties continuata multiplicatio quoties est in indice unitas; et producitur radicatum quaesitum.
Ut dicuntur 2 radicaliter multiplicari ter, cum 8 inde producuntur, quia ternarius index tres unitates continet; et ex prima ab unitate multiplicatione fiunt 2, secunda 4, tertia 8, quae quidem sunt radicatum quaesitum.

Radicalis multiplicationis species infinitae sunt: Ut duplicatio, quae est multiplicatio duorum aequalium invicem, aut datae bis positae; triplicatio, quae est datae ter positae, aut trium aequalium datae.

In his radicata dicuntur duplicatum, triplicatum, quadruplicatum: Indices,– duo, tria, quatuor: Radices, – bipartiens,tripartiens, quadripartiens.

Ut posito binario pro radice, et binario pro indice, bini binarii faciunt 4 duplicatum: Et indice ternario totidem bina, scilicet bis duo bina efficiunt 8 triplicatum: Sic indice quaternario totidem bina, scilicet bis duo bina bis faciunt 16 quadruplicatum.
Et ita in infinitum posito binario pro radice, ut in sequente tabella, cuius prior series est indicum, posterior radicatorum.

	I.	II.	III.	IIII.	V.	VI.	VII.	&c.
1.	2.	4.	8.	16.	32.	64.	128.	&c.

RADICALIS PARTITIO est radicati per radicem partitio in unitatem usque continuata, et numerus partitionum est index quaesitus.

Ut radicatum 8 partitum per radicem 2, usque in unitatem, facit prima partitione 4, secunda partitione 2, et tertia 1. Unde ternarius, numerus scilicet partitionum, est index quaesitus.

Hic indicum et radicum species sunt, ut supra in multiplicatione radicali.

CAPUT IV.

DE RADICALI EXTRACTIONE.

RADICIS EXTRACTIO, dato indice, est inventio quantitatis quae datum radicatum radicali multiplicatione restituit; idemque radicali partitione dividit.

Ut radicati 8 si quaeratur radix tripartiens, ea, per regulas suo loco tradendas, invenietur esse 2; binarius enim radicali triplicatione restituit primo 2, secundo 4, tertio 8 radicatum: Idemque radicatum contra radicali partitione primo in 4, deinde in 2, tertio in unum redigit.

Radicis extractio aut est perfecta aut imperfecta.

Perfecta, ubi nullae supersunt reliquiae.

Ut in superiore exemplo.

Imperfecta vero, ubi aliquae supersunt reliquiae irresolubiles.

Ut si radix tripartiens fuerit extrahenda ex radicato 9, ea quam proxime erit binarius, qui radicaliter triplicatus restituit 8, et non 9; relicti ergo unitate non extracta, imperfecta dicitur extractio.

Hic etiam radicati, radicis, et indicis species sunt, ut supra in multiplicatione radicali. Et quae post extractionem remanserint reliquiae irresolubiles dicuntur.

Quod ex imperfecta extractione provenit est minor terminus, cui si unitatem adjeceris erit major terminus, inter quos vera et perfecta continetur et latet radix.

Ut in superiore exemplo provenit binarius radix imperfecta novenarii, cui si unitatem adjeceris, fiet ternarius, inter quos latet vera et perfecta tripartiens radix novenarii. Verum Geometrae, majoris accurationis studiosi, ipsum radicatum signa indicis praenotare malunt, quam radicem inter terminos includere.

Ut in superiore exemplo radicem tripartientem novenarii ita notant, $\sqrt{c}9$: quam radicem cubicam novenarii pronuntiant. Nos autem sic notamus, L9, et radicem tripartientem novenarii appellamus: De quibus signis amplius suo loco dicemus.

Hic numeri Geometrici seu concreti, quos irrationales et surdos vocant, ortum habent.

In his computationibus radicalibus, indices alii sunt pares, alii impares, alii rursus primi, id est unitate sola dividui, alii compositi, id est numero aliquo perfecte dividui.

Ut indices 2, 4, 6, pares sunt; 3, 5, 7, impares: Indices autem 2, 3, 5, 7, 11, primi sunt, nec ullo numero dividui: 4, autem, 6, 8, 9, 10, &c. sunt compositi ex numeris; nimirum 4 ex duobus binariis; 6 ex duobus ternariis, aut tribus binariis; 8 ex duobus quaternariis, aut quatuor binariis; 9 ex tribus ternariis ; 10 ex duobus quinariis.

Hinc fit radicalis multiplicationis atque extractionis compendium ubi indices sunt compositi, facilius enim per componentes sigillatim multiplicantur, aut extrahuntur, quam per compositos.

Exempli gratia, radix quadripartiens difficilius uno opere extrahitur ex radicato dato, quam si ejusdem radicem bipartientem primo extraxeris atque hinc deinde aliam bipartientem. Sic sextuplicare radicem, vel sextupartientem radicem extrahere, non tam facile sit quam si primo triplicaveris, aut tripartientem extraxeris, inde duplicaveris aut bipartientem extraxeris. Similis est similium ratio. Exemplum in numeris: Radix sextupartiens e 64 facilius extrahitur primo radicem tripartientem extrahendo, ut fiant 4, inde bipartientem extrahendo, ut fiant 2 ; vel primo radicem bipartientem extrahendo e 64, ut fiant 8, inde extrahendo radicem tripartientem e 8, ut fiant 2, radix sextupartiens quaesita.

Ex praemissis colligitur radicalis multiplicationis, partitionis, et extractionis singulas, duo habere examina; nimirum multiplicatio probatur vel partitione vel extractione; partitio probatur vel multiplicatione vel extractione; extractio, vel multiplicatione vel partitione.

Ut in superiore exemplo radicati 32, indicis 5, et radicis 2, si dubites an 32 sint quintuplicatum binarii, radicaliter partire 32 per 2 et incides in 5, indicem pristinum; vel extrahe radicem quintupartientem 32 et incides in 2, pristinam radicem; unde arguitur 32 esse radicatum verum: Item si dubites an 2 sint radix, per eam divide radicaliter 32, et incides in indicem 5, aut eundem binarium quintuplicato, et incides in 32: Denique ut probetur an 5 sint verus index, extrahe radicem quintupartientem e 32, et incides in 2, aut radicaliter quintuplicato 2, et in cides in 32.

Habes itaque ex radicati, indicis, et radicis, duabus quibuscunque datis, tertiam per radicales multiplicationem, partitionem, et extractionem.

CAPUT V.

DE COMPUTATIONIBUS COMPOSITIS.

HUCUSQuae computationes simplices; sequuntur compositae seu regulae.

Composita est computatio quae ex pluribus quantitatibus datis, atque pluribus et diversimodis operationibus, quaesitam producit.
Compositae computationes, seu regulae, vel sunt proportionalium, vel disproportionalium.

Regulae proportionalium sunt, quae per solas computationes simplices proportionatas, scilicet multiplicationes et partitiones, quantitatem quaesitam ex pluribus datis inveniunt.

Ut si quaeratur, is qui tribus horis quatuor miliaria incedit, sex horis quot miliaria incedet? Item si sex boves nutriantur tribus mensuris foeni quatuor diebus, quaeraturque quot boves nutriri possunt quinque mensuris foeni duobus diebus? Item 20 solidi Scotiae sunt una libra, 2 librae sunt tres marcae, 5 marcae valent unicum coronatum; quot ergo solidos valebunt 9 coronati? Quaestiones proportionalium sunt absque una additionum vel substractionum introductione. Multiplicationes enim et partitiones sunt per consectaria suarum definitionum proportionales.

In his spectantur situs et operatio.
Situs quatuor praecepta sunt.
Primum, ut ducta linea, quantitati quaesitae cum suis collateralibus praeparetur sub linea locus.
Ut in exemplis superius propositis subsequitur.

Primum exemplum.

Pro secundo et tertio praecepto, promptior et facilior esset modus, subjungere lineae numeros minores datos una cum quantitate quaesita ; sed an is modus sit tutior nondum liquet.

6 horae, 4 miliaria;
3 horae, quaesita miliaria.

Secundum exemplum.

6 bo. 5 mens. 4 dieb.
quot bo. 3 mens. 2 dieb.

Tertium exemplum.

20 sol. 2 lib. 5 marc. 9 coron.
quot sol. 1 lib. 3 marc. 1 coron.

Secundum, ut duae quantitates, quarum alteri crescente altera decrescit, ex eodem laterae lineae collaterales statuantur.
Ut in primo exemplo, quanto plures fuerint priores horae, tres nempe, tanto pauciora. erunt miliaria quaesita. Sic, crescente numero boum (ut secundo exemplo), decrescit numerus dierum quibus eadem pabulo nutriantur. Unde 3 horae et quaesita miliaria, atque 6 horae et 4 miliaria; itemque 6 boves et 4 dies, atque quaesiti boves et 2 dies, in eodem latere linearum collocantur.

Tertium, ut duae quantitates, simul crescentes vel simul decrescentes, ex adversis lineae lateribus statuantur.
Ut crescentibus 3 horis, crescere etiam 4 miliaria erit necesse, et contra. Sic crescentibus 6 horis, crescere miliaria quaesita, et contra. Item aucto numero boum, augeri et eorum pabulum necesse est, et contra, minuto minui. Sic auctis diebus augeri foenum, et minutis minui. Tertio exemplo, auctis vel minutis solidis, simul et libras priores eis aequales augeri vel minui est necesse. Sic cum posterioribus libris, 2 scilicet marcae priores 3 eis aequales crescent et decrescent; atque cum marcis 5 posterioribus coronatum unum. Tandemque cum coronatis 9 posterioribus, quaesitum solidorum numerum simul augeri et

minui necesse est. Unde ex singulis hisce binis altera quantitas sub, et altera supra lineam constituitur, ut superius cernere licet.

Quartum, ut binae quantitates cognomines linea illi semper sejungantur.

Prout in superioribus exemplis, 3 horae a 6 horis, et 4 miliaria a miliaribus quaesitis in primo exemplo: Et 6 boves a bobus quaesitis, 5 mensurae foeni a 3 mensuris, et 4 dies a 2 diebus in secundo exemplo: Et in tertio exemplo 20 solidi a solidis quaesitis, et 2 librae ab 1 libra, et 5 marcae a 3 marcis, et 9 coronati ab 1 coronato, interposita linea disjunguntur.

His observatis, ad omnium ejusmodi quaestionum solutionem unicum inserviet generale hoc operationis praeceptum :

Multiplica quantitates superiores invicem, item et inferiores invicem, deinde multiplum superiorum partire per multiplum inferiorum, et quotus erit quaesitum quaestioni satisfaciens.

Ut in primo exemplo, multiplica superiores 6 et 4 invicem, fient 24; quae partire per inferiorem numerum 3, et fient 8, numerus miliarium quaesitus. In secundo exemplo, multiplica superiores 6, 5, et 4 invicem, et fient 120; inde multiplica 8 per 2, fient 6; per quae partire 120, producentur 20, numerus boum satisfaciens secundae quaestioni. Item in tertia quaestione multiplica superiores 20, 2, 5, et 9 invicem, fient 1800; deinde multiplica 1, 3, et 1 invicem, fient tantum 3; per quae partire 1800, fient 600, numerus solidorum valentium 9 coronatos.

Itaque omnes species regularum proportionalium unica generali methodo, et operatione, comprehendimus.

De hac doctrina infinitas, – ut regulae trium seu aureae, simplicis, duplicis, quinque quantitatum, sex quantitatum, directae, inversae, &c.,– species et formas tradunt authores; nec tamen triplices, aut alias ejus multiplices formas attigerunt, quas omnes hic breviter habes.

Atque hae sunt proportionalium; sequerentur disproportionalium regulae: Sed quia hae, praeter computationes proportionatas, additiones etiam et substractiones, et alias computationes proportionem disturbantes immistas habent, has ideo omnes missas facimus, quod unica pro eis omnibus inserviet nobis Algebra.

Ut sunt potissima pars omnium arithmeticarum regularum, alligationis, societatis, falsi, simpli, dupli, et aliarum plurimarum, itemque Geometricarum propositionum, problematum, theorematum, &c. quae, confusa tum varietate tum multitudine, memoriam disturbant ;– has ergo relinquimus, Algebram tractaturi.

CAPUT VI.

DE QUANTITATIBUS ABUNDANTIBUS ET DEFECTIVIS.

HACTENUS computationes quantitatum in genere; sequuntur suarum specierum.

Primo, ergo, quantitates aut sunt abundantes, aut defectivae.

Abundantes sunt quantitates majores nihilo, et augmentum prae se ferunt.

Hae, aut nullo, aut hoc signa +, quod copula augmenti dicitur, praenotantur.

Ut si nihil debentis opes aestimentur 100 coronatorum; eae aut sic, 100 cor., aut sic, +100 cor., praenotantur; et sic pronuntiantur, auctae centum coronatis, commodum semper et lucrum significando.

Harum computationes tam ex praemissis quam subsequentibus habentur.

Defectivae sunt quantitates minores nihilo, et minutionem prae se ferunt.

Hae, semper hoc signa –, quod minutionis copula dicitur, praenotantur.

Ut si ejus opes aestimentur cujus debita excedunt bona 100 coronatis, merito ejus opes sic praenotantur, –100 coronatis; et sic pronuntiantur, minutae centum coronatis, damnum semper et defectum significando.

Defectivarum ortum et originem superius ex substractione majoris a minore provenire ostendimus.

Adduntur abundantes et defectivae, si copulae sunt similes, aggregato eorum praeponendo communem copulam.

Ut addendo +8 et +2, fient +5: Item addendo –4 et –6, fiunt –10.

Adduntur vera, si copulae sunt dissimiles, eorum differentiae praeponendo copulam majoris quantitatis.

Ut addendo +6 et –4, fiunt +2: Sic –6 et +4 addita, fiunt –2, copulam majoris, scilicet senarii, semper praeponendo differentiae.

Substrahuntur autem, si substrahendi copulam mutaveris, eamque ad alteram datarum addideris per praecedentes regulas.

Ut sint substrahenda +5 ex +8; muta +5 in –5, et per praemissam adde –5 ad +8, et fiunt +3 pro residuo substractionis quaesito: Item sint ex –5 substrahenda +8; muta +8 in -8, et ea adde ad –5, et fiunt –13, residuum quaesitum: Sic substrahendo –5 ex +8; fiunt +13 ;et +5 ex –8, fiunt –13; et –5 ex –8, fiunt –3; et +8 ex +5, fiunt –3; et –8 ex +5, fiunt +13; et –8 ex –5, fiunt +3.

Abundantes et defectivae multiplicantur et partiuntur, si copulae sint similes, praeponendo multiplo vel quoto copulam pluris; et si copulae sint dissimiles, praenotando copulam minutionis.

Ut sint multiplicanda +3 per +2, aut –3 per –2 producitur multiplum +6; et si dividenda sint +6 per +3, vel –6 per –3, producitur quotus +2; si vera +3 per –2, aut–3 per +2 multiplicaveris, producentur –6, multiplum quaesitum ; et si diviseris +6 per –3, aut –6 per +3, producetur quotus –2.

Radices, tam abundantes quam defectivae, pari indice multiplicatae, producunt radicatum abundans.

Ut sit radix +2, quam ad indicem 4 multiplicabis, et fient primo +2, secundo +4, tertio +8, quarto + 16 abundans; similiter, –2 multiplicata facient primo –2, secundo +4, tertio –8, quarto item abundans +16 ut supra.

Hinc sequitur, radicati abundantis indice pari duas esse radices, alteram abundantem, alteram defectivam; deficientis vero radicati, nullam.

Ut superiore exemplo radicati +16 abundantis, tam abundans +2, quam defectiva –2, erant radices quadripartientes, ut ex superioribus et utriusque examine patebit. Unde nulla restat, sive abundans sive defectiva, quae sit radix quadripartiens defectivae –16.

Radices abundantes indice impari reddunt (multiplicatione radicali) radicata abundantia, et defectivae, defectiva.

Ut radix abundans +2, indice impari 5, radicaliter multiplicata reddit +32; scilicet, primo +2, secundo +4, tertio +8, quarto + 16, quinto +32, radicatum abundans. Sic radix deficiens –2, indice 5, radicaliter multiplicata facit –32; scilicet, primo –2, secundo +4, tertio –8, quarto +16, quinto denique –32, radicatum defectivum dictae radicis.

