GRADUATE DISSERTATION: SHOWING CERTAIN SQUARABLE CIRCULAR LUNES.

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§I.

A Lune in the most general sense can signify a bounded figure, described by two curved lines crossing each other on some surface; and to be called either planar or spherical (*a*) according to the differing natures of the surface (*b*).

(*a*) Leibniz proposed the squaring of certain *Spherical Lunes* Act. Erud. Lips. A. 1692. p. 277.

(*b*) Also it is possible now to refer the curved surface of *Ungulates* as a class of Lunes.

§II.

From the nature of the curves(§I.), plane Lunes in turn are sorted into various denominations (*c*) . But now only circular Lunes are to be considered.[p.2]

(*c*) Squarable Elliptic Lunes are shown in *Kongl. Vet. Acad. Handl.* 1757, following p. 218. Mention was made of cyclo-parabolic lunes, i.e. of lunes contained by circular and parabolic arcs, by WOLF in *Act. Erud. Lips*. 1715, p. 213, 217.

§III.

For us the circular lune (*) is the area ABCD described in a common plane by two circular arcs, with the convex arc ACB and the concave arc (*d*) ADB taken to be intersecting each other (*e*).
[Note: we are looking inwards from the outside of the lune for this definition to apply.]
Also sometimes accustomed to be called by some the *Meniscus*.

(*) See KRAFFT (*in Instit. Geom. Sublimior*, T. I. § 165.) the general theory of lunes, and formerly he gave an elegant theory in *Exercitationibus Mathematicis*, Venetiis A.1724 edition; and amongst others, a method of determining all the innumerable squarable lunes was proposed by DAN. BERNOULLI.
A general account that is also contained in §§ 5 – 7, 21 of a certain outstanding dissertation (see *Comment.Petrop.T.IX*.p.207, and following) of L. EULER is occupied in solving a certain geometrical problem about lunes formed from circles, only that a particular solution of this was given before in the name of BERNOULLI.

(*d*) It is evident that a lune arises from one convex circle pointing outwards and the other in turn concave; for I know that it is not indeed the usual custom for a lune to be said to have both figures convex ; though I may have more to say about the construction of the square of any space from two segments of circles.

(*e*) There is a need for all these to be considered jointly, in order that the planar lune can be understood, so that some ring-like area can then be discerned from the whole circle. And though sometimes we may have seen the name lune arrived at as the area, [p. 3] as between the peripheries of two whole circles, of which the one lies as a tangent within the other (as in *Act.Erud.Lips*. 1709. p. 81, & KRAFFT. *Instit. Geom.Sublim*. T. I. §. 181): yet now we remain with the said accepted definition.

Everything remaining which is mentioned in §§. 4 – 9 about these lunes, we refrain from relating everything or giving a complete demonstration; since partially these come from the first elements of geometry, and partially they pertain to a chapter of our demonstration not yet reached.

§IV.

Let (*f*) E, F (Fig. 1, 3, 4, 5, 6) be the centres of circles ACB and ADB, respectively: the right line FEDC passing through these centres deserves to be called the *Axis of the Lunes* ; indeed the arcs of the whole figure are clearly bisected into equal parts, so that the angles AEB and AFB, which can be called the *angles of convexity* & *concavity* evidently can be put in place and these divide the lune and the arcs of this into similar parts on both sides (*g*); the sectors are AEBCA and AFBDA; the segments are ABCA and ABDA [A note in translation: the order of these has been inverted from these in the text]; and finally the common chord for the circles is seen to be the line AB or the *Base* of the lune, and which is perpendicular to this axis. Therefore the area formed from the arcs and the right lines ACD or BCD may be called the *Semilunes*.

(*f*) See EUCL. Elem. Book. III. Prop. 5.

(*g*) And as regards the other items, *the angles of sections*, see EUCL. Elem. VI.33. For those remaining, it is possible that for these angles, either one or both can exceed two right angles, indeed such angles are occasionally called gibbous or convex (French : *Angle rentrant* : returning angle). Moreover the halves of these, or the angles AEC and AFC, or the angles considered the most relevant, are less than two right angles; such angles are occasionally specified by a name and called *concave* (*Angles saillans*: projecting angles). [p. 4]

§V.

The curvilinear angle CAD (or CBD), which can be called the angle of the lune, must be considered equal to the rectilinear angle EAF that the radii EA and FA of the circles make at the point of intersecting each other at (§3) A (*b*).

(*b*) See EUCL. Elem.Book.III.Prop.16.

§VI.

The arcs of lunes cannot be whole and are unequal and dissimilar to each other. Clearly the convex arc is greater in length than the concave arc: and has a greater proportion to the whole periphery of the circle, or (§4) the angle of convexity is greater than the angle of concavity (*i*) ; & (*k*) half the difference of these is equal to the angle of the lune (§5). Hence the sectors EACB & FADB are also dissimilar, and the halves of these are EAC & FAD.

(*i*) Clearly then, with the centres E and F, the circular arcs AC and AD are described through A within the angle AFD [E in the original] or AEC : then (EUCLID.Elem.III.7.) FA or FD < FC; hence it is apparent that the first of the whole arcs (§4) of the lune to be convex outwards and the following arc is concave. And there the external angle AEC of the Triangle AEF is put in place, to this the internal angle AFE. Hence, &c. see EUCL.Elem.I.16. VI.33.Cor.1.

(*) See EUCL.Elem.I.32.

§VII.

If the radius of convexity were greater (less) than the radius of concavity; then the maximum height of the lune CD, which more simply is called [p. 5] the height of the lune, is equal to the distance between the centres EF, on taking (adding) the difference of the radii. Or generally: $CD = EF + EA - FA,$ i. e. the height of the lune is produced, if the separation of the centres is added to the radius of the convexity, and the radius of the concavity is taken away.

§VIII.

But if both radii should be equal, in which case what can be called the lune of equal curvature is obtained : the width of this is equal to (§7) the separation of the centres, and the sum of the arcs or the perimeter of the lune, is equal to the total perimeter of the circle; thus so that one is the supplement of the other for the total periphery of the circle : indeed the convex arc is the greater part of the circle and the concave arc the smaller part (*l*).

(*l*) Cf. §6.

§IX.

Two whole circles on cutting each other make two opposite lunes ; hence for which they have in common the centres of the arcs, the axis and base, and equal angles (§5). But the convex arc of one is the supplement of the concave arc of the other according to the whole circumference [of the lune], and half the sum of the widths is equal to the separation of the centres (§7) ; and these lunes are unequal and dissimilar, unless they have the same curvature (§8).

§X.

If it is possible to construct equal circular sectors EAC and FAD; it follows either by subtraction as in (Fig. 1, 3, 4), or by addition as in (Fig. 5) that if the area from the mixed lines EAD that lies between these sectors and the axes, is then either taken away or added to each, then the semilune ACD becomes equal to the rectilinear triangle EAF; and thus a perfect square is given of the lune itself. It is apparent that it is not indeed possible for any lune of equal curvature to be squared in this way §6.

§XI.

For a given radius, the circular sectors are (*m*) as the angles of these ; moreover, for a given angle, as the squares of the radii (*n*); whereby with neither given, or generally they are as the angles and as the squares of the radii together, i. e. the sectors EAC and FAD are in the composite ratio :

$$\text{ang.}AEC \times AE^2 : \text{ang.}AFC \times AF^2.$$

(*m*) EUCL.Elem.L.VI.Prop.33.Cor.1.

(*n*) *Cf.* EUCL.Elem.L.XII.Prop.2.

§XII.

Therefore in obtaining equal circular sectors from dissimilar circles (§10) it has to be brought about that (§6) the angles of these are as the inverse squares of the radii (§11). Thus finally the sectors EAC and FAD are equal if :

$$\text{ang. AEC : ang. AFC :: AF}^2 : \text{AE}^2,$$

clearly the angles of convexity and concavity are inversely as the square of the radii of the sectors.

Schol. Now also the arcs AC and AD, or ACB and ADB, are then inversely as the radii of these AE and AF. For indeed the sectors are always composed in the ratio of the radii and the arcs.

Cf. ARCHIMED. *de Circulo & adscr*.

§XIII.

Therefore with some ratio of the angles of convexity and concavity given or taken (§4), which ratio shall be equal to $m : n$ and indeed (§6) $m > n$: the working there returns the rectilinear triangle *AEF* [e.g. see Fig. 1, 3, ..6] as set out in (§12), the external angle of this *AEC* shall be to the opposite internal angle *AFE* as m: n, or the sides

$$AF : AE :: \sqrt{m} : \sqrt{n} :: m : \sqrt{mn} :: \sqrt{mn} : n.$$

Or since the same is returned (*o*) : a kind of triangle AEF is found, in which there are two internal angles $EAF : AFE :: m - n : n$. Thus there is obtained a perfectly squarable lune, of this indeed the angle (§5, §6) is to half the angle of the concavity as $m - n : n$.

(*o*) EUCL.Elem.L.V.17.I.32.

Schol. We devise constructions for the brevity of the student, but only lunes of this kind that can be completed with a ruler and a circle, and thus indeed to be of merit to geometry; or only these rational squarable lunes are to be treated, of which the delineation requires on more than the postulates of Euclid. Above all and in particular, we will consider these five cases here where $m : n = 2:1, 3:1, 3:2, 5:1, 5:3$. For neither fewer nor, I think, more squarable lunes can be constructed from the straight edge and a pair of compasses (*). Now [we ask] whether all these have been treated in turn either by BERNOULLI [p.8] (§3.note*) or by someone else ? This we do not know. Although there is nothing further remaining from the given ratio m : n, if indeed I probe, for the proposition (§10) must be satisfied; as the ratio *AF* : *AE* is determined arbitrarily in every way: yet we propose the most general constructions of our triangle *AEF*, by taking some first ratio of the sides *AF*: *AE*, and not depending on the ratio m : n. Moreover synthetic demonstrations from analysis are not difficult to be assembled from our geometry, and we omit diverse modifications of the construction to shorten the study.

(*)KRAFFT asserts that with enough elegance with these words (Geomet.Subl.Book.I. §167) : *More lunes of this kind can be found* (from these first two that he had consider)*, if other ratios are assumed expressed by the numbers*(such as 2 : 1 & 3 : 1) *; but the equations emerging thus for* NB, *always also rising to higher dimensions, and thus less can be put together*. What has been said can be judged to be more or less correct from our following sections §§ 20, 22, 24, 27.

CASE I. Fig. 1.

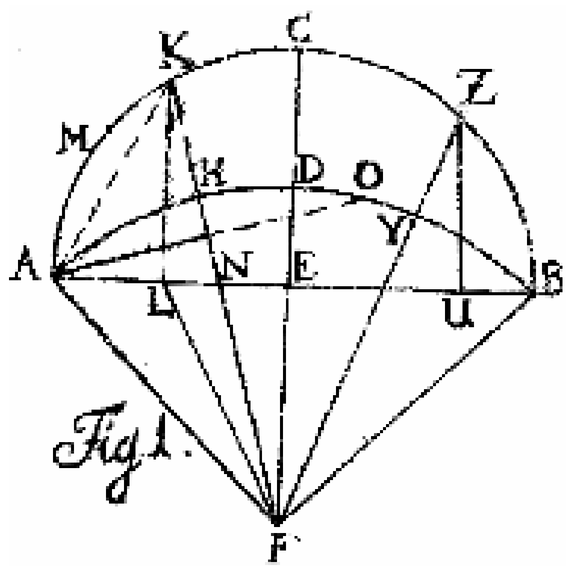


Fig. 1.

§XIV. If [the ratio] is put (§13) as *m* : *n* :: 2 : 1 then the angle EAF is equal to the angle AFE; and thus the triangle AEF is isosceles. And as generally the isosceles triangle has been constructed most easily, from a given ratio of the base AF and of the side AE or EF : thus and then in particular the ratio AF:AE has to be as the square root of (*m* : *n* =) 2 : 1. Clearly (§12.13) $AF^2 = 2AE^2 =$

(as $AE = EF$) $AE^2 + EF^2$,
hence (*p*) the angle AEF is right. Hence AE of given magnitude or of arbitrary size, is made perpendicular and equal to the line EF; then from the centres E & F or with the intervals EA & FA, are to described through [p.9] A the circular arcs (*) ACB & ADB that define the simplest and neatest squarable lune ACBDA, which is called after the discoverer HIPPOCRATUS, and indeed it is a special squarable lune. Clearly the convex arc ACB is a semicircle, the concavity ADB is the quadrant of a circle, and the lune itself is equal [in area] to the triangle AFB = AE^2.

Coroll. On putting the radius of convexity AE = 1: then the radius of concavity AF = $\sqrt{2}$; the distance between the centres EF = 1; the base of the lune AB = 2, and the height CD = 2 - $\sqrt{2}$ (§7).

(*p*) EUCL. Elem.BookI.prop.48.

(*) Clearly so, as each arc and the point F fall on opposite sides of the line AB ; since also everywhere in the sequence concerning the construction of the remaining lunes this has to be understood, lest the construction of the lune is confused with that opposite *cf.* §9. 28.

§XV.

Besides, for this squarable lune [the following] is worthy of mention (*): as any right line FHK drawn from the centre F of the concavity cuts [the arcs] proportionally, thus so that the parts AMK & AH, or KB & HB, are proportionals of the total ACB, ADB, then from the part of the lune AMKH or KBH a squarable quantity is cut off. For the line ANO is made perpendicular to FK, and AK is joined, also it is understood that EK and FO are to be joined. On account of AE = EF, F lies on the periphery of the circle ACB continued; whereby (*q*) the angle *AEK* =2AFK = AFO (*r*), the arcs AMK and AHO (= 2AH) are similar [*i. e.* they subtend the same angle; p.10] and thus they are in proportion to the radii of these arcs AE and AF. Hence AMK : AH :: 2AE : AF :: arc ACB : arc ADB. Again, the circular segments

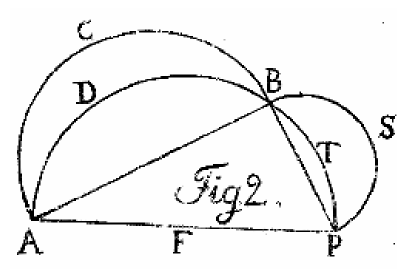
AMK and AHO, on account of the similitude of the arcs (dem.), are similar, and are as the squares of the radii $AE^2 : AF^2$ :: (§14) 1 : 2, or Segm. AMK = $\frac{1}{2}$AHO = AHN; and therefore by adding to each the area AHK, the portion of the lune AMKH = rectilinear triangle ANK. Moreover on account of (*q*) ang. AKF = $\frac{1}{2}$AEF for the semi–right angle, the triangles ANK [note that in the diagram, N lies on AO] and AFB are similar right angled triangles, and indeed isosceles; hence (*S*) triang. ANK : tr.AFB :: $AK^2 : AB^2$ :: ((s) with KL drawn perpendicular to AB) $AB \times AL : AB^2 :: (t)$ AL : AB :: tr. AFL : tr. AFB. Hence also the portion of the lune AMKH = tr. ALF, & the portion BHK = tr.BLF. [a number of el. theorems are used here.]

