APPENDIX.

With the construction of a more outstanding kind of logarithm than the other; in which it is agreed that the logarithm of unity is 0.

***Among the various progressions of logarithms, that kind is more outstanding in which zero is put in place for the logarithm of one, and* 10,000,000,000 i*s put in place for the logarithm of a one over ten-fold, or alternatively of a ten-fold increase : moreover the logarithms of all the other numbers necessarily follow as a consequence from these, and there are several methods by which these can be found*.**

[One should note that the methods of producing the two kinds of logarithms are completely different : Napier's original tables are analytical in nature, and can even be taken as the solution of a first order differential equation, in which the speed of a particle moving towards a fixed point varies with its distance from that point, and it is evidently an exponential decay, as with the swings of a damped pendulum, with the logarithm given as the number of constant periods from release [related to the log-decrement of physics], according to the 'decay constant' and initial amplitude, both taken as 10^7. The translator has a paper in the *Amer. J. Phys.* (Feb. 2000), in which this view is investigated.

On the other hand, the new base 10 logarithms rely on the logarithms of primes being found, and all the other numbers have their logarithms found by addition or subtraction of these logarithms. Finding the logarithms of the small primes takes one right into the heart of the behaviour of numbers that have had their square root extracted a large number of times. The first part deals with taking repeated fifth and then square roots of ten, of which the logarithms are known. Briggs subsequently did a similar sort of thing using only square roots in Ch. III of the *Arithmetica*. These successive fifth roots are not presented here numerically, but their logarithms are easy to write down. Subsequently the method of taking successive square roots of a number is set out, starting with 10, used extensively by Briggs in Ch. VI of the *Arithmetica.* This approach is knocking on the door of the limit idea, and is the definition of the natural logarithm to this day, first used here to evaluate what we now see to be the natural logarithm of 10; non of this was of course part of the reasoning of either Napier or Briggs, but is shows how close one can get by reasoning intuitively. The method ends abruptly and is obviously part of a manuscript.

After a summary that includes some intriguing thoughts for the time, another method for finding the logarithms of small primes is pulled out of the hat, as it were. This other approach mentioned, in which very large numbers are used, is briefly summarised here; and this is set out by Briggs in Ch. V of the *Arithmetica*, translated in this series. Finally note that in the first case of Napier's original logarithms, the logarithms are given and the amplitudes or numbers agreeing with them are calculated; in the second case, for whatever manner of construction, the numbers are given and their logarithms to base 10 are found. One may also wonder at the cessation of the use of the decimal point in the appendix, which might indicate that the son Robert has something to do with putting it in order, and he was not familiar with Napier's notation; Briggs of course never used the decimal point, and relied on the positions of the digits in his ubiquitous tables to indicate where the decimal point had to lie.]

The given ten-fold of the given logarithm, clearly* 10,000,000,000, *has its figures divided by five, and thus in turn makes the sequence of logarithms* 2000000000, 400000000, 80000000 16000000, 3200000, 640000, 128000, 5120, 1024. *The final ten-fold part of these can be divided by two a number of times, and hence form the sequence of numbers* 512, 256, 128, 64, 32, 16, 8, 4, 2, 1. *And all these numbers are logarithms. [Thus, for example, $10^{0.2} = A; 10^{0.04} = B$; *etc.*]

Hence we can now look for the ordinary numbers that correspond to these logarithms, in the same order. Hence between a tenth of or ten times a given logarithm, four mean proportions can be taken [p. 41] ***(the increase to be calculated with some number of zeros, such as 12, as you wish), or rather of the smallest of these (by the extraction of the fifth root), which for argument's sake is called A. Similarly, between A and one, take the smallest of four mean proportions, which is B. Between B and one, take the first or smallest proportion, which is C. And thus to progress by the extraction of the 5th root*** [by successively taking the fifth root, the index is successively divided by 5]***, dividing the interval between that recently found and unity into five parts in proportion or four means; the smallest or the fourth of the four proportions is to be noted, until finally you arrive at the smallest of the tenth order means; these means altogether are called* A, B, C, D, E, F, G, H, I, K.** [Thus only one proportional number is to be taken from each power of 10 until 1024 is reached.] ***Now, go on calculating with these means, and between K and* 1 *find the mean proportion*** [now by taking the square root]***, which shall be L. Thus, take the mean proportion between L and* 1*, which shall be M. Thus similarly the mean between* M *and one shall be* N*. By the same stratagem, (by extracting the square root) the rest of the logarithms arise between any most recent number and unity, with these designated* O, P, Q, R, S, T, V*: for any logarithm in the above series there corresponds a proportional number in this series ; unity shall be the logarithm of the number V,***[as the decimal point is not present, this really means that $10^{0.00000000001} = V$] ***whatever it should be; and* 2 is *the logarithm of the number* T*; and of the number* S*, the logarithm shall be* 4*; and of the number* R, 8*; of the number* Q , 16*; of the number* P, 32*, of the number* O, 64*; of the number* N, 128 *, of the number* M , 256*; of the number* L , 512*; of the number* K, 1024*: All is apparent from the above construction. Moreover, from these now constructed, so the proportions of other logarithms can be constructed, as also the logarithms of other proportions. For as in statics, by the addition of weights of one amount, of double that amount, its quadruple, eightfold, and so on for all the other equal parts of weight, we are able to make a weight of any size, which are here represented for us by numbers and logarithms: Thus from the proportions* V, T, S, R*, etc, which correspond to these logarithms, and also from the others created in the duplicate ratio, etc.,*** [p. 42] ***the proportions of all the logarithms present can be constructed by the multiplication of these in turn, as found by experience. But a particular difficulty of this work is for the extraction of the 5th root with 12 places from the sixty figures needed : but this method apart from this difficulty is so much better in finding the logarithms of the proportions and the proportions of the logarithms.***