Simili modo hinc sequitur, quod radicatum impari indice radicem habeat unicam tantum; abundans, abundantem; et defectivum, defectivam.

Ut superiore exemplo radicatum abundans +32, indice 5, habebit radicem abundantem +2. Sic radicatum defectivum —32, indica eodem, habebit radicem defectivam –2, quod ex praecedente exemplo, et utriusque examine liquido constat.

Regula proportionis non est hic repetenda, quod ae ex multiplicationibus et partitionibus componatur, et per praemissa acquiratur.

CAPUT VII.

DE QUANTITATIBUS FRACTIS.

HACTENUS prima quantitatum divisio; sequitur secunda.

Secundo, etiam, quantitates aut sunt integrae, aut fractae

Quantitates integras hic dicimus, quae aut unitatem, aut nullum habent denominatorem. Integrarum autem per se computationes ex praemissis, cum fractis, vero, ex sequentibus habemus.

Fractas vero dicimus, quae denominatorem diversum ab unitate habent numeratori suppositum.

Denominator est quantitas supposita lineae, quae per quot partes dividenda sit tota indicat.

Numerator autem est quantitas superposita lineae, quae quot ex illis partibus sumendae sint denotat.

V.g. Haec quantitas, 3*ab*, est integra quantitas. Sic (quod idem est) $\frac{3ab}{1}$ est etiam integra, sub specie tamen fractionis. Item $\frac{3ab}{2bc}$, et $\frac{5a}{2}$, et $\frac{3a}{2a}$, sive quod idem est, $\frac{3}{2}$ fractiones sunt, seu fractae quantitates, quarum termini superiores sunt numertores, inferiores vero, denominatores.

Quantitates fractas majores unitate, ex imperfectis partitionibus majoris per minorem, et fractas quantitates unitate minores, ex partitition minoris per majorem, ortum suum ducere superius in divisione ostendimus.
Ut divisis 9 per 2 producuntur $4\frac{1}{2}$, seu, si mavis, $\frac{9}{2}$ unitate majores. Item divisis 3 per 5 oriuntur $\frac{3}{5}$, ut in divisione superius ostendimus.

Unde omnis numerator vicem gerit quantitatis partiendae; denominator vero, quantitatis partientis eam.
Ut superiore exemplo $\frac{3ab}{2bc}$ idem significant quod *3ab* divisa per 2*bc;* sic $\frac{3a}{2a}$ idem valent quod 3*a* divisa per 2*a*, seu brevius, 3 divisa per 2; sive tandem idem valet quod tres partes unitatis divisae in duas; sic $\frac{3}{4}$ sunt tres quartae unitatis, vel tres partitae per quatuor, quod idem est.

Atque omnis quantitas, numeratorem et denominatorem habens, pro fracta habetur, et ut fractio computatur.
Hinc prudenter unitatem pro denominatore integris subjicimus, ut integrae cum fractis, quasi fractae, computentur.
Facilius autem computantur fractae, si earum termini contrahantur, et abbrevientur, priusquam inter operandum accreverint.
Abbreviantur, autem, et contrahuntur, partiendo terminos accretos per suum maximum communem divisorem.
Est autem maximus communis divisor, quo major dari nequeat perfecte dividens utrumque terminum.

Hic habetur, partiendo terminum majorem per minorem, primo, et deinde semper partiendo praecedentem partitorem per suas reliquias, donec tandem nihil remanserit; et ultimus ille divisor (spretis quotis) est maximus communis partitor quaesitus.

Ut terminorum 55 et 15 maximus communis partitor sic habetur;
partire 55 per 15, remanebunt 10; partire 15 per 10, remanebunt 5; partire 10 per 5 et nihil remanebit: 5 ergo sunt maximus communis partitor, partiens 15 per 3, et 55 per 11.

Verum si ad unitatem partitorem perveneris, inabbreviabiles, discreti tamen sunt termini, aut se invicem habentes ut discreti.
Ut sint termini 5*a* et 3*a;* partitis 5*a* per 3*a*, remanent 2*a;* inde partitis 3*a* per 2*a*, remanet 1*a;* per quod partitis 2*a*, nihil remanet. Unde 5*a* et 3*a* non habent majorem partitorem unitate, seu 1*a;* per quod si partiantur, habebunt se invicem ut discreti numeri 5 et 3, ut postea suo loco amplius dicetur.

Verum hic summopere cavendum est a partitione incommensurabilium quantitatum, cujus nullus in aeternum erit finis, ut suo loco perspicuum evadet.

Ut denarii, et suae bipartientis radicis, quam radicem quadratam vocant, nulla reperietur in aeternum communis mensura; multo minus partitor ille maximus, ut suo loco.

Cum terminis inabbreviabilibus tanquam abbreviatis eorum defectu est operandum.

Habito maximo communi divisore, et per eum partito utroque termino, oriuntur novi termini abbreviati; et haec operatio abbreviati dicitur.

CAPUT VIII.

DE COMPUTATIONIBUS QUANTITATUM FRACTARUM.

ADDITIONES et substractiones sunt fractionum ejusdem denominationis.

Si diversae sunt denominationis, ad eandem reducantur.

Reducuntur autem, partitis utriusque denominatoribus per suum maximum communem partitorem, notatis quotis.

Inde multiplicando utrosque terminos prioris in quotum posterioris denominatoris, et fit nova prior; et utrosque terminos posterioris per quotum prioris denominatoris, et fit nova posterior ejusdem denominationis.

Ut sint fractae $\frac{2}{3}$ et $\frac{7}{9}$ ad eandem denominationem reducendae ; denominatorum 3 et 9 communis maximus partitor est 3, per quae divisis denominatoribus, oriuntur1 pro priore, et 3 pro posteriore; deinde, multiplicatis $\frac{2}{3}$, utroque termino, per posteriorem quotum 3, oriuntur $\frac{6}{9}$ & pro priore nova. Sic multiplicando $\frac{7}{9}$ per unitatem (prioris scilicet quotum), fiunt $\frac{7}{9}$ ejusdem denominationis cum $\frac{6}{9}$.

Harum jam ejusdem denominationis, addantur et substrahantur numeratores novi, retento novo et communi denominatore, et habebis additionis totam, et substractionis residuam.

Ut superiore exemplo addantur novi numeratores 6 et 7, et oriuntur 13, quae, cum denominatore communi 9, faciunt $\frac{13}{9}$ pro additionis tota. Sic si substraxeris $\frac{6}{9}$ ex $\frac{7}{9}$, remanebit $\frac{1}{9}$ substractionis residua.

Multiplicantur etiam fractae, partiendo singulas binas, quarum altera est numerator, altera denominator, in suum communem divisorem maximum, notatis omnium ultimis quotis, deinde multiplicando quotos numeratorum invicem, et fiunt novus numerator; et quotos denominatorum invicem, et fiunt novus denominator multipli quaesiti.

Ut sint invicem multiplicandae $\frac{18}{20}$ et $\frac{35}{231}$; primo partiantur 18 et 20 per suum communem divisorem maximum 2, fient $\frac{9}{10}$ et $\frac{35}{231}$; inde partiantur 10 et 35 per 5, fient 2 et 7, hoc situ, $\frac{9}{2}$ et $\frac{7}{231}$; inde partiantur 9 et 231 per 3, fient 3 et 77, hoc situ, $\frac{3}{2}$ $\frac{7}{77}$; denique partiantur 7 et 77 per suum communem divisorem maximum 7, et fient 1 et 11, hoc situ, $\frac{3}{2}\ \frac{1}{11}$: His peractis, duc hos numeratorum ultimos quotos 3 et 1 invicem, sic ultimos denominatorum 2 et 11, fientque illi 3, hi 22, hoc situ, $\frac{3}{22}$ multiplum quaesitum. Item sint multiplicandae fractiones inabbreviabiles hae $\frac{2a}{3}$ et $\frac{4}{5}$ invicem; multiplicentur primo numeratores $2a$ et 4 invicem, et fient $8a$, novus numerator; deinde denominatores 3 et 5 invicem, et fient 15, novus denominator; sunt ergo $\frac{8a}{15}$ multiplum quaesitum.

Hac multiplicatione fractiones fractionum, imo, et fractiones fractionum iterum atque iterum fractarum, ad simplices fractiones reducuntur.

Ut dum quintae trium quartarum, sic notatae, $\frac{2}{5}$ ex $\frac{3}{4}$, per praemissam fiunt primo $\frac{1}{5}\frac{3}{2}$ per abbreviationem; inde, per numeratorum invicem ac denominatorum invicem multiplicationem, fiunt $\frac{3}{10}$, unica fractio simplex, idem valens quod superior fractio fractionum. Sic tres quartae duarum tertiarum unius dimidii, sicnotatae, $\frac{3}{4}$ ex $\frac{2}{3}$ ex $\frac{1}{2}$, fient per abbreviationem, primo $\frac{1}{4}$ ex $\frac{2}{1}$ ex $\frac{1}{2}$, inde $\frac{1}{4}\frac{1}{1}\frac{1}{1}$ seu $\frac{1}{2}\frac{1}{1}\frac{1}{2}$; tandem ductis superioribus quotis invicem, et inferioribus invicem, fiunt $\frac{1}{4}$, idem valens quod $\frac{3}{4}$ ex $\frac{2}{3}$ ex $\frac{1}{2}$.

Partiuntur, autem, invertendo terminos divisoris, et inversos per partiendum multiplicando omnimodo ut superius in multiplicatione.

Ut sint $\frac{3}{10}$ penultimi exempli partiendae per $\frac{3}{4}$; hujus divisoris inverte terminos, et fient $\frac{4}{3}$, quae per $\frac{3}{10}$ multiplicatae fient primo per abbreviationem $\frac{1}{10}\frac{4}{1}$, deinde $\frac{1}{5}\frac{2}{1}$, deinde per multiplicationem superiorum invicem, et inferiorum invicem, fient $\frac{2}{5}$, quotus optatus, et superioris multiplicationis examen.

Extractio autem fit extrahendo indicatam radicem tam ex numeratore quam ex denominatore, sive ex quibuscunque terminis ejusdem rationis, et fient radicis quaesitae termini.

Ut sit extrahenda radix bipartiens ex fractione $\frac{16}{25}$; extrahatur radix bipartiens 16, estque 4; inde radix bipartiens 25, estque 5; ex illo fit numerator 4, ex hoc denominator 5, hoc situ, $\frac{4}{5}$ radix bipartiens $\frac{16}{25}$ optata. Item sit extrahenda radix bipartiens ex $\frac{3}{4}$, seu, quod melius est, ex $\frac{48}{64}$; quanta enim majores sunt quantitates ejusdem rationis, tanto exactiores sunt radices, nisi perfecte extrahantur. Extrahe ergo radicem bipartientem e 48, quae non habetur; ergo e 49, et ea erit 7; sic e 64 extrahatur eadem radix, estque 8; sunt ergo $\frac{7}{8}$ radix bipartiens $\frac{3}{4}$, veritati quam proxima.

Radicales multiplicationes, et partitiones, et regulam proportionis, quia nihil aliud sunt quam multiplicationes et partitiones repetitae, ad praemissa referimus, ex quibus facile habentur.

Completis computationibus his, restituendae sunt fractiones mixtae, quarum scilicet numerator excedit denominatorem, ad suas integras et fractiones. Fit autem restitutio haec, partiendo numeratorem per denominatorem, et emerget in quotiente integra quantitas, et reliquiae erunt numerator, et divisor erit denominator, fractioni illi mixtae et adjunctae.

Ut si ex computationibus completis provenerint $\frac{11}{4}$; dividantur 11 per 4, et fiunt 2 in quoto, et 3 supererunt; unde dum integrae, et tres quartae unitatis, hoc situ, $2\frac{3}{4}$, sunt idem quod superiores $\frac{11}{4}$, reformatae et magis perspicuae.

FINIS LIBRI PRIMI.

LIBER SECUNDUS.

DE LOGISTICA ARITHMETICA.

CAPUT I.

DE INTEGRORUM NOMINATIONE ET NOTATIONE.

HACTENUS computationes quantitatum omnibus Logisticae speciebus communium; sequuntur propriarum.

Tertio, itaque, computationes vel sunt verinomiarum, vel fictinomiarum seu hypotheticarum quantitatum. Unde Logistica vel est verinomiarum, de quibus Lib. II. et III.; vel fictinomiarum seu algebraicarum, de quibus Lib. IV. agetur.

Verinomiae sunt quantitates veris nominibus definitae, quibus, quotae sint multitudine, vel quantae sint magnitudine, explicatur.

Verinomiae aut sunt discretae numero discreto, aut concretae numero concreto nominatae.

Unde, Logistica verinomiarum vel est discretarum quantitatum, quae Arithmetica dicitur, de qua hoc Lib. II.; vel concretarum, quae Geometrica, de qua Lib. III. agetur.

ARITHMETICA, ergo, est Logistica quantitatum discretarum per numeros discretos.

Numerus discretus est, quem suum unicum individuum numeratum metitur.

Numerus discretus aut est integer, aut fractus; unde Arithmetica est integrorum et fractorum.

Integri sunt, quos individua unitas numerata metitur.

Vocales integrorum nominationes quodque suppeditat idioma; ut Latinum, – unum, duo, tria, quatuor, &c.

Scripta autem integrorum nomina., seu notae, novem sunt significativae hae: 1 unum, 2 duo, 3 tria, 4 quatuor, 5 quinque, 6 sex, 7 septem, 8 octo, 9 novem.

Hae, diversis locis, diversos significant numeros.

Praeter has novem notas, seu figuras, est circulus 0, qui nullibi locorum quicquam significat, sed locis vacuis supplendis destinatur.

Locorum series a dextra in sinistram consideratur, in quorum primo, figura suo jam dicto valore nominatur; secundo, decuplo; tertio, centuplo; quarto, millecuplo; quinto, decies millecuplo; sexto, centies millecuplo; septimo loco, millies millecuplo; octavo, decies millies millecuplo valore nominatur. Et ita deinceps in infinitum, per decuplum incrementum semper progrediendo.

Ut 7 sunt septem; at 70 sunt septuaginta; 700, vero, sunt septingenta: Sic 8000, octo millia; 60000, autem, sexaginta millia. Unde 68777, sexaginta octo millia, septingenta septuaginta septem, indicant. Item 90680, nonaginta millia, sexcenta triginta, significant: Et ita de aliis.

Hinc fit majorum numerorum facilis nominatio, si, post tertiam quamque figuram constituto puncto, primum punctum millia appelles; secundum, millia millium; tertium, millena millia millium; quartum, millies millena millia millium; et ita de reliquis punctis. Figurae, vera, primo loco a punctis constitutae, suo valore nominentur; secundo, vera, a punctis loco, decuplo valore; tertio denique, centuplo sui valore nominentur.

Ut hic numerus, 4734986205048205, sic pungatur, 4. 784. 986. 205. 048. 205. Et haec erit ejus nominatio: quatuor millies mille millena millia millium, septingenta triginta quatuor millies millena millia millium, nongenta octoginta sex millena millia millium, ducenta quinque millia millium, quadraginta octo millia, ducenta et quinque: Et ita de allis.

CAPUT II.

DE ADDITIONE ET SUBSTRACTIONE INTEGRORUM.

HACTENUS integrorum nominatio et notatio; sequitur computatio ; et prima de additione et substractione.

130
105
90
70
65
162
65
187
182
600
———
1656

In additione spectatur situs, et operatio seu praxis.

Situs est, ut numeri numeris subscribantur; ita ut, a dextris incipiendo, figurae primae primis, secundae secundis, et reliquae reliquis directe substituantur, ducta sub numero infimo linea.

Operationis tria sunt praecepta.

Primum, ut primi loci figurae, omnes in unam summam colligantur, et, hujus summae, prima tantum figura eis, infra lineam, subscribatur; caeteris, si quae sint, animo reconditis.

Secundum, ut caeterae hae, animo reconditae, una cum omnibus figuras sequentis loci, in unam summam colligantur, et, hujus etiam summae, prima tantum figura eis, infra lineam, subscribatur; caeteris ejus summae figuras, si quae sint, animo reconditis: Et hae operatio in ultimam omnium figuram repetenda est.