(*) Please compare, WHISTON *Schol*.2. with EUCL.E*l*.*Lib*.XII.prop.2.
(*q*) EUCL.El.III.20.
(*r*) EUCL.El.III.3.30.27.vel IV.33.
(*S*) VI.19.
(*s*) III.31. VI.Cor.8.17.
(*t*) VI.I.

Coroll. Whatever portion of this lune defined, either BYZ or KHYZ, from one or from two lines FK FZ, drawn from the centre F of the concavity, and both squarable and divisible in some given ratio (by straight lines), or in parts of given magnitudes (in truth not greater). Clearly with the base cut in a given ratio BU or UL, from the points of the division perpendiculars are defined drawn to the convex arc, and the points are to be joined with F. Moreover I call the base of the portion that part of the lune placed between the perpendiculars KL and ZU, drawn from the ends K, Z of the convex arc KZ.

§XVI.

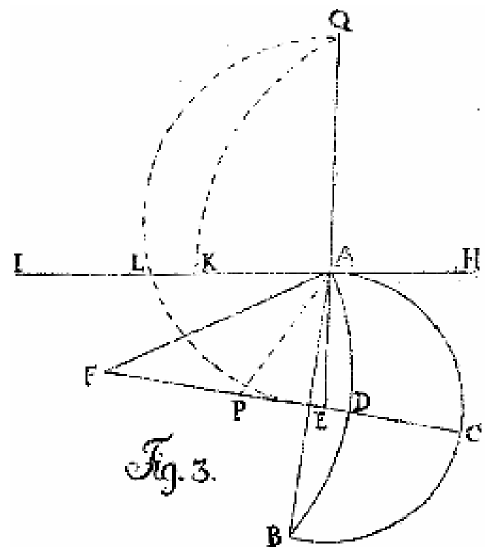
So that they are not omitted here, as [p.11] convex arcs also have semicircular conjugate lunes, of which the first quadrature to be found is the same one brought forwards by HIPPOCRATUS of Chios. Let (Fig. 2) ADTP be a semicircle on the hypotenuse AP of the right–angled triangle ABP constituted (*u*): again let ACB & BSP be the semicircles with diameters AB & BP or described on the perpendicular lines. Thus (*v*) the area of the semicircle ADP = area of the semicircles ACB + BSP. Therefore from each side with the common areas taken away, clearly of the segments ADB & BTP, the two lunes present ACBDA and BSPTB taken together are equal to the rectilinear triangle *ABP* (*).



(*u*) *Cf.* EUCL.El.III.19.
(*v*) EUCL.El.VI.31.XII.2.
(*) Yet these do not admit to individual squares : for when the triangle ABP is scalene, these lunes are dissimilar, and therefore cannot be compared with each other. Since moreover if the triangle ABP should be isosceles, then the lunes are conjugate and they are equal and similar and indeed squarable. §14.
[See : Sir Tomas Heath, *A History of Greek Mathematics* : Vol. 1, p.183 onwards. Dover.]

CASE II. Fig. 3.

§XVII. If $m : n :: 3 : 1$, this PROBLEM must be resolved by: *For two given sides AF, AE or in the ratio of these, to construct the triangle AFE, the external angle of which AEC, to either of the interior opposite sides, shall be the triple of the internal opposite F and adjoining to the base EF; or* (§13) *the angle of this FAE to the vertical shall be twice the other angle F to the base.*



Analysis. Since (by hyp.) the angle FAE = 2AFE: the angle FAE is understood to be bisected by the line AP ; then the angle PAE = PAF = AFE; AP (w) = PF; the angle APE = (x) 2AFE = FAE; [p.12] Triangle APE is similar to triangle FAE, hence in each there is the same ratio of the sum of the sides to the base, clearly : AF + AE : EF :: (AP + PE or) EF : AE; and thus in the sought triangle AFE , the base EF is the mean proportional between the sum of the sides AF + AE and the side AE opposite the angle F ; thus the construction easily arises. Hence also

(w) EUCL.El.I.6 (x) I.32.

§XVIII.

Squarable lunes, of which the angle of convexity is three times the angle of concavity, can be constructed in the following manner (*). To the line AE, which is the radius of convexity either given or taken arbitrarily, there is produced the perpendicular HI at A, on which on both sides from the point A there is taken AH = AK = AE; with centre H and with radius HK a circle is described, which cuts at Q the line EA produced towards A. Upon the diameter EQ describe a semicircle which cuts HI in L. Then with centre A and with radius AQ, and with centre E and with radius EL with circles described cutting each other in F (**), and finally with centres E & F circles are described through A, the arcs of which ACB and ADB form the desired lune ACBDA.

Indeed (y) EL is the mean proportional between AE & EQ, *i.e.* (by construction : [on joining EL and LQ, EA/EL = EL/EQ]) , and EF is the mean proportional between AE & AF + AE and thus (§17) the exterior angle AEC of the triangle AFE is the triple of the interior angle F. Again since, (by construction) HQ = HK = 2AH; then $HQ^2 = 4AH^2$; hence $HQ^2 - AH^2$ or (z) $AQ^2 = 3AH^2 = 3AE^2$, *i. e.* $AF^2 = 3AE^2$. Hence (§13) this lune is twice the area of the rectilinear triangle AEF.

(*) KRAFFT *Geom. Subl. T.I.* §167 offers another construction elicited by an algebraic calculation but hardly set out neatly.

(*) That these circles by necessity cut each other can be readily proven. For since AE < EL, then in the first place AE < AQ + EL, and again AE + EL > 2AE; but AQ < (HQ = HK =) 2AE; consequently AE + EL > AQ or in the second place, AE > AQ + EL. Therefore since the distance between the centres AE is less than the sum and greater than the difference of the radii AQ and EL, it is necessary that the circles mutually cut each other.

(*y*) EULC. EI,III.31. VI.8.

(*z*) EUCL.EI.I.47.

Coroll. 1. Since (*a*) AL is the mean proportional between AE & AQ, & AQ > AE (since $AQ^2 = 3AE^2$) : then AL > AE, hence $AL^2 > AE^2$, and thus $AL^2 + AE^2$ or $EL^2 > 2AE^2$, thus $EL^2 + AE^2 > 3AE^2$. But $AQ^2 = 3AE^2$, hence $AQ^2 < EL^2 + AE^2$, this is $AF^2 < AE^2 + AF^2$. Hence (*b*) the angle AEF is acute ; and hence the half of the convex angle AEC is obtuse & the convex arc ACB is greater than the semicircle. *cf.* Cor. 2.

Coroll. 2. On putting AE = 1, then $AF^2 = (AQ^2 = 3AE^2 =)\ 3$, & $EF^2 = (EL^2 = AE.EQ =)\ 1+\sqrt{3}$; therefore the ratio of the sides of the triangle AEF is given in terms of numbers. Hence in a different way the angles can be found by a trigonometric calculation. But in the present case what follows it is most convenient and simple. In the isosceles triangle PAF the whole sine, [p. 14] that is always put equal to 1, is to *Cos* F :: 2PF : AF :: (*c*) 2EF : AF + AE :: (§ 17) 2AE : EF. Hence, $CosF = \frac{EF}{2AE} = \frac{1}{2}\sqrt{1+\sqrt{3}}$. The logarithm of this thus can both easily and exactly found this. Since (*) generally

$\text{Sin}\,\mathfrak{A} + \text{Sin}\,\mathfrak{B} = 2\text{Sin}\,(\mathfrak{A}+\mathfrak{B})/2 \times \text{Cos}\,(\mathfrak{A}-\mathfrak{B})/2$;

And $1 = 2.\sin 30^0$ & $\sqrt{3} = 2.\sin 60^0$: then

$$\text{Cos}^2\text{F} = \frac{1+\sqrt{3}}{4} = \frac{\text{Sin}30^0+\text{Sin}60^0}{2} = \text{Sin}45^0 \times \text{Cos}15^0;$$

and thus

$$\text{Log Cos } F = \tfrac{1}{2}(\text{LogSin}45^0 + \text{LogCos}15^0) = 9.9166977,$$ on putting as is usual in tables, the logarithm of the whole sine equal to 10. Hence F = 34^0 22' (**) nearly, 3F or AEC = 103^0 6', the convex arc ACB = 206^0 11', 2F or FAE = 68^0 44' = ADB for the concave arc. Concerning the rest, $\overline{\text{Sin}}^2 = 1 - \overline{\text{Cos}}^2$, then $\text{Sin F} = \frac{1}{2}\sqrt{3-\sqrt{3}}$.

Coroll. 3. The height for the lune of this (§ 7) CD = $\sqrt{1+\sqrt{3}} - \sqrt{3}$. And since the line AB : AF :: 2SinF : 1, then the base of the lune AB $= \sqrt{3}.\sqrt{3-\sqrt{3}} = \sqrt{9-3\sqrt{3}} = \sqrt{9-\sqrt{27}}$. Finally the area of the lune =

$$2\,\Delta\,\text{AEF} = \tfrac{1}{2}\text{AB}\times\text{EF} = \tfrac{1}{2}\sqrt{3}.\sqrt{3-\sqrt{3}}.\sqrt{3+\sqrt{3}} = \tfrac{1}{2}\sqrt{6\sqrt{3}} = \tfrac{1}{2}\sqrt[4]{108} = \sqrt[4]{\tfrac{27}{4}}.$$

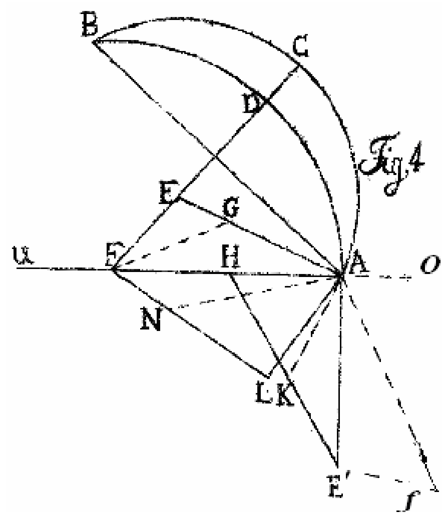
(*a*) EUCL. VI.13. (*b*) I.48.II.12 (*c*) VI.3.

(*) *Inledn. til Trigon. Plana* Prop.7 [*Introduction to Plane Trigonometry* : author anonymous, but most likely to have been Mårtin Strömer, Swedish astronomer and mathematician (note supplied by Johan Sten)]
& since the angles $\mathfrak{A}$, $(\mathfrak{A}+\mathfrak{B})/2$, $\mathfrak{B}$ are equally different.

(**) Here the angle or the arc measuring that is taken by KRAFFT (Geom.Subl.T.I §167) to be 34^0 16', thus the arc ACB is in excess by an error of around 35'.

CASE III. Fig. 4. [p.15]

§XIX. When $m : n :: 3 : 2$, there is indeed another PROBLEM related to the previous one:
To construct a triangle AEF, the two sides of which AF and AE are in the given ratio [of the radii]*, and the angle AFE to the base is twice as great as the angle to the vertex A* (*).



Analysis. With the line FG bisecting the angle AFE, then GFE = GFA = A, FGE = 2A = AFE, and AG = GF; triangle FGE is similar to triangle AFE and hence $\mathrm{AF}+\mathrm{FE}:\mathrm{AE}::(\mathrm{FG}+\mathrm{GE}\text{ or})\,\mathrm{AE}:\mathrm{EF}$; whereby the rectangle $\overline{\mathrm{AF}+\mathrm{FE}}\times\mathrm{FE}=\mathrm{AE}^2$. Therefore by assuming one or other of the lines AF and AE for argument's sake, in which case on account of the given ratio, the other of these is given (by hypothesis): the rectangle is said to be given with a magnitude; clearly (as demonstrated) = AE^2 . Thus by [this construction], the question has been reduced to that of finding the sides of a given rectangle with magnitudes FE & AF + FE, of which the difference AF is given. Thus, this is deduced.

Construction. The lines AF and AE' are taken in the given ratio and joined at right angles ; with AF bisected in H, there is taken HU = HE' on HF produced, then with the centres A & F with the intervals AE' & FU circles are to be described, and if they cross each other at some point E, the sought triangle AFE is obtained. Or if it pleases, join HE, from which there is cut off HK = HA, and then from the lines AF AE' E'K (upon AE' as base) there is constructed the equal triangle A*f*E'.

Now, if with centre H and radius HU = HE' , upon UA produced there is taken the described semicircle UEO; then [p.16] (*d*) $\mathrm{AU}\times\mathrm{AO}=\mathrm{AE}^2$ i.e. (by construction.) AO = FU = FE, $\overline{\mathrm{AF}+\mathrm{FE}}\times\mathrm{FE}=\mathrm{AE}^2$, as required. Thus it follows that (*e*) : $\overline{\mathrm{A}f+f\mathrm{E}'}\times f\mathrm{E}'=\mathrm{AE}^2$. The remaining identity of triangles AFE and A*f*E' or the agreement of the construction of each either thence or soon becomes apparent, because each of these E'K (or E'*f*) & FU (or FE) is equal to the difference of HE' & HA. These two constructions recommend themselves by their equal simplicity, except perhaps that the first or the second should be seen to be preferred a little over the other, if it is wished to construct the triangle either on the line AF or AE'. This is now understood from what has been said.

(*) With this problem taken more general than in EUCL.Elem.Book.IV.Prop.10, where the ratio of the sides is put equal.

(*d*) *cf*. EUCL.II.14. (*e*) ex III.16.36.

§XIX.

The lune is to be constructed, of which the angle of convexity to the angle of concavity is as 3 : 2.

With right lines AL and LN of equal arbitrary length drawn at right angles, on LN produced there is taken LF equal to AN; with AF joined, the perpendicular AE' is made equal to AN. Bisect AF in H, on HF produced there is taken HU = HE'; then with centre F and radius FU & and with centre A and with radius AN or AE', circles are described which cut each other at E (*). Finally with centres F and E there are described circles through A: the arcs of which, that fall from F towards E, make the lune desired ACBDA (**). [p.17] For indeed $AE^2 = LF^2 = AL^2 + LN^2 = 2AL^2$, & $AF^2 = AL^2 + LF^2 = 3AL^2$; and thus $AF^2 : AE^2 :: 3 : 2$. Besides, from the construction it can be gathered from §20, that the angle AFE = 2FAE or AEC : AFE :: 3 : 2 :: $AF^2 : AE^2$. Hence the proposition is agreed upon (§§12.10).

