A Warning.

Since these tables have been perfected by means of many careful arithmetical calculations, and completed by the work and industry of one person, it would not to be a cause of wonder if many errors had crept into these same calculations. Therefore, for the errors that have arisen, either from calculator fatigue, or from careless typographical work, I beg that they may be forgiven by you, gentle readers: indeed my uncertain health has made it more difficult for me to attend to these matters. Truly, if I am led to understand that the learned are pleased to make use of this discovery, then perhaps (God willing) I will give a brief account of the reasoning behind them, and the method of their construction. I would do this so that either this cannon could be amended, or it could be made even better by putting in its place, a new set of tables brought forth by the labours of many, and which would be even more elegant and accurate than it is possible to have from the labours of one person.

Nothing arises complete.

[*Taken from the last page of the Demonstratio*.]

The Construction of the Wonderful Canon of LOGARITHMS.

And the relations of these to the natural numbers;
Together with an Appendix,
Concerning another kind of logarithms to be put in place.

TO WHICH ARE ADDED

Propositions for the easier calculation by which spherical triangles can be solved.

Together with some annotations of the most learned Henry Briggs, in these and in the appendix mentioned.

By the author and inventor, ***John Napier***, Baron of ***Merchiston, &c***. Scotland.

EDINBVRGI,
Excudebat ANDREAS HART.
ANNO DOMINI 1619.

To the Reader Devoted to Mathematics.

A few years ago (Reader and Lover of Mathematics), my father, whose memory I always cherish, brought to the attention of the public, the use of the ***Wonderful Canon of Logarithms*** *; and indeed he himself had advised on Page 7 and at the end of that book, how he was unwilling at that stage to commit himself to type with a deliberate plan of the construction and method of preparation of logarithms. He was of that opinion regarding these tables, at least until there was evidence of their success, and an appraisal of the canon had been made upon their examination by those versatile in this kind of learned matter.*

However, after his departure from this life, it became clear to me that the most skilled mathematicians of the discipline do indeed make the most use of this new discovery; and nothing would please them more than if the construction of this **Wonderful Canon***, or even some part of it, could be put forward for public scrutiny. Therefore, though it appears to be the case that the final touches had not been put on this little work by the author himself, I have done as much as I can, in order that the most worthy desires of those mathematicians can be satisfied, and even for those who are less strong in these studies and who tend to get stuck on the threshold, I have taken this aspect of the work into consideration too. I doubt not, that this posthumous work would have been produced in a far more perfect and elegant manner, if God had granted a longer enjoyment of life to the author, my dear father, who in this book, (according to the most learned opinion, among his other outstanding achievements, brought the theory of logarithms by an easy and reliable method to perfection from the smallest of beginnings). You have, therefore, (kind reader) in this little book, the theory for the construction of Logarithms most fully explained, (which he calls here artificial numbers; for indeed he was in the habit of writing in this little tract for a number of years before the invention of the word 'logarithm'); in which their nature and properties, and the various relations of these to numbers, are clearly demonstrated.*

It is also noted that a certain appendix is added for the syntax of another more outstanding kind of logarithm, (that the inventor of logarithms recalls himself in an Epistle in the introduction to his own book ***Rabdologiæ****),* [Thus Napier, and not Briggs, was the discoverer of base 10 logarithms.] *and in which the logarithm of one is taken as zero. Here also in the last part of the book there follows the final fruits of his labour, regarding the perfection of his logarithms of trigonometry; surely some of the most outstanding propositions for spherical triangles, and at this stage by looking at certain cases, non quadrantal triangles are to be solved, without being divided into quadrantal or right angles triangles: in which indeed certain Propositions had been set out to order the material, and which he had decided to prove in that order, except that the work was suddenly brought to a close by his death.*

Also we have seen fit to put into print some lucubriations by the most excellent Henry Briggs*, Public Professor in London, on these hitherto untold Propositions, and this new kind of logarithm; and who has accepted the enormous labour of constructing this new Canon with the greatest of willingness, from the special friendship that existed between himself and my father ; the method of creation of the current tables, and the explanations of the uses are left to the inventor. For now with himself called away from this life, it seems that by chance the whole onerous task has fallen on the shoulders of the most* *learned Briggs to adorn this Sparta . And meanwhile dear Reader, enjoy whatever your labours are with these, and consult with good people of your race. Farewell.* Robert Napier, Son.

The Construction of the Wonderful Canon of LOGARITHMS.

(Which in the table are henceforth called artificial numbers by the author)

and the relations these have with natural numbers.

First Proposition.

A table of Artificial Numbers [that we will henceforth call logarithms, as the word was coined at a late stage in the development by Napier]***, is a small table, with the aid of which all geometrical dimensions can be calculated with ease, and an acquaintance with celestial motions can be had.***

This deservedly may be called the smallest table, since it does not exceed the volume of a table of sines: but with the greatest ease, since by its use all multiplications, divisions, and more serious root extractions are avoided: indeed with these tables alone, and by a few of the easiest additions, subtractions, and dividing by two, all the figures are measured and in turn the motion is measured.

These numbers are selected from numbers in continued progressions.

Prop. 2. ***Of continued progressions, some are called Arithmetic and which progress in equal intervals*** [p. 6] ***; Others are called Geometric, which increase or decrease proportionally in unequal steps.***

For an Arithmetic progression, thus 1, 2, 3, 4, 5, 6, 7, &c. or 2, 4, 6, 8, 10, 12, 14, 16, &c. For a Geometric progression, thus 1, 2, 4, 8, 16, 32, 64, &c. or 243, 81, 27, 9, 3, 1.

3. ***Accuracy is required in progressions, and ease in working. For the accuracy shall be greater by taking a larger fundamental number : moreover larger numbers are easily made from small numbers by adding zeros.***

As for 100000, that the more rudimentary take for the maximum value of the sine, the more learned put 10000000, from which all of the sines can be expressed with better discrimination. Thus we are using the same number for the maximum sine and for the maximum of the Geometrical proportions.

4. ***In computed Tables, the maxima are also made from larger numbers, with a period placed between the number itself and added ciphers.***

As from 10000000, we make the first number of the computation 10000000.0000000, lest the least error by frequent multiplication will grow large. [This would appear to be the first recorded systematic use of the decimal point, which was not to acquire general acceptance for another hundred years or so.]

5. ***With numbers thus distinct between themselves by the period, whatever number is after the period is called the fraction, the denominator of which is one with a number of zeros after it, which are the figures after the period.***

As from 10000000.04 the same prevails, since $10000000\frac{4}{100}$ is the same. Likewise 25.803, is the same as $25\frac{803}{1000}$. Likewise 9999998.0005021, is the same as $99999998\frac{5021}{10000000}$, and thus for the rest.

6. ***From numbers now computed in the tables, fractions located after the period can be rejected, without sensible error. For our large numbers are insensitive to errors, and appear to be zero, which do not exceed unity.***

As completed from the table, for 9987643.8213051, which is $9987643\frac{8213051}{100000000}$, [p.7.] this can be taken as 9987643 without sensible error.

7. ***There is besides another rule for accuracy ; when of course there is an unknown quantity included, or an irrational number, lying between numerical terms with many digits which are the same.***

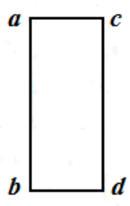
As with the diameter of the circle of 497 parts put in place; since it is not known precisely how many parts there shall be around the circumference, thus the wise, following the thoughts of Archimedes, will put the distance to lie between the boundaries 1562 and 1561. Likewise if the square of each of the sides of some square is 1000 parts, then the diagonal is the square root of the number 20000000; which since it is an irrational number, thus by the extraction of the square root the boundaries of this are sought, either $1414\frac{604}{2828}$ which is greater, and $1414\frac{604}{2829}$ which is less: it is apparent that when the difference of the terms is so much less, then the accuracy is so much greater.

In turn, the bounding numbers of the unknown quantities are themselves to be added, subtracted, multiplied, or divided, as the need arises, in place of the unknown quantity.

8. ***The two boundaries of any quantity are added to the two boundaries of another quantity, where the smaller of the one is added to the smaller of the other, and the larger of the one are added to the larger of the other***

Thus let the line *abc* be divided into two quantities *ab* and *bc* ; let *ab* lie between the greater boundary123.5, and the lesser 123.2 : and let *bc* lie between the greater number 43.2 , and the smaller number 43.1. Hence with the larger added to the larger, and the smaller to the smaller, the whole quantity *ac* then lies between the boundaries 166.7 and 166.3.

9. ***The two boundaries of any quantity are multiplied by the two boundaries of some other quantity, with the smaller of the one by the smaller of the other, and the larger of the same one by the larger of the same other.***



As let there be some quantity *ab*, which is between the larger boundary 10.501, and the smaller boundary 10.500; while the other, *ac*, which is between the larger boundary 3.216, and the smaller boundary 3.215. Hence taking 10.502 by 3.216, and 10.500 by 3.315, the boundaries 33.774432 and 33.757500 arise; between which lies the area *abcd*.

10. ***The subtraction of the boundaries shall be : the greater boundary of the smaller quantity from the lesser of the greater quantity, and the lesser of the smaller quantity from the greater of the larger quantity, are to be taken away for the greater and lesser boundaries of the difference.***

As in the first diagram, if from the boundaries of *ac*, which are 166.7 and 166.3, you take the boundaries of *bc*, which are 43.2 and 43.1, then the boundaries for *ab* become 123.6 & 123.1; and not 123.5 & 123.2. For it is also possible to add the latter to 43.2 & 43.1, to produce 166.7 & 166.3, (by 8) yet this does not follow from the converse, for it may be possible that between 166.7 and 166.3, for if you should subtract some number that lies between 43.2 and 43.1, there may remain a number that is not between 123.5 and 123.2: and indeed this is not possible if the boundaries are 123.6 and 123.1

[Thus, according to Napier, we imagine the average value of a quantity to lie between its 'error bars' , thus for : $ac-\Delta ac < ac < ac+\Delta ac$ and $bc-\Delta bc < bc < bc+\Delta bc$. The boundaries or limits set on the difference, or $ac > bc$, are taken to be : $(ac-\Delta ac)-(bc+\Delta bc) < ac-bc < (ac+\Delta ac)-(bc-\Delta bc)$; essentially widening the boundaries.]

11. ***Division shall be, with the part of the greater boundary divided by the smaller boundary of the divisor, and the lesser boundary to be divided by the greater boundary of the divisor.***

As in the preceding figure, the square *abcd* set up between the boundaries 33.774432 and 33.757500 is divided by the boundaries *ac*, which are 3.216 and 3.215, there comes about $10.505\frac{857}{3215}$ and $10.496\frac{2364}{3216}$ for the boundaries of *ab*; and not 10.502 and 10.500, from the same reason, as we said for subtraction.

12. ***Vulgar fractions can be removed by adding one to the greater boundary.***

As for the preceding boundary *ab*, obviously $10.505\frac{857}{3215}$ and $10.496\frac{2364}{3216}$, these can be taken as 10.506 and 10.496 [p. 9.]

Up to this point we have been concerned with accuracy, that which follows is concerned with ease of working.

13. ***The arithmetical progression is the easiest to construct of all progressions, but this is not so for geometrical progressions.***

This is apparent, since an arithmetical progression can easily be constructed by addition and subtraction; however, a geometric progression is harder with multiplications, divisions, and root extractions to be continued.

Only these geometric progressions can be easily continued, that are continued by parts of the number easily arising from the whole by subtraction.
[This boils down to the repeated subtraction of a small part from the whole number.]

14. ***We say that the easy parts of the number, are any parts the denominator of which is unity with some number of zeros noted : moreover these parts, by rejecting as many figures from the end of the principle number as there are zeros in the denominator.***

As the tenth part, the hundredth, thousandth, 10000th , 100000th, 1000000th, 10000000th, are easy to be called, since the tenth part of any number is found by deleting the final digit, and thus for the others, always by deleting from the whole the final figures which are zeros in the parts from the denominator. As the tenth part of 99321 is 9932, moreover the hundredth part is 993, the thousandth 99, &c.

15. ***The mean is also easily had from half the sum of the parts, the twenti^{eth}, the two hundredth, and the other parts from two and the zeros of the denominator; by rejecting as many zeros from the end of the number, as there are ciphers in the denominator, and dividing the remainder by two.***

As the 2000th part of the number 9973218045 is 4986609, the 20000th part is 498660.

16. ***Hence it follows, if from the total sine augmented to seven ciphers, and with the rest thus arising from the subtraction of its own* 10000000th *part, the series is able to be continued easily as far as a hundred numbers, in that geometric proportion that there is between the total sine and the sine to that less by one, as can be seen,* 10000000 & 9999999; *and this series of proportions we call the first Table.*** [p. 10.]

As from the total sine augmented by seven ciphers, (for the sake of greater accuracy) thus from 10000000.0000000 take 1.0000000, and they make 9999999.0000000; from which take .9999999, and they make 9999998.0000001; and thus you progress, as from the side, until you have made a hundred numbers in proportion, the last of which, if you have calculated correctly, will be 9999900.0004950.

First Table.
10000000.0000000
1.0000000
9999999.0000000
.9999999
9999998.0000001
.9999998
9999997.0000003
.9999997
9999996.0000006
continued as far as
9999900.0004950.

17. ***The second table progresses from the total sine augmented by six ciphers, through fifty other diminishing numbers in proportion, from that proposition which is the easiest, and that is close to the proportion that exists between the first and last proportion of the first Table.***

Secunda Tabula.
100000000.000000
<u>100.000000</u>
9999900.000000
<u>99.999000</u>
9999800.001000
<u>99.998000</u>
9999700.003000
<u>99.997000</u>
9999600.006000
&c. as far as
9995001.222927

As the first and last members of the first Table are 100000000.000000 and 9999900.0004950 ; and fifty proportional numbers are established in this proportion. And thus the nearest proportion that can be calculated with ease is 10000 to 99999 ; which can be continued well enough with the six adjoining ciphers of the total sine, and by taking away from the previous value its 100000th part , as is made subsequently, as you see from the side : and in this Table besides the first enter, which is the total sine, also should contain fifty numbers in proportion, of which the final, if you have not erred, you find to be 9995001.22<u>2927</u>

[Unfortunately, Napier had himself erred, and the true value of the final number in this series of proportions should be 9995001.224804; however, as the whole term 1.222927 is to be ignored subsequently, this would appear to be a small matter.]

18. ***The Third Table contains sixty nine columns, and in any column twenty one numbers are put in place, progressing from that proportion which is easiest to calculate with, and that is closest to that proportion that exists between the first and last of the second Table.***

Thus the first column of this table is to have the total sine augmented by five ciphers, and so for the others thus arisen with their 2000th part taken away.

First column of the third table
10000000.00000
<u>5000.00000</u>
9995000.00000
<u>4997.50000</u>
9990002.50000
<u>4995.00125</u>
9985007.49875
<u>4992.50374</u>
9980014.99501
&c. as far as
9900473.57803.

Since as between the first and last members of the second Table, that is 10000000.000000 and 9995001.222927, the proportion of the progression is hard to use ;thus by the easy to use proportion 10000 to 9995 (which is close enough to that proportion), the twenty one numbers of the column are to be established; the last of which, if you have not erred, will be 9900473.57803. From which with the numbers now in place, with the final individual figures rejected without sensible error, by which those others that follow can be produced more easily.

19. ***The first numbers of all the columns are to progress from the total sine augmented by four ciphers, by that easiest proportion, and near to that proportion, which exists between the first and final numbers of the first column.***

Since the first and last numbers of the first column are 10000000.0000 and 9900473.57803 : for these the proportion with the greatest ease of calculation that is nearby is 100 to 99. Therefore from the total sine, 68 numbers are to be continued in the ratio 100 to 99, the next formed from any of these by taking away a one hundredth part of its self.

[An error is introduced by this approximation; but a little search with a spreadsheet shows that there is no more convenient number to choose that gives a realistic ratio, easy to use in calculations. One presumes that a manual search was made for such a ratio.]

20. ***There is the same progression to be made in the same proportion, with the second number from the first column, and through all the numbers of the second column*** **[p.12]** ***: and from the third through all the third column : and from the fourth, through the fourth: and for the rest respectively, through the rest.***

As from some number in the preceding column, a number of the same order in the following column is established, by taking its one hundredth part, and the numbers that follow are thus in this order.

Prime Column.	*Second Col.*
10000000.0000	9900000.0000
9995000.0000	9895050.0000
9990002.5000	9890102.4750
9985007.4988	9885157.4238
9980014.9950	9880214.8451
as far as	as far as
9900473.5780	9801468.8422

Third Col.	*Thus 4th, 5th,&c.*	*to 69th column.*
9801000.0000	until	5048858.8879
9796099.5000	until	5046334.4584
9791201.4503	until	5043811.2912
9786305.8495	until	5041289.3856
9781412.6966	until	5038768.7409
as far as	as far as	as far as
9703454.1538	until finally	4998609.4019

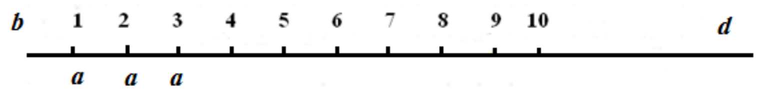
21. [p.13.] ***Hence in the third Table, you have between the total sine and half of the total sine, sixty eight numbers in the proportion 100 to 99; and again between an individual pair of these numbers, you have twenty numbers inserted in the proportion 10000 to 9995: and again in the second Table between two of these numbers, obviously between 10000000 to 9995000, you have 50 numbers inserted in the proportion 100000 to 99999: and finally in the first Table, between the first two of these, you have 100***

numbers in the proportion as the total sine 10000000 to 9999999; the difference of which shall be as great as one, and thus there is no need for smaller parts (from the means being put in place). Thus these three tables, upon completion, are sufficient to compute the table of logarithms.