Tertium, ut ultimi loci figurae, cum novissime animo reconditis, (si quae sint) in unam summam collectae, loca sinistima compleant.

Ut quintum caput Geneseos exhibet annos a creatione Adami et mundi 130 ad Sheth; hinc 105 ad Enosch, hinc ad Kenan 90, hinc ad Mehalalelem 70, ad Jered 65 annos, hinc ad Henoch 162, inde ad Methuselach 65, ad Lamech Un,
hinc ad natum Noah 182, a nato Noah ad initium diluvii 600. Quaeritur, his jam additis, summa annorum a condito orbe ad diluvium ? Anni ergo omnes recto situ, ut a margine, constituantur; inde, primo, dextimae seu primi loci figurae 5, 5, 2, 5, 7, 2, addantur, et fient 26; subscribantur 6 directe, 2 autem in mente reserventur; secundo, haec 2 una cum secundi loci figuras 3, 9, 7, 6, 6, 6, 8, 8, adde, fient 55, quorum priorem notam 5 scribe, posteriorem mente reserva; tertio, hunc reservatum quinarium, una cum 1, 1, 1, 1, 1, 6, ultimi loci figuras, adde, fient 16, quae locis sinistimis ponantur, fitque tota summa 1656.

Additio, tamen, proprie est duorum numerorum, ut tertius inde producatur.

9754862
863556
10618418

Ut sint addendi 9754862 atque 868556; hi, ex superioribus praeceptis, producunt totum 10618418, situ quo a margine. Hujus examen habes in substractione sequente.

In substractione est situs, et operatio.

Situs, ut in additione, a dextris incipit; substrahendi autem et minuendi minor infimo loco, major medio loco, et residuus summo constituitur, linea inter summum et medium interposita; ita ut a dextra figurae primae primis, secundae secundis, et reliquae reliquis, directe superscribantur.

Operatio, contra, a sinistris aptissime inchoatur, aut si mavis a dextris, hoc utcunque servato unico praecepto :

Ut scilicet figura quaeque inferior, aut nihil ubi nihil subest, ex superposita illi figura illa non minore, aut minore additis 10, substrahatur; et residua integra illi superscribatur, si modo inferioris summa, hinc dextrorsum extensa, non excedit summam ei superscriptam; alioquin, si excedit, tunc praefata residua figura minuta unitate, si sit numerus, vel aucta novenario, si sit 0, illi superscribatur.

2690997393
2738154098
47156705

Ut sint 47156705 substrahenda ex 2738154098 numero minuendo: Sit situs hujus supra, illius infra, ut in margine, et dextimae dextimis figurae respondeant: Proinde primo a sinistimis 27 aufer nihil (nihil enim subest), remanent 27 minuenda unitate, quia 4, &c. subscripta excedunt 3, &c. superscripta; et ita 26, residua, sunt superscribenda prioribus 27; deinde aufer 4 ex 3, additis tamen prius his 10, ut 4 inde auferri possint, et supersunt 9, quae absque minutione sunt superscribenda tribus, quia 7 minora sunt quam 8; deinde aufer 7 ex 8, remanet unitas unitate minuenda, et ita 0 est superscribendum, quia dextrorsum 156, &c. exuperant eis superposita 154, &c.; deinde auferatur 1 ab 1, remanet 0; sed quia sequentia 56, &c. exuperant superposita 54, &c., ideo pro 0, 9 sunt superscribenda; similiter aufer 5 ex 5, superest 0, sunt ergo 9 superscribenda, quia 6 exuperant 4 superscripta; inde 6 ex 14 aufer, manent 8, scribe tantum 7, quia subjecta figura 7 excedit 0 superpositum ei; deinde aufer 7 a 10, remanent 3, integre superscribenda, quia subjectum 0 minus est quam 9 superscripta; deinde aufer 0 ex 9, restant 9, integre superscribenda, quia 5 inferius minora sunt 8 suprapositis; tandem

aufer 5 ex 8, et supersunt 3, integre superscribenda, quia nihil post extimam figuram subjectum sequitur quod exuperat superpositum :
Et ita habes residuum quaesitum, 2690997893.

Hujus operis examen fit per, –addendo 47156705 ad 2690997898, et restituentur 288154098; vel per, –substrahendo 2690997893 ex 288154098, et restituentur 47156705: Et ita in aliis.

Aliud exemplum, et superioris additionis examen.

Sint substrahenda 868556 ex 10618418, residuus numerus erit 9754862, ut superius in additione. Idem proveniet si a dextris sinistrorsum operatus fueris.

Si offeratur major numerus ex minore substrahendus, minorem nihilominus semper ex majore, per praemissam, substrahes; verum, residuum minutionis copula praenotabis, et inde numerus defectivus orietur.

Ut sint substrahenda 10618418 ex 868556, fiat, per praemissam, substractio horum ex illis, et provenient (ut in praecedente exemplo) 9754862, quae, conversa in –9754862 defectivum, sunt residuus numerus quaesitus; ut de quantitatibus defectivis, in genere, superius diximus, Lib. I. cap. 6.

CAPUT III.

DE MULTIPLICATIONE INTEGRORUM.

MULTIPLICATIO aut est per unicam, aut plures figuras.
Multiplicatio per unicam, aut est unius aut plurium.

Multiplicatio unius per unicam, seu singularum per singulas, memoriter, ex subjecta tabula, discenda est quam promptissime.

	9	8	7	6	5	4	3	2	1
1	9	8	7	6	5	4	3	2	1
2	18	16	14	12	10	8	6	4	
3	27	24	21	18	15	12	9		
4	36	32	28	24	20	16			
5	45	40	35	30	25				
6	54	48	42	36					
7	63	56	49						
8	72	64							
9	81								

Ut si, quid ex 7 et 8 invicem multiplicatis oriatur, interrogatur? Quaere majorem 8, scilicet in suprema, et minorem 7 in sinistima linea, et in angulo communi offendes 56, multiplum optatum.

Oblitus multipli producti ex duabus majoribus quibuscunque figuras, earum a denario defectus invicem multiplica, provenietque figura dextimi loci; deinde minorem defectum a minore figura, aut majorem a majore, auferto, remanebit figura sinistima, complentes quaesitum.

Ut si oblitus sis quantum septies 8, vel (quod idem est) octies 7, effecerint, earum a denario defectus 3 et 2 invicem multiplica, et provenient 6, figura dextima; deinde aufer 2 a 7, vel 3 ab 8, et restabunt 5, figura sinistima. ltaque 56 sunt quaesitum multiplum ex 7 et 8.

$$\begin{array}{cc} 8 \times & 2 \\ 7 & 3 \\ \hline 5 & 6 \end{array}$$

Multiplicationis plurium per unicam tam situs quam operatio a dextris orditur, procedendo ordine laevorsum; ac multiplicando, etiam, sive supra sive ante multiplicantem constituto, sub utroque ducitur linea. Multiplicatio plurium per unicam tria habet in operatione praecepta.

Primum est, ut dextima figura per datam unicam multiplicetur, et multipli unica figura, vel dextra (si duae sint), subnotetur; sinistra, autem, si quae sit, animo reservetur.

Secundum est, ut figura haec, animo reservata (si quae sit), addatur ad multiplum sequentis figurae, in unicam multiplicatae; aggregati, vero, unica figura, vel dexterior (si duae sint), subnotetur, sinisterior autem (si quae sit) animo reservetur, addenda ut prius; et haec operatio in sinistimam seu ultimam figuram est repetenda.

Tertium est, ut ultimae figurae multiplum, cum novissime animo recondita (si quae sit), in unicam summam collectum, sinistimis locis integrum subnotetur.

Ut sint 865091372 quintuplicanda, seu per 5 multiplicanda: Eorum situs, utcunque sit, ut in margine: Primo, itaque, figura dextima 2, in unicam multiplicatricem 5, multiplicetur; fient 10 (duae figurae), quorum dextram 0 subnoto, 1 autem reservo; deinde duco, seu multiplico, 7 in 5, fient 35; quibus addo reservatum, fiunt 36; quarum dextram 6 sub noto, tribus mente reconditis: Pergo ;–ter 5, seu quinquies 3, sunt 15, et 3, mente reservata, fiunt 18, subnoto 8, et 1 reservo; sequitur quinquies 1, fiunt 5, et 1 reservatum fiunt 6, unica figura subnotanda, nihilo mente reservato; deinde, quinquies 9 sunt 45; subscribo 5, mente autem 4 reservo; sequitur quinquies 0, quod quidem nihil est; hoc, cum 4 reservatis, facit 4 subnotanda: Prosequor;—quinquies 5 sunt 25; noto 5, reservo 2; inde duco 6 in 5, proveniunt 30, et cum reservatis 2 fiunt 32; scribo 2, reservo 3; tandem, duco 8 in 5, fiunt 40, quibus addo 3 reservata, jam omnium ultimo, fiuntque 43, quae quidem integre, in locis omnium sinistimis, scribo. Totum, itaque, multiplum quaesitum est 432545860.

$$\begin{array}{r} 865091372 \\ 5 \\ \hline 4325456860 \end{array}$$

$$\begin{array}{l} \underline{\ 865091372(5} \\ 4325456860 \end{array}$$

Superest, jam, plurium figurarum per plures multiplicatio.

Multiplicatio plurium figurarum per plures, ter praemissa, tria habet praecepta.

Primum, ut totum multiplicandum (per jam praemissa) multiplicetur inquamque figuram multiplicantis, sive adextris, sive sinistris incipere libuerit.

Secundum, ut cujusque figurae multiplum habeat suam dextimam figuram directe notatam sub figura multiplicante, et caeteras ordine sinistrorsum sequentes.

$$\begin{array}{l} 865091372 \\ \underline{92105} \\ 4325456860 \\ 865091372 \\ 1730182744 \\ \underline{7785822348} \\ 79679240818060 \end{array}$$

$$\begin{array}{l} \underline{865091372(92105} \\ 7785822348 \\ 1730182744 \\ 865091372 \\ \underline{4325456860} \\ 79679240818060 \end{array}$$

Tertium praeceptum est, ut sub his particularibus multiplis ducatur alia linea, atque omnis multipla in unum aggregatum addantur, quod quidem erit totale multiplum, et quaesitum totius multiplicationis productum.
Ut sit praecedens numerus 865091372 multiplicandus in 92105:
Collocentur, utrolibet situ, ut a margine; deinde aquovis termino multiplicantis incipe, sive dextro sive sinistro ordine; exempli gratia, a dextro: Duc itaque totum multiplicandum per 5; fient, per praemissa, 4325456860, quae ita locentur ut figura dextima 0 sub figura multiplicante 5 directe statuatur,
caeteris sinistrorsum sequentibus: deinde, idem multiplicandum per 0, quod nihil est, multiplicetur, et nihil proveniet; nihil, ergo, hujus multiplicationis, notari opus habet: sequitur multiplicandum per 1 ducere; proveniet idem ex eodem, viz. 865091372, quarum figurarum dextima 2 subsua multiplicante figura 1 collocetur, caeteris sinistrorsum sequentibus: deinde idem multiplicandum duplicetur, seu per 2 ducatur, et fient 1730182744, quarum situs incipiat sub suo multiplicante 2, et inde sinistrorsum progrediendo: deinde totum multiplicandum duc in 9; fient 7785822348, quae, sub 9 situs initium habentes, hinc ordine laevorsum notentur: Tandem, omnis invicem particularia multipla, lineis interclusa, adde, eo quem habent situ, et fient 79679240818060, pro completo et quaesito multiplicationis multiplo. Nec secus eveniet ex secundo situ, incipiendo, nimirum, multiplicationem per sinistimam figuram multiplicantia 9, ut ex secundo schemate patet. Similiter etiam in similibus.

Hujus examen habes in partitione sequente, ut et illius hoc est examen.

Multiplicatio trium, aut plurium numerorum, perficitur multiplicando primum in secundum, et horum multiplum in tertium, et horum rursus multiplum in quartum; et ita in ultimum.

Ut sint multiplicanda invicem 5, 4, 2, et 3; duc 5 in 4, fient 20; secunda, duc 20 in 2, fient 40; tertia, duc 40 in 8, fient 120, omnium multiplum.

CAPUT IV.

DE PARTITIONE INTEGRORUM.

PARTITIO minoris per majorem non fit aliter quam interponendo lineam, inter superpositum partiendum et infrapositum partitorem, et totus hic quotus fractio est, unitate minor.

Ut sint 8 partienda per 5, sit quotus $\frac{3}{5}$, quae pronunciantur, tres quintae unitatis, vel, tria divisa per quinque; et fractio est.

Unde hinc ortum habent numeri fracti, ut de quantitatibus fractis in genere, cap. 7, docuimus. De his tractabit secunda Arithmeticae pars.

Partitionem aequalis per aequalem, in omnibus quantitatibus, unitatem producere superius diximus.

In partitione, tandem, majoris numeri per minorem, producitur quotus semper unitate major; et proprie huc spectat hinc partitio.

In hac majoris per minorem partitione, spectatur situs et operatio, quorum uterque incipit a laeva, dextrorsum tendens. Situs aptissimus est, ut sinistima partitoris figura sub sinistima partiendi ea minore, aut ante sinistimam partiendi ea majorem, constituatur, caeteris figuras ordine dextrorsum sequentibus, post quarum dextimam immediate incipiat parenthesis, quoto capiendo destinata; et sub omnibus ducatur linea.

Operationis quatuor sunt praecepta.

Primum, ut diligenter exquiratur, quoties partitor subduci potest ab eis supremarum figurarum sinistimis, quae supra vacuum quoti locum, parenthesi proximum, terminantur: sumpta, plerumque, hujus quoti conjectura, ex primis ad primas relatis; deinde, ut figura hunc quotum notans, proximo, post partitorem et parenthesin, vacuo loco constituatur.

Secundum, ut totus partitor, in hanc figuram quoti recens acquisitam, multiplicetur, et multiplum suo debito loco, per cap. 3, constituatur.

Tertium, ut hoc recens multiplum, ex supremis figuras ei directe superpositis, substrahatur, atque deletis etiam figuras, et substractis et ex quibus substrahuntur, supernotentur residuae, per cap. 2 hujus.

Quartum, ut repetantur hae tres operationes, usque quo totum partiendum deleatur, et residua, seu reliquiae, aut nullae aut partitore minores prodeant, dumque etiam nulla quoti loca, usque in dextimam partiendi figuram, vacua relinquantur; atque tandem, post quotum dextrorsum ducatur linea, cui superpositae reliquiae, et suppositus divisor, quoti fragmenta unitate denotabunt.

3 6 6
7(5

Exempli gratia: Sint partiendi dies anni bissextilis 366, per dies hebdomadis 7, ut sciatur quot sint in anno bissextili hebdomades: Statuantur ut a margine; deinde primo inquirendum est, quoties auferri possunt 7 ex 36 (etenim 36 sunt sinistimae illae figurae superpositae, quarum terminus, videlicet, figura 6, desinet supra locum vacuum quoti, proximum parenthesi), et invenies quinquies 7 in 36; statue, ergo, 5 in primo quoti loco vacuo; secundo, multiplica 7 per 5, fient 35 debito loco reponenda, (videlicet 5 sub 5, et 8 sinistrorsum) ; 0 tertio, aufer 35 ex 3 et 6 superpositis, et, eis deletis, relinquitur unitas superius notanda.

0
1
3 6 6
7(5 2 $\frac{2}{7}$
3 5
1 4

Sequuntur ergo reliquae 16, ad et supra sequentum locum vacuum quoti, ex quibus 16, (repetendo opus ut prius), recole quoties 7 auferri possint, et hic bis fieri posse, relictis 2, conspicies; sequente, ergo, et ultimo quoti loco, notentur 2; deinde ducantur 2 per 7, fiunt 14, quid rite collocata cadunt sub 16, ex quibus aufer 14, et, utrisque inde deletis, relinquentur 2 dies, seu $\frac{2}{7}$ hebdomadis, quoto annectendae, ut fiat verus quotus, 52 hebdomadarum et $\frac{2}{7}$ duarum scilicet septimarum hebdomadis, in anno bissextili.

Aliud exemplum.