The logarithms of numbers can be easily found from the given logarithms of their primes.

If two given numbers with given logarithms are in turn multiplied to make a third number, the sum of the logarithms of the two is the logarithm of the third.

Likewise if one number is divided by another to produce a third, with the logarithm of the second taken from the logarithm of the first, there is left the logarithm of the third number.

If from some number squared, cubed, or raised to some higher power to produce some other number; from the logarithm of the first doubled, tripled, or multiplied by five, the logarithm of the other number is produced.

Likewise if from a given number, the square, cube, fourth, etc, root is extracted; and the logarithm of the given number is divided by two, three, or five, and the logarithm of that root is produced.

Hence any common number composed from common numbers by multiplication, division, or by root extraction; the logarithm of this number is composed respectively from addition, subtraction, duplication or triplication of their logarithms. Thus the only difficulty arises with finding the logarithms of the prime numbers [p. 43] ***, which are generally to be found by the following art.***

In order that the logarithms of all numbers can be found, it is necessary that the logarithms of some two common numbers are given, or even to assume from the beginnings of the work, as in the above first construction above, that **0** ***or the cipher is taken for the logarithm of the ordinary number one, and*** **10,000,000,000** ***for the logarithm of ten or*** **10*****. Thus with these given, the logarithm of the fifth part is sought (which is the first prime number)*** **in this way.** ***Between 10 and 1 the mean proportion is sought, which is*** $\frac{316227766017}{100000000000}$ ***. Thus the arithmetical mean is sought between*** **10,000,000,000** ***and*** **0**, ***which*** **5,000,000,000.** ***Then the geometric mean is sought between*** **10** ***and*** $\frac{316227766017}{100000000000}$ ***, which is*** $\frac{562341325191}{100000000000}$ ***. And the arithmetical mean is sought between*** **10,000,000,000 and 5,000,000,000,** ***which is*** **7500000000.**

[This work is evidently unfinished as it ends abruptly; this is the start of Briggs' Ch. VI.]

For all continued proportions :

As the sum of the means and of the one extreme is to the same extreme, thus the difference of the extremes is to the difference of the same extreme and the nearest mean.

[If $\frac{a}{b} = \frac{c}{d}$ then $\frac{\mathrm{a+b+c}}{\mathrm{a}} = \frac{a-d}{a-c}$; this does not appear to be a correct statement.]

A Saving of the half Table of LOGARITHMS.

Of two given complementary arcs of the quadrant : as the sine of the major arc to the sine of twice the arc ; Thus the sine of* 30 *degrees to the sine of the smaller arc. Hence with the logarithm of the sine of twice the arc added to the logarithm of* 30 *degrees; and from the product, by taking the logarithm of the sine of the greater arc, there is remains the logarithm of the sine of the smaller arc. [p. 44]

The reciprocal relations between logarithms and their natural numbers.

1. ***Two sines and their logarithms are given. If just as many numbers arise by multiplying the smaller sine into itself, as there are units in the logarithm of the larger sine; and in turn, if there are just as many numbers arising from multiplying the larger sine a certain number of times by itself, as there are units in the logarithm of the smaller sine; then two equal numbers are produced, and the number produced from the logarithm of the sine is the number made from both logarithms multiplied in turn.***

[The sines of two arcs A and B are given, with $A < B$, together with their logarithms : *i. e.* sin A and log sin A; sin B and log sin B are given.