Hitherto we have shown how the whole sine or natural numbers can be easily inserted into tables in a geometric progression.

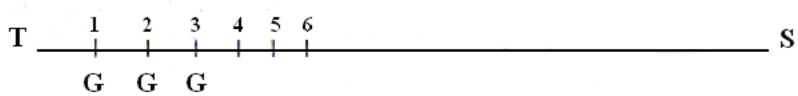
22. ***It still remains at least for the third Table, to insert the logarithms increasing arithmetically for the sines or natural numbers decreasing geometrically.***

23. ***To increase arithmetically is always to increase by the same amount in equal times.***



As from the fixed point *b,* a line is produced indefinitely towards *d* : in which, the point *a* proceeds from *b* towards *d*, moving according to the principle, so that in equal moments of time it is carried by equal intervals : which are *b* 1, 1 2, 2 3, 3 4, 4 5, &c. I say that these increments through *b* 1, *b* 2, *b* 3, *b* 4, *b* 5, &c. are arithmetical. Moreover, with the intervals set equal to the numbers *b* 1, 10: *b* 2, 20: *b* 3, 30: *b* 4, 40: *b* 5, 50, &c. these intervals increase arithmetically : since in equal time intervals, they are understood to always increase by ten. [p.14.]

24. ***To decrease geometrically, is in equal intervals of time from the initial total, thus one, then another, then another part is put in place, always to be diminished by the same proportional part.***

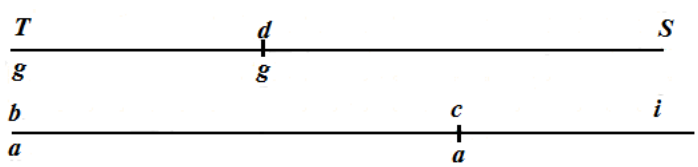


As the line of the total sine shall be TS, on this the point G is moved from T by 1 towards S, and in so much time it is carried from T to 1, which shall be, for argument's sake, the tenth part of TS : in a like time G is moved from 1 to 2, which shall be in the tenth part of 1 S: and from 2 in 3, which shall be the tenth part of 2 S: and from 3 in 4, which shall be the tenth part of 3 S: and thus for the others. I say that these sines TS, 1 S, 2 S, 3 S, 4 S, &c. are said to decrease geometrically: since they are diminished in equal times by unequal similar intervals in proportion. With numbers the intervals are TS, 10000000: 1 S, 9000000: 2 S, 8100000: 3 S, 7290000: 4 S, shall be 6561000, etc. I say these numbers for the sines are to be diminished in the same proportion in equal intervals of time, and are said to be decreasing geometrically.

25. ***Thus the moving point approaching the fixed point geometrically has its velocities as the distances from the fixed point.***

As by repeating the preceding diagram, I say that when the point G moving geometrically is at T, its velocity is as the distance TS : and when G is at 1, its velocity is as 1S: and when at 2, its velocity is as 2S, and thus for the rest of the intervals. And thus the proportions of the distances T S, 1 S, 2 S, 3 S, 4 S, &c., will also be the proportions of the velocities of G at the points T, 1, 2, 3, 4, 5, &c., in turn. For the speed of the point is proven to be greater of less as the distances moved forwards are greater or less in the same time intervals. And thus the ratio of such advances, such also by necessity, is the ratio of the velocities : for such is the ratio of the advances in equal times : T 1, 1 2, 2 3, 3 4, 4 5, &c., and of the [p. 15.] distances :T S, 1 S, 2 S, 3 S, 4 S, &c., as we will soon show. Thus by necessity, by having such distances of G from S, as are seen to be T S, 1 S, 2 S, 3 S, 4 S, &c. in turn; such also is the velocity G at the points T, 1, 2, 3, 4, &c. which was to be shown. But it is apparent that for the ratio of the continued advances T 1, 1 2, 2 3, 3 4, 4 5, &c. , such also is the ratio of the distances T S, 1 S, 2 S, 3 S, 4 S, &c: since the differences of a quantity continued proportionally are also continued in the same proportion. But these distances by hypothesis are continued in proportion, and these advances are the differences of these : whereby the advances or distances gone in equal time intervals are surely to be continued in the same ratio as the distances from S.

26. ***The number given to the logarithm of the sine is produced by a point always moving with some given velocity, the same amount the total sine had when it began to decrease geometrically, until it has decreased to the size of the sine given, in the same time.***

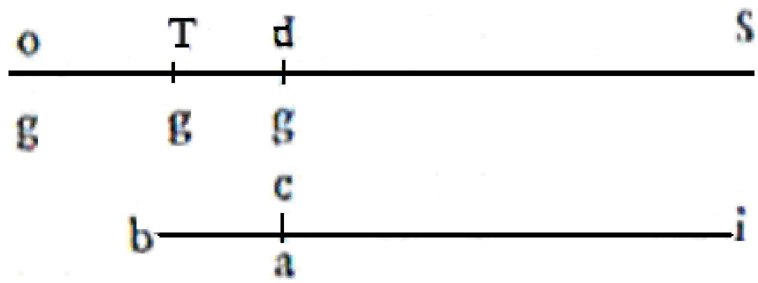


Let the total sine be the line TS, the given sine in the same line dS : surely in certain moments of time *g* has moved geometrically from to T to *d*. And there shall be another line *b i* towards *i* indefinitely, in which *a* moves arithmetically from *b*, with the same velocity which g first had when it was at T : and in the same instances of time *a* proceeds from the fixed point *b* towards *i* as far as to the point *c* : I say that the number measuring the line *bc* is the logarithm of the given sine dS.

27. ***Thus, the logarithm of the total sine is zero.***

For from the diagram, when *g* is at T, making its distance from S the total sine, the arithmetical point *a* begins from *b*, thus has proceeded nowhere. Thus from the definition of the distance, the logarithm of the total sine is zero. [p.16.]

28. ***Hence it also follows that the logarithm of any given sine is greater than the difference between the given sine and the total sine; and less than the difference between the total sine, and a quantity greater to that [given sine] in the same ratio, which is the ratio of the total sine to the given sine. And these differences are said to be the boundaries of the logarithm.***



As by repeating the preceding diagram, and with the line ST extended in o; In order that thus : oS is to TS, as TS to dS. I say that the number bc of the logarithm of the sine dS is greater than Td, and less than oT. For in the amount of time in which *g* is carried from o to T, and with as much time to be carried from T to d (by 24), since the part oT is to o S, as the amount Td is to the line TS, and in this amount of time (by the definition of the logarithm), *a* is taken from b to c: Thus so that oT, Td, and bc are distances traveled in equal amounts of time. But when *g* is moving between T and o, there is a greater velocity than at T, and a lesser velocity between T and d; moreover at T, *g* has the same velocity as *a* (by 26.) It follows that the interval traversed oT is the greater length : and the interval traversed Td is the lesser length that the slower g makes : and since the mean for the interval bc (that a point from its average motion carries out the same motion in the same time) is between these two motions in the same time, or distances, which was to be shown. And thus for the logarithm that bc designates, the boundary oT is greater, and Td is less.

[Note : Thus, using Harriot's inequality signs, invented at about this time, we can now write : oT > bc > Td. Presumably, Harriot only became aware of Napier's existence on the publication of the *Demonstratio* ; being the man he was, he had already come across a version of the log function in his work on navigation, when he had integrated numerically the log of a sine. He had also observed the small error in the tables, about which he kept quiet, as he did about all his other discoveries.

One might be inclined to imagine that the major source of errors in the tables came about from the rounding process that has been used, mainly to save computational time, in going from the second to the third much larger table, in which the logarithm is seen as a counting number, corresponding to a step length on the geometrical line of size 1×10^{-7}; later developments will see interpolations whereby the sines are given equal steps of increase of 10', in which case the logarithms may fall between counting numbers. At present the setting of bounds on the logarithms may seem a rather academic procedure, however, the error margins grow, and eventually the interpolation of the sines can take place from the boundaries of the known logarithms. Incidentally, it was pointed out to me by D. T. Whiteside, some years ago now, that Napier's tables are actually tables of antilogarithms, with the word used in the modern sense (and not as used by Napier to denote the logs of the complements of sines), since the logarithms are given, and the corresponding ratios on the geometric number line are evaluated approximately.

One may also speculate that the occurrence of the number 9900473.5780 in the 21st row of the first column of the third table acted as a god-send for Napier, for he was able to use this number rounded as the first number of the second column : and subsequently for the other numbers in the first column in jumping to the second and subsequent columns. It seems unlikely, viewing his state of health, that Napier would have been able to finish off his tables using the much shorter division of the 1/2000th part as in the second table, which would have taken far longer, without this special number. Indeed, if you look forwards with a spreadsheet as mentioned above, you will not find another number nearly as convenient as this one, that allowed this large jump of almost 1/100 to be possible. Thus, 20 steps each diminishing the sine by 0.0005 takes you from 10,000,000 to 9,900,473.5780 ; this is approximated by the single jump to 9900000.0000, or diminished by 0.01, which introduces an error of around 0.005%, and this limits the accuracy of the tables to about 5 places rather than 7.

We may also consider the role played by the logarithms of the first and second tables : these have been incremented in the simplest possible way. However, a more error free set of tables would result if, instead of rounding table 2 to 99999900.0000000, the first process was again used to reduce the sine almost to the value given, after say 51 passes. In the same way, the first table scheme could be applied to increase the logarithm a little and to bring the argument of the sine closer to 9900000.00000 than 9900473.57803 for the third table. The downside of this is that one no longer has easy numbers to add for the logarithms, which is again time consuming, but not perhaps too much extra an effort, considering the benefit of greater accuracy. One must presume that Napier had considered all of this, and had 'gone for broke' with his original scheme, and stuck with the easy multiples for logarithms and put all the errors onto the sines. Neither the original translator Macdonald or anyone one else that I know of, has expressed these views, which seem remarkable in their simplicity.]

29. ***Thus to show the boundaries of the logarithm of a given sine.***

From what has been proved above, the lesser boundary remains on taking the given sine from the total sine; and the greater boundary is to be found, by multiplying the total sine by the lesser boundary, and with the product divided by the given sine, as in the following example. [p. 17.]

30. ***Thus the first proportional number of the first table, which is* 9999999*, has it logarithm between the boundaries* 1.0000001 and 1.0000000.**

For from above, take 9999999 from the total sine increased by the ciphers, when it becomes unity with its own ciphers for the lesser boundary : this unity increased by its ciphers, and multiplied by the total sine, and divided by 9999999, becomes 1.0000001; or (if you require more accuracy) 1.00000010000001 is found for the greater boundary.

31. ***The boundaries are not sensitive to the different distances, or between these truly any number may be put in place for the logarithm.***

As in the above example, the logarithm of this sine 9999999, thus has 1.0000000 or 1.00000010, or perhaps optimally this number 1.0000005 as the upper boundary : since indeed 1.0000001 and 1.0000000, which in turn are not sensitive to a fraction such as $\frac{1}{10000000}$: therefore these numbers themselves, and whatever may be put between them which are much less, and which are much less sensitive to error, and indeed can be put in place from different logarithms within these bounds.

32. ***Whatever the departure of the sine of the geometric proportion from the total sine, with the logarithm or of one boundary of one sine given, the logarithms of the rest are also given.***

This by necessity is a consequence of the arithmetical increment, the geometrical decrement, and the logarithm definition : accordingly by these, as the sines continue to decrease geometrically thus meanwhile the logarithms continue to increase in equal steps. Thus by decreasing to any sine of the geometric progression, there corresponds its own logarithm of the arithmetic progression by increasing : the first with the first, the second with the second, and so on henceforth.

Thus so that if the first logarithm corresponding to the first sine after the total sine can be given, then the second logarithm will be the double of this, the third the triple, and thus for the others : thus in the end everything becomes known about the logarithms, as will become apparent from the following example. [p. 18.]

33. ***Hence it is possible as a consequence to give the exact logarithms of all the sines in the first table, to be included between the nearby boundaries.***

Since the logarithm of the total sine is 0 (by 27) and the logarithm of the first sine after the total sine, which is 9999999 in the first table, the logarithm (by 30) lies between the boundaries 1.0000001 and 1.0000000 : by necessity the logarithm of the second after the total sine, which is 9999998.0000001, will be continued between the double of these boundaries : between 2.0000002 and 2.0000000 as it were: and the logarithm of the third 9999997.0000003, between the triple of these, obviously 3.0000003 and 3.0000000. And thus for the others, always by being increased equally : the boundaries for the interval of the first boundaries : then you will have completed the logarithms boundaries of all the proportions of the first table. You will be able, by progressing in the same way, if it is desired, to put in place those logarithms with the small insensitive errors according to this order: for the logarithm of the total sine is 0; for the first logarithm after the total sine is 1.00000005 (by 31); for the second, 2.00000010 : for the third, 3.00000015. And so on henceforth.

34. ***The difference of the logarithm of the total sine and of the given sine, is the logarithm of this given sine.***

This is apparent, since the logarithm of the total sine is indeed zero by 27, this with nothing taken from the given logarithm, it is necessary that the whole logarithm itself remains.

35. ***The difference of two logarithms must be added to the logarithm of the greater sine of the same, in order that you can have the logarithm of the lesser sine : and must be taken from the logarithm of the smaller sine, in order that you can have the logarithm of the greater sine.***

This is necessary, as the logarithms increase according to the decrease in the sines, and the logarithm of the greater sine is the smaller. And thus the difference of the logarithms is added to the smaller logarithm, in order that you can have the larger logarithm, though of the smaller sine : and conversely, by taking the difference [p. 19.] from the greater logarithm, then you can have the smaller logarithm though of the greater sine.

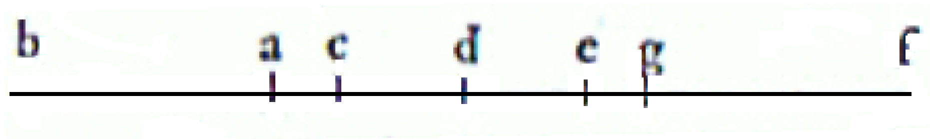
36. ***Similarly, equidistant logarithms are of sines in proportion.***

This follows by necessity the definitions of logarithms and of motion : For since through these, by geometrically decreasing in the same ratio, the arithmetical increment always responds with an equal increment : by necessity similarly with the sines of the proportion, there corresponds equal differences of the logarithms and numbers, and we can conclude the boundaries of the numbers. As with the above example of the first Table, since the proportion is the same between the proportion after the total sine 9999999.0000000, and the third 9999997.0000003 : to that which there is between the fourth 9999996.0000006, and the sixth 9999994.0000015. Thus the logarithm of the first is thus 1.00000005, that differs from that logarithm of the third 3.00000015, by the same difference, by which the logarithm of the fourth 4.00000030, differs from that logarithm of the sixth proportional 6.00000030. Also there is the same equal ratio between the differences of the boundaries of the logarithms in turn : it is seen as with the smaller and larger between these, to which the sines are similarly to be in proportion.

37. ***As of three sines continued in geometric proportion, the square of the mean is equal to the product of the extremes taken in turn : Thus with their own logarithms, double the mean is equal to the sum of the extremes. Thus for any two given logarithms, the third will become known.***

Since the ratio of these three sines is the same between the first and the second, as between the second and the third: Thus, by what has been said above, there is the same

difference between the first and the second, as between the second and the third. For example, let the first logarithm be expressed by the line b c, the second by the line bd, and the third by the line be : and to be understood in this way by the single line b c d e, and the differences c d and d e are equal : the mean of these, b d, is doubled, with this line being produced from e to f, in order that b f is thus the double of b d. I say that b f is equal to the sum of both the lines; that is, of the first logarithm b c and of the third b e: Indeed from the equality, b d and d f are equal, and by taking c d from b d, and d e from d f : then by necessity there is left b c and e f equal to each other. And thus the total length b f is equal to both b e and e f : and hence equal to the sum of b e and b c, which had to be shown. [Or, b c = b d – cd and b e = b d + d e, and hence b c + b e = b d + d f = b f, as required; the same kind of argument is used in the next prop.] Thus the rule follows : if, of three given logarithms you should double the mean, and hence take the extreme given, then the other extreme sought will become known : and if you add the given extremes, and you halve this sum, then you will produce the mean of the extremes.