Coroll. 1. Since (by the demonstration)

$$\mathrm{FL}^2 = AN^2 = 2AL^2, AF^2 = 3AL^2, \mathrm{HU}^2 = \mathrm{HE'}^2 = \mathrm{HA}^2 + \mathrm{AE'}^2 = \tfrac{1}{4}\mathrm{AF}^2 + \mathrm{AN}^2$$

$$= \tfrac{1}{4}\mathrm{AL}^2 + 2\mathrm{AL}^2 = \tfrac{11}{4}\mathrm{AL2}, \mathrm{HF}^2 = \tfrac{1}{4}\mathrm{AF}^2 = \tfrac{3}{4}\mathrm{AL}^2 :$$

then $\mathrm{AF} = \mathrm{AL}.\sqrt{3}, \mathrm{AN}$ or $AE = \mathrm{AL}.\sqrt{2}, \mathrm{HU} = \mathrm{HF}$ *i. e.* FU or $\mathrm{EF} = \mathrm{AL}.\frac{\sqrt{11}-\sqrt{3}}{2}$*;* and thus *AF, AE* , *EF* are between themselves as $2\sqrt{3}, 2\sqrt{2}, \sqrt{11}-\sqrt{3}$; or as $\sqrt{12}, \sqrt{8}, \sqrt{11}-\sqrt{3}$*;* & the squares of these are as $12, 8, \&\, (11+3-2\sqrt{33}=)\, 14-2\sqrt{33}\, (=14-\sqrt{132}) < (14-\sqrt{121} = 14-11 =)\, 3.$ Hence since 12 > 8 + 3, it then follows from a stronger position, $\mathrm{AF}^2 > \mathrm{AE}^2 + \mathrm{EF}^2$; consequently the angle AEF is obtuse, AEC is acute & the convex arc ACB is less than a semicircle.

Coroll. 2 On account of the isosceles Triangle GAF, then :

$Cos\,\mathrm{FAE} = \frac{\mathrm{AF}}{2\mathrm{AG}} = (c)\frac{\mathrm{AF+FE}}{2\mathrm{AE}} = (Cor.1)\frac{\sqrt{11}+\sqrt{6}}{8}$, to this cosine there corresponds 26^0 49' approximately. Hence 2FAE or AFC = 53^0 38' and 3FAE or AEC = 80^0 26'; the convex arc ACB = 160^0 53', and the concave arc ADB = 107^0 16'. From the rest there is produced $sin\,FAE = \frac{\sqrt{9-\sqrt{33}}}{4}$.

Coroll. 3. On putting the radius of convexity AE = 1 : the radius of concavity is $\sqrt{\frac{3}{2}}$; the separation of the [p.18] centres $EF = \frac{\sqrt{11}-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{22}-\sqrt{6}}{4}$;the height of the lune (§7) $CD = 1 - \frac{3\sqrt{3}-\sqrt{11}}{2\sqrt{2}} = 1 - \frac{3\sqrt{6}-\sqrt{22}}{4}$; the base

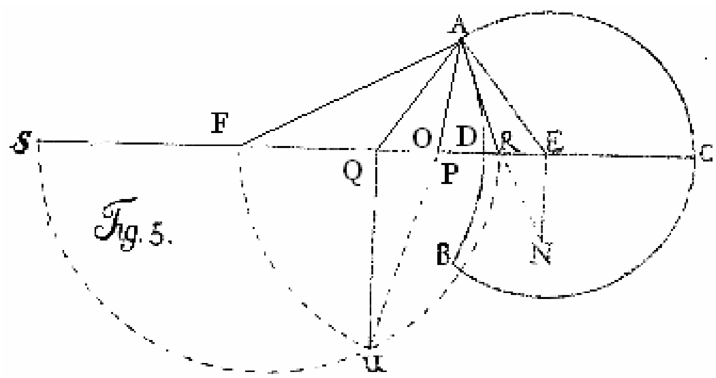
$$AB = (2AF \times sin\,AFE = 4AF.Sin\tfrac{1}{2}\mathrm{AFE}.Cos\tfrac{1}{2}\mathrm{AFE} = \mathrm{Cor.2})\frac{3+\sqrt{33}}{8}.\sqrt{9-\sqrt{33}}$$; Area

=2 Triangle AFE = $\frac{1}{2}AB \times EF = \frac{1}{8}\sqrt{6(9-\sqrt{33})}$.

(**) And it is apparent at once that the same or the twin of this lune is soon found, if from the centres U and F, with the intervals FL and FA circles are described towards FU, thus as use has not been made of the point E or of the construction the triangle AEF, as this corresponds better to the preceding section.

(*c*) EUCL.Elem.VI.3.

CASE IV. Fig. 5.



§XXI. Let the ratio be $m : n :: 5 : 1$ or the angle FAE = 4 × AFE, hence there arises this PROBLEM: *With the given ratio of the sides AF:AE =* a:b*, to construct the triangle AFE, of which the angle to the vertex FAE is four times the angle AFE to the base EF.* [p.19]
Analysis. The right lines AQ, AP, AR divide the angle FAE into four parts equal to each other and to the particular angle (by hypothesis) AFE : thus is follows that (*d*) AQ = FQ, AR = QR; triangle AER is similar to triangle FEA, and triangle AEP is similar to QEA, thus :
$\div\div\mathrm{EF, EA, ER}$ & $\div\div\mathrm{EQ, EA, EP}$,and thus (*e*) $EF:EP::EQ:ER :: (\mathrm{EF - EQ : EP - ER}, \textit{i.e.})$ $FQ : PR :: a+b : b :: FR \times PR$, which therefore is the given ratio. But, as triangle ARP is similar to triangle FRA, then $FR \times PR = (AR^2 =) QR^2$. Therefore the given ratio $FR \times PQ : QR^2 = a+b : b.$ Hence, if QR is assumed as given arbitrarily, the rectangle $FR \times FQ$ is given in magnitude, and the difference of the sides QR thus (*cf.* §19) is itself given, and finally the whole figure and triangle FAE.

Construction. The right lines $QR = a$ and $RE = b$ are put in place; on *EQ* produced the is taken *QS* the fourth proportional of ER, RQ, and EQ ; on the diameter RS make a semicircle, which cuts the line RS perpendicular to Q at U in Q. Bisect QR at O, there is taken OF = OU. Finally with centre Q and radius QF , and with centre R and radius RQ circles are described, the intersection of which is A (moreover these circles intersect each other if FQ < 2QR) : then on joining AF and AE, or AR and AE : the triangle AEF or REA satisfies the proposition.

For : (*f*)

$$\mathrm{SQ \times QR = QU^2 = OU^2 - OQ^2 = OF^2 - OQ^2 = \overline{OF + OQ} \times \overline{OF - OQ} = FR \times FQ.}$$

But $\mathrm{SQ \times QR : QR^2 :: SQ : QR ::}$ (by construction) $\mathrm{EQ : ER ::} a+b:b.$ [p.20] Hence $FR \times FQ : QR^2 :: a+b : b$ as required. The remainder is apparent from analysis. Now easily it gives : §XXII.

(*d*) EUCL.El.I.5.32. VI.4. (*e*) or by VI.3. $\sqrt{18}$. (*f*) II.14.

§XXII.

To construct a squarable lune having the angle of convexity five times the angle of concavity.

Clearly, for the line ER taken as you please, put in place the perpendicular EN = 2ER, and on ER produced, there is taken RQ = RN. Thus, for the given points QRE, the points F and A (*) can be determined as in the preceding section. Then with centres E and F circles are to be described through A that form the required lune ACBDA (within the angle AFE and the twin to this and from the other line FE with the similar part put BFE). And indeed

$QR^2 = RN^2 = ER^2 + EN^2 = ER^2 + 4ER^2 = 5ER^2$*;* and the angle ABC = AFE + FAE = (§21) AFE + 4AFE = 5AFE. Hence (§13) the lune = 2TriangleAFE.

Coroll. 1. On putting ER = 1, then AR = QR =

$\sqrt{5}, \frac{EQ.QR}{ER}$ or $QS = 5+\sqrt{5}, RQ.QS = 5+5\sqrt{5} = QU^2, OU^2 = QU^2 + QO^2 = QU^2 + \frac{1}{4}QR^2 =$

$\frac{25}{4} + 5\sqrt{5} = OF^2, OF - \frac{1}{2}QR$ or $FQ = \frac{1}{2}\sqrt{25+20\sqrt{5}} - \frac{1}{2}\sqrt{5} = \frac{1}{2}\sqrt{5}(\sqrt{5+4\sqrt{5}} - 1),$

$ER + \frac{1}{2}QR + OF = 1 + \frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{25+20\sqrt{5}} = EF \times ER = AE^2.$

Thus, there are the sides ER AR AE of triangle ARE, or the sides *AE AF EF* of the triangle of the similar triangles FAE, in the ratio between themselves as

$$1, \sqrt{5}, \sqrt{1 + \frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{25+20\sqrt{5}}}$$

respectively.

Coroll. 2. Since (Cor.1.) $EF^2 : AE^2 + AF^2 :: 1 + \frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{25+20\sqrt{5}} : 6;$ but

$\frac{1}{2}\sqrt{5} > (\frac{1}{2}\sqrt{4} =) 1, \& \frac{1}{2}\sqrt{25+20\sqrt{5}} > (\frac{1}{2}\sqrt{25+20.2}) = \frac{1}{2}\sqrt{65} > \frac{1}{2}\sqrt{64} = \frac{8}{2} =) 4$: then $EF^2 > AE^2 + AF^2$, and thus the angle AEC is much greater than the obtuse angle FAE, and the convex arc ACB is greater than a semicircle. And on account of FE > FA or FD, the centre E falls within the lune itself, otherwise as in the above cases.

Coroll. 3. Since in triangle ARQ with equal sides, Cos RQA $= \frac{AQ}{2QR} = \frac{FQ}{2QR}$: of this angle which is (Cor.1.)

$= 2AFE = \frac{1}{2}FAE = \frac{2}{5}AEC.$ The cosine $= \frac{1}{4}\sqrt{5+4\sqrt{5}} - \frac{1}{4} = \text{Cos } 46^0 52'$ approximately. Thus the convex arc ACB is approximately equal to 234^0 20' & the concave arc ADB is equal to 46^0 52'.

Coroll. 43. By taking for example the radius of convexity AE = 1: also the height of the lune, and the length of the base, as well as the area, can be expressed exactly by certain surds, or approx. by nearby rational numbers.

(*) By *Cor.* 1, then the ratio is $FQ : 2QR :: \sqrt{5+4\sqrt{5}} - 1 : 4.$ But $4\sqrt{5} = \sqrt{80} < 9$, whereby $\sqrt{5+4\sqrt{5}} < (\sqrt{14} <) 4,$ and thus $\sqrt{5+4\sqrt{5}} - 1 < 3 < 4.$ Hence FQ < 2QR, and consequently (preceding section) the circles, with their point of intersection A to be defined, cut each other entirely.

CASE V. Fig. 6.

§XXIII. The final case $m : n :: 5 : 3$ is resolved by this [p.22]

PROBLEM : *To show the triangle AEF, the angle of which EFA to the base EF is one and a half times the angle EAF to the vertex, with the given ratio of the sides AF:AE = a:b.*

Analysis. Consider it done and the angle AFE is divided into three equal parts by the lines FO and FP, therefore the individual parts are (by hyp.) $= \frac{1}{2}$EAF. Thus, there is the angle AFO = OAF = EFP; also OF (*g*) = OA; triangle EAF is similar to EFP and thus *AF:AE* ::PF:FE:: (*h*) *PO:OE :: a:b.* In addition (*h*) OF or AO:AF :: PO:PA. Hence (*i*) $AO : AE :: PO^2 : EO \times AP$. Now if EO, OP, of which (by demonstration) the ratio is given, are given ; the third proportion of this is given, which is PS, and the point S is given; on account of which (*k*) $OP^2 = EO \times PS$, then (by demonstration) $AO : AE ::$ $(\mathrm{EO} \times \mathrm{PS} : \mathrm{EO} \times \mathrm{AP} ::) PS:AP$ and

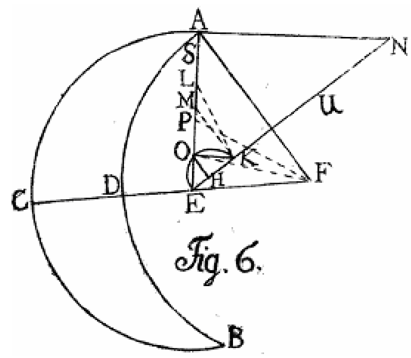


Fig. 6.

thus $\mathrm{AO} : (\mathrm{AE} - \mathrm{AO} =)\mathrm{EO} :: \mathrm{PS} : (\mathrm{AP} - \mathrm{PS} =) \mathrm{AS}$, whereby $AO \times AS = EO \times PS = OP^2$. Therefore from the magnitude the rectangle under the lines AO and AS, of which (by dem.) the difference OS is given; thus there are given the point A (*cf.*§19), OF = OA, AE and thus (on account of the given ratio AF:AE,) *AF*, and EF, clearly which is the mean proportional between the given AE and EP (as Triangles EFP and EAF are similar), consequently the point F and triangle AEF or FEP. Hence the ratio of the construction is apparent, that clearly we will adapt at once to our case.

(*g*) EUCL.I.5. (*b*) VI.3. (*i*) VI.22.1. (*k*) VI.17.

§XXIV. [p.23]

The squarable lune, of which the angle of convexity is as 5 : 3 to the angle of concavity, thus can be described in this way : On the indefinite right line EU there is taken the ratio EH:HK ::3:5; on EK, to which there are erected the perpendiculars HO and KL, there is described the semicircle which crossed HO in O; through the points of EO there is drawn the indefinite line EOLA, on which there is taken OK equal to OP and LS; OS or PL is bisected in M and there is put in place MA = MK; through the point A thus found put in place a perpendicular to AE or (*l*) parallel to the line OK, that crosses EU in N; then with centres O and A, with radii OA and AN circles are described; these by themselves crossing (*) define the point F; finally with the centres E and F there are circles to be described through A that form the desired lune ACBDA.