1 1 8
1 4 1
4 0 2
4 2 9
8 6 1 0 9 4
4 3 2(1 9 9 3 $\frac{118}{432}$
4 3 2
3 8 8 8
3 8 8 8
1 2 9 6

Sint 861094 partienda per 432: constituta ut a margine; videlicet, 4 ante 8, quia 4 minora sunt quam 8; deinde consideratur quoties 432 e 861 auferri possint, conjecturam ex 4 et 8 sumendo, vel ex 43 et 86, et invenies 4 bis ab 8, et 3 bis a 6 auferri posse; sed tamen 2 non posse ab 1 auferri; unde (fallente hac conjectum) pones omnes ab omnibus semel tantum auferri posse; scribatur ergo 1, pro prima quoti figura, per quod multiplica 432, et fient 432, ex 861 auferenda, et restabunt 429. Itaque, a 4290 perscrutare quoties poteris auferae 432, sumpta conjectura ex 4, quae in 42

novies habentur, et remanent 6; 69 etiam novies continent 3, et satis multa praeterea remanebunt, quibus novies contineantur 2 ; ideo 9 sequente loco vacuo, pro secunda figura, quoti, ponitur; deinde, ductis 432 per 9, fient 8888, substrahenda ex 4290, et remanent 402. Itaque, perscrutare quoties a 4029 auferri possint 432, simili qua prius conjectura, et invenies rursus novies hoc etiam fieri posse; per alia ergo 9, quota adjecta, multiplica
432, et fient, ut supra, 3888 substrahenda ex 4029, et supersunt 141. Considera, ergo; ex 1414 quoties 432 auferri possint, et id ter fieri posse reperies.

Ultimo ergo quotientis loco 3 ponantur, per quae multiplica 432, fient 1296, auferenda ex 1414 suprapositis, et tandem supersunt reliquiae 118, sive $\frac{118}{432}$, quae quoto adjiciantur, et perfectum quotum $1993\frac{118}{432}$ reddent.

Tertium exemplum, et superioris multiplicationis examen.

```
            0
    4325456 8
   90884594
  182101733
 79679240818060
 865091372(92105
 ---------------
 7785822348
  1730182744
   865091372
    4325456860
```

Sint 79679240818060 partienda per 865091372; constituantur ut a margine; deinde considera quoties 8650 auferri possint ex 79679, seu quoties 8 ex 79, et perspicies novies id fieri posse, relictis etiam quot sufficiunt pro 6, et 5, et reliquis sequentibus, etiam novies auferendis ; deinde e per 9, loco quoti posita, multiplica partitorem 865091372, et fient 7785822348, debito loco subnotanda, et ex illis superpositis auferenda, et remanebunt 1821017338 partienda per oblatum partitorem 86509, &c., et percipies hinc ex illis (nimirum 8 ex 18, et caetera ex caeteris) bis tantum auferri posse; per 2, ergo, quoto apposita, multiplica partitorem, et fient 1730182744, ex superioribus reliquiis auferenda, et relinquentur 908345940, ex quibus, datus ille partitor semel tantum auferri potest; per 1, ergo, quoto appositum, multiplica partitorem, et exsurget ipse idem partitor 865091372; quo ex 908345940 subducto, relinquuntur 432545686, ex quibus partitor 865091372 ne semel quidem auferri potest; ideo, posito 0 in quoto, perges, et quoties partitor ille ex 4325456860 auferri possit, quaeras, et invenies id quinquies fieri (facta conjectura per 8, quae ex 43 quinquies auferri possunt); per 5, ergo, quoto adjecta, multiplica partitorem, et producuntur 4325456860, ex superioribus etiam his aequalibus auferenda, et nihil superest reliquium: Perfecta est ergo haec partitio, et 92105 sunt quotus integer; atque approbat penultimum exemplum multiplicationis, et ab eo approbatur. Ita est de aliis.

CAPUT V.

DE MULTIPLICATIONIS ET PARTITIONIS COMPENDIIS MISCELLANEIS.

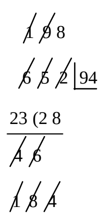
MULTIPLICATIONES per 10, 100, 1000, aut per alia ex unitate et circulis quotvis conflata, facile fiunt adjiciendo, solum, tot circulos multiplicando a dextra, quot habet multiplicator.

Ut sint 865091372 decuplicanda, fientque 8650913720; aut centuplicanda, fientque 86509137200; aut per 10000000 multiplicanda, et provenient 8650913720000000.

Contra, partitio 10, 100, 1000, aut alia ex unitate et circulis quotvis conflata, facile fit abscindendo tot, ex dextimis figuras partiendi, quot circulos habet partitor: abscissae tamen figurae supra lineam, et partitor infra lineam, sunt constituendae, et quoto adjiciendae ad quoti fragmenta notandum.

Ut sint 865091372 partienda per 100, fient $8650913\frac{72}{100}$ pro quoto. Item sint 8650913720000089 partienda per 10000000, fit quotus $865091372\frac{89}{10000000}$

Hinc sequitur, multiplicationes et partitiones per quoscunque numeros auctos circulis facile fieri, spretis primo in multiplicatione circulis, donec multiplicandi et multiplicantis reliquae figurae invicem ducantur; inde multiplo tot circulos restituendo; atque pro partitione abscindendo et circulos a partitore et tot figuras a partiendo; atque reliquum partiendi per reliquum partitoris dividendo, et reliquias cum abscissis, posita inter eas linea, adjiciendo.



Ut sint multiplicanda 65294 per 2300, fient primo 1501762 (ex ductu 65294 in 28), deinde adjiciendo 00 fient 150176200, pro toto multiplo. Item sint partienda 65294 per 2300, constituantur ut a margine, ita ut pro 00 abscindantur gnomone 94, hoc modo |94; deinde partiantur 652 per 23, et fiet quotus, cum fragmentis residuis, $28\frac{894}{2300}$, seu (quod idem est) $28\frac{447}{1150}$.

Primo intuitui facilem se objicit bipartitio.

Ut sint bipartienda 65294; quis non primo intuitu conspicet 3 esse dimidium 6, et 2 quinarii, relicto 1, constituente cum sequente binario 12, quorum dimidium est 6; deinde sequuntur 9, quorum dimidium sunt 4, relicto 1 (denario scilicet); sequuntur ergo omnium ultimo 14, quorum dimidium est 7; ex quibus omnibus 3, 2, 6, 4, 7, fiunt 32647, dimidium numeri oblati, dicto citius repertum, ut a margine notatur.

66294
32647

Hinc sequitur facillima quintuplicatio, nimirum, decuplum bipartiendo.

Ut sint 865091372 quintuplicanda; ejus decuplum (per primam sectionem hujus) est 8650913720, eujus dimidium 432456860 est quintuplum oblati facillime inventum.
Et contra, facilis est quinquepartitio, duplum decupartiendo.

Ut sint 4325456860 quinquepartienda; primo duplentur, fientque 8650913720; deinde (ex sectione 2 hujus) partiantur per 10, fientque 865091372, quinta pars oblati quaesita.

Nec difficilis erit nonuplatio, numerum a suo decuplo auferendo; aut quadruplatio, numerum a suo quintuplo abstrahendo; aut sextuplatio, numerum ad suum quintuplum adjiciendo; aut ipsa duplatio, numerum ad sibi aequalem addendo. Siquidem hae multiplicationes, facillima additione, aut substractione, perficiuntur.

Simili	092105	Simplum
	184210	Duplum
	276315	Triplum
	368420	4^m
	460525	5^m
	552630	6^m
	644735	7^m
	736840	8^m
	828945	9^m
	921050	Decuplum, examinis gratia

Atque ex his rursus, simili facilitate, habetur triplicatio, septuplicatio, et octuplicatio, atque ita per omnes novem figuras multiplicatio.

Exemplorum loco, sint praecedentia et sequentia.

Ubi autem multiplicator omnes novem figuras, aut earum potissimam partem, complexus fuerit, vel partitoris quotus prolixus fore videtur, praestat singularium multipla, sive per praemissa compendia, sive per additionem continuatam, sive, omnium facillime, per ossa Rhabdologiae nostrae, impromptu habere, atque, ex eis sic acquisitis, integram multiplicationem aut partitionem contexere.

```
865091 372
736840|276315
552630| 644735
 460525| 184210
   8289  45
    092  105
7 9 6 7 9 240818060
```

Ut, pro examine ultimi exempli cap. de multiplicatione, videamus quid 92105 multiplicata per 865091372 producant: Duc ergo 92105 per 2, 3, 4, 5, 6, 7, 8, 9, sive per praemissa compendia, sive per continuam additionem simpli ad simplum, ad duplum, triplum, quadruplum, et ad caetera sub decuplo multipla; et fient ordine multipla ut a margine. Ita tamen, ut singula multipla eodem figurarum numero constent, praeponendo deficientibus 0 sinistrorsum; deinde, pro multiplicationis compendio, haec singularia multipla eo ordine collocentur, quem multiplicans 865091372 indicat, nimirum, sub 8 incipiat octuplum e tabella extractum, sub 6 sextuplum, sub 5 quintuplum, et ita de caeteris ut subsequitur; quibus tandem omnibus additis, proveniunt 79679240818060, multiplum quaesitum, et illi consonum quod supra repertum est.

```
          5 6 3 1 5
        3 4 2 6 3
      1 2 6 3 8
    3 4 1 5 8
  4 6 8 9 4           0
 5 9 9 5 2     1 8 4 2 1
7 9 6 7 9  2 4  0 8 1 8 0 6 0
9 2 1 0 5) 8 6 5 0 9 1 3 7 2
7 3 6 8 4 0     6 4 4 7 3 5
  5 5 2 6 3 0     1 8 4 2 1 0
    4 6 0 5 2 5
      8 2 8 9 4 5
        0 9 2 1 0 5
          2 7 6 3 1 5
```

Similiter pro examine superiorum, tam multiplicationis quam partitionis, sint 79679240818060 partienda per 92105: Constituantur situ quo a margine (per cap. 4 hujus) ; deinde ex 796792, numerum tabulae proxime minorem, viz. 736840, aufer, et figuram quoti ei in tabula respondentem, 8 viz., pro quoto pone, et supererunt 599524; ex quibus aufer sextuplum partitoris, numerum scilicet tabulae illi proxime minorem, qui est 552630, et 6 adjiciantur quoto, relictis 468940; ex quibus aufer 460525, quintuplum partitoris, et fiet quotus jam 865, relictis 84158, ex quibus ne simplex partitor 092105 auferri vel semel possit. Ideo erit jam quotus 8650, et reliquiae jam 841581 supererunt; ex quibus aufer 828945, partitoris noncuplum tabulatum, et fiet jam quotus 86509, relictis 126368; ex quibus aufer 092105, unicum scilicet et simplum partitorem; et concreto jam quoto 865091, supersunt 342630; ex quibus 876315, triplum scilicet partitoris, aufer, et erit jam quotus 8650913, et supererunt 663156; ex quibus aufer 644735, et ex quoto jam 86509137 supersunt 184210; ex quibus aufer partitoris duplum tabulatum, et nihil supererit, producto toto et integro quoto 865091372 superioribus conveniente.

Hactenus multiplicatio et partitio simplex, cum suis compendiis : sequuntur multiplicatio et partitio radicalis.

CAPUT VI.

DE RADICALI MULTIPLICATIONE ET PARTITIONE INTEGRORUM.

MULTIPLICATIONIS praxis ex ipsa ejusdem definitione patet. Si enim unitatem in radicem multiplicaveris, fit ipsa radix; quam si secundo in radicem duxeris, fit duplicatum; quod si tertio duxeris duplicatum in radicem, fit triplicatum; quod, quidem, si quarto in radicem duxeris, oritur inde quadruplicatum; atque sic deinceps quintuplicatum, sextuplicatum, etc., secundum qualitatem indicis.

Ut sint 235 multiplicanda ad indicem 4, seu quadruplicanda :

Primo multiplica unitatem per 235 radicem, et fiunt 235; quae secundo duc in radicem 235, fientque 55225, oblatae radicis duplicatum; quod et per radicem tertii multiplicato, et fient12977875, oblatae triplicatum; quod adhuc per radicem 2 quarta vice multiplicato, et fient 3049800625, optatum oblatae radicis quadruplicatum.

Unde, radicem aliquoties ab unitate ducere, idem est quod tot aequales radici invicem ducere.

Ut superiore exemplo, radix 235 quatuor multiplicationibus simplicibus ab unitate ducitur, et fit quadruplicatum 3049800625 ; et si 235, 235, 285, et 235, invicem duxeris, idem emerget radicatum, seu quadruplicatum; et ita de aliis.

Radices quarum indices sunt compositi, facilius per componentes multiplicari et extrahi posse, quam per compositos indices, superius diximus.

Ut superiore exemplo, facilius est radicaliter duplicare 235, fientque 55225; atque haec rursus duplicare, seu in 55225 ducere, fientque 3049800625, superius quadruplicatum: nam quadruplicare idem est quod duplicatum duplicare. Item sint 10 radicaliter sextuplicanda; ea fient continuata ab unitate multiplicatione, – primo 10; secundo 100, duplicatum; tertio triplicatum, 1000; quarto quadruplicatum, 10000; quinto quintuplicatum, 100000; sexto, tandem, sextuplicatum, 1000000.

Verum, per dictum compendium, si radicaliter triplicaveris duplicatum 100, seu duplicaveris triplicatum 1000, proveniet inde sextuplicatum 1000000; quam antea aliquantum expeditius.

Idem enim est sextuplicare radicaliter quod triplicare duplicatum, aut duplicare triplicatum radicis. Bis enim tria, aut ter duo, idem sunt quod sex.

Est et alia radicalis duplicationis praxis, quam per continuatam multiplicationem ; nimirum, radicem multiplicandam in duas partes secando, et duplum multipli, quod fit ex ductu prioris in posteriorem, una cum duarum partium duplicatis, addita faciunt duplicatum quaesitum.

Ut si radix oblata 35 radicaliter duplicanda, secentur in partes, priorem 30, posteriorem 5; duc 30 in 5, fient 150, quorum duplum est 300; quibus adde duplicatum 30 et duplicatum 5, quae sunt 900, et 25, fientque 1225, duplicatum totius numeri 35.

Nec secus eveniet ex ductu 35 in 35. Item si prorogetur radix, sit 352; sint partes, prior 350, posterior 2; duc invicem, fient 700, quorum duplum est 1400; quibus adde duplicatum 2, quod est 4, et duplicatum 350 superius inventum, quod est 122500 ;

et ex his tribus summis, 122500, 1400, et 4, additis, fiunt l23904, duplicatum quaesitum radicis 3522 Experire multiplicando 352 in 352, et idem invenies. Eodem modo operandum foret in quartam usque figuram, foretque 3521. Ex anterioris partis 3520 duplicato jam habito l2390400, et posteriore parte 1, habebitur 12397441, duplicatum totius numeri, seu radicis, 3521.

Est etiam et alia radicalis triplicationis praxis; nimirum, radicem in duas partes secando, et triplum partis cujusque in alterius duplicatum ducendo, et singulas partes triplicando; ex his enim quatuor summis additis, producitur radicis oblatae triplicatum.

Ut sit radix oblata 35 radicalitcr triplicanda, secentur ut supra in partes 30 et 5; duc triplum 5, quod est 15, in duplicatum 30, quod est 900; et triplum 30, quod est go, in duplicatum 5, quod est 2; et fiunt illa 13500, haec vera 2250; deinde radicaliter triplicentur 30, et fiunt 27000; radicaliter etiam triplicentur 5, et fient 12; ex his ergo quatuor summis, 13500, 2250, 27000, et 125, additis, fiunt 42875, oblatae radicis triplicatum quaesitum. Experire per multiplicationem continuatam 35, 35, et 35, invicem, et in idem incides. Similiter si hic prorogetur radix, sitque 351; partiantur in anteriorem partem 350, et posteriorem 1, ex quibus, modo praefato, habebis has quatuor summas, 42875000, 367500, 1050, et 1; quibus additis, fient 43243551, triplicatum quaesitum radicis oblatae 351.