38. ***The product of the means of four geometrical proportions is equal to the product of the extremes: Thus regarding their logarithms, the sum of the means is equal to the sum of the extremes. Thus with any three of these logarithms given, the fourth becomes known.***

Since the ratio of four numbers in proportion is the same between the first and the second as between the third and the fourth : Thus, (from the second but last proposition) the difference between the first and second logarithms is equal to the difference between the third and the fourth logarithms of the numbers. Thus such quantities written above are taken on the line b f, as here, of which ba refers to the first logarithm, bc the second, be the third, and bg to the fourth; with the differences ac and eg equal: Thus as d put in the middle of ce, by necessity is also put in the middle of ag. I now say that the sum of the second and third logarithms, bc and be: is equal to the sum of the first and fourth logarithms, ba and bg. For since (by previous propositions) twice bd or bf is equal to the sum of bc and be : since the differences of these taken with bd, are seen to be the equal amounts cd and de. By the same reason, likewise bf is equal to the sum of ba and bg : since the differences of these from bd, are seen to be also the equal amounts ad and dg. Since the sum of ba and bg, and the sum of bc and be, are thus both equal to twice bd, which is equal to bf: and hence they are equal to each other, which was to be shown. Thus the rule follows, if of four logarithms, from the sum of the given extremes, you take away either of the known means, there is left the logarithm of the mean that is sought: and if from the sum of the logarithms of the known means you take away either of the known logarithms of the extremes, then there is left the sought logarithm of the extreme sought.

39. ***The difference of the logarithms of two given sines lies between two boundaries: the total sine is to the greater boundary as the lesser sine is to the difference of the sines; & the total sine is to the lesser boundary as the greater sine is to the difference of the sines.***

Let TS be the total sine, while dS is the larger sine and eS the smaller sine of the two sines given. Beyond ST the distance TV is marked by the point V to the left, according to the ratio : ST is to TV, as the lesser sine eS is to the difference of the sines de. Then on the right side of T towards S, the point c designates the distance Tc, according to the ratio : TS is to Tc, as the larger sine dS is to the difference of the sines de. I say that the difference of the logarithms corresponding to the sines dS and eS, are to be set up between the major and minor boundaries VT and Tc. [Thus, ST/TV = eS/de; and TS/Tc = dS/de;]

For since by hypothesis, as eS to de, thus TS is to TV; and as dS is to de, thus TS is to Tc: likewise also (from the nature of the proportions) two conclusions follow :
In the first place, that V S has the same ratio to T S, [p. 22.] as TS has to cS.
In the second place, that TS has the same ratio to cS that dS has to eS.
[Hence, (1) VS/TS = TS/cS and in turn, (2) TS/cS = dS/eS.
For, from ST/TV = eS/de or TV/ST + 1 = de/eS + 1, we have VS/TS = dS/eS = TS/cS;
Likewise on taking 1, we have : VT/TS = de/eS; dS/de = TS/Tc .]
And hence (by 36) the difference of the corresponding logarithms for the sines dS and eS, is equal to the difference of the corresponding logarithms of the total sine T S, and the sine cS. [i. e. log dS – log eS = log TS – log cS; or log eS – log dS = log cS, as the logarithm of the total sine is zero] But this difference (by 34) is the logarithm of the sine cS ; and this logarithm lies between the major boundary VT, and the minor boundary Tc (by 28): since by the first conclusion now mentioned, the greater boundary VS has the same ratio with the total sine TS, as TS to cS. Hence by necessity the difference of the corresponding logarithms for the sines dS and eS, is established to lie between the greater boundary VT, and the lesser boundary Tc, which had to be shown.

40. ***To show the different boundaries between the logarithms of two given sines.***

When by the preceding proposition, the smaller sine is to the difference of the sines, in the same ratio as the total sine is to the greater boundary of the difference of the logarithms [*i. e.* TS/VT = eS/de]: and the greater sine has the same ratio to the difference of the sines as the whole sine has to the smaller boundary [*i. e.* dS/de = TS/Tc] : then it follows from the nature of the proportions, that by multiplying the total sine by the difference of the given sines, the greater boundary TS arises from the product on division by the smaller given sine : and from the product on division by the larger sine, there arises the smaller boundary Tc.

EXAMPLE.

For, let the larger of the given sines be 9999975.5000000, and the smaller moreover be 9999975.0000300 : the difference of which 0.49997000 is multiplied by the total sine (first with eight ciphers added to both after the decimal point for the sake of demonstration, although seven will suffice.) that hence will produce 0.49997122, if it is divided by the larger sine, to wit by 99999975.5000000, with 8 figures after the decimal point for the lesser boundary. Thus there arises 0.49997124 for the greater boundary, when the same product is divided by the lesser sine, to wit, by 9999975.0000300 : [p. 23.] between which, as has been shown, the difference of the given logarithms have been established to lie. But since the extension of this fraction to 8 places beyond the decimal is to a greater accuracy than that required, especially with these sines only 7 places are put after the decimal point : thus by deleting the eighth or final figure from both boundaries, each boundary can be established by the fraction 0.4999712 for the one difference of the logarithms, without the smallest scruple of an error.

41. ***To exhibit the logarithms of sines, or of natural numbers, that are not present in the first tables of proportions, but falling nearby or between the values of these; or to at least find the boundaries of these within negligible distances.***

For a given sine, first note down the nearest sine in the tables that is either greater or smaller: whereby according to 33 the logarithms of the boundaries of this tabulated sine are to be found and noted : then (according to the above propositions) the boundaries of the differences between the logarithms of the given sines and the tabulated sine, or both, or (since they are almost equal, as the above example makes clear) either of these is sought. Now these are found, or add either of these to those noted boundaries, or subtract from these (according to 8, 10, and 35) since the given sine will be less or greater than that closest to it in the given table: and hence the numbers produced are the closest boundaries within which is included the logarithm of the given sine.

EXAMPLE.

For let the given sine be 9999975.5000000, to which the nearest sine in the Table is 9999975.0000300 less than that given : the boundaries of the logarithm of this sine are (by 33) 25.00000025 and 25.0000000 : then (from the previous propositions) the difference between the logarithms of the given line and of the table is 0.4999712 : as (by 35) take from these boundaries, which are the boundaries of the smaller sine, and there comes abut 24.5000313 and 24.5000288, the boundaries sought of the given sine [p. 24.] 9999975.5000000 : whose logarithm can be any number within the boundaries, but optimally can be taken as 24.5000300 (by 31) without appreciable error.

ANOTHER EXAMPLE.

Let the given sine be 9999900.0000000,and the tabulated sine closest to that is 9999900.0004950: and the boundaries of the logarithm of this sine are 00.00000100 and 100.0000000 (by 33) : Then the difference between the logarithms of these sines is, by the preceding, 0.0003950 , that is added (by 35) to the above boundaries, and these make 100.0005050 for the greater boundary and 100.0004950 for the smaller boundary, and the logarithm of the sine sought lies between these boundaries.

42. ***Hence it follows that all the logarithms of all the proportional numbers of the second table can be given precisely well enough: or can be included between known boundaries with fractional differences with negligible errors.***

As in the preceding example, when the logarithm of this sine 9999900 (which is the first proportional of the second Table) can be shown to be between these boundaries 100.0005050 and 100.0004950 : by necessity the logarithms of the second proportionals, by 32, will be between these boundaries 200.0010100 and 200.0009900 : and of the third proportionals, between these boundaries 300.0015150 and 300.0014850, etc. And at last the logarithm of the final sine of the second Table, which is 9995001.222927, lies between these boundaries 5000.0252500 and 5000.0247500. Now from these boundaries that you are now found, by 31, you can find the logarithms of these sought.

43. ***To show how to find the logarithms of these sines or natural numbers not in the second Table of proportions, but nearby or falling between these values: or to be included between known boundaries with negligible fractional differences.***

For the sine given, the sine of the second Table nearest, either greater or less is noted : and the logarithms of the boundaries of this tabulated sine [p. 25.] are sought by the previous propositions : then by the proportion rule, to seek the fourth with a proportion to the total sine itself, having the same ratio as the lesser to the greater of the given sine and of the tabulated sine. In one way that can be brought about by multiplying the smaller of the given and tabulated sines by the total sine, and by dividing by the greater of the given and tabulated sines. With another easier way, by multiplying the difference of the given and tabulated sines by the total sine, and by dividing the product by the greater of the given and tabulated sines, and the quotient taken from the total sine.

But since the logarithm of this fourth proportional (by 36) still differs from the logarithm of the total sine, as much in turn as the logarithms of the given and tabulated sines differ : And since also the difference of these, is the same as the logarithm of the fourth proportion by 34 : Thus the logarithms of the fourth proportional, by the penultimate proposition are to be sought from the first Table, and add to the logarithms of the tabulated boundaries, or subtract from these, according to 8, 10, and 35; according to whether the tabulated sine were greater or smaller than that given, and the boundaries of the logarithms of the given sine are produced.

EXAMPLE.

For let the given sine be 9995000.000000, and the sine of the second Table nearest to this sine is 9995001.222927 : and the logarithms of the boundaries of this by the preceding propositions are 5000.0252500 and 5000.024750. Hence the fourth difference can be found by either of the methods discussed above, and this becomes 999999.7764614; the logarithms of the boundaries are found from the first Table by 41, and they are 1.2235387 and 1.2235386 : which on being added to the above boundaries, by 8 and 35, to give the boundaries for the given logarithms 5001.2487888 & 5001.2482886. Hence the mean between these, which is 5001.2485387, can best be taken (from 31) for the logarithm of the given sine 9995000 and can be put in place without sensible error.

44. ***Hence it follows, that the logarithms of all the proportions of the first column of the third Table can be given with enough accuracy :*** [p. 26.] ***or that can be placed between the differences of known boundaries with negligible fractions.***

For since by the preceding, the logarithm of the first sine after the total sine, which is of 9995000.000000 (which is the first sine below the total sine, from the proportional numbers of the first column of the third Table) is 5001.2485387, without appreciable error : of the second proportional number, to wit 9990002.5000, the logarithm (by 32) is 10002.4979774. And thus henceforth for the rest, by progressing as far as the final sine of the column, 9900473.57808 : of this, by comparison with the ratio, the logarithm will be 100024.9707740 , of which the boundaries are 100024.9657720 and 100024.9757760.

45. ***To find the logarithms of the natural numbers, or of the sines that are not themselves in the proportions of the first column of the third Table, but are nearby or fall between those : or that can be placed between the differences of known boundaries with a negligible fractions.***

For the given sine, the nearest sine of the first Column of the third Table, either smaller or the larger is noted ; the boundaries of the logarithms of this tabulated sine are sought according to the preceding propositions : then the fourth proportion is found, that has the same ratio to the total sine as the given sine to the larger or smaller of the tabulated sines, by one of the methods described in the penultimate proposition: the logarithms of the boundaries of this fourth proportion thus are found from the second Table, then either add those found to the boundaries of the above boundaries of the tabulated sine, or subtract from these (by 8, 10, and 35) and the logarithms of the boundaries of the given sine are produced.

EXAMPLE.

For let the given sine be 9900000, and the proportion of the sine in the first column of the third Table closest to this is 9900473.57808, the logarithms of the boundaries of this by the preceding are 100024.9657720 and 100024.9757760. Hence the fourth proportion is 9999521.6611820, and the logarithms of the boundaries of this [p. 27.] (by 43, taken from the second Table) are 478.3502290 and 478.3502812 : with which boundaries added to the boundaries of the above tabulated sine (by 8 and 35), the boundaries 100503.3260572 and100503.3160010 arise, between which by necessity the logarithm sought falls. Thus the mean number between these, which is 100503.3210291, can be put in place for the logarithm of the given sine 9900000, without sensible error.

46. ***Hence it follows that the logarithms of all the proportions of the third Table can be given without appreciable error.***

For since, (from the preceding), 100503.3210291 is the logarithm of the first sine of the second column, which is 9900000, and the other first sines of the remaining columns are progressing in the same proportion; by necessity (from 32 and 36) the logarithms of the same numbers always increase by the same difference, and hence by adding 100503.3210291 to the preceding logarithm, the following logarithm is made. Hence by thus having the first logarithms of any column, and by the preceding penultimate proposition, with all the logarithms of the first column given ; then you are able, if you wish, to put in place all the logarithms of the same column at the same time, by always adding to the above logarithm, this difference of logarithms 5001.2485387, in order that the lower nearby of this column of logarithms is made : Or you may wish to put in place all the orders of the logarithms at the same time, obviously all the second logarithms for each column ; then all the third, then all the fourth, and thus the remaining are to be set up, by always adding 100503.3210291 to any of the logarithms of the preceding column, in order that the logarithm of the same following order comes about. Or indeed by the other method, all the logarithms of all the table of proportions is put in place; of which the last, 6934250.8007528 agrees as the logarithm of the sine 4998609.4034 .

47. ***The logarithms are to be ascribed to all their natural numbers of the third Table, in order that the third Table can be completed :*** [p. 28.] ***and that henceforth we will always call the Radical table.***

This account of the Table is done by setting up the columns with the number and order described by 20 and 21: and by dividing every column into two series :

Tables of Radicals.

First column.		Second column.	
Natural nos.	Logarithms	Natural nos.	Logarithms
10000000.0000	0	9900000.0000	100503.3
9995000.0000	5001.2	9895050.0000	105504.6
9990002.5000	10002.5	9890102.4750	110505.8
9985007.4988	15003.7	9885157.4238	115507.1
9980014.9950	20005.0	9880214.8451	120508.3
&c. as far as	&c. as far as	&c. as far as	&c. as far as
....	....	....	....
9900473.5780	100025.0	9801468.8422	200528.2

and so on for the others, until

Column 69.

Natural nos.	Logarithms.
5048858.8900	6834225.8
5046334.4605	6839227.1
5043811.2932	6844228.3
5041289.3879	6849229.6
5038768.7435	6854230.8
&c. as far as	&c. as far as
....	....
4998609.4034	6934250.8

[p. 29.] the first of which contains the geometrical proportions that we call the sines and natural numbers; while the second contains their logarithms progressing in arithmetical intervals. However, two things have to be kept in mind (for the sake of putting these together) : In the first place, since for all of these logarithms, one digit past the decimal point is sufficient, and the other six more recent are now rejected : yet which if in the beginning you might have ignored : then the error by frequent multiplication in the first table, would have increased to an intolerable level in this third table. In the second place, if the second figure after the decimal point is greater than four, then the first figures, which alone remains after the point, is to be increased by one. As for 10002.48, etc. it is more correct to put 20002.5 than 10002.4 : and for 10003.5001, more correctly we put 1000.4, rather than 1000.3. And thus with that put in place, the Table of Radicals can now proceed to be made with what is allowed.

48. ***Now with the table of radicals completed, we can excerpt the logarithm numbers from this little table.***

Since indeed the two previous tables were of use in setting up the third table ; Thus this third table of radicals can be put to use as the principal table of logarithms, with great ease and without sensible error.

49. ***To most easily find the logarithms of the sines greater than 9996700.***

This can be done, by the subtraction alone for the given sine from the total sine. For by 29, the logarithm of the sine 9996700, is between the boundaries 3300 and 3301; which boundaries indeed (since they are in turn different only by one) truly from its own logarithm cannot differ by a sensible error, as it is seen that at most the boundaries differ by one. Thus the lesser boundary 3300, which is obtained by subtraction alone, can be taken for the logarithm. By necessity the same reasoning applies for all sines greater than this.

50. ***To find the logarithms of all the sines taken within the boundaries of the table of radicals.***

Of the given sine, and of that tabulated nearest to it, multiply the difference [p.30.] by the total sine, and divide the product by the easiest divisor, which is either the given sine, or the closest tabulated sine, or some other made up between the one and the other; and the difference of the logarithms is produced of either the greater or lesser or of some intermediate boundaries (by 39), of which there is a negligible difference between the logarithms of the boundaries and the true value, on account of the closeness of the numbers of the Table. Therefore add this produced by any of these (by 35), to the logarithm found tabulated in the table, if the given sine is less than the tabulated sine; otherwise any number produced is taken from the logarithm of the tabulated sine, and there arises the logarithm of the given sine sought.

EXAMPLE.

For, if the given sine is 7489557, whose logarithm is sought. The tabulated sine closest to this is 7490786.6119, from this take that with the ciphers added, thus : 74895557.0000, giving the remainder 1229.6119; which multiplied by the total sine, and on division by the easiest number, which is either 7489557.0000, 7490786.6119, or some optimal number situated between these, such as by 7490000, and from the easiest division there comes about 16401 : which (since the given sine is less than the tabulated one) add to the tabulated logarithm of the sine, as is seen to be 2889111.7, and there is produced 2890751.8, which has the same value as $2890751\frac{4}{5}$: but since by the rules of the table no fractions are admitted, nor any number after the point, and we put 2890752 for that number, which is the logarithm sought.

ANOTHER EXAMPLE.

Let the given sine be 7071068.0000; and the sine in the table closest to that is 7970084.4434 ; the difference of the sines is 983.5566; for which multiplied by the total sine, and the product to be optimally divided by 7071000, which is between the given sine and the tabulated sine, and there hence arises 1390.0 : which (since the given sine exceeds the tabulated sine nearest to it) subtracted from the logarithm found for the number tabulated and found in the table, [p. 31] obviously from 3467125.4, and there remains 3465734.5. Hence 3454735 is put for the logarithm sought for the given sine 7071068. Thus this wonderful ability of choosing the divisor eases the calculations.