For $OP^2 = OK^2 = (m)\ EK \times KH, \& EO^2 = EK \times EH$, and thus $OP^2 : OE^2$ :: KH : HE :: 5 : 3. Again (*n*)
$\because$ EO, OK or OP, OL and (as LS = OP) PS = OL, hence $\because$ EO, OP, PS ,

& (*o*) $OA \times AS = (MA^2 - MO^2 = MK^2 - MO^2 = OK^2 =) OP^2$;
and finally AE : AF (= AN) :: EO : (OK=) OP, & OF = OA: whereby (from the preceding section) the angle EFA $= \frac{3}{2}$EAF. Hence the angle AEC: AFE :: 5:3 :: (by dem. $OP^2 : OE^2$ ::) $AF^2 : AE^2$, which had to be done (§13) [p.24]

Coroll. 1. In the triangle with equal legs AOF, there is
total sine 1: 2 Cos EAF :: AO or OF : AF :: (*p*) OP : AP = AO – OP thus

$$1 + 2\cos EAF = \frac{AO}{OP} = \frac{AO}{OK} = \frac{OM+MK}{OK} =$$

$$\frac{OM+\sqrt{OM^2+OK^2}}{OK} = \frac{OS+\sqrt{OS^2+4OP^2}}{2OP} =$$

(because EO : OP :: OP : PS :: EO + OP : OP + PS or OS and thus $\frac{OS}{OP} = \frac{EO+OP}{EO}$,)

$\frac{1}{2} + \frac{\sqrt{(EO+OP)^2+4EO^2.}}{2EO}$ Hence, since EO : OP :: $\sqrt{3} : \sqrt{5}$, there is produced

$$\cos EAF = \frac{\sqrt{5}+\sqrt{20+2\sqrt{15}}}{4\sqrt{3}} - \frac{3}{4} = \frac{\sqrt{15}+\sqrt{60+6\sqrt{15}}-3}{12} = 0.8330387,$$

whereby EAF = 33^0 35' 16", AFE = 50^0 22' 54", AEC = 83^0 58' 10".

Coroll. 2. This is also the case in Cor. 4 §22.

(*) By Coroll. 1 then, AF or AN : 2AO :: $\sqrt{15} + \sqrt{60+6\sqrt{15}} - 3 : 12$. But $\sqrt{15} < 4$ and thus $\sqrt{15} + \sqrt{60+6\sqrt{15}} - 3 < (4 + \sqrt{84} - 3 = 1 + \sqrt{84} < 1 + 10 = 11 < 12.$ Hence AN < 2AO, on account of which the said circle cut each other. [p.25]

(*l*) EUCL.III.31. (*m*) III.31. VI.8.17 (*n*) VI.8 (*o*) II.6 (*p*) VI.3.

§XXV.

Therefore we have five squarable lunes, constructed from circles and straight lines. We have proposed these in this order, in which the numbers expressing the ratio *m* : *n* are smaller, and we call these I, II, III, IV, and V. But if rather the ratio used is the magnitude of the quantity $\frac{m}{n}$, which sets out the proportion *m : n* in terms of the denominator of the squarable lune, then the lunes are collected together in this order: III, V, I, II, IV. We have not seen any mention made of lunes III, IV and V [in other works]. Moreover, in order that a comparison of the individual dimensions of these five lunes can be presented, that is easily put in place if it is so desired : here we add a small table which shows the approximate sizes of the numbers, that follows on putting the radius of convexity AE in all these lunes to be common and equal to 1.
[p.26]

Lunes	I.	II.	III.	IV.	V.
Convex arc ACB	180^0	206^0 11'	$160^0$53'	$234^0$20'	$167^0$56'
concave arc ADB	90^0	$68^0$44'	$107^0$16'	$46^0$52'	$100^0$46'
Concave radius AF	1.414	1.732	1.225	2.236	1.291
Distance of centres EF	1.000	1.651	0.560	2.509	0.718
Base AB	2.000	1.948	1.960	1.779	1.989
Width CD	0.586	0.919	0.336	1.273	0.427
Area	1.000	1.613	0.552	2.232	0.714

§XXVI. [p.27]

Since (El.Trig. Pl.) in all rectilinear triangles such as AEF,

AF : AE :: Sin AEC : Sin AFC :

if generally two angles could be defined which were in the ratio of the squares of their sines, then there would be innumerable circular squarable lunes (§§ 10.12). Now this is seen to be far from the truth in any assigned case of interest. Because if the proposed ratio were again (§13) *m : n,* which is clearly the ratio of the convex and concave angles : (also taking *m, n* to be whole numbers prime relative to each other) : then so far the problem is soluble, then a single equation might be possible to be show or to be determined, and indeed to be finite or algebraic, from which construction the delineation of the lune itself depends. As indeed on calling the angles or the circular arcs $\mathfrak{A}$ and $\mathfrak{B}$, *s* the sine and *z* the cosine of $\mathfrak{A}$,
then (*) :

$$\mathrm{Sin}(\mathfrak{A}+\mathfrak{B}) = s.\mathrm{Cos}\mathfrak{B} + z.\mathrm{Sin}\mathfrak{B},\ \&\ \mathrm{Cos}(\mathfrak{A}+\mathfrak{B}) = z.\mathrm{Cos}\mathfrak{B} - s.\mathrm{Sin}\mathfrak{B} :$$

it is apparent, on putting successively $\mathfrak{B} = \mathfrak{A}, 2\mathfrak{A}, 3\mathfrak{A}$, &c. that the sines and cosines of $2\mathfrak{A}, 3\mathfrak{A}, 4\mathfrak{A}$ &c., clearly of twice, three, four, and finally of any multiple of the angle $\mathfrak{A}$, are expressed in terms of *s* and *z* (†). And indeed, as it is sufficient for these only to be indicated, thus these expressions of the sines of any multiple of the angle can be obtained as powers of s rising in steps unless they are odd, on account of which, since any smaller nearby odd power that arises can be multiplied by s thus to give an even power [p.28], and all the even powers of this also are worthy of note, (strictly such that they are whole numbers and not fractions), since for which it is possible to substitute $1 - zz$ in place of *ss*: it follows that the sine of the multiple angles $m\mathfrak{A}$ & $n\mathfrak{A}$ can be set up in this form :
$\mathrm{Sin}\, m\mathfrak{A} = Z.s$ and $\mathrm{Sin}\, n\mathfrak{A} = Z'.s$ (**) thus in order that $Z = \mathrm{Sin}\, m\mathfrak{A}/s$ and $Z' = \mathrm{Sin}\, n\mathfrak{A}/s$ are functions of *z* and certain rational numbers and integers. Now there is set up the measure of the sought angles AEC and AFC (by hypothesis) in terms of the common $\mathfrak{A}$, or AEC = $m\mathfrak{A}$ and AFC = $n\mathfrak{A}$: the size of these, which has been said, is required that $Z : Z' :: \sqrt{m} : \sqrt{n}$; and thus (Æ) $Z\sqrt{n} = Z'\sqrt{m}$, which is the desired equation to be solved for the single unknown quantity *z*, and if it is able to be constructed, by construction *z* is given, the cosine of the angle $\mathfrak{A}$; from which is given $\mathfrak{A}$ itself, clearly taken to be acute lest $m\mathfrak{A}$ is made greater than two right angles, on which account the negative roots of this equation are to be rejected. Thus by this agreement also the multiples $m\mathfrak{A}$ and $\overline{m-n}.\ \mathfrak{A}$ of $\mathfrak{A}$ are given *i. e.* the angles AEC and EAF, of which either with the ratio given AF: AE = $\sqrt{m} : \sqrt{n}$ defines the

kind of the triangle AEF, thus in order that the lune sought is easily constructed. Certainly the equation Æ except in few, and I think in no more than in the five most simple cases above can be handled, in which either cubic or higher grades are to be taken ; hence these [possible higher power equations] are to be constructed with more difficulty, and not only with a ruler and a pair of compasses. [p.29]

(*) See anywhere in *Scriptores Trigonometiae*, e.g.. De la CAILLE Elem.Math.§.742, *cf.* if you please with, *Inledn.til.Trigonom.Plan*.prop.4.

(†) The general law thus of expressing the sine or cosine of any multiple angle, is to be found explained in all texts, as in WOLF *Elem.Math*.T.1.p.m.322. KÆSTNER *Analendl.Grössen*.§176.

(**) Thus there are found : $\sin 2m\mathfrak{U}/s = 2z$, $\sin 3m\mathfrak{U}/s = 4zz - 1$, $\sin 4m\mathfrak{U}/s = 8z^3 - 4z$, $\sin 5m\mathfrak{U}/s = 16z^4 - 12zz + 1$, $\sin 6m\mathfrak{U}/s = 32z^5 - 32z^3 + 6z$, and thus so on.

§XXVII.

In order that these (§26) can be illustrated by examples : take

I : for the ratio $m : n :: 2 : 1$, then it becomes (above **) Z = 2z, Z' = 1, thus the equation III. $2z = \sqrt{2}$ or $z = \sqrt{\frac{1}{2}} = \cos 45^0$.

II: for the ratio $m : n :: 3 : 1$. whereby $Z = 4zz - 1$, $Z' = 1$, and thus $4zz - 1 = \sqrt{3}$, hence $z = \frac{1}{2}\sqrt{1+\sqrt{3}}$ as in §18, Cor.2, above.

III: for the ratio $m : n :: 3 : 2$ as anew $Z = 4zz - 1$, but $Z' = 2z$, hence $\overline{4zz-1}.\sqrt{2} = 2z\sqrt{3}$, or $zz - \frac{z\sqrt{3}}{2\sqrt{2}} = \frac{1}{4}$, by taking the positive root of this equation, [p.30] then $z = \frac{\sqrt{11}+\sqrt{3}}{4\sqrt{2}}$ as before in §20. Cor.2.

IV: for the ratio $m : n :: 5 : 1$; the equation is found : $16z^4 - 12zz + 1 = \sqrt{5}$, of which the positive roots are $\frac{1}{2}\sqrt{\frac{3}{2} \pm \frac{1}{2}\sqrt{5+4\sqrt{5}}}$; of which the latter is rejected as it is imaginary, then $z = \frac{1}{2}\sqrt{\frac{3}{2} + \frac{1}{2}\sqrt{5+4\sqrt{5}}} = \cos\mathfrak{U}$; thus also $2\cos^2\mathfrak{U} - 1$ or $\cos 2\mathfrak{U} = \frac{\sqrt{5+4\sqrt{5}}-1}{4}$, as above in §22.Cor.3.

V: for the ratio $m : n :: 5 : 3$ gives the biquadratic equation $16z^4 - (3 + \sqrt{\frac{5}{3}})zz + 1 + \sqrt{\frac{5}{3}} = 0$, but equally, can be resolved in terms of quadratics.

But if $m : n :: 4 : 1$, then $4z^3 = 2z + 1$.

If $m : n :: 4 : 3$; then $4z^3\sqrt{3} - 4zz = 2z\sqrt{3} - 1$.

If $m : n :: 5 : 2$; there is produced $16z^4 - 12zz = 2z\sqrt{\frac{5}{2}} - 1$.

If $m : n :: 5 : 4$, a biquadratic equation arises, no term of which is absent.

If $m : n :: 6 : 1$, Æ is an equation of the fifth order.

§XXVIII.

But it is more difficult to solve this problem about squarable circular lunes, if the ratio *m : n* is not given, but some other condition, e.g. [p.31] either the angle of convexity or convexity of the lune, or the ratio of the base to the width. Now nothing is attained by lingering over these. Therefore Claudat observed following our little work : *Opposite lunes* (§9) *are unable likewise to be squarable*, then the square of the circle is to be defined. If indeed the square of each lune is given, from that the square of the difference of these is given, which as it is the difference of the whole circles (indeed by adding to each of the lunes the intervening area there is produced the circle itself) and a third circle can easily be described equal to the difference of these two circles (*) : the third circle is entirely squarable, contrary to the hypothesis. Which is known to be the case if the said difference vanishes, or the lunes are equal curves : as we have advised before (§10).

(*) *Cf*. EUCL.El.XII.2 & VI.31.

DISSERTATIO GRADUALIS, LUNULAS QUASDAM CIRCULARES QUADRABILES EXHIBENS.

Martinus Johan. Wallenius.

§I.

Lunula sensu generalissimo denotare potest figuram, duabus lineis curvis, in superficie quacunque descriptis & sibi occurentibus, terminatam; atque, pro diversa hujus superficiei indole, vel *plana*, vel *sphaerica* (*a*) &c. vocati (*b*).

(*a*) Quadraturam quarundam *Lunularum Sphaericarum* proposuit LEIBNITIUS in Actis Erud. Lips. A. 1692. p. 277.

(*b*) Eatenus quoque superficies *Ungularum* curvatas, ad classem Lunularum referre licet.

§II.

Pro varia curvarum (§I.) natura, Lunulae planae varias iterum sortiunur denominationes (*c*) . Non autem nisi Lunulae circulares nostrae nunc erunt considerationis. [p.2]

(*c*) *Lunulae Ellipticae quadrabiles* exhibentur in *Kongl. Vet. Acad. Handl*. 1757, p. 218 seqq. *Lunararum Cyclo-parabolicarum* i. e. arcu circulari & arcu parabolae contentarum, mentionem fecit WOLFIUS in *Act. Erud. Lips*. 1715, p. 213, 217.

§III.

Lunula Circularis (*) nobis est Spatium ABCD duobus arcubus circularibus, convexo ACB & concavo (*d*) ADB, in plano communi descriptis atque se mutuo secantibus (*e*) comprehensum. Vocari etiam a nonnullis interdum solet *Meniscus*.

(*) Teste KRAFFT (*in Instit. Geom. Sublimior*, T. I. § 165.) theoriam Lunularum generalem & elegantem dedit quondam in Exercitationibus Mathematicis, Venetiis A. 1724 editis, atque, inter alia, methodum determinandi omnes innumeras Lunulas quadrabiles proposuit DAN. BERNOULLI. Ratio generalis id praestandi continetur quoque in §§ 5 – 7, 21. dissertationis cuiusdam (vid. *Comment.Petrop.T.IX*.p.207, seqq.) L. EULERI occupati in solvendo Problemate quodam Geometrico circa Lunulas a circulis formatas, cuius particularem solutionem antea dederat modo nominatus BERNOULLI.

(*d*) Altero scil. convexitatem, altero concavitatem, extrorsum vertente; neque enim, quod sciam, figura utrinque convexa, sensu maxime proprio Lunula dici suevit; ut taceam nondum adparvisse quadraturam ullius spatii ex duobus segmentis circularibus constati.