We are given that $(\sin A)^{sinB} = \log \sin B$; and similarly, $(\sin B)^{sinA} = \log \sin A$;

hence, $\sin B \log \sin A = \log \log \sin B$; and $\sin A \log \sin B = \log \log \sin A$; then on adding,

$\log (\log \sin A \times \log \sin B) = \sin B \log \sin A + \sin A \log \sin B$; or,

$\log[(\sin A)^{sinB} \times (\sin B)^{sinA}] = \log(\log \sin A \times \log \sin B)$; giving,

$(\sin A)^{sinB} \times (\sin B)^{sinA} = \log \sin A \times \log \sin B$.]

2. ***As the greater sine to the smaller sine; thus the velocity of increase or decrease of the logarithms of the smaller, to the velocity of increase or decrease of the logarithms of the greater.***

3. ***Two sines in the ratio of the square, cube, quadruplicate, etc, have their logarithms in the double, triple, quadruple ratio, &c.***

4. ***And two sines in the same ratio as from one order to another order, (that is, triplicate to quadruplicate, or cube to fourth order) have their logarithms, in the same ratio as the indices of the order, such as 3 to 5.***

5. ***If the first sine multiplied by the second sine produces a third sine, then the logarithm of the first sine added to the logarithm of the second sine produces the logarithm of the third sine. Thus in division, the logarithm of the divisor from the logarithm of the dividend, leaves the logarithm of the quotient.***

6. ***And if so many numbers equal to the first sine are to be multiplied in turn, to produce the second; just as many equal to the logarithm of the first likewise are to be added together to produce the logarithm of the second sine.*** [Thus, if $(\sin A)^n = \sin B$ then $n \log \sin A = \log \sin B$.]

7. ***Any geometric mean between two sines, has the logarithm of its mean equal to the arithmetic mean of the logarithms of the sines.***

8. ***If the first sine divides a third as many times as there are units in A, and a second sine divides the same third as many times as there are units in B: Likewise the same***

first sine divides a fourth, in as many units as there are in C; and likewise the same second divides the same fourth as many times as there are units in D. I say, that the ratio A to B, is the same as C to D, and likewise for the logarithm of the second to the logarithm of the first. [sinA3/sinA1 = A; sinA3/sinA2 = B; sinA4/sinA1 = C; sinA4/sinA2 = D. Hence, A/B = sin A2/sin A1= C/D.]

9. ***Hence it shall be that the logarithm of the number offered is the number of places or figures, that can be taken from the product of the number multiplied by itself as many times as there are places in*** **10,000,000,000 multiplied in the same manner.**

[Rather than explain this method here, you are referred to Briggs' *Arithmetica*, Ch. V as mentioned earlier, if you have not met this method before and you would like to know more about it. Some hints are now provided, as a final flourish by John Napier, that grand old Scottish philosopher : for whom you may look in vain to find a monument to honour his existence, in his 'ain toon of Auld Reekie' : shame on you, Edinburghians! You have honoured lesser men far more. Perhaps the 400th aniversary of the birth of logarithms will stir some action.....]

10. ***Likewise if the index*** [one] ***is the logarithm of ten, the number of figures less one will be the logarithm of the root*** [of the product, or of the original number] **.**

It is asked, what number shall be the logarithm of two? I reply, the number of places of the number made by multiplying 2 by itself in turn 10,000,000,000 times.

But you say, here the number made from 2 by doubling it 10,000,000,000 is beyond counting. I reply, yet the number of places of this that I seek is countable. Thus from the given root [i.e. base number] **two and the index (10,000,000,000) you must find the number of places to be multiplied, and not the actual number itself that arises from the multiplication ; and by our rule you find 301029995 &c. for the number of places sought, and for the logarithm of two.**

FINIS.

APPENDIX.

De alia eaque præstantiore LOGARITHMORUM *specie construenda*; in qua scilicet, unitatis Logarithmus est 0.