51. ***All the sines in the duplicate proportion have 6931469.22 for the difference of their logarithms.***

Since indeed the sine of any quantity to its own half is the same ratio that the total sine has to 50000000 : therefore (by 36) the difference of the logarithms of any sine and its half, is the same as the difference of the logarithms of the total sine and of its half 50000000. But the difference of the logarithm of the total sine, and the logarithm of the sine 50000000, is the same as the logarithm of the sine 50000000 (by 34), and of this 50000000, the logarithm (by the preceding propositions) is 6931469.22, and this difference is the difference of all the logarithms, the sines of which are in the duplicate ratio : and as a consequence the double of this, truly 13862938.44, is the difference of all the logarithms, the sines of which are in the quadruple ratio; and the triple of this, clearly 20794407.66, is the difference of all the logarithms, the sines of which are in the octuple ratio.

52. ***All the sines in the decuplet ratio have 23025843.34 for the difference of their logarithms.***

For from the penultimate proposition, the sine 8000000 has the logarithm 2231434.68 : and the given difference between the logarithms of the sines 8000000, and of its eighth part 10000000, is 20794407.66 : Hence by addition they make 23025842.34, for the logarithm of the sine 10000000 : and since this sine is the tenth part of the total sine, all the sines in the decuplet ratio have this same difference 23025843.34 between their logarithms, by the same cause and reason, that now we have set out in the penulimate proposition, which had to be proven. And as a consequence, to the hundredth proportion there corresponds the duplicate of this logarithm, which is 46051684.68, for the difference of the logarithms [p. 32]: And the triple of this, which is 69077527.02, is the difference of all the logarithms, the sines of which are in the times one thousand ratio. And thus for the 10000th ratio, and the others, as below.

[Thus, we note that as the sines decrease in size, their logarithms increase in size : a consequence of the exponential decay nature of Napier's logarithms; a criticism leveled at the logarithms at the time was that small numbers had large logarithms, which was not always convenient; again, the calculation of a logarithm not present in the table might consume more time than an ordinary calculation of the problem. Napier was aware of all this, and to paraphrase what he said : Nothing can be created perfect at first.]

53. ***Thus all the sines in the ratio composed from the duplicate and the decuplet, their logarithms have the differences formed from the difference 6931469.22, and the difference 23025842.34, respectively.***

As can be seen in the following table. the given proportions of sines

Given proportions of the sines.	Corresponding differences of the logarithms.	Given proportions of the sines.	Corresponding differences of the logarithms.
× 2	6931469.22	× 8000	8987134.68
× 4	13862938.44	× 10000	92103369.36
× 8	20794407.66	× 20000	99034838.58
× 10	23025843.34	× 40000	105966307.02
× 20	29957311.56	× 80000	112897777.70
× 40	36888780.78	× 100000	115129211.70
× 80	43820250.00	× 200000	122060680.92
× 100	46051684.68	× 400000	128992150.14
× 200	52983153.90	× 800000	135923619.36
× 400	5991462312	× 1000000	138155054.04
× 800	66846092.34	× 2000000	145086523.26
× 1000	69077527.02	× 4000000	152017992.48
× 2000	76008996.24	× 8000000	158949461.70
× 4000	82940465.46	× 10000000	161180896.38

54. ***To find the logarithms of all the sines excluded beyond the boundaries of the table of radicals.***

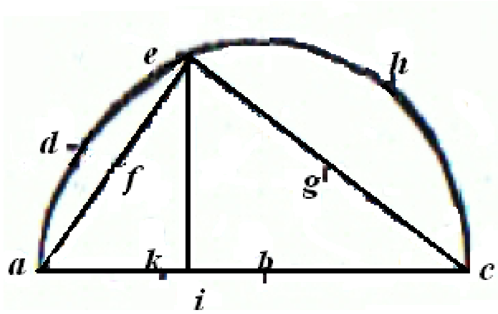
This can easily be done, by multiplying the given sine by 2, 4, 6, 8, 10, 20, 40, 80, 100, 200 : or by any other proportional number expressed from this table, until a number is produced, which is contained within the limits of the radical table. Now the logarithm of this is sought from the table (by 50), to this found add finally the difference of the logarithms that the table indicates in agreement by the previous multiplication.

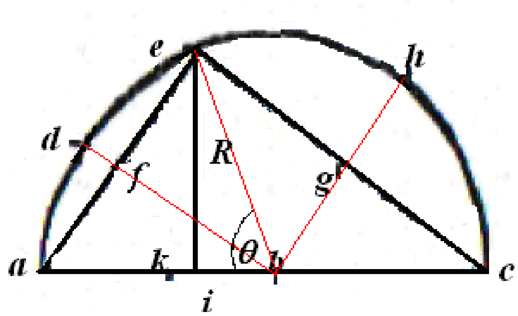
EXAMPLE.

The logarithm of the sine 378064 is sought; since this lies outside the boundaries of the table of radicals, it is multiplied by some other proportion such as 20 from the preceding table, and it becomes 7561280, now the logarithm of this can be found within the table (by 50), and it becomes 2795444.9, to which add the difference found in the table agreeing with the 20th proportion [which effectively divides by 20 again], which is 29957311.56, and they make 32752756.4. Thus 32752756 is the logarithm sought for the given sine 378064.

55. ***As half the total sine is to the sine of half of some given arc; thus the sine of the complement of the same half arc is to the sine of the whole arc.***

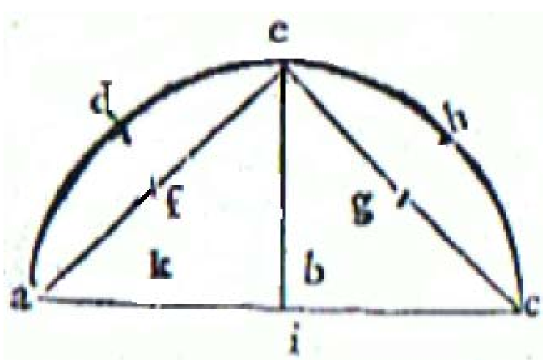
Let a b be the total sine, it is doubled and becomes a b c; on this diameter make the semi-circle, in which the arc designated by a e is divided in two equal parts at d: hence the complement of the half of this arc d e, is extended from e towards c, and which is the arc e h, and to this the arc h c by necessity is equal : since the quadrant d e h is equal to the quadrant from the sum of the arcs a d and h c. Hence, the line e i is drawn perpendicular to a i c, which likewise is the sine of the arc a d e; and the line a e of which the half f e is the sin of the arc d e, which is half of the arc a d e ; and the line e c, the half of which e g, is the sine e h, and likewise is the sine of the complement of the arc d e ; moreover the half of the total sine a b is a k. I say that, as a k has the ratio to e f, thus e g has the same ratio to e i : indeed the two triangles c e a and c i e are equal-angled : since i c e, or a c e is a common angle for both, and the other c i e and c e a is right, with the one by hypothesis, and the other as it is an angle lying on the circumference of the semi-circle. Thus, as a c the hypotenuse of the triangle c e a, is to the lesser side a e of that triangle ; thus c e the hypotenuse of the triangle c i e is to the lesser side e i of that triangle. [Thus, a c/a e = c e/e i.] And it now follows that since the whole a b is to the whole a e, as e g is to e i;





[Thus, a b/a e = e g/e i, on dividing each hypotenuse in half]: we can conclude, by necessity, that a k which is half a b, has same ratio to the half of a e or f e, as e g has to e i, which was to be proven. [It is required to show that a k/f e = e g/e i; which is the ratio of a b/2 on a e/2; from which the result follows. In modern terms, we can write this proportionality as : $\frac{R/2}{R\sin\theta/2} = \frac{R\cos\theta/2}{R\sin\theta}$.]

56. ***Twice the logarithm of the arc of 45 degrees is equal to the logarithm of half the total sine.***



With the above diagram repeated, as in this case a e and e c are equal to each other; i falls on b, and e i is the total sine, and e f is equal to e g : and the sine of any of these is the sine of 45 degrees. And since (by the preceding), the proportion a k (that is half the total sine) to e f , (which is the sine of 45 degrees) : that also is the proportion e g (also the sine of 45 degrees), to e i (now the total sine). Hence (by 37, [relating the mean and extremes of a ratio in terms of logarithms]) twice the logarithm of the sine of 45 degrees, is equal to the logarithms of the extremes, obviously of the total sine, and of the half of this. But the sum of these logarithms is only the logarithm of one of these, obviously of the half of the total sine, since the logarithm of the total sine is zero, by 27. Therefore by necessity twice the logarithm of the sine of the arc of 45 degrees is equal to the logarithm of half the total sine, which was to be shown. [And which is a very circuitous way of showing that the sine of 45 degrees is $1/\sqrt{2}$! Note that in this and the following propositions, Napier talks about the logarithm of an arc, where he really means the logarithm of the sine of the arc, which is the logarithm of a straight line and not of the arc, which is part of the circumference of the circle. We have put in the word 'sine' in what follows.]

57. ***The sum of the logarithms of half the total sine and the logarithm of the sine any other arc, is equal to the sum of the logarithms of the sine of half of this arc and of the sine of the complement of half of this.*** [p. 35] ***Hence it is possible to find the logarithm of the sine of half of this arc, from the logarithms of the other three given.***

Since by the last but one proposition, half the total sine is proportioned to the sine of half of some other arc, as the sine of the complement of the same half arc is to the sine of the whole arc: Thus, (by 38) the sum of the logarithms of the two extremes of the ratio, obviously equal to the logarithm of half the whole sine and the logarithm of the sine of the given arc, is equal to the sum of the logarithms of the means, clearly the logarithm of the sine of half the given arc and the logarithm of the sine of half the complement of the given arc. Thus by the same proposition 38, if you add the logarithm of half the total sine (by 51, or as found previously) to the logarithm of the sine of any given whole arc, and then you take away the logarithm of the sine of half the complement of the previously given arc, then there remains the logarithm of the sine of half the same arc sought : which were to be shown.

EXAMPLE. Let the logarithm of half the total sine be 6931469 (by 51), and let the whole arc be 69 degrees and 20 minutes, the logarithm of which 665143, has been given : half of the whole arc is 34 degrees and 40 minutes, and the logarithm of the sine of this arc is sought. The complement of half of this arc is 55 degrees and 20 minutes, and the logarithm of the sine of this is given as 1954370 : Thus by adding 6931469 to 665143, the sum becomes 7596612 : from which take 1954370, and there is left the logarithm sought 5642242, the sine of the arc 34 degrees and 40 minutes.

58. ***With the logarithms of the sines of all the arcs not less than 45 degrees given, all the sines of the arcs less than 45 degrees can be found easily.***

From the logarithms of all the sines of the arc not less than 45 degrees given by hypothesis, you can find by the above, the logarithms of all the remaining sines of the arcs decreasing as far as twenty two and a half degrees. From this now may be found similarly, all the logarithms of the sines of the arcs as far as 11 degrees and 15 minutes. And from these again, the logarithms of all the sines of the arcs as far as 5 degrees and 38 minutes. And thus henceforth as far as the first minute.

59. ***To put in place a Table of Logarithms.***

Forty five rather long pages are to be prepared, in order that besides the upper and lower margins sixty lines of numbers are to be accommodated. All of the pages are to be divided by 20 equally spaced transverse lines : and all of the distances are divided to take three rows of numbers. From this all of the pages are divided into seven columns by vertical lines, with two lines put in place between the second and third column, and between the fifth and the sixth : between the others only a single line is put in place. The first page at the top left hand side, above the three columns, has this title written : **| 0 *Degrees* |**, and there is written below at the right-hand side under the three final columns thus **| 89 *Degrees*** |. The second page has written on top to the left-hand side thus **| 1 *Degree* |**, and written underneath thus **| 87 *Degrees* |**. And by proceeding thus with the other pages, in order that one less than the quadrant or 89 degrees is always filled up by the addition of the degrees above and below. Then the first column has this title for the individual pages written above : **| *Minutes of the above degrees* |**. The second column has this title written above: **| *Sines of left-hand arcs*|**. The third column has this title written above **| *Logarithms of the left-hand sines*|**. The fourth column has this title written above : **| *Differences between the logarithms of the complements*|**. The fifth column has this title written below :**| *Logarithms of the right-hand arcs*|**. The sixth column has this title written below : **| *Sines of the right-hand arcs*|**. The seventh column has this title written below : **| *Minutes of the degrees written below*|**. [p. 37.] There upon the number of the minutes can be put in place in the first column progressing from 0 to 60. Also in the seventh column the number of minutes can be put in place, decreasing from 60 to 0 : with these according to the rule, in order that any of the two numbers in the first and seventh column in the same line opposite each other, make a sum of one degree or 60 minutes. E. g. , 0 to 60, 1 to 59, 2 to 58, 3 to 57, etc. are put in place opposite each other. Also; between any two of the twenty horizontal lines, three numbers are contained in any interval of any column. In the second column are put in place the numbers of the sines corresponding to the above degrees and minutes from the left-hand side put in the same line. Also in the sixth column are put the numbers for the sines corresponding to the sines and minutes below from the right-hand side, according to RHINHOLD'S Table, or from some table more precise. With this completed, with the reckoning of the logarithms of all the sines between the total sine and half of this, according to 49 and 50 : the rest of the sines are computed by prop. 54. Or otherwise, and much more easily and precisely, the logarithms of all of the sines between the whole sine and the sine of 45 degrees is computed by the same propositions 49 and 50 : from which you can now find all of the logarithms of the remaining sines of the arcs less than 45 degrees, you can easily find from what has gone before. With all the logarithms whatever computed, in the third column you arrange the logarithms corresponding to the above sines, with the minutes from the left side, and with these in the fifth column the logarithms are arranged corresponding to the degrees below, and with the minutes from the right side, and put on the same line as these for the right-hand sines. Finally the middle column is thus done : where the logarithm of some right number on the right, is taken from the logarithm of some number on the left on the same line, with the difference of these noted on the same line between these two, then you will have completed the whole middle column. We have ourselves computed this table to the nearest minute, [p. 38.] and we leave the correction and improvement of the Table of sines to the learned who have more leisure.

Summary of the Table of Logarithms,
constructed otherwise.

60. ***Since occasionally the logarithms found by 54, are different from those found those logarithms found by 58; as the logarithm of the sine of* 378064*, is found by the one method to be* 32752756*, and by the other indeed to be* 32752741 *;it proves that there is corruption in certain places in the table. On account of which I consult with the learned (with whom perhaps there will be an abundance of pupils and of computers*** [of the human variety] ***) in order that a more exact and with more numbers can be produced, in order that the total sine shall be 100000000, obviously with eight ciphers after the figure of unity, as the former total sine contains only seven digits. Then in order that our first table may contain one hundred numbers, progressing in that proportion, which is thus between the new total sine, and the sine one less than that, as between* 100000000 *and* 99999999.**

***The second Table also should contain one hundred numbers, in that proportion, which is between this new sine, and the number less to that by one hundred , obviously between* 100000000 *and* 99999900.**

The third table which is called the table of radicals, has thirty five columns, and one hundred numbers are contained in every column. One hundred numbers can progress in that ratio, which is the ten thousandth, to the number less by one to that, clearly* 100000000 *to* 99990000*. Thirty-five numbers in that proportion, which is 100 to 99 or* 100000000 *to* 99000000*, standing in the first row of the first column, or the second, or third, etc, are to progress among themselves to fill all the columns. And with these logarithms found and put in place, the rest of the preceding rules are observed. And from the completed radical table, all the logarithms of the sines between the total sine and the sine of 45 are found most precisely by 49 and 50 : and from the logarithm of the sine of the arc of 45 degrees doubled, you can have the logarithms of half the total sine by 56. And finally from these now found, the other logarithms from the last but one preceding propositions you can now find ; which can be set down in the row of the table by the preceding, and the table is made, surely the most outstanding of all Mathematical Tables, and prepared for the most important uses.

End of the construction of the table of logarithms.

Admonitio.

Quum hujus Tabulæ calculus, qui plurimum Logistarum ope & diligentia perfici dibuisset, unius tantum opera & industria absolutus sit, non mirum est si plurumi errores in eam irrepserint. Hisce igitur sive à Logistæ laßitudine, sive Typographi incuria profectis ignoscant, obsecro, benevoli Lectores : me enim tum infirma valetudo, tum rerum graviorum cura præpedivit, quo minus secundam his curam adhiberem. Verùm si huius inventi usum eruditis gratum fore intellexero, dabo fortasse brevì (Deo aspirante) rationem ac methodum aut hunc canonum emendandi, aut emendatiorem de novo condendi, ut ita plurium Logistarum diligentia, limatior tandem & accuratior, quàm unius fieri potuit, in lucem prodeat.

Nihil in ortu perfectum.

MIRIFICI
LOGARITHMORUM
Canonis Constructio*;*
Et eorum ad naturales ipsorum numeros habitudines;
UNÀ CUM
Appendice, de alia eaque præstantiore Logarithmorum specie condenda.