(*e*) His omnibus opus est determinationibus junctim sumendis, ut haec Lunula & plana esse intelligatur, & cum a circulo integro tum a spatio quovis annulari discernatur. Et quamvis viderimus etiam Lunulae nomine nonnunquam (ut in Act.Erud.Lips. 1709. p. 81, & KRAFFT. Instit. Geom.Sublim. T. I. §. 181) [p. 3] venire spatium, inter peripherias integras duorum circulorum, quarum altera alteram intus tangit, interjectum : in definita tamen vocis acceptione nunc manebimus. Caeterum quae in §§. 4 – 9 de his Lunulis dicentur, eorum vel omni vel plenaria demonstratione supersedemus; quia partim perfacile ex primis Geometriae elementis fluunt, partim ad caput tractationis nostrae haud pertinent.

§IV.

Sunto (*f*) E, F (Fig. 1, 3, 4, 5, 6) centra circulorum ACB, ADB, respective : recta linea FEDC per haec centra transiens *Axis Lunulae* merito dicatur; quippe quae bifarium, in partes scil. aequales, similes & utrinque similiter positas dividit Lunulam ipsam eiusque arcus, immo totam figuram, ut angulos AEB, AFB, qui *anguli convexitatis & concavitatis* dicentur (*g*); Sectores AEBCA, AFBDA; Segmenta ABDA, ABCA; chordam denique circulis communem videlicet rectam AB seu *Basin* Lunulae, atque huic perpendicularis est. Spatium igitur mixtilineum ACD vel BCD *Semilunula* vocari poterit.

(*f*) Vid. EUCL. Elem. Lib. III. Prop. 5. (*g*) Ut &, alio quidem respectu, *anguli Sectorum* EUCL. Elem. VI.33. Caeterum fieri potest ut horum angulorum vel alteruter vel uterque duos rectos superet, qualis quidem *gibbus* seu *convexus* passim dicatur (Gallice *Angle rentrant*). Semisses autem eorum AEC, AFC non possunt non singulae esse duobus rectis minores, seu anguli sensu maxime proprio; quales specifico nomine *concavi* (*Angles saillans*) nonnunquam vocantur. [p. 4]

§V.

Angulus curvilineus CAD (vel CBD), qui angulus Lunulae vocari poterit, aequalis censendus est angulo rectilineo EAF quem efficiunt radii EA, FA circulorum ad alterutrum intersectionis (§3) punctum A ducti (*b*).

(*b*) Confr. EUCL. Elem.Libr.III.Prop.16.

§VI.

Arcus Lunulae integri non possunt non & *inaequales* esse & *dissimiles*. Scilicet arcus *convexus* & longitudine maior erit *concavo*; & maiorem ad totam circuli peripheriam habebit proportionem, seu (§4) *angulus convexitatis* maior erit *angulo concavitatis* (*i*) ; & (*k*) semidifferentia eorum ipse *angulus Lunalae* (§5). Dissimiles ergo etiam erunt Sectores EACB & FADB horumque dimidii EAC & FAD.

(*i*) Scilicet dum centris E, F per A describuntur intra angulum AFE vel AEC arcus circulares AC, AD : erit (EUCLID.Elem.III.7.) FA vel FD < FC; unde patet istorum arcuum adeoque & (§4) integrorum Lunulae arcuum priorem fore extrorsum convexum, posteriorem concavum. Ac ibi ad centrum insistit externus Triangulis AEF angulus AEC, huic internus AFE. Ergo &c. EUCL.Elem.I.16. VI.33.Cor.1. (*) EUCL.Elem.I.32.

§VII.

Si *radius concavitatis* fuerit maior (minor) radio *convexitatis*; erit maxima Lunulae latitudo CD, [p. 5] quae simpliciter *Latitudo Lunulae* dicatur, aequalis distantiae centrorum EF, demta (addita) radiorum differentia. Vel generatim $CD = EF + EA - FA,$ i. e. prodibit latitudo Lunulae, si distantiae centrorum adiiciatur radius convexitatis & subtrahatur radius concavitatis.

§VIII.

Quod si ambo radii fuerint aequales, quo in casu *Lunula Aequicurva* vocari posset : erit *latitudo* eius (§7) aequalis *distantiae centrorum*, & summa arcuum, seu *perimeter Lunulae*, toti *peripheriae circuli*; adeo ut alter alterius sit supplementum ad totam peripheriam; convexus quidem semicirculo maior, concavus semicirculo minor (*l*).

(*l*) Confr. §6.

§IX.

Duo circuli integri semet secando duas efficiunt *Lunulas oppositas*; quibus ergo communia sunt arcuum centra, axis, basis & aequales anguli (§5). At arcus convexus alterius est suplementum arcus concavi alterius ad totam circumferentiam, & semisumma latitudinum (§7) aequalis distantiae centrorum; suntque hae Lunulae & inaequales & dissimiles, nisi (§8) aequicurvae fuerint.

§X.

Si aequales construi possint Sectores circulares EAC, FAD; sequitur, utrinque aut (Fig. 1, 3, 4) ablato communi, aut (Fig. 5) addito, si quod Sectores illos & axem interiacet, spatio mixtilineo EAD, aut denique, si res tulerit, facta tum subtractione, tum additione, fore Semilunulum ACD = Triangulo rectilineo EAF; adeoque dabitur perfecta *Quadratura* huius ipsius *Lunulae*. Non posse igitur ullam Lunulam aequicurvam (§8) hoc quidem pacto quadrati, patet, §6 .

§XI.

Dato radio, Sectores circulares sunt (*m*) ut anguli ipsorum; dato autem angulo, ut quadrata radiorum (*n*); quare neutro dato, seu generatim erunt ut anguli & quadrati radiorum coniunctim, i. e. Sectores EAC & FAD in ratione composita

$\text{ang.}AEC \times AE^2 : \text{ang.}AFC \times AF^2$.

(*m*) EUCL.Elem.L.VI.Prop.33.Cor.1. (*n*) Cfr.EUCL.Elem.L.XII.Prop.2.

§XII.

Ad obtinendaam igitur (§10) *aequalitatem Sectorum circularium* dissimilium (§6) efficiendum est ut (§11) *anguli* eorum sint in *ratione duplicata inversa radiorum*. Sic aequates tum demum erunt Sectores EAC & FAD si ang. AEC : ang. AFC :: AF^2 : AE^2, scilicet anguli convexitatis & concavitatis inverse ut quadrata ex radiis Sectorum.

Schol. Sunt vero tunc etiam *Arcus* AC & AD, seu ACB & ADB, *inverse* ut radii eorum AE, AF. Sunt enim semper Sectores in ratione compositaq radiorum atque arcuum. Cfr. ARCHIMED. *de Circulo & adscr*.

§XIII.

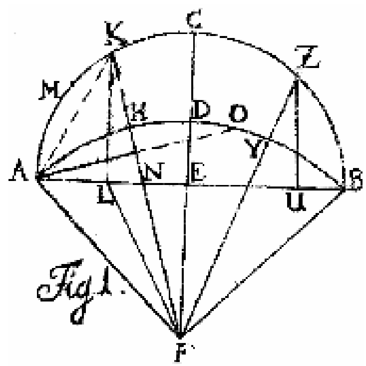
Data ergo seu assumta ratione aliqua angulorum convexitatis & concavitatis (§4) quae ratio sit = *m*: *n* & quidem (§6) *m* > *n* : eo redit negotium ut (§12) constituatur Triangulum rectilineum *AEF*, cuius angulus externus *AEC* sit ad oppositum internum *AFE* ut *m*: *n*, vel latera ipsa $AF : AE :: \sqrt{m} : \sqrt{n} :: m : \sqrt{mn} :: \sqrt{mn} : n.$ Vel quod eodem recidit (*o*) : inveniendum est specie Triangularum AEF, in quo sint duo anguli interni $EAF : AFE :: m-n : n$. Sic obtinebitur Lunulae perfecte quadribilis, cuius quidem angulus (§5, §6) sit ad semissem anguli concavitatis ut $m-n:n$.

(*o*) EUCL.Elem.L.V.17.I.32.

Schol. Brevitati studentes, eiusmodi tantum, quae recta & circulo absolvi possunt, ideoque praecipuo quodam iure Geometricae vocari merentur, constructiones comminiscemur; seu eas tantum Lunulas quadrabiles describendi rationem trademus, quarum delineatio non plura quam Euclidea ista supponit Postula. Considerabimus scilicet potissimum & speciatim quinque hos casus ubi $m:n = 2:1, 3:1, 3:2, 5:1, 5:3.$ Neque enim pauciores (*) neque, puto, plures Lunulae quadrabiles construi regula & circino possunt. Utrum vero eas omnes sigillatim pertractaverit vel BERNOULLI [p.8] (§3.not.*) vel alius quispiam ? nobis huad innotuit. Caeterum quamvis data ratione *m* : *n*, non amplius, si quidem scopo proposito (§10) satisfieri debet, arbitraria sed omnimode determinata sit ratio *AF* : *AE* : constructiones tamen Trianguli nostri *AEF* proponemus generaliores, assumendo primum rationem *AF*: *AE* laterum quamcunque, & a ratione *m* : *n* haud pendentem. Demonstrationes autem syntheticas, ex Analysibus nostris Geometricis haud difficulte concinnandas, ut & diversas constructionum modificationes, brevitatis studio omittemus.

(*) Id tamen fatis diserte adserit KRAFFT (Geomet.Subl.T.I. §167) his verbis : *Plures* (duabus illis prioribus, quas consideraverat) *ejusmodi Lunulae inveniri possunt*, *si assumantur rationes aliae* (quam 2 : 1 & 3 : 1) *numeri expsressa ; sed aequationes prodeunt sic ad* NB, *altiores etiam perpetuo dimensiones assurgentes, atque adeo minus concinna.* Quam minus recte vel adposite haec dixerit, iudicari peterit ex sequentibus nostris § § 20, 22, 24, 27.

CASUS I. Fig. 1.



§XIV. Si ponatur (§13) *m* : *n* :: 2 : 1 erit ang. EAF = AFE; adeoque Triangulum AEF aequicrurum. Et quemadmodum generatim constructu facillimum est Triangulum aequicrurum, data ratione baseos AF & cruris AE vel EF : sic & speciatim dum rationem AF:AE oportet esse subduplicatam ipsius (*m*:*n* =) 2:1. Scilicet (§12.13) $\mathrm{AF}^2 = 2\mathrm{AE}^2 = (\text{ob AE} = \text{EF})\ \mathrm{AE}^2 + \mathrm{EF}^2$, ergo (*p*) ang. AEF rectus. Fiat ergo rectae AE, magnitudinis datae vel arbitrariae, perpendicularis & aequalis EF; tum centris E & F describendi [p.9] per A seu intervallis EA & FA, arcus circulares (*) ACB & ADB comprehendent Lunulam *quadrabilem* simplicissimam & maxime concinnam ACBDA, quae, ab inventore, HIPPOCRATIS cognominatur, & quidem speciatim *Lunula Quadrantalis*. Est nimirum arcus convexus ACB Semicirculus, concavus ADB Quadrans circuli, Lunula ipsa = Triangulo AFB = AE*q*.

Coroll. Posito *radio convexitatis* AE = 1: erit *radius concavitatis* AF = $\sqrt{2}$, *Distantia centrorum* EF = 1, *Basis Lunulae* AB = 2, *Latitudo* CD = 2 - $\sqrt{2}$ (§7).

(*p*) EUCL. Elem.Lib.I.prop.48.

(*) Ita scilicet, ut arcus uterque & punctum F cadant ad oppositas rectae AB partes; quod etiam ubique in sequentibus circa constructiones caeterarum Lunularum est intelligendum, ne confundatur Lunula construenda cum altera ipsi opposita cfr. §9. 28.

§XV.

De hac Lunula Quadrantali praeterea memoratu dignum est (*): quod recta quaecunque e centro F concavitatis per eam ducta FHK tum arcus proportionnaliter secet, ita ut partes resectae AMK & AH, vel KB & HB, sint totis ACB, ADB proportionales, tum ab ipsa Lunula portionem AMKH vel KBH quadrabilem abscindat. Fiat enim recta ANO perpendicularis ad FK, iungatur AK, iungi etiam intelligantur EK, FO. Ob AE = EF, cadit F in peripheriam circuli ACB continuatam; quare (*q*) ang.*AEK* =2AFK = AFO (*r*), arcus AMK AHO (= 2AH) similes [p.10] ideoque proportionales radiis suis AE, AF. Ergo AMK : AH :: 2AE : AF :: arc. ACB : ADB. Porro segmenta circularia AMK, AHO, ob similitudinem (dem.) arcuum, similia, sunt ut quadrata radiorum AE*q* : AF*q* :: (§14) 1 : 2, seu Segm. AMK =

$\frac{1}{2}$ AHO = AHN; adiecto igitur utrinque spatio AHK, erit portio Lunulae AMKH = triangulo ANK rectilineo. Caeterum ob (*q*) ang. AKF = $\frac{1}{2}$ AEF semirecto, similia sunt triangula ANK AFB rectangula & quidem aequicrura; proinde (*S*) triang. ANK : tr.AFB :: AKq : AB*q* :: ((*s*) ducta KL perpendiculari ad AB) $AB \times AL : ABq ::$ ((*t*) AL : AB :: tr. AFL : tr. AFB. Ergo quoque portio Lunulae AMKH = tr. ALF, & BHK = BLF.

(*) Cfr. sis, WHISTON *Schol*.2.ad.EUCL.E*l*.*Lib*.XII.prop.2. (*q*) EUCL.El.III.20. (*r*) EUCL.El.III.3.30.27.vel IV.33. (*S*) VI.19. (*s*) III.31. VI.Cor.8.17. (t) VI.I.

Coroll. Quaelibet huius Lunulae portio, ut BYZ vel KHYZ, aut una aut duabus rectis FK FZ, e centro F concavitatis ductis definita, & quadrabilis est & divisibilis in ratione (rectis lineis) data quacunque, vel in partes datae magnitudinis (iusto non maiores). Secta videlicet basi BU vel UL in data ratione, e punctis divisionum ducenda ad eam perpendicula definient in arcu convexo puncta iungenda cum ipso F. *Basin* autem portionis voco partem baseos Lunulae interiacentem perpendicula KL ZU in eam ab extremis K, Z, arcus convexi KZ ducta.

§XVI.

Neque praetermittendae hoc loco sunt, utpote [p.11] quae etiam arcus convexos habent Semicirculares, Lunulae coniugatae, quarum quadraturam primus detexisse fertur idem HIPPOCRATUS Chius. Sit (Fig. 2) ADTP semicirculus super hypotenusa AP trianguli rectanguli ABP constitutus (*u*): sint porro ACB & BSP semicirculi diametris AB & BP seu cathetis descripti. Est itaque (*v*) area semicirculi ADP = semicirculis ACB + BSP. Ablatis igitur utrinque spatiis communibus nim. segmentis ADB & BTP, binae Lunulae superstites ACBDA BSPTB simul sumtae aequales sunt Triang. rectilineari *ABP* (*).