Sunt et quadruplicationis, quintuplicationis, sextuplicationis, et aliarum radicalium multiplicationum, particulares praxes ad inveniendum radicatum indicis cujusque. Sed quia tum modus per continuatam multiplicationem prima sectione descriptus generalis est, et satis facilis, tum ex converso regularum extractionum haberi possunt omnes hae particulares praxes, has ideo hoc loco praetermittimus.

Partitionis etiam praxis ex ipsa ejusdem definitione patet. Si enim radicatum per radicem in unitatem usque partitus fueris, numerus partitionum est radicalis quotus, seu index quaesitus.

Ut sint 55225 radicatum dividendum per radicem 285, emerget primus quotus 285, quibus per 285 divisis, emerget 1, secundus et ultimus quotus. Est ergo binarius hujus partitionis index, et radix est bipartiens radicati. Item si radicatum 12977875 per radicem 235 partiendum sit, emergit, prima partitione, 55225, secunda, 285, tertia, 1. Unde partitio haec tripartitio est, et index est ternarius, et radix tripartiens dicitur. Numerus indicis, seu qualitas radicis, habetur, tam descendendo a radicato ad unitatem, per partitionem, quam ascendendo ab unitate ad radicatum, per multiplicationem: In utroque enim, numerus operationum est index, et radicis qualitas.

Ut superiore exemplo, sicut radicatum 12977875, partitum per 285 radicem, tertia partitione exhibet unitatem; ita, ab unitate continuata multiplicatio 285 radicis, tertia multiplicatione in idem radicatum incidet; unde utroque modo arguitur ternarius esse index.

Hinc fit, quod partitionis radicalis rarus sit usus in computationibus, cum suo munere etiam fungatur multiplicatio. In his autem, multiplicatio radicalis, quae ab unitate in radicatum, aut partitio, quae a radicato in unitatem praecise non incidit, inutiles sunt. Arguunt enim oblatam radicem non esse perfectam radicem radicati.

CAPUT VII.

DE INVENIENDIS REGULIS EXTRACTIONUM RADICALIUM.

UNIUSCUIUSQuae radicis propria est, et particularis, extrahendi regula.

Regula quaeque extractionis consistit in resolutione radicati in sua supplementa. Supplementum est, differentia duorum radicatorum ejusdem speciei.

Ut sint radicata ejusdem speciei 100 et 144; videlicet, illud

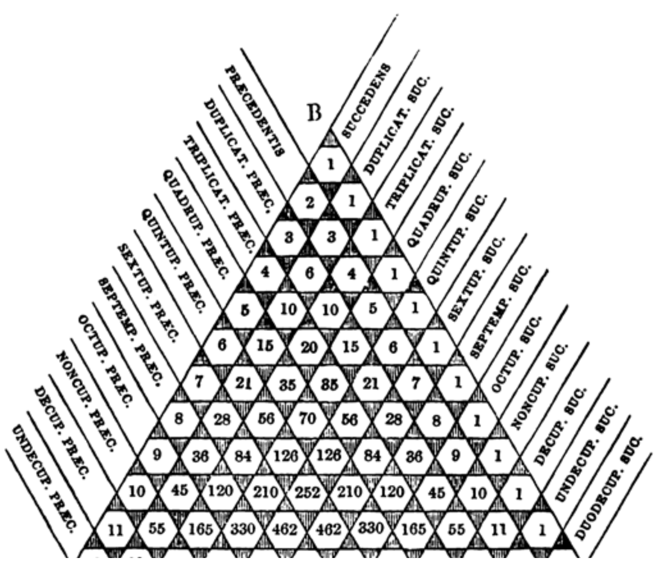
duplicatum 10, et hoc duplicatum 12; differentia duplicatorum 100 et 144 est 44, quae

sunt verum supplementum praefatorum radicatorum.

Supplementa ergo variantur, prout radicatorum et radicum species variae fuerint. Alia enim est regula inveniendi supplementa duplicationis, et extractionis radicis bipartientis, alia triplicationis, et extractionis radicis tripartientis, alia quadruplicationis, et extractionis quadrupartientis radicis; et ita reliquarum omnium.

Regulas, autem, inveniendi omnium radicatorum et radicum supplementa, docet Tabella nostra triangularis, areolis hexagonis referta, dextrorsum inscriptis unitate sola, sinistrorsum vera numeris ab unitate unitatis incremento crescentibus, et a vertice descendentibus; et quae introrsum habet singularum areolarum numerum, quemque aequalem duobus numeris proxime superpositis.

TABULA SUPPLEMENTORUM.



Sit triangulum *ABC*, angulos, *A* sinistrum, *B* verticalem, et *C* dextrum, habens. Quot autem radicum species desideras tabellam comprehendere, per bis tot partes, plus una, dividatur quodque latus; v.g., ad duodecim extractionum species continendum, dividatur quodque latus in 25 aequales partes; et, incipiendo a basi *AC*, per singula alterna puncta laterum ducantur, intra triangulum, linea duodecim basi parallelae; simili modo incipies a latere *AB*, et ei duodecim parallelas, a singulis alternis punctis basis, per singula alterna puncta lateris *BC*, tam intra triangulum quam ultra lineam *BC*, digiti spatio extendes; eodem prorsus modo, lateri *BC*, duodecim parallelas a singulis alternis punctis basis, et per singula alterna puncta lateris *BA*, ultra triangulum digiti spatio extendes. Hinc habes triangulum areolis hexagonis refertum, quarum 12 dextimae, et linea *BC* proximm, 12 unitatibus sigillatim inscribuntur; sinistimae vero numeris 1, 2, 3, 4, 5, etc., usque ad 18, ordine, a vertice *B* ad angulum sinistrum *A* descendentibus, inscribuntur; deinde hexagonum quodque interius vacuum inscribatur aggregato duorum numerorum ei proxime superpositorum; ut sub 2 et 1 scribantur 3; sub 3 et 3, 6; sub 3 et 1, 4; et ita in calcem usque tabulae. Tandem, ascribantur tituli sinistrorsum, 'praecedentis,' supra secundum hexagonum 2; supra tertium 3 scribe, 'duplicatum praecedentis;' supra quartum scribe, 'triplicatum praecedentis;' et sic de caeteris radicatis praecedentis usque ad duodecuplicatum: dextrorsum vero scribe supra primum hexagonum 'succedens;' supra secundum, 'duplicatum succedentis ;' supra tertium, 'triplicatum succedentis;' et sic de reliquis radicatis succedentis, usque ad tredecuplicatum, prout in tabulae ipsius diagrammate subscripto habes.

Cuique supplemento respondent dum partes radicis: alteri constante una vel pluribus figuras sinistimis jam inventis, quae praecedens dicitur ; alteri constante unica dextra figura. jam proxime invenienda, quae succedens nuncupatur.

Atque hae radicis partes, et supplementum, mutuo se componunt et condunt ad invicem; ut postea patebit.

Ut duplicati 144 radix bipartiens est 12, quorum duae partes, 1 et 2, respondent supplemento 44, et anterior 1 dicitur 'praecedens,' posterior vero 2, 'succedens;' at si duplicatum 15129 sumpseris, cujus radix bipartiens est 123, anterior pars jam nuper inventa, scilicet 12, dicitur 'praecedens,' et unica figura 3, 'succedens.'

Ex hac itaque tabella, cujusvis supplementi inveniendi regula sic colligitur et legitur. Primo, quaeratur juxta crus sinistrum numerus indicis propositae radicis; hic enim, una

cum reliquis arealibus numeris directe in eadem linea. ordine sequentibus, quaesitum supplementum legendum offert; modo una cum eorum singulo legantur tituli ei oblique, tam sinistrorsum quam dextrorsum, superpositi invicem multiplicandi.

Exempli gratia:–Quaeritur, e tabula, regula inveniendi supplementum duplicationis et extractionis radicis bipartientis. Haec in secunda linea duobus numeris, 2 et 1, et suis titulis invicem multiplicandis, exprimitur hoc modo, – praecedentis 2 succedens, et 1 duplicatum succedentis, quae ita legenda et pronuntianda sunt: supplementum duplicationis constat praecedentis duplo multiplicato per succedens, et unico seu ipso duplicato succedentis; ut exemplis sequentibus patebit. Sic pro supplemento triplicationis, in tertia linea reperies tres numeros, 3, 3, 1, qui, cum suis titulis, ita leguntur: supplementum triplicationis tribus constat numeris; primus est, duplicati praecedentis triplum multiplicatum per succedens; secundus est, praecedentis triplum multiplicatum per duplicatum succedentis; tertius est, ipsum triplicatum succedentis.

Similiter, pro supplemento quintuplicationis, vel extractionis radicis quinquepartientis: In quinta linea reperies quinque numeros hos, 5, 10, 10, 5, 1, qui, cum suis titulis, indicant supplementum quintuplicationis, aut quintupartientis radicis, quinque constare partibus; quarum prima est quadruplicati praecedentis quintuplum ductum per succedens; secunda est, triplicati praecedentis decuplum ductum per duplicatum succedentis; tertia est, duplicati praecedentis decuplum ductum per triplicatum succedentis; quarta est, praecedentis quintuplum ductum in quadruplicatum succedentis; quinta est, ipsum quintuplicatum succedentis: Atque ita quadruplicationis, sextuplicationis, et reliquarum supplementa ex hac tabella invenire poteris.

Ex sinistimarum figurarum radice quam maxima. praecedente, et singulorum supplementorum figuras succedentibus in unum collectis, emerget radix plene extracta.

Supplementa autem omnium radicum hactenus descripta habes; superest ergo jam de earundem extractione disserere.

CAPUT VIII.

DE RADICUM EXTRACTIONE.

OMNIS radicati radix aut est unius figurae, aut plurium.

Radices omnes unicae figurae, infra tredecupartientem, exhibet tabella subsequens.

Ut si radicem quintupartientem 16809 quaeras: In linea quintuplicatorum quaerendus est oblatus numerus, aut ei saltem proxime minor, 16807; et loco eis directe supremo reperies figuram quaesitam, nimirum, radicem quintupartientem oblati, vel oblato proximi, quae est 7, et supersunt 2, extractione irresolubiles.

TABULA RADICATORUM ET RADICUM.

Duplicatum	4	9	16	25	36	49	64	81
Triplicatum	8	27	64	125	216	343	512	729
Quadruplicatum	16	81	256	625	1296	2401	4096	6561
Quintuplicatun	32	243	1024	3125	7776	16807	32768	59049
Sextuplicatum	64	729	4096	15625	46656	117649	262144	531441
Septemplicatum	128	2187	16364	78125	279936	823643	2097162	4782969
Octuplicatum	256	6561	65536	390625	1679616	5764801	16777216	43046721
Noncuplicatum	512	19683	262144	1953125	10077696	40353607	134217728	387420489
Decuplicatum	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401
Undecuplicatum	2048	1771471	4194304	48828125	362797056	1977326743	8589934592	31381059609
Duodecuplicatum	4096	531141	16777216	244140625	2176782336	13841287201	68719476736	282429536481

Construitur haec tabula, ex continuatis multiplicationibus, per singulas novem figuras, seriebus ab eisdem figuras dcscendentibus. Ut in fronte tabulae, habes figuras, octo multitudinis, 2, 3, 4, 5, &c. usque ad 10; et sub quaque figura, suum duplicatum, triplicatum, quadruplicatum, &c. in tredecuplicatum usque: ut, sub 2 habes 4, 8, 16, 32, 64, 128, 256, 512, 1004, 2048, 4096, cum suis inscriptionibus a margine sinistrorsum; nimirum, duplicatum, triplicatum, quadruplicatum, &c. et duodecuplicatum ultimo; sub 3, autem, habes sua radicata 9, 27, 81, 243, &c. continuata ratione descendentia; sub 4, item, habes 16, 64, 256, &c.: et ita ad novenarium usque; sub quo habes 81, 729, 6561, &c.; ut in tabula ipsa perspici possunt.

In extractione radicis constantis pluribus figuras, spectandi sunt situs, et operatio.

55225
1
5 5225
2
4

20
2
40
3
120. Prior pars prioris suppleminti. 9. Posterior pars prioris supplementi.
129. Totum primus supplementum.

1 23
5 5 2 25
2 3
4
1 2 9

230
2
460 5
5 5
2300. Prior pars posterioris suppleminti. 25. Posterior pars posterioris supplementi.
2325. Totum posterius supplementum.

0
1 2 3
5 5 2 2 5
2 3 5

Situs est, ut sub radicato oblato duae ducantur linea parallelae, quarum intervallum sit radicis quaesitae capax, inter quas, sub figura radicis dextima, signetur punctum; abhinc autem sinistrorsum, sub secunda quaque figura. pro radice bipartiente, et sub tertia quaque pro tripartiente, et sub quinta quaque pro quintupartiente, &c., notentur puncta.

In haec puncta, inter lineas posita, cadunt figurae singulae radicis quaesitae.

Operationis autem duo sunt praecepta.

Primum, ut a figuras, in sinistrum punctum terminantibus, auferas radicatum propositae speciei, in tabula secunda inventae, quam maximum auferae poteris, notatis supra reliquiis, et loco hujus–sinistimi puncti ponatur assumpti radicati figura tabularis, pro prima radicis figura.

Secundum, ut, facta conjectura ex primis supplementi numeris, succedat in sequenti puncto, seu periodo ejusmodi, recens et nova figura, cujus supplementum propositae speciei, quam maximum sit, non tamen excedat figuras illi superiores in ipsa hac periodo terminantes; ex quibus inde supplementum idem auferatur, reliquiis supra notatis.

Atque haec secunda operatio toties, seu tot vicibus, repetatur, quot supersunt periodi vacum versus dextram; et figurae in periodos incidentes sunt radix quaesita.

Ut, exempli gratia, sit extrahenda radix bipartiens e 55225 :

Constituantur situ quo a margine; deinde primo, a 5, sinistimae periodi, aufer maximum duplicatum tabulae quod auferri possit, scilicet 4, et restat 1, deletis 5 supra et 4 infra; quaternarii radix bipartiens tabularis, quae est 2, ponatur primae periodi loco; 2 ergo dicuntur praecedens figura; cui succedens est quaerenda, et in secunda periodo constituenda, cujus supplementum non excedat suprascriptas figuras 152, facta

conjectura ex prima supplementi parte, scilicet ex 2 jam inventis (quae 20 sunt in hoc loco) duplicatis et multiplicatis per inveniendam, id est, 40 multiplicatis per succedentem. Si autem quaternarius foret succedens, prior supplementi pars foret 160, quae excedunt 152; rejecto ergo quaternario, eligimus ternarium in hac periodo collocandum; cujus supplementum duplicationis, ex priore tabula, est 129 (scilicet ter 40, et ter 3); quo supplemento ex 152 oblato, supersunt hoc loco reliquiae 23, seu ad dextram usque 2325, et pars praecedens radicis jam est 23, vel 230; iterum ergo repetatur secunda operatio, ad inveniendam novam succedentem figuram, dextimae periodi loco ponendam; hujus novae succedentis supplementum, quodcunque sit futurum, constat (per primam tabulam) praecedentium 280 duplo, scilicet 460, multiplicato in succedentem, et succedentis duplicato; at si 6 multiplicaveris per 460, excedunt 232; rejectis ergo 6, computetur supplementum quinarii; eritque, ex tabulis,
232, (scilicet quinquies 460, et quinquies 5) quibus ablatis ex totidem suprapositis, nihil superest; et radix bipartiens 23.5, jam plena et perfecta, emersit oblati numeri 55225.

Aliud exemplum extractionis bipartientis radicis.

1 6 4860
4 0
1 6

400
2
800
6
4800. Prior pars huius supplementi.

6
6
36. Posterior pars hujus supplementi.
4836. Totum supplementum.

0 24
1 6 4 8 6 0
4 0 6
1 6
4 8 6 0

Sit extrahenda radix bipartiens e 164860, collocatis ut a margine: Figurarum primae periodi, scilicet 16, radix bipartiens est 4, in illi periodo locanda; horum 4 duplicatum, 16, est ex superioribus 16 auferendum, et supersunt 48 in secundam periodum usque, et 40 sunt figurae praecedentes; jam si vel unitem pro succedente elegeris, erit ejus supplementum 81, quae non possunt ex 48 auferri; ergo figura succedens et nova, in secunda periodo ponenda, erit 0, cujus supplementum est 0; quo ex 48 ablato, supersunt in ultimam periodum 4860: et jam 40 sunt figurae praecedentes inventae; et quaeritur nova succedens, ultimae periodi loco statuenda; ea erit necessaria 6, cujus supplementum (scilicet sexies duplum 400, et sexies 6) est 4836; quibus ex 4860 sublatis, restant 24, reliquiae irresolubiles; et radix bipartiens quaesita, imperfecta tamen, est 406, oblati radicati 164860.