QUIBUS ACCESSERE
Propositiones ad trianguli sphærica faciliore calculo resolvenda :
Unà cum Annotationibus aliquot doctissimi* D. Henrici Briggii*, in eas memoratum appendicem.
Autore & Inventore ***Ioanne Nepero,*** Barone ***Merchistoni, &c***. Scoto.

EDINBVRGI,
Excudebat ANDREAS HART.
ANNO DOMINI 1619.

LECTORI MATHESEOS STUDIOSOS.

Ante aliquot annos (Lector Philomathes) Mirifice Logarithmorum Canonis usum, memoriæ semper colendæ parens publici Iuris fecerat; ejus verò syntaxin ac creandi methodum, ut ipse monuit Pag.7 & ultima Logarithmorum, certo consilio Typis committere noluit; donec quodnam esset eorum, qui in hoc doctrinæ genere versati sunt, de hoc Canone Indicium ac censura exploratum habuisset. Mihi verò, post ipsius ex hac vita commigrationem certis tecmeriis constat, Methematicarum disciplinarum peritissimos novum hoc inventum plurimi facere; & nihil iis gratius accidere posse, quàm si Mirifice hujus Canonis constructio, aut ea saltem, quæ ipsi aliquid lucis affere possint, publicæ utilitatis gratia in lucem prodeant. Quamvis igitur mihi probè perspectum sit, ipsum authorem huic opusculo extremam manum non imposuissè; feci tamen quantum in me fuit, ut horum honestissime desiderio satisfieret, eorumque studiis præsertim qui imbecilliores sunt, & in ipso limine hærere solent, hac in parte consuleretur. Nec dubito, quin hoc opus posthumum multò perfectius ac elimatius in lucem prodisset, si ipsi authori patri charissimo (in quo, ex optimorum hominum sententia, inter alia præclara hoc eximii eminebat, res difficillimas methodo certa & facili, quam paucissimis expedire) Deus longiorem vitæ usuram concessisset, Habes igitur (Lector benevole) in hoc libello, doctrinam constructam Logarithmorum (quos hic numeros artificialles appellat; hunc enim tractactù, ante inventam Logarithmorum vocem, apud se per aliquot annos conscriptum habuerat) copiosissimè explicatam; in qua eorum natura, symptomata, ac variæ ad naturales eorum numero habitudines perspicuè demonstrantur. Visum est etiam ipsi syntaxi subnectere Appendicem quandam, de alia Logarithmorum specie multò præstantiore condenda, (cujus, ipse inventor in Epistola Rabdologiæ *suæ præfixa meminit)& in qua Logarithmus unitatis est 0. Hanc loco ultimo ultimus ejus labor excipit, ad ulteriorem Trigonometriæ suæ Logarithmicæ perfectionem spectans; nempe propositiones quædam eminentissimæ, in Triangulis sphæricis non quadrantalibus resolvendis, absque eorum in quadrantalia aut rectangul divisione, & absque casuum observatione : quas quidem Propositiones in ordinem redigere, & ordine demonstrare statuerat, nisi nobis morte præproprera præreptus fuisset. Lucubrationens etiam aliquot, Mathematici excellentissimi* D.Henrici Briggii *publici apud* Londinenses *Professoris, inmemorotas Propositiones, & novam hanc Logarithmorum speciem, Typis mandari curavimus; qui novi hujus Canonis supputandi laborem gravissimum, pro singulari amicitatia quæ illi cum Patre meo* L.M. *intercessit, animo libentissimo in se suscepit ; creandi methodi, & usuum explanatione inventori relictis. Nunc autem ipso ex hac vita evocato, totius negotii onus doctissimi Briggii humeris incumbere, & Sparta hæc ornanda illi sorte quadam obtigisse videtur. Hisce interim (Lector)laboribus quibuscunque fruere, & pro humanitate tua boni consulito. Vale.*

Robertus Neperus, F.

MIRIFICI
LOGARITHMORUM
Canonis Constructio;
(QUI ET TABULA ARTIFACIALIS
ab autore denceps appellatur)
eorumque ad naturales ipsorum numeros habitudines.

POSITIO PRIMA.

Tabula Artificialis, est minima Tabula, cujus opera facillimo computu omnium Geometricarum dimensionum, motuumque sublimium habetur notitia.

Hæc meritò minima dicitur, quia Tabulum sinuum volumine non exsuperat : facillima, quia per eam omnes multiplicationes, divisiones, extractionesque radicum graviores evitantur : solis enim & per paucis facillimisque additionibus, subtractionibus, & bipartitationibus omnes generaliter figuras motusque metitur.

Hæc è numeris proportione continuâ progredientibus excerpitur.

Pos. 2. ***Continuarum progressionum, alia Arithmetica quæ per æqualia*** [p. 6] ***intervalla progreditur : Alia Geometrica, quæ per inæqualia & proportionaliter crescentia, aut desicienta incedit.***

Arithmetica progressio, ut 1, 2, 3, 4, 5, 6, 7, &c. vel 2, 4, 6, 8, 10, 12, 14, 16, &c. Geometrica verò, ut 1, 2, 4, 8, 16, 32, 64, &c. vel 243, 81, 27, 9, 3, 1.

3. ***In progressionibus requiritur accuratio, & operis facilitas. Accuratio sit, pro fundamento numeros majores accipiendo : majores autem numeri ex minoribus facillimè fiunt adjectis cybris.***

Ut pro 100000, quem rudiores sinum maximum faciunt, eruditiores ponunt 10000000, quò melius omnium sinuum discrimen exprimatur. Unde & eodem nos pro sinu toto & maximo proportionalium Geometricorum utimur.

4. ***In Tabulis computandis etiam ex numeris majoribus maximi fiant, interposita periodo inter numerum ipsum & cyphras adjectas.***

Ut ex 10000000, nos initio computationis facimus 10000000.0000000, ne minutissimus error frequenti multiplicatione in immensum crescet.

5. ***In numeris periodo sic in se distinctis, quicquid post periodum notatur fractio est, cujus denominator est unitas cum tot cyphras post se, quod sunt figuræ post periodum.***

Ut ex 10000000.04 valet idem, quod $10000000\frac{4}{100}$. Item 25.803, idem quod $25\frac{803}{1000}$. Idem quod 9999998.0005021, idem valet quod $99999998\frac{5021}{10000000}$, & sic de cæteris.

6. ***E Tabulis jam computatis, rejici possunt fractiones post periodum locatæ, absque ullo sensibilis errore. In magnis enim nostris numeris errore insensibilis, & quasi nullus habetur, qui unitatem non exsuperat.***

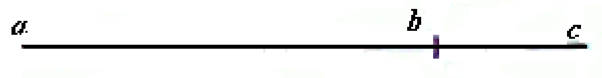
Ut completa Tabula pro 9987643.8213051, qui sunt $9987643\frac{8213051}{100000000}$, [p.7.] accipi possunt hi 9987643 absque sensibili errore.

7. ***Est præterà alia accurationis formula ; quum scilicet quantitas ignota, seu numero inexplicabilis, inter terminos numerales pluribus unitatibus non differentes includitur.***

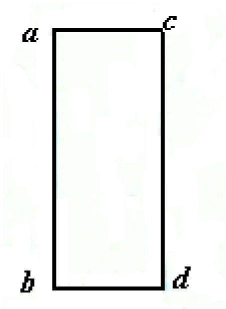
Ut positâ Diametro circuli partium 497; quia nescitur præcisè quot partium sit ambitus, ideo eruditiores ex Archimedis sententiâ, eum inter terminos 1562, & 1561 incluserunt. Item si costarum quadrati quælibet sit partium 1000, erit diagonalis radix quadrata numeri 20000000; quæ cum sit numero inexplicabilis, ideò per extractionem radicis quadratæ quæruntur ejus terminus, vel $1414\frac{604}{2828}$ major, & $1414\frac{604}{2829}$ minor: videlicet quanto minor terminorum differentia sit, tanto major accuratio.

Vice ipsarum quantitatum ignotarum, termini earum sunt addendi, subtrahendi, multiplicandi, aut dividendi prout opus fuerit.

8. ***Adduntur bini alicuius quantitatis termini ad binos terminos alterius, quum minor illius minori huius, & major illius majori huius additur.***



Ut sit linea *abc*, in duas quantitates *ab*, & *bc* divisa ; sit ab inter terminos 123.5 majorem, & 123.2 minorem : sitque *bc* inter 43.2 majorem, & 43.1 minorem. Addita ergo majore ad majorem, & minore ad minorem, fiet tota quantitas *ac* inter terminos 166.7 & 166.3.



9. ***Multiplicantur bini alicuius quantitatis termini per binos terminos alterius, quum minor illius in minorem unius, & major illius in majorem huius ducitur.***

Ut sit altera quantitas *ab*, quæ sit inter terminos 10.501 majorem, & 10.500 minorem; altera verò *ac*, quæ sit inter terminos 3.216, majorem, & 3.215 minorem. Unde ductis 10.502 in 3.216, & 10.500 in 3.315, provenient termini 33.774432, & 33.757500; inter quos erit area *abcd*.

10. ***Terminorum subtractio sit, terminum maiorem minoris quantitatis à minore maioris, & minorem minoris à maiore majoris auferendo.***

Ut in schemate primo, si ex terminis *ac*, qui sunt 166.7 & 166.3, subduxeris terminos *bc*, qui sunt 43.2 & 43.1, fient 123.6 & 123.1 pro terminis *ab*; & non 123.5 & 123.2. Nam licet etiam horum addito ad 43.2 & 43.1, producebat 166.7 & 166.3, (per octvam) non tamen è converso sequetur, quin aliquid possit esse inter 166.7 & 166.3, ex quo si subtraxeris aliquid quod sit inter 43.2 & 43.1, remaneret id quod non sit inter 123.5 & 123.2: verùm non esse id inter terminos 123.6 & 123.1 est impossibile.

11. ***Divisio sit, partiendo terminum maiorem dividendi per minorem divisoris, & minorem dividendi per maiorem divisoris.***

Ut in præcedente figura, quadratum *abcd* inter terminos 33.774432 & 33.757500 constitutum dividatur per terminos *ac*, qui sunt 3.216 & 3.215, provenient $10.505\frac{857}{3215}$ & $10.496\frac{2364}{3216}$ for terminis ab; & non 10.502 & 10.500, eadem ratione, quam in subtractione diximus.

12. ***Rudes terminorum fractiones delendœ sunt addita unitate ad terminum maiorem.***

Ut pro terminis *ab* præcenentibus, scilicet $10.505\frac{857}{3215}$ & $10.496\frac{2364}{3216}$, capiantur 10.506 & 10.496 [p. 9.]

Hactenus de accuratione, sequitur de facilitate operis.

13. ***Omnis progreßionis Arithmeticœ facilis est constructio, Geometricœ autem non omnis.***

Patet hoc, siquidem additione & subtractione sit facillimè Arithmetica progressio : geometrica verò, difficillimis multiplicationibus, divisionibus, & radicum extractionibus continuatur.

Solæ Geometricæ illæ progressiones facilè continuantur, quæ per subrectionem facilis partis numeri à toto oriuntur.

14. ***Partes numeri faciles dicimus, partes quaslibet cujus denominationes unitates & cyphris quotcunque notantur: habentur autem hæ partes, rejiciendo tot figuras ultimas principalis numeri, quot sunt cyphræ in denominatore.***

Ut partes decima, centisima, millesima, 10000^{a} , 100000^{a}, 1000000^{a}, 10000000^{a}, faciles dicuntur, quia cujuslibet numeri decima pars habetur delendo ejus ultimam figuras, & sic de cæteris, semper delendo tot figuras ultimas quot sunt cyphræ in denominatione partis. U t decima hujus 99321 est 9932, ejus autem centesima est 993, millesima 99, &c.

15. ***Mediocriter etiam facilè habentur partes dimidia, vigesima, ducentesima, & aliæ per binarium & cyphras denominatæ; rejiciendo tot figures ultimas principalis numeri, quot sunt cyphræ in denominatore, & reliquum bipartiendo.***

Ut numeri 9973218045 pars 2000^{a} est 4986609, pars 20000^{a} est 498660.

16. ***Hinc sequitur, si à sinu toto septem cyphris aucto, cæterisque inde ortis suam* 10000000^{am} *partem substraxeris, continuari possunt quam facillimè centum numeri, in ea proportione Geometrica, quæ est inter sinum totum & sinum eo minorem unitate, scilicet* 10000000 & 9999999; *hancque seriem proportionas ilium primam Tabulam nominamus.*** [p. 10.]

Prima Tabula.
10000000.0000000
<u>1.0000000</u>
9999999.0000000
<u>.9999999</u>
9999998.0000001
<u>.9999998</u>
9999997.0000003
<u>.9999997</u>
9999996.0000006
continuando usque ad
9999900.0004950.

Ut ex sinu toto aucto septem cyphris (majoris accurationis gratiâ) sic 10000000.0000000 aufer 1.0000000, fient 9999999.0000000; ex quibus aufer .9999999, fient 9999998.0000001; & sic prosequaris ut à latere, donec centum creaveris proportionalia, quorum ultimum (si rectè computaveris) erit 9999900.0004950.

17. ***Tabula secunda progreditur à sinu toto sex cyphris aucto, per quinquaginta numeros alios deficientes proportionaliter, ea propositione quæ facillima est, & quàm proxima proportioni, quæ est inter primum & ultimum primæ Tabulæ.***

Secunda Tabula.
100000000.000000
<u>100.000000</u>
9999900.000000
<u>99.999000</u>
9999800.001000
<u>99.998000</u>
9999700.003000
<u>99.997000</u>
9999600.006000
&c. usque ad
9995001.222927

Ut primæ Tabulæ primum & ultimum sunt 100000000.000000, & 9999900.0004950 ; in quorum proportione est constituere quinqaginta proportionales numeros. Proxima itaque facilis proportio, est 10000 ad 99999 ; quæ continuatur satis exactè adjiciendo sinui toti sex cyphras, & auferendo ab antecedente suam partem 100000^{am} , ut fiat subsequens, ut à latere vides : & hæc Tabula præter primum qui est sinus totus, etiam quinquaginta contineat proportionales numeros, quorum ultimum (si erraveris) reperies esse 9995001.222927

18. ***Tertia Tabula sexaginta novem columnis constat, & in qualibit columna ponuntur numeri viginti & unus, progredientes ea proportione quæ facillima est, & quàm proxima illi proportioni quæ est inter primum & ultimum secundæ Tabulæ.***

Unde hujus prima columna facillimè habetur à sinu toto quinque cyphris aucto, & à cæteris inde ortis suam 2000^am^ partem auferendo.

Prima Columna Tertiæ Tabulæ

10000000.00000
5000.00000
9995000.00000
4997.50000
9990002.50000
4995.00125
9985007.49875
4992.50374
9980014.99501
&c. usque ad
9900473.57803.

Ut quia inter 10000000.000000 primum secundæ tabulæ, & 9995001.222927 ejusdem ultimum, proportio difficilis est progressionis ; ideò in proportione facili 10000 ad 9995 (quæ illi propinqua satis est) constituendi sunt numeri viginti & unus; quorum ultimus (ni erraveris) erit 9900473.57803. A quibus jam creatis, rejici potest ultima singulorum figura absque sensibili errore, quò facilius ab iis alii postea creentur.

19. ***Primi numeri omnium columnarum, progrediuntur, à sina toto quatuor cyphris aucto, eâ proportione facillimâ, & proximâ proportioni, quæ est inter primum & ultimum primæ columnæ .***

Ut primæ columnæ primus & ultimus sunt 10000000.0000, & 9900473.57803 : his proportio facillima maximè propinqua est 100 ad 99. A sinu igitur toto continuandi sunt 68 numeri in ratione 100 ad 99, auferendo à quolibet eorum suam centesimam partem.

20. ***Eâdem proportione, à primæ columnæ numero secundo, per omnium* [p.12] *columnarum secundos : & à tertio, per tertios : & à quarto, per quartos: & à cæteris respectivè, per cæteros sit progressio.***

Ut ex antecedentis columnæ numero aliquo sit numerus ejusdem ordinis in sequenti columna, subtrahendo suam centesimam partem, numerosque hoc qui sequitur ordine constituendo.