Inter *varios* Logarithmorum *progressus, is est præstantior, qui cyphram pro* Logarithmo pro *unitatis statuit, &* 10,000,000,000 *pro Logarithmo denarii seu decupli instituit: cæterorum autem omnium* Logarithmi*, ex his stabilitis necessariò consequentur, & modus inveniendi eos varius est, quorum sic se habet.*

Logarithmorum *decupli datum, videlicet* 10,000,000,000, decies partire per quinque; & fient inde numeri sequentes 2000000000, 400000000, 80000000 16000000, 3200000, 640000, 128000, 5120, 1024. *Horum ultimum decies etaim bipartire, & fient inde numeri sequentes* 512, 256, 128, 64, 32, 16, 8, 4, 2, 1. *Atque hi omnes numeri sunt* Logarithmi*. Quæramus igitur singulorum numeros vulgares, qui iis ordine respondent. Inter denarium ergo seu decuplum* [p. 41] 10 *atque unitatem (auctos calculi gratiâ quotvis cyphris, uspote duodenis)capiantur quatuor media proportionalia, seu potiùs (per extractionem radicis super solidæ)eorumdem minimum, quod sit doctrinæ gratiâ A. Inter A & unitatem, capiatur similiter ex quatuor proportionalibus minimum medium, quod sit B. Inter B & unitatem, capiatur medium quartum seu minimum, quod sit C. Et ita progredere per extractionem supersolidæ radicis, dividendo intervallum inter recens inventum & unitatem, in quinque intervalla proportionalia seu in quatuor media; quorum omnium quartum seu minimum semper notetur, usque dum ad decimum medium minimum perveneris; quæ his notis signentur* D, E, F, G, H, I, K. *Computatis jam exactè hisce proportionalibus, perge, & inter* K *& unitatem quare medium proportionale, quod sit* L. *Sic inter* L *& unitatem cape medium proportionale, quod sit* M. *Sic simile medium inter* M *& unitatem, quod sit* N. *Eodem articificio (per extractionem quadratam)creentur inter quemque recentem numerum & unitatem, reliqua intermedia proporitonalia, his notis signanda* M, N, O, P, Q, R, S, T, V*: Quorum proportionalium cuilibet, respondes ordine suis* Logarithmus *superioris seriei; & unitas erit* Logarithmus *numeri V, quicunque fuerit; &* 2 *erit* Logarithmus *numeri* T, *&* 4 *numeri* S, *&* 8 *numeri* R, 16 *numeri* Q, 32 *numeri* P, 64 *numeri* O, 128 *numeri* N, 256 *numeri* M, 512 *numeri* L, 1024 *numere* K*: Quæ omnia ex superiore constructione patent. Ex his autem jam constructis, construi possunt aliorum tum* Logarithmorum *proportionalia, tum proportionalium* Logarithmi. *Nam sicuti in staticis ex additione ponderum unitatis, binarii, quaternarii, 8tii, & aliorum pariter parium numerorum, omnis creari potest ponderum numerus, qui apud nos jam* Logarithmi *sunt: Ita ex proportionalibus* V, T, S, R, *&, quæ illis respondent, & ex cæteris etiam duplicatâ ratione creandis, constitui* [p. 42] *possunt omnium* Logarithmorum *oblatorum respondentia proportionalia, per eorundem invicem multiplicationem respectivè, ut docebit experientia. Hujus autem operis præcipua difficultas, est in denis proportionalibus duodecim figurarú è sexaginta figuris supersolido more extrahendis : sed quanto major hæc difficultas, tanto exactior est hic modus in* Logarithmis *proportionalium, &* Logarithmorum *proportionalibus inveniendis.*

Alius modus facilè creandi LOGARITHMOS numerorum compositorum, ex datis LOGARITHMOS suorum primorum.

***Si duo numeri datorum* Logarithmorum, *invicem multiplciati componunt tertium; eorum* Logarithmorum *aggregatum erit tertii* Logarithmus.**

***Item si numerus per numerum divisus producit tertium, è primi* Logarithmo *secundi subtractus; relinquit tertii* Logarithmorum.**

***Si ex numero in se quadratè, cubicè, supersolidè, &c. ducto, producitur alter quivis;ex primi* Logarithmo *duplato, triplato, aut quintplato, producitur illius alterius* Logarithmus.**

Item si ex dato per extractionem quadratam, cubicam, supersolidam, &c. extractionem quadratam, cubicam, supersolida, &c., extrahatur radix; datique* Logarithmus *bisecetur, trisecetur, aut per quinque secetur, prodecetur* Logarithmus *ejusdem radicis.