Exemplum extractionis tripartientis radicis.

20
20
400
3
1200
3
3600. Prima pars prioris supplementi.

20 3
3 3
60 9
9
540. Secunda pars prioris supplementi.

4
1 2 977875
2
8

Sit extrahenda radix tripartiens e 12977875: punctis et situ collocentur ut a margine; inde, maximum triplicatum tabulae non excedens 12, scilicet 8, ex 12 aufer, et supersunt 4, superscribenda, deletis 12; octonarii autem radix tripartiens, scilicet 2, primae periodi loco statuatur, et 20 sunt jam praecedens; cuius

3
3
9
3
27. Tertia pars prioris supplementi.

3600
540
27
4167. Totam prius supplementum.

4 810
12977875
2 3
8
4167

succedens, superscripta, dividerentur per triplum duplicati praecedentis, scilicet per cum suo supplemento, sic quaeratur: supplementum tripartientis radicis (ex prima tabula) constat tribus partibus, quarum prima fit ex triplo duplicati praecedentis multiplicato per succedentem; unde sequitur, si 4977, suprascripta, dividerentur per triplum duplicati praecedentis, scilicet per triplum 400, quod est 1200, de succedante dabit quotus conjecturam; succedens enim plerumque est aut huic quoto aequalis, aut unitate eo minor; si itaque 4977 per 1200 partirentur, 4 foret quotus et succedens; at quaternarii supplementum est 5824, quae excedunt 4977; rejecto ergo 4, eligamus 3, quorum supplementum est 4167 (nam 1200 ducta per 3, et ter 3 ducta per ter 20, at triplicatum 3, aggregata faciunt 4167); his ex 4977 ablatis, supersunt hic 810, et in periodum tertiam 810875, et 230 jam sunt figurae praecedentis. Quaerenda restat succedens in dextimum punctum locanda: ea, simili modo quo praemissa, invenietur; estque 5, cujus supplementum est 810875 (nam triplum duplicati 230 ductum per 5, scilicet 793500, et radix tripartiens triplum 230 ductum in 25, videlicet 17250, et triplicatum 5, quod est 125, fiunt 810875); his ergo ex superpositis eis aequalibus ablatis, nihil superest; et radix tripartiens quaesita, 235, perfecta et integra, oblati triplicati 12977875, producitur. Atque ita in omnibus extractionibus radicum, ubi indices sunt primi et incompositi, ut quintupartientis, et sextupartientis, &c., operaberis.

230	
230	
69	230
46	3
52900	690
3	25
158700	345
5	138
793500. Prima pars posterioris suppleminti.	17250. Secunda pars posterioris supplementi.

5	
5	793500
25	17250
5	125
125. Tertia pars posterioris supplementi.	810875. Totum posterius supplementum.

4810
12977875
2 3 5
8
4167
810875

Exemplum extractionis radicum ubi indices sunt compositi.

Sit extrahenda radix quadrupartiens e 3049800625; quod hoc expeditius fiat, bis extrahendo radicem bipartientem, quam semel extrahendo radicem quadrupartientem, console cap. 4 Lib. I. Eorum ergo extrahatur radix bipartiens, situ et operatione debitis, ut a margine, et producentur 55225, radix bipartiens oblatorum; rursus hujus radicis bipartientis extrahe jam secundo radicem bipartientem, ut, etiam a margine vides, et reperies huius bipartientis bipartientem radicem esse 235, quae est radix quadrupartiens optata, radicati 3049800625 oblati. Atque ita in extactione radicum sextupartientium, octupartientium, nonupartientium, et aliarum quarumcunque compositarum operaberis.

```
      5 5 2 2
    2 7 6       0
  5 2 4
3 0 4 9 8 0 0 6 2 5
-------------------
  5   5   2   2   5
-------------------
2 5     5 5 2 2 2 5
  5 2 5
   2 2 0 4
     2 2 0 8 4
```

```
    0
   2 3
 1
 5 5 2 2 5
 ---------
 2  3  5
 ---------
 4
 1 2 9
   2 3 2 5
```

```
50
 2
100        5
5          5
500        25. Supplementum primum.

550
  2
1100       2
 2         2
2200       4. Supplementum secundum.

5520
  2
11040      2
   2       2
22080      4. Supplementum tertium.

55220
    2
110440     5
     5     5
552200     25. Supplementum quartium.
```

```
20
2
40     3
 3     3
120    9. Supplementum primum.

230
 2
460    5
 5     5
2300   25.  Supplementum secundum.
```

CAPUT IX.

DE RATIONE EMENDANDI EXTRACTIONES IMPERFECTAS.

Quae post extractionem supersunt reliquiae, quantumcunque irresolubiles quoad totum, tamen ex parte seu partim reformari possunt.

Hae autem superstites reliquiae, nisi ex parte seu partim reformentur, errorem plerumque sensibilem pariunt.

Sensibilis radicis et suarum reliquiarum error duobis modis praecipuis emendatur.

Prior est, post radicem imperfectam ducendo lineam, supra quam scribantur reliquiae, infra vero supplementum unitatis, integer pro minore termino, et unitate minutus pro majore; inter hos enim terminos latet vera quantitas radicis, quae numero nulli definiri possit.

Ut sit extrahenda radix bipartiens e 164860, in secundo praecedentium exemplo reperies eorundem radicem bipartientem 406 imperfectam, et reliquias 24 superstites. Constitue ergo lineam post 406, et supra lineam scribantur 24, ut a margine ; inde ex inventa jam praecedente 406, quaere supplementum unitatis succedentis respondens duplicationi; hoc integrum erit 818, et unitate minutum erit 812, quorum illud pro minore, hoc pro majore termino bis sub linea scribatur. Erit ergo quaesitae radicis bipartientis minor terminus $406\frac{24}{813}$, et major $406\frac{24}{812}$, inter quos terminos continetur et latet praecisa radix bipartiens numeri 164860 oblatio. Ita ut absque sensibili errore (praecipue in mechanicis) radix bipartiens 164860 possit dici $406\frac{24}{813}$, vel $406\frac{24}{812}$.

$406\frac{24}{813}$

$406\frac{24}{812}$

406	
2	
812	1
1	1
812	1. Unitatis supplementum.

Radicis tripartientis emendandi exemplum.

Sit extrahenda radix tripartiens numeri 998, hinc (ex praecedente tabula) deprehenditur proxime esse 9, et supersunt adhuc 269 reliquiae emendandae. Illa 9 ante lineam, haec 269 supra eam, ut a margine statuantur. Ex figura autem 9, pro praecedente inventa, unitatis supplementum triplicationis integer reperietur 271, et unitate minutus 270, ut a margine conspiceae licet, quae sub lineis scribantur. Et ita radix tripartiens numeri 998 oblati praecise inter terminos $9\frac{269}{271}$ minorem, et $9\frac{269}{270}$ majorem, constituitur; ita ut pro ea, hinc vel illa absque sensibili errore capi possit.

$9\frac{269}{271}$

$9\frac{269}{270}$

9	9	1	
9	3	1	
81	27	1. Supplementi tertia pars.	
3	1		
243	27.	27. Secunda pars.	
1			
243		243. Prima pars.	
		271. Totum supplementum unitas.	

Posterior modus est, ut totum radicatum propositae speciei oblatum per cujusvis electi numeri radicatum ejusdem speciei multiplicetur; producti autem radix ejusdem speciei (spretis reliquiis) per numerum illum electum partiatur. Nam quotus, cum suis fragmentis linea distinctis, erit minor terminus; et si ad eorundem fragmentorum numaratorem addideris unitatem, erit major terminus, inter quos continetur vera radix.

Ut sit extrahenda radix bipartiens emendata admodum, e numero 50. Electus numerus sit 1000, ejus duplicatum erit 1000000, per quae multiplica 50, fient 50000000 producta; quorum radicem ejusdem speciei, bipartientem nempe, quaere, situ et operatione quibus a margine ; et erit haec radix bipartiens 7071 (neglectis reliquiis 959), per electum 1000 partire, et fiet inde quotus, cum fragmentis, $7\frac{71}{1000}$, pro minore termino, et $7\frac{72}{1000}$ pro majore termino. Ita ut alterutrum, pro ipsa radice bipartiente numeri 50 oblati, capi possit, absque errore perceptibili.

959
1 5 1
1
5 0 0 0 0 0 0 0
7 0 7 1
1 9
9 8 4 9
1 4 1 4 1

700	
2	
1400	7
7	7
9800	49. Supplementum primum.
7070	
2	
14140	1
1	1
14140	1. Supplementum secundum.

Exemplum praecedentis tripartientis radicis hoc modo emendatae.

Sit extrahenda tripartiens radix emendata, e numero 998.

Sit alius numerus electus 100; ejus triplicatum erit 1000000 ; per haec multiplica 998, provenient 998000000, quarum extrahatur tripartiens radix, ut a margine, et ea producetur quam proxime 999, quae (spretis jam reliquiis 997001) per electum numerum 100 partiantur; proveniet inde $9\frac{99}{100}$, pro minore termino, et (numeratore unitate aucto) $9\frac{100}{100}$, quae sunt 10, pro termino majore. Itaque, absque perceptibili errore, $9\frac{99}{100}$, vel 10, dici possunt radix tripartiens numeri 998 oblati; inter hos enim terminos arctissime includitur.

9 9 7 0 0 1
2 7 7 0 1
2 6 9
9 9 8 0 0 0 0 0 0
9 9 9
7 2 9
2 4 1 2 9 9
2 6 7 0 3 9 9 9

	90	
90	3	
90	270	81
8100	81	9
3	270	729
24300	216	
9	21870	

218700. Supplem. primi prima pars.
21870. Secunda pars.
729. Tertia pars.
241299. Totum suppl. prius.

990		
990	990	
89100	3	
891	2970	81
980100	81	9
3	2970	729
2940300	2376	
9	240570	

26462700. Posterioris supp. prima pars.
240570. Secunda pars.
729. Tertia pars.
26703999. Totum posterius suppl.

Hi modi, quia radices imperfectas non perficiunt, sed nimis imperfectas reddunt, Mechanicis magis quam Mathematicis placent. Ut cap. 4 Lib. I. diximus.

Geometrae ergo his radicatis numeris, radices non habentibus, praeponunt signum radicis debitae. Unde ex his ita signatis oritur prima species geometricorum numerorum, quam uninomia vocant.

Ut superiorum duplicatorum 164860, et 50, non extrahunt radices bipartientes (quia nullas praecise habent in numeris), nec extractas emendant, sed signum radicis extrahendae, quam quadratam vocant, numero praeponunt, hoc modo, $\sqrt{Q}164860$, et, $\sqrt{Q}50$; vel sic, $\sqrt{q}164860$, et $\sqrt{q}$ 50; quae sic pronuntiant, radix quadrata numeri 164860,–et, radix quadrata numeri 50. Nos autem ita ⊔164860 notamus, et ⊔50; et sic pronuntiamus, radix bipartiens 164860,–et, radix bipartiens ⊔50. Ita, radicem tripartientem numeri 998 nec extrahunt (quod non sit in numeris) nec emendant, sed ita signant, 2998, et ita pronuntiant, radix cubica 998. Nos autem ita notamus, $\sqrt{C}$ 998; proferimus autem sic, radix tripartiens └998, ut amplius suo loco dicemus. Utcunque, hinc uninomia seu medialia dicuntur, et sunt Geometriae Logisticae fundamentum: sequenti ergo Libro tractabuntur; hinc autem ea oriri monuisse satis est.

Hucusque computationes simplices numerorum integrorum; sequuntur compositae, seu regulae.

CAPUT X.

DE REGULIS PROPORTIONIS INTEGRORUM.

PROPORTIONIS integrorum numerorum regulae, sub generali quantitatum methodo, et praeceptis initio fuse satis explicantur.

Prout in tribus quaestionibus exemplaribus, simplicis, duplicis, et triplicis regulae proportionis, cap.5 Lib. I. propositis, et unica generali methodo solutis, cernere poteris.

Particularia tamen numerorum exempla specialia habent compendia, quibus expediantur: Si enim in majusculis numeris, partitor aliquot habeat versus dextram circulos, poteris, compendii gratia, tot fere multipli loca dextrorsum a figuras vacua, aut circulis referta relinquere, multiplicationem a sinistris ordiendo.

Ut (exemplo e sinibus) si 10000000 dederint 9925461, quantum dabunt 7986354 ? At quia hinc omnes fere figuras complectitur, ideo, per cap. 5 Lib. II. compendiose multiplicentur 9925461 per singulas novem figuras, ut in tabula a margine factum est; deinde sub singua figura multiplicantis 7986854 incipiat, et inde procedat numerus tabulae figurae illi respondens, neglectis tamen et omissis sex dextimorum locorum figuras omnibus, propterea quod septem circuli partitoris 10000000 eas abscinderent, si exprimerentur et non omissae fuissent:

09925461	1
19850922	2
29776383	3
39701844	4
49627305	5
59552766	6
69478227	7
79403688	8
89329149	9
99254610	decuplet.

7986354
69478227
8932914 ·
794036 · ·
59552 · · · ·
2977 · · · · ·
496 · · · · · ·
39 · · · · · · · · ·
7926824 | 1 · · · · · · · · ·
10000000

Haec igitur particularia multipla, praeter sua sex loca dextima vacua, in unum addita fiunt 79268241 a quibus, tam locis quam figuras dextimis, si abscindantur septem circulorum partitoris otiosa loca (ut partitionis compendium exigit), restabunt 7926824, responsum quaesitum. Ubi ergo 10000000 dant 9925461, sequetur, quod 7986854 dabunt 7926824.

Quae autem, in hujus compendii multiplis, omittuntur figurae versus dextram, etiamsi omnes novenariae essent, non vel unica unitate augerent responsum. Merito igitur eae omnes negligi possunt in his majusculis numeris in quibus ne unitatis quidem, totius et integri, error est sensibilia.

.
. .
. . .
. . . .
. . . . .
. . . . . .
5888889

Esto enim quod vacua illa puncta, a margine posita, novenariae essent (quod supra possibilitatem est), nihilominus ad 5888889 tantum accrescerent, quae ad 79268241...... addita facerent 79268246888889, quibus per 10000000 partitis proveniunt ad summum $7926824\frac{6888889}{10000000}$, quae non extenduntur ad 7926825, nec superius productum unitate exuperant. In maximis itaque numeris, maxima laude dignum est hoc compendium regulae trium.

Est etiam aliud compendium hujus regulae, absque omissione figurarum, omnes oblatos quaestionis numeros debitis locis supra vel infra lineam constituendo, ut in generali methodo, cap. 5 Lib. I. proposita praecepimus; deinde singuli bini numeri, quorum alter est superior et quasi numerator, alter inferior quasi denominator, partiantur per suum maximum communem divisorem, donec singuli numeratores ad singulos denominatores fuerint in ratione prima seu minima ad invicem, notatis omnium ultimis quotis ; tandem, multiplum quotorum superiorum omnium partiatur per multiplum quotorum inferiorum, et hic quotus erit responsum quaesitum, quaestioni satisfaciens.