Prima Columna.	*Secunda Col.*
10000000.0000	9900000.0000
9995000.0000	9895050.0000
9990002.5000	9890102.4750
9985007.4988	9885157.4238
9980014.9950	9880214.8451
&c. descendendo ad	&c. descendendo ad
9900473.5780	9801468.8422

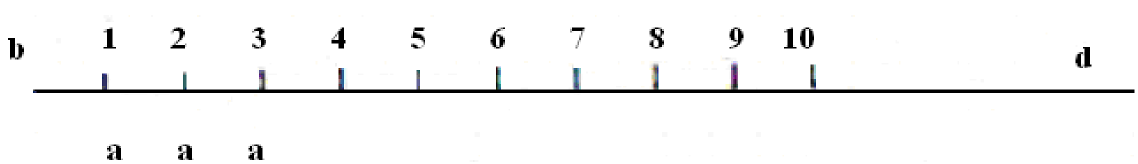
Tertia Col.	*Inde 4^a, 5^a,&c. usq.*	*ad 69^{am} column.*
9801000.0000	&c. usque ad	5048858.8879
9796099.5000	&c. usque ad	5046334.4584
9791201.4503	&c. usque ad	5043811.2912
9786305.8495	&c. usque ad	5041289.3856
9781412.6966	&c. usque ad	5038768.7409
&c. descen-dendo ad	&c. descen-dendo ad	&c. descen-dendo ad
9703454.1538	&c. usque tandem ad	4998609.4019

21. [p.13.] ***In tertio ergo Tabula, habes inter siunum totum, & medium sinus totius, interjectos sexaginta octo numeros in proportione ut 100 ad 99; & rursus inter singulos binos horum, interjectos viginti numeros in proportione ut 10000 ad 9995 : & rursus inter binos primos horum, scilicet inter10000000 ad 9995000, habes in secunda Tabula interjectos 50 numeros, in proportione ut 100000 ad 99999: & tandem inter binos primos horum, habes in prima Tabula interiectos centum numeros, in proportione ut 10000000 sinus totus ad 9999999; quorum differentia quum sit tantum unitatis, non est opus eam (interiectis mediis) minutiùs partiri. Unde hæ tres tabulæ (postquam completæ fuerint) ad tabulam artificialem (Logarithm.) computandam sufficient.***

Hucusque sinus seu numeros naturales proportione geometrica progredientes tabulis facillimè inserere docuimus.

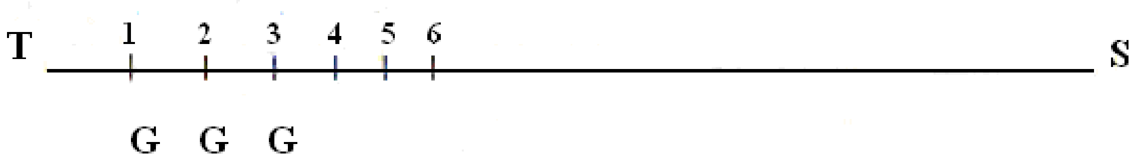
22. ***Superest Tabulæ saltem tertiæ, apud sinus sive numeros naturales geometicè decrescentes, suos numeros artificiales Arithmeticè crescentes inserere.***

23. ***Arithmeticè crescere, est æqualibus temporibus æquali semper quantitate augeri.***



Ut puncto *b* fixo versus *d*, infinitè producatur linea : in qua, ex b versus *d* procedat punctus *a*, movens ea lege, ut æqualibus temporis momentis æqualibus feratur spatiis : quæ sint *b* 1, 1 2, 2 3, 3 4, 4 5, &c. Dico hoc incrementum per *b* 1, *b* 2, *b* 3, *b* 4, *b* 5, &c. Arithmeticum dici. In numeris autem sint *b* 1, 10: *b* 2, 20: *b* 3, 30: *b* 4, 40: *b* 5, 50, &c. Arithmicè crescere : quia æqualibus momentis, æquali numero denarii semper augeri intelliguntur. [p.14.]

24. ***Geometricè decrescere, est æqualibus temporibus quantitatem primò totam, inde aliam atque aliam ejus partem superstitem, simili semper proportionali parte diminui.***

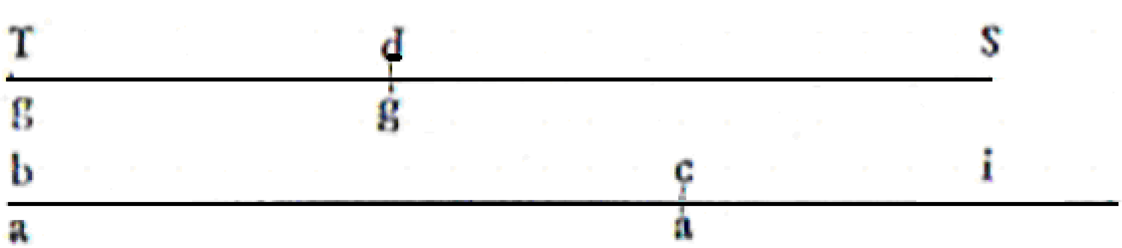


Ut sit linea sinus totius TS, in hac moveatur punctus G, à T in 1 versus S, quantoque tempore defertur à T in 1, quæ sit (exempli gratiâ) decima pars TS : tanto idem G tempore moveatur ab 1 in 2, quæ sit decima pars 1 S: & à 2 in 3, quæ sit decima pars 2 S: & à 3 in 4, quæ sit decima pars 3 S: & sic cæteris. Dico hos sinus TS, 1 S, 2 S, 3 S, 4 S, &c. dici Geometricè decrescere : quia inæqualibus spatiis proportione similibus & tempore æqualibus diminuuntur. In numeris sit TS, 10000000: 1 S, 9000000: 2 S, 8100000: 3 S, 7290000: 4 S, sit 6561000, etc. Dico hos sinuum numeros, æqualibus temporibus proportione diminutos, dici Geometricè decrescere.

25. ***Unde punctus mobilis Geometricè ad fixum accedens, velocitates suas prout distantias, à fixo proportionatas habet.***

Ut repetito præcedenti Schemate, dico quum mobilis punctus geometricus G est in T, ejus velocitas est ut distantia TS : & quum G est in 1, ejus velocitas est ut 1 S: & quum in 2, ejus velocitas est ut 2 S, & sic decæteris. Atque ita quæ est proportio distantiarum T S, 1 S, 2 S, 3 S, 4 S, &c. ad invicem. Nam magis minúsve velox punctus arguitur, prout magis minúsve longè sub æqualibus temporibus ferri conspicitur. Qualis itaque processus ratio, talem etiam & velocitatem esse necesse est : at talis est sub æqualibus temporibus ratio processuum T 1, 1 2, 2 3, 3 4, 4 5, &c. qualis [p. 15.] distantiarum T S, 1 S, 2 S, 3 S, 4 S, &c.ut mox docimus. Unde necessariò qualis habitudo distantiarum G ab S, videlicet T S, 1 S, 2 S, 3 S, 4 S, &c. invicem; talis etiam est velocitatum G in punctis T, 1, 2, 3, 4, &c. quod erat demonstrandum. At quòd processuum T 1, 1 2, 2 3, 3 4, 4 5, &c. talis sit ratio, qualis distantiarum T S, 1 S, 2 S, 3 S, 4 S, &c. patet : quia quantitatum proportionaliter continuatarum differentiæ etiam in eadem proportione continuantur. At hæ distantiæ (per hypothesin) proportionaliter continuantur, & illi processus sunt harum differentiæ : quare eadem processus quâ distantias ratione continuari certum est.

26. ***Numerus artificialis sinus dati, est qui Arithmeticè crevit tantá semper velocitate, quantâ sinus totus incepit Geometricè decrescere, tantoque tempore, quanto sinus totus in sinum illum datum decrevit.***

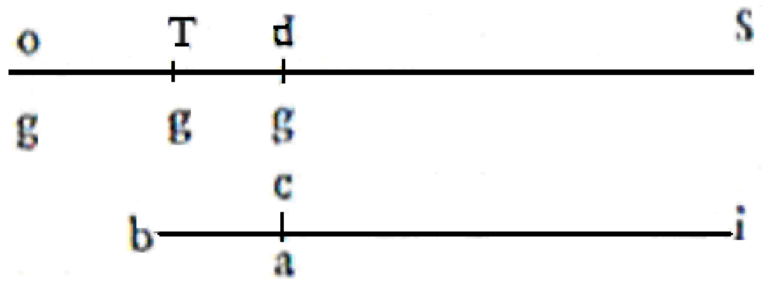


Sit sinus totus linea TS, sinus datus in eadem linea dS : certis quibusdam momentis moveatur g Geometricè à T in d. Sitque alia linea bi versus i infinita, in qua ex b moveatur a Arithmeticè, eadem velocitate quà g primò cum erat in T : totidemque temporis momentis procedat a ex b fixo versus i usque in c punctum : dicetur numerus metiens bc lineam numerus artifialis sinus dS dati.

27. ***Unde sinus totius nihil est pro artificili.***

Nam ex Schemate, cum g est in T faciens suam distantiam ab S sinum totum, punctus Arithmeticus a incipiens in b, nusquam inde procedit. Unde ex definitione distantiæ, sinus totius nullus erit artificialis. [p.16.]

28. ***Hinc etiam sequitur, quod cujuslibet dati sinus numerus artificialis, major est differentia inter sinum totum, & sinum datum; & minor differerentia quæ est inter sinum totum, & quantitatem eo majorem in eadem ratione, quæ est sinus totius ad datum. Atque hæ differentiæ dicuntur ideò termini artificialis.***



Ut repetito præcedenti Schemate, protractaque; lineâ ST in *o*, Ita ut S*o* se habeat ad TS, ut TS ad dS. Dico sinus dS numerum artificialem bc, majorem esse quam Td, & minorem quam oT. Quanto enim tempore *g* ab *o* in T fertur, tanto & g à T in d feretur (per 24) quia oT est tanta pars o S, quanta Td est lineæ TS, tantoque tempore (per definitionem artificialis) feretur & a à b in c: Ita ut oT, Td, & bc sint æqualium temporum processus. At quia *g* inter T & *o* movens, velocior est quàm in T, & inter T & d tardior, in T autem g æquivelox est atque a (per 26.) Sequetur processum *o*T processum quem g tardus facit, minorem esse : & bc processum (quem punctus a mediocri suo motu totidem etiam temporis momentis perficit) medium quodam esse inter utrumque, quod erat demonstrandum. Numeri itaque artificialis quem *bc* designat, dicitur oT terminus major, & T*d* terminus minor.

29. ***Dati itaque sinus artificiales terminos exhibere.***

Ex præmissa probatur minorem terminum relinqui, ablato sinu dato à sinu toto; & majorem terminum produci, multiplicato sinu toto in terminum minorem, & producto diviso per sinum datum, ut sequenti exemplo. [p. 17.]

30. ***Unde primæ Tabulæ primum proportionale, quod est* 9999999*, habet suum artificialem numerum inter termines* 1.0000001 & 1.0000000.**

Nam per præmissam aufer 9999999 à sinu toto cyphris aucto, fiet unitas cum suis cyphris pro minore termino: hanc unitatem cyphris auctam, & multiplicatam in sinum totum, divido per 9999999, & fient 1.0000001; sive (si majorem accurationem requitis) 1.00000010000001 pro majore termino.

31. ***Insensibili differentia distantes termini ipsi, sive inter eos quidvis pro numero artificiali vero habetur.***

Ut in superiore exemplo, sinus hujus 9999999. artificialis numerus habetur hic 1.0000000, sive hic 1.00000010, sive omnium optimè hic 1.0000005 : quia enim ipsi termini 1.0000001 & 1.0000000, insensibili fractione utpote $\frac{1}{10000000}$ differunt ab invicem : ideò & ipsi, & quicquid inter eas est, multò minùs, multoque insensibiliore errore, à vero different artificiali inter hos terminos constituto.

32. ***Quotcunque sinuum Geometricá proportione à sinu toto deficientium, unius artificiali numero aut terminis datis, cæterorum etiam dare.***

Consequitur hoc necessariò incrementi Arithmetici, decrementi Geometrici, & numeri artificialis definitiones : siquidem per illas, ut sinus Geometricâ proportione decrescunt continuò ita interim artificiales, continuo Arithmico progressu per æqualia accrescunt. Unde cuilibet sinui Geometricæ progressionis decrescendo, respondet suus artificialis Arithmeticæ progressionis crescendo : primo scilicet primus, & secundo secundus, & deinceps.

Ita ut si primus artificialis, respondens primo sinui post sinum totum detur, secundus artificialis erit ejus duplum, tertius triplum, & sic de cæteris : donec omnes omnium artificiales innotescant, ut sequenti exemplo patebit. [p. 18.]

33. ***Hinc omnium sinuum proportionalium primæ Tabulæ, numeri artificiales inter terminos propinquos includi, & per consequens exactè satis dari possunt.***

Ut cum sinus totius artificialis sit 0 (per 27) & primi post sinum totum, que est 9999999 in prima Tabula, artificialis sit (per 30) inter terminos 1.0000001. & 1.0000000 : necessariò secundi post sinum totum, qui est 9999998.0000001, artificialis continebitur inter dupla illorum terminorum : scilicet inter 2.0000002 & 2.0000000 : & tertii 9999997.0000003, inter eorundem triplat scilicet inter 3.0000003 & 3.0000000. Et sic in cæteris, æqualiter semper augendo : terminos intervallo primorum terminorum : donec omnium proportionalium primæ Tabulæ artificiales terminos compleveris. Poteris consimili progressu, si libet, numeros ipsos artificiales exiguo & insensibili error continaure hoc ordine, pro sinus totius artificiali, erit 0; pro primi post sinum totum artificiali, erit 1.00000005 (per 31) pro secundi, 2.00000010 : pro tertii, 3.00000015. Et ita deinceps.

34. ***Differentia artificialium sinus totius & sinus dati, est ipsius dati artificialis.***

Patet hoc, quum enim sinus totius artificialis sit nihil per 27, hoc nihilo ex artificiali dati subducto, ipsum integrum artificialem dati remanere necesse est.

35. ***Duorum artificialium differentia, addenda est ad artificialem maioris sinus eorundem, ut habeas artificialem minoris: & subtranenda ab artificili minoris sinus, ut habeas artificialem maioris.***

Necessariò hoc sit, siquidem crescunt artificiales decrescentibus sinibus, atque minor est artificialis majoris sinus, & major minoris. Ideóque æquum est differentium addere minori artificiali, ut habeas artificialem majorem licet minoris sinus : & contrà, auferre differentiam [p. 19.] à majore artificiali, ut habeas minorem artificialem licet majoris sinus.

36. ***Similiter proportionatorum sinuum sunt æquidifferentes artificiales.***

Consequitur hoc necessariò definitiones artificialium & motuum : Nam cum per eas, Geometrico decremento similiter proportionato, respondet Arithmeticum incrementum æquale semper : necessariò similiter proportionatis sinibus, respondere æqui-differentes artificiales & numeros, & numerorum terminos concludimus. Ut insuperiori exemplo primæ Tabulæ, quia similis est proportion inter primum proportionale post sinum totum 9999999.0000000, & tertium 9999997.0000003 : ei quæ est inter quarti 9999996.0000006, & sextum 9999994.0000015. Ideò numerus artificilis iste 1.00000005 primi, differt ab artificiali isto 3.00000015 tertii, eadem differentiâ, quâ artifilialis iste 4.00000020 quarti, differt ab artificiali isto 6.00000030 sexti proportionalis. Eadem etiam est æqualitatis ratio inter differentias terminorum artificialium adinvicem : videlicet tam minorum inter se, quam etiam majorum inter se, quorum sinus sunt similiter proportionati.

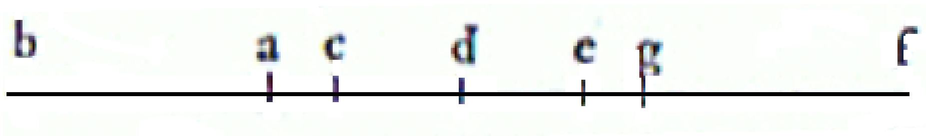
37. ***Ut trium sinuum in proportione Geometrica continuatorum, quadratum medii æquatur facto ex ductis invicem extremis : Ita in suis artificialibus numeris, duplum medii æquatur aggregato extremorum. Unde horum artificialium duobus quibuscunque datis, tertius innotescit..***

Quia horum trium sinuum, ratio quæ est inter primum & secundum, similis est rationi

quæ est inter secundum et tertium : Ideò (per præmissam) suorum artificialium ea est differentia inter primum & secundum, quæ est inter secundum & tertium. Sit (verbi gratiâ) primus artificialis lineà bc expressus, secundus lineâ bd, tertius lineâ be : sintque unicâ lineâ bcdc comprehensi hoc modo sintque differentiæ cd & de æquales : horum medium bd dupletur, productá lineâ hac à b ultra e in f, ita ut bf sit duplum bd. Dico bf æquari utrisque lineis, bc primi artificialis, & be tertii : [p. 21.] ab æqualibus enim bd & df, aufer æqualia cd & de : scilicet cd, à bd, & de, à df : & remanebunt bc & ef necessariò æqualia. Cum itaque tota bf, æqualis sit utrisque be & ef : ergo & utrisque be & bc æquabitur, quod erat demonstrandum. Unde sequitur canon : si trium horum artificilium medium datum duplaveris, & hinc subtraxeris extremum datum, reliquum extremorum quæsitum innotescet : & si extrema data conjunxeris, & aggregatum hoc bipartiveris, medium fiet notum.