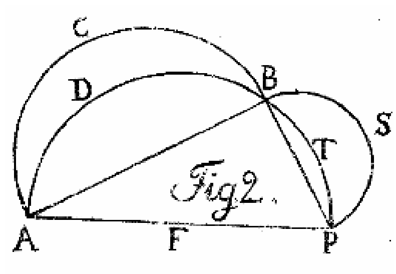


Fig2.

(*u*) Cfr. EUCL.El.III.19. (*v*)EUCL.El.VI.31.XII.2. (*)Non ideo tamen singulae quadraturam admittunt: quando enim scalenum est Triangulum ABP, Lunulae hae dissimiles sunt, nec proinde comparari inter se possunt. Quod si autem Isosceles fuerit Triang. ABP, Lunulae coniugatae & aequales & similes erunt & quidem quadrantales §14.

CASUS II. Fig. 3.

§XVII. Si *m* : *n* :: 3 : 1, resolvendum est hoc PROBLEMA. *Datis duobus lateribus AF, AE vel ratione eorum, construere Triangulum AFE, cuius angulus externus AEC, alterutri istorum laterum oppositus, sit triplus interni oppositi F & basi EF adiacentis; seu* (§13) *cuius angulus FAE ad verticem sit duplus alterutrius anguli F ad basin.*

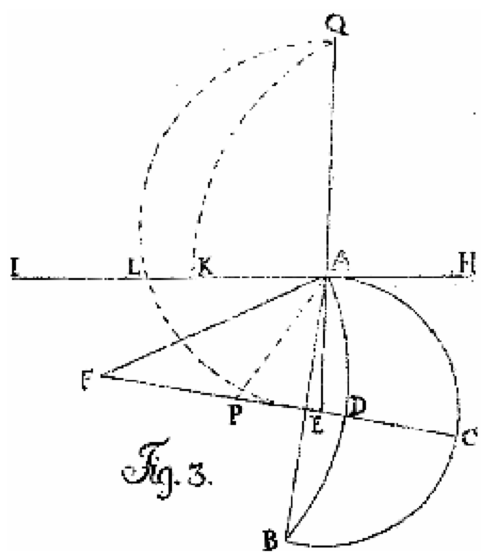


Fig. 3.

Analysis. Quia (hyp.) ang.FAE = 2AFE: bisectus intelligatur ang. FAE recta AP; erit ang. PAE = PAF = AFE; AP (*w*) = PF; ang. APE = (*x*) 2AFE = FAE; [p.12] Triang. APE sim. FAE, unde in utroque eadem erit ratio summae laterum ad basin, scilicet AF + AE : EF :: (AP + PE seu) EF : AE; adeoque in quaesito triangulo AFE basis EF erit media proportionalis inter summam laterum AF + AE atque latus angulo F oppositum AE; unde facilis emergit Constructio. Proinde etiam

(*w*) EUCL.El.I.6 (*x*) I.32.

§XVIII.

Lunula quadrabilis, cuius angulus convexitatis triplus sit anguli concavitatis, construi sequentem in modum poterit (*). Rectae AE, quae sit radius convexitatis vel datus vel pro arbitrio sumemdus, fiat in A perpendicularis HI, in qua utrinque a puncto A capiantur AH = AK = AE; centro H radio HK describatur circulus, qui rectam EA versus A producendam secabit in Q. Super diametro EQ describe semicirculum qui secet ipsam HI in L. Tum centro A radio AQ & centro E radio EL descriptis circulis sibi occursuris (**) in F, denique centris E & F describantur per A circuli, quorum arcus ACB ADB formabunt Lunulam ACBDA desideratam.

Est enim (*y*) EL media proportionalis inter AE & EQ, i. e. (Constr.) EF media proportionalis inter AE & AF + AE ideoque (§17) exterior Tr:li AFE angulus AEC triplus interioris F. Porro quia (Constr.) HQ = HK = 2AH; erit

$HQ^2 = 4AH^2$; hinc $HQ^2 - AH^2$ seu (z) $AQ^2 = 3AH^2 = 3AE^2$, i. e. $AF^2 = AE^2$.

Ergo (§13) Lunula haec erit dupla Triangula rectilinei AEF.

(*) Aliam constructionem calculo Algebraico elicitam sed vix aeque concinnam affert KRAFFT *Geom. Subl. T.I.* §167.

(*) Hos circulos se necessario secturos esse facile probatur. Nam quia AE < EL, erit 1:mo AE < AQ + EL. & porro AE + EL > 2AE; at AQ < (HQ = HK =) 2AE; consequenter AE + EL > AQ seu 2:do AE > AQ + EL. Cum ergo distantia centrorum AE sit minor quam summa & major quam differentia radiorum AQ EL, oportet circulos se mutuo secare. (*y*) EULC. EI,III.31. VI.8. (*z*) EUCL.EI.I.47.

Coroll. 1. Quia (*a*) AL est media proportionalis inter AE & AQ, & AQ > AE (quia $AQ^2 = 3AE^2$) : erit AL > AE, hinc $AL^2 > AE^2$, ideoque $AL^2 + AE^2$ seu $EL^2 > 2AE^2$, unde $EL^2 + AE^2 > 3AE^2$. At $AQ^2 = 3AE^2$, ergo $AQ^2 < EL^2 + AE^2$, h.e. $AF^2 < AE^2 + AF^2$. Ergo (*b*) angulus AEF acutus; adeoque dimidius ang. convexitatis AEC erit obtusus & arcus convexus ACB semicirculo maior. cfr. Cor. 2.

Coroll. 2. Posito AE = 1, est AF*q* = ($AQ^2 = 3AE^2 =$) 3,

& EF*q* = (EL*q* = AE.EQ =) $1+\sqrt{3}$; datur ergo in numeris ratio laterum Trianguli AEF. Hinc vario modo per calculum Trigonometricam inveniri possunt anguli. In praesenti autem casu maxime commodus & simplex est, qui sequitor. In triangulo PAF aequicruto est Sinus totus, quem semper ponimus = 1, [p. 14]

ad *Cos* F :: 2PF : AF :: (*c*) 2EF : AF + AE :: (§ 17) 2AE : EF. Ergo

$CosF = \frac{EF}{2AE} = \frac{1}{2}\sqrt{1+\sqrt{3}}$. Huius logarithmus quam fieri potest tum facile tum exacte sic invenietur. Quia (*) generatim

Sin $\mathfrak{A}$ + Sin $\mathfrak{B}$ = 2Sin ($\mathfrak{A} + \mathfrak{B}$)/2 × Cos ($\mathfrak{A} - \mathfrak{B}$)/2;

Atqui 1 = 2sin 30^0 & $\sqrt{3}$ = 2sin 60^0 : erit

$Cos^2F = \frac{1+\sqrt{3}}{4} = \frac{Sin30^0+Sin60^0}{2} = Sin45^0 \times Cos15^0$;

adeoque

$\text{Log Cos}\, F = \frac{1}{2}(\text{LogSin}45^0 + \text{LogCos}15^0) = 9.9166977$, posito, ut in Tabulis solet, Sinus totius Logarithmo 10. Ergo F = 34^0 22' (**) fere, 3F seu AEC = 103^0 6', arcus convexus ACB = 206^0 11', 2F seu FAE = 68^0 44' = ADB arcui concavo. De caetero quia $\overline{\text{Sin}}^2 = 1 - \overline{\text{Cos}}^2$, erit $\text{Sin F} = \frac{1}{2}\sqrt{3-\sqrt{3}}$.

Coroll. 3. Latitudo Lunuale huius (§ 7) CD = $\sqrt{1+\sqrt{3}} - \sqrt{3}$. Et quia recta AB : AF :: 2SinF : 1, sit *Basis Lunulae* AB $= \sqrt{3}.\sqrt{3-\sqrt{3}} = \sqrt{9-3\sqrt{3}} = \sqrt{9-\sqrt{27}}$. Denique Area Lunulae =

$$2\,\Delta\,\text{AEF} = \frac{1}{2}\text{AB}\times\text{EF} = \frac{1}{2}\sqrt{3}.\sqrt{3-\sqrt{3}}.\sqrt{3+\sqrt{3}} = \frac{1}{2}\sqrt{6\sqrt{3}} = \frac{1}{2}\sqrt[4]{108} = \sqrt[4]{\frac{27}{4}}.$$

(*a*) EUCL. VI.13. (*b*) I.48.II.12 (*c*) VI.3.

(*) Inledu, til Trigon. Plana Prop.7 & quia anguli $\mathfrak{A}$, $(\mathfrak{A}+\mathfrak{B})/2$, $\mathfrak{B}$ sunt aequidifferentes.

(**) Hic angulus vel eum metiens arcus a KRAFFT (Geom.Subl.T.I §167) assumitur 34^0 16', unde in arcum ACB redundaret error circiter 35'.

CASUS III. Fig. 4. [p.15]

§XIX. Quando *m* : *n* :: 3 : 2, en aliud priori affine
PROBLEMA *Triangulum AEF constituere, cuius bina latera AF, AE sint in data ratione, atque angulus AFE ad basin duplus anguli ad verticem A* (*).

Analysis. Bisecante recta FG angulum AFE, est GFE = GFA = A, FGE = 2A = AFE, AG = GF; Tr: lum FGE sim. AFE ideoque
$\text{AF}+\text{FE} : \text{AE} :: (\text{FG}+\text{GE seu})\ \text{AE} : \text{EF};$ quare rectangulum
$\overline{\text{AF}+\text{FE}}\times\text{FE} = \text{AE}q.$ Assumpta igitur rectarum AF AE alterutra pro lubitu, quo ipso ob datam (hyp.) earum rationem dabitur & altera: datur dictum rectangulum magnitudine ; quippe (dem.) = AE*q* dato. Eo itaque reducta est quaestio, ut inveniantur latera FE & AF + FE rectanguli magnitudine dati, quarum datur differentia AF. Unde haec deducitur.

Constructio. Iungantur ad angulos rectos vel datae vel in data ratione sumendae rectae AF AE'; Bisecta AF in H, in HF producta capiatur HU = HE', tum centris A & F intervalis AE' & FU describendi circuli, si sibi occurrant in puncto aliquo E, obtinetur quaestum Triangulum AFE. Vel si placet, iungatur HE, a qua abscindatur HK = HA, tumque ex rectis AF AE' E'K (super AE' tanquam basi) construetur par Triangulum A*f*E'.

Nam si centro H radio HU = HE', super UA producta descriptus concipiatur semicirculus UEO; erit [p.16] (*d*) $\text{AU}\times\text{AO} = \text{AE}q$ i.e. (constr.) AO = FU = FE,
$\overline{\text{AF}+\text{FE}}\times\text{FE} = \text{AE}q,$ ut oportuit. Sic & (*e*) sequitur esse $\overline{\text{A}f + f\text{E}'}\times f\text{E}' = \text{AE}^2$.
Caeterum identitas Tr:lorum AFE A*f*E' seu consensus constructionis utriusque vel inde mox patet, quod utraque ipsarum E'K (seu E'f) & FU (seu FE) sit aequalis differentiae ipsarum HE' & HA. Pari simplicitate se commendant binas hae constructiones, nisi forte quod vel prior vel posterior tantillum praeferenda videatur, prout super vel recta positione data vel AF vel AE' construendum sit desideratum Triangulum. Intelligitur ex iam dictis quomodo

(*) Hoc problemate tanquam magis generali comprehenditur EUCL.Elem.L.IV.Prop.10, ubi ratio laterum ponitur ratio aequalitatis. (*d*) cfr. EUCL.II.14. (*e*) ex III.16.36.

§XIX.

Construenda sit Lunula, cuius angulus convexitatis sit ad angulum concavitatis ut 3 : 2.

Ductis ad angulum rectum duabus rectis aequalibus AL LN longitudinis arbitrariae, sumatur in LN producta ipsi AN aequalis LF; iunctae AF fiat perpendicularis AE' = AN. Bisecta AF in H, in HF producta capiatur HU = HE'; dein centro F radio FU & centro A radio AN vel AE' describantur circuli, qui se mutuo secabunt (*) in E. Denique centris F, E per describantur circuli: horum arcus, qui ab F versus E cadunt, *Lunulam* ACBDA desideratam comprehendent (**). [p.17] Est enim AE*q* = LF*q* = AN*q* + LN*q* = 2AL*q*, & AF*q* = AL*q* + LF*q* = 3AL*q*; adeoque AF*q* : AE*q* :: 3 : 2. Praeterae ex constructione collata cum §20 sequitur esse ang. AFE = 2FAE seu AEC : AFE :: 3 : 2 :: AF*q* : AE*q*. Constat ergo propositum (§§12.10).

Coroll. 1. Quia (dem)

$\mathrm{FL}^2 = AN^2 = 2ALq, AF^2 = 3AL^2, \mathrm{HU}^2 = \mathrm{HE'}^2 = \mathrm{HA}^2 + \mathrm{AE'}^2 = \frac{1}{4}\mathrm{AF}^2 + \mathrm{AN}^2$

$= \frac{1}{4}\mathrm{AL}^2 + 2\mathrm{AL}^2 = \frac{11}{4}\mathrm{AL}2, \mathrm{HF}^2 = \frac{1}{4}\mathrm{AF}^2 = \frac{3}{4}\mathrm{AL}^2$:

erunt $\mathrm{AF} = \mathrm{AL}.\sqrt{3}, \mathrm{AN}$ seu $AE = \mathrm{AL}.\sqrt{2}, \mathrm{HU} = \mathrm{HF}$ *i. e.* FU seu $\mathrm{EF} = \mathrm{AL}.\frac{\sqrt{11}-\sqrt{3}}{2}$*;* adeoque *AF, AE* , *EF* inter se ut $2\sqrt{3}, 2\sqrt{2}, \sqrt{11}-\sqrt{3}$; seu ut $\sqrt{12}, \sqrt{8}, \sqrt{11}-\sqrt{3}$*;* & quadrata earum ut

$12, 8, \&(11+3-2\sqrt{33} =)14-2\sqrt{33}(=14-\sqrt{132}) < (14-\sqrt{121} = 14-11 =)3.$ Ergo quia 12 > 8 + 3, a fortiori erit $\mathrm{AF}^2 > \mathrm{AE}^2 + \mathrm{EF}^2$; consequenter ang. AEF obtusus, AEC acutus & arcus convexus ACB semicirculo minor.