Deinque quicunque numerus vulgaris ex vulgaribus componitur per multiplicationem, divisorem, aut extractionem: ejus* Logarithmus *componitur respectivè per additionem, subtractionem, duplationem, seu triplicationem, &c. suorum* Logarithmorum. *Unde sola difficultas est in numerorum primorum [p. 43] ***inveniendis; qui hac sequenti arte generali invenientur.***

***Ad omnes* Logarithmos *inveniendos, oportet duorum aliquorum vulgarium numerorum* Logarithmos *dari, aut saltem assumi pro fundamento operis, ut in superiore prima constructione,* 0 *seu cyphra assumebatur pro* Logarithmo *vulgaris unitatis, &* 10,000,000,000 *pro* Logarithmus *denarii seu* 10. *His itaque datis, quæratur quinarii (qui primus numerus est)* Logarithmus *hoc modo. Inter 10 & 1 quæratur medium proportionale, quod est* $\frac{316227766017}{100000000000}$. *Sic inter* 10,000,000,000 *&* 0 *quæratur medium Arithmeticum, quod est* 5,000,000,000. *Deinde inter* 10 *&* $\frac{316227766017}{100000000000}$ *capiatur medium Geometricum, quod est* $\frac{562341325191}{100000000000}$. *Et similiter inter* 10,000,000,000 & 5,000,000,000, *quod est 7*500000000.**

In continuè proportionalibus universis.

Ut summa mediorum & alterutrius extrmi, ad eundem extremum; sic differentia extremorum, ad differentiam extremi & medii proximi.

Compendium dimidii Tabulæ LOGARITHMORUM.

Duorum arcuum quadrantem complentium, ut sinus majoris, ad sinum dupli arcus ; Ita sinus* 30 *graduum, ad sinum minoris. Unde addito* Logarithmo *dupli arcus ad* Logarithmum 30 *graduum; & à producto, subducto* Logarithmo *majoris, relinquitur* Logarithmus *minoris. [p. 44]

Habitudines LOGARITHMORUM & suorum naturalium numerorum invicem.

1. ***Dentur duo sinus & sui* Logarithmi. *Si totidem numeri æquales sinui minori in se ducantur, quot sunt unitates in majoris* Logarithmo : *& contrà, totidem æquales sinui majori in se ducantur, quot sunt unitates in minoris* Logarithmo; *erunt duo producta æqualia, & producti sinus* Logarithmus, *erit numerus factus ex ambobus* Logarithmis *invicem multiplicatis.***

2. ***Ut sinus major ad minorem; Ita velocitas incrementi, aut decrementi* Logarithmorum *apud minorem, ad velocitatem incrementi aut decrementi* Logarithmorum *apud majorem .***

3. ***Duo sinus in ratione duplicata, triplicata, quadruplicata, &c. habent suos* Logarithmos *in ratione dupla, tripla, quadrupla, &c.***

4. ***Et duo sinus in ratione ut orde ad ordem, (id est ut triplicatum ad quintuplicatum, vel cubus as supersolidum) habent suos* Logarithmos, *in ratione ut eorundem ordinum indices, id est, ut 3 ad 5.***

5. ***Si primus sinus in secundum ductus producit tertium;* Logarithmus *primi additus secundi* Logarithmo *producit tertii* Logarithmum. *Sic in divisione, divisoris* Logarithmus *ex dividendi* Logarithmo *subductus, relinquit quotientis* Logarithmum.**

6. ***Et si quot æqules primo, invicem ducti producunt secundum; totidem, æqules primi* Logarithmo, *simul additi producunt* Logarithmum *secundi.***

7. ***Medium quodvis Geometricum inter duos sinus, habet suum* Logarithmum *medium tale Arithmeticum inter sinuum* Logarithmos.**

8. ***Sinus primus dividis tertium, quoties sunt unitates in A; numerus secundus dividit eundem tertium, quoties sunt unitates in B : Item idem primus dividit quartum, quoties sunt unitates in C; & idem secundus dividis eundem quartum, quoties sunt unitates in D. Dico, quæ est ratio A ad B, eadem est C ad D, &* Logarithmi *secundi ad* Logarithmum *primi.***

9. ***Hinc sit quod numeri oblati* Logarithmus, *est numerus locorum seu figurarum, quas comprehendit factum ex oblato toties in se ducto quoties sunt unitates in* 10,000,000,000.**

10. ***Idem si index ordinis sit* Logarithmus *denarii, numerus figurarum (unâ demptâ) ordinis scilicet multipli, erit* Logarithmus *radicis.***

Quæritur, quis numerus sit Logarithmus binarii. Respondeo, numerus locorum numeri facti ex 10,000,000,000 binariis invicem ductis.

At dices, hic numerus factus ex 10,000,000,000 binariis invicem ductis est innumerabilis. Respondeo, numerus tamen locorum ejus (quem quæro) est numerabilis. Ex data itaque radice (binario) & indice (10,000,000,000) quære numerum locorum multipli, & non numerum ipsius multipli; & per regulam

nostram invenies 301029995 &c. pro numero locorum quæsito, & Logarithmo binarii.

FINIS.