38. ***Quatuor Geometricè proportionalium, sicut factum ex ductu mediorum,æquatur facto ex ductu extremorum : Ita suorum artificialium, aggregatum mediorum æquator aggregato extremorum. Unde horum artificialiumn tribus quibuscunque datis, quartum innotescit.***

Quia horum quatuor proportionalium, ratio quæ est inter primum & secundú, similis est rationi quæ est inter tertium & quartum : Ideò (per penultimatè præmissam) suorum artificialium, ea est differentia inter primum & secundum, quæ est inter tertium &



quartum. Tales itaque quantitates in linea b f suprascripta sumantur, ut hic, quarum b a primum artificialem, b c secundum, b c tertiam, & b g quartum referat, factis differentiis a c & e g æqualibus : Ita ut d in medio c e positum, in medio a g etiam poni necesse est, Iam dico aggregatum b c secundi, & b e tertii : æquari aggregato b a primi, & b g quarti. Nam quia (per præmissam) duplum b d, quod est b f, æquatur utrisque b c & b e : quia differentiæ eorum à b d, vidilicet c d & d e sunt æquales. Eadem ratione, & idem b f æquabitur utrisque b a & b g : quia eorum differentiæ à b d, videlicet a d & d g sunt etiam æquales. Quum itaque & aggregatum ex b a & b g, & aggregatum ex b c & b e, sint iidem duplo b d, quod est b f æqualia : ergo & inter se æquabuntur, quod erat demonstrandum. Unde sequitur canon, si quatuor artificialium, ab aggregato extremorum datorum, subduxeris alterum mediorum cognitum, relinquetur reliquum medium quod quærebatur : & si ab aggregato mediorum cognitorum subduxeris alterum extremorum cognitum, relinquetur extremum quæsitum.

39. ***Duarum artificialium differentia, est inter duos terminos, ad quorum majorem se habet sinus totus, ut eorum artificialium minor sinus ad sinuum differentiam : & ad minorem terminum se habet sinus totus, ut artificialium sinus maior ad sinuum differentia.***

Sit sinus totus TS, sinus duo dati d S major, & e S minor : Ultra S T signetur punct V distantia T V, ea lege, ut S T se habeat ad T V, ut e S minor sinus, ad d e differentiam sinuum. Deinde citra T versus S, signetur puncto e distantia T c, ea lege, ut T S se habeat ad T c, ut d s sinus major, ad d e differentiam sinuum. Dico differentiam artificialium respondentium sinibus d S & e S, constitui inter terminos V T majoreml, & T c minorem. Nam quia ex hypothesi, ut e S ad d e, ita T S ad T V; & ut d S ad d e, ita T S ad T c se habent : ideò etiam (ex natura proportionalium) sequuntur duæ conclusiones : Primò, quod V S se habet ad T S, [p. 22.] ut idem T S ad c S. Secundò, quod similis est ratio T S ad c S, rationi quæ est d S ad e S. Et propterea (per 36) differentia artificialium respondentium sinibus d S & e S, æqualis est differentiæ artificialium respondentium sinui toto T S, & sinui e S. At hæc differentia (per 34) est artificialis ipsius sinus c S; & hic artificialis inter terminos V T majorem, & T c minorem (per 28) includitur : quia per primam conclusionem jam dictam, V s major sinu toto se habet ad sinum totum T s, ut idem T s ad e S. Unde necessatiò differentia artificialium respondentium sinibus d S & e S, constituitur inter terminos V T majorem, & T c minorem, quod erat demonstrandum.

40. ***Terminos differentiæ inter artificiales numeros duorum datorum sinuum exhibere.***

Quum per præmissam, sinus minor se habeat ad differentiam sinuum, ut sinus totus ad majorem terminum differentiæ artificialium : & sinus major se habeat ad differentiam sinuum, ut sinus totus ad minorem terminum : sequetur ex natura proportionalium, quod ducto sinu toto per differentiam datorum sinuum, orietur ex producto diviso per minorem datorum, maior terminus : & ex producto diviso per majorem sinuum, orietur minor terminus.

EXEMPLUM.

Vt sit sinuum datorum major 9999975.5000000, minor autem 9999975.0000300 : quorum differentia .49997000 ducta in sinum totum (adjectis priùs octo cyphris utrique post punctum demonstrationis gratiâ, licet alioquin septem sufficiant) quod hinc producitur, si per maiorem sinum scilicet per 99999975.5000000 diviseris, provenient .49997122 octa figuratum post punctum pro minore termino. Sic quod producitur, per minorem sinum,scilicet per 9999975.0000300 diviseris, provenient .49997124 pro maiore termino : [p. 23.] inter quos (ut demonstratum est) constituitur differentia artificialium sinuum datorum. Sed quia protractio huius fractionis in octavam figuram ultra punctum, est accuratio plusquá requisita, præsertim cum in ipsis sinibus septem tantum ponantur figuræ post punctum : ideò deletâ octavâ illâ sive ultimâ utrisque termini figurâ, uterque terminus unà ipsa artificialium differentia, in fractione .4999712 stabiliri potest, absque vel minimo scrupulo sensibilis erroris.

41. ***Sinuum vel numerum naturalium, non in ipsos proportionales primæ Tabulæ, sed prope vel inter eos cadentium : numeros artificiales, eorumve saltem terminos insensibili differentia distantes exhibere.***

Sinui dato primum primæ Tabulæ proximum, sive minorem sive maiorem nota : huius tabulati sinus terminos artificiales (per 33) quare, & inventos reserva : deinde (per præmissam) terminos differentiæ inter artificiales numeros sinus dati & sinus tabulati, sive ambos, sive (qui ferè æquales sunt, ut superiori exemplo patet) eorum alterutrum quære. Hos iam inventos, horumve alterutrum adde ad illos nuper refervatos terminos,aut ab illis subtrahe (per 8, 10. & 35) prout sinus datus fuerit minor aut maior tabulato ei proximo : & qui hinc producuntur numeri, erunt termini propinqui inter quos includetur artificialis numerus sinus dati.

EXEMPLUM.

V t sit sinus datus 9999975.5000000, cui sinus in Tabula proximus, est 9999975.0000300 minor dato : huius termini artificiales (per 33) sunt 25.00000025 & 25.0000000 : deinde (per præmissam) differentia inter artificiales numeros linuum dati & tabulati, est .4999712 : quam (per 35) aufer ab illis terminis, quia sunt termini minoris sinus, & provenient 24.5000313 & 24.5000288, termini quæsiti sinus dati [p. 24.] 9999975.5000000 : cuius ipse artificialis numerus in quovis terminorum, sive optimè in 24.5000300 (per 31) constitui potest absque sensibili errore.

ALIUD EXEMPLUM.

Sit sinus datus 9999900.0000000, sinus tabulatus ei proximus 9999900.0004950: huius termini artificiales sunt 100.00000100 & 100.0000000 (per 33) : Deinde differentia inter artificiales numeros illorum sinuum, erit (per præmissum) .0003950 , quam (per 35) ad superiores terminos adde, & fient 100.0005050 pro maiore termino, & 100.0004950 pro minore termino, inter quos artificialis quæsitus sinus dati includitur.

42. ***Hinc sequitur, omnium proportionalium secundæ Tabulæ numeros artificiales exactè satis dari : seu inter terminos cognitios insensibili fractione differentes includi posse.***

Ut in præcedenti exemplo, quum numerus artificialis hiuis sinus 9999900 (qui est primum proportionale secundæ Tabulæ) demonstretur inter terminos hos 100.0005050 & 100.0004950 esse : necessariò secundi proportionalis, numerus artificialis (per 32) erit inter hos terminos 200.0010100 & 200.0009900 : & tertii proportionalis, inter hos terminos 300.0015150 & 300.0014850. &c. Et tandem ultimi sinus secundæ Tabulæ, que est 9995001.222927, artificialis numerus includetur inter terminos hos 5000.0252500 & 5000.0247500. Quibus iam habitis terminis (per 31) poteris ipsos artificiales numeros eorum exquirere.

43. ***Sinuum vel numerorum naturalium non in ipsos proportionales secundæ Tabulæ, sed prope vel inter eos cadentium, numeros artificiales exhibere : seu inter terminos cognitos insensibili fractione differentes includere.***

Sinui dato sinum secundæ Tabulæ proximum, sive maiorem sive minorem nota : huius tabulati sinus terminos [p. 25.] artificiales per præmissam quære : deinde per regulam proportionis, quære quartum proportionale se habens ad sinum totum : ut sinuum dati & tabulati minor se habet ad majorem. Quod uno modo perfici poterit, ducendo dati & tabulati minorem in sinum totum, & productum in majorem dividendo. Altero modo faciliore, ducendo sinuum dati & tabulati differentiam in sinum totum, & productum in dati & tabulati majorem dividendo, atque quotientem ex sinu toto auferendo. At quia hujus quarti proportionalis, numerus artificialis (per 36) tantum differt ab artificiali sinus toti, quantum invicem artificiales sinuum dati & tabulati differunt : Et quia etiam illorum differentia, eadem est cum ipso artificiali quarti per 34 : Ideò artificiales terminos quarti, per punultimè præmissam è Tabulæ prima quære, & inventos adde ad artificiales terminos tabulati, aut ab illis subtrahe per 8, 10. & 35. prout tabulatus sinus fuerit major aut minor dato, & producentur artificiales termini sinus dati.

EXEMPLUM.

Vt sit sinus datus 9995000.000000, sinus Tabulæ secundæ ei proximus est 9995001.222927 : huius termini artificiales (per præmissum) sunt 5000.0252500 & 5000.024750. Quartum deinde proportionale alterutro modorum suprascriptorum quære,

& fiet 999999.7764614, cujus terminos artificiales (per 41) è prima Tabula quære , eruntque 1.2235387 & 1.2235386 : quos ad superiores terminos per 8 & 35 adde, fientque pro terminis artificialibus dati 5001.2487888 & 5001.2482886. Unde & numerus inter hos medius, qui est 5001.2485387, optimè (per 31 pos.) pro ipso artificiali numero sinus 9995000 dati statuitur absque sensibili errore.

44. ***Hinc sequitur, omnium proportionalium primæ Columnæ tertiæ Tabulæ,*** [p. 26.] ***numeros artificiales exactè satis dari : seu inter terminos cognitos insensibili fractione differentes includi posse.***

Nam quum per præmissam, huius 9995000.000000 (qui est primus sinus infra sinum totum, ex proportionalibus primæ Columnæ tertiæ Tabulæ) numerus artificialis sit 5001.2485387 absque errore sensibili : secundi proportionalis scilicet 9990002.5000, numerus artificialis (per 32) erit 10002.4979774. Et sic cæteris, progrediendo usque ad ultimum ejus columnæ sinum 9900473.57808 : cujus, pari ratione artificialis numerus erit 100024.9707740 eiusque termini 100024.9657720 & 100024.9757760 erunt.

45. ***Numerorum naturalium, seu sinuum non in ipsos proportionales primæ Columnæ tertiæ Tabulæ, sed prope vel inter eos cadentium, numeros artificiales exhibere : seu inter cognitos terminos insensibili fractione differentes includere.***

Sinui dato sinum primæ Columnæ tertiæ Tabulæ proximum, sive minorem sive maiorem nota; huius tabulati terminos artificiales per præmissam quære : deinde quartum proportionale se habens ad sinum totum, ut sinuum dati & tabulati minor ad maiorem, per unum ex modis in penultimè præcedente descriptis quære : huius quarti ita inventi terminos artificiales (per penultimè præmissam) è secunda Tabula quære, & inventos adde ad terminos tabulati sinus superius inventos, aut ab illis subtrahe (per 8, 10, & 35) & producentur artificiales termini sinus dati.

EXEMPLUM.

Vt sit sinus datus 9900000, proportionalis sinus primæ Columnæ tertiæ Tabulæ ei proximus, est 9900473.57808, cuius termini artificiales per præmissam sunt 100024.9657720 & 100024.9757760. Quartum inde proportionale erit 9999521.6611820, cuius [p. 27.] termini artificiales (per 43 è secunda Tabula desumpti) sunt 478.3502290 & 478.3502812 : quibus terminis ad terminos superiores tabulati (per 8 & 35) additis, provenient termini 100503.3260572 & 100503.3160010, inter quos necessario cadit artificialis numerus quæsitus. Unde numerus inter hos medius, qui est 100503.3210291, pro vero artificiali numero sinus 9900000 dati, statui absque sensibili errore potest.

46. ***Hinc sequitur, omnium proportionalium tertiæ Tabulæ numeros artificiales exactè satis dari.***

Nam quum (per præmissam) 100503.3210291, sit artificialis primi sinus secundæ Columnæ, qui est 9900000, cæterique primi reliquarum columnarum sinus eadem proportione progrediantur; necessario (per 32 & 36) eorum numeri artificiales eadem semper differentia crescunt, additis 100503.3210291 anticedenti artificiali, ut fiat sequens. Habitis ergo sic primis artificialibus cuiusque columnæ, atque per penultimè præcedentem omnibus artificialibus primæ columnæ datis; elige tibi, an mavis simul eiusdem columnæ omnes artificiales condere, addendo semper ad superiorem artificialem cuiuslibet columnæ, hanc artificialium differentiam 5001.2485387, ut fiat proximè inferior eiusdem columnæ artificialis : An mavis simul eiusdem ordinis omnes artificiales, scilicet omnes secundos singularum columnarum artificiales ; inde omnes tertios, inde quartos, & sic reliquos constituere, addendo semper 100503.3210291 cuilibet artificiali præcedentis columnæ, ut eiusdem ordinis sequentis columnæ artificialis proveniat. Utrovis enim modo, omnes omnium huius Tabulæ proportionalium habentur artificiales ; quorum ultimus, & ad sinum 4998609.4034 congruens, est 6934250.8007528.

47. ***Omnibus tertiæ Tabulæ naturalibus numeris, ascribendi sunt sui artificiales, ut tertia Tabula integra fiat & perfecta:*** [p. 28.] ***quam posthàc semper radicalem vocabimus.***

Hæc hujus Tabulæ conscriptio sit constituendo columnas numero & ordine quibus per 20 & 21 describuntur : & divisâ unaquâque columnâ in duas series;

RADICALIS TABULÆ.

Columna prima		Columna secunda	
Naturales	Artificiales	Naturales	Artificiales
10000000.0000	0	9900000.0000	100503.3
9995000.0000	5001.2	9895050.0000	105504.6
9990002.5000	10002.5	9890102.4750	110505.8
9985007.4988	15003.7	9885157.4238	115507.1
9980014.9950	20005.0	9880214.8451	120508.3
&c. usqua ad	&c. usqua ad	&c. usqua ad	&c. usqua ad
....	....	....	....
9900473.5780	100025.0	9801468.8422	200528.2

& cæteri usque ad

Columna 69.

Naturales	Artificiales
5048858.8900	6834225.8
5046334.4605	6839227.1
5043811.2932	6844228.3
5041289.3879	6849229.6
5038768.7435	6854230.8
&c. usqua ad	&c. usqua ad
....	....
4998609.4034	6934250.8

[p. 29.] quarum prima, proportionalia illa Geometrica, quæ sinum numerosque naturales nominamus ; secunda, hos suos artificiales Arithmeticæ per æqualia progredientes contineat. Duobus tamen (compendii gratiâ) animadversis : Primò, quod illis omnibus artificialibus, unam post punctum relinqui figuram satis sit, cæteris sex novissimis jam refectis : quas tamen si initio neglexisses : error inde frequenti multiplicatione priorum tabularum, accrevisset in hac tertia intollerabilis. Secundò, si secunda post punctú figura excedat quaternarium : figura prima, quæ sola post punctum relinquitur, est unitate augèda. Ut pro 10002.48, &c. rectius est ponere 20002.5, quam 10002.4 : & pro 10003.5001, aptius ponimus 1000.4, quam 1000.3. Itaque eo situ procedat jam radicalis Tabula quo præmittitur.

48. ***Perfecta jam radicali Tabulâ, ex ea sola Tabula artificialis numeros excerpimus.***
Ut enim priores duæ Tabulæ ad constitutionem tertiæ inserviebant; Ita tertia hæc radicalis ad principalem artificialem Tabulam, quam facillime & absque errore sensibili condendam inservit.

49. ***Sinuum majorum quam 9996700, artificiales numeros facillimè exhibere.***
Fiet hoc, sola subtractione sinus dati à sinu toto. Nam per 29, artificialis numerus sinus 9996700, est inter terminos 3300 & 3301; qui quidem termini (quia invicem unitate tantuem differunt) à suo artificiali vero, non possunt errore sensibili, videlicet majore unitate differre. Unde ipse terminus minor 3300, qui sola subtractione habetur, pro ipso artificiali capi potest. Eadem necessario ratio est de omnibus sinibus hoc majoribus.

50. ***Sinuum omnium intra limites Tabulæ radicilis comprehensorum, artificiales exhibere.***
Sinuum dati, & tabulati ei proximi, differerntiam duc [p.30.] in sinum totum; productum partire per facillimum divisorem, qui vel sit sinus datus, vel tabulatus ex proximus, vel inter utrumque utcunque constitutus; & producetur differentiæ artificialium aut terminus major, aut minor, aut intermedium quidpiam (per 39) quorum nullus â vera artificialium differentia errore sensibili differet, propter propinquitatem numerorum Tabulæ. Et ideò hunc eorum quemcunque productum (per 35) adde, ad artificialem tabulati in Tabula repertum, si sinus datus sit minor tabulato sinu : alioquin ullum productum ex hoc tabulati artificiali subtrahe, & proveniet dati sinus numerus artificialis quæsitus.

EXEMPLUM.