Coroll. 2 Ob Isosceles Tr. GAF, est $Cos\mathrm{FAE} = \frac{\mathrm{AF}}{2\mathrm{AG}} = (\mathrm{c})\frac{\mathrm{AF+FE}}{2\mathrm{AE}} = (\mathrm{Cor.1})\frac{\sqrt{11}+\sqrt{6}}{8}$, cui Cosinui respondent 26^0 49' circiter. Hinc 2FAE seu AFC = 53^0 38' & 3FAE vel AEC = 80^0 26'; arcus convexus ACB = 160^0 53', concavus ADB = 107^0 16'. De caetero prodit $\sin FAE = \frac{\sqrt{9-\sqrt{33}}}{4}$.

Coroll. 3. *Posito radio convexitatis AE = 1 : sit radius concavatatis* $\sqrt{\frac{3}{2}}$ *; distantia* [p.18] *centrorum* $EF = \frac{\sqrt{11}-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{22}-\sqrt{6}}{4}$ *;Latitudo Lunulo* (§7)

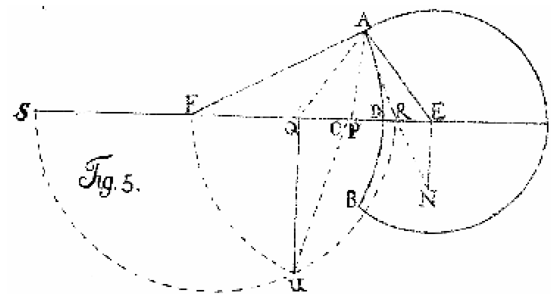
$CD = 1 - \frac{3\sqrt{3}-\sqrt{11}}{2\sqrt{2}} = 1 - \frac{3\sqrt{6}-\sqrt{22}}{4}$ *; Basis*

$AB = (2AF \times \sin AFE = 4AF.Sin\frac{1}{2}\mathrm{AFE}.\mathrm{Cos}\frac{1}{2}\mathrm{AFE} = \mathrm{Cor.2})\frac{3+\sqrt{33}}{8}.\sqrt{9-\sqrt{33}}$; *Area* =2 Triang.AFE = $\frac{1}{2}AB \times EF = \frac{1}{8}\sqrt{6(9-\sqrt{33})}$.

(**) Sponte patet eandem seu geminam huic Lunulam mox obtineri, si centris U, F. intervallis FL, FA describaritur versus FU circuli, adeo ut inventione puncti E seu constructione Tr:li AEF opus non sit, quam ideo tantum adhibuimus, ut §phus haec praecedenti melius responderet.

(*c*) EUCL.Elem.VI.3.

CASUS IV. Fig. 5.



§XXI. Esto $m : n :: 5 : 1$ seu ang FAE = 4AFE, unde exsurgit hoc PROBLEMA. *Data ratione laterum AF:AE =* a:b, *construere Tr:lum AFE, cuius angulus FAE ad verticem sit quadruplus anguli AFE ad basin EF.*

[p.19] *Analysis.* Rectae AQ, AP, AR dividant ang FAE in quatuor partes inter se adeoque (hyp.) singulas angulo AFE aequales : erit sic (*d*) AQ = FQ, AR = QR; Tr.
AER **S** FEA & Tr. AEP **S** QEA, unde

∺ EF, EA, ER & ∺ EQ, EA, EP, ideoque (*e*) *EF:EP::EQ:ER* :: (EF - EQ : EP - ER, i.e.) $FQ : PR :: a+b : b :: FR \times PR$, quae igitur ratio datur. At, ob Tr. ARP **S** FRA, est $FR \times PR = (AR^2 =) QR^2$. Datur itaque ratio $FR \times PQ : QR^2 = a+b : b$. Hinc, si QR pro arbitrio assumatur ut data, datur rectangulum $FR \times FQ$ magnitudine, & laterum differentia QR, unde (cfr. §19) dabuntur ipsa, & denique tota figura atque Triang. FAE.

Constructio. Statuantur rectae *QR = a & RE = b* in directum; in *EQ* producta capiatur *QS* quarta proportionalis ipsis ER, RQ, EQ ; super diametro RS fiat semicirculus, qui in U secet rectam ipsi RS in Q perpendicularem. Bisecta QR in O, capiatur OF = OU. Denique centro Q radio QF atque centro R radio RQ describantur circuli, quorum intersectio sit A (fecabunt autem sese hi circuli si fuerit FQ < 2QR) : tum iunctis AF, AE, vel AR, AE : Triangullum AEF vel REA proposito satisfaciet.

Nam (*f*)

$$SQ \times QR = QU^2 = OU^2 - OQ^2 = OF^2 - OQ^2 = \overline{OF + OQ} \times \overline{OF - OQ} = FR \times FQ.$$

At $SQ \times QR : QR^2 :: SQ : QR ::$ (constr.) $EQ : ER :: a+b : b.$ [p.20] Ergo

$FR \times FQ : QR^2 :: a+b : b$ ut oportuit. Caetera ex analysi patent. Iam facile erit

(*d*) EUCL.El.I.5.32. VI.4. (*e*) vel per VI.3.V.18. (*f*) II.14.

§XXII.

Construere Lunulam quadrabilem havituram angulum convexitatis quintuplum anguli concavitatis.

Nimirum rectae ER pro lubitu sumendae fiat perpendicularis EN = 2ER, & in ER producta, capiatur RQ = RN. Datis sic QRE punctis, determinentur, ut in § praecedenti, puncta F, A (*). Tum centris E & F describendi per A circuli formabunt (intra angulum AFE & huic geminum atque ab altera rectae FE parte similiter positum BFE) requisitam *Lunulam* ACBDA. Etenim

$QR^2 = RN^2 = ER^2 + EN^2 = ER^2 + 4ER^2 = 5ER^2$; atque angul. ABC = AFE + FAE = (§21) AFE + 4AFE = 5AFE. Ergo (§13) Lunula = 2Tr.AFE.

Coroll. 1. Posito ER = 1, est AR = QR = $\sqrt{5}, \frac{EQ.QR}{ER}$ seu $QS = 5+\sqrt{5}, RQ.QS = 5+5\sqrt{5} = QU^2, OU^2 = QU^2 + QO^2 = QU^2 + \frac{1}{4}QR^2 = \frac{25}{4} + 5\sqrt{5} = OF^2, OF - \frac{1}{2}QR$ seu $FQ = \frac{1}{2}\sqrt{25+20\sqrt{5}} - \frac{1}{2}\sqrt{5} = \frac{1}{2}\sqrt{5}(\sqrt{5+4\sqrt{5}} - 1),$

$ER + \frac{1}{2}QR + OF = 1 + \frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{25+20\sqrt{5}} = EF \times ER = AEq.$

Sunt itaque Δli ARE latere ER AR AE, vel illi similis Δli FAE latera *AE AF EF* inter se ut $1, \sqrt{5}, \sqrt{1+\frac{1}{2}\sqrt{5}+\frac{1}{2}\sqrt{25+20\sqrt{5}}}$ respective.

Coroll. 2. Quia (Cor.1.) $EF^2 : AE^2 + AF^2 :: 1 + \frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{25+20\sqrt{5}} : 6;$ atqui $\frac{1}{2}\sqrt{5} > (\frac{1}{2}\sqrt{4} =) 1, \& \frac{1}{2}\sqrt{25+20\sqrt{5}} > (\frac{1}{2}\sqrt{25+20.2}) = \frac{1}{2}\sqrt{65} > \frac{1}{2}\sqrt{64} = \frac{8}{2} =) 4$: erit $EF^2 > AE^2 + AF^2$, adeoque ang FAE multoque magis AEC obtusus, & arcus convexus ACB semicirculo maior. Atque ob FE > FA vel FD, cadet centrum E intra ipsam Lunulam, secus quam in casibus superioribus.

Coroll. 3. Cum in Tr:lo aequicruro ARQ sit Cos RQA $= \frac{AQ}{2QR} = \frac{FQ}{2QR}$: prodit (Cor.1.) huius anguli $= 2AFE = \frac{1}{2}FAE = \frac{2}{5}AEC.$ Cosinus $= \frac{1}{4}\sqrt{5+4\sqrt{5}} - \frac{1}{4} = \text{Cos } 46^0 52'$ circiter. Sunt itaque proxime arcus convexus ACB = 234^0 20' & concavus ADB = 46^0 52'.

Coroll. 4. Assumto ex gr. radio convexitatis AE = 1: exprimi possent, numeris quidem surdis exacte, rationalibus autem quam proxime, huius etiam Lunulae Latitudo, Basis atque Area.

(*) Per *Cor.* 1, est $FQ : 2QR :: \sqrt{5+4\sqrt{5}} - 1 : 4.$ At $4\sqrt{5} = \sqrt{80} < 9$, quare $\sqrt{5+4\sqrt{5}} < (\sqrt{14} <) 4,$ ideoque $\sqrt{5+4\sqrt{5}} - 1 < 3 < 4.$ Ergo FQ < 2QR, consequenter (§ praeced.) circuli, suo occusu punctum A definituri, se omnino secabunt.

CASUS V. Fig. 6.

§XXIII. Denique casus *m* : *n* :: 5 : 3 resolvitur in hoc [p.22]

PROBLEMA. *Exhibere Triangulum AEF, cuius angulus EFA ad basin EF sit sesquialter anguli EAF ad verticem, data ratione laterum AF:AE = a:b.*

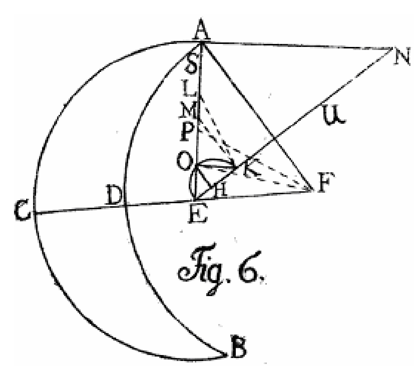


Fig. 6.

Analysis. Puta factum & angulum AFE rectis FO FP divisum in tres partes aequales, singulas ergo (hyp.) $= \frac{1}{2}$EAF. Erit sic ang AFO = OAF = EFP, OF (*g*) = OA, Tr. EAF **S** EFP ideoque *AF:AE* ::PF:FE:: (*h*) *PO:OE :: a:b.* Praeterea (*h*) OF seu AO:AF :: PO:PA. Hinc (*i*) $AO : AE :: POq : EO \times AP.$ Si iam EO, OP, quarum (dem.) datur ratio, ponantur datae ; datur ipsis tertia proportionalis, quae sit PS, atque punctum S; & ob (*k*) $OPq = EO \times PS$, erit (dem.) $AO : AE :: (EO \times PS : EO \times AP ::) PS:AP$ ideoque $AO : (AE - AO =) EO :: PS : (AP - PS =) AS$, quare $AO \times AS = EO \times PS = OPq.$ Datur ergo magnitudine rectangulum sub rectis AO & AS, quarum (dem.) datur differentia OS; unde (cfr.§19) dabuntur ipsae, A punctum, OF = OA, AE adeoque (ob datam rationem AF:AE,) *AF*, vel & EF, quippe quae (ob Tr. EFP **S** EAF) erit media proportionalis inter datas AE & EP, consequenter F punctum & Tr:lum AEF vel FEP. Hinc ratio patet *Constructionis*, quam ad specialem nostrum casum statim aptabimus, videlicet

(*g*) EUCL.I.5. (*b*) VI.3. (*i*) VI.22.1. (*k*) VI.17.

§XXIV. [p.23]

Lunula quadrabilis, cuius angulus convexitatis sit ad angulum concavitatis ut 5 : 3, describi hoc modo poterit: In recta indefinata EU sumantur EH:HK ::3:5; super EK, ad quam erigantur perpendiculares HO & KL, describatur semicirculus cui occurrat HO in O; per puncta EO ducatur recta indefinita EOLA, in qua capiantur ipsi OK aequales OP, LS; bisecetur OS vel PL in M & abscindatur MA = MK; per punctum A sic inventum fiat ipsi AE perpendicularis vel (*l*) ipsi OK parallela, occursura ipsi EU in N; tum centris O & A, radiis OA & AN describantur circuli; hi occursu suo (*) definient punctum F; centris denique EF describendi per A circuli formabunt Lunulam desideratam ACBDA.

Nam OPq = OKq = (m) $EK \times KH$, & $EOq = EK \times EH$, ideoque OPq : OEq :: KH : HE :: 5 : 3. Porro (n)
∺ EO, OK *v*el OP, OL atque (ob LS = OP) PS = OL, ergo ∺ EO, OP, PS,
& (o) $\mathrm{OA} \times \mathrm{AS} = (\mathrm{MA}q - \mathrm{MO}q = \mathrm{MK}q - \mathrm{MO}q = \mathrm{OK}q =)\ OPq$;
denique AE : AF (= AN) :: EO : (OK=) OP, & OF = OA: quare (§praec.) ang EFA $= \frac{3}{2}$ EAF. Ergo ang AEC: AFE :: 5:3 :: (dem. OPq : OEq ::) AFq : AEq, quod erat faciendum (§13) [p.24]

Coroll. 1. In aequicruro Triangulo AOF est Sinus totus 1: 2 Cos EAF :: AO vel OF : AF :: (p) OP : AP = AO – OP unde 1 + 2 Cos EAF = $\frac{\mathrm{AO}}{\mathrm{OP}} = \frac{\mathrm{AO}}{\mathrm{OK}} = \frac{\mathrm{OM+MK}}{\mathrm{OK}} =$

$$\frac{\mathrm{OM}+\sqrt{\mathrm{OM}q+\mathrm{OK}q}}{\mathrm{OK}} = \frac{\mathrm{OS}+\sqrt{\mathrm{OS}q+4\mathrm{OP}q}}{2\mathrm{OP}} =$$

(quia EO : OP :: OP : PS :: EO + OP : OP + PS seu OS adeoque $\frac{\mathrm{OS}}{\mathrm{OP}} = \frac{\mathrm{EO+OP}}{\mathrm{EO}}$,)

$\frac{1}{2} + \frac{\sqrt{(\mathrm{EO+OP})q+4\mathrm{EO}q.}}{2\mathrm{EO}}$ Hinc, quia EO : OP :: $\sqrt{3} : \sqrt{5}$, prodit Cos EAF =

$\frac{\sqrt{5}+\sqrt{20+2\sqrt{15}}}{4\sqrt{3}} - \frac{3}{4} = \frac{\sqrt{15}+\sqrt{60+6\sqrt{15}}-3}{12} = 0.8330387$, quare EAF = 33^0 35' 16'', AFE = 50^0 22' 54'', AEC = 83^0 58' 10''.

Coroll. 2. Valet hic quoque Cor. 4 §22.