Ut 4 aedificantes construxerunt murum 6 pedes altum, 48 ulnas longum, diebus 42; quaeritur, quot diebus 5 aedificantes aedificabunt murum 9 pedes altum, 50 ulnas longum? Per cap. 5 Lib. I. disponantur omnes numeri debito situ, et stabunt ut a margine;

Aedific.	Pedes	Ulnae.	Dies.
4	9	50	42
5	6	48	Quot dies?

deinde numerum superiorem 4, et inferiorem 6, abbrevia per 2, eorum maximum divisorem, et fient $\frac{2}{3}$ hoc situ, $\frac{2\cdot 9\cdot 50\cdot 42}{5\cdot 3\cdot 48}$; deinde partire 2 superiorem et 48 inferiorem, per communem partitorem 2, et fient 1, et 24, hoc situ, $\frac{1\cdot 9\cdot 50\cdot 42}{5\cdot 3\cdot 24}$ deinde partiantur superiora 9 et inferiora 3, per 3, et fient 3 supra, et 1 infra, hoc situ, $\frac{1\cdot 3\cdot 50\cdot 42}{5\cdot 1\cdot 24}$; deinde partiantur 50, et 5, per 5, et fient superius 10, inferius 1, hoc situ, $\frac{1\cdot 3\cdot 10\cdot 42}{1\cdot 1\cdot 24}$; deinde partiantur 10 superiora, et 24 inferiora, per suum maximum communem divisorem 2, et fient 5 supra, et 12 infra, hoc situ, $\frac{1\cdot 3\cdot 5\cdot 42}{1\cdot 1\cdot 12}$; tandem, partiantur superiora 42, et inferiora 12, per eorundem maximum divisorem 6, et fient 7 superius, et 2 inferius, hoc situ, $\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 1\cdot 2}$. Ecce jam habes familiares et tractabiles numeros 1, 3, 5, 7, atque 1, 1, 2, invicem multiplicandos, pro majusculis numeris oblatis. Ducantur ergo invicem 1, 3, 5, 7, et fient 105; ducantur item inferiora 1, 1, 2, invicem, et fient 2; per quae partiantur 105, et proveniet quotus $52\frac{1}{2}$, numerus dierum quaestioni satisfaciens, absque magnis aut laboriosis multiplicationibus et divisionis operibus.

Haec proportionum regula plurimas habet tacitas et latentes species, non negligendas, quae beneficio etiam hujus compendii fruuntur, et eo expediuntur.

Ut regula reducendi fractiones fractionum, iterum atque iterum fractarum (de qua generaliter superius, et particulariter inferius tractabitur), est species hujus regulae proportionis, ex definitione regulae proportionis. Unde eodem compendio illa, quo hinc, abbreviatur, ut in exemplis cap. 8 Lib. I. ostensum est, et inferius ostendetur.

De integrorum numerorum Arithmetica satis; superest fractorum, seu fractionum.

CAPUT XI.

DE FRACTIONIBUS, SEU NUMERIS FRACTIS.

FRACTIO, seu numerus fractus, est quem minima et individua pars, unica scilicet unitatis numerata, metitur.

Ut apud astronomos gradus pro unitate habetur, et 7 scrupula prima sunt ejus fractio; nam hoc loco unicum scrupulum primum est ejus minima et individua pars, quae eam metitur, et ab ea numeratur per 7. Sic 5 solidi sunt librae fractio, quia 1 solidus hic est minima et individua unius librae pars, quae fractionem metitur, et ab ea numeratur per 5. Sic 17 trigesimae unitatis partes sunt fractio; quia unica trigesima est hoc loco minima et individua unitatis pars, quae 17 trigesimas metitur, et ab eis numeratur septemdecies.

Fractionum, aliae vulgares, aliae physicae dicuntur.
Vulgares sunt, quae varias et liberas habent denominationes.

Ut unum dimidium, duae tertiae, quatuor undecimae, quae ortum habent in partitione, cap. 4 Lib. II.

In Arithmetica fractionum vulgarium spectantur nominatio et operatio.
Nominatio, quae voce fit, dicitur pronuntiatio; quae autem scriptis, notatio.
Vulgarium fractionum nominatio duobus terminis exprimitur, numeratore et denominatore.
Denominator est, qui nominat in quot partes aequales unitas sit distribuenda.
Numerator vero est, qui numerat quot ex his partibus sint sumendae.

Atque hic prius, et cardinali numero, –ille posterius, et ordinali, pronuntiatur; hic quoque supra lineam, ille vera infra, notatur.

Ut si unitatem, aut rem unicam propositam, in partes quotvis, ut v. g. in 11, partitus fueris, et harum 5 partes sumendas esse statuas– 11 sunt hujus fractionis denominator, et 5 sunt ejusdem numerator; quorum, haec prius, et per numerum cardinalem (quinque), profertur; et illa 11 posterius, et per numerum ordinalem (undecimas), pronuntiatur; hoc modo, 'quinque undecimas partes.' Notantur vera haec 5 supra lineam, et illa 11 infra, hoc modo, $\frac{5}{11}$; et ita de allis.

Huic constat quod fractio, seu fractus numerus, sit pars, aut partes, unitatis divisae in partes aliquot.

Constat etiam quod fractio idem valet quod numerus numeratoris, divisus per numerum denominatoris.

Ut in superiore fractione $\frac{5}{11}$, idem est eam pronuntiare, 'quinque undecimas unitatis,' quod, 'quinque partita per 11 ;' ut in cap. 7 Lib. I. generaliter diximus.

Sunt et quaedam improprie fractiones, quae non sunt unitatis pars, aut partes, expresse, sed sunt partes fractionum; et hae fractiones fractionum nominantur; quas nos notamus interposita particula 'ex,' alii notant per omissionem posterioris lineae, aut linearum.

Ut duae quintae partes, trium quartarum partium unitatis, non est proprie, nec immediate, unitatis fractio, sed fractio fractionis unitatis; significant enim unitatem in quatuor partes dividi, et harum tres partes rursus in 5 partes partiri, quarum duas tandem sumendas esse. Ideo sic notamus, $\frac{2}{5}$ ex $\frac{3}{4}$, alii sic notant, $\frac{2}{5}\frac{3}{4}$; et sic pronuntiamus, 'duae quintae ex tribus quartis,' aut, 'duae quintae trium quartarum.'

Vulgarium fractionum nominationem sequuntur operationes, et computationum praxes.

Ergo, quoad praxin, reducimus integrum ad fractionis speciem, subjiciendo unitatem pro denominatore, et abbreviamus fractionum terminos quum accreverint, partiendo eos per suum maximum communem divisorem, quem etiam invenimus continua partitione partitoris per suas reliquias, donec nihil remanserit; atque hujus beneficio diversas denominationes ad eandem reducimus; reductas tandem addimus et substrahimus, in omnibus et per omnis, juxta canones quantitatum in genere cap. 7 Lib. I. conscriptos.

Ut binarius, numerus integer, reducitur ad speciem fractionis cum fit $\frac{2}{1}$. Sic 3 ad $\frac{3}{1}$, et 4 ad $\frac{4}{1}$, et 5 ad $\frac{5}{1}$, idem enim valent. Abbreviamus etiam terminos hujus fractionis, $\frac{6}{10}$, partiendo et numeratorem et denominatorem per suum maximum communem divisorem 2, et fient $\frac{3}{5}$. Sic $\frac{35}{49}$, partitae per suum maximum communem divisorem 7, fiunt $\frac{5}{7}$. Quem etiam maximum divisorem ibidem sic invenies: partire 49 per 35, remanent 14; partire 35 per 14, remanent 7; partire 14 per 7, remanent nihil; 7 ergo sunt maximus communis divisor terminorum 35 et 49. Sic etiam per eosdem canones, duas fractiones diversarum denominationum, ut $\frac{11}{132}$ et $\frac{7}{128}$, ad eandem denominationem reduces hoc modo: partire 132 et 128 per suum maximum divisorem communem, scilicet per 4, et fient 33 et 32; duc 7 per 33, et fient 231; et duc 128 per 33, et fient 4224, hoc situ, $\frac{231}{4224}$ pro fractione $\frac{7}{128}$ oblata: Similiter, duc 22 per 11, et 32 per 132, et fient 352 et 4224, hoc situ, $\frac{352}{4224}$, pro fractione $\frac{11}{132}$ oblata. Habes ergo duas oblatas fractiones, $\frac{11}{132}$ et $\frac{7}{128}$ ad has, $\frac{352}{4224}$, et $\frac{231}{4224}$, ejusdem denominationis, reductas, per dictos generales canones. Quibus jam reductis, per eosdem canones addimus dictos numeratores, et fiunt, cum communi denominatore, $\frac{583}{4224}$, pro additionis toto. Sic, substrahimus numeratorem novum 21 ex novo 352, et supersunt, cum novo denominatore, $\frac{121}{4224}$, pro substractionis residuo. Item, sint reducendae ad eandem denominationem: $\frac{4}{12}$ et $\frac{7}{15}$: primo abbreviantur, fiuntque $\frac{1}{3}$ et $\frac{7}{15}$; deinde partiantur 3 et 15 per maximum communem divisorem 3, et fient 1 et 5; duc $\frac{1}{3}$, tam numeratorem quam denominatorem, per 5, et fient $\frac{5}{15}$; sic duc $\frac{7}{15}$ per 1, et manent $\frac{7}{15}$ ejusdem denominationis cum $\frac{5}{15}$; quibus jam additis, fient $\frac{12}{15}$, sive, per abbreviationem, $\frac{4}{5}$, pro additionis toto. Simili modo, pro substractionis residuo, aufer $\frac{5}{15}$ a $\frac{7}{15}$ et relinquentur $\frac{2}{15}$, substractionis residuum. Item, sint addendae, seu potius uniendae, 66 et $\frac{2}{3}$: primo fient $\frac{66}{1}$ et $\frac{2}{3}$, per unitatis subscriptionem; deinde, per reductionem ad communem denominatorem, fient $\frac{198}{3}$ et $\frac{2}{3}$; tandem, additione numeratorum, fient $\frac{200}{3}$. Item, sint addendae fractiones $\frac{1}{2}$ et $\frac{2}{3}$ et $\frac{3}{4}$: hae, per reductionem ad eandem denominationem, fient primo $\frac{6}{12}$ et $\frac{8}{12}$ et $\frac{9}{12}$; deinde, suprascribendo aggregatum numeratorem, et subscribendo communem denominatorem, fient $\frac{23}{12}$, quae sunt 1 et $\frac{11}{12}$. Item, sint auferendae $\frac{8}{12}$ a $\frac{11}{12}$: restant primo, per substractionem, $\frac{3}{12}$; deinde, per abbreviationem, fient $\frac{1}{4}$, residuum quaesitum.

CAPUT XII.

DE MULTIPLICATIONIBUS ET PARTITIONIBUS SIMPLICIBUS ET RADICALIBUS FRACTORUM NUMERORUM.

MULTIPLICANTUR fracti, atque etiam fractiones fractionum iterum atque iterum fractarum ad simplices fractiones reducuntur, singulum quemque numeratorem in singulum quemque denominatorem, per cap.7 Lib. I., ita abbreviando ut nulla remaneat inter eos composita ratio. Inde, ductis novis superioribus invicem, fit novus numerator; et novis inferioribus invicem, fit novus denominator, multipli aut fractionis quaesitae.
Ut sint $\frac{1078}{1768}$ et $\frac{3705}{1449}$ et $\frac{1455}{2090}$ invicem multiplicanda: abbrevientur superiora 1078, et inferiora 1768, per 2, et fient 539 supra, et 884 infra. Item, abbrevientur 3705 superiora, et 1449 inferiora, per 3, et fient 1235 superius, et 483 inferius. Inde, abbrevientur 1455 et 2090, per 5, et fient 291 superius, et 418 inferius; hoc situ, $\frac{539}{884}\frac{1235}{483}\frac{291}{418}$: deinde abbrevientur 539 superius, et 483 inferius, per suum communcae divisorem 7, et fient 77 supra, et 69 infra; deinde abbrevia 1235 superius, et 418 inferius, per suum communem divisorem 19, et fient 65 supra, et 22 infra; deinde abbrevia 291 superius, et 884 inferius, per suum maximum communem divisorem 17, et fient 23 superius, et 52 inferius; hoc situ, $\frac{77}{52}\frac{65}{69}\frac{23}{22}$: His factis, abbrevia 77 superius, et 22 inferius, per suum communem maximum divisorem 11, et fient 7 superius, et 2 inferius; sic abbrevia 65 superius, et 52 inferius, per suum communem divisorem maximum 13, et fient 5 supra, et 4 infra; tandem, abbrevia 23 superius, et 69 inferius, per suum communem partitorem maximum, videlicet per 23, et fient 1 superius, et 3 inferius, hoc situ, $\frac{7\cdot 5\cdot 1}{4\cdot 3\cdot 2}$: hos itaque faciles et tractabiles numeros superiores, 7, 5, 1, invicem multiplica, et fient 35 pro novo numeratore ; similiter, multiplicentur invicem inferiora 4, 3, 2, et fient 24 pro novo denominatore: Est ergo fractio $\frac{35}{24}$ multipla quaesita omnium oblatarum fractionum. Item, sint invicem multiplicandae hae inabbreviabiles fractiones $\frac{3}{4}$ et $\frac{5}{7}$: multiplicentur ergo 3 in 5, et fient 15, novus numerator; multiplicentur etiam inferiores 4 in 7, et fient 28, novus denominator: Unde $\frac{15}{28}$ sunt fractio quaesita multupla. Item, sint reducendae $\frac{2}{3}$ ex $\frac{4}{5}$ ex $\frac{6}{7}$ ex $\frac{8}{9}$ ad eandem simplicem fractionem: abbrevientur primo 6 superius, et 9 inferius, per suum communem maximum divisorem 3, et fient 2 supra, et 3 infra, hoc situ, $\frac{2}{3}\frac{4}{5}\frac{2}{7}\frac{8}{3}$; ulterius enim non est inter has superior qui sit cum inferiore aliqua abbreviabilis : multiplicentur ergo, tandem, superiores 2, 4, 2, 8, invicem, et fient 128; itemque, multiplicentur 3, 5, 7, 8, inferiora invicem, et fient 315; ex his fiet novus denominator, ex illis vero numerator, hoc situ, $\frac{128}{315}$, fractio simplex quaesita, idem valens quod $\frac{2}{3}$ ex $\frac{4}{5}$ ex $\frac{6}{7}$ ex $\frac{8}{9}$ oblatae.

Pro partitione autem transponantur termini partitoris, et transpositos per partiendum multiplica omnino, ut in praecedente, et cap. 7 Lib. I., praecepimus.

Ut si partiendae sint $\frac{4}{5}$ per $\frac{2}{3}$: retentis illis, inverte has, et ita multiplicato $\frac{4}{5}$ et $\frac{3}{2}$; fientque, primo, per abbreviationem, $\frac{2}{5}$ et $\frac{3}{1}$; deinde fient, per multiplicationem, $\frac{6}{5}$, sive, quod idem est, $1\frac{1}{5}$ pro quoto quaesito.

Aliud exemplum mistarum.

Sint $66\frac{2}{3}$ partiendae per $2\frac{1}{5}$, seu, quod idem est, per reductionem partiantur $\frac{200}{3}$ per $\frac{11}{5}$: has inversas hoc modo, $\frac{5}{11}$ et illas, $\frac{200}{3}$, invicem multiplica, et provenient $\frac{1000}{33}$, quae sunt $10\frac{10}{33}$, pro quoto quaesito.

Aliud exemplum, pro examine multiplicationis.

Sint, ex praecedentibus exemplis, $\frac{15}{28}$ partiendae per $\frac{3}{4}$: invertatur hinc, fiet $\frac{4}{3}$ et illa manet $\frac{15}{28}$; abbrevientur, et fient primo $\frac{5}{28}$ et $\frac{4}{1}$, deinde fient $\frac{5}{7}$ et $\frac{1}{1}$; hos numeratores invicem, et denominatores invicem duc, et fient $\frac{5}{7}$; ut superius in multiplicatione.
Multiplicatio radicalis in fractis numeris, prout in integris, per continuatam multiplicationem perficitur: Si enim fractionem oblatam in se, aut sibi aequalem, multiplicaveris, fit duplicata; si hanc duplicatam in eandem primam multiplicaveris, fit triplicata; quam triplicatam si adhuc in primam multiplicaveris, proveniet inde quadruplicata radicis et fractionis primo oblatae: Et sic deinceps quintuplicata, sextuplicata, et ceterae radicatae componuntur.

Ut sit radix haec, $\frac{2}{3}$ radicaliter multiplicanda ad indicem 5, seu quintuplicanda: Primo, multiplica $\frac{2}{3}$ in $\frac{2}{3}$ fient $\frac{4}{9}$ pro duplicata ; deinde multiplica hanc duplicatam $\frac{4}{9}$ per easdem $\frac{2}{3}$ fient $\frac{8}{27}$, triplicata oblatae fractionis; hanc triplicatam $\frac{8}{27}$, multiplica per easdem $\frac{2}{3}$ et proveniet oblatae radicis quadruplicata $\frac{16}{81}$; quam tandem per $\frac{2}{3}$ multiplica, et producentur $\frac{32}{243}$.