Vt, sit sinus datus 7489557, cujus quaritur artificialis. Sinus tabulatus ei proximus est 7490786.6119, hinc aufero illam adjectis cyphris sic 74895557.0000, relinquentur 1229.6119; quæ ducta in sinum totum, divido per numerum facillimum, qui sit vel 7489557.0000 vel 7490786.6119; vel optimè per quippiam inter eos constitutum, utpote per 7490000, & facillima divisione provenient 16401 : quæ (quia datus sinus minor est tabulato) adde ad artificialem tabulati, videlicet ad 2889111.7, & fient 2890751.8, quæ idem valent quod $2890751\frac{4}{5}$: sed quia Tabula principalis nec fractiones admitttit, nec quicquam ultra punctum, ponimis pro illo 2890752, qui est artificialis quæsitis.

ALIUD EXEMPLUM.

Sit sinus datus, 7071068.0000; sinus tabulæ ei proximus erit 7970084.4434 ; quorum differerentia est 983.5566; quibus ductis in sinum totum, productum divide optimè per 7071000, quæ sunt inter sinus datum & tabulatum, provenient inde 1390.0 : quæ (quia sinus datus excedit tabulatum ei proximum) subtrahatur ex artificiali numero tabulati in tabula reperto, [p. 31] scilicet à 3467125.4, remanebit 3465734.5. Unde 3454735 ponitur pro artificiali quæsito sinus 7071068 dati. Itaque hæc libertas divisorem elegendi miram parit facilitatem.

51. ***Omnes sinus in proportione dupla, habent 6931469.22 pro differentia suorum artificialium.***

Quia enim omnis sinus ad suum dimidium eadem est ratio, quæ est sinus totius ad 50000000 : ideò (per 36) differentia artificialium cujusque sinus & sui dimidii, est eadem cum differentia artificialium sinus totius, & sui dimidii 50000000. At eadem est differentia artificialium sinus totius, & sinus 50000000, cum ipso artificiali sinus 50000000 (per 34) cuius 50000000, artificialis (per præmissam) erit 6931469.22 erit differentia omnium artificialium, quorum sinus sunt in proportione dupla : & per consequens duplum ejus, scilicet 13862938.44, erit differentia omnium artificialium, quorum sinus sunt in ratione quadrupla : & triplum ejus, videlicet 20794407.66, erit differentia omnium artificialium, quorum sinus sunt in ratione octupla.

52. ***Omnes sinus in proportione decupla, habent 23025843.34 pro differentia suorum artificialium.***

Nam per penultimè præmissam, sinus 8000000 habet artificialem sinum 2231434.68 : & præmissam differentia inter artificiales sinuum 8000000, & suæ octavæ partis 10000000, est 20794407.66 : Unde per additioné fiunt 23025842.34, pro artificiali sinus 10000000 : & quum ad hunc sinus totus sit decuplus, omnes sinus in ratione decupla, eandem illam differentiam 23025843.34, inter suos artificiales habebunt, eadem causa & ratione, quam jam in dupla proportione per præcedentem exposuimus, quod probandum erat. Et per consequens, centuplæ proportioi respondebit hujus artificialis duplum, quod est 46051684.68, pro differentia artificialium [p. 32]: Et ejusdem triplum, quod est 69077527.02, erit differentia omnium artificialium, quorum sinus sunt in ratione millecupla. Et sic de ratione 10000[a], & alia, ut infra.

53. ***Unde omnes sinus in ratione composita ex duplo & decuplo, habent artificiales suos differerentiâ 6931469.22, & differentiâ 23025843.34 respectivè differences.***

Ut in tabella subsequenti conspicere licet.

Sinuum proportionales datæ.	Artificialium respondentes differentiæ.	Sinuum proportionales datæ.	Artificialium respondentes differentiæ.
Dupla	6931469.22	8000pla	8987134.68
Quadrupla	13862938.44	10000pla	92103369.36
Octupla	20794407.66	20000pla	99034838.58
Decupla	23025843.34	40000pla	105966307.02
20oupla	29957311.56	80000pla	112897777.70
40oupla	36888780.78	100000pla	115129211.70
80oupla	43820250.00	200000pla	122060680.92
Centupla	46051684.68	400000pla	128992150.14
200pla	52983153.90	800000pla	135923619.36
400pla	5991462312	1000000pla	138155054.04
800pla	66846092.34	2000000pla	145086523.26
Millecupla	69077527.02	4000000pla	152017992.48
2000pla	76008996.24	8000000pla	158949461.70
4000pla	82940465.46	10000000pla	161180896.38

54. ***Omnium sinuum ultra limites radicalis Tabulæ exclusorum, numeros artificiales investigare.***

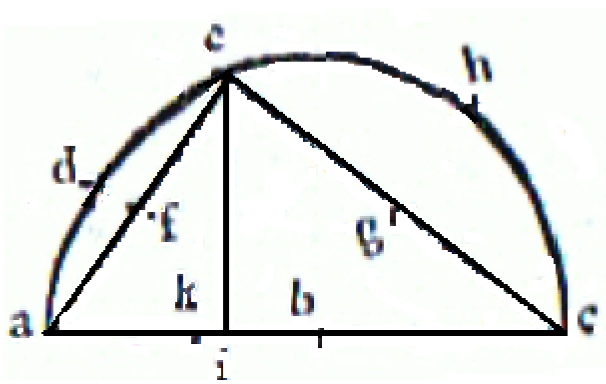
Hoc facilè fit, sinum datú multiplicando per 2, 4, 6, 8, 10, 20, 40, 80, 100, 200 : vel per alium quemvis proportionis numerum hac tabella expressum, donec producatur numerus, qui intra limites radicalis tabulæ contineatur. Hujus jam sub Tabula comprehensi artificialem (per 50) quære, cui acquisito adde tandem differentiam artificialem, quam Tabulla indicat priori convenisse multiplicationi.

EXEMPLUM.

Quæratur, quem artificialem sinus 378064 habeat; is cum ultra limites Tabulæ radicalis excludatur, per numerum aliquem proportionem præcedentis tabellæ, utpote per 20 ducatur, fietque 7561280, cujus jam infra Tabulam cadentis artificialem (per 50) quære, fietque 2795444.9, ad quem adde differentiam in Tabulla inventam convenientem vigecuplæ proportioni, quæ est 29957311.56, fientque 32752756.4. Unde 32752756 est artificialis quæsitus, sinus 378064 dati.

55. ***Ut dimidium sinus totius, se habet ad sinum dimidii alicujus arcus ; Ita sinus complementi ejusdem dimidii, ad sinum totius arcus.***

Sit sinus totus a b, dupletur & sit a b c; hac diametro fiat semi-circulus, in quo signetur arcus ille a e, bifarium in d divisus : ejus ergo dimidii quod est d e, extendatur complementum ab e versus c, quod sit arcus e h, cui & h c necessariò æquatur : quia d e h quadrans reliquo quadranti arcuum a d & h e. Proinde ducantur linea e i perpendicularis ad a i c, quæ ideò sinus est arcus a d e : & linea a e cuius dimidium f e, est sinus arcus d e, qui est dimidium arcus a d e : & linea e c, cujus dimidium e g est sinus e h, & ideò est sinus complementi arcus d e: dimidium autem sinus totius a b sit a k. Dico ut a k se habet ad e f, ita e g ad e i se habeat : duo enim trianguli c e g, & c i e, æqui-anguli sunt : quia i c e, vel a c e angulus utrique communis est, & uterque c i e, & c e a rectus est, ille ex hypothesi, hic, quia in peripheria est, & semi-circulum occupat. Ideoque ut ac hypotenusa trianguli c e a, ad ejus minus latus a e; ita se habet c e hypotenusa triang. c i e, ad ejus minus latus e i. Et denique cum jam totum a b est ad totum a e, ut e g ad e i : Concludimus necessariò dimidium a b, quod est a k, se habere ad dimidium a e, quod est f e : ut e g se habet ad e i, quod erat demonstrandum.



56. ***Duplum artificialis arcus 45 graduum, est artificialis dimidii sinus totius.***

Repetito præcedenti Schemate sit casus talis, quod a e, & e c, sint æquales. In hoc casu cadet i in b, eritque e i sinus totus, atque e f, & e g æquabuntur : eorumque quivis sinus est 45 graduum. Et quia (per præcedentum) quæ est proportio dimidii sinus totius a k, ad e f sinum 45 graduum : ea est etiam proportio e g sinus quoque 45 graduum, ad e i iam sinum totum. Ideò (per 37) duplum artificialis sinus 45 graduum, æquale est artificialibus extremorum, scilicet sinus totius, & eius dimidii. At horum amborum artificiales, sunt tantum artificialis alterius eorum,scilicet dimidii sinus totius : quia reliqui scilicet ipsius totius (per 27) artificialis nullus est. Necessariò igitur duplum artificialis arcus 45 graduum, est artificialis dimidii sinus totius, quod erat demonstrandum.

57. ***Aggregatum ex artificiali dimidii sinus totius, & artificiali*** [p. 35]***cuiusque arcus, æquatur aggregato artificialium dimidii ejus arcus, & complementi hujus dimidii. Unde artificialis huius dimidii arcus haberi potest, cæterorum trium artificialibus datis.***

Quia per penultimè præmissam, dimidium sinus totius proportionatur ad sinum dimidii alicuis arcus, ut sinus complementi eiusdem dimidii arcus, ad sinum totius arcus : Ideò (per 38) aggregatum artificialium duorum extremorum, scilicet artficialis dimidii sinus totius, & artificialis cuiusvis totalis arcus, æquabitur aggregato artificialium mediorum, videlicet artificialis dimidii eiusdem arcus, & artificialis complementi hujus dimidii. Unde & per eandem 38, si addideris artificialem dimidii sinus totius (per 51, vel per præmissam inventum) ad artificialem cuiusvis totalis arcus datum : & hinc subtraxeris artificialem complementi dimidii prioris arcus datum, relinquetur ipde artificialis petitus eiusdem dimidii arcus : quæ erant demonstranda.

EXEMPLUM. Sit artificialis dimidii sinus totius (per 51) 6931469, sitque arcus totalis 69 graduum & 20 minutorum, cuius artificialis sit 665143 datus : totalis arcus dimidium est 34 graduum & 40 minutorum, huius artificialem quæro. Complementum huius dimidii arcus est 55 graduum, & 20 minutorum, cuius artificialis sit 1954370 datus : Addo itaque 6931469 ad 665143, & fiet aggregatum 7596612 : ex quo aufero 1954370, & relinquentur 5642242 artificialis quæsitus, arcus 34 graduum & 40 minutorum.

58. ***Datis artificialibus omnium arcuum non minorum 45 gradibus, omnium arcuum minorum artificiales facillemè habentur.***

Ex artificialibus arcuum omnium non minorem 45 gradibus per hypothesin datis, habebis per præmissam, artificiales reliquorum omnium arcuum decrescentium usque ad vigesimum secundum gradum cum semisse. Ex quibus iam habitis, artificiales similiter reliquorum arcuum usque ad 11 gradus & 15 minuta habebuntur. Et ex his rursus, artificiales omnium arcuum usque ad 5 gradus & 38 minuta. Et ita deinceps in primum usque minutum.

59. ***Tabulam Artificialem condere.***

Paginæ præparentur quadraginta quinque longiusculæ, ut præter margines superiorem & inferiorem, sexaginta etiam lineæ numerales capere valeant. Paginarum quælibet lineamentis transversis in 20 spatia æqualia dividatur : spatiorum quodvis tres lineas numerales capere valeat. Inde aliis lineis descendentibus dividatur pagina quævis in columnas septem, interposita duplici linea inter columnas secundam & tertiam, & inter quintam & sextam : inter cæteras verò simplex ponatur linea. Prima pagina in fronte superiore lævorsum, supra tres primas columnas superscribatur hoc titulo **| 0 *Gradus* |**, & subscribatur inferiùs & dextrorsum sub tribus ultimis columnis sic **| 89 *Gradus*** | Secunda pagina superscribatur lævorsum sic **| 1 *Gradus* |** , & subscribatur sic **| 87 *Gradus* |**. Et ita cum cæteris paginis procedendo, ut suprà scripti infrà scriptis additi, quadrantem uno minus sive 89 gradus semper compleant. Inde prima columna per singulas paginas titulum hunc suprascriptum habeat **| *Minuta graduum suprascriptorum* |.** Secunda columna hoc titulo superscribatur **| *Sinus arcuum sinistorum*|.** Tertia columna hoc titulo superscribatur **| *Artificiales arcuum sinistorum*|.** Quarta columna hoc titulo & superscribatur **| *Differentia inter Artificiales complementorum*|.** Quinta columna subscribatur hac subscriptione **| *Artificiales arcuum dextrorum*|.** Sexta columna subscribatur hac subscriptione **| *Sinus arcuum dextrorum*|.** Septima columna subscribatur hac subscriptione **| *Minuta graduum infra scriptorum* |.** [p. 37.] Primæ diende columnæ inserantur numeri minutorum ab 0 ad 60 progriendo. Septimæ etiam columnæ inserantur numeri minutorem à 60 ad 0 decrescendo: ea lege, ut primæ & septimæ columnæ bina quævis minuta in eadem linea opposita, gradum integrum seu 60 minuta perficiant. Exempli gratia, 0 ad 60, & 1 ad 59, & 2 ad 58, & 3 ad 57, &c. opponantur. Atque; inter bina quæque viginti lineamentorum transversorum, tres numeri in quolibet intervallo cujuslibet columnæ contineantur. In secunda columna ponantur numeri sinuum, respondentium gradibus suprà, & minutiis à latere lævorsum in eadem linea positis. In sexta etiam columna ponantur numeri sinuum respondentium gradibus infra, & minutiis à latere dextrorsum in eadem sinuum RHINHOLDI Tabula, vel si qua exactior. His peractis, omnium sinuum inter sinum totum & suum dimidium, artificiales per 49 & 50 : cæterorum verò sinuum artificiales per 54 computato. Sive aliter, multoque & exactius & facilius, omnium sinuum inter sinum totum & sinum 45 graduum artificiales, per easdem 49 & 50 computato : ex quibus jam habitis, omnes reliquorum arcuum minorum 45 gradibus artificiales, per præmissam minorum 45 gradibus artificales, per præmissam quam facillimè acquires. Quibus omnibus artificialibus utcunque computatis, in tertia columna locabis artificiales numeros respondentes gradibus suprà, & minutiis à latere sinistro, suisque & in quinta columna locabi numeros artificiales respondentes gradibus infrà, & minutiis à latere dextro, suisque sinibus dextrorsum in eadem linea positis. Media tandem columna sic perficitur : numerum quemque artificialem dextrum, ex artificiali sinistrorsum in eadem linea posito aufer, notat â differentiâ in eadem linea linea inter utrumque, donec totam mediam columnam compleveris. Hanc Tabulam nos ad singula minuta computavimus, [p. 38.] atque eruditis (quibus plus sit otii) ejus exactiorem elimationem, ut & Tabulæ sinuum emendationem relinquimus.

Epitome Tabulæ artificialis
aliter condendæ.

60. ***Quia nonnunquam artificiales per 54 inventi, differunt ab artificialibus per 58 inventu; ut huius sinus* 378064, *numerus artificialis per illam est* 32752756, *per hanc verò est* 32752741 *; arguitur quibusdam in locis Tabula sinuum vitiosa esse. Quapropter consulo eruditis (quibus forsan dicipularum & computistarum copia sit) ut Tabulam sinuum exactiorem &maioris numeri edant, utpote cuius sinus totus sit 100000000, scilicet octo cyphrarum prater unitatis figuram, cum prior sinus totus septem tantum constet. Deinde ut Tabula nostra prima contineat centum numeros, progredientes in ea proportione, quæ est inter hunc novum sinum totum, & sinum eo minorem unitate, utpote inter* 100000000, *&* 99999999.**

***Secunda Tabula contineat etiam centum numeros, in ea proportione, quæ est inter hunc novum sinum, & numerum eo minorem centerario,scilicet inter* 100000000, *&* 99999900.**

Tabula Tertia quæ & radicalis dicitur, trigintaquinque colummnas, & centum numeros in qualibet columna continet. Centum numeri eiusdem columnæ progrediantur in ea proportione, quæ est decem millium, ad numerum eo minorem unitate, videlicet* 100000000 *ad* 99990000. *Trigintaquinque primi inter se, aut secundi, aut tertii, aut cæteri eiusdem ordinis omnium columnarum inter se progrediuntur ea proportione, quæ est 100 ad 99, aut sinus totius* 100000000 *ad* 99000000. *In his suisque artificialibus inveniendis & continuandis, observentur regulæ cæteræ præcedentes. Atque ex completa sic radicali Tabula, omnium sinuum inter sinum totum & sinum 45 graduum artificiales, exactißimè per 49 & 50 reperies : atque ex artificiali arcus 45 graduum duplato, habebis artificialem dimidii sinus totius per 56. Et tandem ex his iam habitis,

cæteros artificiales per penultimè præcedentem exquires; quos in ordinem Tabulæ per præcedentem rediges, & fiet Tabula, omnium certè Mathematicarum Tabularum præstantißima & ad usus præclarisimos parata.

Finis constructionis Tabulæ Artificialis.