(*) Per Coroll. 1 est AF seu AN : 2AO :: $\sqrt{15} + \sqrt{60+6\sqrt{15}} - 3 : 12$. At $\sqrt{15} < 4$ ideoque $\sqrt{15} + \sqrt{60+6\sqrt{15}} - 3 < (4 + \sqrt{84} - 3 = 1 + \sqrt{84} < 1 + 10 = 11 < 12.$ Ergo AN < 2AO, quapropter secabunt se dicti circuli. [p.25]

(*l*) EUCL.III.31. (*m*) III.31. VI.8.17 (*n*) VI.8 (*o*) II.6 (*p*) VI.3.

§XXV.

Habemus ergo quinque Lunulas quadrabiles, recta & circulo construendas. Has eo ordine proposuimus, quo minores sunt numeri rationem $m:n$ exprimentes, atque I :mam, II:dam, III: tiam, IV:tam, V:tam vocabimus. Quod si autem ratio habenda sit quantitatis $\frac{m}{n}$, quae exponens est proportionis m : n & *Denominator Lunulae* quadrabilis dici posset : hoc ordine III, V, I, II, IV essent collocandae. Ullam III:tiae, IV: tae & V: ae mentionem facam non vidimus. Caeterum ut quinque harum Lunularum comparatio, qua singulas dimensiones, eo facilius, si placet, instituatur: subiungimus hoc loco Tabellam, quae istas dimensiones numeris prope veris exhibet, posito quidem radio convexitatis AE omnibus illis Lunulis communi & = 1.

[p.26]

Lunulae	I.	II.	III.	IV.	V.
Arcus convexus ACB	180^0	206^0 11'	$160^0$53'	$234^0$20'	$167^0$56'
Arcus concavus ADB	90^0	$68^0$44'	$107^0$16'	$46^0$52'	$100^0$46'
Radius concavit. AF	1.414	1.732	1.225	2.236	1.291
Distantia Centror. EF	1.000	1.651	0.560	2.509	0.718
Basis AB	2.000	1.948	1.960	1.779	1.989
Latitudo CD	0.586	0.919	0.336	1.273	0.427
Area	1.000	1.613	0.552	2.232	0.714

§XXVI. [p.27]

Quoniam (El. Trig. Pl.) in omni Tri:lo rectilineo ut AEF : AE :: Sin AEC : Sin AFC : si generaliter definiri possent bini anguli qui sint in ratione duplicata Sinuum suorum, haberentur (§§ 10.12) innumerae Lunulae circulares quadrabiles. Permultum vero abest ut hoc in quovis casu assignabili praestari queat. Quod si tamen proposita fuerit ratio (§13) *m : n,* angulorum scilicet convexitatis & concavitatis, rationalis : (sunto etiam *m, n* numeri integrari inter se primi) : eatenus solubile est Problema, ut exhiberi posset aequatio solitaria seu determinanda & quidem finita seu Algebraica, a cuius constructione pendebit ipsa Lunulae delineatio. Cum enim notantibus $\mathfrak{A}$ & $\mathfrak{B}$ angulos vel arcus circulares, atque *s* ipsius $\mathfrak{A}$ Sinum, *z* Cosinum,
sit (*) $\text{Sin}(\mathfrak{A}+\mathfrak{B}) = s.\text{Cos}\mathfrak{B} + z.\text{Sin}\mathfrak{B}$, & $\text{Cos}(\mathfrak{A}+\mathfrak{B}) = z.\text{Cos}\mathfrak{B} - s.\text{Sin}\mathfrak{B}$, : patet, ponendo successive $\mathfrak{B} = \mathfrak{A}, 2\mathfrak{A}, 3\mathfrak{A}$, &c. prodituros Sinus atque Cosinus ipsorum $2\mathfrak{A}, 3\mathfrak{A}, 4\mathfrak{A}$ &c. scilicet anguli dupli, tripli, quadrupli & demum utcunque multipli per *s* & *z*, nempe Sinum & Cosinum anguli simpli, expressos (†). Et quidem, quod indicasse tantum sufficiat, has sic obtinendas Sinuum pro angulus multiplis expressiones ingrediuntur potestates ipsius *s* non nisi impares. quamobrem, cum quaelibet potestas impar proveniat ex multiplicatione ipsius *s* in potestatem eius proxime [p.28] inferiorem adeoque parem, & omnis potentia eius par sit etiam dignitas (stricte talis, scilicet integra, non fracta) ipsius *ss*, pro quo *ss* substitui poterit $1 - zz$: sequitur, Sinus angulorum multiplorum $m\mathfrak{A}$ & $n\mathfrak{A}$ hae forma sisti posse :
$\text{Sin}\, m\mathfrak{A} = Z.s$ & $\text{Sin}\, n\mathfrak{A} = Z'.s$ (**) ita ut $Z = \text{Sin}\, m\mathfrak{A}/s$ & $Z' = \text{Sin}\, n\mathfrak{A}/s$ sint ipsius *z* functiones & quidem rationales atque integrae. Statuatur iam angulorum quaerundorum AEC, AFC mensura (hyp.) communis $\mathfrak{A}$, seu AEC = $m\mathfrak{A}$ &

AFC = $n\mathfrak{A}$: vi eorum, quae dicta funt, requiritur ut $Z : Z' :: \sqrt{m} : \sqrt{n}$; ideoque (Æ) $Z\sqrt{n} = Z'\sqrt{m}$, quae est aequatio desiderata unicam quantitatem incogitam z insolvens, &, si construi queat, constructa dat z Cosinum anguli $\mathfrak{A}$; quo ipso datur $\mathfrak{A}$, acutus scilicet sumendus ne fiat $m\mathfrak{A}$ maior duobus rectis, quamobrem reiiciendae sunt istius aequationis radices negativae. Dantur itaque hoc pacto etiam ipsius $\mathfrak{A}$ multipli $m\mathfrak{A}$, $\overline{m-n}.\ \mathfrak{A}$ i. e. AEC,EAF, quarum alteruter una cum ratione data AF: AE = $\sqrt{m}:\sqrt{n}$ definit speciem Tr:li AEF, adeo ut Lunula quaesita iam constructu sit facilis. Enimvero aequatio Æ praeterquam in paucis neque, puto, pluribus quam quinque illis casibus simplicioribus supra pertractatis, vel cubica vel superioris adhuc gradus fieri deprehenditur; proinde difficilius, atque regula tantum & circino omnino non, construenda. [p.29]

(*) Vid. passim Scriptores Trigonometiae, ex. gr. De la CAILLE Elem.Math.§.742, cfr. sis, Insedn.til Trigonom.Plan.prop.4.

(†) Lex generalis sic exprimendi Sinum vel Cosinum anguli quamcunque multipli, passim exposita reperitur, ut in WOLFII Elem,Math.T.1.p.m.322. KÆSTNER Anal.endl.Grössen.§176.

(**) Sic reperiuntur Sin $2m\mathfrak{A}/s = 2z$, Sin $3m\mathfrak{A}/s = 4zz - 1$,
Sin $4m\mathfrak{A}/s = 8z^3 - 4z$, Sin $5m\mathfrak{A}/s = 16z^4 - 12zz + 1$, Sin $6m\mathfrak{A}/s = 32z^5 - 32z^3 + 6z$, & sic porro.

§XXVII.

Ut haec (§26) Exemplis illustrentur: sunto

I: mo $m : n :: 2 : 1$, erit tunc (§ cit. not. **) Z = 2z, Z' = 1, unde aequat. III. $2z = \sqrt{2}$ seu $z = \sqrt{\frac{1}{2}}$ =Cos 45^0.

II: do $m : n :: 3 : 1$. quare $Z = 4zz - 1$, $Z' = 1$, adeoque $4zz - 1 = \sqrt{3}$, unde $z = \frac{1}{2}\sqrt{1+\sqrt{3}}$ ut supra §18.Cor.2.

III: tio $m : n :: 3 : 2$ ut denuo $Z = 4zz - 1$, at Z' = 2z, hinc $\overline{4zz-1}.\sqrt{2} = 2z\sqrt{3}$, seu $zz - \frac{z\sqrt{3}}{2\sqrt{2}} = \frac{1}{4}$, cuius aequationis radicem affirmativam sumendo, [p.30] sit $z = \frac{\sqrt{11}+\sqrt{3}}{4\sqrt{2}}$ ut antea §20. Cor.2.

IV: to $m : n :: 5 : 1$; reperitur aequatio $16z^4 - 12zz + 1 = \sqrt{5}$, cuius radices affirmativae sunt $\frac{1}{2}\sqrt{\frac{3}{2} \pm \frac{1}{2}\sqrt{5+4\sqrt{5}}}$; quarum posteriore , quia imaginaria, reiecta, est $z = \frac{1}{2}\sqrt{\frac{3}{2} + \frac{1}{2}\sqrt{5+4\sqrt{5}}}$ = Cos$\mathfrak{A}$;unde etiam $2\text{Cos}^2\mathfrak{A} - 1$ seu cos$2\mathfrak{A} = \frac{\sqrt{5+4\sqrt{5}}-1}{4}$, ut supra. §22.Cor.3.

V: to $m : n :: 5 : 3$ dat aequatio $16z^4 - (3 + \sqrt{\frac{5}{3}})zz + 1 + \sqrt{\frac{5}{3}} = 0$ biquadraticam, sed pariter, instar quadraticae, resolubilem.

Ast si $m : n :: 4 : 1$, sit $4z^3 = 2z + 1$.

Si $m : n :: 4 : 3$; est $4z^3\sqrt{3} - 4zz = 2z\sqrt{3} - 1$.

Si $m : n :: 5 : 2$; prodit $16z^4 - 12zz = 2z\sqrt{\frac{5}{2}} - 1$.

Si $m : n :: 5 : 4$, oritur aequatio biquadratica, cuius nullus terminus deest.

Si $m : n :: 6 : 1$, Æ erit aequatio quinti gradus.

§XXVIII.

Difficilius autem evadit hoc Problema de invenienda Lunula Circula quadrabili, si data fuerit non ratio $m : n$ sed alia aliqua conditio, ex. gr.[p.31] angulus aut convexitatis aut concavitatis aut Lunulae, vel ratio baseos ad latitudinem, &c. Verum his immorari nil attinet. Claudat igitur opellam nostram sequens observatio : *Lunulae Oppositae* (§9) *simul quadrabiles esse nequeunt*, dum defineratur quadratura circuli. Si enim daretur quadratura Lunulae utriusque, data eo ipso foret quadratura differentiae earum, quae cum sit ipsa integrorum circulorum differentia (adjecto enim utrique Lunulae seorsim spatio eas interiacente, prodeunt ipsi circuli) atque facile (*) describi queat tertius circulus aequalis binorum illorum circulorum differentiae : daretur omnino tertii huius circuli quadratura, contra hypothesin. Quid sentiendum sit, si evanuerit dicta differentia, seu Lunulae fuerent aeqicurvae : verbo antea (§10) monuimus.

(*) Cfr. EUCL.El.XII.2 & VI.31.

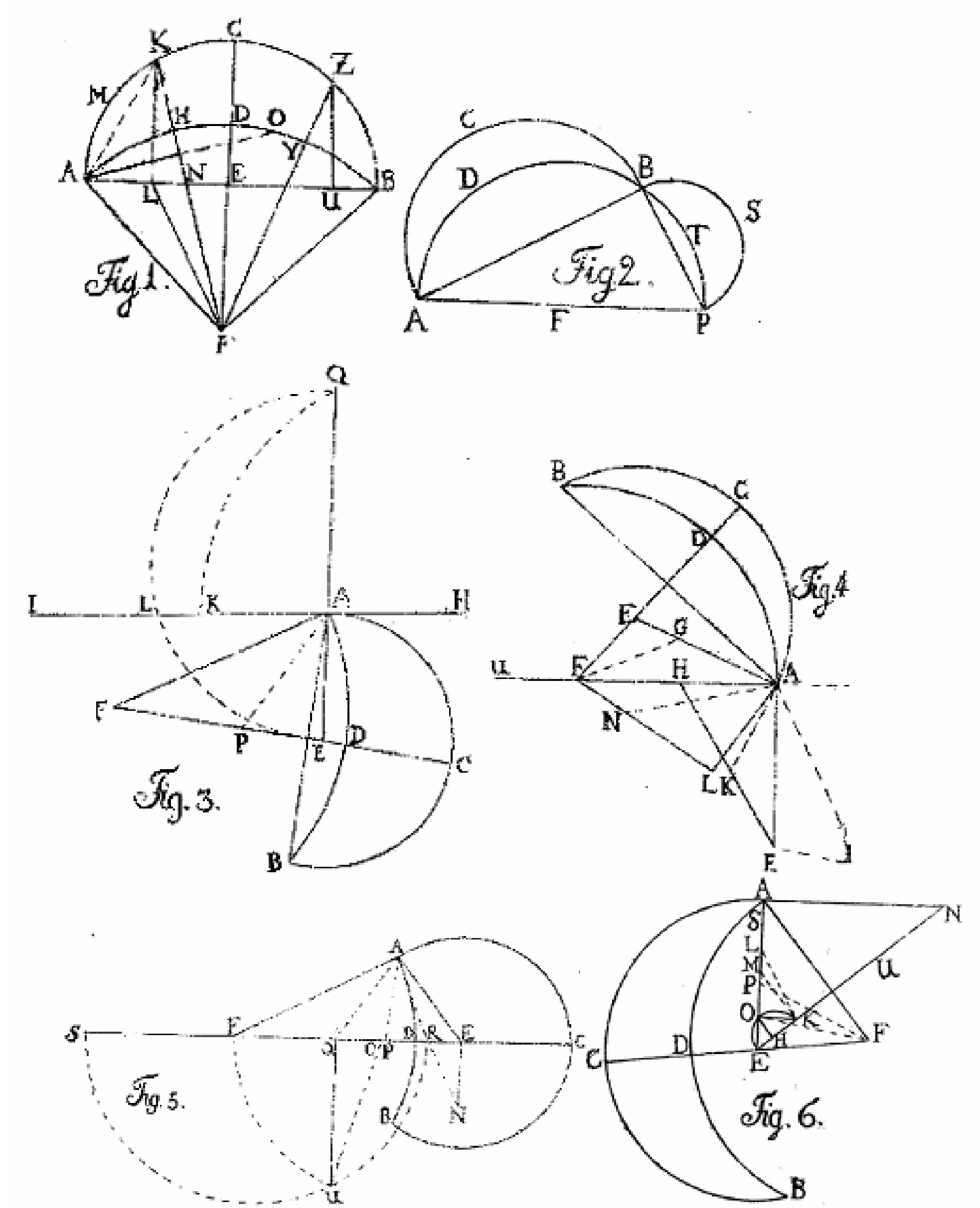


Fig.1.
Fig.2.
Fig.3.
Fig.4
Fig.5.
Fig.6.