Partitio etiam radicalis fractionum, ex continuata partitione simplici, per regulas superiores, et integrorum more, pcrficitur, partiendo fractionem radicatam propositam per suam radicem datam, continue, in unitatem usque; et numerus partitionum dabit indicem, ut superius saepe dictum est.

Sint, pro exemplo et examine praecedentis, radicaliter partiendae $\frac{32}{243}$ per $\frac{2}{3}$: ea fient, primo, $\frac{16}{81}$; secundo, $\frac{8}{27}$; tertio, $\frac{4}{9}$; quarto, fiet ipsa radix $\frac{1}{3}$; quinto, devenit ad unitatem: Itaque, quinque hae partitiones exhibent indicem quinarium pro radicali quoto quaesito, et arguunt praecedentem quintuplicationem legitimam esse.

CAPUT XIII.

DE EXTRACTIONE RADICUM E NUMERORUM FRACTIONIBUS.

SI, fractionis reductae ad minimos terminos, uterque terminus habuerit radices propositae speciei, eae extrahendae sunt integrorum more, et fient numerator et denominator radicis quaesitae.

Sin autem uterque terminus fractionis non habeat talem radicem, tunc ejus radix, aut geometricae more aut mechanico, est extrahenda.

Ut sit radix tripartiens extrahenda e $\frac{54}{16}$, cujus termini minimi sunt $\frac{27}{8}$; in hujus termino utroque clare constat radix tripartiens numeratoris 3, et denominatoris 2; sunt ergo $\frac{2}{3}$ radix tripartiens quaesita oblati radicati $\frac{54}{16}$, seu $\frac{27}{8}$.

Aliud exemplum.

Contra vero sit fractio oblata $\frac{290}{40}$, cujus termini minimi sunt $\frac{29}{4}$; sit ex his extrahenda radix bipartiens: Dico, quia uterque terminus, 29 et 4, non habeat radicem bipartientem, (quamvis alter, scilicet 4, habeat) ideo, fractionis oblatae $\frac{290}{40}$, seu $\frac{29}{4}$ non est arithmetice extrahenda radix perfecta, sed aut geometrice aut mechanice.

Geometricus modus est, praeponere radicatae fractioni toti signum radicis extrahendae, aut signa radicis utrique termino.

Ut superioris exempli fractio $\frac{290}{40}$, seu $\frac{29}{4}$ veram radicem bipartientem non habet; ideo, toti fractioni aut praeponendum est signum radicis, hoc modo, ⊔ $\frac{29}{4}$ aut utrique termino, hoc modo $\frac{⊔29}{⊔4}$; aut, tandem, eadem radix est mechanice admodum inter terminos includenda, ut sequitur.

Mechanicus autem modus est, radicem quaesitam, quae numeris exprimi nequit, inter terminos majorem et minorem quam arctissime includere.

Termini quanto majoris denominationis, et minoris differentiae fuerint, tanto radicem arctius includunt, et praecisius definiunt.

Ut ergo termini sint denominationis optatae, numerum optatae denominationis multiplica radicaliter, secundum indicem et speciem radicis quaesitam; inde, hujus radicatum multiplica per fractionis oblatae numeratorem, productum partire per ejusdem denominatorem; quoti radix propositae speciei, tam proxime minor quam major, extrahatur integrorum more, et utrique denominator electus subscribatur: Erunt enim termini optati, ille minor, hic major, radicem quaesitam includentes.

Ut sit fractionis superioris $\frac{290}{40}$ seu $\frac{29}{4}$, seu extrahenda radix bipartiens: Sit denominatio optata 200 partium; harum radicatum propositae speciei (duplicatum scilicet) est 40000; multiplica ergo 40000 per 29, numeratorem oblatum, et productum partire per 4, ejusdem denominatorem, provenient 290000 quotus; eujus radix bipartiens, nempe proxime minor, est 538, et proxime major est 539, (ex superioribus integrorum regulis) quibus subscribendo denominatorem 200 optatum, fient $\frac{538}{200}$, minor terminus, et $\frac{539}{200}$, major terminus, differentes sola unica ducentesima unitatis, et ita arcte satis includentes ⊔ $\frac{29}{4}$ seu radicem bipartientem hujus fractionis $\frac{29}{4}$ quaesitam.

Exemplum radicis tripartientis.

Item, sit extrahenda radix tripartiens e $\frac{2}{3}$ seu potius (quia haec radix non est numero explicabilis) quaerantur ejus termini major et minor, denominationem quam volueris, exempli gratia, mille partium habentes; inter quos vera tripartiens radix duarum tertiarum, seu L $\frac{2}{3}$, contineatur. Triplicatum ergo mille partium, quod est 1000000000, multiplica per 2, fient 2000000000; quae partire per 3, et fient 66666666$\frac{2}{3}$; horum radix tripartiens minor est 873, et major est 874, quibus subscribatur electus denominator 1000, fientque $\frac{873}{1000}$ minor, et $\frac{874}{1000}$ major terminus radicis tripartientis duarum tertiarum, seu $\llcorner\frac{2}{3}$ quaesitae.

Ultimo omnium, fractiones, quarum numerator excedit denominatorem, ad integros numeros restituendae sunt partiendo numeratorem per denominatorem, ut fuse satis, in fine Libri Primi, explicatum est.

Ut si ex praecedentibus operibus provenerit fractio $\frac{562}{18}$, continet haec fractio integros per quos danda est ultima responsio potius quam per fractos. Dividantur ergo 562 per 18, et fient 31 in quoto, et relinquuntur 4, quae sunt $\frac{4}{18}$, seu $\frac{2}{9}$; unde, ex tota reductione proveniunt $31\frac{2}{9}$, quae sunt 31 integri, et duae novenae unitatis.

CAPUT XIV.

DE REGULIS PROPORTIONIS FRACTIONUM.

IN regulis proportionis fractionum, observandae sunt situs et operatio.

In his, situs numeratorum idem est, qui et generaliter quantitatum capitis quinti Libri Primi; denominatorum vera contrarius; ita ut si numerator hac lege infra lineam ceciderit, denominator invertetur, et cadet supra; et contra, si ille supra, hic (ut par est) cadet infra.

$$\frac{\cdot 5\cdot 5}{3.}$$

Ut si proponatur e cisterna aliqua aquae congii $\frac{3}{7}$ partes efffuere in $\frac{5}{12}$ horae; quaeratur autem quanto tempore $2\frac{1}{2}$, seu $\frac{5}{2}$, congii effluent ? Primo, ducta linea ut a margine,

$$\frac{7\cdot 5\cdot 5}{3\cdot 12\cdot 2}$$

prioris temporis numerator 5 supra lineam constituatur (per hanc et praeceptum generale cap. 5 Lib. I.), quia $\frac{5}{12}$ & horae et horae quaesitae sunt cognomines, atque simul crescunt et decrescunt: Sic $\frac{5}{2}$ congii simul cum quaesitis horae partibus crescunt et decrescunt; ideo, et illarum numerator 5 supra lineam collocatur: sed prioribus congii $\frac{3}{7}$ partibus, crescentibus decrescunt temporis quaesiti partes, et decrescentibus crescunt; ideo illarum numerator 3 infra lineam statuitur. Contra vero denominatores omnes, scilicet, 7, 12, et 2, singuli singulis suis numeratoribus ex adverso latere linam opponuntur, ut a margine.

Operatio sic perficitur. Multiplica superiores numeros invicem, et producetur numerator; item et inferiores invicem, et producetur denominator, praefatum numeratorem dividens, et optatum productum exhibens. Ut praefati exempli superiores numeri 7, 5, 5, invicem multiplicati, faciunt 175 numeratorem; et inferiores 3, 12, 2, invicem multiplicati, producunt 72 denominatorem; unde $\frac{175}{72}$, sive (per 175 partitionem) $2\frac{31}{72}$, horae sunt tempus quaesitum, quo nimirum $2\frac{1}{2}$ congii aquae effluent.

$$\frac{7\cdot 5\cdot 5}{3\cdot 12\cdot 2} \quad \frac{175}{72}$$

Hic etiam memor sis secundi compendii cap. 10. Lib. II. quotiescunque inter superiorum aliquem, et aliquem inferiorum, inciderit ratio composita quae abbreviari possit.

Ut pro terminis superioris exempli $\frac{7}{3}$ $\frac{5}{12}$ et $\frac{5}{2}$, qui abbreviari nequiunt, si capiantur termini hi, $\frac{7}{6}$ et $\frac{10}{21}$ et $\frac{15}{8}$, qui rationes habent compositas, et abbreviari quidem possunt : Ideo, collocatis ut a margine, dividantur 7 superiora, et 21 inferior &, per communem maximam mensuram 7 et fient $\frac{1}{3}$ hoc situ, $\frac{7\cdot10\cdot15}{6\cdot3\cdot8}$; deinde, simili , modo abbrevientur 10 superius, et 6 inferius, et fient eorum loco 5 superius et 3 inferius, hoc situ $\frac{7\cdot5\cdot15}{3\cdot3\cdot8}$; abbrevientur tandem 15 superius, et alterutra 3 inferius, et fient 5 superius, et 1 inferius hoc situ $\frac{1\cdot5\cdot5}{3\cdot1\cdot8}$ vel hoc, $\frac{1\cdot5\cdot5}{1\cdot3\cdot8}$ 5· vel aliter $\frac{1\cdot5\cdot5}{6\cdot1\cdot4}$; quorum multiplicatis superioribus 1, 5, 5, invicem fient 25; deinde, similiter, inferioribus 3, 1, 8, vel 6, 1, 4, invicem, et fient 24: Itaque, tota fractio, quaestioni satisfaciens, erit $\frac{25}{24}$, seu, per abbreviationem, $1\frac{1}{24}$. Atque ita in omnibus similibus compendiose operandum est.

$\frac{7\cdot10\cdot15}{6\cdot21\cdot8}$

CAPUT XV.

DE FRACTIONIBUS PHYSICIS.

PERFECIMUS jam fractionum vulgarium computationes omnes; ordine sequerentur physicae.

Physicas vocant partem aut partes integri partiti per statutum et vulgo receptum partitorem, denominatoris loco, a suis authoribus impositum.

Ut monetariis nostratibus placuit monetae libram non in quotvis sed in 20 partes dividere, eisque denominationem solidi imponere; et solidum in 12 partes subdividere, quas denarios denominant. Sic medici, ponderis libram in partes 12, quas uncias nominant, et unciam in 8 drachmas, et drachmam in 3 scrupulos, etc., partiuntur. Chronologi, annum in 12 menses, mensem in 30 (aut id circa) dies, diem in 24 horas, &c., distribuunt. Astronomi, gradum in 60 scrupula prima, seu minutias primas, primam in 60 secundas, secundam in 60 tertias, et ita deinceps, partiuntur.

In fractionibus physicis, hoc cum vulgaribus commune est, quod quoties ultra suum denominatorem accreverint, superiorem locum unitate augent; et quoties major fractio eo minore subducenda sit, praestanda est ei superioris loci unitas ad defectum supplendum. Ut si fuerint 14 horae addendae ad 19 horas, quae constituunt 33 horae, pro quibus non scribuntur 33 horae, sed, sub titulo dierum et horarum, scribuntur dies 1 et 9 horae; quia ultra 9 horas accreverunt 24 horae in diem. Ita, pro 7 et 9 denariis addendis, non scribuntur 16 denarii, sed 4 denarii, et, pro accretis reliquis 12 denariis, 1 sub titulo solidi inscribenda. Item, pro 48 scrupulis primis addendis ad 15 scrupula prima, non scribuntur 63 scrupula prima, sed 3 tantum, et 1 gradus, pro reliquis 60 scrupulis. Haud secus, prout tam in integris quam in fractis vulgaribus numeris, fieri solet; ut cum addenda sunt quinque centena ad octo centena, non scribuntur, nec pronunciantur, 13 centena, sed mille trecenti, hoc modo, 1300. Et in fractis vulgaribus addendis, veluti sex septenae seu $\frac{6}{7}$, et quinque septenae seu $\frac{5}{7}$ additae, et debito modo abbreviatae, non sunt dicendae undecim septenae, sed quatuor septenae, et, pro reliquis septem septenis, scribenda est unitas, hoc situ, $1\frac{4}{7}$. Sic, in substractione, si fuerint substrahendae 14 horae, superioris exempli, ex uno die et 9 horis,– quia 14 horae excedunt 9 horas, ideo resolvendus est dies ille unicus in 24 horas, quae cum 9 horis efficiunt 33 horas; e quibus subductis 14 horis, relinquuntur, ut supra, 19 horae quaesitae. Sic, ex 1 solido et 4 denariis, si substrahendi sint 9 denarii, qui excedunt 4 denarios, et ideo 4 denariis praestandum est unum solidum, seu 12 denarii, ut fiant inde 16 denarii, e quibus jam subductis 9 denariis, remanent, ut supra, 7 denarii quaesiti. Sic, si e gradu et 3 scrupulis primis fuerint 15 scrupula prima substrahenda, reducenda sunt unus gradus et tria scrupula ad 63 scrupula prima, ut inde 15 auferri possint, et remanebunt 48 scrupula prima Haud secus ac in vulgaribus integris et fractis, unitas praecedentis loci praestanda est sequenti loco ad defectum ejus supplendum. Ut si e mille trecentum auferantur quinque centena, ter centenis praestandum est unum millenarium, ut hinc fiant tredecim centena, equibus jam aufer quinque centena, supersunt octo centena, ut supra. Sic, ex uno et quatuor septenis, auferendae sunt sex septenae; ut hoc fiat, reducenda est unitas in septem septenas, quae praestandae sunt quatuor septenis, ut fiant undecim septenae ; e equibus sex septenis ablatis, remanent quinque septena, ut supra. Vides itaque communem horum omnium concentum.

Est enim omnium, tam fractorum quam integrorum, par ratio et respectus ad suas denominationes, tam ascendendo quam descendendo.

Ut integrorum unitatum ad suas denas, centenas, et millenas superiores; et ad suas decimas, centesimas, et millesimas inferiores. Sic, scrupulorum primorum ad suos sexagenos (qui gradus sunt), et sexagenos gradus (qui duo signa sunt), ascendendo; et ad suas sexagesimas partes (quae scrupula secunda sunt), et scrupulorum secundorum sexagesimas descendendo (quae scrupula tertia sunt). Item, solidi ad sua vigecupla ascendendo (quae librae dicuntur), et ad subs duodecimas descendendo (quae denarii dicuntur). Par est respectus, et computationis similitudo, in omnibus.

Hinc fit quod otiosum et superfluum foret, hisce particularem calculum texere, cum harum computatio facilius expediatur per vulgares integros et fractos numeros, quam per suas particulares formas calculi, praesertim astronomici, ad quem etiam tabula prolixa sexagenaria requiritur, una cum molestia duplicium figurarum unicuique denominationi subjectarum, ubi arithmetica vulgaris requirit tantum unicam figuram unicuique loco.

Ut patet expertis: Nam facilius et citius astronomicum thema vulgato more arithmetico computatur, et resolvitur, quam ipso astronomico calculo; praeter etiam tabulae praefatae, et duplicium figurarum, sub quaque denominatione, taedium. Namque vulgaris arithmetica loco primo unitatum, et loco secundo denorum, et loco tertio centenorum, et reliquorum singulorun unicam tantum figuram in singulis locis constituit. Astronomicus autem calculus in singulis suis locis, tam ascendendo quam descendendo, sic notatis, ′ ″ ‴ ⁗, etc. binas plerumque figuras subjectas complectitur, quae omnis confusionem pariunt.

Unde, omnibus fractionibus praeter jam expositas vulgares omissis, huic Arithmeticae Logisticae finem imponimus.

DEO autem OPT. MAX., et suis Numeris omnibus Infinito, Immenso, et Perfecto, retribuatur omnis laus, honor, et gloria in aeternum.

Amen.

FINIS LIBRI SECUNDI.